AI for Predictive Maintenance, Reliability and Warranty

– CH22M536
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Modules:

- Statistical and Industrial Predictive Maintenance
- ML/AI for PM and Reliability and Warranty

Prof. KALLOL

Typical Industrial Complex/ Process Plant / Transportation Vehicle

- Process Industries
- Shipping, Maritime Industries
- Locomotives, Automobiles
- Aerospace
- Nuclear

These can be broken down wrt specific targetted systems & equipments :

Mechanical :	Electrical :	Instrumentation :	Computer Based Systems :
Structures, Piping, Roto-Dynamic Eqpt., DGs, Process Eqpt., etc	Power supplies / VFDs, Power Devices, DGs, etc.	Process sensors, Chemical-Electro-Chemical, Domain-Specific Inst.	Embedded Real-time Sys. & OS, DAC, Historians, Alarm-Trip Annunciations, etc.

If these equipments needs to maintained, then :

- How to maintain
- Can we reduce manpower on job maintenance
- Individual to System-Oriented Maintenance
- Prompts & Guidelines to follow for Targetted Repair & Maintenance Actions.

× Mechanical :

- Roto-Dynamic Eqpt - (I/P-O/P) ODE/PDE Models, Eigen-structures of shafts, Thrust / Radial Bearing Models, Hydro-Dynamic & Hydro-Static Bearing Models

  Eg: Motor Pumps

- Process Eqpt - Heat & Mass Transfer Models

  Eg: Heat Exchangers (Shell & Tube, Plate Type)

- Piping - N/W Models, Thermal Expansion & Pipe Support Models

  Eg: Pipe Supports, Pipe Restrainers

- DGs, Process Eqpt - Electro-Chemistry, Ion-X Models, pH, K, etc., Corrosion Models, Air Compressor Eqpt. Models, Inter-cooler Models, etc., IC Engine Models, Heat-Transfer with JCW of DGs, Synchronous Gen. D-Q Axis Models.

  Eg: Ion Exchangers, Conductivity Sensors, Air Compressors, DGs, Turbo-Visory Systems

- Electrical, Power supplies / VFDs, Power Devices, etc - D-Q Axes Framework / State-Space, Transfer Fn., State-space, Freq. Domain Models.

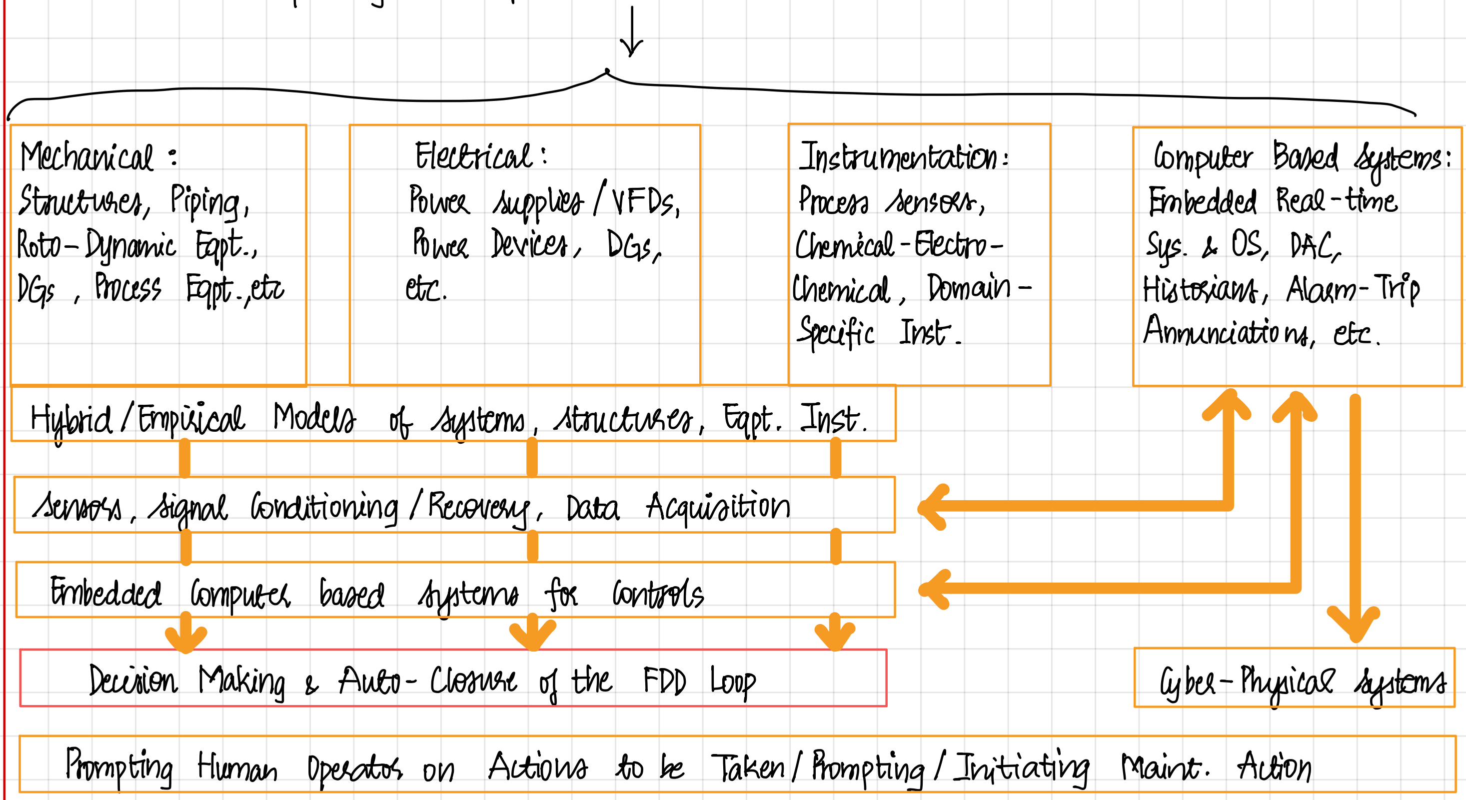
  Eg: Precision Power supplies that are well regulated, Switched mode devices, Converter-Inverter sets, Variable Frequency Drives, Transformers.

- Electrical, switchgear & Protection Relays – Transfer Fn., State-space, Freq. Domain Models.
- Instrumentation, Actuators – Transfer Fn., State-space, Freq. Domain Models.

  Eg: Control Valves
- Instrumentation, Process sensors

  Keywords – Rise-time, BW, Tr. Char., ss-Char., etc

Incorporating the Information Back-bone in Industrial Facilities



Digital Twins

Continuous Updating with Real-time Data (using digital twin)

× Dynamics of Structures, Mech. & Process sys., Power sys.; Inst., Eqpt. & Find Control Elements

Mechanical (typical):

- Structures — FEM Models / stress - strain Models of Mech. Structures
- Piping — N/W Models, Thermal Expansion & Pipe Support Models.
- Roto-Dynamic Eqpt — Simplified I/P - O/P ODE/PDE Models, Eigen-structures of Shafts
  Hydro-Dynamic & Hydro-Static Bearing Models
- Process Eqpt, etc. — Heat & Mass Transfer Models
- Control Eqpt. — Transfer Fn. Models, Frequency Domain Models.

Electrical (typical):

- Power Supplies / VFDs / Power Devices, etc. — Transfer Fn., State-space, Freq. Domain Models
- Motors / Drives / servo Devices etc. — D-Q Axes Framework / State-Space

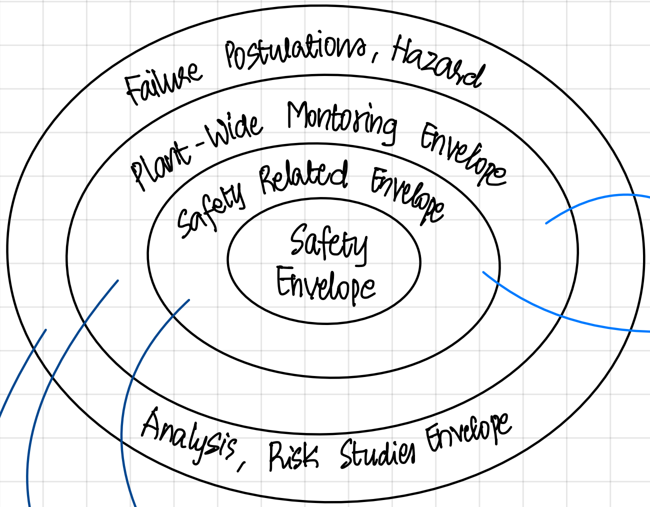
Instrumentation (typical):

- Process Sensors & Inst. Systems — Transfer Function & Freq. Domain Models
- Chemical-Electro-Chemical Sensors — State Space Models (derived from electro-chemical phen.)
- Domain-specific Inst. — First-principle, Phenomenological, Data-based.

Computer-based systems (typical):

- Embedded Real-time sys. & OS. — Boolean Models, Finite-State Machines
- Historians, Alarm-Annunciation, etc. — Data-Flow Structures, Automatons, Petri-Nets

× Safety Concepts in Critical Infra Projects:



Safety Envelope - Should only do the demanded task like Shutting of the valve and nothing else
Very simple μcontroller / trippers

Optimizing the plant performances
Sophisticated design, SCADAs etc.

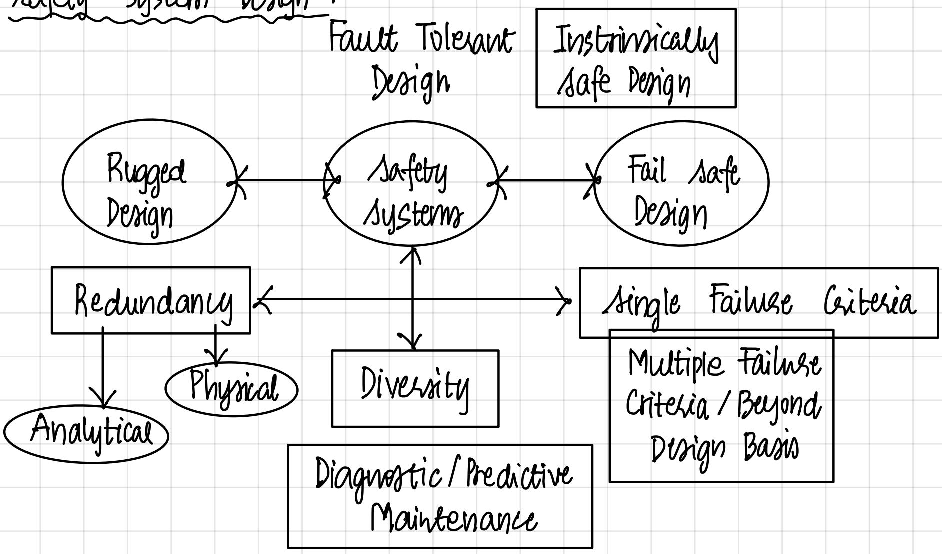
This is required to keep the system running and shouldn't fail

Apriori Computation of System Characteristics.
(Transient & Steady-State) & Safety Limits

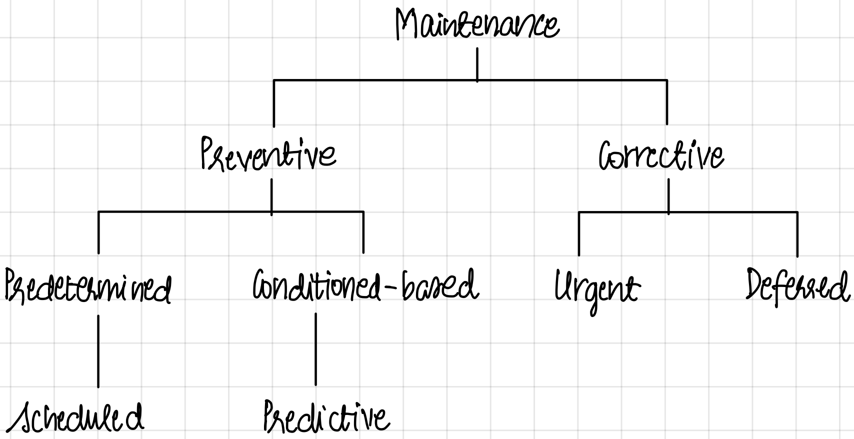
Computer Infrastructure & Data Acquisition

Domain Knowledge, Simulations, Experiments on Prototypes.

× Safety System Design :



× The Present-day Paradigms of Total Maintenance Management :



Reliability, Availability, Maintainability, Inspectability

Preventive Maintenance :
- Individual Knowledge
- Maint. logs of Sys. / Eqpt.
- Experience Feedback
- Trade-based Skill-Sets
- Procedures (OEM Guidelines)

Predetermined :
- Reliability Models
- Redundant Sys. in Opn.
- Hot / Cold Standby
- Arrhenius / Eyrring Models

Condition-based :
- Condition based Maint.
- In-service Inspection
- Data Analysis of On-Line Measurements

Corrective Maintenance :
- Individual Knowledge
- Maint. logs of Sys. / Eqpt.
- Experience Feedback
- Trade-Based Skill-sets
- Procedures (OEM Guidelines)

Data Base / Data Structure / Big-Data Analytics

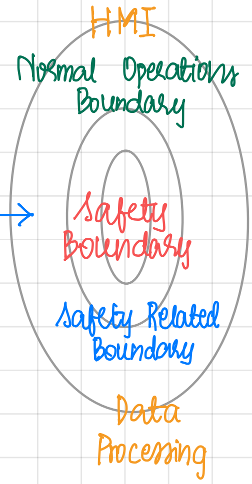
Algorithms, Math-Stat Models

Urgent, Deferred :
- OEM Guidelines, Skill-sets, Graceful Degradation

⇒ **Predictive Maintenance :**
- Hybrid Models (Mechanistic & Data Oriented)
- Transient & Steady State Data
- Filtering, Estimation, De-Noising
- Fault Detection / Diagnostics Algorithms
- Inferential Statistics
- Boolean & Discrete Event System Models
- Prognostics & Forecasting.

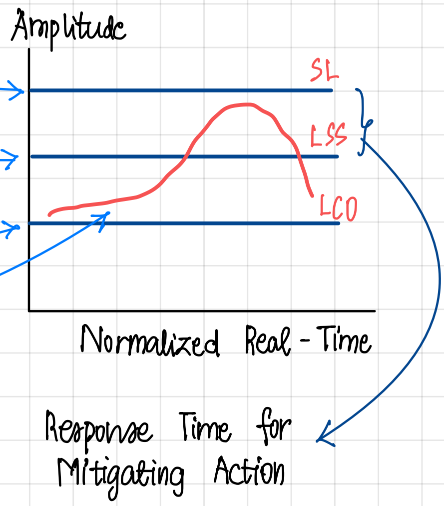
× Systems & Plant Operating Boundaries :

- Assessment of the Safety Envelope / Boundaries
- Formulation of Postulated Initiating Events (PIE)
- Dynamics of PIEs
  - Fundamental Analysis, CFD, FEM, etc.
- Choice of Sensor & Inst. Systems
- Adequacy of Sensor Type & Position / Mounting
- Sensor Accuracy & Time Constants
- Signal Conditioning Error & Processing Time
- Safety Assessment of Network Protocols / WSN
- Data Processing, Data Mining, Big-Data Analytics
- Plant-wide EMI Assessment & Inst. Design for EMC.
- Human Machine Interface Perspective.



× Indicative Safety Paradigms for a Process :

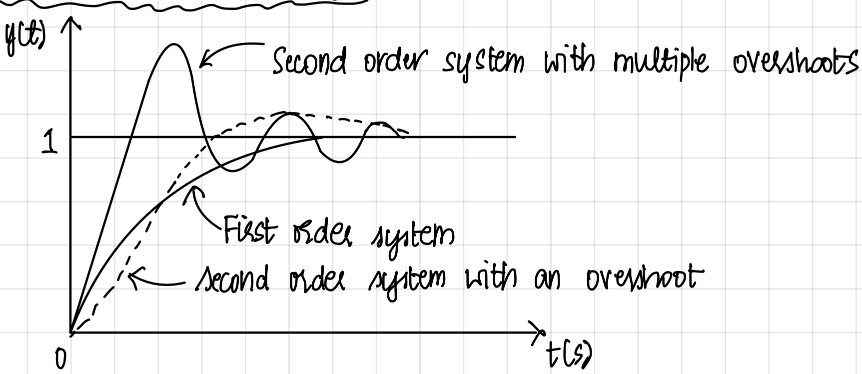
- Safety Limits (SL)
- Limiting Safety Settings (LSS)
- Limiting Conditions for Operation (LCO)
- Plant Transients



If we have a safety limits, say 15 kg/cm², then the LSS must be set lesser than that, say 12, then the LCO must be set at 10. Why?
These are the safety boundaries.
The Plant transient if spikes away from the LCO to the SL. There needs to be a balance of plant safety and original performance. There should be a mitigating action to bring down the spike to LCO so that the productivity is not totally affected and prevent a large scale damage / plant shut down.

× Sensors, Actuators & System Characteristics & Plant Safety :

- Response Characteristics
- Order of Sensor / Process
- Uncertainity



Non-homogenous equation : $\frac{d^2y}{dt^2} + P_1 \frac{dy}{dt} + P_0 y = Q(t)$

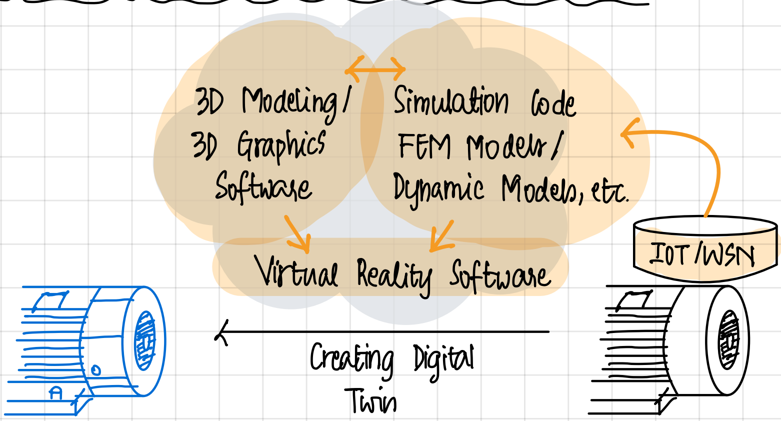
H.O. ODE/PDE

Homogenous equation : $\frac{d^2y}{dt^2} + P_1 \frac{dy}{dt} + P_0 y = 0$

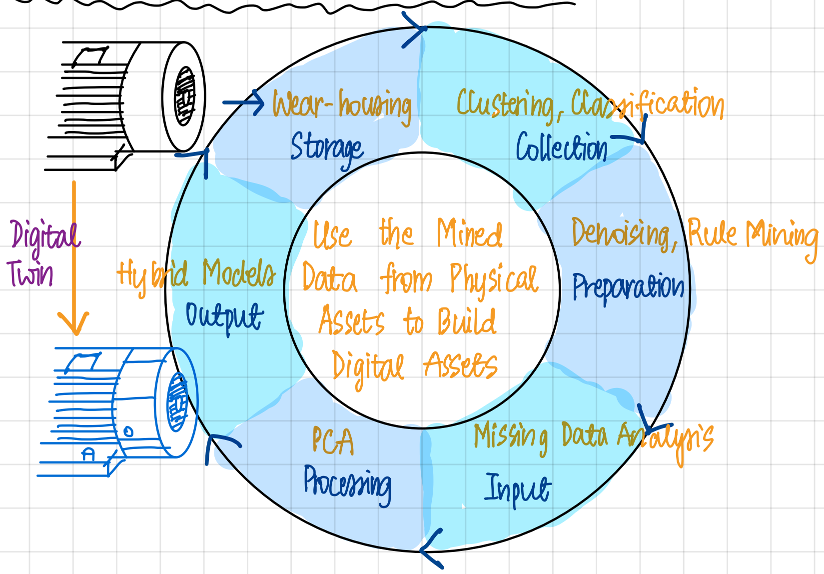
× Condition Based Maintenance :

- Calibrate only if Drifts are there → • Physically Redundant systems
- Capture similar data from similar sensors / inst. correlate data from dissimilar inst. → • Analytically Redundant systems • Temporal Redundancy
- Compare Measured Data from Estimated Data → • Math Model of systems
  Mechanistic
  Data Oriented
  Model Reference systems
  Bayesian Estimators

× Creating Digital Twins - A Model Based Approach :



× Building Digital Twins from Data Mining Paradigms :



× Functional Responsibilities in Maintenance Management:

- Multi-skill capabilities
- Specialized production or processing knowledge
- Troubleshooting knowledge and skill
- Communications with users of maintenance services
- Construction or fabricating skills
- Planning maintenance work
- Purchasing maintenance parts and materials
- Directing or coordinating contractor work
- Preventive Maintenance recommendations

Traditional Ways:
Pre-detection of incipient failures
Post-failure remedial-action decisions
Repetitive-job standardization

Computerized Maintenance Management Systems (CMMS)

Inventory Control
Supply Chain Management

RPN (Measure) = Severity × Occurrence × Detection
Risk Priority Number

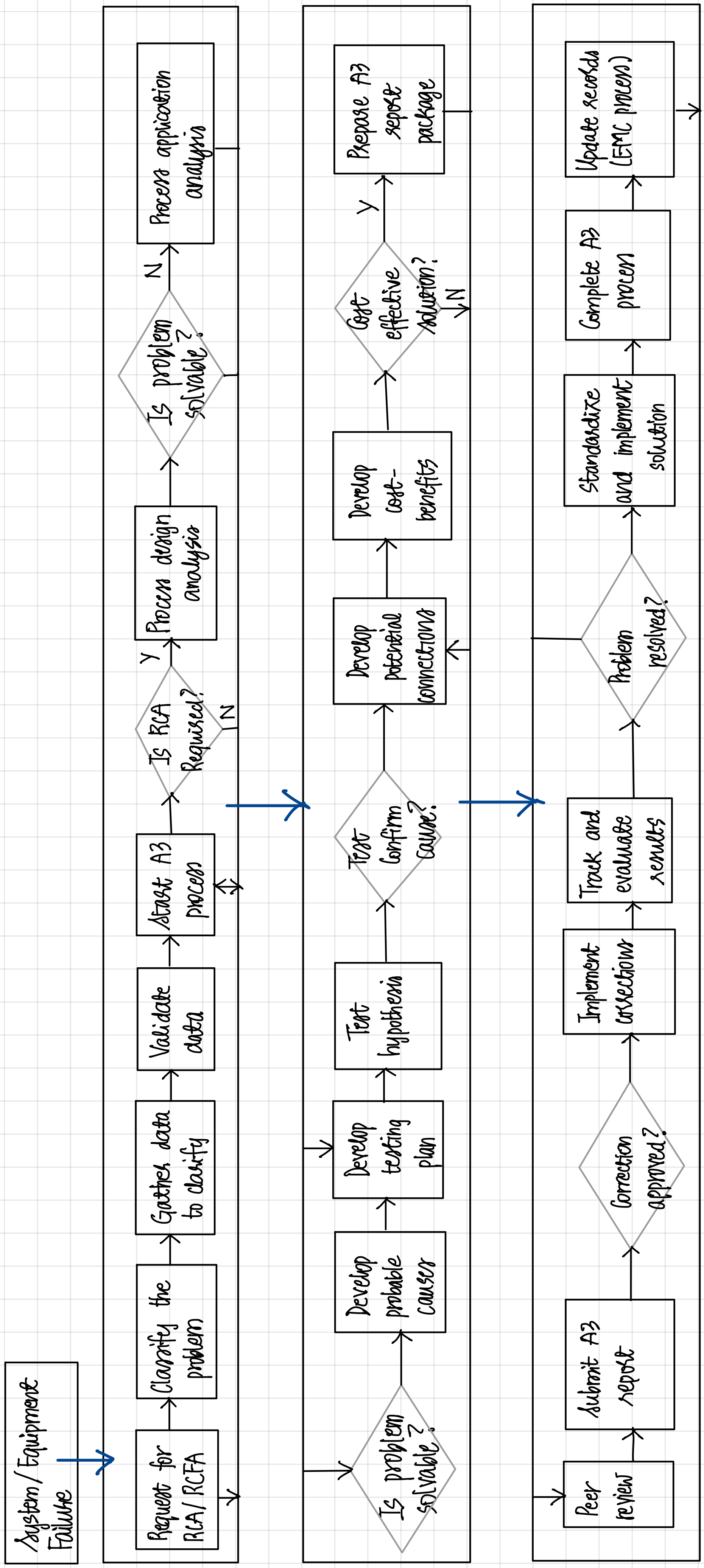
× Maintenance Personnel Toolbox:

Repetitive repair or replacement of specific items such as belts, bearings, filters and screens
Scheduled routine work, such as oiling, cleaning, housekeeping and inspection
Spare-parts production and overhaul
Planned equipment overhaul
Building and facility repairs
Assigned area service
Planned non-repetitive replacement and repairs
Relocations
Modifications
Equipment improvements
Repairs on non-critical or lightly loaded equipment that can be economically shut down pending scheduled repair

Taken care by:
CMMS

Plan	Do	Check	Act
Identify Problem	Perform root cause analysis	Implement action plan	Standardize & implement solution

× Root Cause Analysis (RCA) - Concepts :



Automated RCA - Data Oriented, Missing Data Handling, Fitting Hybrid Models

CMMS

× Making Simple Models – A step towards Predictive Maintenance

Large-size pumps:
Vibration models

Inadequate heat dissipation in Heat Exchanger:
Estimation Models

Electrical Equipments:
Frequency Domain models

Final Control Element:
Control Valve Model

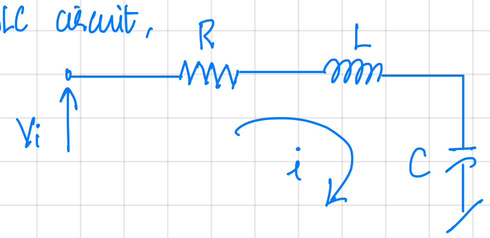
× Understanding Linear systems:

Recall simple linear functions starting with $y = mx + C$ curve

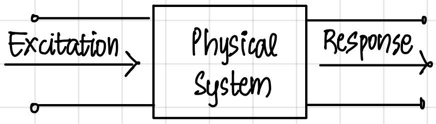
Second order DE: $y'' + r(x)y' + y = r(x)$;
Second order DE with a forcing function are called non-homogeneous DE

Any system that has got an energy storage part is only governed by dynamics or a differential element / integro differential function.
Any part of a system which is translational or mechanical, which does not store energy will essentially have an alegrabic equation.

