ID-5004W: AI in Predictive Maintenance, Reliability, and Warranty
April – July 2023

Home Assignment - 04

Due: Thursday, June 1st

Instructions

- Make any additional assumptions if needed and justify your assumption.
- Please submit your own work.
- Attach the codes and results for the respective questions if MATLAB or any software is used.

Problem - 1: Derive the covariance relationship

Let x and y be jointly Gaussian n and m vectors. A and B are known p by n and p by m matrices, respectively. Then the random p vector z is defined as z = Ax + By is Gaussian characterized by:

Mean $\mu_z = A\mu_x + B\mu_z$ and Covariance $P_{zz} = AP_{xx}A^T + AP_{xy}B^T + BP_{yx}A^T + BP_{yy}B^T$

Derive the above covariance relationship.

Problem - 2: Derivation

To determine matrix K_k (Kalman Gain), we used the following relationship:

$$E[\tilde{x_k}^T \tilde{x_k}] = \text{Trace } E[\tilde{x_k} \tilde{x_k}^T]$$

Derive the above relationship

Problem - 3: Kalman Gain Derivation

Derive the expression for Kalman Gain (K_k).

Assignment ---- To Derive all the Expressions, for Covariance Minimization

An Optimal State Estimator

- Estimating the State
- Minimizes the Error
- Obtaining the Mean & Covariance

$$E[x_k] = \hat{x}_k$$

$$E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] = P_k$$

Kalman Gain & Covariance

To determine matrix K_k, to be termed as Kalman Gain

In order to minimize

$$E[(\hat{x}_k - x_k)^T(\hat{x}_k - x_k)]$$

OR

$$E[(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)]$$

If $\tilde{x_k} = (x_k - \hat{x}_k)$, then

$$E[(\hat{x}_k - x_k)^T(\hat{x}_k - x_k)] = E[\tilde{x_k}^T \tilde{x_k}]$$

Since $E[\tilde{x_k}^T \tilde{x_k}] = \text{Trace } E[\tilde{x_k}\tilde{x_k}^T]$

Define a Matrix P_k as

$$P_k = E[\tilde{x_k}\tilde{x_k}^T]$$

To Obtain the Expression for Covariance

- **Step 1 :**

Formulating $\tilde{x_k}$

$$\tilde{x}_k = [A_{k|k-1}\hat{x}_{k-1} + K_k(z_k - H_k A_{k|k-1}\hat{x}_{k-1})] - A_{k|k-1}x_{k-1}$$

Grouping terms

$$\tilde{x}_k = A_{k|k-1}x\tilde{_{k-1}} - K_k H_k A_{k|k-1}\hat{x}_{k-1} + K_k(H_k x_k + v_k)$$

Again grouping the terms, we obtain

$$\tilde{x}_k = (I - K_k H_k)A_{k|k-1}x\tilde{_{k-1}} + K_k v_k$$

Obtaining the Covariance Matrix

- **Step 2 :** Expanding the term $P_k = E[\tilde{x}_k \tilde{x}_k{}^T]$

Formulating the matrix P_k

$$P_k = E\{[(I - K_k H_k)A_{k|k-1}\tilde{x}_{k-1} + K_k v_k][(I - K_k H_k)A_{k|k-1}\tilde{x}_{k-1} + K_k v_k]^T\}$$

Grouping the terms

$$T_1 = (I - K_k H_k)A_{k|k-1}E[\tilde{x}_{k-1}\tilde{x}_{k-1}^T]A^T{}_{k|k-1}(I - H_k{}^T K_k{}^T)$$

$$T_2 = K_k E[v_k \tilde{x}_{k-1}^T]A^T{}_{k|k-1}(I - H_k{}^T K_k{}^T$$

$$T_3 = (I - K_k H_k)A_{k|k-1}E[\tilde{x}_{k-1} v_k{}^T]K_k{}^T$$

$$T_4 = K_k E[v_k v_k{}^T]K_k{}^T$$

Going by the earlier definition

$$E[\tilde{x}_{k-1}\tilde{x}_{k-1}^T] = P_{k-1}$$

AND

$$E[v_k {v_k}^T] = R_k$$

It follows that

$$E[v_k \tilde{x}_{k-1}^T] = 0 = E[\tilde{x}_{k-1} {v_k}^T]$$

Since

$$E[v_k {v^T}_{k-1}] = 0$$

AND

$$E[v_k {x^T}_0] = 0$$

Obtaining Apriori P

Thus P_k can be re-written as :

$$P_k = (I - K_k H_k) P_k^{-} (I - K_k H_k)^T + K_k R_k K_k{}^T$$

where

$$P_k^{-} = A_{k|k-1} P_{k-1} A^T{}_{k|k-1}$$

Expanding the above equation for P_k gives

$$P_k = P_k^{-} - K_k H_k P_k^{-} - P_k^{-} H_k{}^T K_k{}^T + K_k (H_k P_k^{-} H_k{}^T + R_k) K_k{}^T$$

trace of a matrix is equal to the trace of its transpose.

$$T[P_k] = T[P_k'] - 2T[K_k H P_k'] + T[K_k (H P_k' H^T + R) K_k^T]$$

$$T[P_k] = T[P'_k] - 2T[K_k H P'_k] + T[K_k (H P'_k H^T + R) K_k^T]$$

Obtaining *K*

$$\frac{dT[P_k]}{dK_k} = -2(H P'_k)^T + 2K_k (H P'_k H^T + R)$$

$$(H P'_k)^T = K_k (H P'_k H^T + R)$$

$$K_k = P'_k H^T (H P'_k H^T + R)^{-1}$$

$$S_k = H P'_k H^T + R$$

Obtaining the **P** Matrix

$$\begin{aligned} P_k &= P'_k - P'_k H^T \left(H P'_k H^T + R\right)^{-1} H P'_k \\ &= P'_k - K_k H P'_k \\ &= (I - K_k H)\, P'_k \end{aligned}$$

$$\begin{aligned} P'_{k+1} &= E\left[e'_{k+1} e^{T\prime}_{k+1}\right] \\ &= E\left[\Phi e_k \left(\Phi e_k\right)^T\right] + E\left[w_k w_k^T\right] \\ &= \Phi P_k \Phi^T + Q \end{aligned}$$

The Kalman Filter