# Reinforcement Learning: Bellman Equations and Dynamic Programming

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# (Recall) Returns

Suppose the sequence of rewards after step $t$ is :

$$r_{t+1}, r_{t+2}, r_{t+3}, \cdots$$

What do we want to maximize?

We want to maximize the **return**, $G_t$, for each step $t$.

**Episodic tasks**: interaction breaks naturally into episodes, e.g., plays of a game, trips through a maze.

$$G_t = r_{t+1} + r_{t+2} + \cdots + r_T,$$

where $T$ is a final time step at which a **terminal state** is reached, ending an episode.

# (Recall) Returns for Continuing Tasks

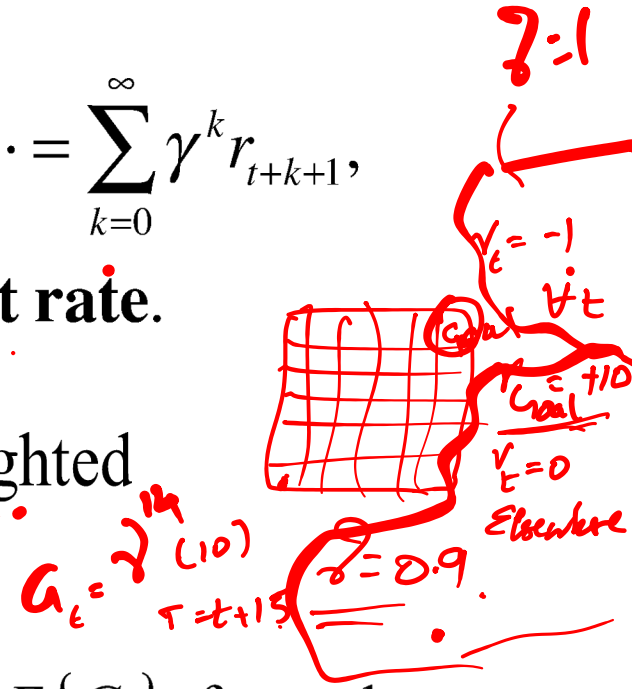
**Continuing tasks**: interaction does not have natural episodes.

**Discounted return**:

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1},$$

where $\gamma, 0 \le \gamma \le 1$, is the **discount rate**.

shortsighted $0 \leftarrow \gamma \rightarrow 1$ farsighted



In general,

we want to maximize the **expected return**, $E\{G_t\}$, for each step $t$.

# Value Functions

- Expected future rewards starting from a state (or state-action pair) and following policy $\pi$

**State-value function for policy $\pi$ :**

$$v_\pi(s) = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

**Action-value function for policy $\pi$ :**

$$q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a\right]$$

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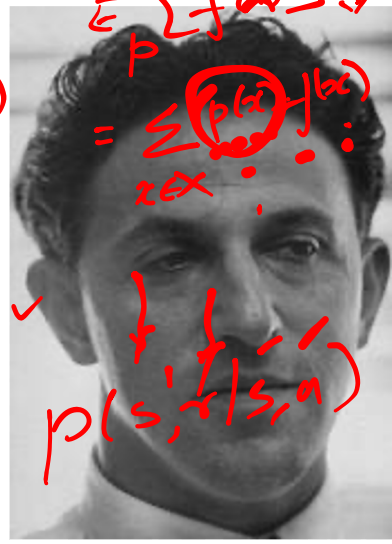
$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$$

# Bellman Equation for a Policy $\pi$

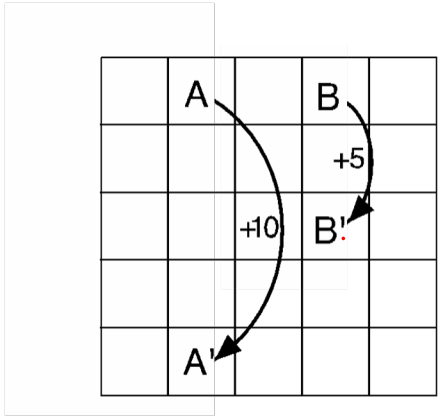
$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r|s, a) \Big[ r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s'] \Big] \\
&= \sum_a \pi(a|s) \sum_{s', r} p(s', r|s, a) \Big[ r + \gamma v_\pi(s') \Big], \quad \text{for all } s \in \mathcal{S},
\end{aligned}
$$

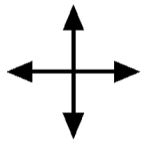


- Linear equation in $|S|$ variables
- Unique solution exists

# An Example

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = –1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.