Taking a RLC circuit,

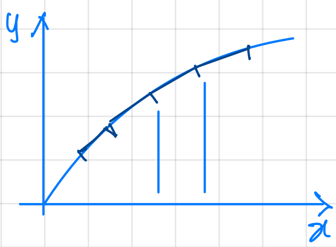


$$\Rightarrow Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt + q(0)$$



Physical could be defined by ODE/PDE models

Piece-wise linear system in a simple graph: In a process, we have a operating regime which is linear in a given region. Non-linear if you exceed the region



$$e_1(t) \longrightarrow w_1(t),$$
$$e_2(t) \longrightarrow w_2(t)$$

For a linear system,
$$e_1(t) + e_2(t) \longrightarrow w_1(t) + w_2(t)$$

$$e_1(t) = e_2(t) = \dots = e_n(t)$$

For a linear system,
$$\sum_{k=1}^{n} e_k(t) = ne_1(t) \longrightarrow \sum_{k=1}^{n} w_k(t) = nw_1(t)$$

$$e(t) \longrightarrow w(t)$$

$$e(t-T) \rightarrow w(t-T)$$

If superpositioning and scalability holds for a system, it is a linear system.

Conceptualizing Linear Models:

$$\frac{d^2w}{dt^2} + a_1 \frac{dw}{dt} + a_0 w = e(t)$$

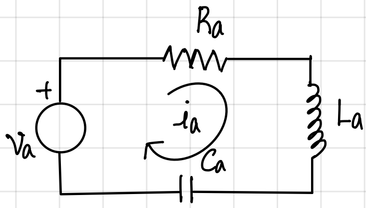
$$\frac{d^2w_1}{dt^2} + a_1 \frac{dw_1}{dt} + a_0 w_1 = e_1 ,$$

$$\frac{d^2w_2}{dt^2} + a_1 \frac{dw_2}{dt} + a_0 w_2 = e_2 ;$$

$$\Downarrow$$

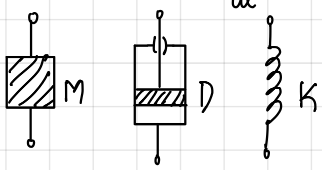
$$\frac{d^2}{dt^2}(w_1+w_2) + a_1 \frac{d}{dt}(w_1+w_2) + a_0 (w_1+w_2) = (e_1+e_2)$$

Analogous systems:

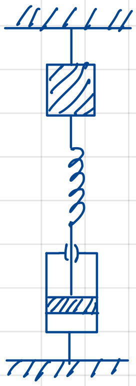


$$T_I = I_\theta \alpha = I_\theta \frac{d\Omega}{dt} = I_\theta \frac{d^2\theta}{dt^2}$$

$$f_M = Ma = M.\frac{du}{dt} = M.\frac{d^2x}{dt^2}$$



Mass Dashpot Spring



The total force in this system is similar to the voltage given to a RLC

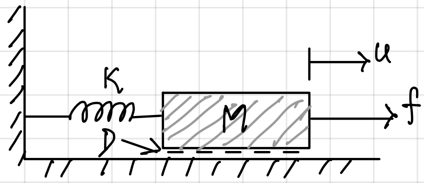
$$M\frac{d^2x}{dt^2} + \frac{1}{K}x + \mu \frac{dx}{dt}$$

Taking the RLC circuit, $iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt + q(0)$

Converting current into charge, $i = \frac{dq}{dt}$

$$\Rightarrow R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{1}{C}q$$

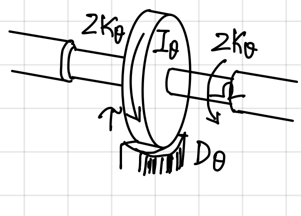
similar



$$f_D = Du = D\frac{dx}{dt} \qquad f_K = \frac{1}{K}x = \frac{1}{K}\int u\,dt = \frac{1}{K}\left[\int_0^t u\,dt + x(0)\right]$$

;

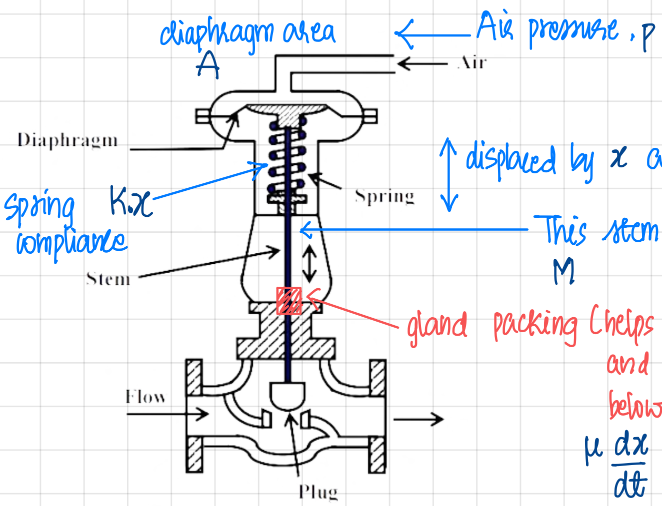
$$T_D = D_\theta \Omega = D_\theta \frac{d\theta}{dt} \qquad T_K = \frac{1}{K_\theta}\theta = \frac{1}{K_\theta}\int \Omega\,dt = \frac{1}{K_\theta}\left[\int_0^t \Omega\,dt + \theta(0)\right]$$



Electrical system (f-v analogy) —	Translational	Rotational
Voltage, v	Force, f	Torque, T
Current, i	Acceleration, a	Angular acceleration, α
Charge, q	Velocity, u	Angular velocity, Ω
	Displacement, x	Angular displacement, θ
Inductance, L	Mass, M	Moment of inertia, I_θ
Resistance, R	Damping coefficient, D	Rotational damping coefficient, D_θ
Capacitance, C	Compliance, K	Torsional compliance, K_θ

× **Understanding a Control Valve:**

Force = pA



displaced by x as it needs to close or open to allow the fluid to go through

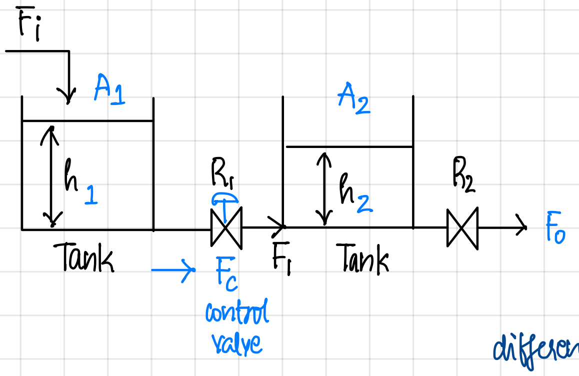
This stem would have a mass ; $M\frac{d^2x}{dt^2}$

gland packing (helps the stem to move freely and prevents the liquid from the below part from leakage)

$\mu\frac{dx}{dt}$

$$pA = M\frac{d^2x}{dt} + \mu\frac{dx}{dt} + K.x + W$$

× **Understanding a Process Tank:**



different type,

$$A_1\frac{dh_1}{dt} = F_i - F_c \qquad A_1\frac{dh_1}{dt} = F_i - K_1(h_1 - h_2)$$

$$A_2\frac{dh_2}{dt} = F_c - F_0 \qquad A_2\frac{dh_2}{dt} = K_1(h_1 - h_2) - K_2h_2$$

$$F_c = K_1(h_1 - h_2) \qquad F_c = K_1(h_1 - h_2)$$

$$F_0 = K_2h_2 \qquad F_0 = K_2h_2$$

× Understanding Maintenance Paradigms from Data :

Input Variables
U-1
U-2
⋮
U-n

→

System
- Parameters
- Internal variables
- Measured variables
- Disturbances

System Behaviour
- Rule-Based
- Static
- Dynamic

Domain Knowledge Inputs

Knowledge Inadequacies
- Uncertainity
- External Noise

→

Output Variables
Y-1
Y-2
⋮
Y-m

Objectives
- Availability
- Productivity
- Profitability

Even after all this, there is Aleatory, Uncertainity, Epistemic, Knightian

Data Requirements, to meet the Objectives

Can it be Managed with Data & Qualitative Domain Knowledge

We need to understand
- Maintainability
- Testability
- Repairability

Internal Disturbance Process

External Disturbance

Maintenance Methodologies / Paradigms
- Rule / Procedure Based
- Reliability Models
- Static Simulation of SSE
- Dynamic Simulation of SSE

Use Data Processing / Analytic Tools → System Identification from Data →
- System / Eqpt. I/P-O/P Relationships
- Qualitative Models
- Governing ODEs/PDEs
- Key Parameter Indicators

× Appreciation of Model Building for Maintenance Automation :

The Industrial Landscape :

Procedures, from CMMS, PM Schedules from Test Data

Failure Mdls. / Reliability Mdls., Fault Tree, Accelerated Test Data

Boolean Models, RTOS Mtrg.

Procedures, Maint. Plans, Look-up Tables, Data-structures, Logical Test Models, Logic Analyser Outputs

Mechanical : Structures, Piping, Roto-Dynamic Eqpt., DGs, Process Eqpt., etc

Electrical : Power supplies / VFDs, Power Devices, DGs, etc.

Instrumentation : Process sensors, Chemical-Electro-Chemical, Domain-Specific Inst.

Computer Based Systems : Embedded Real-time Sys. & OS, DAC, Historians, Alarm-Trip Annunciations, etc.

Computerized Maintenance Management System (CMMS)

ODEs/PDEs, Laplace Transforms, Z-Transforms, Freq. Domain Anal. (1)

(3) Uncertainities/Noise, Stat. Mdls. & Linear Comb. with SSE Mdls.

Dynamic Simulation Models, of Individual systems & Components for Continuous Variable systems (Time-Driven) & Discrete Event systems (Event Driven)

Multi-variate Anal., Matrix Operations, Least Squares (2)

(6) Power Spectral Density, Statistical Distance Measures

Coupled Dynamics with system Interactions, Hardware in Loop simulations, Multi-Input-Multi-Output systems, Complex Dynamical systems

Kalman Filters & Particle Filter Intro. (5)

(4) Estimation, Linearization, Concepts of Monte Carlo

Linear/Non-Linear Estimation Models, Bayesian Estimation Paradigms, Missing Data Handling, Monte-Carlo simulations

Sequential Stat., Inferential Stat., Multiple Model, GLR, etc. (7)

(8) Data Analysis of Healthy & Faulty Data, Big Data

Fault Detection and Diagnosis Models, sequential / Inferential statistics

Prompting Human Operator on Actions to be Taken / Prompting / Initiating Maint. Action

Restructing / Reconfiguring system Dynamics, based on New Data / Events

(*) Flow of lecture

× The ODE for understanding the Transient & Steady state Characteristics of a Dynamic system:

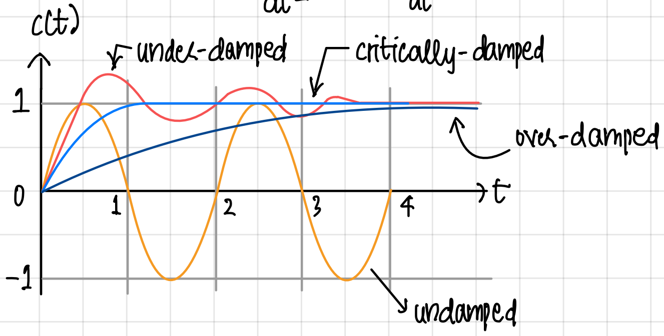
Response Characteristics :

Two major measures of performance are apparent :
- (The short period of time immediately after the system is turned on) The transient response
- The steady-state error (The diff. between the actual output & desired output as time tends to infinity)

Non-homogenous equation : $\frac{d^2y}{dt^2} + P_1 \frac{dy}{dt} + P_0 y = Q(t)$ → particular integral

$y'' + p(x)y' + q(x)y = r(x)$

Homogenous equation : $\frac{d^2y}{dt^2} + P_1 \frac{dy}{dt} + P_0 y = 0$ → complimentary part



* The need for Transforms:

Helps in converting DE to Algebraic equations

Most popular: Laplace Transform

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt \longrightarrow F(s) = \mathcal{L}[f(t)]$$

- Simplifies Operations
- Simplifies Functions
- Converts Integro-differential equations to Algebraic equations
- Effectively makes use of step and impulse responses.

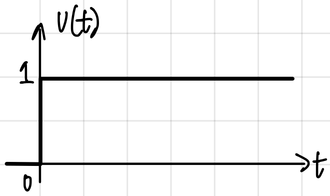
- Basics:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$f(x) = \frac{u(x)}{v(x)} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

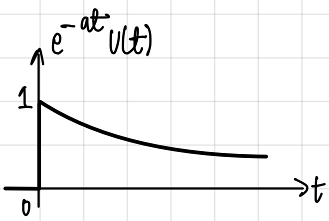
$$\int u\,dv = uv - \int v\,du$$

- $$F(s) = \int_0^\infty f(t) e^{-st}\, dt \longrightarrow F(s) = \mathcal{L}[f(t)]$$



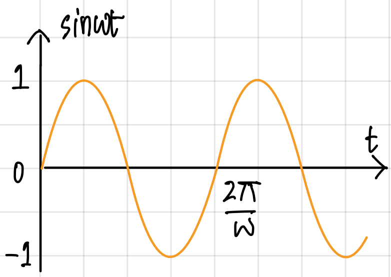
$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \quad ; \quad F(s) = \int_0^\infty 1 . e^{-st} dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s}$$

$$\therefore \boxed{\mathcal{L}[u(t)] = \frac{1}{s}}$$



$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} . e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$\therefore \boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}}$$



$$\sin \omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$\mathcal{L}[\sin \omega t] = \frac{1}{2j}\mathcal{L}[e^{j\omega t} - e^{-j\omega t}] = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right)$$

$$\therefore \boxed{\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2}}$$

Important Properties of Laplace Transforms:

$$\mathcal{L}[f_1(t)+f_2(t)] = \int_0^\infty [f_1(t)+f_2(t)]e^{-st}\,dt$$
$$= \int_0^\infty f_1(t)e^{-st}\,dt + \int_0^\infty f_2(t)e^{-st}\,dt$$
$$= \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$

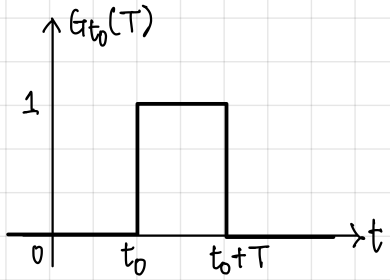
$$\mathcal{L}[cf(t)] = \int_0^\infty [cf(t)]e^{-st}\,dt$$
$$= c\int_0^\infty f(t)e^{-st}\,dt$$
$$= c\,\mathcal{L}[f(t)]$$

$$\mathcal{L}[c_1f_1(t)+c_2f_2(t)] = c_1F_1(s)+c_2F_2(s)$$

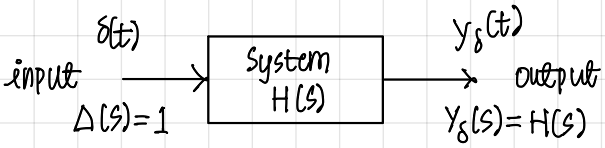
$f(t)$	$F(s)=\mathcal{L}[f(t)]=\int_0^\infty f(t)e^{-st}dt$
$U(t)$	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin wt$	$\frac{w}{s^2+w^2}$
$\cos wt$	$\frac{s}{s^2+w^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$e^{-at}f(t)$	$F(s+a)$
t^n	$\frac{n!}{s^{n+1}}$

- The Shifting Theorem,

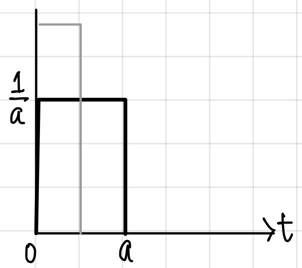
If $\mathcal{L}[f(t)]=F(s)$, then $\mathcal{L}[f(t-t_0)U(t-t_0)] = e^{-t_0s}F(s)$

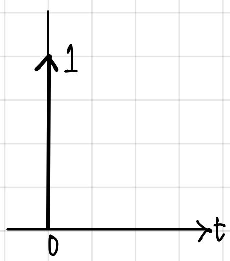


$$G_{t_0}(T) = U(t-t_0)-U(t-t_0-T)$$



- Impulse Response:





$$\delta(t) = \lim_{a\to 0}\frac{1}{a}[U(t)-U(t-a)]$$

$$\mathcal{L}[\delta(t)] = \lim_{a\to 0}\frac{1}{as}\cdot\left[1-\left(1-as+\frac{1}{2}a^2s^2-\ldots\right)\right]=1 \quad ; \quad \mathcal{L}[k\,\delta(t)]=k$$

- Convolution Theorem :

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau).d\tau\right] = \mathcal{L}\left[\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right] = F_1(s) F_2(s)$$

$$\mathcal{L}\left[f_1(t) * f_2(t)\right] = \mathcal{L}\left[f_2(t) * f_1(t)\right] = F_1(s) F_2(s)$$

- Differential Theorem : If a function f(t) and its derivative are both Laplace transformable and if $\mathcal{L}[f(t)] = F(s)$, then

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \mathcal{L}\left[Df(t)\right] = sF(s) - f(0+)$$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

Integrating by parts, we let

$$u = f(t), \qquad du = \left[\frac{df(t)}{dt}\right] dt,$$

$$dv = e^{-st} dt, \qquad v = -\frac{1}{s} e^{-st}$$

Hence,

$$F(s) = -\frac{1}{s} f(t) e^{-st} \Big|_0^\infty + \frac{1}{s} \int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

$$= \frac{f(0+)}{s} + \frac{1}{s} \mathcal{L}\left[\frac{df(t)}{dt}\right]$$

$$\mathcal{L}\left[D^2 f(t)\right] = s\mathcal{L}\left[Df(t)\right] - f'(0+)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt}\right] = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \dots - sf^{(n-2)}(0+) - f^{(n-1)}(0+)$$

$$= s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0+)$$

- Integration Theorem :

If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \frac{F(s)}{s}$,

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \int_0^\infty \left[\int_0^t f(t)\, dt\right] e^{-st} dt$$

Integrating by parts, we let

$$u = \int_0^t f(t)\, dt, \qquad du = f(t)\, dt$$

$$dv = e^{-st} dt, \qquad v = -\frac{1}{s} e^{-st}$$

Hence,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = -\frac{e^{-st}}{s}\int_0^t f(t)\,dt\,\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$

$$\frac{1}{s}\mathcal{L}[f(t)] = \frac{F(s)}{s}$$

$$\int f(t)\,dt = \int_0^t f(t)\,dt + f^{(-1)}(0+)$$

$$\mathcal{L}\left[\int f(t)\,dt\right] = \mathcal{L}\left[\int_0^t f(t)\,dt\right] + \mathcal{L}\left[f^{(-1)}(0+)\right]$$

$$= \frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}$$

× Solution of linear different equations :

$$\frac{d^2w}{dt^2} + b_1\frac{dw}{dt} + b_0 w = e(t)$$

Second ODE, one of the building blocks for all engineering system models
Solving this in the form of complimentary part + particular integral.
particular integral part essentially dictates the steady state part of the system.
so solving this in Laplace form makes the equation simpler.

$$\mathcal{L}[D^2w] + b_1\mathcal{L}[Dw] + b_0\mathcal{L}[w] = \mathcal{L}[e(t)]$$

$$\mathcal{L}[w] = W(s) \qquad \mathcal{L}[e(t)] = E(s)$$

$$\mathcal{L}[Dw] = s.W(s) - w(0),$$

$$\mathcal{L}[D^2w] = s^2.W(s) - sw(0) - w'(0)$$

$$[s^2W(s) - sw(0) - w'(0)] + b_1[sW(s) - w(0)] + b_0W(s) = E(s)$$

$$(s^2 + b_1 s + b_0)W(s) = E(s) + (s+b_1)w(0) + w'(0)$$

$$W(s) = \frac{1}{s^2+b_1s+b_0}\left[E(s) + (s+b_1)w(0) + w'(0)\right]$$

$$w(t) = \mathcal{L}^{-1}[W(s)] = \mathcal{L}^{-1}\left[\frac{E(s) + (s+b_1)w(0) + w'(0)}{s^2+b_1s+b_0}\right]$$

Book to Refer: Analysis of Linear Systems – D.K. Cheng

$$W(s) = H(s)\,E_0(s)$$

(Response transform) = (Transfer function) × Total excitation transform)

An Understanding of System Dynamics:

- Proper Rational Polynomial Fractions:

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}$$

$$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

System Type - No. of Free Poles / Integrators

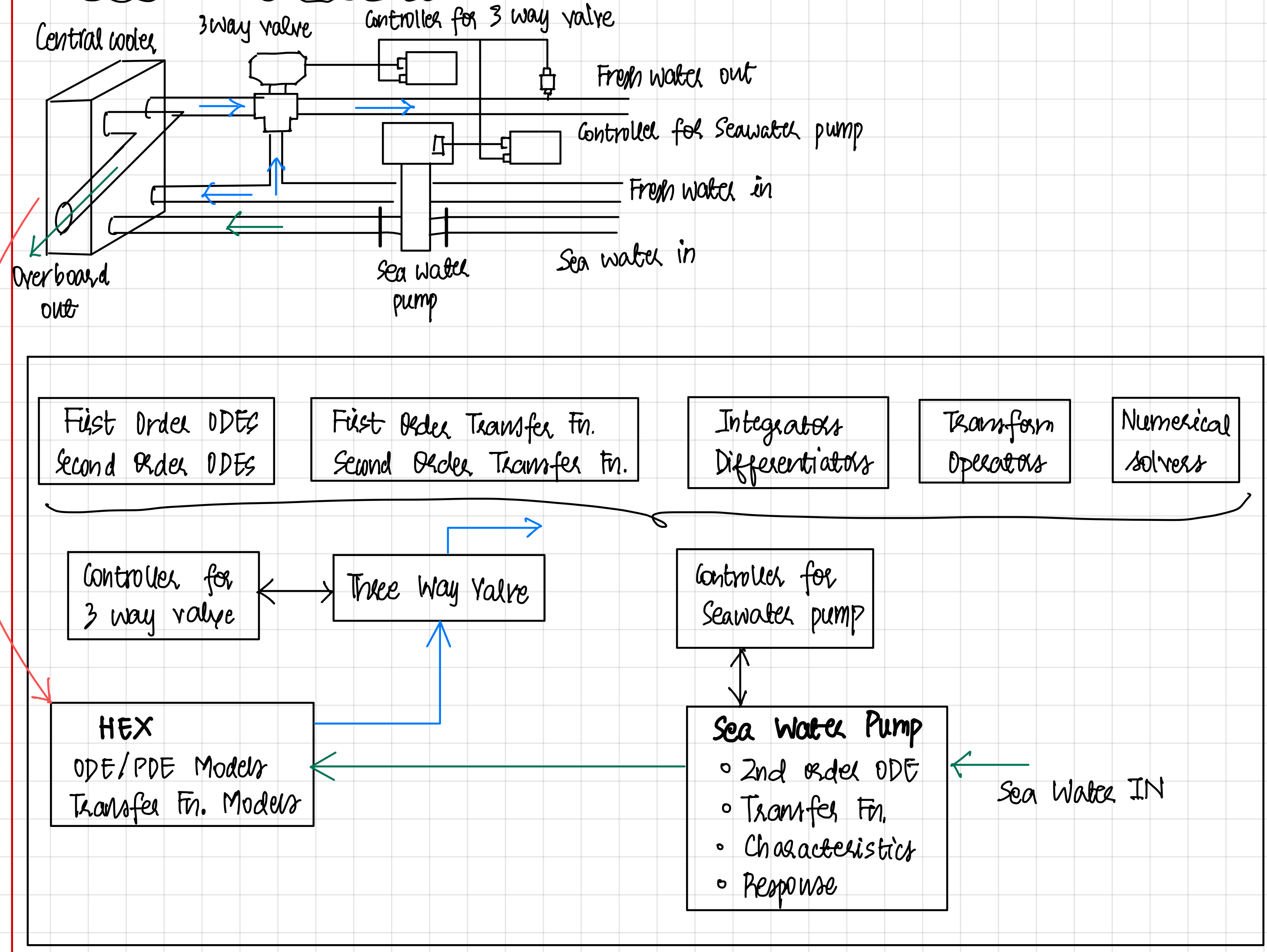
$$\frac{[a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0]}{[b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0]}$$

s^k, k is # integrators

poles

Those parts of the dynamics which will cause the dynamic system to go to rest are called zeroes and ones that cause the system to go to indeterminant / unstable states are called poles.

