ID5004: AI in Predictive Maintenance, Reliability, and Warranty : End Sem

Part 1

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Part 2

Q1 **Consider the deterministic system, where**

$$x_k = A_{k|k-1}x_{k-1} + B_{k-1}u_{k-1}\ ,$$

where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

It is desired to take the state from $\boldsymbol{X_o}$ to $\boldsymbol{X_f}$, where

$$X_0 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix},\quad X_f = \begin{pmatrix} 6 \\ -8 \\ 2 \end{pmatrix}$$

If the input sequence is begun at **step 0**, and the system is completely controllable, then ;

(a) How many steps are required to move the system to the **desired** ***state***? (*1 mark*)

(b) Obtain the equation, governing the input state, ***x(0)*** and the inputs, ***u(i)*** *(1 mark)*

[Hint: Write down the discrete steps for controllability]

(c) Calculate the ***inputs u(i),*** required to move the system to the ***desired state***. (*1 mark*)

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1\. a) $x_0 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ & $x_f = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix}$

$$\Delta x = x_f - x_0 = \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -1 \end{bmatrix}$$

~~Since Δx is represents the state's change after one step i.e. 1 the number of steps required to make the system desired state is N = 1~~

The number of steps required (N) is the maximum absolute change in any element of Δx

$$\therefore N = \max \left[6, |-7|, |-1| \right] = 7$$

So steps is 7 required.

b) $x(7) = A x(6) + B u(6)$

$x_f = A x(6) + B u(6)$

$$\therefore \begin{bmatrix} 6 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(6) \\ x_2(6) \\ x_3(6) \end{bmatrix} + \left[u(6) \right]$$

c) Now we can derive three equation,

$x_1(6) + u(6) = 6$

$2x_2(6) + u(6) = -8$

$3x_3(6) + u(6) = 2$

Solving we got, $u(6) = 6 - x_1(6)$

$u(6) = -8 - 2x_2(6)$

$u(6) = 2 - 3x_3$

Since all these expressions represents same value $u(6)$, we can equate & get

$x_1(6) = 6$ & $x_2(6) = -4$ & $x_3(6) = 2/3$

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so initial state $x(6) = \begin{bmatrix}0\\-1\\3\end{bmatrix}$

& input at step 6 is $u(6) = 6 - x_1(6) = 0$

Therefore the input $u(i)$ required to move the system to the desired state are $u(i) = 0$ for all i from 1 to 7.

Q2 (a) **Considering an overdetermined set of equations, represented by**

$$y = Hx + v$$

Fill in the blanks, for the expressions (*1 mark*)

(b) Using the expression, obtained above, find x, for the set of equations, given below *(1 mark)*

$3x_1 - x_2 = -4$

$2x_1 + x_2 = 1$

$x_1 - 2x_2 = -5$

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Recursive Least square estimation :

> Let us consider the following linear system defined by,

$$y = Hx + v \quad \ldots (i)$$

if we rewrite the equation (i) in matrix term it will be shown as,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & & \vdots \\ H_{n1} & \cdots & H_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

So, we also can write, Now we know that $J = v^T v$.

$$v = y - Hx ;$$

So, $J(\hat{x}) = v^T v = (y - H\hat{x})^T (y - H\hat{x}) \quad \ldots (ii)$

Expanding the equation (ii) we got,

$$J(\hat{x}) = y^T y - y^T H\hat{x} - \hat{x}^T H y + \hat{x}^T H^T H \hat{x}$$

Now, we know also that partial derivatives with respect to $\hat{x}$ of J must vanish in order for there to be minima.

$$\frac{\partial J}{\partial \hat{x}} = -2y^T H + 2\hat{x}^T H^T H = 0$$

$$\therefore \hat{x} = (H^T H)^{-1} H^T y$$ (this's equivalent to the formulation of pseudo inverse)

At a particular instance K, above equation can be written as,

$$\hat{x}(K) = \left[H^T(K) H(K)\right]^{-1} H^T(K) Z(K) \quad \ldots (iii)$$

The least square estimate based on $(K+1)$ measurements is,

$$H(K+1) = \left[\frac{H(K)}{h^T(K+1)}\right],$$ ∵ where $H(K+1)$ is $H(K)$ with an additional row $h^T(K+1)$

