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VERSION 2019 – REVISION A

by W. Keith Nicholson

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Base Text Revision History

Current Revision: Version 2019 — Revision A

2019 A	• New Section on Singular Value Decomposition (8.6) is included. • New Example 2.3.2 and Theorem 2.2.4. Please note that this will impact the numbering of subsequent examples and theorems in the relevant sections. • Section 2.2 is renamed as *Matrix-Vector Multiplication*. • Minor revisions made throughout, including fixing typos, adding exercises, expanding explanations, and other small edits.
2018 B	• Images have been converted to LaTeX throughout. • Text has been converted to LaTeX with minor fixes throughout. Page numbers will differ from 2018A revision. Full index has been implemented.
2018 A	• Text has been released with a Creative Commons license.

Contents

Foreward

Mathematics education at the beginning university level is closely tied to the traditional publishers. In my opinion, it gives them too much control of both cost and content. The main goal of most publishers is profit, and the result has been a sales-driven business model as opposed to a pedagogical one. This results in frequent new "editions" of textbooks motivated largely to reduce the sale of used books rather than to update content quality. It also introduces copyright restrictions which stifle the creation and use of new pedagogical methods and materials. The overall result is high cost textbooks which may not meet the evolving educational needs of instructors and students.

To be fair, publishers do try to produce material that reflects new trends. But their goal is to sell books and not necessarily to create tools for student success in mathematics education. Sadly, this has led to a model where the primary choice for adapting to (or initiating) curriculum change is to find a different commercial textbook. My editor once said that the text that is adopted is often everyone's third choice.

Of course instructors can produce their own lecture notes, and have done so for years, but this remains an onerous task. The publishing industry arose from the need to provide authors with copy-editing, editorial, and marketing services, as well as extensive reviews of prospective customers to ascertain market trends and content updates. These are necessary skills and services that the industry continues to offer.

Authors of open educational resources (OER) including (but not limited to) textbooks and lecture notes, cannot afford this on their own. But they do have two great advantages: The cost to students is significantly lower, and open licenses return content control to instructors. Through editable file formats and open licenses, OER can be developed, maintained, reviewed, edited, and improved by a variety of contributors. Instructors can now respond to curriculum change by revising and reordering material to create content that meets the needs of their students. While editorial and quality control remain daunting tasks, great strides have been made in addressing the issues of accessibility, affordability and adaptability of the material.

For the above reasons I have decided to release my text under an open license, even though it was published for many years through a traditional publisher.

Supporting students and instructors in a typical classroom requires much more than a textbook. Thus, while anyone is welcome to use and adapt my text at no cost, I also decided to work closely with Lyryx Learning. With colleagues at the University of Calgary, I helped create Lyryx almost 20 years ago. The original idea was to develop quality online assessment (with feedback) well beyond the multiple-choice style then available. Now Lyryx also works to provide and sustain open textbooks; working with authors, contributors, and reviewers to ensure instructors need not sacrifice quality and rigour when switching to an open text.

I believe this is the right direction for mathematical publishing going forward, and look forward to being a part of how this new approach develops.

W. Keith Nicholson, Author
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Preface

This textbook is an introduction to the ideas and techniques of linear algebra for first- or second-year students with a working knowledge of high school algebra. The contents have enough flexibility to present a traditional introduction to the subject, or to allow for a more applied course. Chapters 1–4 contain a one-semester course for beginners whereas Chapters 5–9 contain a second semester course (see the Suggested Course Outlines below). The text is primarily about real linear algebra with complex numbers being mentioned when appropriate (reviewed in Appendix A). Overall, the aim of the text is to achieve a balance among computational skills, theory, and applications of linear algebra. Calculus is not a prerequisite; places where it is mentioned may be omitted.

As a rule, students of linear algebra learn by studying examples and solving problems. Accordingly, the book contains a variety of exercises (over 1200, many with multiple parts), ordered as to their difficulty. In addition, more than 375 solved examples are included in the text, many of which are computational in nature. The examples are also used to motivate (and illustrate) concepts and theorems, carrying the student from concrete to abstract. While the treatment is rigorous, proofs are presented at a level appropriate to the student and may be omitted with no loss of continuity. As a result, the book can be used to give a course that emphasizes computation and examples, or to give a more theoretical treatment (some longer proofs are deferred to the end of the Section).

Linear Algebra has application to the natural sciences, engineering, management, and the social sciences as well as mathematics. Consequently, 18 optional "applications" sections are included in the text introducing topics as diverse as electrical networks, economic models, Markov chains, linear recurrences, systems of differential equations, and linear codes over finite fields. Additionally some applications (for example linear dynamical systems, and directed graphs) are introduced in context. The applications sections appear at the end of the relevant chapters to encourage students to browse.

SUGGESTED COURSE OUTLINES

This text includes the basis for a two-semester course in linear algebra.

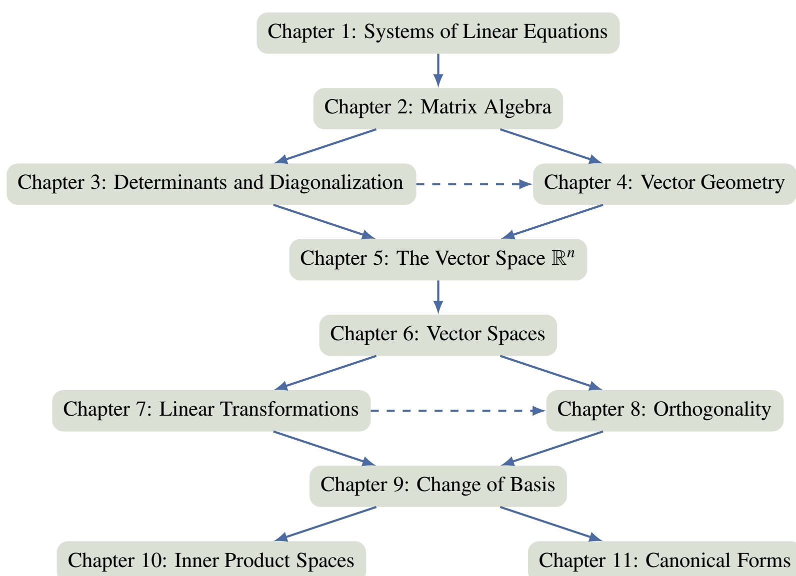
- Chapters 1–4 provide a standard one-semester course of 35 lectures, including linear equations, matrix algebra, determinants, diagonalization, and geometric vectors, with applications as time permits. At Calgary, we cover Sections 1.1–1.3, 2.1–2.6, 3.1–3.3, and 4.1–4.4 and the course is taken by all science and engineering students in their first semester. Prerequisites include a working knowledge of high school algebra (algebraic manipulations and some familiarity with polynomials); calculus is not required.

- Chapters 5–9 contain a second semester course including $\mathbb{R}^n$, abstract vector spaces, linear transformations (and their matrices), orthogonality, complex matrices (up to the spectral theorem) and applications. There is more material here than can be covered in one semester, and at Calgary we cover Sections 5.1–5.5, 6.1–6.4, 7.1–7.3, 8.1–8.7, and 9.1–9.3 with a couple of applications as time permits.

- Chapter 5 is a "bridging" chapter that introduces concepts like spanning, independence, and basis in the concrete setting of $\mathbb{R}^n$, before venturing into the abstract in Chapter 6. The duplication is balanced by the value of reviewing these notions, and it enables the student to focus in Chapter 6 on the new idea of an abstract system. Moreover, Chapter 5 completes the discussion of rank and diagonalization from earlier chapters, and includes a brief introduction to orthogonality in $\mathbb{R}^n$, which creates the possibility of a one-semester, matrix-oriented course covering Chapter 1–5 for students not wanting to study the abstract theory.

CHAPTER DEPENDENCIES

The following chart suggests how the material introduced in each chapter draws on concepts covered in certain earlier chapters. A solid arrow means that ready assimilation of ideas and techniques presented in the later chapter depends on familiarity with the earlier chapter. A broken arrow indicates that some reference to the earlier chapter is made but the chapter need not be covered.



HIGHLIGHTS OF THE TEXT

- **Two-stage definition of matrix multiplication.** First, in Section 2.2 matrix-vector products are introduced naturally by viewing the left side of a system of linear equations as a product. Second, matrix-matrix products are defined in Section 2.3 by taking the columns of a product AB to be A times the corresponding columns of B. This is motivated by viewing the matrix product as composition of maps (see next item). This works well pedagogically and the usual dot-product definition follows easily. As a bonus, the proof of associativity of matrix multiplication now takes four lines.

- **Matrices as transformations.** Matrix-column multiplications are viewed (in Section 2.2) as transformations $\mathbb{R}^n \to \mathbb{R}^m$. These maps are then used to describe simple geometric reflections and rotations in $\mathbb{R}^2$ as well as systems of linear equations.

- **Early linear transformations.** It has been said that vector spaces exist so that linear transformations can act on them—consequently these maps are a recurring theme in the text. Motivated by the matrix transformations introduced earlier, linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ are defined in Section 2.6, their standard matrices are derived, and they are then used to describe rotations, reflections, projections, and other operators on $\mathbb{R}^2$.

- **Early diagonalization.** As requested by engineers and scientists, this important technique is presented in the first term using only determinants and matrix inverses (before defining independence and dimension). Applications to population growth and linear recurrences are given.

- **Early dynamical systems.** These are introduced in Chapter 3, and lead (via diagonalization) to applications like the possible extinction of species. Beginning students in science and engineering can relate to this because they can see (often for the first time) the relevance of the subject to the real world.

- **Bridging chapter.** Chapter 5 lets students deal with tough concepts (like independence, spanning, and basis) in the concrete setting of $\mathbb{R}^n$ before having to cope with abstract vector spaces in Chapter 6.

- **Examples.** The text contains over 375 worked examples, which present the main techniques of the subject, illustrate the central ideas, and are keyed to the exercises in each section.

- **Exercises.** The text contains a variety of exercises (nearly 1175, many with multiple parts), starting with computational problems and gradually progressing to more theoretical exercises. Select solutions are available at the end of the book or in the Student Solution Manual. There is a complete Solution Manual is available for instructors.

- **Applications**. There are optional applications at the end of most chapters (see the list below). While some are presented in the course of the text, most appear at the end of the relevant chapter to encourage students to browse.