| | | | | |
|---|---|---|---|---|
| 8.8 | 4.4 | 5.3 | 1.5 |
| 1.5 | 3.0 | 2.3 | 1.9 | 0.5 |
| 0.1 | 0.7 | 0.7 | 0.4 | -0.4 |
| -1.0 | -0.4 | -0.4 | -0.6 | -1.2 |
| -1.9 | -1.3 | -1.2 | -1.4 | -2.0 |

State-value function for equiprobable random policy; $\gamma = 0.9$

$v(s) = \frac{1}{4}[0 + \gamma 8.8] + \frac{1}{4}[0 + \gamma 0.7] + \frac{1}{4}[0 + \gamma 2.3] + \frac{1}{4}[0 + \gamma 1.5]$

$= 3.0$

# Solving RL problems

- Learn an optimal *policy* – a mapping from states to actions such that no other policy has a higher long term reward

# Solving RL problems

- Learn an optimal *policy* – a mapping from states to actions such that no other policy has a higher long term reward
- Can learn such a policy directly
- Or through estimating an optimal *value* function
- Optimal Value function: The estimated long term reward that you would get starting from a state and behaving optimally

# Optimal Value Functions

- For finite MDPs, policies can be partially ordered:

$\pi \geq \pi'$ if and only if $v_\pi(s) \geq v_{\pi'}(s)$ for all $s \in S$

$\pi_1 \geq \pi_2$ OR $\pi_1 \leq \pi_2$ $\forall \pi_1, \pi_2$

↳ Total order.

$\exists\ \pi_1 \& \pi_2$ s.t. $\pi_1 \not\geq \pi_2$ AND $\pi_1 \not\leq \pi_2$

Partial order.

$s' \leq s$

$s - s'$

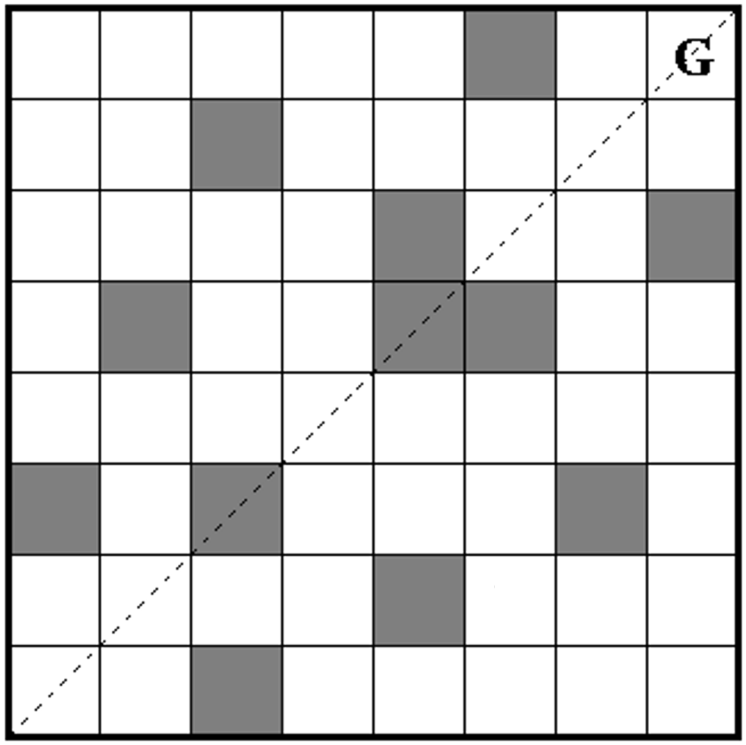
$v_{\pi'}(s) \not\geq v_\pi(s)$

$v_\pi(s) \geq v_{\pi'}(s)$

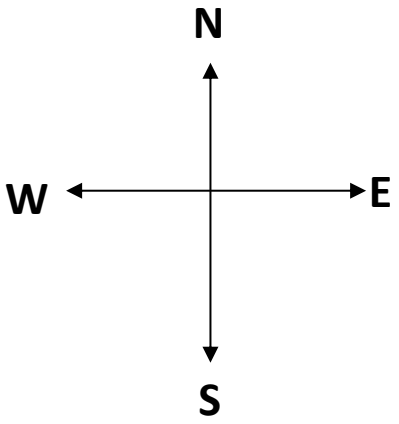
$\exists\ \pi^*$ s.t. $v_{\pi^*}(s) \geq v_\pi(s)$ $\forall s$ $\forall \pi$