Similarly, $Z(K+1) = \left[\frac{Z(K)}{Z_{K+1}}\right]$ ∴ then,

$$H^T(K+1) H(K+1) = \left[H^T(K) \mid h(K+1)\right] * \left[\frac{H(K)}{h^T(K+1)}\right]$$

Let us define,

$P(K+1) = \left[H^T(K) H(K)\right]^{-1}$ $\Rightarrow H^T(K+1) H(K+1) = H^T(K) H(K) + h(K+1) h^T(K+1) \quad \ldots (iv)$

then equation (iv) can be rewritten as,

$$\rightarrow P(K+1) = \left[H^T(K+1) H(K+1)\right]^{-1} = \left[H^T(K) H(K) + h(K+1) h^T(K+1)\right]^{-1}$$

As, we know from inverse property,

$$(\Gamma + uv^T)^{-1} = \Gamma^{-1} - \frac{\Gamma^{-1} u v^T \Gamma^{-1}}{1 + v^T \Gamma^{-1} u}$$, Now applying the same property in $P(K+1)$ term

we got, $P(K+1) = \left[H^T(K) H(K)\right]^{-1} \quad \ldots (v)$

$$= P(K) - \frac{P(K) h(K+1) h^T(K+1) P(K)}{1 + h^T(K+1) P(K) h(K+1)} \quad \ldots (vi)$$

Defining, $\delta(K+1) = 1 + h^T(K+1) P(K) h(K+1) \quad \ldots (vii)$

$$\mathcal{K}(K+1) = P(K) h(K+1) \delta^{-1}(K+1)$$

Kappa

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Using the defined value, update equation will become,

$$P(K+1) = P(K) - P(K)\,h(K+1)\,\delta^{-1}(K+1)\,h^T(K+1)\,P(K)$$

$$= [I - \mathbf{K}(K+1)\,h^T(K+1)]\,P(K) \quad \text{--- (viii)}$$

since $\delta(K) = [(H^T(K)\,H(K))^{-1}\,H^T(K)\,Z(K)]$

then equation of step $(K+1)$ will become,

$$\hat{x}(K+1) = P(K+1)\,H^T(K+1)\,Z(K+1)$$

$$= P(K+1)\,[H^T(K) \mid h(K+1)] \begin{bmatrix} Z(K) \\ \hline Z_{K+1} \end{bmatrix}$$

Now applying $P(K+1)$ value we got,

$$\hat{x}(K+1) = P(K+1)\,[H^T(K)\,Z(K) + h(K+1)\,Z_{K+1}]$$

$$= [P(K) - P(K)\,h(K+1)\,\delta^{-1}(K+1)\,h^T(K+1)\,P(K)]$$

$$* [H^T(K)\,Z(K) + h(K+1)\,Z_{K+1}]$$

$$= P(K)\,H^T(K)\,Z(K) + P(K)\,h(K+1)\,Z_{K+1} - P(K)\,h(K+1)\,\delta^{-1}(K+1)\,h^T(K+1)\,P(K)\,H^T(K)\,Z(K)$$

$$- P(K)\,h(K+1)\,\delta(K+1)\,h^T(K+1)\,P(K)\,h(K+1)\,Z_{K+1}. \quad \text{--- (ix)}$$

Now,

$$\hat{x}(K+1) = \hat{x}(K) + \delta^{-1}(K+1)\,P(K)\,h(K+1)\,Z_{K+1} - P(K)\,h(K+1)\,\delta^{-1}(K+1)\,h^T(K+1)\,P(K)\,H^T(K)\,Z(K)$$

taking common on RHS we got,

$$\Rightarrow \hat{x}(K+1) = \hat{x}(K) + P(K)\,h(K+1)\,\delta^{-1}(K+1)\,[Z_{K+1} - h^T(K+1)\,\hat{x}(K)]$$

$$\therefore \hat{x}(K+1) = \hat{x}(K) + \mathbf{K}(K+1)\,[Z_{K+1} - h^T(K+1)\,\hat{x}(K)]$$

[Proved]