- **Appendices.** Because complex numbers are needed in the text, they are described in Appendix A, which includes the polar form and roots of unity. Methods of proofs are discussed in Appendix B, followed by mathematical induction in Appendix C. A brief discussion of polynomials is included in Appendix D. All these topics are presented at the high-school level.

- **Self-Study.** This text is self-contained and therefore is suitable for self-study.

- **Rigour.** Proofs are presented as clearly as possible (some at the end of the section), but they are optional and the instructor can choose how much he or she wants to prove. However the proofs are there, so this text is more rigorous than most. Linear algebra provides one of the better venues where students begin to think logically and argue concisely. To this end, there are exercises that ask the student to "show" some simple implication, and others that ask her or him to either prove a given statement or give a counterexample. I personally present a few proofs in the first semester course and more in the second (see the Suggested Course Outlines).

- **Major Theorems.** Several major results are presented in the book. Examples: Uniqueness of the reduced row-echelon form; the cofactor expansion for determinants; the Cayley-Hamilton theorem; the Jordan canonical form; Schur's theorem on block triangular form; the principal axes and spectral theorems; and others. Proofs are included because the stronger students should at least be aware of what is involved.

CHAPTER SUMMARIES

Chapter 1: Systems of Linear Equations.

A standard treatment of gaussian elimination is given. The rank of a matrix is introduced via the row-echelon form, and solutions to a homogeneous system are presented as linear combinations of basic solutions. Applications to network flows, electrical networks, and chemical reactions are provided.

Chapter 2: Matrix Algebra.

After a traditional look at matrix addition, scalar multiplication, and transposition in Section 2.1, matrix-vector multiplication is introduced in Section 2.2 by viewing the left side of a system of linear equations as the product $A\mathbf{x}$ of the coefficient matrix A with the column $\mathbf{x}$ of variables. The usual dot-product definition of a matrix-vector multiplication follows. Section 2.2 ends by viewing an $m \times n$ matrix A as a transformation $\mathbb{R}^n \to \mathbb{R}^m$. This is illustrated for $\mathbb{R}^2 \to \mathbb{R}^2$ by describing reflection in the x axis, rotation of $\mathbb{R}^2$ through $\frac{\pi}{2}$, shears, and so on.

In Section 2.3, the product of matrices A and B is defined by $AB = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array} \right]$, where the $\mathbf{b}_i$ are the columns of B. A routine computation shows that this is the matrix of the transformation B followed by A. This observation is used frequently throughout the book, and leads to simple, conceptual proofs of the basic axioms of matrix algebra. Note that linearity is not required—all that is needed is some basic properties of matrix-vector multiplication developed in Section 2.2. Thus the usual arcane definition of matrix multiplication is split into two well motivated parts, each an important aspect of matrix algebra. Of course, this has the pedagogical advantage that the conceptual power of geometry can be invoked to illuminate and clarify algebraic techniques and definitions.

In Section 2.4 and 2.5 matrix inverses are characterized, their geometrical meaning is explored, and block multiplication is introduced, emphasizing those cases needed later in the book. Elementary matrices are discussed, and the Smith normal form is derived. Then in Section 2.6, linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ are defined and shown to be matrix transformations. The matrices of reflections, rotations, and projections in the plane are determined. Finally, matrix multiplication is related to directed graphs, matrix LU-factorization is introduced, and applications to economic models and Markov chains are presented.

Chapter 3: Determinants and Diagonalization.

The cofactor expansion is stated (proved by induction later) and used to define determinants inductively and to deduce the basic rules. The product and adjugate theorems are proved. Then the diagonalization algorithm is presented (motivated by an example about the possible extinction of a species of birds). As requested by our Engineering Faculty, this is done earlier than in most texts because it requires only determinants and matrix inverses, avoiding any need for subspaces, independence and dimension. Eigenvectors of a 2×2 matrix A are described geometrically (using the A-invariance of lines through the origin). Diagonalization is then used to study discrete linear dynamical systems and to discuss applications to linear recurrences and systems of differential equations. A brief discussion of Google PageRank is included.

Chapter 4: Vector Geometry.

Vectors are presented intrinsically in terms of length and direction, and are related to matrices via coordinates. Then vector operations are defined using matrices and shown to be the same as the corresponding intrinsic definitions. Next, dot products and projections are introduced to solve problems about lines and planes. This leads to the cross product. Then matrix transformations are introduced in $\mathbb{R}^3$, matrices of projections and reflections are derived, and areas and volumes are computed using determinants. The chapter closes with an application to computer graphics.

Chapter 5: The Vector Space $\mathbb{R}^n$.

Subspaces, spanning, independence, and dimensions are introduced in the context of $\mathbb{R}^n$ in the first two sections. Orthogonal bases are introduced and used to derive the expansion theorem. The basic properties of rank are presented and used to justify the definition given in Section 1.2. Then, after a rigorous study of diagonalization, best approximation and least squares are discussed. The chapter closes with an application to correlation and variance.

This is a "bridging" chapter, easing the transition to abstract spaces. Concern about duplication with Chapter 6 is mitigated by the fact that this is the most difficult part of the course and many students welcome a repeat discussion of concepts like independence and spanning, albeit in the abstract setting. In a different direction, Chapter 1–5 could serve as a solid introduction to linear algebra for students not requiring abstract theory.

Chapter 6: Vector Spaces.

Building on the work on $\mathbb{R}^n$ in Chapter 5, the basic theory of abstract finite dimensional vector spaces is developed emphasizing new examples like matrices, polynomials and functions. This is the first acquaintance most students have had with an abstract system, so not having to deal with spanning, independence and dimension in the general context eases the transition to abstract thinking. Applications to polynomials and to differential equations are included.

Chapter 7: Linear Transformations.

General linear transformations are introduced, motivated by many examples from geometry, matrix theory, and calculus. Then kernels and images are defined, the dimension theorem is proved, and isomorphisms are discussed. The chapter ends with an application to linear recurrences. A proof is included that the order of a differential equation (with constant coefficients) equals the dimension of the space of solutions.

Chapter 8: Orthogonality.

The study of orthogonality in $\mathbb{R}^n$, begun in Chapter 5, is continued. Orthogonal complements and projections are defined and used to study orthogonal diagonalization. This leads to the principal axes theorem, the Cholesky factorization of a positive definite matrix, QR-factorization, and to a discussion of the singular value decomposition, the polar form, and the pseudoinverse. The theory is extended to $\mathbb{C}^n$ in Section 8.7 where hermitian and unitary matrices are discussed, culminating in Schur's theorem and the spectral theorem. A short proof of the Cayley-Hamilton theorem is also presented. In Section 8.8 the field $\mathbb{Z}_p$ of integers modulo p is constructed informally for any prime p, and codes are discussed over any finite field. The chapter concludes with applications to quadratic forms, constrained optimization, and statistical principal component analysis.

Chapter 9: Change of Basis.

The matrix of general linear transformation is defined and studied. In the case of an operator, the relationship between basis changes and similarity is revealed. This is illustrated by computing the matrix of a rotation about a line through the origin in $\mathbb{R}^3$. Finally, invariant subspaces and direct sums are introduced, related to similarity, and (as an example) used to show that every involution is similar to a diagonal matrix with diagonal entries ± 1.

Chapter 10: Inner Product Spaces.

General inner products are introduced and distance, norms, and the Cauchy-Schwarz inequality are discussed. The Gram-Schmidt algorithm is presented, projections are defined and the approximation theorem is proved (with an application to Fourier approximation). Finally, isometries are characterized, and distance preserving operators are shown to be composites of a translations and isometries.

Chapter 11: Canonical Forms.

The work in Chapter 9 is continued. Invariant subspaces and direct sums are used to derive the block triangular form. That, in turn, is used to give a compact proof of the Jordan canonical form. Of course the level is higher.

Appendices

In Appendix A, complex arithmetic is developed far enough to find *n*th roots. In Appendix B, methods of proof are discussed, while Appendix C presents mathematical induction. Finally, Appendix D describes the properties of polynomials in elementary terms.

LIST OF APPLICATIONS

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As we undertake this new publishing model with the text as an open educational resource, I would also like to thank my previous publisher. The team who supported my text greatly contributed to its success.

Now that the text has an open license, we have a much more fluid and powerful mechanism to incorporate comments and suggestions. The editorial group at Lyryx invites instructors and students to contribute to the text, and also offers to provide adaptations of the material for specific courses. Moreover the LaTeX source files are available to anyone wishing to do the adaptation and editorial work themselves!

W. Keith Nicholson
University of Calgary

1. Systems of Linear Equations

1.1 Solutions and Elementary Operations

Practical problems in many fields of study—such as biology, business, chemistry, computer science, economics, electronics, engineering, physics and the social sciences—can often be reduced to solving a system of linear equations. Linear algebra arose from attempts to find systematic methods for solving these systems, so it is natural to begin this book by studying linear equations.

If a, b, and c are real numbers, the graph of an equation of the form

$$ax+by=c$$

is a straight line (if a and b are not both zero), so such an equation is called a *linear* equation in the variables x and y. However, it is often convenient to write the variables as $x_1, x_2, \dots, x_n$, particularly when more than two variables are involved. An equation of the form

$$a_1x_1+a_2x_2+\cdots+a_nx_n=b$$

is called a **linear equation** in the n variables $x_1, x_2, \dots, x_n$. Here $a_1, a_2, \dots, a_n$ denote real numbers (called the **coefficients** of $x_1, x_2, \dots, x_n$, respectively) and b is also a number (called the **constant term** of the equation). A finite collection of linear equations in the variables $x_1, x_2, \dots, x_n$ is called a **system of linear equations** in these variables. Hence,

$$2x_1-3x_2+5x_3=7$$

is a linear equation; the coefficients of x_1, x_2, and x_3 are 2, -3, and 5, and the constant term is 7. Note that each variable in a linear equation occurs to the first power only.

Given a linear equation $a_1x_1+a_2x_2+\cdots+a_nx_n=b$, a sequence $s_1, s_2, \dots, s_n$ of n numbers is called a **solution** to the equation if

$$a_1s_1+a_2s_2+\cdots+a_ns_n=b$$

that is, if the equation is satisfied when the substitutions $x_1=s_1, x_2=s_2, \dots, x_n=s_n$ are made. A sequence of numbers is called **a solution to a system** of equations if it is a solution to every equation in the system.

