Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Problem - 1: Derive the covariance relationship

Let x and y be jointly Gaussian n and m vectors. A and B are known p by n and p by m matrices, respectively. Then the random p vector z is defined as z = Ax + By is Gaussian characterized by:

Mean $\mu_z = A\mu_x + B\mu_z$ and Covariance $P_{zz} = AP_{xx}A^T + AP_{xy}B^T + BP_{yx}A^T + BP_{yy}B^T$

Derive the above covariance relationship.

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Problem 1 :- Derive the covariance relationship:-

Answer:-

Below information is given in the problem statement,

i) x & y be jointly Gaussian n & m vector

ii) x is an n-dimensional Gaussian vector with mean μ_x & covariance matrix P_{xx}

iii) y is an n-dimensional Gaussian vector with mean μ_y & covariance matrix P_{yy}

(iv) A & B are known matrix & there dimension are as follows,

$A \rightarrow p \times n$

$B \rightarrow p \times m$

Vector Z is defined by,

$$Z = Ax + By$$

• Calculate the Mean of Z :-

Let consider, μ_Z is the mean of vector Z

$$\therefore \mu_Z = E[Z]$$
$$= E[Ax + By] \quad \because \text{substituting } Z = Ax + By$$
$$= A\,E[x] + B\,E[y]$$
$$= A\mu_x + B\mu_y$$

$\because E[x] = \mu_x$ = Mean of x

$E[y] = \mu_y$ = Mean of y

$$\therefore \mu_Z = A\mu_x + B\mu_y$$

• Calculate the covariance matrix of Z :-

Let consider, μ_Z is the mean & P_{ZZ} is the covariance matrix of Z

$$\therefore P_{ZZ} = E\left[(Z - \mu_Z)(Z - \mu_Z)^T\right]$$

$\because$ substituting $Z = Ax + By$ we got,

$$\Rightarrow P_{ZZ} = E\left[(Ax + By - A\mu_x - B\mu_y)(Ax + By - A\mu_x + B\mu_y)^T\right]$$

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

$$\Rightarrow P_{zz} = E\left[(Ax - A\mu_x)(Ax - A\mu_x)^T\right] + E\left[(By - B\mu_y)(By - B\mu_y)^T\right] + E\left[(Ax - A\mu_x)(By - B\mu_y)^T\right] + E\left[(By - B\mu_y)(Ax - A\mu_x)^T\right]$$

$$\Rightarrow P_{zz} = A \cdot A^T \cdot E\left[(x-\mu_x)(x-\mu_x)^T\right] + B \cdot B^T \cdot E\left[(y-\mu_y)(y-\mu_y)^T\right] + A \cdot B^T \cdot E\left[(x-\mu_x)(y-\mu_y)\right] + B \cdot A^T\, E\left[(y-\mu_y)(x-\mu_x)^T\right]$$

substituting the covariance matrix value of x & y vector in previous equation we got,

$$\Rightarrow P_{zz} = A \cdot A^T \cdot P_{xx} + B \cdot B^T \cdot P_{yy} + A \cdot B^T\, E\left[(x-\mu_x)(y-\mu_y)^T\right] + B \cdot A^T\, E\left[(y-\mu_y)(x-\mu_x)^T\right]$$

Rearanging the equation based on matrix multiplication format we got,

$$\Rightarrow P_{zz} = A \cdot P_{xx} \cdot A^T + B \cdot P_{yy} \cdot B^T + A \cdot E\left[(x-\mu_x)(y-\mu_y)^T\right] B^T + B \cdot E\left[(y-\mu_y)(x-\mu_x)^T\right] \cdot A^T$$

As per the problem statement, x & y are jointly gaussian vector, hence their covariance will be defined by,

$$\mathrm{cov}[x,y] = E\left[(x-\mu_x)(y-\mu_y)^T\right]$$

substituting the value in P_{zz} equation, we got,

$$\Rightarrow P_{zz} = A \cdot P_{xx} \cdot A^T + B \cdot P_{yy} \cdot B^T + A \cdot \mathrm{cov}[x,y] \cdot B^T + B \cdot \mathrm{cov}[y,x] \cdot A^T$$

$\because$ covariance matrix can be written as,

$$\mathrm{cov}[x,y] = P_{xy}$$

$$\mathrm{cov}[y,x] = P_{yx}$$

substituting the symbolic representation in P_{zz} equation we got,

$$P_{zz} = A \cdot P_{xx} \cdot A^T + B \cdot P_{yy} \cdot B^T + A \cdot P_{xy}\, B^T + B\, P_{yx}\, A^T$$

[Proved]

Hence the covariance relation of gaussian vector Z is $P_{zz} = A P_{xx} A^T + B P_{yy} B^T + A P_{xy} B^T + B P_{yx} A^T$.