$\Rightarrow \pi^* \geq \pi$ $\forall \pi$

# Example

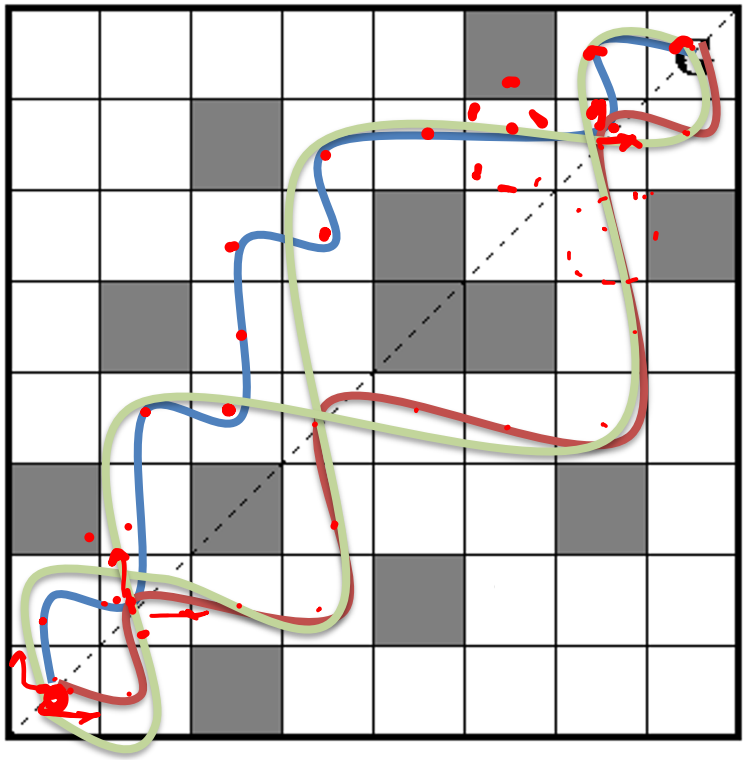



$$M = \langle S, A, p, r \rangle$$

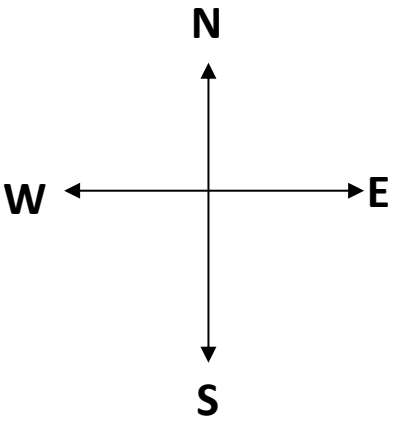


# Example

$r_t = -1$



$$M = \langle S, A, p, r \rangle$$

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Many optimal policies but only one optimal value function

# Optimal Value Functions

- For finite MDPs, policies can be partially ordered:

  $\pi \geq \pi'$ if and only if $v_\pi(s) \geq v_{\pi'}(s)$ for all $s \in S$

- There is always at least one (and possibly many) policies that is better than or equal to all the others. This is an optimal policy. We denote them all $\pi_*$
- Optimal policies share the same optimal state-value function:

$$v_*(s) = \max_\pi v_\pi(s) \text{ for all } s \in \mathcal{S}$$

$$q_*(s,a) = \max_\pi q_\pi(s,a) \text{ for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}$$

# Bellman Optimality Equation for $v_*$

The value of a state under an optimal policy must equal the expected return for the best action from that state:

$$
\begin{aligned}
v_*(s) &= \max_{a \in \mathcal{A}(s)} q_{\pi_*}(s,a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \left[ r + \gamma v_*(s') \right].
\end{aligned}
$$

# Bellman Optimality Equation for $q_*$

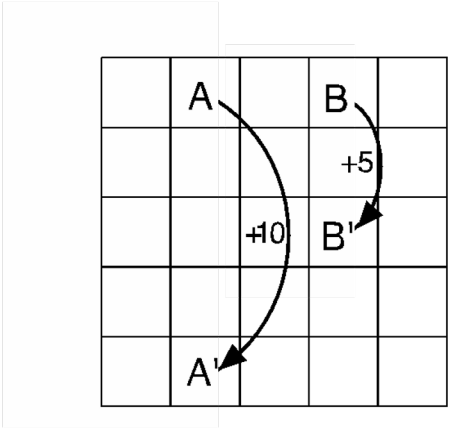
$$
\begin{aligned}
q_*(s,a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \;\Big|\; S_t = s, A_t = a\Big] \\
&= \sum_{s',r} p(s', r \,|\, s, a)\Big[r + \gamma \max_{a'} q_*(s', a')\Big].
\end{aligned}
$$

# Why Optimal State-Value Functions are Useful

Any policy that is greedy with respect to $v_*$ is an optimal policy.