For example, $x=-2$, $y=5$, $z=0$ and $x=0$, $y=4$, $z=-1$ are both solutions to the system

$$\begin{aligned} x+y+\ \ z&=3\\ 2x+y+3z&=1 \end{aligned}$$

A system may have no solution at all, or it may have a unique solution, or it may have an infinite family of solutions. For instance, the system $x+y=2$, $x+y=3$ has no solution because the sum of two numbers cannot be 2 and 3 simultaneously. A system that has no solution is called **inconsistent**; a system with at least one solution is called **consistent**. The system in the following example has infinitely many solutions.

Example 1.1.1

Show that, for arbitrary values of s and t,

$$\begin{aligned} x_1 &= t - s + 1 \\ x_2 &= t + s + 2 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

is a solution to the system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & = & -3 \\ 2x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & 0 \end{array}$$

Solution. Simply substitute these values of x_1, x_2, x_3, and x_4 in each equation.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 &= (t - s + 1) - 2(t + s + 2) + 3s + t = -3 \\ 2x_1 - x_2 + 3x_3 - x_4 &= 2(t - s + 1) - (t + s + 2) + 3s - t = 0 \end{aligned}$$

Because both equations are satisfied, it is a solution for all choices of s and t.

The quantities s and t in Example 1.1.1 are called **parameters**, and the set of solutions, described in this way, is said to be given in **parametric form** and is called the **general solution** to the system. It turns out that the solutions to *every* system of equations (if there *are* solutions) can be given in parametric form (that is, the variables x_1, x_2, $\dots$ are given in terms of new independent variables s, t, etc.). The following example shows how this happens in the simplest systems where only one equation is present.

Example 1.1.2

Describe all solutions to $3x - y + 2z = 6$ in parametric form.

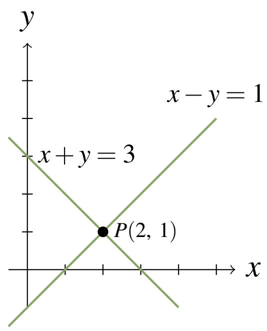
Solution. Solving the equation for y in terms of x and z, we get $y = 3x + 2z - 6$. If s and t are arbitrary then, setting $x = s$, $z = t$, we get solutions

$$\begin{aligned} x &= s \\ y &= 3s + 2t - 6 \qquad s \text{ and } t \text{ arbitrary} \\ z &= t \end{aligned}$$

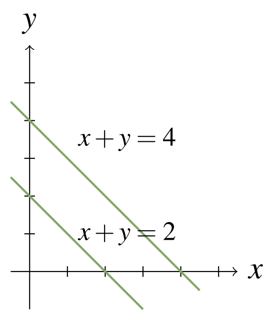
Of course we could have solved for x: $x = \frac{1}{3}(y - 2z + 6)$. Then, if we take $y = p$, $z = q$, the solutions are represented as follows:

$$\begin{array}{rcl} x & = & \frac{1}{3}(p - 2q + 6) \\ y & = & p \qquad\qquad\qquad p \text{ and } q \text{ arbitrary} \\ z & = & q \end{array}$$

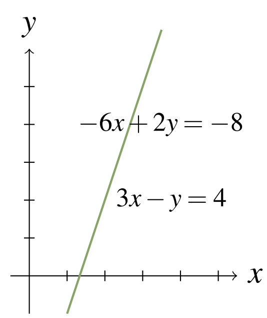
The same family of solutions can "look" quite different!



(a) Unique Solution
$(x = 2, y = 1)$



(b) No Solution



(c) Infinitely many solutions
$(x = t, y = 3t - 4)$

Figure 1.1.1

When only two variables are involved, the solutions to systems of linear equations can be described geometrically because the graph of a linear equation $ax + by = c$ is a straight line if a and b are not both zero. Moreover, a point $P(s,\ t)$ with coordinates s and t lies on the line if and only if $as + bt = c$—that is when $x = s$, $y = t$ is a solution to the equation. Hence the solutions to a *system* of linear equations correspond to the points $P(s,\ t)$ that lie on *all* the lines in question.

In particular, if the system consists of just one equation, there must be infinitely many solutions because there are infinitely many points on a line. If the system has two equations, there are three possibilities for the corresponding straight lines:

1. *The lines intersect at a single point. Then the system has a* unique solution *corresponding to that point.*
2. *The lines are parallel (and distinct) and so do not intersect. Then the system has* no solution.
3. *The lines are identical. Then the system has* infinitely many solutions—*one for each point on the (common) line.*

These three situations are illustrated in Figure 1.1.1. In each case the graphs of two specific lines are plotted and the corresponding equations are indicated. In the last case, the equations are $3x - y = 4$ and $-6x + 2y = -8$, which have identical graphs.

With three variables, the graph of an equation $ax + by + cz = d$ can be shown to be a plane (see Section 4.2) and so again provides a "picture" of the set of solutions. However, this graphical method has its limitations: When more than three variables are involved, no physical image of the graphs (called hyperplanes) is possible. It is necessary to turn to a more "algebraic" method of solution.

Before describing the method, we introduce a concept that simplifies the computations involved. Consider the following system

$$\begin{array}{rcrcrcrcr}
3x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & -1 \\
2x_1 & & & - & x_3 & + & 2x_4 & = & 0 \\
3x_1 & + & x_2 & + & 2x_3 & + & 5x_4 & = & 2
\end{array}$$

of three equations in four variables. The array of numbers[1]

$$\left[\begin{array}{rrrr|r}
3 & 2 & -1 & 1 & -1 \\
2 & 0 & -1 & 2 & 0 \\
3 & 1 & 2 & 5 & 2
\end{array}\right]$$

occurring in the system is called the **augmented matrix** of the system. Each row of the matrix consists of the coefficients of the variables (in order) from the corresponding equation, together with the constant

[1] A rectangular array of numbers is called a **matrix**. Matrices will be discussed in more detail in Chapter 2.

term. For clarity, the constants are separated by a vertical line. The augmented matrix is just a different way of describing the system of equations. The array of coefficients of the variables

$$\left[\begin{array}{rrrr} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{array}\right]$$

is called the **coefficient matrix** of the system and $\left[\begin{array}{r} -1 \\ 0 \\ 2 \end{array}\right]$ is called the **constant matrix** of the system.

Elementary Operations

The algebraic method for solving systems of linear equations is described as follows. Two such systems are said to be **equivalent** if they have the same set of solutions. A system is solved by writing a series of systems, one after the other, each equivalent to the previous system. Each of these systems has the same set of solutions as the original one; the aim is to end up with a system that is easy to solve. Each system in the series is obtained from the preceding system by a simple manipulation chosen so that it does not change the set of solutions.

As an illustration, we solve the system $x+2y=-2$, $2x+y=7$ in this manner. At each stage, the corresponding augmented matrix is displayed. The original system is

$$\begin{array}{rl} x+2y = & -2 \\ 2x+\ \ y = & 7 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 2 & 1 & 7 \end{array}\right]$$

First, subtract twice the first equation from the second. The resulting system is

$$\begin{array}{rl} x+2y = & -2 \\ -3y = & 11 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 0 & -3 & 11 \end{array}\right]$$

which is equivalent to the original (see Theorem 1.1.1). At this stage we obtain $y=-\frac{11}{3}$ by multiplying the second equation by $-\frac{1}{3}$. The result is the equivalent system

$$\begin{array}{rl} x+2y = & -2 \\ y = & -\frac{11}{3} \end{array} \qquad \left[\begin{array}{rr|r} 1 & 2 & -2 \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Finally, we subtract twice the second equation from the first to get another equivalent system.

$$\begin{array}{rl} x = & \frac{16}{3} \\ y = & -\frac{11}{3} \end{array} \qquad \left[\begin{array}{rr|r} 1 & 0 & \frac{16}{3} \\ 0 & 1 & -\frac{11}{3} \end{array}\right]$$

Now *this* system is easy to solve! And because it is equivalent to the original system, it provides the solution to that system.

Observe that, at each stage, a certain operation is performed on the system (and thus on the augmented matrix) to produce an equivalent system.

Definition 1.1 Elementary Operations

The following operations, called ***elementary operations****, can routinely be performed on systems of linear equations to produce equivalent systems.*

I. *Interchange two equations.*

II. *Multiply one equation by a* nonzero *number.*

III. *Add a multiple of one equation to a* different *equation.*

Theorem 1.1.1

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

The proof is given at the end of this section.

Elementary operations performed on a system of equations produce corresponding manipulations of the *rows* of the augmented matrix. Thus, multiplying a row of a matrix by a number k means multiplying *every entry* of the row by k. Adding one row to another row means adding *each entry* of that row to the corresponding entry of the other row. Subtracting two rows is done similarly. Note that we regard two rows as equal when corresponding entries are the same.

In hand calculations (and in computer programs) we manipulate the rows of the augmented matrix rather than the equations. For this reason we restate these elementary operations for matrices.

Definition 1.2 Elementary Row Operations

The following are called ***elementary row operations*** *on a matrix.*

I. *Interchange two rows.*

II. *Multiply one row by a* nonzero *number.*

III. *Add a multiple of one row to a* different *row.*

In the illustration above, a series of such operations led to a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$$

where the asterisks represent arbitrary numbers. In the case of three equations in three variables, the goal is to produce a matrix of the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

This does not always happen, as we will see in the next section. Here is an example in which it does happen.