[Proved].

Assignment 4: ID5004W **-** AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Problem - 2: Derivation

To determine matrix K_k (Kalman Gain), we used the following relationship:

$$E[\tilde{x_k}^T \tilde{x_k}] = \text{Trace } E[\tilde{x_k} \tilde{x_k}^T]$$

Derive the above relationship

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Problem 2:— Derivation:—

Assumption while deriving Kalman Gain (K_K) is as below,

$$E[\tilde{x}_K^T \tilde{x}_K] = \text{Trace } E[\tilde{x}_K \tilde{x}_K^T]$$

Answer →

Let consider,

$\tilde{x}_K$ is a column vector defined as,

$$\tilde{x}_K = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1}$$

then $\tilde{x}_K^T$ will be a row vector as,

$$\tilde{x}_K^T = [x_1 \ x_2 \ x_3 \ \cdots \ x_N]_{1\times N}$$

Now,

RHS → Let multiply the column & row vector,

$$\tilde{x}_K \cdot \tilde{x}_K^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1} \cdot [x_1 \ x_2 \ \cdots \ x_N]_{1\times N}$$

Resultant will be a $N \times N$ matrix,

$$= \begin{bmatrix} x_1^2 & x_2 x_1 & \cdots & x_N x_1 \\ x_1 x_2 & x_2^2 & & \vdots \\ \vdots & \vdots & & \\ x_1 x_N & \vdots & \cdots & x_N^2 \end{bmatrix}_{N\times N}$$

So now if we take Trace of above matrix we got,

$$\text{Trace}[\tilde{x}_K \cdot \tilde{x}_K^T] = [x_1^2 + x_2^2 + x_3^2 + \cdots x_N^2]$$

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Now take,

LHS → Let multiply row vector with column vector,

$$\tilde{x}_K^T \cdot \tilde{x}_K = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}_{1\times N} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1}$$

Resultant will be a 1×1 matrix (scaler),

$$\tilde{x}_K^T \cdot \tilde{x}_K = \left[x_1^2 + x_2^2 + x_3^2 + \cdots x_N^2 \right]$$

∵ By comparing the LHS & RHS we got,

$$\tilde{x}_K^T \cdot \tilde{x}_K = \text{Trace}\left[\tilde{x}_K \cdot \tilde{x}_K^T \right].$$ [Proved].

Problem - 3: Kalman Gain Derivation

Derive the expression for Kalman Gain (K_k).

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

Problem 3: Kalman Gain Derivation:—

Answer:— We are following the least square derivation of Kalman Gain.

Kalman Gain is defined by K_K or $K(K)$.

Plant equation:—

$$x_{K+1} = Ax_K + Bu_K + \epsilon_K$$

$$y_K = Cx_K + v_K$$

Where,

$K \rightarrow$ Discrete time instant

$x_K \rightarrow \in \mathbb{R}^n$ is the state vector at the discrete time instant K

$u_K \rightarrow \in \mathbb{R}^m$ is the control input vector

$\epsilon_K \rightarrow \in \mathbb{R}^n$ is the disturbance or process noise vector, We assume that ϵ_K is white in nature with zero mean & uncorrelated with the covariance matrix. given by $E[\epsilon_K \epsilon_K^T] = R_1$

$A \in \mathbb{R}^{n \times m}$ & $B \in \mathbb{R}^{n \times m}$ are the state & input matrices

$C \in \mathbb{R}^{r \times n}$ is the output matrix

$y_K \in \mathbb{R}^r$ is the output vector (observed measurement)

$v_K \in \mathbb{R}^r$ is the measurement noise vector, we assumes that v_K is also white in nature with zero mean & uncorrelated with covariance matrix defined by $E[v_K v_K^T] = R_2$

For Deterministic Observer:— Open loop prediction equation will be given by,

$$\hat{x}_{K+1} = A\hat{x}_K + Bu_K + L[y_K - C\hat{x}_K]$$

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

For Stochastic observer,

(i) Prediction Equation :—

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$

(ii) filter Equation :—

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K[y_k - C\hat{x}_{k|k-1}]$$

Error equation,

(i) filter Error :—

$$e_k = x_k - \hat{x}_{k|k}$$

(ii) One step prediction error :—

$$x_{e_k} = x_k - \hat{x}_{k|k-1}$$

Covariance Matrix :—

$$P_k = E\{e_k e_k^T\}$$

$$\bar{P}_k = E\{x_{e_k} x_{e_k}^T\}$$

Both P_k & $\bar{P}_k$ are $n \times n$ symmetric matrices.