Typical Simulation Model Approach:



× **Building a Transfer Function:**

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \dots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \dots + b_{m-1}\dot{x} + b_m x$$

$$G(s) = \left.\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_0 s^{n-1} + \dots + a_{n-1} s + b_n} \quad ; m \leq n$$

Taking the control valve as an example,

$$F(t) \longrightarrow \boxed{F(t) = m \cdot \frac{d^2x(t)}{dt^2} + c \cdot \frac{dx(t)}{dt} + kx(t)} \longrightarrow x(t)$$

$$F(t) \longrightarrow \boxed{H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}} \longrightarrow x(t)$$

× **Differential Equations to Transfer Functions:**

$$\boxed{\text{ODE}} \underset{\text{Inverse Laplace Transform}}{\overset{\text{Laplace Transforms}}{\rightleftarrows}} \boxed{\text{Transfer Function}}$$

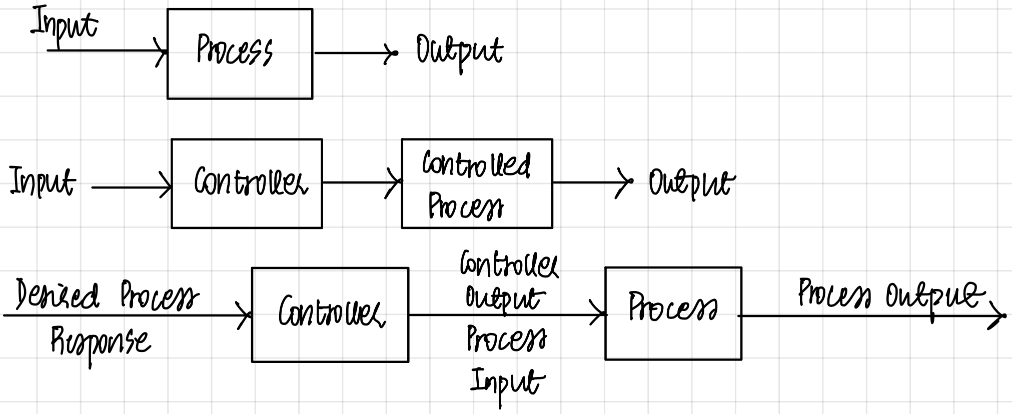
Given: $2\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 5\frac{dx}{dt} - 2x$

Transfer function : $H(s) = \dfrac{5s - 2}{2s^2 - 4s + 3}$

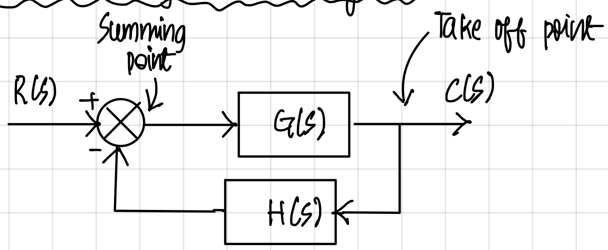
° Laplace Transformation of Operations:

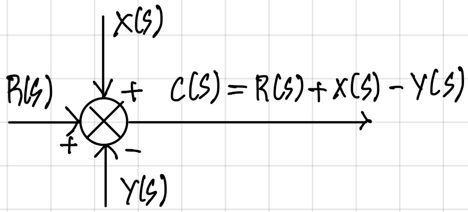
	$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1.	$f(t - t_0)\, U(t - t_0)$	$e^{-t_0 s} F(s)$
2.	$\frac{d}{dt} f(t)$	$sF(s) - f(0+)$
3.	$\int_0^t f(t)\,dt$	$\frac{F(s)}{s}$
4.	$\int f(t)\,dt$	$\frac{F(s)}{s} + \frac{f^{(-1)}(0+)}{s}$
5.	$t \cdot f(t)$	$-\frac{d}{ds} F(s)$
6.	$\frac{1}{t} f(t)$	$\int_0^\infty F(s)\,ds$

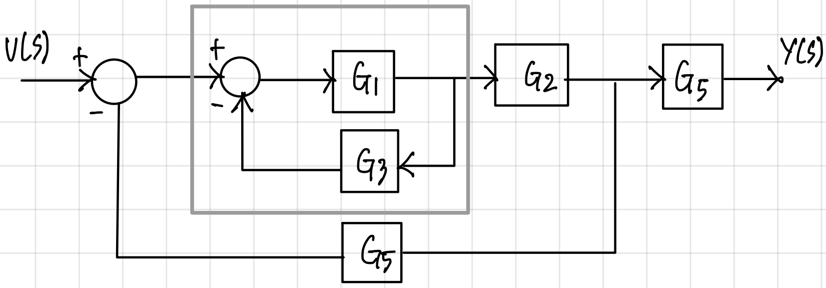
× Understanding systems as a Control Paradigms :

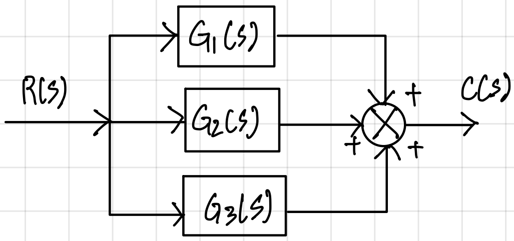


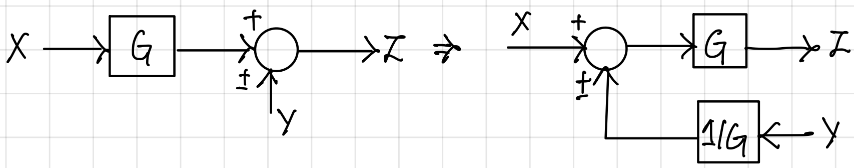
× Understanding Block Diagrams :

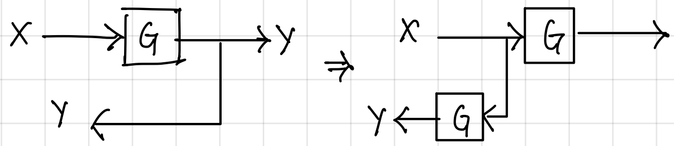


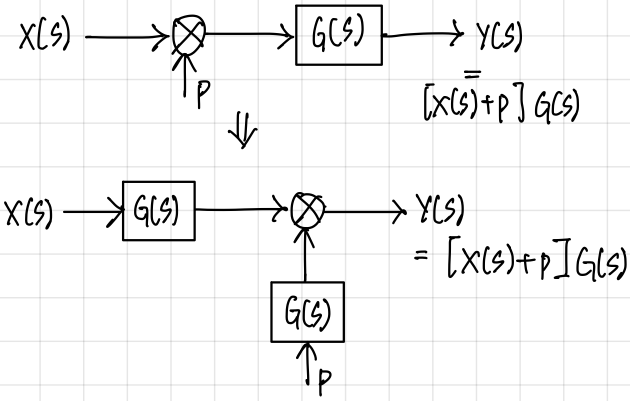


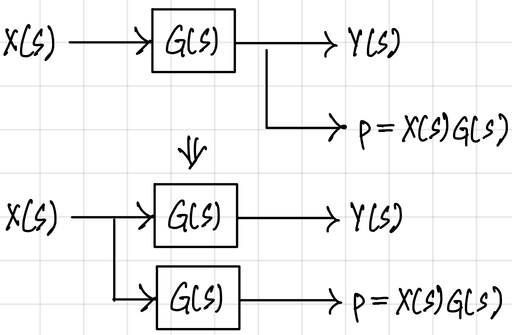




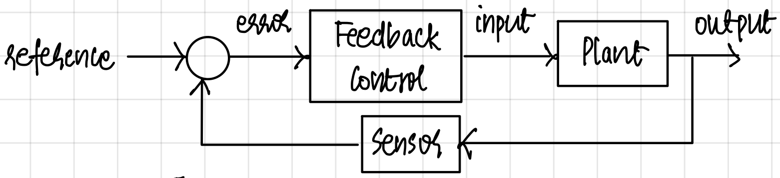


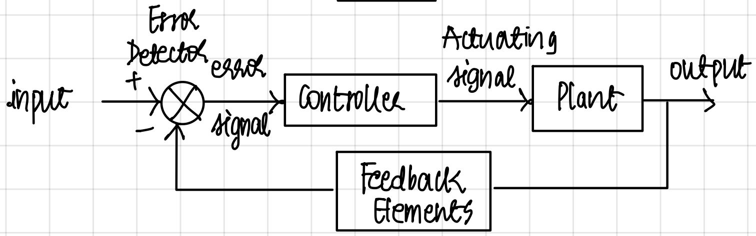




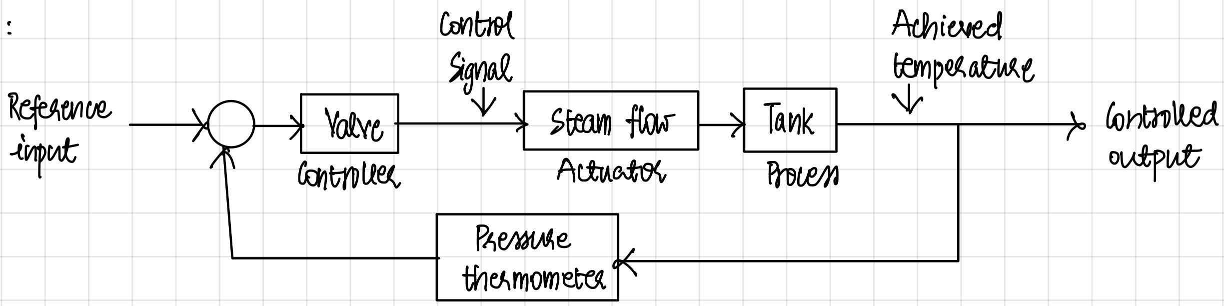


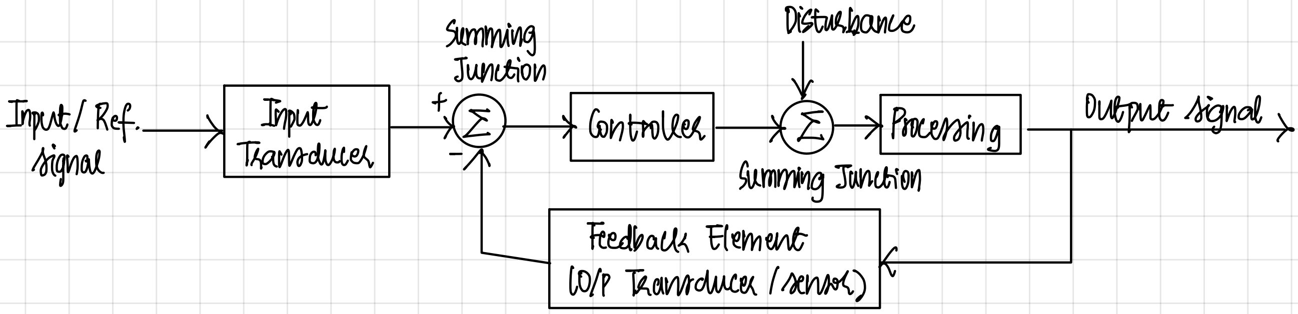
× The Concept of Feedback :



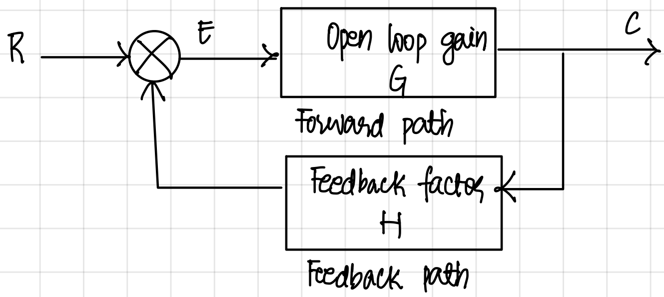


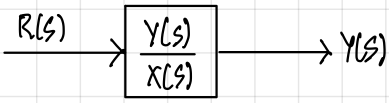
Eg:





× How to analyse Feedback loops - The Transfer function approach :



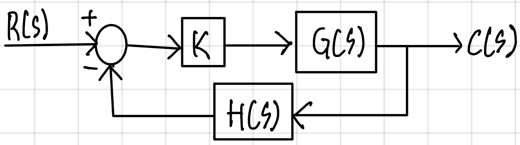


$$R(s) - H(s)\,Y(s) = E(s)$$

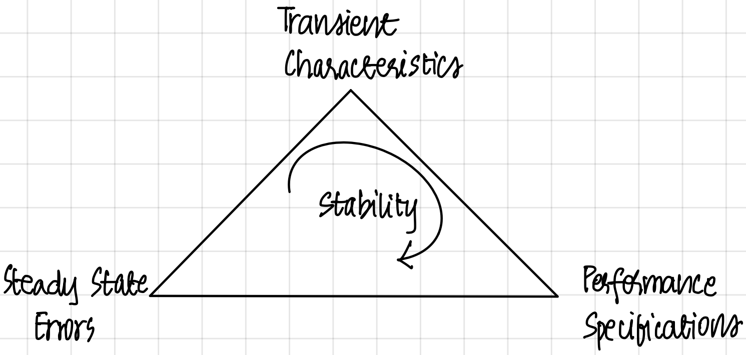
$$Y(s) = E(s)\,G(s)$$

$$\Rightarrow Y(s) = R(s)\,G(s) - H(s)\,G(s)\,Y(s)$$

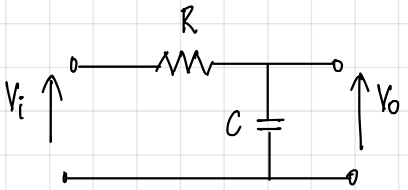
$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)}$$

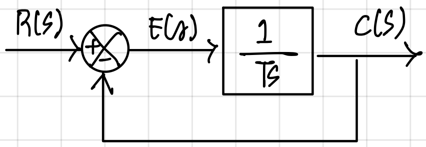


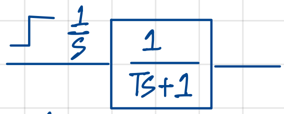
System Behaviour :



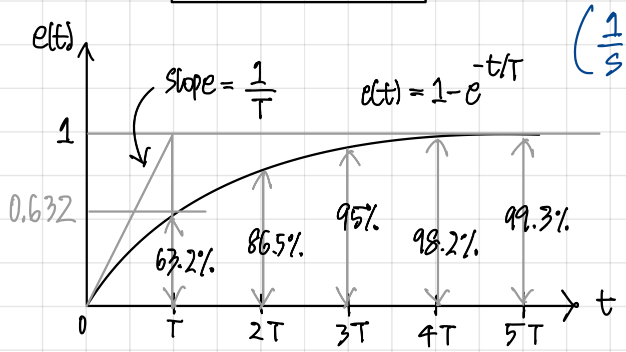
The Order of Systems :







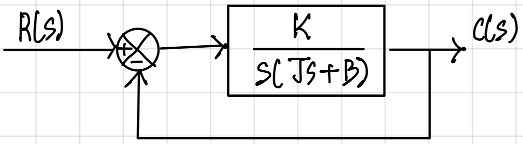
$$\left(\frac{1}{s}\right)\left(\frac{1}{Ts+1}\right)$$



output to step input

Goodness of the Control system
Time taken for a system to response

Second Order Systems :



$$\frac{C(S)}{R(S)} = \frac{K}{JS^2 + BS + K}$$

roots can be real or imaginary

$$\frac{C(S)}{R(S)} = \frac{\frac{K}{J}}{\left[S + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[S + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

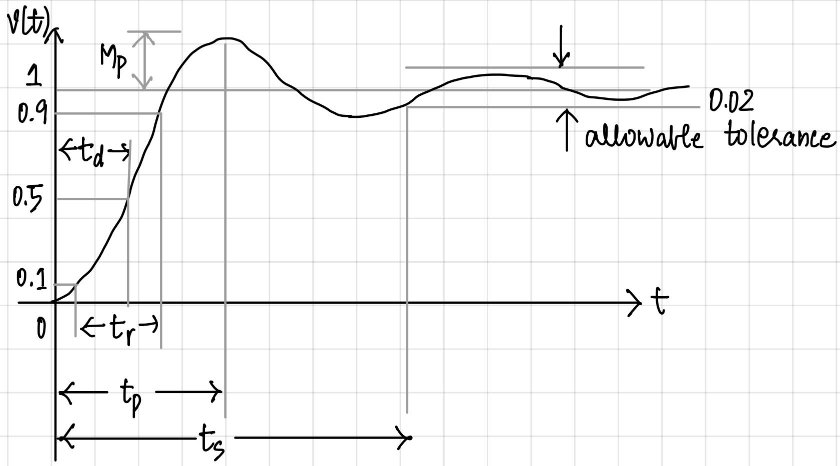
$$\frac{K}{J} = \omega_n^2 \ , \quad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

ω_n: undamped natural co-efficient

ζ: damping factor

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

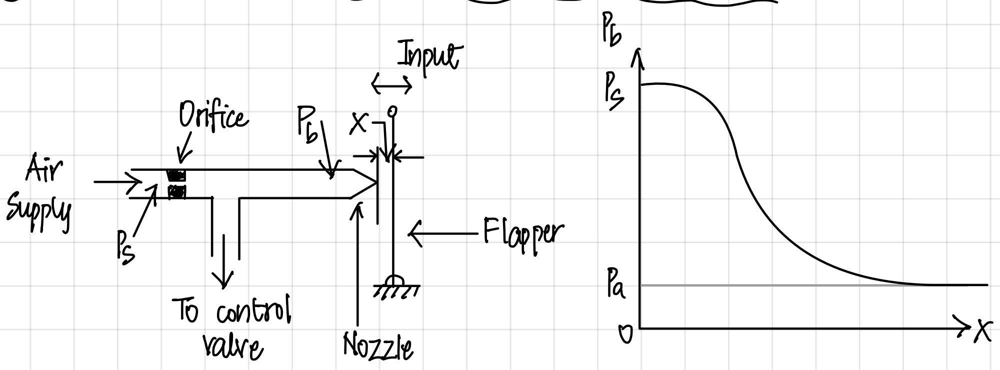
$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$



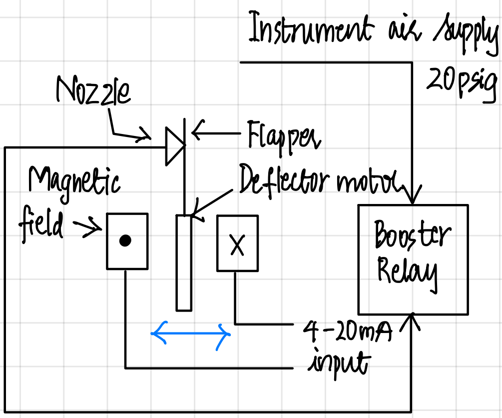
t_r – Rise time. – The lesser the rise time, the better the response and the better the control eqpt.
If t_r is lesser, the M_p gets higher, the settling time t_s goes higher.

t_p – Peak time
M_p – Peak Overshoot
t_d – Delay time

× Pneumatic, Mechanical, Hydraulic, Electrical Controllers: (Just for examples)

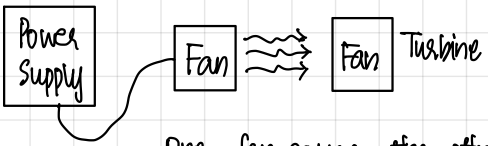


Flapper Nozzle System



Control valves are still pneumatically operated since the response time is very low as compared to electrical systems
Controlled High Speed Systems are Pneumatics

- Fluid Coupling:



One fan causes the other to become a generator

Similarly, have an impeller instead of fan, by changing the level of fluid using coupling coefficient
Fluid Coupling Devices.

- Variable Frequency Drives

- Servo Motors, Stepper Motors

× Response Characteristics:

1. Delay time, t_d : The delay time is the time required for the response to reach half the final value the very first time.

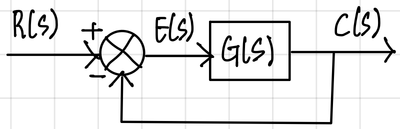
2. Rise time, t_r : The t_r is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

3. Peak time, t_p : The peak time is the time required for the response to reach the first peak of the overshoot.

4. Maximum (percent) overshoot, M_p : The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then its common to use the maximum percent overshoot. It is defined by the amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

5. Settling Time, t_s : The settling time is the time required for the response curve to reach and stay within a range about the final value of the size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.

× Steady State Error:



$$R(s) - C(s) = E(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(s) = \frac{1}{1+G(s)} R(s)$$

Due to final value theorem,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Consider a step response,

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s} = \frac{1}{1+G(0)}$$

Type of a system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s+1)(T_b s+1)\ldots}{(T_1 s+1)(T_2 s+1)\ldots} = K$$

$$e_{ss} = \frac{1}{1+K_p}$$

$$K_p = \lim_{s \to 0} \frac{K(T_a s+1)(T_b s+1)\ldots}{s^N(T_1 s+1)(T_2 s+1)\ldots} = K$$

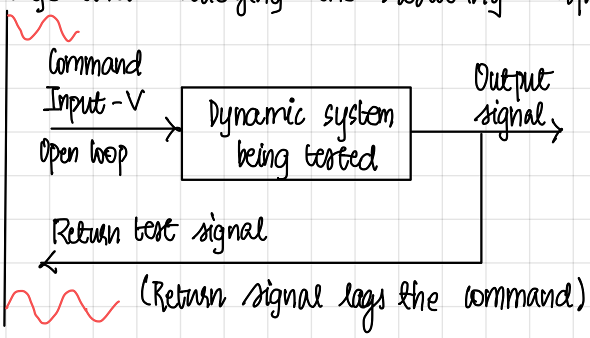
$$e_{ss} = \frac{1}{1+K'} \text{, for type 0 systems}$$

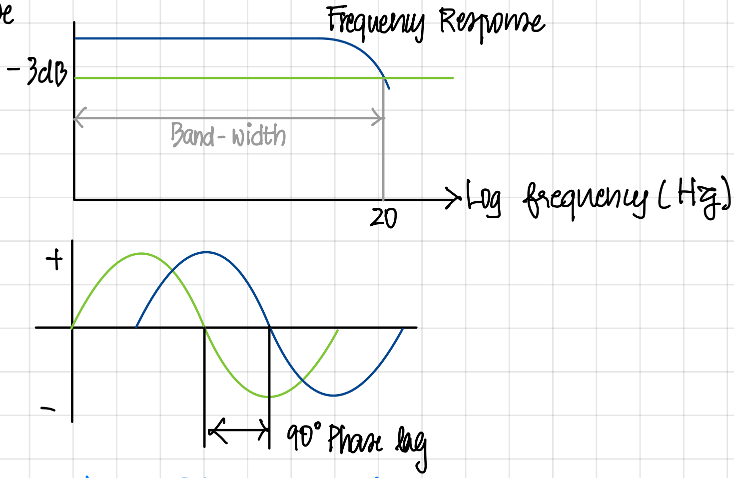
$e_{ss} = 0$, for type 1 or higher systems

Ramp Input, $e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)}$

× Conceptualizing Frequency Response :

- By the term frequency response, we mean the steady-state response of a system to a sinusoidal input.
- In frequency-response methods, we vary the frequency of the input signal over a certain range and studying the resulting response





when giving 0.001 Hz to 150Hz AC current to a DC galvanometer.
At 0.001Hz, the galvanometer swings left to right as the frequency is almost a DC.
As the frequency increases, the galvanometer swing reduces and at 150Hz it stops.
The amplitude vs frequency graph droops.