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Q3 Consider the System Dynamics, with uncertainties, given by

$$x_k = A_{k|k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}$$

Starting with the expression:

$$P_k = E[\tilde{x}_k\tilde{x}_k^T]$$

w.r.t. covariance minimization

(a) Obtain the expression for the a posteriori Covariance, by completing the in-between steps, in the blanks provided : *(1 mark)*

$$P_k = E\{[(I-K_kH_k)A_{k|k-1}\tilde{x}_{k-1}+K_kv_k][(I-K_kH_k)A_{k|k-1}\tilde{x}_{k-1}+K_kv_k]^T\}$$

$$P_k = P_k^- - K_kH_kP_k^- - P_k^-H_k^TK_k^T + K_k(H_kP_k^-H_k^T + R_k)K_k^T$$

(b) And obtain thereon the expression for Kalman Gain : (*1 mark*)

$$K_k = P_k'H^T\left(HP_k'H^T + R\right)^{-1}$$

Stating all the assumptions, w.r.t. the ***statistical properties of the noise***

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3. a) considering the system dynamics, with uncertainties

$$x_k = A_{k|k-1} x_{k-1} + B_{k-1} u_{k-1} + w_{k-1}$$

this equation can be written with one step ahead prediction,

$$x_{k+1} = A_{k+1|k} x_k + B_k u_k + w_k$$

output: $y_k = C x_k + v_k$

Also. $A \in R^{n \times n}$ & $B \in R^{n \times m}$ are the state & input matrices

$C \in R^{r \times n}$ is the output matrix

$y_k \in R^r$ is the output vector (observed measurement)

$v_k \in R^r$ is the measurement noise vector, we assume that v_k is also white in nature with zero mean & uncorrelated with covariance matrix defined by $E[v_k v_k^T] = R_2$

where, $k \to$ discrete time instant

$x_k \to \in R^n$ is the state vector at the discrete-time instant k

$u_k \to \in R^m$ is the control input vector

$w_k \to \in R^n$ is the disturbance or process noise vector & we assume that w_k is white in nature with zero mean & uncorrelated with covariance matrix given by $E[w_k w_k^T] = R_1$

For deterministic observer, open loop prediction equation will be given by,

$$\hat{x}_{k+1} = A\hat{x}_k + B u_k + L[y_k - C\hat{x}_k]$$

For stochastic observer,

a) Prediction equation :- $\hat{x}_{k+1|k} = A\hat{x}_{k|k} + B u_k$

b) Filter equation :-

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K[y_k - C\hat{x}_{k|k-1}]$$

Error equation,

a) Filter Error :-

$$e_k = x_k - \hat{x}_{k|k}$$

b) One step prediction Error :

$$\varepsilon_k = x_k - \hat{x}_{k|k-1}$$

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covariance matrix,

$$P_k^- = E\left[x_{e_k} x_{e_k}^T\right] \quad \& \quad P_k = E\left[e_k e_k^T\right]$$

both P_k & P_k^- are $n \times n$ symmetric matrices

Now if we subtract the plant equation & prediction eq. we get,

$$x_{k+1} - \hat{x}_{k+1|k} = A x_k - A\hat{x}_{k|k} + B u_k - B u_k + w_k$$
$$= A x_k - A\hat{x}_{k|k} + w_k$$

$$\therefore x_{e(k+1)} = A e_k + w_k$$

Now,

$$x_k - \hat{x}_{k|k} = x_k - \tilde{x}_{k|k-1} + K\left[C x_k + v_k - C\hat{x}_{k|k-1}\right]$$

After the problem statement H is used,

$$= x_k - \tilde{x}_{k|k-1} + K\left[H x_k + v_k - H\hat{x}_{k|k-1}\right]$$
$$= x_k - \tilde{x}_{k|k-1} - KH\left[x_k - \hat{x}_{k|k-1}\right] - K v_k$$

$\therefore$ Above equation can be written in terms of filter equation, writing this equation we get,

$$e_k = x_{e_k} - KH\, x_{e_k} - K v_k$$
$$\Rightarrow e_k = \left[I - KH\right] x_{e_k} - K v_k$$

Now we will work with below equation,

$$\left.\begin{aligned} x_{e_{k+1}} &= A e_k + w_k \\ e_k &= \left[I - KH\right] x_{e_k} - K v_k \end{aligned}\right] \Rightarrow \text{overall error propagating overtime}$$

The main objective of Kalman filter is to find the value K, such that the error is minimized in least square sense.