Example 1.1.3

Find all solutions to the following system of equations.

$$\begin{array}{rcr} 3x+4y+z &=& 1 \\ 2x+3y &=& 0 \\ 4x+3y-z &=& -2 \end{array}$$

Solution. The augmented matrix of the original system is

$$\left[\begin{array}{rrr|r} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

To create a 1 in the upper left corner we could multiply row 1 through by $\frac{1}{3}$. However, the 1 can be obtained without introducing fractions by subtracting row 2 from row 1. The result is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

The upper left 1 is now used to "clean up" the first column, that is create zeros in the other positions in that column. First subtract 2 times row 1 from row 2 to obtain

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 4 & 3 & -1 & -2 \end{array}\right]$$

Next subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -5 & -6 \end{array}\right]$$

This completes the work on column 1. We now use the 1 in the second position of the second row to clean up the second column by subtracting row 2 from row 1 and then adding row 2 to row 3. For convenience, both row operations are done in one step. The result is

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \end{array}\right]$$

Note that the last two manipulations *did not affect* the first column (the second row has a zero there), so our previous effort there has not been undermined. Finally we clean up the third column. Begin by multiplying row 3 by $-\frac{1}{7}$ to obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right]$$

Now subtract 3 times row 3 from row 1, and then add 2 times row 3 to row 2 to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{8}{7} \end{array}\right]$$

The corresponding equations are $x = -\frac{3}{7}$, $y = \frac{2}{7}$, and $z = \frac{8}{7}$, which give the (unique) solution.

Every elementary row operation can be **reversed** by another elementary row operation of the same type (called its **inverse**). To see how, we look at types I, II, and III separately:

Type I *Interchanging two rows is reversed by interchanging them again.*

Type II *Multiplying a row by a nonzero number k is reversed by multiplying by* $1/k$.

Type III *Adding k times row p to a different row q is reversed by adding* $-k$ *times row p to row q (in the new matrix). Note that* $p \neq q$ *is essential here.*

To illustrate the Type III situation, suppose there are four rows in the original matrix, denoted R_1, R_2, R_3, and R_4, and that k times R_2 is added to R_3. Then the reverse operation adds $-k$ times R_2, to R_3. The following diagram illustrates the effect of doing the operation first and then the reverse:

$$\left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 + kR_2 \\ R_4 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1 \\ R_2 \\ (R_3 + kR_2) - kR_2 \\ R_4 \end{array}\right] = \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}\right]$$

The existence of inverses for elementary row operations and hence for elementary operations on a system of equations, gives:

Proof of Theorem 1.1.1. Suppose that a system of linear equations is transformed into a new system by a sequence of elementary operations. Then every solution of the original system is automatically a solution of the new system because adding equations, or multiplying an equation by a nonzero number, always results in a valid equation. In the same way, each solution of the new system must be a solution to the original system because the original system can be obtained from the new one by another series of elementary operations (the inverses of the originals). It follows that the original and new systems have the same solutions. This proves Theorem 1.1.1. □

Exercises for 1.1

Exercise 1.1.1 In each case verify that the following are solutions for all values of s and t.

a. $x = 19t - 35$
$y = 25 - 13t$
$z = t$

is a solution of

$$\begin{aligned} 2x + 3y + z &= 5 \\ 5x + 7y - 4z &= 0 \end{aligned}$$

b. $x_1 = 2s + 12t + 13$
$x_2 = s$
$x_3 = -s - 3t - 3$
$x_4 = t$

is a solution of

$$\begin{aligned} 2x_1 + 5x_2 + 9x_3 + 3x_4 &= -1 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

Exercise 1.1.2 Find all solutions to the following in parametric form in two ways.

a. $3x + y = 2$

b. $2x + 3y = 1$

c. $3x - y + 2z = 5$

d. $x - 2y + 5z = 1$

Exercise 1.1.3 Regarding $2x = 5$ as the equation $2x + 0y = 5$ in two variables, find all solutions in parametric form.

Exercise 1.1.4 Regarding $4x - 2y = 3$ as the equation $4x - 2y + 0z = 3$ in three variables, find all solutions in parametric form.

Exercise 1.1.5 Find all solutions to the general system $ax = b$ of one equation in one variable (a) when $a = 0$ and (b) when $a \neq 0$.

Exercise 1.1.6 Show that a system consisting of exactly one linear equation can have no solution, one solution, or infinitely many solutions. Give examples.

Exercise 1.1.7 Write the augmented matrix for each of the following systems of linear equations.

a. $\begin{aligned} x - 3y &= 5 \\ 2x + y &= 1 \end{aligned}$

b. $\begin{aligned} x + 2y &= 0 \\ y &= 1 \end{aligned}$

c. $\begin{aligned} x - y + z &= 2 \\ x - z &= 1 \\ y + 2x &= 0 \end{aligned}$

d. $\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ z - x &= 2 \end{aligned}$

Exercise 1.1.8 Write a system of linear equations that has each of the following augmented matrices.

a. $\left[\begin{array}{rrr|r} 1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & -1 & 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$

Exercise 1.1.9 Find the solution of each of the following systems of linear equations using augmented matrices.

a. $\begin{aligned} x - 3y &= 1 \\ 2x - 7y &= 3 \end{aligned}$

b. $\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned}$

c. $\begin{aligned} 2x + 3y &= -1 \\ 3x + 4y &= 2 \end{aligned}$

d. $\begin{aligned} 3x + 4y &= 1 \\ 4x + 5y &= -3 \end{aligned}$

Exercise 1.1.10 Find the solution of each of the following systems of linear equations using augmented matrices.

a. $\begin{aligned} x + y + 2z &= -1 \\ 2x + y + 3z &= 0 \\ -2y + z &= 2 \end{aligned}$

b. $\begin{aligned} 2x + y + z &= -1 \\ x + 2y + z &= 0 \\ 3x - 2z &= 5 \end{aligned}$

Exercise 1.1.11 Find all solutions (if any) of the following systems of linear equations.

a. $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= -20 \end{aligned}$

b. $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= 16 \end{aligned}$

Exercise 1.1.12 Show that the system

$$\left\{ \begin{array}{rcrcrcr} x & + & 2y & - & z & = & a \\ 2x & + & y & + & 3z & = & b \\ x & - & 4y & + & 9z & = & c \end{array} \right.$$

is inconsistent unless $c = 2b - 3a$.

Exercise 1.1.13 By examining the possible positions of lines in the plane, show that two equations in two variables can have zero, one, or infinitely many solutions.

Exercise 1.1.14 In each case either show that the statement is true, or give an example[2] showing it is false.

a. If a linear system has n variables and m equations, then the augmented matrix has n rows.

b. A consistent linear system must have infinitely many solutions.

c. If a row operation is done to a consistent linear system, the resulting system must be consistent.

d. If a series of row operations on a linear system results in an inconsistent system, the original system is inconsistent.

Exercise 1.1.15 Find a quadratic $a+bx+cx^2$ such that the graph of $y=a+bx+cx^2$ contains each of the points $(-1, 6)$, $(2, 0)$, and $(3, 2)$.

Exercise 1.1.16 Solve the system $\left\{\begin{array}{l} 3x+2y=5 \\ 7x+5y=1 \end{array}\right.$ by changing variables $\left\{\begin{array}{l} x = 5x'-2y' \\ y=-7x'+3y' \end{array}\right.$ and solving the resulting equations for x' and y'.

Exercise 1.1.17 Find a, b, and c such that

$$\frac{x^2-x+3}{(x^2+2)(2x-1)} = \frac{ax+b}{x^2+2} + \frac{c}{2x-1}$$

[*Hint*: Multiply through by $(x^2+2)(2x-1)$ and equate coefficients of powers of x.]

Exercise 1.1.18 A zookeeper wants to give an animal 42 mg of vitamin A and 65 mg of vitamin D per day. He has two supplements: the first contains 10% vitamin A and 25% vitamin D; the second contains 20% vitamin A and 25% vitamin D. How much of each supplement should he give the animal each day?

Exercise 1.1.19 Workmen John and Joe earn a total of \$24.60 when John works 2 hours and Joe works 3 hours. If John works 3 hours and Joe works 2 hours, they get \$23.90. Find their hourly rates.

Exercise 1.1.20 A biologist wants to create a diet from fish and meal containing 183 grams of protein and 93 grams of carbohydrate per day. If fish contains 70% protein and 10% carbohydrate, and meal contains 30% protein and 60% carbohydrate, how much of each food is required each day?

1.2 Gaussian Elimination

The algebraic method introduced in the preceding section can be summarized as follows: Given a system of linear equations, use a sequence of elementary row operations to carry the augmented matrix to a "nice" matrix (meaning that the corresponding equations are easy to solve). In Example 1.1.3, this nice matrix took the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The following definitions identify the nice matrices that arise in this process.

[2]Such an example is called a **counterexample**. For example, if the statement is that "all philosophers have beards", the existence of a non-bearded philosopher would be a counterexample proving that the statement is false. This is discussed again in Appendix B.

Definition 1.3 Row-Echelon Form (Reduced)

A matrix is said to be in ***row-echelon form*** *(and will be called a* ***row-echelon matrix****) if it satisfies the following three conditions:*

1. *All* ***zero rows*** *(consisting entirely of zeros) are at the bottom.*
2. *The first nonzero entry from the left in each nonzero row is a* 1*, called the* ***leading 1*** *for that row.*
3. *Each leading* 1 *is to the right of all leading* 1*s in the rows above it.*

A row-echelon matrix is said to be in ***reduced row-echelon form*** *(and will be called a* ***reduced row-echelon matrix****) if, in addition, it satisfies the following condition:*

4. *Each leading* 1 *is the only nonzero entry in its column.*

The row-echelon matrices have a "staircase" form, as indicated by the following example (the asterisks indicate arbitrary numbers).

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1s proceed "down and to the right" through the matrix. Entries above and to the right of the leading 1s are arbitrary, but all entries below and to the left of them are zero. Hence, a matrix in row-echelon form is in reduced form if, in addition, the entries directly above each leading 1 are all zero. Note that a matrix in row-echelon form can, with a few more row operations, be carried to reduced form (use row operations to create zeros above each leading one in succession, beginning from the right).

Example 1.2.1

The following matrices are in row-echelon form (for any choice of numbers in $*$-positions).

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

The following, on the other hand, are in reduced row-echelon form.

$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The choice of the positions for the leading 1s determines the (reduced) row-echelon form (apart from the numbers in $*$-positions).

The importance of row-echelon matrices comes from the following theorem.

Theorem 1.2.1

Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

In fact we can give a step-by-step procedure for actually finding a row-echelon matrix. Observe that while there are many sequences of row operations that will bring a matrix to row-echelon form, the one we use is systematic and is easy to program on a computer. Note that the algorithm deals with matrices in general, possibly with columns of zeros.