The frequency Domain :

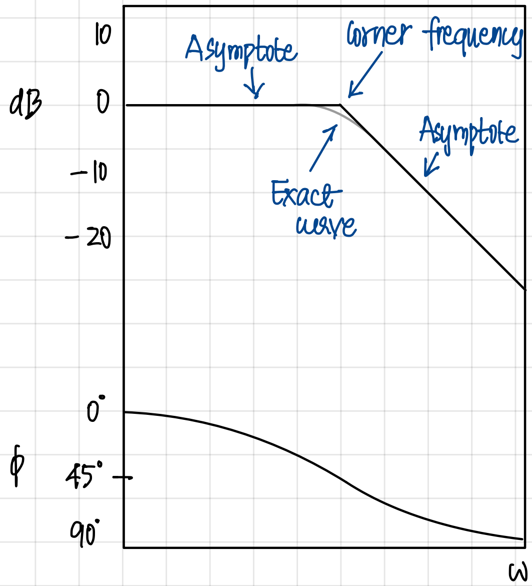
$$G(s) = \frac{K(T_a s+1)(T_b s+1)\dots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\dots(T_p s+1)}$$

$$G(jw) = \frac{K(T_a jw+1)(T_b jw+1)\dots(T_m jw+1)}{(jw)^N(T_1 jw+1)(T_2 jw+1)\dots(T_p jw+1)}$$

1. Gain K
2. Integral and derivative factors $(jw)^{\mp 1}$
3. First order factors $(1+jwT)^{\mp 1}$
4. Quadratic factors $[1+2\zeta(jw/w_n)+(jw/w_n)^2]^{\mp 1}$

$$G(jw) = \frac{1}{1+2\zeta\left(j\frac{w}{w_n}\right)+\left(j\frac{w}{w_n}\right)^2}$$

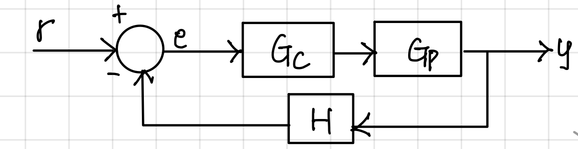
Bode plot :



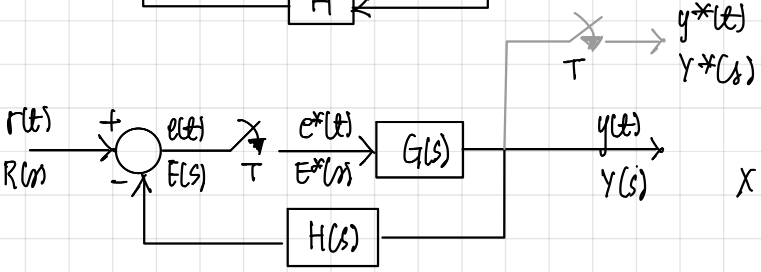
Relating Bode Poles with Damping factor :

$\frac{0.35}{t_r}$ is related to corner frequency.

× Sampling the Control loop : Z-Transforms :



$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$



$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^{k}$$

× Z-transforms:

$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^{k}$$

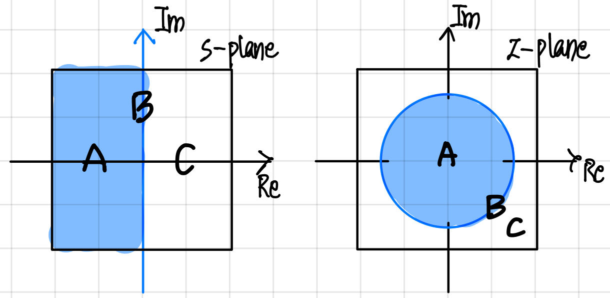
$$x(t) = \begin{cases} 1(t), & 0 \le t \\ 0, & t < 0 \end{cases}$$

$$X(z) = Z[1(t)] = \sum_{k=0}^{\infty} 1z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

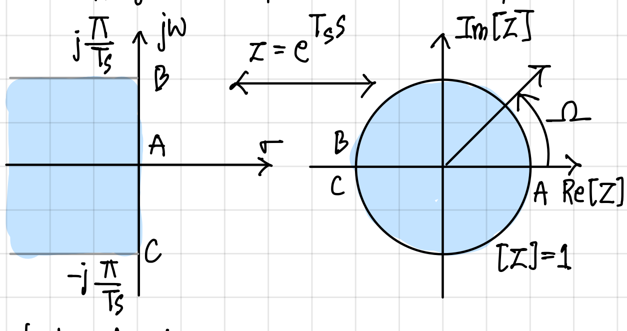
$$x(t) = \begin{cases} a^k, & k = 0,1,2\ldots \\ 0, & k < 0 \end{cases}$$

$$X(z) = Z[a^k] = \sum_{k=0}^{\infty} x(k)z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots$$

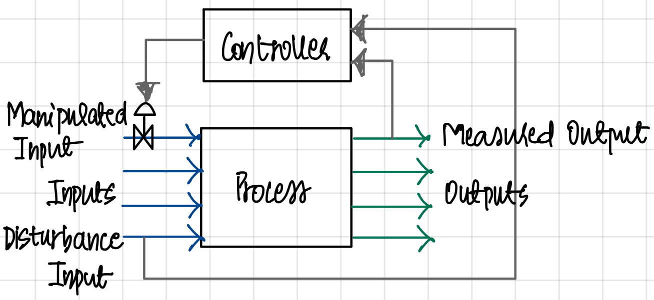


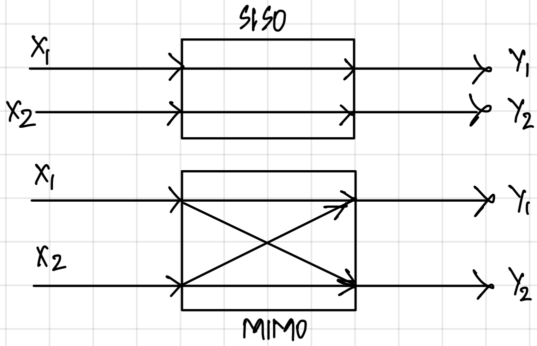
Mapping the s-plane into the z-plane



Laplace domain

× Limitations of the Transfer Function Approach for Multi Variable Problem Solving :





High Order DE

↓

A Set of First Order DEs

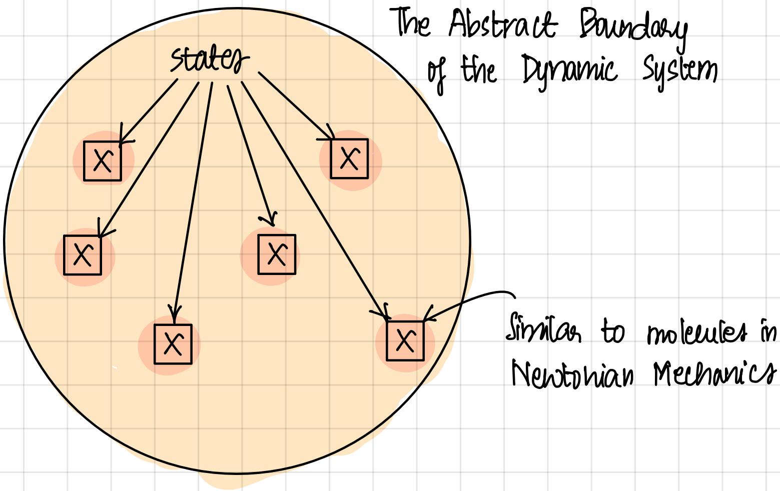
↙ ↓ ↘

Numerical solution | Matrix Equation | State Space (Control System)

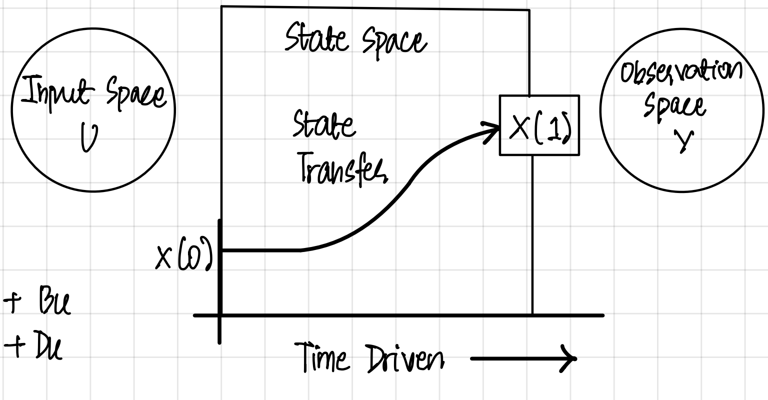
× Visualizing the State space Framework :

Any Dynamic System comprises of internal variables, termed as states

No. of States,
same as Order of the system
– effectively the no. of energy storing elements in the system



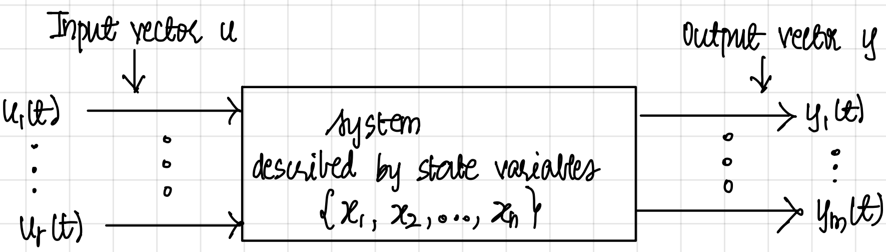
∘ Concept of State Transfer :



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

× State Space Equations :

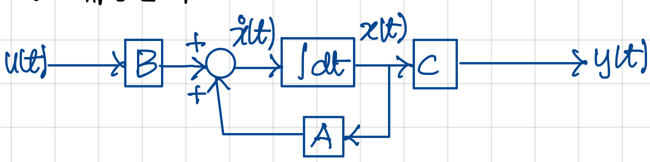
MIMO Systems :



$\dot{x} = Ax + Bu$; $y = Cx + Du$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} + \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \\ b_{n1} & \cdots & b_{nr} \end{bmatrix}}_{n\times r} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}}_{r\times 1}$$

$$\underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}}_{m\times 1} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1}$$



Selection of state variables from an nth order DE :

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = u(t)$$

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad \dots, \quad x_n = y^{n-1}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_4, \dots, \text{ and } \dot{x}_n$$

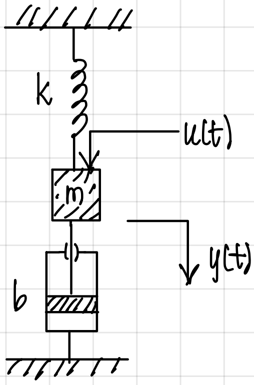
$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = [1 \ 0 \ 0 \dots 0]\, x$$

Formulating the State Space Equations :

Eg1: Taking the same control value eg.,



$$m\ddot{y} + b\dot{y} + ky = u$$

where, y is the linear travel / displacement

This system is of second order. This means that the system involves two integrators, states space variables $x_1(t)$ and $x_2(t)$

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

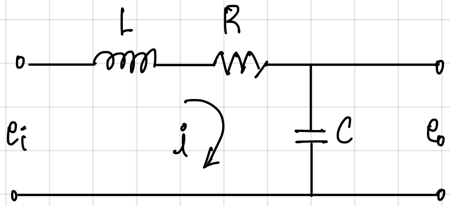
$$\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

substituting $x_1 = y$, $x_2 = \dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \Leftarrow \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$[y] = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = x_1$$

Eg2:



$$L\frac{di}{dt} + Ri + \frac{1}{c}\int i\,dt = e_i$$

$$\frac{1}{c}\int i\,dt = e_o$$

$$sL\,i(s) + R\,i(s) + \frac{1}{Cs}\,i(s) = E_i(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

Differential equation for the system

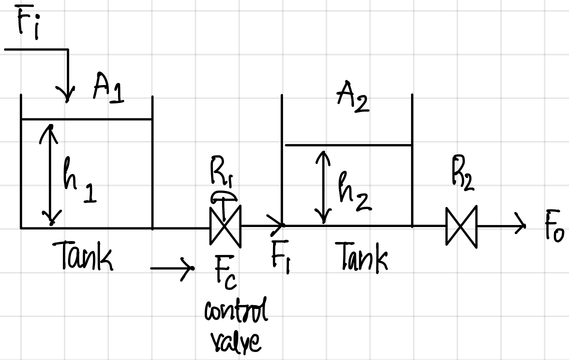
$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Taking state variables by, $x_1 = e_o$, $x_2 = \dot{e}_o$
&
$u = e_i$, $y = e_o$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eg3:



$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \end{bmatrix} u$$

$A_1\frac{dh_1}{dt} = F_i - F_c$	$A_1\frac{dh_1}{dt} = F_i - K_1(h_1 - h_2)$
$A_2\frac{dh_2}{dt} = F_c - F_o$	$A_2\frac{dh_2}{dt} = K_1(h_1 - h_2) - K_2h_2$
$F_c = K_1(h_1 - h_2)$	$F_c = K_1(h_1 - h_2)$
$F_o = K_2h_2$	$F_o = K_2h_2$

Eg4:

$$2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 10u(t)$$

$x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$

$\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + 5u(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 5u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

× Transfer function to State space :

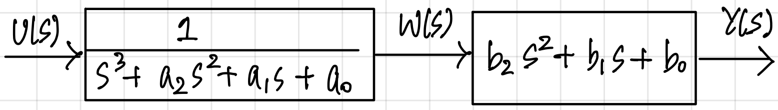
$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n}$$

$$\begin{aligned}
x_1 &= y & \dot{x}_1 &= x_2 \\
x_2 &= \dot{y} & \dot{x}_2 &= x_3 \\
&\vdots & &\vdots \\
& & \dot{x}_{n-1} &= x_n \\
x_n &= y^{(n-1)} & \dot{x}_n &= -a_n x_1 - \dots - a_1 x_n + u \quad (Ax + Bu)
\end{aligned}$$

$$y = Cx = x_1$$

× Transfer Functions with Numerator :

$$\frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$



$$W(s) = \frac{U(s)}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t)$$

$$\dddot{w} = -a_2 \ddot{w} - a_1 \dot{w} - a_0 w + u(t)$$

$$Y(s) = b_2 s^2 W(s) + b_1 s W(s) + b_0 W(s)$$

$$y(t) = b_2 \ddot{w} + b_1 \dot{w} + b_0 w$$

$$\Rightarrow y = b_0 x_1 + b_1 x_2 + b_2 x_3$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

× <u>Solution of the State Equation :</u>

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu$$

$$e^{-At}x(t) = x(t_0) + \int_0^t e^{-At}Bu(T)\,dT$$

$$x(t) = e^{At}x(t_0) + \int_0^t e^{A(t-T)}Bu(T)\,dT$$

× <u>Solution of the State Equation by Laplace Transform :</u>

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$sX(s) - AX(s) = X(0) + BU(s)$$

$$X(s) = (sI-A)^{-1}X(0) + (sI-A)^{-1}BU(s)$$

$$X(s) = \mathcal{L}[e^{At}]X(0) + \mathcal{L}[e^{At}]BU(s) \longrightarrow$$

Initial time is given by t_0 instead of 0

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-T)}Bu(T)\,dT$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-T)}Bu(T)\,dT$$

∘ Discretizing the State Equation :

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-T)}Bu(T)\,dT$$

$$\downarrow$$

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^t e^{A(t-s')}Bu(s')\,ds'$$

$$\phi(t,t_k)\,x(t_k) + \Gamma(t,t_k)\,u(t_k)$$

$$x(t_{k+1}) = \phi(t_{k+1},t_k).\,x(t_k) + \Gamma(t_{k+1},t_k)\,u(k)$$

× State Transition Matrix:

$\phi(t)$ of the following system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\phi(t)$ is given by,

$$\phi(t) = e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$$

$$sI-A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$\therefore \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi^{-1}(t) = \phi(-t)$$

$$\phi^{-1}(t) = e^{-At} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

× Properties of the State Transition Matrix:

$$\phi(t) = e^{At}$$

$$\phi(0) = e^{A0} = I$$

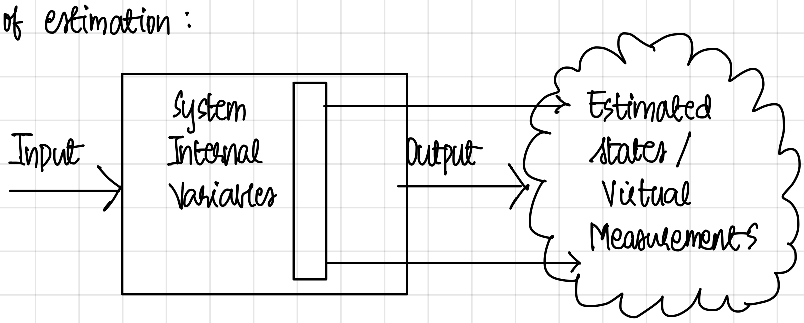
$$\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1} \text{ or } \phi^{-1}(t) = \phi(-t)$$

$$\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$$

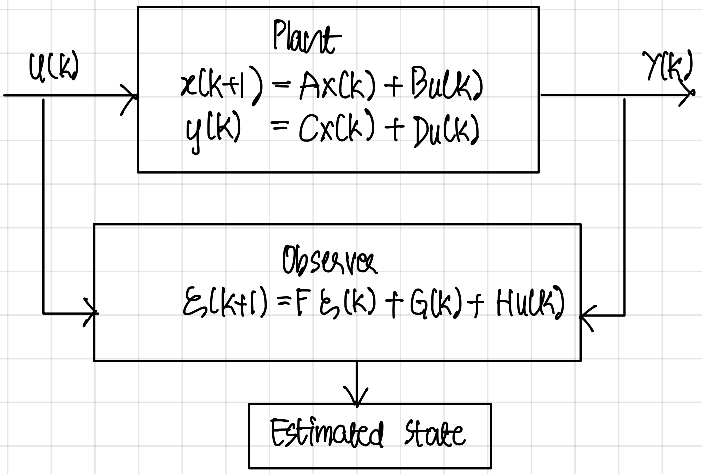
$$[\phi(t)]^n = \phi(nt)$$

$$\phi(t_2-t_1)\,\phi(t_1-t_0) = \phi(t_2-t_0) = \phi(t_1-t_0)\,\phi(t_2-t_1)$$

× Basis of estimation :



- Uncertainities in Process
- Uncertainities in Measurement



× Factoring in Uncertainities of Process and Measurements :

System Model : $x_k = Ax_{k-1} + Bu + w$

- The matrix A is state transition matrix
- The matrix B is input matrix
- The vector w represents additional noise, assumed to have covariance Q.

Measurement model : $Z_k = Hx_k + v$

- Matrix C is measurement matrix
- The vector v is measurement noise, assumed to have covariance R.

Best estimate of state $\hat{x}$ with covariance P.

× Linear Transformations on a Gaussian Random Variable :

Let x be a Gaussian Random n vector with mean m_x and covariance $P_{x}x$ and A is a known $m \times n$ matrix, then the random vector y is defined by

$$y = Ax$$

is a Gaussian with mean and covariance :

$$m_y = Am_x$$
$$P_{yy} = AP_{xx}A^T$$

Let x and y be jointly Gaussian n and m vectors,
A and B are known $p\times n$ and $p\times m$ matrices, then the random p vector z is defined as,

$$z = Ax + By$$

is Gaussian characterized by Mean and Covariance:

$$m_z = Am_x + Bm_y$$

$$P_{zz} = AP_{xx}A^T + AP_{xy}B^T + BP_{yx}A^T + BP_{yy}B^T$$

× Linear Combinations of Jointly Gaussian Random Variables & Non-Random Vectors are also called Gaussian Random Variables

If we modify the earlier equation and write:

$$z = Ax + By + c$$

where c is a known Non-Random p vector,
then z is a Gaussian Random p vector, with mean and covariance,

$$m_z = Am_x + Bm_y + c$$

$$\underbrace{P_{zz} = AP_{xx}A^T + AP_{xy}B^T + BP_{yx}A^T + BP_{yy}B^T}$$

$$x_k = \underbrace{A_{k|k-1}}_{\text{discretized version}} x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}$$

$$x_k = A_{k|k-1}x_{k-1} + B_{k-1}u_{k-1}$$

$$\boxed{P_{zz} = AP_{xx}A^T + \cancel{AP_{xy}B^T} + \cancel{BP_{yx}A^T} + BP_{yy}B^T}$$

$$P_{zz} = AP_{xx}A^T + BP_{yy}B^T$$

$$P_{zz} = AP_{xx}A^T$$

× System Dynamics with Forcing Function & Noise:

State equation

$$x_k = A_{k|k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}$$

Alternatively,

$$x_k = \phi_{k|k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + w_{k-1}$$

Measurement Equation,

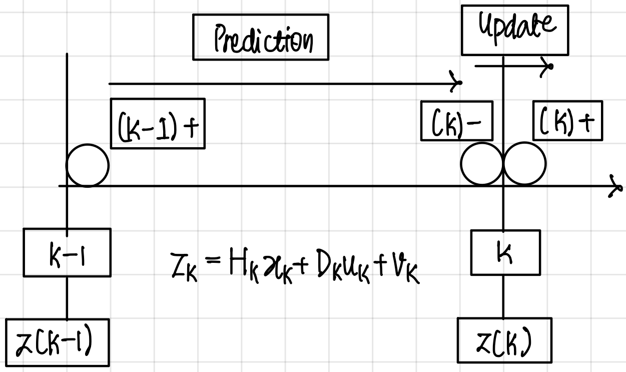
$$z_k = H_k x_k + D_k u_k + v_k$$

$$E[x_0] = \mu_0^x$$

$$E[w_k] = 0 \ \forall k$$
$$E[v_k] = 0 \ \forall k$$
$$\text{cov}\{w_k, w_j\} = Q_k \delta_{kj}$$
$$\text{cov}\{v_k, v_j\} = R_k \delta_{kj}$$
$$\text{cov}\{x_0, x_0\} = P_0$$
$$\text{cov}\{w_k, v_j\} = 0 \ \forall k$$
$$\text{cov}\{x_0, w_k\} = 0 \ \forall k$$
$$\text{cov}\{x_0, v_j\} = 0 \ \forall j$$