Therefore, given $v_*$, one-step-lookahead search produces the long-term optimal actions.

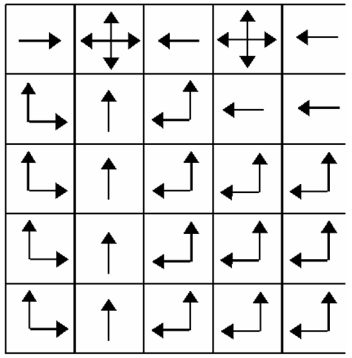
E.g., back to the gridworld:




a) gridworld

| | | | | |
|---|---|---|---|---|
| 22.0 | 24.4 | 22.0 | 19.4 | 17.5 |
| 19.8 | 22.0 | 19.8 | 17.8 | 16.0 |
| 17.8 | 19.8 | 17.8 | 16.0 | 14.4 |
| 16.0 | 17.8 | 16.0 | 14.4 | 13.0 |
| 14.4 | 16.0 | 14.4 | 13.0 | 11.7 |

b) $V^*$



c) $\pi^*$

# What About Optimal Action-Value Functions?

Given $q_*$ , the agent does not even have to do a one-step-ahead search:

$$\pi_*(s) = \arg\max_{a \in \mathcal{A}(s)} q_*(s, a)$$

# Dynamic Programming

- DP is the solution method of choice for MDPs
  - Requires complete knowledge of system dynamics (transition matrix and rewards)
  - Computationally expensive
  - Curse of dimensionality
  - Guaranteed to converge!

# Policy Evaluation

- For a given policy $\pi$, compute the state value function $v_\pi$
- Recall Bellman equation for $v_\pi$:

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a) \left[ r + \gamma v_\pi(s') \right]$$

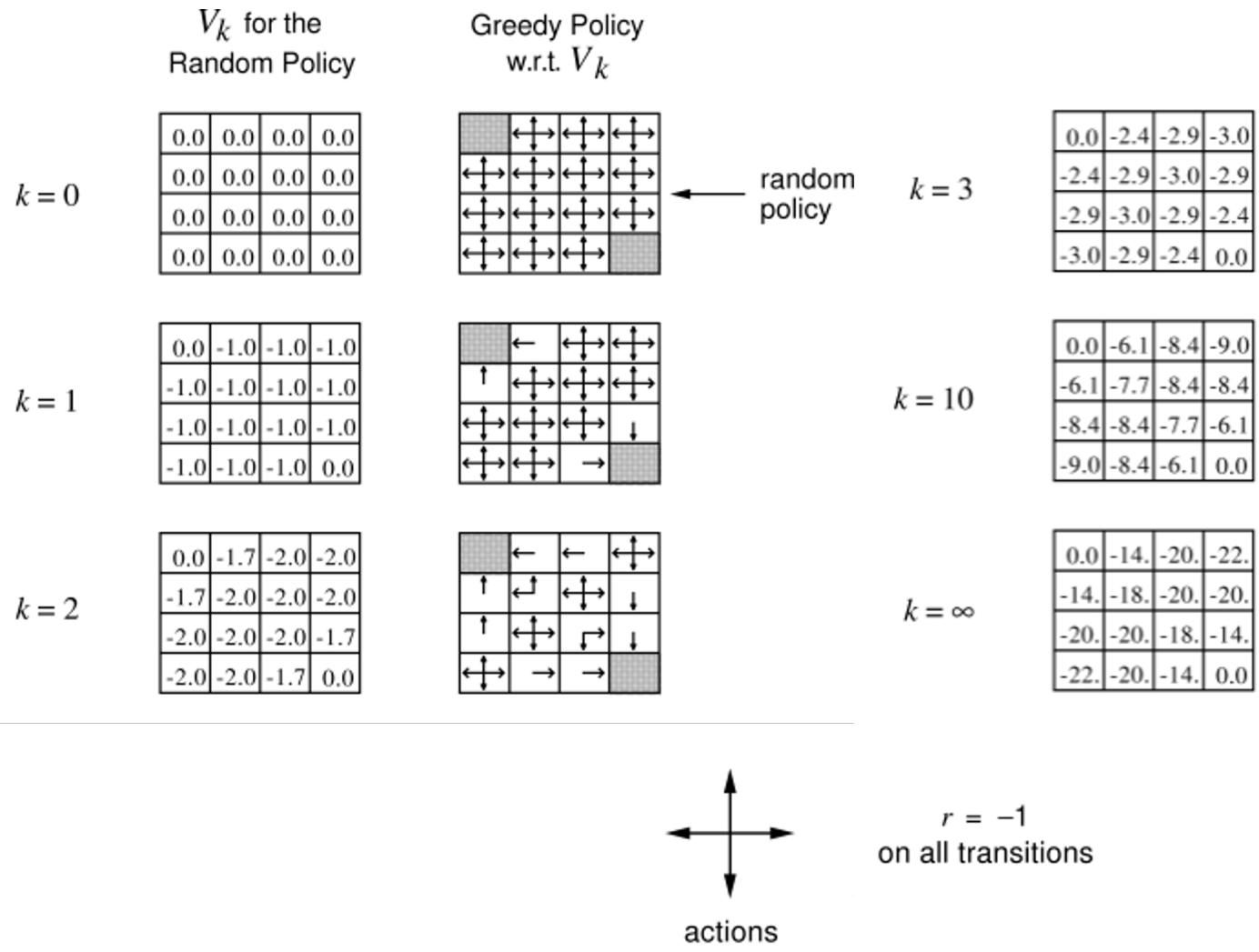
  - a system of $|S|$ simultaneous linear equations
  - solve iteratively