Gaussian[3] Algorithm[4]

Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Now multiply the new top row by $1/a$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Observe that the gaussian algorithm is recursive: When the first leading 1 has been obtained, the procedure is repeated on the remaining rows of the matrix. This makes the algorithm easy to use on a computer. Note that the solution to Example 1.1.3 did not use the gaussian algorithm as written because the first leading 1 was not created by dividing row 1 by 3. The reason for this is that it avoids fractions. However, the general pattern is clear: Create the leading 1s from left to right, using each of them in turn to create zeros below it. Here are two more examples.

---

[3]Carl Friedrich Gauss (1777–1855) ranks with Archimedes and Newton as one of the three greatest mathematicians of all time. He was a child prodigy and, at the age of 21, he gave the first proof that every polynomial has a complex root. In 1801 he published a timeless masterpiece, *Disquisitiones Arithmeticae*, in which he founded modern number theory. He went on to make ground-breaking contributions to nearly every branch of mathematics, often well before others rediscovered and published the results.

[4]The algorithm was known to the ancient Chinese.

Example 1.2.2

Solve the following system of equations.

$$\begin{array}{rcrcrcr} 3x & + & y & - & 4z & = & -1 \\ x & & & + & 10z & = & 5 \\ 4x & + & y & + & 6z & = & 1 \end{array}$$

Solution. The corresponding augmented matrix is

$$\left[\begin{array}{rrr|r} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Create the first leading one by interchanging rows 1 and 2

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{array}\right]$$

Now subtract 3 times row 1 from row 2, and subtract 4 times row 1 from row 3. The result is

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 1 & -34 & -19 \end{array}\right]$$

Now subtract row 2 from row 3 to obtain

$$\left[\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

This means that the following reduced system of equations

$$\begin{array}{rcrcr} x & + & 10z & = & 5 \\ y & - & 34z & = & -16 \\ & & 0 & = & -3 \end{array}$$

is equivalent to the original system. In other words, the two have the same solutions. But this last system clearly has no solution (the last equation requires that x, y and z satisfy $0x + 0y + 0z = -3$, and no such numbers exist). Hence the original system has no solution.

Example 1.2.3

Solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Solution. The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Subtracting twice row 1 from row 2 and subtracting row 1 from row 3 gives

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

Now subtract row 2 from row 3 and multiply row 2 by $\frac{1}{3}$ to get

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is in row-echelon form, and we take it to reduced form by adding row 2 to row 1:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding reduced system of equations is

$$\begin{aligned} x_1 - 2x_2 \phantom{{}+x_3} + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

The leading ones are in columns 1 and 3 here, so the corresponding variables x_1 and x_3 are called leading variables. Because the matrix is in reduced row-echelon form, these equations can be used to solve for the leading variables in terms of the nonleading variables x_2 and x_4. More precisely, in the present example we set $x_2 = s$ and $x_4 = t$ where s and t are arbitrary, so these equations become

$$x_1 - 2s + t = 2 \quad \text{and} \quad x_3 - 2t = 1$$

Finally the solutions are given by

$$\begin{aligned} x_1 &= 2 + 2s - t \\ x_2 &= s \\ x_3 &= 1 + 2t \\ x_4 &= t \end{aligned}$$

where s and t are arbitrary.

The solution of Example 1.2.3 is typical of the general case. To solve a linear system, the augmented matrix is carried to reduced row-echelon form, and the variables corresponding to the leading ones are called **leading variables**. Because the matrix is in reduced form, each leading variable occurs in exactly one equation, so that equation can be solved to give a formula for the leading variable in terms of the nonleading variables. It is customary to call the nonleading variables "free" variables, and to label them by new variables s, t, $\dots$, called **parameters**. Hence, as in Example 1.2.3, every variable x_i is given by a formula in terms of the parameters s and t. Moreover, every choice of these parameters leads to a solution to the system, and every solution arises in this way. This procedure works in general, and has come to be called

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. *Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.*
2. *If a row* $\left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$ *occurs, the system is inconsistent.*
3. *Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.*

There is a variant of this procedure, wherein the augmented matrix is carried only to row-echelon form. The nonleading variables are assigned as parameters as before. Then the last equation (corresponding to the row-echelon form) is used to solve for the last leading variable in terms of the parameters. This last leading variable is then substituted into all the preceding equations. Then, the second last equation yields the second last leading variable, which is also substituted back. The process continues to give the general solution. This procedure is called **back-substitution**. This procedure can be shown to be numerically more efficient and so is important when solving very large systems.[5]

Example 1.2.4

Find a condition on the numbers a, b, and c such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in terms of a, b, and c).

$$\begin{aligned} x_1 + 3x_2 + x_3 &= a \\ -x_1 - 2x_2 + x_3 &= b \\ 3x_1 + 7x_2 - x_3 &= c \end{aligned}$$

Solution. We use gaussian elimination except that now the augmented matrix

$$\left[\begin{array}{rrr|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right]$$

[5]With n equations where n is large, gaussian elimination requires roughly $n^3/2$ multiplications and divisions, whereas this number is roughly $n^3/3$ if back substitution is used.

has entries a, b, and c as well as known numbers. The first leading one is in place, so we create zeros below it in column 1:

$$\left[\begin{array}{rrr|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & c-3a \end{array}\right]$$

The second leading 1 has appeared, so use it to create zeros in the rest of column 2:

$$\left[\begin{array}{rrr|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

Now the whole solution depends on the number $c-a+2b = c-(a-2b)$. The last row corresponds to an equation $0 = c-(a-2b)$. If $c \neq a-2b$, there is *no* solution (just as in Example 1.2.2). Hence:

The system is consistent if and only if $c = a-2b$.

In this case the last matrix becomes

$$\left[\begin{array}{rrr|c} 1 & 0 & -5 & -2a-3b \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, if $c = a-2b$, taking $x_3 = t$ where t is a parameter gives the solutions

$$x_1 = 5t-(2a+3b) \quad x_2 = (a+b)-2t \quad x_3 = t.$$

Rank

It can be proven that the *reduced* row-echelon form of a matrix A is uniquely determined by A. That is, no matter which series of row operations is used to carry A to a reduced row-echelon matrix, the result will always be the same matrix. (A proof is given at the end of Section 2.5.) By contrast, this is not true for row-echelon matrices: Different series of row operations can carry the same matrix A to *different* row-echelon matrices. Indeed, the matrix $A = \left[\begin{array}{rrr} 1 & -1 & 4 \\ 2 & -1 & 2 \end{array}\right]$ can be carried (by one row operation) to the row-echelon matrix $\left[\begin{array}{rrr} 1 & -1 & 4 \\ 0 & 1 & -6 \end{array}\right]$, and then by another row operation to the (reduced) row-echelon matrix $\left[\begin{array}{rrr} 1 & 0 & -2 \\ 0 & 1 & -6 \end{array}\right]$. However, it *is* true that the number r of leading 1s must be the same in each of these row-echelon matrices (this will be proved in Chapter 5). Hence, the number r depends only on A and not on the way in which A is carried to row-echelon form.

Definition 1.4 Rank of a Matrix

*The **rank** of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.*

Example 1.2.5

Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$.

Solution. The reduction of A to row-echelon form is

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & -5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because this row-echelon matrix has two leading 1s, rank $A = 2$.

Suppose that rank $A = r$, where A is a matrix with m rows and n columns. Then $r \leq m$ because the leading 1s lie in different rows, and $r \leq n$ because the leading 1s lie in different columns. Moreover, the rank has a useful application to equations. Recall that a system of linear equations is called consistent if it has at least one solution.

Theorem 1.2.2

*Suppose a system of m equations in n variables is **consistent**, and that the rank of the augmented matrix is r.*

1. *The set of solutions involves exactly $n - r$ parameters.*
2. *If $r < n$, the system has infinitely many solutions.*
3. *If $r = n$, the system has a unique solution.*

Proof. The fact that the rank of the augmented matrix is r means there are exactly r leading variables, and hence exactly $n - r$ nonleading variables. These nonleading variables are all assigned as parameters in the gaussian algorithm, so the set of solutions involves exactly $n - r$ parameters. Hence if $r < n$, there is at least one parameter, and so infinitely many solutions. If $r = n$, there are no parameters and so a unique solution. □

Theorem 1.2.2 shows that, for any system of linear equations, exactly three possibilities exist:

1. *No solution. This occurs when a row* $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ *occurs in the row-echelon form. This is the case where the system is inconsistent.*
2. *Unique solution. This occurs when* every *variable is a leading variable.*

3. *Infinitely many solutions. This occurs when the system is consistent and there is at least one nonleading variable, so at least one parameter is involved.*

Example 1.2.6

Suppose the matrix A in Example 1.2.5 is the augmented matrix of a system of $m = 3$ linear equations in $n = 3$ variables. As rank $A = r = 2$, the set of solutions will have $n - r = 1$ parameter. The reader can verify this fact directly.

Many important problems involve **linear inequalities** rather than **linear equations**. For example, a condition on the variables x and y might take the form of an inequality $2x - 5y \leq 4$ rather than an equality $2x - 5y = 4$. There is a technique (called the **simplex algorithm**) for finding solutions to a system of such inequalities that maximizes a function of the form $p = ax + by$ where a and b are fixed constants.