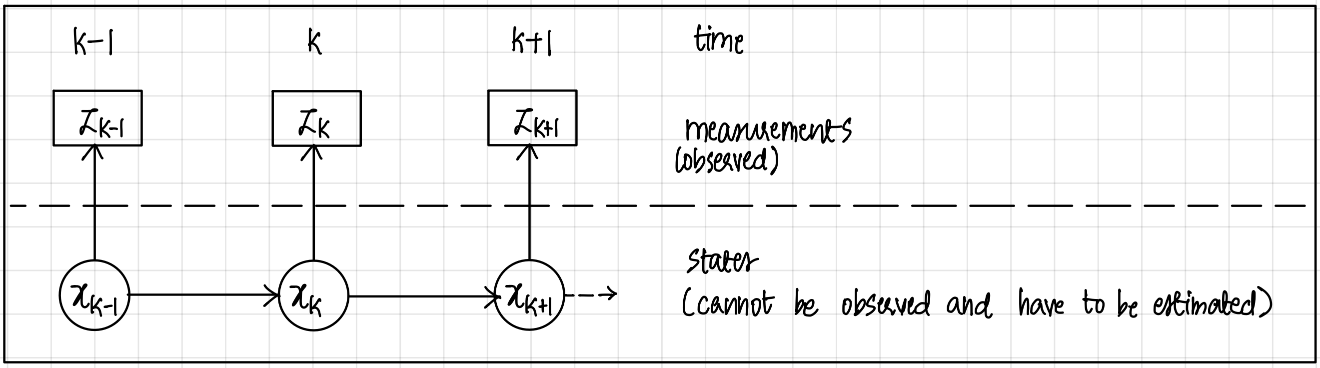
× Understanding the Prediction & Update Process / steps :

$$x_k = A_{k|k-1} x_{k-1} + B_{k-1} u_{k-1}$$



At time k, we have a measurement z_k available, where

$$z_k = H_k x_k + v_k$$



<u>Kalman Gain & Covariance :</u>

To determine matrix K_k to be termed as Kalman Gain

In order to minimize

$$\mathbb{E}\left[(\hat{x}_k - x_k)^T (\hat{x}_k - x_k)\right]$$

or

$$\mathbb{E}\left[(x_k - \hat{x}_k)^T (x_k - \hat{x}_k)\right]$$

If $\tilde{x}_k = (x_k - \hat{x}_k)$, then

$$\mathbb{E}\left[(\hat{x}_k - x_k)^T (\hat{x}_k - x_k)\right] = \mathbb{E}\left[\tilde{x}_k^T \tilde{x}_k\right]$$

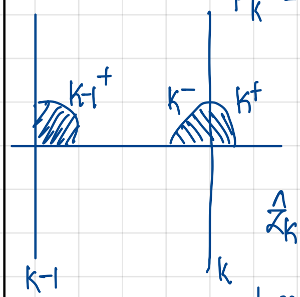
Since $\mathbb{E}\left[\tilde{x}_k^T \tilde{x}_k\right] = \text{Trace } \mathbb{E}\left[\tilde{x}_k \tilde{x}_k^T\right]$

Define a matrix P_k as,

$$P_k = \mathbb{E}\left[\tilde{x}_k \tilde{x}_k^T\right]$$

$\hat{x}_{k^-} = A\hat{x}_{k-1}^+ + Bu_{k-1}$

$P_{k^-} = A P_{k-1^+} A^T$



$\hat{x}_{k^+} = \hat{x}_{k^-} + K\epsilon$

$P_{k^+} = P_{k^-} - K(?)$

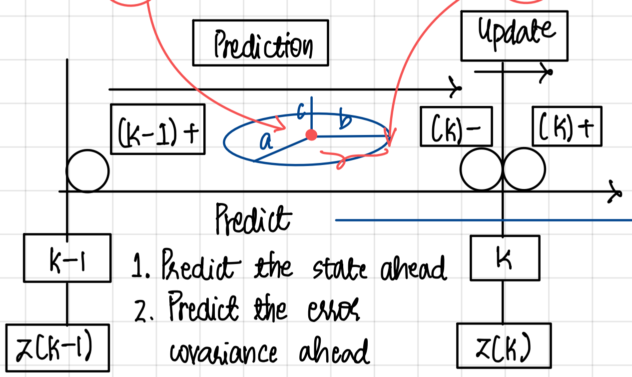
$\hat{z}_k - z_k = \epsilon$

$z_k = h x_k + v$

$y_k = C x_k$

× Understanding the Prediction & Update Process/ steps :

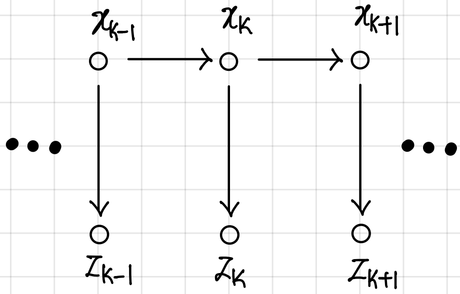
$$x_K = A_{K|K-1} x_{K-1} + B_{K-1} u_{K-1} + w_{K-1}$$

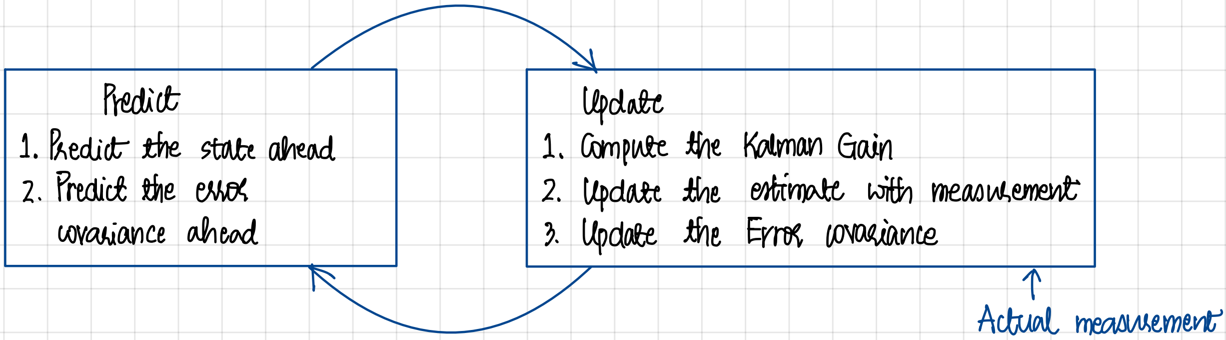


Update
1. Compute the Kalman Gain
2. Update the estimate with measurement
3. Update the Error covariance

$$Z_K = H_K x_K + D_K u_K + V_K$$

- Predict & Update :





- An optimal state estimator :

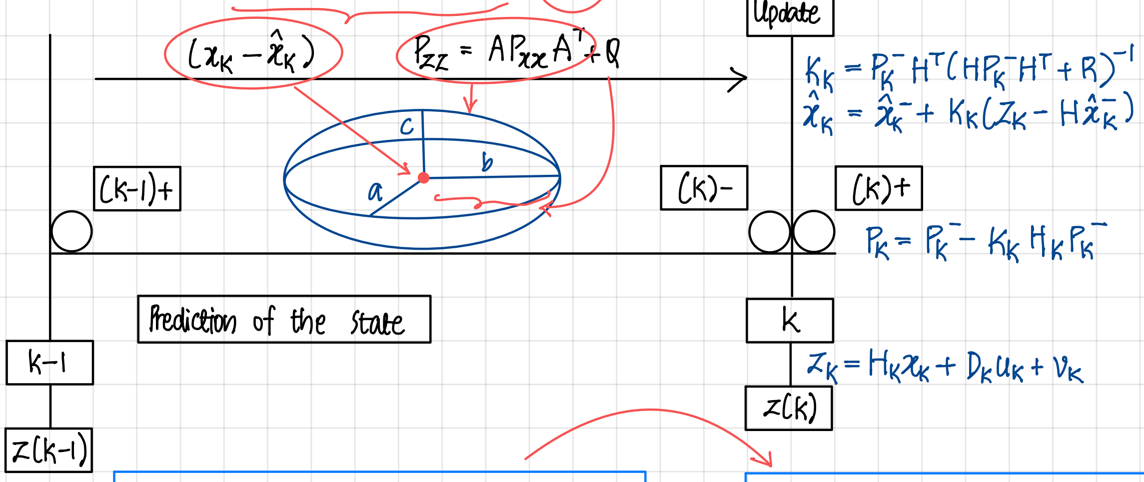
- Estimate the state
- Minimizes the Error
- Obtaining the Mean & Covariance

$$\mathbb{E}[x_K] = \hat{x}_K$$

$$\mathbb{E}\left[(x_K - \hat{x}_K)(x_K - x_K)^T\right] = P_K$$

× Review of the Prediction & Update Process / Steps

$$x_k = \underbrace{A_{k|k-1} x_{k-1} + B_{k-1} u_{k-1}}\; + w_{k-1}$$

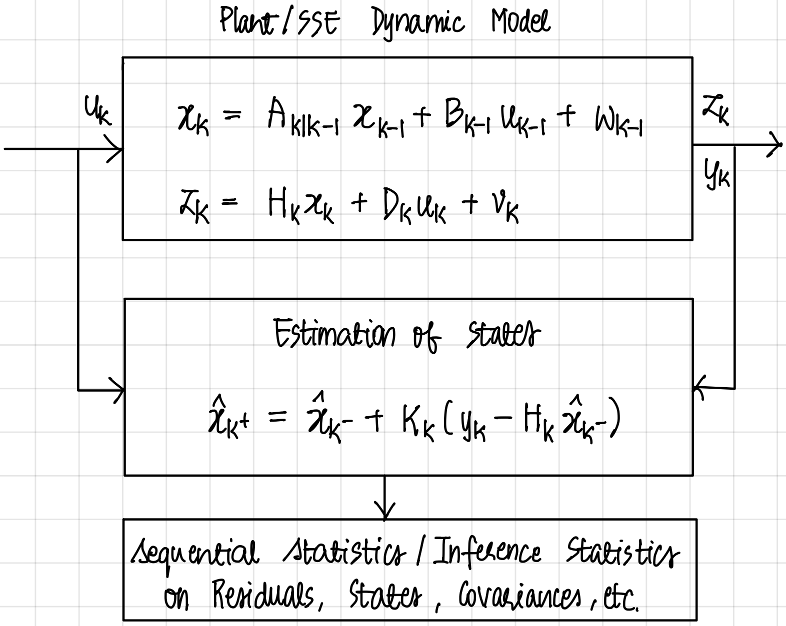


1. Project the state ahead
   $$\hat{x}_k^- = A \hat{x}_{k-1} + B u_k$$
2. Project the error covariance ahead
   $$P_k^- = A P_{k-1} A^T + Q$$

1. Compute the Kalman gain:
   $$K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$$
2. Update estimate with measurement Z_k
   $$\hat{x}_k = \hat{x}_k^- + K_k (Z_k - H \hat{x}_k^-)$$
3. Update the error covariance
   $$P_k = (I - K_k H) P_k^-$$

× Why estimate?

Predictive Maintenance



Fault Detection & Diagnostics
- System Faults
- Actuator Faults
- Sensor Faults

- Monitoring Residues
- Monitoring States
- Monitoring Convergence of Prior & Posterior Covariances

- Multiple Model Methods
- Generalized Likelihood Method

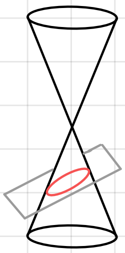
× Building Uncertainty Models:

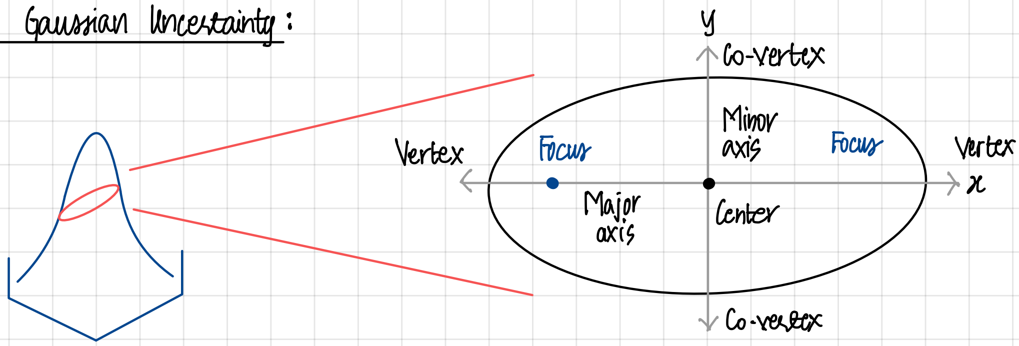
$$f(x\mid\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

$$p(x;\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

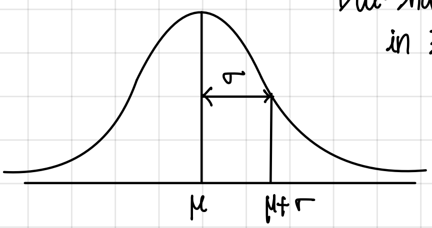
$$f_{x_1,x_2}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} \exp\left\{-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}$$

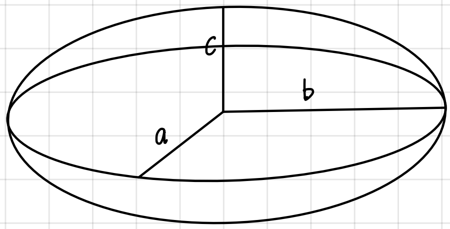
× Ellipsoid as a Measure of Gaussian Uncertainty:





Bell-shaped curve in 3D





$$\sigma^2 = \frac{\Sigma(x-\mu)^2}{N} \; ; \; \operatorname{cov}(x,y) = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n-1}$$

× The Statistical Moments:

The covariance of X & Y, denoted $\operatorname{Cov}(X,Y)$, is the number

$$\operatorname{Cov}(X,Y) = \mathbb{E}[(X-\mu_x)(Y-\mu_y)]$$

where, $\mu_x = \mathbb{E}(X)$ and $\mu_y = \mathbb{E}(Y)$

Computational Formula:

$$\operatorname{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

Correlation Formula:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

$$\rho = \rho_{X,Y} = \rho(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sigma_X\sigma_Y}$$

$$m \triangleq \mathbb{E}[X] \qquad \mathbb{E}[cy] = c\,\mathbb{E}[y]$$

$$\mathbb{E}[y_1+y_2] = \mathbb{E}[y_1] + \mathbb{E}[y_2]$$

$$\begin{aligned}
P = \mathbb{E}[(x-m)(x-m)^T] &= \mathbb{E}[xx^T - xm^T - mx^T + mm^T] \\
&= \mathbb{E}[xx^T] - \mathbb{E}[x]m^T - m\,\mathbb{E}[x^T] + mm^T \\
&= \mathbb{E}[xx^T] - mm^T - mm^T + mm^T
\end{aligned}$$

$$P = \mathbb{E}[xx^T] - mm^T$$

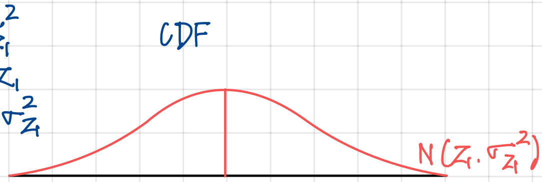
$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

- Explaining using an example:

- Navigating – Propagating a Conditional PDF?
  - Say, you are lost in sea. However, you have some idea on obtaining directions from the pole-star
  - Obtain a measurement

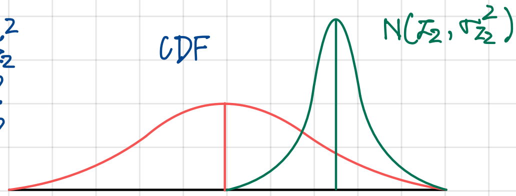
- Access the Measurement
  - The measurement may not be exact.
  - It would have a mean and a variance.

$z_1, \sigma_{z_1}^2$
$\hat{x}_1 = z_1$
$\hat{\sigma}_1^2 = \sigma_{z_1}^2$



- Update the Previous Measurement
  - Now, let's say your friend is better in navigating
  - He takes a second measurement - more accurate?
  - Even this would be a mean and variance, but possibly the variance would be smaller - he has more expertise

$z_2, \sigma_{z_2}^2$
$\hat{x}_2 = ?$
$\hat{\sigma}_2^2 = ?$



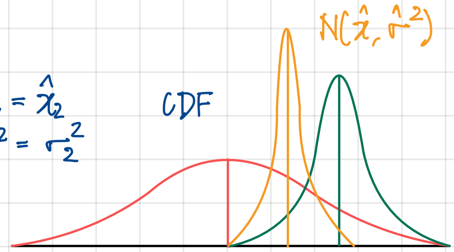
- Access the Results of Two Measurements:
  - $$\hat{x}_2 = \left[\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_1 + \left[\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)\right] z_2$$
    $$= \hat{x}_1 + K_2\left[z_2 - \hat{x}_1\right]$$

where, $K_2 = \sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)$

  - Now combining the variances, we obtain
    $$\frac{1}{\sigma_2^2} = \frac{1}{\sigma_{z_1}^2} + \frac{1}{\sigma_{z_2}^2}$$

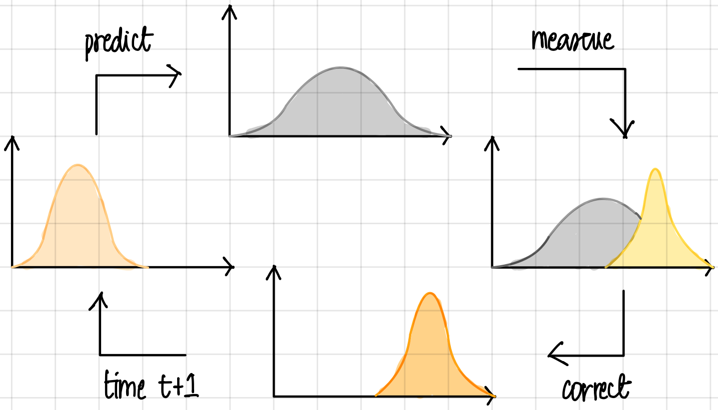
- Result:
  - The result is an improved idea as to how much away you are from the coast
  - Are we propagating a CDF and each time updating it with the latest measurement?

$\hat{x} = \hat{x}_2$
$\hat{\sigma}^2 = \sigma_2^2$



- Thus,
  - Time update (a priori estimates)
    Project state and covariance forward to next time step, i.e. predict
  - Measurement update (a posteriori estimates)
    Update with a (noisy) measurement of the process, i.e. correct

× Kalman Filter - A Bayesian Approach :



State Equation

$$x_k = A_{k|k-1}\, x_{k-1} + B_{k-1} u_{k-1} + w_{k-1}$$

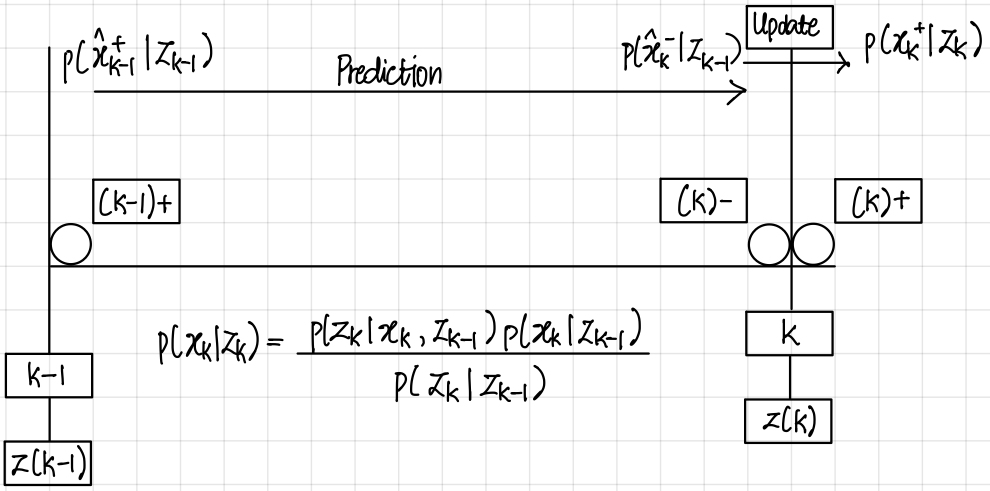
Alternatively,

$$x_k = \Phi_{k|k-1}\, x_{k-1} + \Gamma_{k-1} u_{k-1} + w_{k-1}$$

Measurement Equation,

$$z_k = H_k\, x_k + D_k u_k + v_k$$

× Bayesian Approach :



$$\boxed{p(x_k | Z_{k-1}) = \left[(2\pi)^{n/2} P_k^{-\,1/2}\right]^{-1} \exp\left(-\frac{1}{2}\left[(x_k - \hat{x}_k^-)^T P_k^{-\,-1} (x_k - \hat{x}_k^-)\right]\right)}$$

prior function

$\longrightarrow$ The posterior function To generate $p(x_k | Z_k)$

Baye's theorm, $p(x_k|Z_k) = \dfrac{p(x_k, Z_k)}{p(Z_k)}$

$$\Rightarrow \quad p(x_k | Z_{1:k}) = \frac{p(Z_{1:k} | x_k)\, p(x_k)}{p(Z_{1:k})}$$

seperate $p(Z_{1:k})$ into $p(Z_k, Z_{1:k-1})$

$$= \frac{p(Z_k, Z_{1:k-1} | x_k)\, p(x_k)}{p(Z_k, Z_{1:k-1})}$$

factorize joint probability : $p(a, b|c) = p(a|b, c) . p(b|c)$ and $p(a, b) = p(a|b) . p(b)$

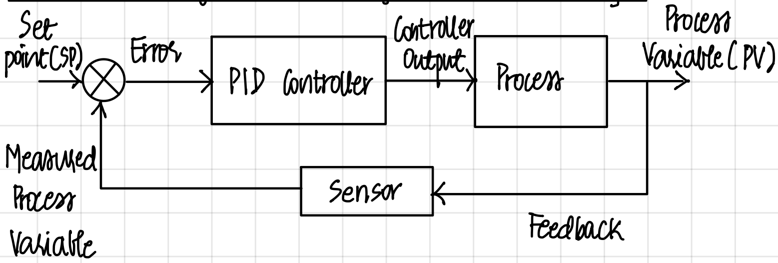
$$= \frac{p(Z_k | Z_{1:k-1}, x_k)\, p(Z_{1:k-1} | x_k)\, p(x_k)}{p(Z_k | Z_{1:k-1})\, p(Z_{1:k-1})} = \frac{p(Z_k | Z_{1:k-1}, x_k)\, p(x_k | Z_{1:k-1})\, p(Z_{1:k-1})\, p(x_k)}{p(Z_k | Z_{1:k-1})\, p(Z_{1:k-1})\, p(x_k)}$$

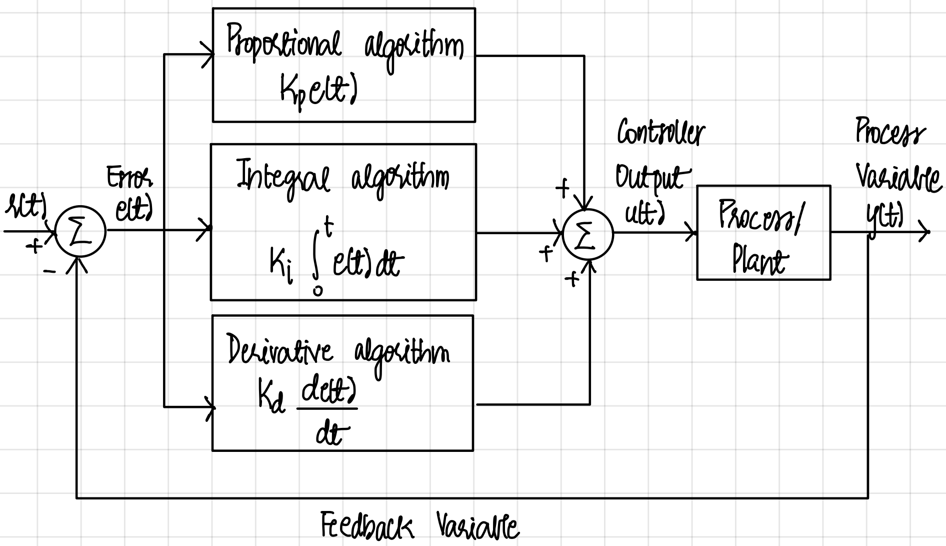
$$= \frac{p(Z_k | x_k)\, p(x_k | Z_{1:k-1})}{p(Z_k | Z_{1:k-1})}$$

Summary of Bayesian Approach:

Formulating the Gaussian Density Function
Establish the Independence of Noise Function
Establish Gaussianity of the Prior Function
Computing the Mean & Covariance of the Prior
Understand the Posterior Function
Plug the Gaussian Expression with mean & variance of each term
By plugging, we obtain PDF for posterior

Maintenance of PID Control Systems in Industry:

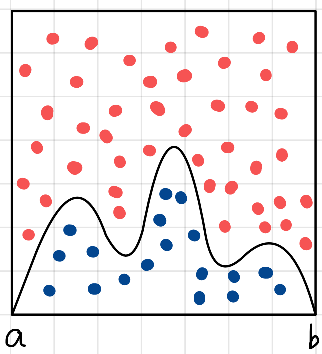




Monte Carlo Method:

- Based on random experiments or trials
  Eg: choose points randomly inside the square
- The area under the curve $f(x)$ is

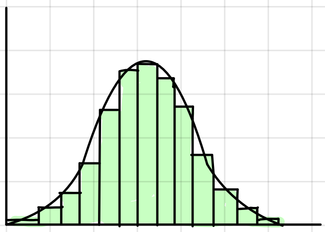
$$= \int_a^b f(x)\,dx \approx \frac{\text{number of blue points}}{\text{total number of points}} \times \text{area of the square}$$



- As total number of random points increase, the accuracy gets better.