# Iterative Policy Evaluation Algo.

Input $\pi$, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

# Example of Policy Evaluation



# Policy Improvement

- Suppose we have computed $v_\pi$ for an arbitrary deterministic policy $\pi$
- For a given state $s$, would it be better to choose an action $a \neq \pi(s)$?
- The value of doing $a$ in state $s$ is:

$$\begin{aligned} q_\pi(s,a) &\doteq \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned}$$

- It is better to switch to action $a$ for state $s$ if and only if

$$q_\pi(s,a) > v_\pi(s)$$

# Policy Improvement Cont.

Do this for all states to get a new policy $\pi'$ that is **greedy** with respect to $v_\pi$:

$$\begin{aligned}\pi'(s) &= \arg\max_a q_\pi(s,a) \\ &= \arg\max_a \sum_{s',r} p(s',r \mid s,a)\,[r + \gamma v_\pi(s')]\end{aligned}$$

Then, $v_{\pi'} \geq v_\pi$

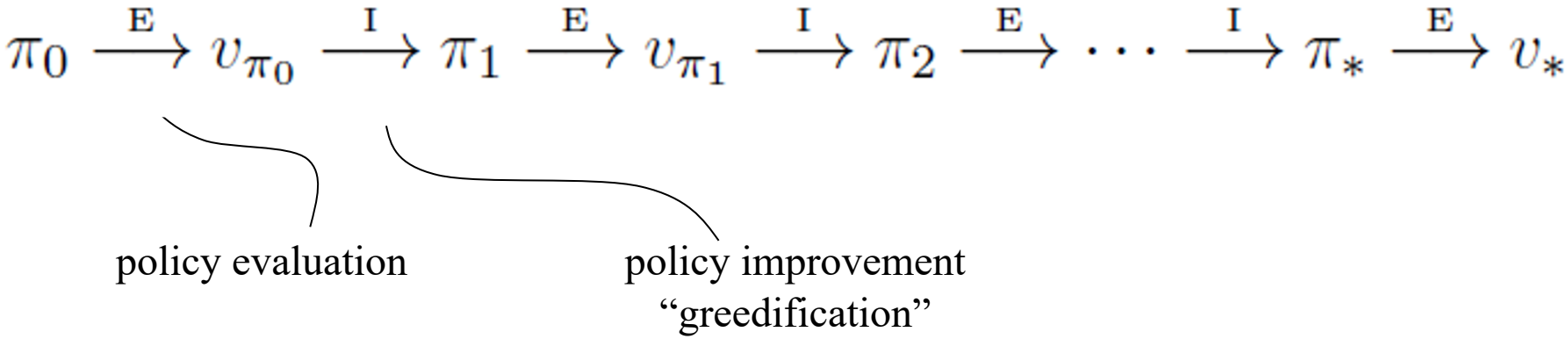
# Policy Improvement Cont.

What if $v_{\pi'} = v_\pi$? Then, for all $s, \in \mathcal{S}$, we have

$$v_{\pi'}(s) = \max_a \sum_{s', r} p(s', r \mid s, a) \left[ r + \gamma v_\pi(s') \right]$$

But this is the Bellman Optimality equation.
So $v_{\pi'} = v_*$ and both $\pi$ and $\pi'$ are optimal policies.