Exercises for 1.2

Exercise 1.2.1 Which of the following matrices are in reduced row-echelon form? Which are in row-echelon form?

a. $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrrr} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{rrrr} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d. $\left[\begin{array}{rrrrr} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

e. $\left[\begin{array}{rr} 1 & 1 \\ 0 & 1 \end{array}\right]$

f. $\left[\begin{array}{rrr} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$

Exercise 1.2.2 Carry each of the following matrices to reduced row-echelon form.

a. $\left[\begin{array}{rrrrrrr} 0 & -1 & 2 & 1 & 2 & 1 & -1 \\ 0 & 1 & -2 & 2 & 7 & 2 & 4 \\ 0 & -2 & 4 & 3 & 7 & 1 & 0 \\ 0 & 3 & -6 & 1 & 6 & 4 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrrrrrr} 0 & -1 & 3 & 1 & 3 & 2 & 1 \\ 0 & -2 & 6 & 1 & -5 & 0 & -1 \\ 0 & 3 & -9 & 2 & 4 & 1 & -1 \\ 0 & 1 & -3 & -1 & 3 & 0 & 1 \end{array}\right]$

Exercise 1.2.3 The augmented matrix of a system of linear equations has been carried to the following by row operations. In each case solve the system.

a. $\left[\begin{array}{rrrrrr|r} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

b. $\left[\begin{array}{rrrrrr|r} 1 & -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

c. $\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 4 & 6 & 2 \\ 0 & 1 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Exercise 1.2.4 Find all solutions (if any) to each of the following systems of linear equations.

a. $\begin{array}{rl} x - 2y = & 1 \\ 4y - x = & -2 \end{array}$

b. $\begin{array}{rl} 3x - y = & 0 \\ 2x - 3y = & 1 \end{array}$

c. $\begin{aligned} 2x + y &= 5 \\ 3x + 2y &= 6 \end{aligned}$

d. $\begin{aligned} 3x - y &= 2 \\ 2y - 6x &= -4 \end{aligned}$

e. $\begin{aligned} 3x - y &= 4 \\ 2y - 6x &= 1 \end{aligned}$

f. $\begin{aligned} 2x - 3y &= 5 \\ 3y - 2x &= 2 \end{aligned}$

Exercise 1.2.5 Find all solutions (if any) to each of the following systems of linear equations.

a. $\begin{aligned} x + y + 2z &= 8 \\ 3x - y + z &= 0 \\ -x + 3y + 4z &= -4 \end{aligned}$

b. $\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$

c. $\begin{aligned} x + y - z &= 10 \\ -x + 4y + 5z &= -5 \\ x + 6y + 3z &= 15 \end{aligned}$

d. $\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$

e. $\begin{aligned} 5x + y &= 2 \\ 3x - y + 2z &= 1 \\ x + y - z &= 5 \end{aligned}$

f. $\begin{aligned} 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$

g. $\begin{aligned} x + y + z &= 2 \\ x + z &= 1 \\ 2x + 5y + 2z &= 7 \end{aligned}$

h. $\begin{aligned} x + 2y - 4z &= 10 \\ 2x - y + 2z &= 5 \\ x + y - 2z &= 7 \end{aligned}$

Exercise 1.2.6 Express the last equation of each system as a sum of multiples of the first two equations. [*Hint*: Label the equations, use the gaussian algorithm.]

a. $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 3 \\ x_1 - 2x_2 + 2x_3 &= 2 \end{aligned}$

b. $\begin{aligned} x_1 + 2x_2 - 3x_3 &= -3 \\ x_1 + 3x_2 - 5x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -35 \end{aligned}$

Exercise 1.2.7 Find all solutions to the following systems.

a. $\begin{aligned} 3x_1 + 8x_2 - 3x_3 - 14x_4 &= 2 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 1 \\ x_1 - 2x_2 + x_3 + 10x_4 &= 0 \\ x_1 + 5x_2 - 2x_3 - 12x_4 &= 1 \end{aligned}$

b. $\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$

c. $\begin{aligned} x_1 - x_2 + x_3 - 2x_4 &= 1 \\ -x_1 + x_2 + x_3 + x_4 &= -1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 2 \\ x_1 - x_2 + 2x_3 + x_4 &= 1 \end{aligned}$

d. $\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 4 \\ 3x_2 - x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - 3x_3 + 5x_4 &= 0 \\ x_1 + x_2 - 5x_3 + 6x_4 &= -3 \end{aligned}$

Exercise 1.2.8 In each of the following, find (if possible) conditions on a and b such that the system has no solution, one solution, and infinitely many solutions.

a. $\begin{aligned} x - 2y &= 1 \\ ax + by &= 5 \end{aligned}$

b. $\begin{aligned} x + by &= -1 \\ ax + 2y &= 5 \end{aligned}$

c. $\begin{aligned} x - by &= -1 \\ x + ay &= 3 \end{aligned}$

d. $\begin{aligned} ax + y &= 1 \\ 2x + y &= b \end{aligned}$

Exercise 1.2.9 In each of the following, find (if possible) conditions on a, b, and c such that the system has no solution, one solution, or infinitely many solutions.

a. $\begin{aligned} 3x + y - z &= a \\ x - y + 2z &= b \\ 5x + 3y - 4z &= c \end{aligned}$

b. $\begin{aligned} 2x + y - z &= a \\ 2y + 3z &= b \\ x - z &= c \end{aligned}$

c. $\begin{aligned} -x + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b \end{aligned}$

d. $\begin{aligned} x + ay &= 0 \\ y + bz &= 0 \\ z + cx &= 0 \end{aligned}$

e. $\begin{aligned} 3x - y + 2z &= 3 \\ x + y - z &= 2 \\ 2x - 2y + 3z &= b \end{aligned}$

f. $\begin{aligned} x + ay - z &= 1 \\ -x + (a-2)y + z &= -1 \\ 2x + 2y + (a-2)z &= 1 \end{aligned}$

Exercise 1.2.10 Find the rank of each of the matrices in Exercise 1.2.1.

Exercise 1.2.11 Find the rank of each of the following matrices.

a. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 2 & a^2 \\ 1 & 1-a & 2 & 0 \\ 2 & 2-a & 6-a & 4 \end{bmatrix}$

Exercise 1.2.12 Consider a system of linear equations with augmented matrix A and coefficient matrix C. In each case either prove the statement or give an example showing that it is false.

a. If there is more than one solution, A has a row of zeros.

b. If A has a row of zeros, there is more than one solution.

c. If there is no solution, the reduced row-echelon form of C has a row of zeros.

d. If the row-echelon form of C has a row of zeros, there is no solution.

e. There is no system that is inconsistent for every choice of constants.

f. If the system is consistent for some choice of constants, it is consistent for every choice of constants.

Now assume that the augmented matrix A has 3 rows and 5 columns.

g. If the system is consistent, there is more than one solution.

h. The rank of A is at most 3.

i. If rank $A = 3$, the system is consistent.

j. If rank $C = 3$, the system is consistent.

Exercise 1.2.13 Find a sequence of row operations carrying

$$\left[\begin{array}{ccc} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{array}\right] \text{ to } \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right]$$

Exercise 1.2.14 In each case, show that the reduced row-echelon form is as given.

a. $\left[\begin{array}{ccc} p & 0 & a \\ b & 0 & 0 \\ q & c & r \end{array}\right]$ with $abc \neq 0$; $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{ccc} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{array}\right]$ where $c \neq a$ or $b \neq a$; $\left[\begin{array}{ccc} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$

Exercise 1.2.15 Show that $\left\{\begin{array}{rl} az+\ by+\ cz &= 0 \\ a_1x+b_1y+c_1z &= 0 \end{array}\right.$ always has a solution other than $x = 0$, $y = 0$, $z = 0$.

Exercise 1.2.16 Find the circle $x^2+y^2+ax+by+c=0$ passing through the following points.

a. $(-2,\ 1)$, $(5,\ 0)$, and $(4,\ 1)$

b. $(1,\ 1)$, $(5,\ -3)$, and $(-3,\ -3)$

Exercise 1.2.17 Three Nissans, two Fords, and four Chevrolets can be rented for \$106 per day. At the same rates two Nissans, four Fords, and three Chevrolets cost \$107 per day, whereas four Nissans, three Fords, and two Chevrolets cost \$102 per day. Find the rental rates for all three kinds of cars.

Exercise 1.2.18 A school has three clubs and each student is required to belong to exactly one club. One year the students switched club membership as follows:

Club A. $\frac{4}{10}$ remain in A, $\frac{1}{10}$ switch to B, $\frac{5}{10}$ switch to C.
Club B. $\frac{7}{10}$ remain in B, $\frac{2}{10}$ switch to A, $\frac{1}{10}$ switch to C.
Club C. $\frac{6}{10}$ remain in C, $\frac{2}{10}$ switch to A, $\frac{2}{10}$ switch to B.

If the fraction of the student population in each club is unchanged, find each of these fractions.

Exercise 1.2.19 Given points $(p_1,\ q_1)$, $(p_2,\ q_2)$, and $(p_3,\ q_3)$ in the plane with p_1, p_2, and p_3 distinct, show that they lie on some curve with equation $y = a+bx+cx^2$. [*Hint*: Solve for a, b, and c.]

Exercise 1.2.20 The scores of three players in a tournament have been lost. The only information available is the total of the scores for players 1 and 2, the total for players 2 and 3, and the total for players 3 and 1.

a. Show that the individual scores can be rediscovered.

b. Is this possible with four players (knowing the totals for players 1 and 2, 2 and 3, 3 and 4, and 4 and 1)?

Exercise 1.2.21 A boy finds \$1.05 in dimes, nickels, and pennies. If there are 17 coins in all, how many coins of each type can he have?

Exercise 1.2.22 If a consistent system has more variables than equations, show that it has infinitely many solutions. [*Hint*: Use Theorem 1.2.2.]

1.3 Homogeneous Equations

A system of equations in the variables $x_1, x_2, \dots, x_n$ is called **homogeneous** if all the constant terms are zero—that is, if each equation of the system has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Clearly $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is a solution to such a system; it is called the **trivial solution**. Any solution in which at least one variable has a nonzero value is called a **nontrivial solution**. Our chief goal in this section is to give a useful condition for a homogeneous system to have nontrivial solutions. The following example is instructive.

Example 1.3.1

Show that the following homogeneous system has nontrivial solutions.

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The leading variables are x_1, x_2, and x_4, so x_3 is assigned as a parameter—say $x_3 = t$. Then the general solution is $x_1 = -t$, $x_2 = t$, $x_3 = t$, $x_4 = 0$. Hence, taking $t = 1$ (say), we get a nontrivial solution: $x_1 = -1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$.

The existence of a nontrivial solution in Example 1.3.1 is ensured by the presence of a parameter in the solution. This is due to the fact that there is a *nonleading* variable (x_3 in this case). But there *must* be a nonleading variable here because there are four variables and only three equations (and hence at *most* three leading variables). This discussion generalizes to a proof of the following fundamental theorem.

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Proof. Suppose there are m equations in n variables where $n > m$, and let R denote the reduced row-echelon form of the augmented matrix. If there are r leading variables, there are $n - r$ nonleading variables, and so $n - r$ parameters. Hence, it suffices to show that $r < n$. But $r \leq m$ because R has r leading 1s and m rows, and $m < n$ by hypothesis. So $r \leq m < n$, which gives $r < n$. □

Note that the converse of Theorem 1.3.1 is not true: if a homogeneous system has nontrivial solutions, it need not have more variables than equations (the system $x_1 + x_2 = 0$, $2x_1 + 2x_2 = 0$ has nontrivial solutions but $m = 2 = n$.)