Less theory, More simulation

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx \longrightarrow E(x) \cong \frac{1}{N}\sum_{i=1}^{N} x_i$$



Bayesian Filtering / Estimation for Non-Linear systems with Non-Gaussian Process & Measurement Uncertainities:

State Variable Representation of Dynamical systems, which are strongly Non-linear

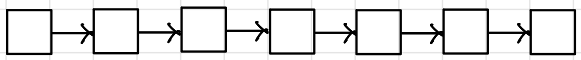
$$x_k = A(r_k | r_{k-1}) x_{k-1} + B(r_k | r_{k-1}) u_{k-1} + F(r_k | r_{k-1}) w_{k-1}$$
$$z_k = C(r_k | r_{k-1}) x_k + D(r_k | r_{k-1}) u_k + G(r_k | r_{k-1}) v_k$$

The state sequence is a Markov random process

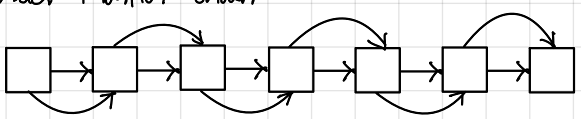
Markov chains and processes

$$x = (x_1, x_2, \dots, x_t, \dots, x_T)$$

1st order Markov chain

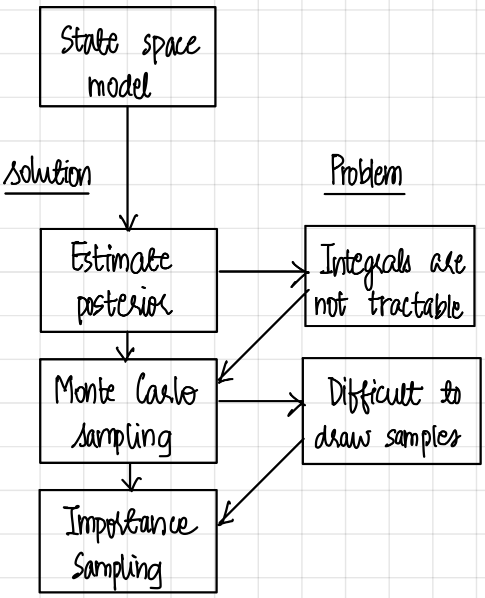


2nd order Markov chain



Components in a Bayesian approach:

- State vector x_k
- Measurement vector z_k
- Initial pdf $p(x_0)$
- Likelihood pdf: (from model) $p(z_k | x_k)$
- Prior pdf $p(x_k | z_{1:k-1})$
- Posterior pdf $p(x_k | z_{1:k})$



The Bayesian Posterior:

$$p(x_k|z_{1:k}) = \frac{p(z_k|x_k)p(x_k|z_{1:k-1})}{p(z_k|z_{1:k-1})}$$

normalizing term

$$\boxed{p(x_k|z_k) = K\, p(z_k|x_k) \int p(x_k|x_{k-1})\, p(x_{k-1}|z_{k-1})\, dx_{k-1}}$$

likelihood

temporal prior

posterior probability at previous time step

$p(x_k|z_k)$ can be an arbitrary, non-Gaussian, multi-modal distribution

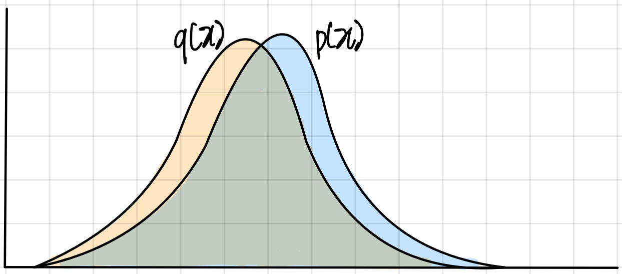
evidence: the normalizing constant in the denominator

$$p(z_k|z_{1:k-1}) = \int p(z_k|x_k)\, p(x_k|z_{1:k-1})\, dx_k$$

Importance Sampling:

- Classical Monte Carlo Integration - Difficult to draw samples from the desired distribution
- Importance sampling solution:
  1. Draw samples from another (proposal) distribution.
  2. Weight them according to how they fit the original distribution

- Free to choose the proposal density
- Important:
  - It should be easy to sample from the proporsal density
  - Proporsal density should resemble the original density as closely as possible.



$$\mathbb{E}[f(x)] = \int f(x)p(x)dx \approx \frac{1}{n}\sum_i f(x_i)$$

$$= \int f(x)\frac{p(x)}{q(x)}q(x)dx \approx \frac{1}{n}\sum_i f(x_i)\frac{p(x_i)}{q(x_i)}$$

× The Bayesian Approach for the posterior :

$$\boxed{\begin{array}{ll} p(x_0) & \\ p(x_t \mid x_{t-1}) & \text{for } t \geq 1 \\ p(y_t \mid x_t) & \text{for } t \geq 1 \end{array}}$$

$$I(f_t) = \mathbb{E}_{p(x_{0:t} \mid y_{1:t})}\left[f_t(x_{0:t})\right] \triangleq \int f_t(x_{0:t})\, p(x_{0:t} \mid y_{1:t})\, dx_{0:t}$$

At any time t, the posterior distribution is given by Baye's theorem

$$\boxed{p(x_{0:t} \mid y_{1:t}) = \frac{p(y_{1:t} \mid x_{0:t})\, p(x_{0:t})}{\int p(y_{1:t} \mid x_{0:t})\, p(x_{0:t})\, dx_{0:t}}}$$

$$\boxed{p(x_{0:t+1} \mid y_{1:t+1}) = p(x_{0:t} \mid y_{1:t}) \frac{p(y_{t+1} \mid x_{t+1})\, p(x_{t-1} \mid x_t)}{p(y_{t+1} \mid y_{1:t})}}$$

Prediction :

To be continued from Lecture 14, 52:24

Prof. NIRAV

× From Traditional tools to AI:

- Data-rich world
- Can we use the data for Predictive Maintenance?
- Predictive Maintenance:
  - Data Acquistion and Processing
  - Diagnostics: Anomaly or Fault Detection and localization
  - Prognostics and forecasting: Monitoring the health and predicting the remaining useful life
  - Decision Support and Human/Software Machine interface
    - Decision-making through control (corrective action) and optimization

× Data-driven Predictive Maintenance:

- Diagnostics: Data-driven Fault detection and localization using key statistics and formulation
  - Unsupervised Approaches to detect, and localize faults with unlabelled data
  - Supervised Approaches: classification to faulty conditions with labelled data
- Prognostics and Forecasting
  - Remaining useful life is an important indicator of health
  - Regression-based formulation (Predict RUL) from time series of futures.
  - Classification-based formulation (Predict failure within given classes)
  - Survival Analysis based approaches
    - Traditional Approaches Kaplan-Meier Model, Cox hazard model
    - Regression models for survival response
    - DL based formulations

Data-driven Fault Diagnostics
Unsupervised ML:
T^2 and Q statistics for application of PCA in fault detection

Video 16

Traditional Approach:
Univariate Statistical Monitoring

In a multivariate data, we have $x = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_n]^T$; m variables, n datapoints

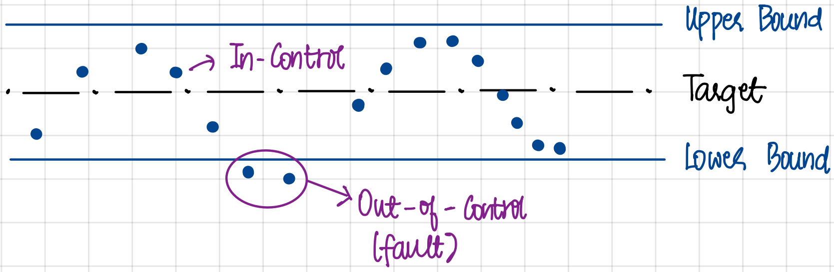
Taking one variable, $(x_1, x_2, \dots, x_n)$
There will be two types of variations:

- Common cause:
  Usually data obtained is associated with some random noise, $x_1 + \underbrace{\epsilon_1 \sim N(0, \sigma^2)}_{\text{random noise}}$
- Special cause:
  Fault(s) in the system

Variations in data are due to special causes

Univariate Data Analysis Methods - Shewhart Chart, Cummulative Chart, Exponentially Weighted Moving Average Chart

Shewhart Chart, Taking i^{th} variable $\tilde{x}_i$



If the data is within upper and lower bound, then the data is in-control
if its outside there bounds, then the data is out-of-control (fault)

W : An event : In-control
W_k : An event : k^{th} fault

$$\tilde{x}_i \begin{cases} \to W \\ \to W_k \end{cases}$$

if $\tilde{x}_i \subset W : H_0$
$\tilde{x}_i \subset W_k : H_1$

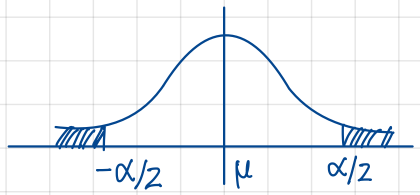
$$p(\tilde{x}_i / W) \ \& \ p(\tilde{x}_i / W_k)$$

wkt, $\tilde{x}_i \sim N(\underset{\text{Target}}{\mu}, \sigma^2)$

$$P(\tilde{x}_i = \mu) \text{ or } P(\tilde{x}_i \neq \mu)$$

Instead of this, we can just find the level of significance.

$$P(\mu - L \leq \tilde{x}_i \leq \mu + U) = 1 - \alpha$$; α - level of significance
(the amount of accuracy we are ready to compromise)

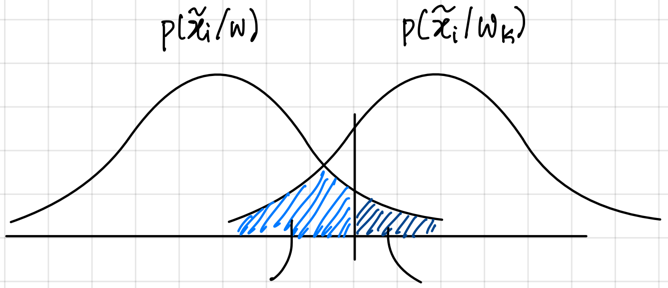


$P(\mu - Z_{\alpha/2}\sigma \le \tilde{x}_i \le \mu + Z_{\alpha/2}\sigma)$

$\underbrace{\mu - Z_{\alpha/2}\sigma}_{\text{Lower}}$, $\underbrace{\mu + Z_{\alpha/2}\sigma}_{\text{Upper}}$

By following this approach, the data is prone to,

- Type I error
- Type II error



The problem with univariate approaches is that, we can't observe what is happening with other variables and if 2 variables are correlated, we can't observe that.

× T^2 Statistic :

- Training dataset $X \in \mathbb{R}^{n \times m}$ (m variables, n observations) (Multivariate)
- Sample covariance matrix S is given by,

$$\boxed{S = \frac{1}{n-1} X^T X}$$

- Eigenvalue decomposition of the sample covariance matrix S is

$$S = V \underbrace{\Lambda}_{\text{singular values}} \overbrace{V^T}^{\text{eigenvector}}$$

× Why T^2 Statistics?

Quantifies the derivation of observed data from the reference point in a multivariate space.

- Also referred to as Hotelling's T^2 statistics
- Follows a Hostelling's T^2 distribution, which is a generalization of the univariate Student's t-distribution to multiple dimensions
- Scaled squared 2-norm of an observation vector from its mean
- Takes into account both mean and covariances of the variables.

× What is T^2 Statistics?

Assuming S is invertible

$$z = \Lambda^{-1/2} V^T x_i$$

Where $x \in \mathbb{R}^m$ is the observation vector

Hotelling's T^2 statistic is given by,
$$T^2 = z^T z$$

- Scaling on x is in the direction of the eigenvectors which is inversely proportional to the standard deviation along the eigenvectors

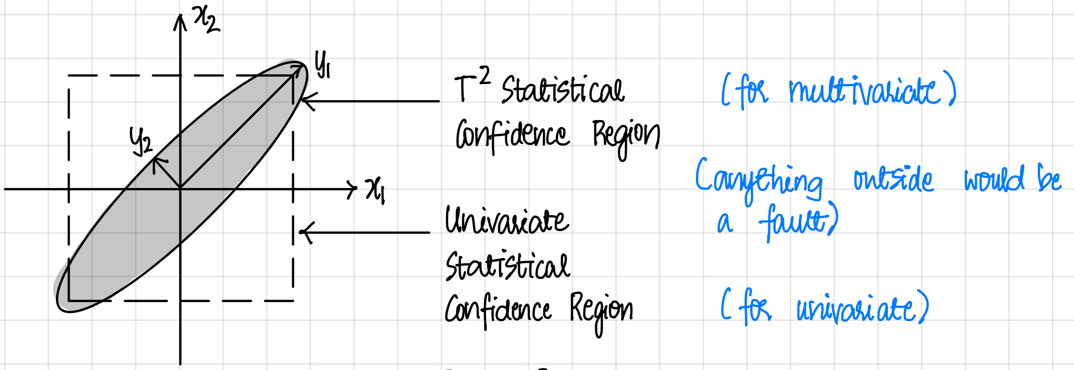
Thresholds for T^2 Statistics :

T^2 statistic follows a χ^2 distribution with m degrees of freedom
$$T_\alpha^2 = \chi_\alpha^2(m)$$

Assumptions :
- Observations are randomly sampled from a multivariate normal distribution
- Sample mean vector and covariance matrix for normal operations are equal to actual mean vector and covariance matrix

$$T^2 = z_1^2 + z_2^2 + \dots + z_m^2$$
$T^2 \leq T_\alpha^2$: confidence region in the training

As degree of correlation between variables increases, elliptical confidence region becomes elongated and the amount of conservatism increases.



For a level of significance α, and $m = 2$, $T^2 \leq T_\alpha^2$ is an elliptical confidence region

Unknown Covariance matrix :
- Covariance matrix can be estimated from the sample covariance matrix
- Faults can be detected using the threshold.

$$T_\alpha^2 = \frac{m(n-1)(n+1)}{n(n-m)} F_\alpha(m, n-m)$$

where, $F_\alpha(m, n-m)$ is upper $100\alpha\%$ critical point of the F-distribution with m and $n-m$ degrees of freedom.

Outlier detection :

$$T_\alpha^2 = \frac{(n-1)^2 (m/(n-m+1)) F_\alpha(m, n-m-1)}{n(1+(m/(n-m-1))}$$

× Data requirements for T^2 Statistic :

- Quality and quantity of the training dataset influences the effectiveness of T^2 statistic
- For a given α, the relative error is calculated by,

$$\boxed{\epsilon = \frac{\frac{m(n-1)(n+1)}{n(n-m)} F_\alpha(m, n-m) - \chi^2_\alpha(m)}{\chi^2_\alpha(m)}}$$

- If ϵ is low, more data should be collected
- Number of observations is approximately 10 times the dimensionality of the observation space.
- In case some diagonal elements of Λ are small, T^2 values will be erratic.
- In that condition, dimensionality reduction is recommended.

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× PCA : Revisit

- Dimensionality Reduction
- Denoising Technique
- Identification Technique

$$X = \lambda_1 t_1 P_1^T + \lambda_2 t_2 P_2^T + \dots + \lambda_m t_m P_m^T$$

$n \times m$

$n \leq m$

loading vector

scores matrix

singular values

$$PCA\left(\frac{1}{\sqrt{n-1}} X\right) = U \Lambda V^T$$

$X \rightarrow$ minimum no. of factors

$$\hat{X} = \lambda_1 \tilde{t}_1 \tilde{P}_1^T + \lambda_2 \tilde{t}_2 \tilde{P}_2^T + \dots + \lambda_a \tilde{t}_a \tilde{P}_a^T \; ; \quad \boxed{\lambda_{a+1} = \dots = \lambda_m \cong 0} \text{ small}$$

$$\text{rank}(\hat{X}) = a < m$$

$$\text{rank}(X) = m$$

$$X = (T \Lambda P^T)_a + (T \Lambda P^T)_{m-a}$$

$$T_{m-a}^T X = 0$$

× Application of PCA in fault detection :

- T^2 statistic can be calculated from the PCA representation

$$\boxed{T^2 = x^T V (\Sigma^T \Sigma)^{-1} V^T x}$$

where, $V \in \mathbb{R}^{m \times m}$ unitary matrix, $\Sigma \in \mathbb{R}^{n \times m}$ contains non-negative real singular values of decreasing magnitude along its main diagonal

$$PCA(X) = \tilde{T} \Lambda P^T = (\tilde{T}\Lambda P^T)_a + \underbrace{(\tilde{T}\Lambda P^T)_{m-a}}_{\approx 0}$$

$S = U \Lambda V^T$; $\quad V = P_a \longrightarrow$ eigen vectors

$\Lambda = \Lambda_a = \Sigma_a^2$

- Smaller singular values are prone to errors because these values contain small signal-to-noise ratio
- Therefore, loading vectors associated with larger singular values should be retained.
- T^2 for the lower-dimensional space by including the loading vector (P_a) with only a largest singular values

$$\boxed{T^2 = x^T P_a \Sigma_a^{-2} P_a^T x}$$

Where, Σ_a contains first a rows and columns of Σ

- The above equation measures the variations in the scores space only
- If the actual mean and covariance are known, then

$$\boxed{T_\alpha^2 = \chi_\alpha^2(a)}$$

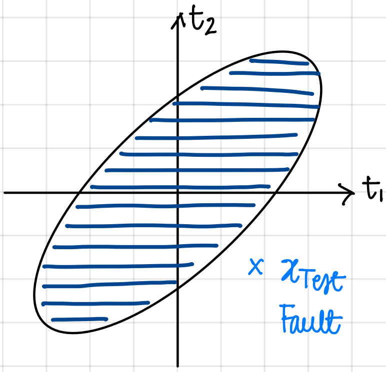
- T^2 statistic threshold when actual covariance matrix is estimated from sample covariance

$$\boxed{T_\alpha^2 = \frac{a(n-1)(n+1)}{n(n-a)} F_\alpha(a, n-a)}$$

Eg: $n = 50$, $a = 2$, $\alpha = 0.05$

$F_{0.05}(2, 50-2) \longrightarrow$ Check the F-Table $= 3.2$
(or call a python fn.)

$T_{0.05}^2 = \frac{2(49) \times 51}{50(48)} \times 3.2 = 6.64$; T^2 should be $\leq T_{0.05}^2$
$T^2 \leq 6.64$



Applying this with data,

x_{Test}, Training: P_a, Σ_a

$T^2 = x_{Test}^T P_a \Sigma_a^{-2} P_a^T x_{Test}$ ≤ 6.64
≥ 6.64

$\underset{2\times1}{t} = \underset{10\times2}{P_a^T} \underset{10\times1}{x_{Test}}$ $\quad a=2, \ m=10$

$T^2 = t^T \Sigma_a^{-2} t \leq 6.64$

Assuming $\Sigma_a^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \Sigma_a^{-2} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \frac{t_1^2}{2} + t_2^2 \leq 6.64$

equation of ellipse

- For detecting outliers

$$\boxed{T_\alpha^2 = \frac{(n-1)^2 (a/(n-a-1)) F_\alpha(a, n-a-1)}{n(1+(a/(n-a-1))}}$$

- T^2 statistic is sensitive to inaccuracies in the PCA space corresponding to the smaller singular values.
- This is because, it directly measures the scores corresponding to smaller singular values.

× Squared prediction error and Q statistics :

x_{new} & observed low-dimensional space by PCA

Observing the contribution of vectors corresponding to the lowest eigen values

$$X = \underbrace{(T\Lambda P^T)_a}_{\text{captures variations in the data}} + (T\Lambda P^T)_{m-a} \Big] \text{ variation in noise}$$

$$\Lambda_{a+1} \cong \ldots \cong \Lambda_m \cong 0$$

Incase of no fault, this space just captures some random noise aka. residual space

$Q = r^T r$; where r is Residual vector of m

× Q statistics :

Measures the sum of squared residuals or the sum of squared standardized residuals

- Used for monitoring the portion of the observation space corresponding to the $(m-a)$ smallest singular values

$$\boxed{\begin{aligned} Q &= r^T r \\ r &= (I - PP^T)x \end{aligned}}$$

where, r is the residual vector, a projection of the observation x into the residual space,
P is the loading matrix

- Does not directly measure the variations along each loading vector
- Measures the total sum of variations in the residual space
- Does not suffer from an over-sensitivity to inaccuracies in the smallest singular values

× Threshold for Q Statistic :

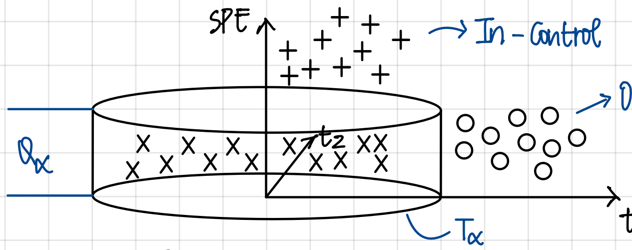
- Also known as squared prediction error (SPE)
- Squared 2-norm measuring the deviation of the observations to the lower-dimensional PCA representation
- Distribution for the Q statistic

$$\boxed{Q_\alpha = \theta_1 \left[\frac{h_0 C_\alpha \sqrt{2\theta_2}}{\theta_1} + 1 + \frac{\theta_2 h_0 (h_0 - 1)}{\theta_1^2} \right]^{1/h_0}}$$

where, $\theta_i = \sum_{j=a+1}^{n} \sigma_j^{2i}$, $h_0 = 1 - \frac{2\theta_1\theta_3}{3\theta_2^2}$, C_α is the normal derivate corresponding to the $(1-\alpha)$ percentile

- The above equation can be used for computing the threshold for a given α.

- When two statistics are utilized along with their respective thresholds, cylindrical in-control region is produced.



'o' indicates T^2 statistic violation, '+' indicates Q statistic violation

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Eg: Data: $X_{n\times m}$; n observations and m variables
: Unlabelled data
: Training set, normal operation regime

$$\text{Covariance } (S) = \frac{1}{n-1} X^T X$$

SVD(X) : compute significant singular values (SV)
Total SVs : m
Significant SVs : a $\rightarrow \lambda_1 > \ldots > \lambda_a > 0$

$\lambda_{a+1} \cong \ldots \cong \lambda_m \cong 0$, but they may not be due to fault

$$\hat{X} = \underbrace{\lambda_1 t_1 P_1^T + \lambda_2 t_2 P_2^T + \ldots + \lambda_a t_a P_a^T}_{\text{use this loading space to compute T statistic}} + \underbrace{\lambda_{a+1} t_{a+1} P_{a+1}^T + \ldots + \lambda_m t_m P_m^T}_{0}$$

$$= T_a \Lambda_a P_a^T$$

$$= \begin{bmatrix} t_1 & t_2 & \ldots & t_a \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_a \end{bmatrix} \begin{bmatrix} P_1^T \\ P_2^T \\ \vdots \\ P_a^T \end{bmatrix}$$

$$T^2 = x^T V (\Sigma^T \Sigma)^{-1} V^T x \quad ; \quad T_\alpha^2 = \frac{a(n-1)(n+1)}{n(n-a)} F_\alpha(a, n-a)$$

where, a : # significant SV, α : level of significance

$$T^2 \leq T_\alpha^2$$

$$\underbrace{x^T P_a \Lambda_\alpha^{-1} P_a^T x}_{\text{Ellipsoid}} \leq T_\alpha^2$$

$$t_a = P_a^T x \Rightarrow t_a^T \Lambda_a^{-1} t_a \leq T_\alpha^2$$

$$\boxed{\frac{t_1^2}{\lambda_1} + \frac{t_2^2}{\lambda_2} + \ldots + \frac{t_a^2}{\lambda_a} \leq T_\alpha^2}$$ equation for ellipse in 2D and ellipsoid in other dimension

When you get x_{New} during testing / operation,

$t_{new} = P_a^T X_{new} = \begin{bmatrix} t_{1,new} \\ t_{2,new} \end{bmatrix}$ check with the region calculated above
if this point outside the ellipsoid then it is a fault
otherwise normal.