Now, if we substract the plant - equation & prediction equation we get,

$$x_{k+1} - \hat{x}_{(k+1|k)} = Ax_k - A\hat{x}_{k|k} + B\cancel{u}_k - \cancel{B}u_k + \epsilon_k$$

$$= Ax_k - A\hat{x}_{k|k} + \epsilon_k$$

$$\therefore x_{e(k+1)} = Ae_k + \epsilon_k$$

Now,

$$x_k - \hat{x}_{k|k} = x_k - \tilde{x}_{k|k-1} - K[Cx_k + v_k - C\hat{x}_{k|k-1}]$$

$$= x_k - \tilde{x}_{k|k-1} - KC[x_k - \hat{x}_{k|k-1}] - Kv_k$$

$\therefore$ Above equation can be written in terms of filter equation. Writing this equation we get,

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

$e_k = x_{e_k} - KCx_{e_k} - Kv_k$

$\Rightarrow e_k = [I - KC]\,x_{e_k} - Kv_k$

Now we will work with below equation,

$$\left.\begin{aligned} x_{e_{k+1}} &= Ae_k + \epsilon_k \\ e_k &= [I-KC]\,x_{e_k} - Kv_k \end{aligned}\right] \rightarrow \text{Overall error propagating over time}$$

The main objective of kalman filter is to find the value of K, such that the error is minimized in "Least Square Sence".

So, far we know that,

Plant equation:- $x_{k+1} = Ax_k + Bu_k + \epsilon_k$

One step prediction equation:- $\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$

Filter equation:- $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K[y_k - C\hat{x}_{k|k-1}]$

Now to achieve the goal we can either minimize $\underline{x_e x_e^T}$ or $\underline{ee^T}$ for finding the value of K_k = Kalman Gain

Now we know that,

$x_{e_k} = Ae_{k-1} + \epsilon_{k-1}$

& $\bar{P} = E\{x_e x_e^T\} = E\{(Ae+\epsilon)(Ae+\epsilon)^T\}$

$= E\{Aee^TA^T\} + E\{\epsilon\epsilon^T\} + AE\{\cancel{e\epsilon^T}\}^0 + E\{\cancel{\epsilon e^T}\}^0 A^T$

[Both the term become zero as we consider ϵ is zero mean gaussian white noise]

$= E\{Aee^TA^T\} + E\{\epsilon\epsilon^T\}$

We know from our definition statement $E\{\epsilon\epsilon^T\} = R_1$

$= AE\{e_{k-1}e_{k-1}^T\}A^T + R_1$

So, $\bar{P}_k = AP_{k-1}A^T + R_1$

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

$$E[e_k e_k^T] = E\{[I-KC]x_{e_k} - Kv]\,[[I-KC]x_{e_k} - Kv]^T\}$$

$$= (I-KC)\,E\{x_{e_k} x_{e_k}^T\}\,(I-KC)^T + K E\{v v^T\} K^T$$

substituting the value we got,

$$= (I-KC)\,\bar{P}_k\,(I-KC)^T + K R_2 K^T$$

$$\bar{P}_k = E\{x_{e_k} x_{e_k}^T\}$$

$$\& \quad R_2 = E\{v v^T\}$$

$$\because P_k = E[e_k e_k^T] = \bar{P} + KC\bar{P}_k C^T K^T - K(\bar{P}_k - \bar{P}_k C^T K^T + K R_2 K^T$$

$$= \bar{P}_k + K[C\bar{P}_k C^T + R_2]K^T - 2\bar{P}_k C^T K^T$$

Assignment 4: ID5004W - AI in Predictive Maintenance, Reliability and Warranty

Author: Aloy Banerjee

Roll. No.: CH22M503

$\because$ $[C\bar{P}_K C^T + R_2] \rightarrow$ Hessian Term (H)

$[\bar{P}_K C^T] \rightarrow$ Gradient Term (G)

Kalman Gain,

$$K_K = H^{-1} g$$

substituting the value of hessian & gradient we got,

K_K = Kalman gain = $[C\bar{P}_K C^T + R_2]^{-1} \bar{P}_K C^T$.

$$\underline{K_K = [C\bar{P}_K C^T + R_2]^{-1} \bar{P}_K C^T.}$$ | proved