So far we know that, system dynamics equation with one step ahead value can be written as,

$$x_{k+1} = A x_k + B u_k + w_k$$

One step prediction equation will be,

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + B u_k$$

filter equation :-

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K\left[y_k - H\hat{x}_{k|k-1}\right]$$

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Now to achieve the goal we can either minimize $x_e x_e^T$ & ee^T; for finding the value of K_k = Kalman Gain

Now again we also know that,

$$x_{e_k} = A e_{k-1} + w_{k-1} \;\&$$

$$P^- = E[x_{e_k} x_{e_k}^T]$$

$$= E\{(Ae + w)(Ae + w)^T\}$$

$$= E\{A e e^T A^T\} + E\{w w^T\} + A E\{e w^T\} + E\{w e^T\} A^T$$

Both the term become zero as we consider w is zero mean gaussian white noise term

$$= E[A e e^T A^T] + E[w w^T]$$

We know from our definition statement

$$E[w_k w_k^T] = R_1$$

so,

$$= A\,E[e_{k-1} e_{k-1}^T] A^T + R_1$$

$$P_k^- = A P_{k-1} A^T + R_1$$

$$E[e_k e_k^T] = E\{[[I - KH] x_{e_k} - Kv][[I - KH] x_{e_k} - Kv]^T\}$$

$$= [I - KH] E\{x_{e_k} x_{e_k}^T\} [I - KH]^T + K E\{v v^T\} K^T$$

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Substituting the value we got,

$$= (I - KH) P_k^- (I - KH)^T + K R_2 K^T$$

$$\because P_k^- = E\{ x_{e_k} x_{e_k}^T \} \;\&\; R_2 = E\{ v v^T \}$$

$$\because P_k = E[e_k e_k^T] = P^- + KHP_k^- C^T K^T - K(P_k^- - P_k^- H^T K^T + K R_2 K^T)$$

$$= P_k^- + K_k[H P_k^- C^T + R_2] K_k^T - 2 P_k^- C^T K^T$$

$\therefore$ so

$$P_k = P_k^- - K_k H_k P_k^- - P_k^- H_k^T K_k^T + K_k (H_k P_k^- H_k^T + R_k) K_k^T$$

[considering $R_k = R_2$]

b) Now we know that

$[H P_k^{-1} H^T + R_k] \rightarrow$ It is a Hessian term ~~[illegible]~~

& $P_k^- C^T \rightarrow$ Gradient-Term (G)

We know that

K_k = Kalman Gain = Hessian^{-1} . Gradient

Substituting the value of Hessian & Gradient we got,

$$K_k = P_k^- H^T [H P_k^- H^T + R]^{-1}$$

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Q5 Considering the Bayesian Posterior, as indicated below,

Explain the steps to establish this relation, using the fundamental postulates of Baye's :

(1 mark)

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})}$$

Q5. According the problem statement, Bayes' fundamental postulates is given,

$$p(x_k|z_{1:k}) = \frac{p(z_k|x_k)\,p(x_k|z_{1:k-1})}{p(z_k|z_{1:k-1})}$$

$$\because\ p(x_k|z_{1:k}) = \frac{p(z_{1:k}|x_k)\,p(x_k)}{p(z_{1:k})}$$

separate $p(z_{1:k})$ into $p(z_k, z_{1:k-1})$

$$= \frac{p(z_k, z_{1:k-1}|x_k)\,p(x_k)}{p(z_k, z_{1:k-1})}$$

factorize joint probability according to below formula,

$p(a,b|c) = p(a|b,c)\cdot p(b|c)$ &

$p(a,b) = p(a|b)\cdot p(b)$

$$= \frac{p(z_k|z_{1:k-1}, x_k)\,p(z_{1:k-1}|x_k)\,p(x_k)}{p(z_k|z_{1:k-1})\,p(z_{1:k-1})}$$

$$= \frac{p(z_k|z_{1:k-1}, x_k)\,p(x_k|z_{1:k-1})\,\frac{p(z_{1:k-1})}{p(x_k)}}{p(z_k|z_{1:k-1})\,p(z_{1:k-1})\,p(x_k)}$$

$$= \frac{p(z_k|x_k)\,p(x_k|z_{1:k-1})}{p(z_k|z_{1:k-1})}$$

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Q.6 7, 8, 9 are done in the jupyter notebook.