PCA Approach for Fault Identification :

Objective :
Determine which observation variables are most relevant for diagonising the fault.

Contribution Plot : (Approach I)

- Used to identify the variables that contribute most to a specific fault.
- Applied in response to a T^2 violation
- Typically, it is based on quantifying the contribution of each process variable to the individual scores.
- For each variable summing the contributions only of those scores responsible for the out-of-control status.

Contribution Plot - Steps :

Check the normalized scores $\left(\frac{t_i}{\sigma_i}\right)^2$ → $\frac{t_1^2}{\sigma_1^2}, \frac{t_2^2}{\sigma_2^2}, \dots, \frac{t_r^2}{\sigma_r^2} > \frac{1}{a} T_\alpha^2$; $r < a$

↓

Determine $r \leq a$ scores responsible for out of control status

↓

Calculate contribution of out of control
$cont_{i,j} = \frac{t_i}{\sigma_i^2} P_{i,j} (x_j - \mu_j)$

↓

When $cont_{i,j}$ is negative, set it equal to zero

↓

Calculate the total contribution of the j^{th} process variable, x_j

$$CONT_j = \sum_{i=1}^{r} cont_{i,j}$$

↓

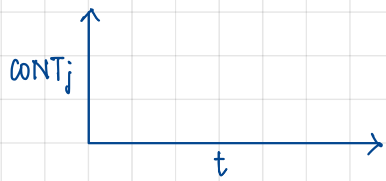
Plot $CONT_j$ for all m variables, x_j

Where, r = Residual vector $P_{i,j} = (i,j)^{th}$ element of the loading matrix

- Variables are prioritized by the total contribution values
- Only sensitive to smaller singular values.

Eg: $\underset{a\times 1}{t} = \underset{a\times m}{P_a^T} \underset{m\times 1}{x}$

$t_1 = P_{11} x_1 + P_{12} x_2 + \dots + P_{a1} x_m$
$t_2 = P_{21} x_1 + P_{22} x_2 + \dots + P_{a2} x_m$
$\vdots$
$t_r = P_{1r} x_1 + P_{2r} x_2 + \dots + P_{ar} x_m$



$cont_{1,2} = \left(\frac{t_1}{\sigma_1}\right) P_{1,2} (x_2 - \mu_2)$ ⟶ calculate for $j = 1, \dots, r$: $cont_{i,j}$ → $CONT_j$ → plot it

× Checking for Change in Variance : (Approach II)

- Based on quantifying the total variation of each of the variables in the residual space.
- Assumption : m−a smallest singular values are all equal.
- Variance of each variable inside the residual space is given by,

$$\boxed{\hat{s}_j^2 = \sum_{i=a+1}^{m} p_{i,j}\,\sigma_i^2}$$

- Given q new observations, out of control variable is indicated by

$$\boxed{s_j^2/\hat{s}_j^2 > F_\alpha(q-a-1,\ n-a-1)}$$

where, s_j^2 and $\hat{s}_j^2$ are variance estimates for new and training set observations, resp.
$F_\alpha(q-a-1,\ n-a-1)$ is $(1-\alpha)$ percentile limit using F distribution

- The above equation is testing the null hypothesis $s_j = \hat{s}_j$
- One side alternative hypothesis = $s_j > \hat{s}_j$
- Null hypothesis is rejected if the equation holds
- In two sided hypothesis testing, the two-sided alternative hypothesis $s_j \neq \hat{s}_j$ is concluded if

$$\boxed{s_j^2/\hat{s}_j^2 > F_{\alpha/2}(q-a-1,\ n-a-1)} \quad \text{or} \quad \boxed{s_j^2/\hat{s}_j^2 > F_{\alpha/2}(n-a-1,\ q-a-1)}$$

× Checking for shift in Mean : (Approach III)

For mean
- In two sided hypothesis testing, the two-sided alternative hypothesis is concluded if

$$\boxed{\frac{\mu_j - \hat{\mu}_j}{\hat{s}_j\sqrt{\frac{1}{q-a}+\frac{1}{n-a}}} > t_{\alpha/2}(q+n-2a-2)} \quad \text{or} \quad \boxed{\frac{\mu_j - \hat{\mu}_j}{\hat{s}_j\sqrt{\frac{1}{q-a}+\frac{1}{n-a}}} < -t_{\alpha/2}(q+n-2a-2)}$$

where, μ_j and $\hat{\mu}_j$ are means of x_i for new and training set observations respectively
$t_{\alpha/2}(q+n-2a-2)$ is $(1-\alpha/2)$ percentile limit using t distribution

- The fault identification approach discussed require a group of $q \gg 1$ observations

× RES : (Approach IV)

- Fault identification measure based on an observation vector at a single time instant is the normalized error.

$$\boxed{RES_j = r_j/\hat{s}_j} \qquad \max(RES_j\ ,\ j=1,\ldots,m)$$

where, r_j is the j^{th} variable of the residual vector

- RES can be used to prioritize the variables where the variable with the highest normalized error is given priority.

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× Identify by the fault class :

- One model based on the data from all fault are stacked into matrix X
- Maximum likelihood classification for an observation x is fault class i with the maximum score discriminant is given by,

$$\boxed{\begin{aligned} g_i(x) &= -\frac{1}{2}\underline{(x-\bar{x}_i)^T P(P^T S_i P)^{-1} P^T (x-\bar{x}_i)} + \ln p_i - \frac{1}{2}\ln\left[\det(P^T S_i P)\right] \\ \bar{x}_i &= \frac{1}{n_i}\sum_{x_j \in X_i} x_j \end{aligned}}$$

aka. scores discriminant in form of $t^T \Sigma^{-1} t$

where, n_i is the number of the data points in fault class i,
X_i is the set of vectors x_j which belong to fault class i
$S_i \in \mathbb{R}^{m \times m}$ is the sample covariance matrix for fault class i

aka. multi-model PCA

i^{th} class fault $i = 1, 2, 3$ & 0 – Normal Condition
New data x_{new}
$\zeta_i \rightarrow$ fault i^{th} event

$P(\zeta_i \mid x_{new})$ means probability that ζ_i fault has occured $\forall i$

Discriminant function at x_{new},
$g_i(X_{new}) = P(\zeta_i \mid x_{new})$ build for each class and compare $g_i(\cdot)$

$$P(\zeta_i \mid x_{new}) = \frac{P(x_{new} \mid \zeta_i)P(\zeta_i)}{P(x_{new})}$$

Taking log on both sides,

$$\ln P(\zeta_i \mid x_{new}) = \ln P(x_{new} \mid \zeta_i) + \ln P(\zeta_i)$$

Naive Bayes Classification,
$P(x \mid \zeta_i) \sim \mathcal{N}(\mu_i, \Sigma_i)$

$$= \frac{1}{(2\pi)^{m/2}\sqrt{\det(\Sigma_i)}} \exp\left(-\frac{1}{2}(x-\mu_i)^T \Sigma^{-1}(x-\mu_i)\right)$$

considering all classes have equal amount of data

$$\ln P(\zeta_i \mid x_{new}) = \cancel{-\frac{m}{2}\ln 2\pi} - \frac{1}{2}\ln\left[\det(\Sigma_i)\right] - \frac{1}{2}(x_{new}-\mu_i)^T \Sigma^{-1}(x_{new}-\mu_i) + \cancel{\ln P(\zeta_i)}$$

constant

$$g_i(X_{new}) = -\frac{1}{2}(x_{new}-\mu_i)^T \Sigma^{-1}(x_{new}-\mu_i) - \frac{1}{2}\ln\left[\det(\Sigma_i)\right]$$

discriminant function for i^{th} fault

$$\underline{g_i(X_{new})} > g_j(X_{new}) \; ; \; i \neq j$$

fault is coming from i, ζ_i

Data : X_0, $X_1, X_2, \ldots, X_p$

$X_0 \to$ Normal operating regime

$X_1, X_2, \ldots, X_p \to$ faulty operations

we need to get $g_i(x)$, for that we need

$$\underset{}{X_1 \to \underset{m\times 1}{\overline{X}_1}, \underset{m\times m}{S_1}} \; ; \; X_2 \to \underset{m\times 1}{\overline{X}_2}, \underset{m\times m}{S_2} \; ; \ldots ; X_p \to \overline{X}_p, S_p$$

then, $$g_1(x) = -\frac{1}{2}(x-\overline{X}_1)^T S_1^{-1}(x-\overline{X}_1) - \ln[\det(S_1)]$$

$$g_2(x) = -\frac{1}{2}(x-\overline{X}_2)^T S_2^{-1}(x-\overline{X}_2) - \ln[\det(S_2)]$$

$$\vdots$$

$$g_p(x) = -\frac{1}{2}(x-\overline{X}_p)^T S_p^{-1}(x-\overline{X}_p) - \ln[\det(S_p)]$$

when x_{new} comes,

$$x_{new} \to \max\{g_1(x_{new}), g_2(x_{new}), \ldots, g_p(x_{new})\}$$

If $S_1 = S_2 = \ldots = S_p = \Sigma$

$$g_i(X) = -(x-\overline{X}_1)^T \Sigma^{-1}(x-\overline{X}_2)$$

is in form $X^T V \Lambda^{-1} V^T X$

- If P is selected to include all of the dimensions of the data and the overall likelihood from all fault classes is same.
Then the previous equation reduces to discriminant function for multivariate statistics

$$\boxed{g_i(x) = -(x-\bar{x}_i)^T (S_i)^{-1}(x-\bar{x}_i) - \ln[\det(S_i)]}$$

- Multivariate statistics serves as a benchmark for the other statistics
- The score discriminant, residual discriminant, and combined discriminant are used with multiple PCA models

- An observation x is classified as a fault class i with maximum score discriminant

$$\boxed{g_i(x) = -\frac{1}{2} x^T P_i \Sigma_{a,i}^{-2} P_i^T x - \frac{1}{2}[\det(\Sigma_{a,i}^{-2})] + \ln(P_i)}$$

where, P_i is the loading matrix for fault class i
Σ_{ai} is the diagonal matrix
Σ_{ai}^2 is the covariance matrix
P_i is the overall likelihood of fault class i

Note: Assumption – observation vector x has been auto scaled according to mean and SD of the training set for fault class i

Weakness: Useful information for other classes is not utilized when each model is derived.

- The previous equation reduces to

$$\boxed{T_i^2 = x^T P_i \Sigma_{a,i}^{-2} P_i^T x}$$

Assumptions: Overall likelihood for all fault classes and sample covariance matrix for all class is the same

- Residual Discriminant represents the observation if important variations in the discriminating between the faults are assumed contained in the residual space.

$$\boxed{Q_i/(Q_\alpha)_i} \quad \text{where, } Q_i = r_i^T r_i \; ; \; r_i = (I - P_i P_i^T)x$$

- Combined discriminant represents the observation if important variations are contained both within the score and residual space

$$\boxed{c_i[T_i^2/(T_\alpha^2)_i] + (1-c_i)[Q_i/(Q_\alpha)_i]}$$

where, c_i is a weighting factor between 0 and 1 for fault class i

- When a fault is diagnosed as fault i, it is not likely to represent a new fault when

$$\boxed{[T_i^2/(T_\alpha^2)_i] \ll 1} \qquad \boxed{[Q_i/(Q_\alpha)_i] \ll 1}$$

- If either of these conditions is not satisfied, it is likely that the observation represents a new fault

- It is recommended to access the likelihood of successful diagnosis before application of pattern classification

$$\boxed{\text{Similarity index} = f = \frac{1}{m}\sum_{j=1}^{m} \tilde{\sigma}_j \text{ is used to quantify similarity between the covariance structures} \quad (\text{Range}: 0-1)}$$

where, $\tilde{\sigma}_j$ is the j^{th} singular value of $V_1^T V_2$, V_1 and V_2 contain all m loading vectors for classes 1 and 2 respectively

× Reduction order and PLS prediction:

- In fault diagnosis, the reduction order is determined by the value that minimizes the information criteria

$$\boxed{f_m(a) + \frac{a}{\tilde{n}}}$$

where, $\tilde{n}$ is the average no. of observations per class, $f_m(a)$ is the misclassification rate for the training set

- Misclassification rate = $\dfrac{\text{No. of incorrectly assigned classes}}{\text{Total no. of classifications made}}$

$$\boxed{\begin{aligned} &\text{Estimated score vector } (\hat{t}_j) = E_{j-1} w_j \\ &\text{Matrix residual } (E_j) = E_{j-1} - \hat{t}_j q_j^T \end{aligned}}$$

q_j is the loading vector

- Prediction of the prediction block $Y_{train2,a}$ of the training set using the PLS1 with a PLS components

$$\boxed{Y_{train2,a} = F_j = \sum_{j=1}^{a} b_j \hat{t}_j q_j^T} \quad \text{or} \quad \boxed{Y_{train2,a} = XB2_a}$$

where, b_j is the regression coefficient, $B2_a$ is the regression matrix

- Prediction of the predicted block $Y_{train1,a}$ of the training set using PLS1 with a PLS components

$$\boxed{\begin{aligned} &Y_{train1,a} = [Y_{train1,a} \; Y_{train2,a} \; \ldots \; Y_{train\,p,a}] \\ &Y_{train\,i,a} = f_{i,j} = \sum_{j=1}^{a} b_{i,j} \hat{t}_i q_{i,j} \end{aligned}}$$

× Fault detection, identification and diagnosis using PLS:

- All fault detection, identification and diagnosis techniques for PCA can be applied for PLS.

> Overestimation: When the element of Ytrain for an in-class member > 1 or element Ytrain for non-class member is > 0

> Underestimation: When the element of Ytrain for an in-class member > 1 or element Ytrain for non-class member is < 0

> Taking underestimation and overestimation of Y into account into a second cycle of PLS algorithm (NIPALS) is done if some of the elements are underestimated while others are overestimated

Eg: 4 faults

		Unestimation
x_{new} → PLS → 0.2		−0.2
0.3		−0.3
1.2 → Nearest to 1 → fault		0.9 → fault
0.1	Maximum value	−0.1

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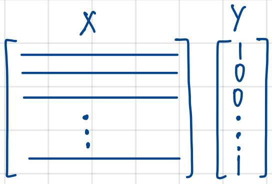
Summary:

Fault Detection / Isolation / Identification using PCA
PCA + discriminant analysis
PLS Model : Supervised Learning

Supervised Learning:

$X^{n \times m}$
set of features



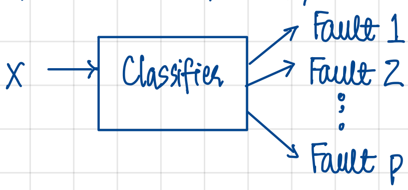
label for each dataset

Eg: Normal operation (1)
Faulty operation (0)

Requires data for each fault, otherwise it is not possible to model
Fault 1, Fault 2, ..., Fault P

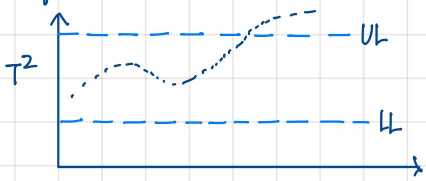
(p+1) classes or in case of unsupervised learning like T^2, we need on p classes.

Create multi-class classifier using simple classification techniques
like RF, Decision trees, SVM, NNs



Now that we have the classifications, we can do supervised fault identification and localization

We can use some Chart (Unsupervised learning) to look at some multivariate statistics,
to figure out Eg: Use T^2 to detect fault when the curve goes out of bounds



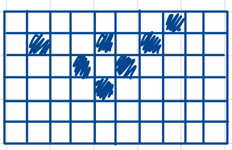
As soon as the system goes out of bounds, we can trigger our supervised classifier built using the data which allows us to

$x \longrightarrow$ Predict class j (j^{th} fault)

We can use any classifier based on the data to classify the data

Eg: X using sensors time domain

Plot the Frequency vs Time


$\longrightarrow$ CNN $\longrightarrow$ Features $\longrightarrow$ Classifier $\rightarrow$ Healthy Condition, F1, F2, F3

Preprocessed like an image
Color coded

Prognostics and Forecasting

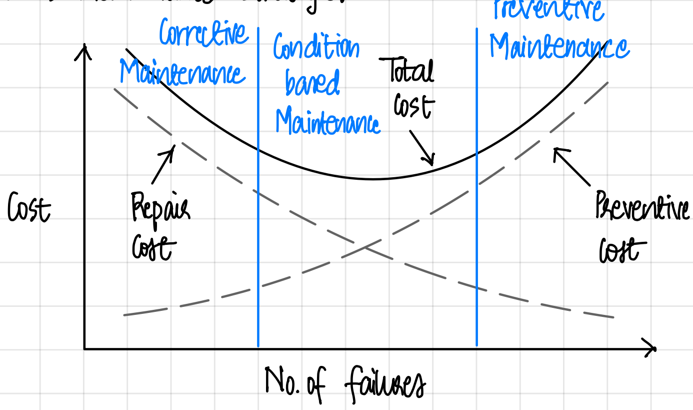
Prognostics :

- Prognostics and health management :
  - Monitoring health of equipment / software / human being using real-time data
  - Forecast the health of equipment / software / human using the historical data.
- Prognostics :
  - Predict actual remaining useful life of equipment while it is in operation
- Condition-based maintenance system:
  - Combination of hardware and software
  - Monitor, detect, isolate and predict performance of equipment and degradation
  - No interruption to the operation.
  - Maintenance based on the current condition.

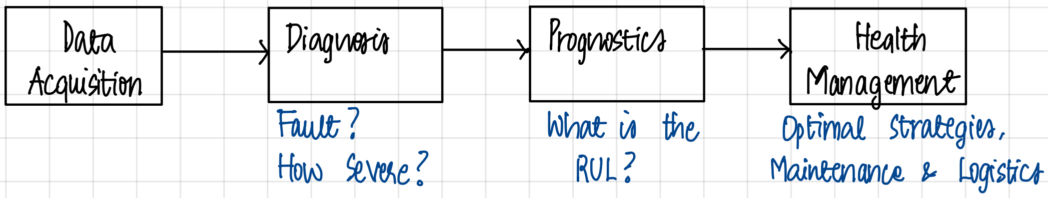
Maintenance Paradigm :

- Corrective Maintenance
  There is fault, failure has occured, correct it and proceed
- Preventive Maintenance
  Irrespective of the condition, performing a scheduled maintenance in regular intervals
  Good but inefficient strategy
  - Periodic Interval
  - Use reliability engineering to decide proper intervals for maintenance and replacements.
  - Pre-scheduled maintenance
- Predictive Maintenance (Condition-based maintenance System)
  Best Strategy and very cost-efficient

Cost vs Maintenance Strategies :



Prognostics and Health Management :



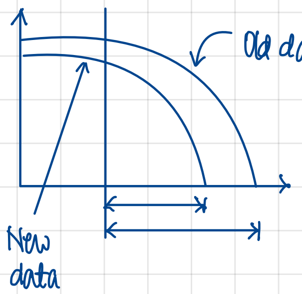
Prognostics (Definition):
1. Predicts how long it will take until failure under the current operating condition
2. An estimation of time to failure and risk for one or more existing and future failure modes.

Prognostics: Algorithms

- Prognostics methods:
  - Physics-based approaches
  - Data-driven approaches
- Objective:
  - To predict the future damage/degradation and the remaining useful life (RUL) based on measured data
- Physics-based approaches
- Data-driven approaches

Data/Degradation data + Usage conditions

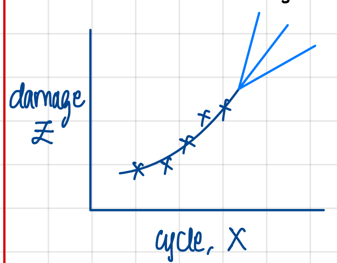
Physics-based model to estimate the parameters and simulate and see how new parameters behave.



Data-driven – (ML/DS/OL) → function – { Regression / Classification } → RUL

- Hybrid

- Paris (or Paris-Erdogan) Model for damage vs cycle



$$z = f(x, \theta)$$

estimated parameter

Data-driven, linear, non-linear

$$\frac{dz}{dx} = \frac{C(\Delta K)^m}{\Delta\sigma\sqrt{\pi z}}$$

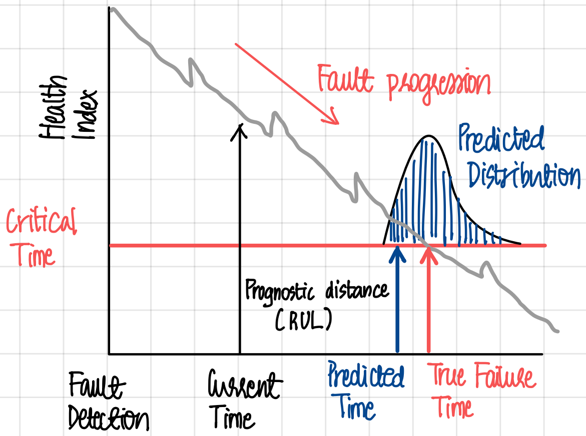
usage conditions, others are parameters

problem with physics-based approach is that the assumptions and equations needs to be precise to get a good model
but that doesn't happen in data-driven approach.

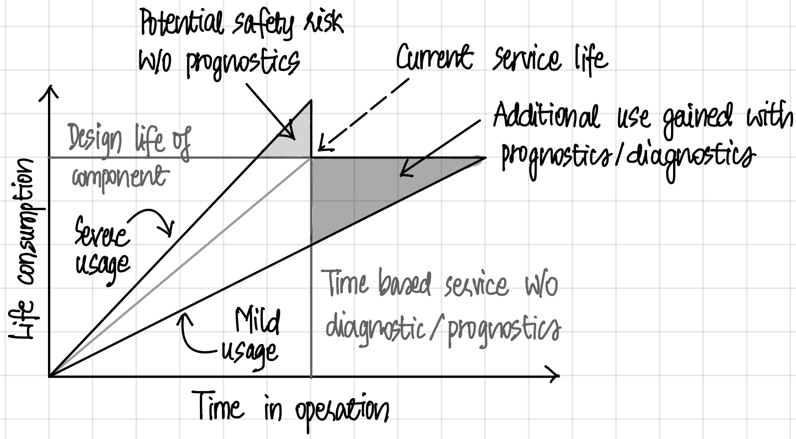
Prognostics : Benefits

- Cost : Reducing operating costs, Increased revenues
- System Design : Optimizing system design in future, Improved reliability predictions
- Production and Operation : Better process control, Safety, Improved operational reliability
- Logistic Support and maintenance : Optimized Supply chain, reduced maintenance-induced faults

Prognostics : Fault to Failure



Prognostics : for improved online reliability



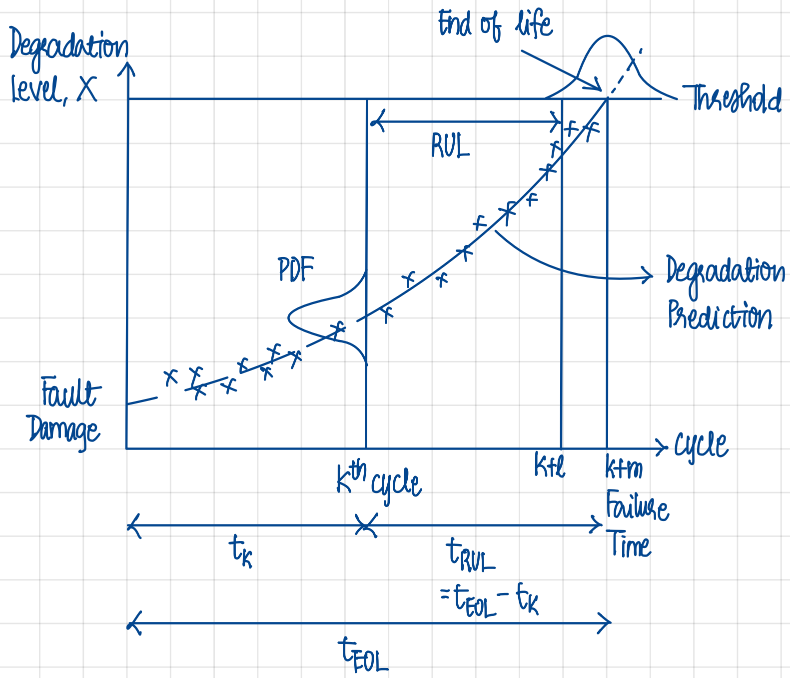
Prognostics : Challenge

- Optimal sensor selection and location
- Feature extraction
- Conditions for prognostics approaches. physics or data-driven
- Uncertainities and accessing its accuracy

Sources uncertainity → Model
- Measurement noise (data) — physics – ODE/PDE discretization – errors
- Disturbances (physics/model) — Degradation model

Physical uncertainity

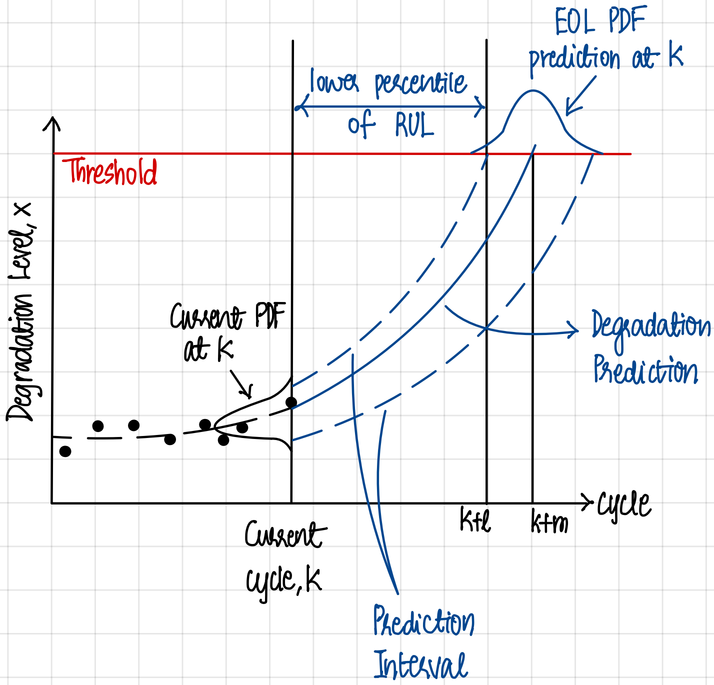
Metrics for prognostics:
- False alarm rate, missed estimation rate, correct rejection rate – Statistical metrics
- RUL (Remaining useful life)
  Prognostic horizon, prediction speed, Relative accuracy, α–λ performance.