# Policy Iteration



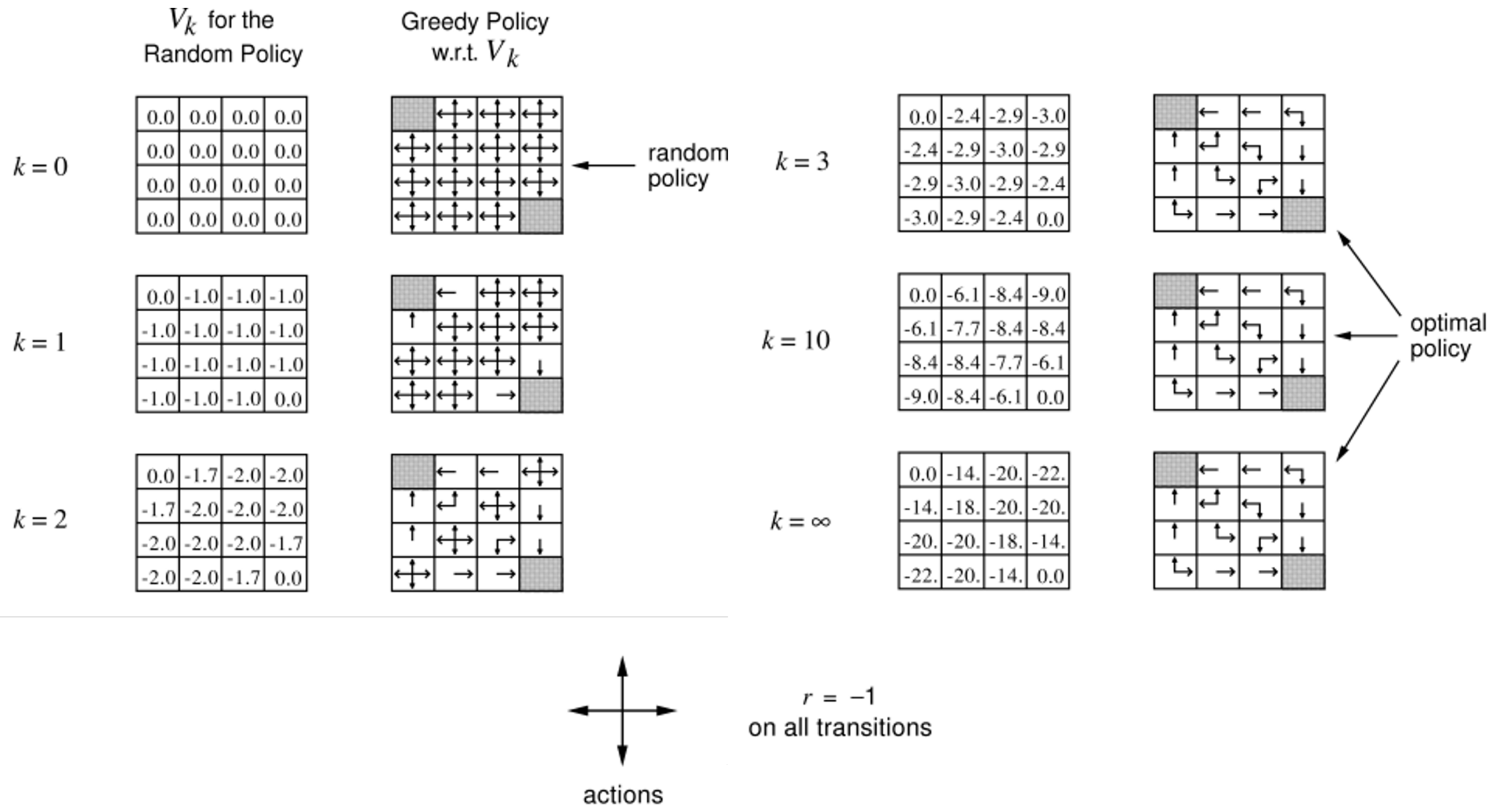
# Policy Iteration Algo.

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \left[ r + \gamma V(s') \right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   *old-action* $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \left[ r + \gamma V(s') \right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

# Example of Policy Evaluation



# Value Iteration

- Turning the Bellman optimality equation into an update rule.

$$
\begin{aligned}
v_*(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \left[ r + \gamma v_*(s') \right].
\end{aligned}
$$

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \left[ r + \gamma v_k(s') \right],
\end{aligned}
$$