Theorem 1.3.1 is very useful in applications. The next example provides an illustration from geometry.

Example 1.3.2

We call the graph of an equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ a **conic** if the numbers a, b, and c are not all zero. Show that there is at least one conic through any five points in the plane that are not all on a line.

Solution. Let the coordinates of the five points be (p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4), and (p_5, q_5). The graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ passes through (p_i, q_i) if

$$ap_i^2 + bp_iq_i + cq_i^2 + dp_i + eq_i + f = 0$$

This gives five equations, one for each i, linear in the six variables a, b, c, d, e, and f. Hence, there is a nontrivial solution by Theorem 1.3.1. If $a = b = c = 0$, the five points all lie on the line with equation $dx + ey + f = 0$, contrary to assumption. Hence, one of a, b, c is nonzero.

Linear Combinations and Basic Solutions

As for rows, two columns are regarded as **equal** if they have the same number of entries and corresponding entries are the same. Let $\mathbf{x}$ and $\mathbf{y}$ be columns with the same number of entries. As for elementary row operations, their **sum** $\mathbf{x} + \mathbf{y}$ is obtained by adding corresponding entries and, if k is a number, the **scalar product** $k\mathbf{x}$ is defined by multiplying each entry of $\mathbf{x}$ by k. More precisely:

$$\text{If } \mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \text{ and } \mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \text{ then } \mathbf{x} + \mathbf{y} = \left[\begin{array}{c} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{array}\right] \text{ and } k\mathbf{x} = \left[\begin{array}{c} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{array}\right].$$

A sum of scalar multiples of several columns is called a **linear combination** of these columns. For example, $s\mathbf{x} + t\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ for any choice of numbers s and t.

Example 1.3.3

$$\text{If } \mathbf{x} = \left[\begin{array}{r} 3 \\ -2 \end{array}\right] \text{ and } \left[\begin{array}{r} -1 \\ 1 \end{array}\right] \text{ then } 2\mathbf{x} + 5\mathbf{y} = \left[\begin{array}{r} 6 \\ -4 \end{array}\right] + \left[\begin{array}{r} -5 \\ 5 \end{array}\right] = \left[\begin{array}{r} 1 \\ 1 \end{array}\right].$$

Example 1.3.4

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, determine whether $\mathbf{v}$ and $\mathbf{w}$ are linear combinations of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

Solution. For $\mathbf{v}$, we must determine whether numbers r, s, and t exist such that $\mathbf{v} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$, that is, whether

$$\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r + 2s + 3t \\ s + t \\ r + t \end{bmatrix}$$

Equating corresponding entries gives a system of linear equations $r + 2s + 3t = 0$, $s + t = -1$, and $r + t = 2$ for r, s, and t. By gaussian elimination, the solution is $r = 2 - k$, $s = -1 - k$, and $t = k$ where k is a parameter. Taking $k = 0$, we see that $\mathbf{v} = 2\mathbf{x} - \mathbf{y}$ is a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Turning to $\mathbf{w}$, we again look for r, s, and t such that $\mathbf{w} = r\mathbf{x} + s\mathbf{y} + t\mathbf{z}$; that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r + 2s + 3t \\ s + t \\ r + t \end{bmatrix}$$

leading to equations $r + 2s + 3t = 1$, $s + t = 1$, and $r + t = 1$ for real numbers r, s, and t. But this time there is *no* solution as the reader can verify, so $\mathbf{w}$ is *not* a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Our interest in linear combinations comes from the fact that they provide one of the best ways to describe the general solution of a homogeneous system of linear equations. When solving such a system with n variables $x_1, x_2, \dots, x_n$, write the variables as a column[6] matrix: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The trivial solution is denoted $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. As an illustration, the general solution in Example 1.3.1 is $x_1 = -t$, $x_2 = t$, $x_3 = t$, and $x_4 = 0$, where t is a parameter, and we would now express this by saying that the general solution is $\mathbf{x} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$, where t is arbitrary.

Now let $\mathbf{x}$ and $\mathbf{y}$ be two solutions to a homogeneous system with n variables. Then any linear combination $s\mathbf{x} + t\mathbf{y}$ of these solutions turns out to be again a solution to the system. More generally:

$$\textit{Any linear combination of solutions to a homogeneous system is again a solution.} \tag{1.1}$$

[6] The reason for using columns will be apparent later.

In fact, suppose that a typical equation in the system is $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, and suppose that $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$, $\mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]$ are solutions. Then $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0$.

Hence $s\mathbf{x} + t\mathbf{y} = \left[\begin{array}{c} sx_1 + ty_1 \\ sx_2 + ty_2 \\ \vdots \\ sx_n + ty_n \end{array} \right]$ is also a solution because

$$\begin{aligned}
a_1(sx_1 + ty_1) &+ a_2(sx_2 + ty_2) + \cdots + a_n(sx_n + ty_n) \\
&= [a_1(sx_1) + a_2(sx_2) + \cdots + a_n(sx_n)] + [a_1(ty_1) + a_2(ty_2) + \cdots + a_n(ty_n)] \\
&= s(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + t(a_1y_1 + a_2y_2 + \cdots + a_ny_n) \\
&= s(0) + t(0) \\
&= 0
\end{aligned}$$

A similar argument shows that Statement 1.1 is true for linear combinations of more than two solutions.

The remarkable thing is that *every* solution to a homogeneous system is a linear combination of certain particular solutions and, in fact, these solutions are easily computed using the gaussian algorithm. Here is an example.

Example 1.3.5

Solve the homogeneous system with coefficient matrix

$$A = \left[\begin{array}{rrrr} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{array} \right]$$

Solution. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

so the solutions are $x_1 = 2s + \frac{1}{5}t$, $x_2 = s$, $x_3 = \frac{3}{5}$, and $x_4 = t$ by gaussian elimination. Hence we can write the general solution $\mathbf{x}$ in the matrix form

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] = \left[\begin{array}{c} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{array} \right] = s \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ 0 \end{array} \right] + t \left[\begin{array}{c} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{array} \right] = s\mathbf{x}_1 + t\mathbf{x}_2.$$

Here $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$ are particular solutions determined by the gaussian algorithm.

The solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Example 1.3.5 are denoted as follows:

Definition 1.5 Basic Solutions

The gaussian algorithm systematically produces solutions to any homogeneous linear system, called ***basic solutions****, one for every parameter.*

Moreover, the algorithm gives a routine way to express *every* solution as a linear combination of basic solutions as in Example 1.3.5, where the general solution $\mathbf{x}$ becomes

$$\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{5}t\begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Hence by introducing a new parameter $r = t/5$ we can multiply the original basic solution $\mathbf{x}_2$ by 5 and so eliminate fractions. For this reason:

Convention:
Any nonzero scalar multiple of a basic solution will still be called a basic solution.

In the same way, the gaussian algorithm produces basic solutions to *every* homogeneous system, one for each parameter (there are *no* basic solutions if the system has only the trivial solution). Moreover every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 1.3.5). If A has rank r, Theorem 1.2.2 shows that there are exactly $n-r$ parameters, and so $n-r$ basic solutions. This proves:

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n-r$ basic solutions, one for each parameter.*
2. *Every solution is a linear combination of these basic solutions.*

Example 1.3.6

Find basic solutions of the homogeneous system with coefficient matrix A, and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where r, s, and t are parameters. In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Exercises for 1.3

Exercise 1.3.1 Consider the following statements about a system of linear equations with augmented matrix A. In each case either prove the statement or give an example for which it is false.

a. If the system is homogeneous, every solution is trivial.

b. If the system has a nontrivial solution, it cannot be homogeneous.

c. If there exists a trivial solution, the system is homogeneous.

d. If the system is consistent, it must be homogeneous.

Now assume that the system is homogeneous.

e. If there exists a nontrivial solution, there is no trivial solution.

f. If there exists a solution, there are infinitely many solutions.

g. If there exist nontrivial solutions, the row-echelon form of A has a row of zeros.

h. If the row-echelon form of A has a row of zeros, there exist nontrivial solutions.

i. If a row operation is applied to the system, the new system is also homogeneous.

Exercise 1.3.2 In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

a. $\begin{array}{rl} x-2y+z &= 0 \\ x+ay-3z &= 0 \\ -x+6y-5z &= 0 \end{array}$

b. $\begin{array}{rl} x+2y+z &= 0 \\ x+3y+6z &= 0 \\ 2x+3y+az &= 0 \end{array}$

c. $\begin{array}{rl} x+y-z &= 0 \\ ay-z &= 0 \\ x+y+az &= 0 \end{array}$

d. $\begin{array}{rl} ax+y+z &= 0 \\ x+y-z &= 0 \\ x+y+az &= 0 \end{array}$

Exercise 1.3.3 Let $\mathbf{x} = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array}\right]$, $\mathbf{y} = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$, and $\mathbf{z} = \left[\begin{array}{r} 1 \\ 1 \\ -2 \end{array}\right]$. In each case, either write $\mathbf{v}$ as a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, or show that it is not such a linear combination.

a. $\mathbf{v} = \left[\begin{array}{r} 0 \\ 1 \\ -3 \end{array}\right]$

b. $\mathbf{v} = \left[\begin{array}{r} 4 \\ 3 \\ -4 \end{array}\right]$

c. $\mathbf{v} = \left[\begin{array}{r} 3 \\ 1 \\ 0 \end{array}\right]$

d. $\mathbf{v} = \left[\begin{array}{r} 3 \\ 0 \\ 3 \end{array}\right]$

Exercise 1.3.4 In each case, either express $\mathbf{y}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, or show that it is not such a linear combination. Here:

$$\mathbf{a}_1 = \left[\begin{array}{r} -1 \\ 3 \\ 0 \\ 1 \end{array}\right], \ \mathbf{a}_2 = \left[\begin{array}{r} 3 \\ 1 \\ 2 \\ 0 \end{array}\right], \text{ and } \mathbf{a}_3 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

a. $\mathbf{y} = \left[\begin{array}{r} 1 \\ 2 \\ 4 \\ 0 \end{array}\right]$

b. $\mathbf{y} = \left[\begin{array}{r} -1 \\ 9 \\ 2 \\ 6 \end{array}\right]$

Exercise 1.3.5 For each of the following homogeneous systems, find a set of basic solutions and express the general solution as a linear combination of these basic solutions.