Video 22

× Degradation and RUL:

Objective: Predict the remaining useful life (RUL) before the damage grows beyond the threshold



- Due to uncertainity, predicted degradation and End of Life (EOL) is represented as distribution
- For a given time, the area under the EOL is considered as failure probability
- EOL signals the time for the system maintenance
- RUL, the remaining time to the maintenance from the current time is given by,

$$\boxed{t_{RUL} = t_{EOL} - t_K}$$

- Lower bound of RUL is considered as the maintenance time.

× Prognostics methods

Categorized based on the availability of

- Physical model describing the behaviour of damage
- Field operating conditions
- Damage degradation data for similar systems

Yes → Physics-based
- Used when usage conditions are available additionally
- Physics-based algorithm can be considered as parameter estimation model

No → Data-driven
- Used when failure phenomenon is too complex to model
- Requires several sets of run-to-failure data (training data)

Prediction of degradation behavior:

Least squares method – Parameter estimation

- Used to find unknown parameter or co-efficients by minimizing the sum of square errors (SSE) between measured (y_k) and simulation (z_k, from model / function) data
- The relationship is given by,

degradation data

$$\boxed{y_k = z_k + \varepsilon_k}$$

where, k is the time index

- The error ε_k can represent measurement error in y_k as well as z_k
- Assumptions:
  - Error comes from measurement only
  - Measurement error does not include bias but unbiased noise
  - Simulation model $z(t;\theta)$ is a linear function of input variable t (time) and parameter (θ)

$$\boxed{z(t;\theta) = \theta_1 + \theta_2 t, \quad \theta = [\theta_1 \ \theta_2]^T}$$

$$y_k = z_k + \varepsilon_k$$
$$= \theta_1 + \theta_2 t_k + \varepsilon_k$$

for n points,

$$y_1 = \theta_1 + \theta_2 t_1 + \varepsilon_1$$
$$y_2 = \theta_1 + \theta_2 t_2 + \varepsilon_2$$
$$\vdots$$
$$y_n = \theta_1 + \theta_2 t_n + \varepsilon_n$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_n \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$Y = X\theta + \varepsilon$$

Sum of squared errors (SSE):

- The pair of input variable and measured degradation at data points is denoted as:

$$\boxed{(t_k, y_k), \quad k = 1, \ldots, n_y \ \text{and} \ y = [y_1 \ y_2 \ \ldots \ y_{n_y}]^T}$$

where, n_y is measured degradation

- The simulation model can be evaluated at data points and corresponding error is given as,

$$\boxed{z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_{n_y} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = X\theta}$$

X → design matrix

$$\boxed{e = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = y - z}$$

$$\boxed{SSE = e^T e = [y - z]^T [y - z] = [y - X\theta]^T [y - X\theta]}$$

$$\boxed{\frac{d(SSE)}{d\theta} = 2\left[\frac{de}{d\theta}\right]^T e = 2X^T[y - X\theta] = 0}$$

$$\boxed{\text{The estimated parameters} = \hat{\theta} = [X^T X]^{-1} [X^T y]}$$

All the assumptions for OLS holds here

Model, $z_k(t;\theta) = \theta_1 + \theta_2 L t_k^2 + \theta_3 t_k^3$

$L \rightarrow$ Loading condition

(t_k, y_k) $y_k \rightarrow$ degradation level

$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ - parameters that can be measured using simple LS approachs

× **Data-driven approach:**

- Functional relationship between input variables and output degradation has to be built from the data
- The quality of prediction depends on:
  - selection of mathematical function
  - Number of data
  - Level of noise in the measurement

× Quality of fitting:

- Co-efficient of determination, R^2, the ratio between the variation of function prediction of the data.

$$\boxed{R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}}$$

- SST (total sum of squares) $= \sum_{k=1}^{n_y} (y_k - \bar{y})^2$
- SSR (regression sum of squares) $= \sum_{k=1}^{n_y} (z_k - \bar{y})^2$
- SSE (residual sum of squares) $= \sum_{k=1}^{n_y} (y_k - z_k)^2$

- SST is the variation of data wrt. the mean of the data $\bar{y}$
- SSR is the variation of the function prediction z_k wrt. the mean of data
- SSE is the sum of square of errors remaining after the fit
- When the sum of y_k is equal to the sum of z_k

$$\boxed{SST = SSR + SSE}$$

- R^2 close to 1 is considered an accurate function
- However, R^2 only measures accuracy in data points which can be unrelated to the true accuracy of the function prediction.
- Therefore, adjusted R^2 denoted as $\bar{R}^2$ is used by penalizing the number of co-efficients as

$$\boxed{\bar{R}^2 = 1 - (1 - R^2)\frac{(n_y - 1)}{(n_y - n_p)}}$$

where, n_y and n_p are no. of data and coefficients respectively.

× Overfitting:

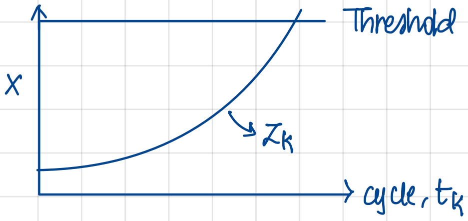
When the no. of unknown coefficients is larger than the no. of data, the least square method tends to fit the noise rather than the trend.

- Overfitting is a modeling error occuring when function is overly complex and when the function has no conformability with the data shape.
- Techniques available to avoid overfitting
  - Cross-validation
  - Regularization — LASSO
  - Early stopping } NN - RF
  - Pruning
- Other approaches:
  - Behavior of degradation is expressed with a simple function
  - More training data and usage conditions are used

× RUL Prediction:

Predicting Degradation level, $Z_K = \theta_1 + \theta_2 t^2 + \theta_3 t^3$

How to get RUL?



specified

$$\boxed{Z_{Threshold} - Z_{EOL}(\theta, t) = 0}$$

Data

$$\boxed{Z_{Threshold} - \hat{\theta}_1 - \theta_2 t_{EOL}^2 - \theta_3 t_{EOL}^3 = 0}$$; Solve for t_{EOL}

Solve using one of the solving methods like Newton-Raphson method

Above equation will give 3 solutions, but it will have a unique solution since it is a monotonically increasing nature of the function.

So, once we have t_{EOL}, $t_{RUL} = \hat{t}_{EOL} - t_K$

- Remaining time until the degradation grows a threshold
- The threshold of degradation is determined so that the system is still safe but needs maintenance
- To find RUL, it is necessary to find the time cycle when the level of degradation reached a threshold
- The non-linear equation has to be solved to find the time cycle t_{EOL}

$$\boxed{y_{threshold} - Z(t_{EOL}; \hat{\theta}) = 0}$$

- The above equation is solved numerically using Newton-Raphson iterative method

- Maintenance has to be ordered when the RUL becomes zero
- The performance of different method in predicting RUL is compared using metrics such as
  - Prognostic horizon (PH)
  - $\alpha - \lambda$ accuracy
  - Relative accuracy (RA)
  - Cummulative relative accuracy (CRA)
  - Convergence

× **Summary for RUL Prediction:**

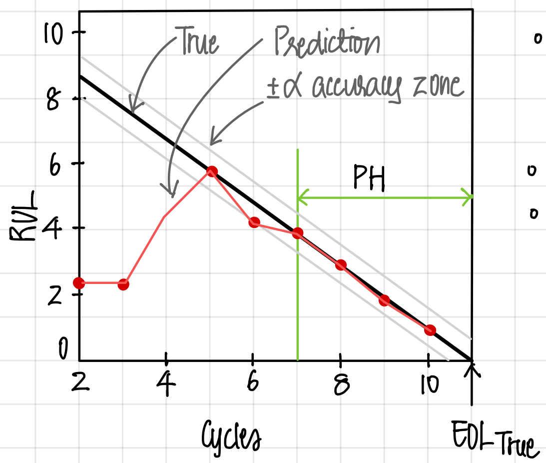
Point 1 : Data (t_k, y_k)
Point 2 : Build a model at the k^{th} cycle $Z_k(t_k; \hat{\theta})$ till t_k
Point 3 : Solve $Z_{Threshold} - Z_k(t_k; \hat{\theta}) = 0$ to obtain an estimate of $\hat{t}_{EOL,k}$

Point 4 : RUL, $\hat{t}_{RUL} = \hat{t}_{EOL,k} - t_k$

We now need metrics to compare different models to get the best prediction model.

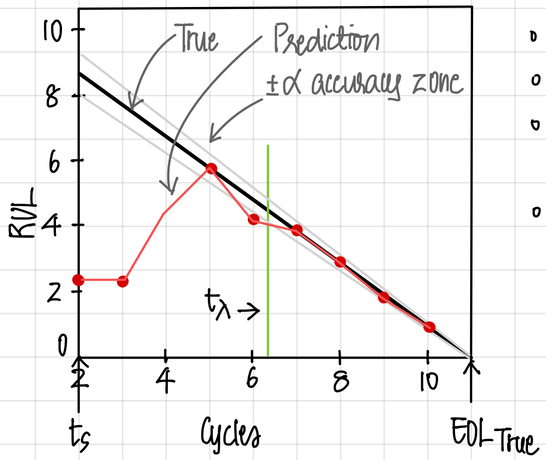
× **Prognostics metrics :**

× Prognostic Horizon (PH) :
Difference between the EOL and the first time when the prediction result continuously resides in the PHA accuracy zone

a) 

- The two parallel grey lines indicate the constant bound of the accuracy zone with a magnitude of ± pha error concerning true EOL
- Here, PHA = 5%
- The prognostics method with a larger PH indicates the better performance.

× $\alpha - \lambda$ accuracy :
Determines whether a prediction result falls within the α accuracy zone at a specific cycle t_λ

b) 

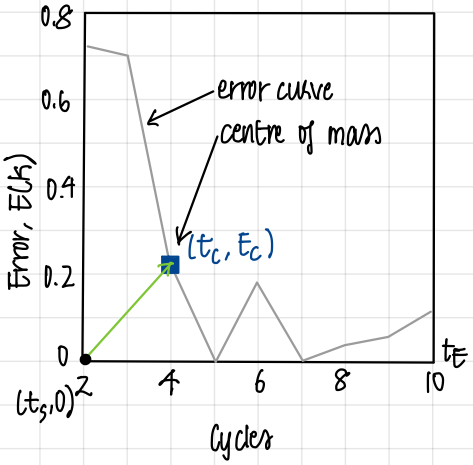
- The accuracy zone varies with $\pm\alpha$ ratio to the true RUL.
- The grey line shows the accuracy zone when $\alpha = 0.05$
- The accuracy zone shrinks with more data suggesting an increase of prediction accuracy
- Specific cycle t_λ is expressed with a fraction of λ between 0 (starting cycle of RUL prediction) and 1 (true EOL)

$$\boxed{t_\lambda = t_s + \lambda(EOL_{True} - t_s)}$$

× (Cumulative) Relative accuracy (RA, CRA):

RA is the relative accuracy between the true and prediction RUL at t_λ

CRA is same as the average of RA values accumulated at every cycle from t_s to t_E

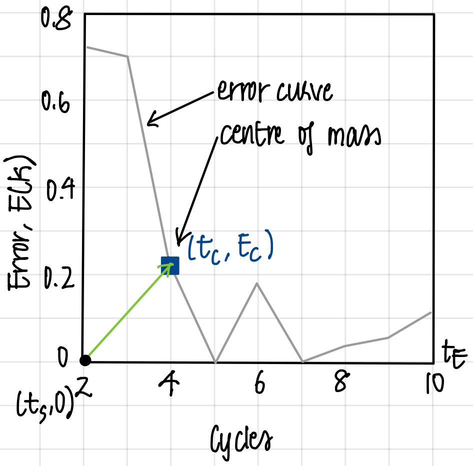
c) 

$$\boxed{RA = 1 - \frac{|RUL_{True} - RUL|}{RUL_{True}} \text{ at } t_\lambda}$$

- The relative error shown in the plot (grey curve) can be used to calculate RA (RA = 1 − relative error)
- When RA and CRA are close to 1, prediction accuracy is high

× Convergence :

Euclidean distance between the point $(t_s, 0)$ and the centre of mass of the area under the error curve (t_c, E_c).



$$\text{Center of mass} = CoM = \sqrt{(t_c - t_s)^2 + E_c^2}$$

where,
$$t_c = \frac{1}{2}\frac{\sum_{k=s}^{E}(t_{k+1}^2 - t_k^2)E(k)}{\sum_{k=s}^{E}(t_{k+1} - t_k)E(k)}$$

$$E_c = \frac{1}{2}\frac{\sum_{k=s}^{E}(t_{k+1} - t_k)E^2(k)}{\sum_{k=s}^{E}(t_{k+1} - t_k)E(k)}$$

- The lower the distance is, the faster the convergence is

× Uncertainty :

- Noise in measured data is caused by measurement variability, which represent uncertain measurement environment.
- Noise is random in nature and thus it is assumed to be Gaussian, i.e., statistical noise having a probability density function equal to the normal distribution

Objective : Estimate the level of uncertainty in estimated model parameters and remaining useful life when measured data have noise

Quantification using LS regression :

- Assumption :
  - Noise is normally distributed with zero mean
  - They are independent and identically distributed (i.i.d.)
- Variance of error is given by,

$$\boxed{\hat{\sigma}^2 = \frac{SSE}{n_y - n_p}}$$

where, $(n_y - n_p)$ represents the degree of freedom

n_y and n_p are the no. of data and unknown parameters respectively

- For the linear model with 2 parameters ($z(x) = \theta_1 + \theta_2 x$), the variance of parameters is derived using the estimated parameter and the variance of error equations

$$\boxed{\text{Design matrix} = X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_{n_y} \end{bmatrix}, \text{ Therefore, } X^T X = \begin{bmatrix} n_y & \sum_{i=1}^{n_y} x_i \\ \sum_{i=1}^{n_y} x_i & \sum_{i=1}^{n_y} x_i^2 \end{bmatrix}, \ X^T y = \begin{bmatrix} \sum_{i=1}^{n_y} y_i \\ \sum_{i=1}^{n_y} x_i y_i \end{bmatrix}}$$

$$\hat{\theta} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \begin{bmatrix} n_y & \sum_{i=1}^{n_y} x_i \\ \sum_{i=1}^{n_y} x_i & \sum_{i=1}^{n_y} x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n_y} y_i \\ \sum_{i=1}^{n_y} x_i y_i \end{bmatrix} = \begin{bmatrix} \bar{y} - \hat{\theta}_2 \bar{x} \\ \sum_{i=1}^{n_y} (x_i - \bar{x})(y_i - \bar{y}) / \sum_{i=1}^{n_y} (x_i - \bar{x})^2 \end{bmatrix}$$

$$\boxed{\hat{\theta} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \begin{bmatrix} \bar{y} - \bar{x} S_{xy} / S_{xx} \\ S_{xy} / S_{xx} \end{bmatrix}}$$

- By using a theorem for variance calculation, the variance of two parameters is obtained as,

$$\boxed{\mathrm{var}(\hat{\theta}_2) = \mathrm{var}\left(\frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{S_{xx}} \right) = \frac{\sum (x_i - \bar{x})^2 \sigma^2}{S_{xx}^2} = \frac{S_{xx} \sigma^2}{S_{xx}^2} = \frac{\sigma^2}{S_{xx}}}$$

$$\boxed{\mathrm{var}(\hat{\theta}_1) = \mathrm{var}(\bar{y} - \hat{\theta}_2 \bar{x}) = \frac{\sigma^2}{n_y} + \bar{x}^2 \frac{\sigma^2}{S_{xx}}}$$

- The terms related with x are constants because the source of uncertainty is from the noise in the data and the terms related with data y_i are random variables.
- The variance of two parameters in a linear model is,

$$\boxed{\mathrm{var}(\hat{\theta}) = \begin{bmatrix} \frac{\sigma^2}{n_y} + \bar{x}^2 \frac{\sigma^2}{S_{xx}} \\ \frac{\sigma^2}{S_{xx}} \end{bmatrix}, \qquad S_{xx} = \sum_{i=1}^{n_y} (x_i - \bar{x})^2}$$

- The variance in model parameters is linearly proportional to the variance of data.
- Large number of data reduces uncertainty in model parameters and eventually makes them deterministic ($n_y \to \infty$, $S_{xx} \to \infty$)
- In case of many unknown parameters, the derivation is complicated and the following can be employed to describe the correlation between the parameters.

$$\boxed{\Sigma_{\hat{\theta}} = \sigma^2 [X^T X]^{-1}}$$

- Since the variation of error in data is unknown, usually estimated value can be used instead.

Compute αx and $[100 - \alpha]\%$
Bounds on degradation level

Issues in practical prognostics:

- In practical cases, simple polynomial functions and Gaussian noise does not suffice
- Bayesian-based approaches are employed instead of linear regression method
- The noise and bias have an effect on the prognostics results
- It is difficult to identify model parameters accurately when they are correlated
- Loading conditions can be correlated with the parameters too
- Physical degradation models are rare in practice
- In case of data-driven approach, it is not easy to obtain several sets of degradation data due to expensive time and costs.
- Degradation data cannot be directly measured in most case and need to be extracted from sensor signals.

Video 23

DEEP LEARNING APPROACHES FOR PREDICTIVE MAINTENANCE

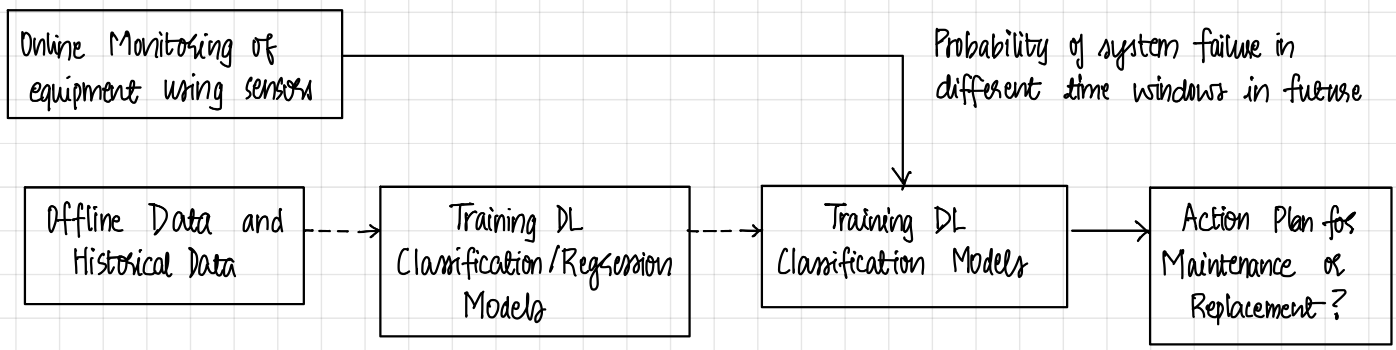
Preventive maintenance:

- Taking the necessary actions to prevent failure of equipment from occuring before it happens.
- Traditional approaches to PM
  - Time-based maintenance
  - Usage-based maintenance
- PM in the age of Big data
  - Preventive Maintenance
  - Prescriptive Maintenance
- Why PM?
  - Less distruption in operation
  - Improved life expentency of equipment
  - Improved efficiency, etc.

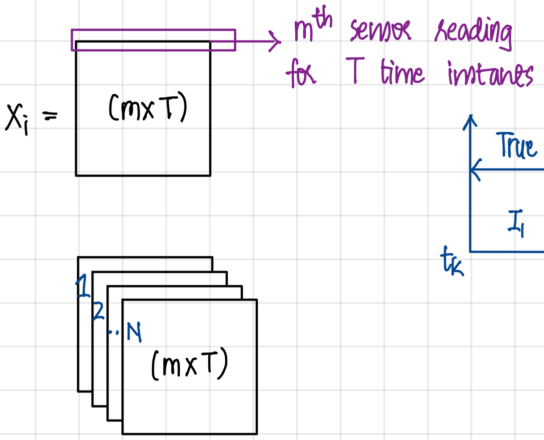
Predicted Maintenance:

- Predict the failure of the equipment and time horizon (Prognostic modeling)
  - Use of data to build models for failure
- Action based on prognostic prediction (Maintenance Optimization)
  - Provide decisions for a maintenance schedule and related decisions for preventing failure.

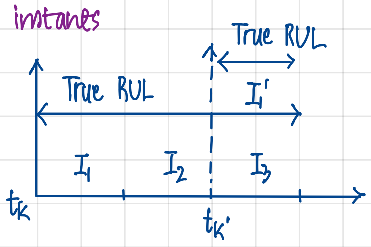
Dynamic Prognostic and Maintenance Optimization from Data:



N components monitored using m sensors for Time instants T

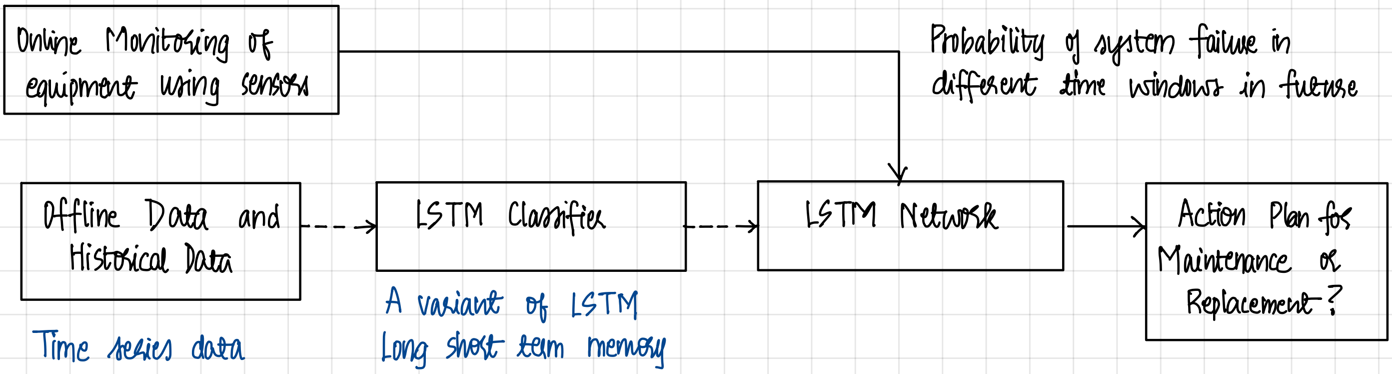


True Residual useful lifetime (RUL) belongs to?



× <u>Dynamic Prognostic and Maintenance Optimization from Data</u> :

1. Predict the system residual useful lifetime (RUL)
2. Action Plan or decisions



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