# Value iteration Algo.

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Loop:
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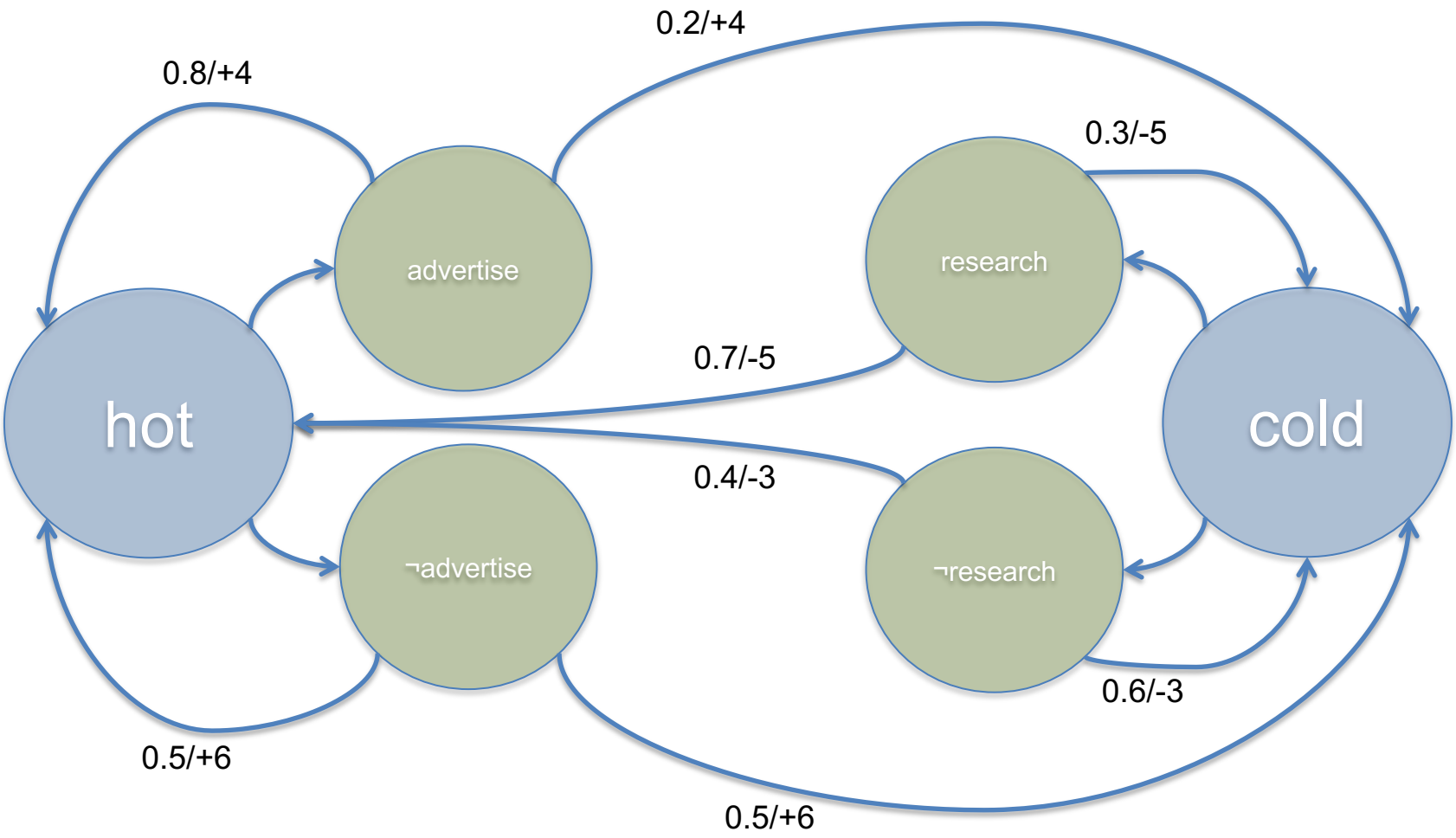
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

# Applying Value Iteration

- Recall the computer manufacturer problem for which we formulated an MDP.
- Consider applying value iteration to find the optimal policy the manufacturer should follow to obtain maximum return.

# Manufacturer Problem MDP



# Applying Value Iteration

- Recall the computer manufacturer problem for which we formulated an MDP.
- Consider applying value iteration to find the optimal policy the manufacturer should follow to obtain maximum return.

$$\text{Initialise } V(\text{hot}) = V(\text{cold}) = 0$$

$$\text{For value iteration :}$$

$$V(s) \leftarrow \max_a \sum_{s'} P_{ss'}^a \left[ R_{ss'}^a + \gamma V(s') \right] (\text{need to select appropriate } \gamma \text{ value})$$

$$\text{Consider state } hot$$

$$A(hot) = \{advertise, \neg advertise\}$$

$$P_{hot,hot}^{advertise} = 0.8, P_{hot,cold}^{advertise} = 0.2, P_{hot,hot}^{\neg advertise} = 0.5, P_{hot,cold}^{\neg advertise} = 0.5$$

$$R_{hot,hot}^{advertise} = R_{hot,cold}^{advertise} = +4, R_{hot,hot}^{\neg advertise} = R_{hot,cold}^{\neg advertise} = +6$$

# Applying Value Iteration

Initialise $V(\text{hot}) = V(\text{cold}) = 0$

For value iteration :

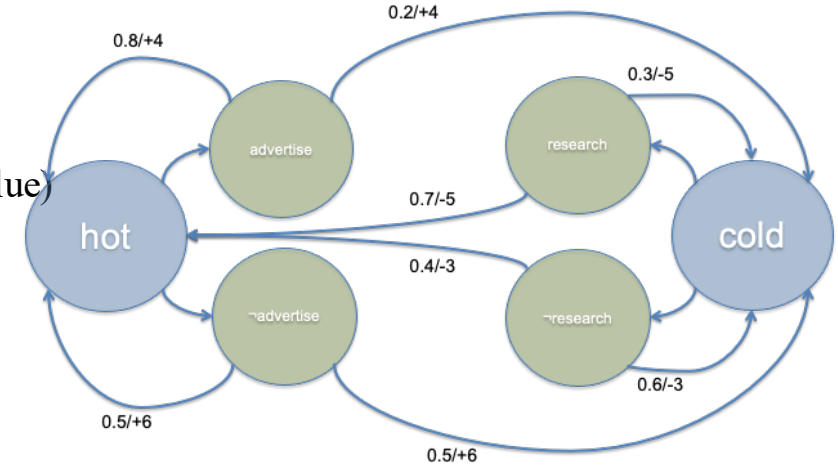
$V(s) \leftarrow \max_a \sum_{s'} P^a_{ss'} \left[ R^a_{ss'} + \gamma V(s') \right]$ (need to select appropriate $\gamma$ value)

Consider state *hot*

$A(hot) = \{advertise, \neg advertise\}$

$P^{advertise}_{hot,hot} = 0.8, P^{advertise}_{hot,cold} = 0.2, P^{\neg advertise}_{hot,hot} = 0.5, P^{\neg advertise}_{hot,cold} = 0.5$

$R^{advertise}_{hot,hot} = R^{advertise}_{hot,cold} = +4, R^{\neg advertise}_{hot,hot} = R^{\neg advertise}_{hot,cold} = +6$



# Asynchronous DP

- Disadvantage of algorithms discussed is we have to do the updates over the entire state set.
- In asynchronous DP, the updates are not done over the entire state set at each iteration.
- Have to ensure that every state is visited sufficiently often for convergence.
- Gives flexibility to choose order of updates.
- Can intertwine real time interaction with the environment and DP updates.
- Can focus updates on parts of state space relevant to agent.

# Realtime Dynamic Programming

- "…the controller performs asynchronous DP *concurrently* with actual process of control…"
- Control decisions are based on the current estimate of the value function
- States selected for updating depend on the sequence visited during the execution

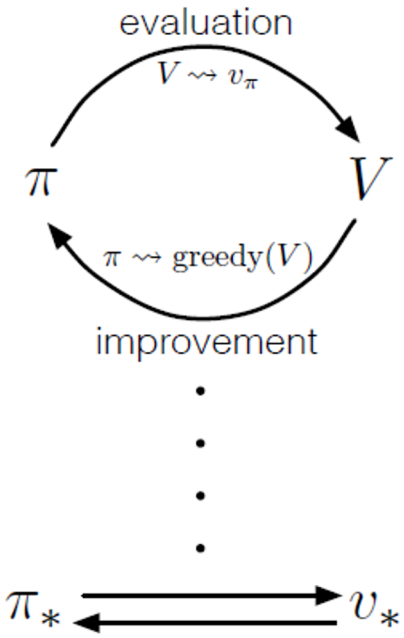
# Value Iteration, Revisited

- If policy evaluation step is stopped after one update of each state, then we apply greedification to that one state, we get value iteration

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[ r + \gamma v_k(s') \Big],
\end{aligned}
$$

# Generalized Policy Iteration

- GPI refers to the idea of letting policy evaluation and policy improvement interact, independent of their granularity.



# GPI

- Almost all RL methods can be viewed as GPI.
- For example, policy iteration has evaluation running to completion before improvement begins. In value iteration, only one step of evaluation is done before the improvement step. In Asynchronous DP, the two are interleaved at a finer granularity.