a. $\begin{array}{rl} x_1+2x_2-x_3+2x_4+x_5 &= 0 \\ x_1+2x_2+2x_3+x_5 &= 0 \\ 2x_1+4x_2-2x_3+3x_4+x_5 &= 0 \end{array}$

b. $\begin{array}{rl} x_1+2x_2-x_3+x_4+x_5 &= 0 \\ -x_1-2x_2+2x_3+x_5 &= 0 \\ -x_1-2x_2+3x_3+x_4+3x_5 &= 0 \end{array}$

c. $\begin{array}{rl} x_1+x_2-x_3+2x_4+x_5 &= 0 \\ x_1+2x_2-x_3+x_4+x_5 &= 0 \\ 2x_1+3x_2-x_3+2x_4+x_5 &= 0 \\ 4x_1+5x_2-2x_3+5x_4+2x_5 &= 0 \end{array}$

d. $$\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 &= 0 \\ 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 &= 0 \\ x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 &= 0 \end{aligned}$$

Exercise 1.3.6

a. Does Theorem 1.3.1 imply that the system $\left\{ \begin{array}{l} -z + 3y = 0 \\ 2x - 6y = 0 \end{array} \right.$ has nontrivial solutions? Explain.

b. Show that the converse to Theorem 1.3.1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

Exercise 1.3.7 In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A. Assume that A has nonzero entries. Give all possibilities.

a. Rank $A = 2$.

b. Rank $A = 1$.

c. A has a row of zeros.

d. The row-echelon form of A has a row of zeros.

Exercise 1.3.8 The graph of an equation $ax+by+cz=0$ is a plane through the origin (provided that not all of a, b, and c are zero). Use Theorem 1.3.1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

Exercise 1.3.9

a. Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax+by+c=0$, where a, b, and c are not all zero.]

b. Generalize and show that there is a plane $ax+by+cz+d=0$ through any three points in space.

Exercise 1.3.10 The graph of

$$a(x^2+y^2)+bx+cy+d=0$$

is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

Exercise 1.3.11 Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r. Show that the system has nontrivial solutions if and only if $n > r$.

Exercise 1.3.12 If a consistent (possibly nonhomogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.

1.4 An Application to Network Flow

There are many types of problems that concern a network of conductors along which some sort of flow is observed. Examples of these include an irrigation network and a network of streets or freeways. There are often points in the system at which a net flow either enters or leaves the system. The basic principle behind the analysis of such systems is that the total flow into the system must equal the total flow out. In fact, we apply this principle at every junction in the system.

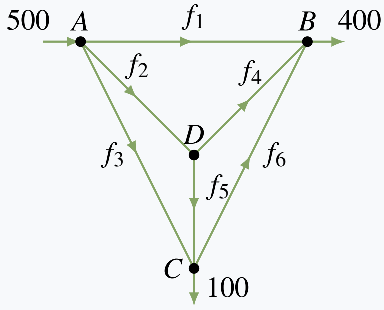
Junction Rule

At each of the junctions in the network, the total flow into that junction must equal the total flow out.

This requirement gives a linear equation relating the flows in conductors emanating from the junction.

Example 1.4.1

A network of one-way streets is shown in the accompanying diagram. The rate of flow of cars into intersection A is 500 cars per hour, and 400 and 100 cars per hour emerge from B and C, respectively. Find the possible flows along each street.



Solution. Suppose the flows along the streets are f_1, f_2, f_3, f_4, f_5, and f_6 cars per hour in the directions shown.
Then, equating the flow in with the flow out at each intersection, we get

$$\begin{array}{ll} \text{Intersection } A & 500 = f_1 + f_2 + f_3 \\ \text{Intersection } B & f_1 + f_4 + f_6 = 400 \\ \text{Intersection } C & f_3 + f_5 = f_6 + 100 \\ \text{Intersection } D & f_2 = f_4 + f_5 \end{array}$$

These give four equations in the six variables $f_1, f_2, \dots, f_6$.

$$\begin{array}{rcrcrcrcrcrcr} f_1 & + & f_2 & + & f_3 & & & & & & & = & 500 \\ f_1 & & & & & + & f_4 & & & + & f_6 & = & 400 \\ & & & & f_3 & & & + & f_5 & - & f_6 & = & 100 \\ & & f_2 & & & - & f_4 & - & f_5 & & & = & 0 \end{array}$$

The reduction of the augmented matrix is

$$\left[\begin{array}{rrrrrr|r} 1 & 1 & 1 & 0 & 0 & 0 & 500 \\ 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrrr|r} 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, when we use f_4, f_5, and f_6 as parameters, the general solution is

$$f_1 = 400 - f_4 - f_6 \qquad f_2 = f_4 + f_5 \qquad f_3 = 100 - f_5 + f_6$$

This gives all solutions to the system of equations and hence all the possible flows.
Of course, not all these solutions may be acceptable in the real situation. For example, the flows $f_1, f_2, \dots, f_6$ are all *positive* in the present context (if one came out negative, it would mean traffic flowed in the opposite direction). This imposes constraints on the flows: $f_1 \geq 0$ and $f_3 \geq 0$ become

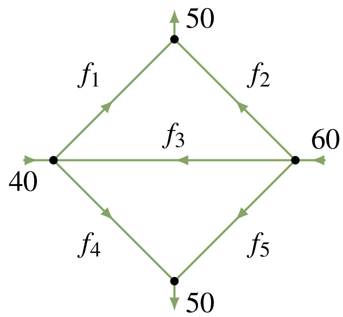
$$f_4 + f_6 \leq 400 \qquad f_5 - f_6 \leq 100$$

Further constraints might be imposed by insisting on maximum values on the flow in each street.

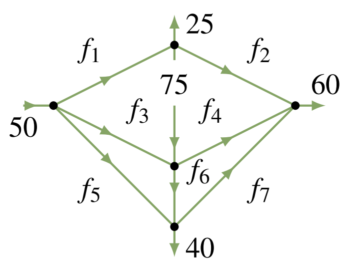
Exercises for 1.4

Exercise 1.4.1 Find the possible flows in each of the following networks of pipes.

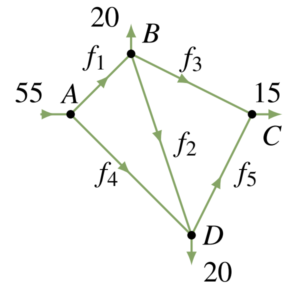
a.



b.



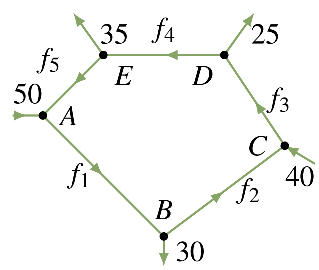
Exercise 1.4.2 A proposed network of irrigation canals is described in the accompanying diagram. At peak demand, the flows at interchanges A, B, C, and D are as shown.



a. Find the possible flows.

b. If canal BC is closed, what range of flow on AD must be maintained so that no canal carries a flow of more than 30?

Exercise 1.4.3 A traffic circle has five one-way streets, and vehicles enter and leave as shown in the accompanying diagram.



a. Compute the possible flows.

b. Which road has the heaviest flow?

1.5 An Application to Electrical Networks[7]

In an electrical network it is often necessary to find the current in amperes (A) flowing in various parts of the network. These networks usually contain resistors that retard the current. The resistors are indicated by a symbol (-\/\/\/-), and the resistance is measured in ohms (Ω). Also, the current is increased at various points by voltage sources (for example, a battery). The voltage of these sources is measured in volts (V),

[7]This section is independent of Section 1.4

and they are represented by the symbol ( ). We assume these voltage sources have no resistance. The flow of current is governed by the following principles.

Ohm's Law

The current I and the voltage drop V across a resistance R are related by the equation $V = RI$.

Kirchhoff's Laws

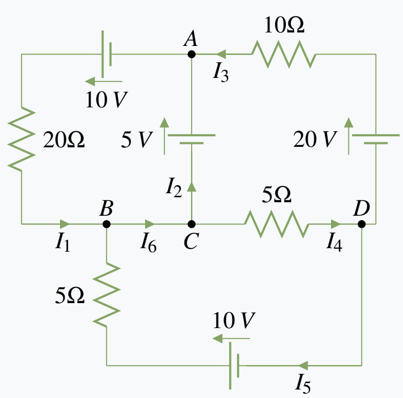
1. *(Junction Rule) The current flow into a junction equals the current flow out of that junction.*
2. *(Circuit Rule) The algebraic sum of the voltage drops (due to resistances) around any closed circuit of the network must equal the sum of the voltage increases around the circuit.*

When applying rule 2, select a direction (clockwise or counterclockwise) around the closed circuit and then consider all voltages and currents positive when in this direction and negative when in the opposite direction. This is why the term *algebraic sum* is used in rule 2. Here is an example.

Example 1.5.1

Find the various currents in the circuit shown.

Solution.



First apply the junction rule at junctions A, B, C, and D to obtain

$$\begin{array}{ll} \text{Junction } A & I_1 = I_2 + I_3 \\ \text{Junction } B & I_6 = I_1 + I_5 \\ \text{Junction } C & I_2 + I_4 = I_6 \\ \text{Junction } D & I_3 + I_5 = I_4 \end{array}$$

Note that these equations are not independent (in fact, the third is an easy consequence of the other three). Next, the circuit rule insists that the sum of the voltage increases (due to the sources) around a closed circuit must equal the sum of the voltage drops (due to resistances). By Ohm's law, the voltage loss across a resistance R (in the direction of the current I) is RI. Going counterclockwise around three closed circuits yields

$$\begin{array}{lrl} \text{Upper left} & 10 + & 5 = 20I_1 \\ \text{Upper right} & -5 + & 20 = 10I_3 + 5I_4 \\ \text{Lower} & & -10 = -20I_5 - 5I_4 \end{array}$$

Hence, disregarding the redundant equation obtained at junction C, we have six equations in the six unknowns $I_1, \ldots, I_6$. The solution is

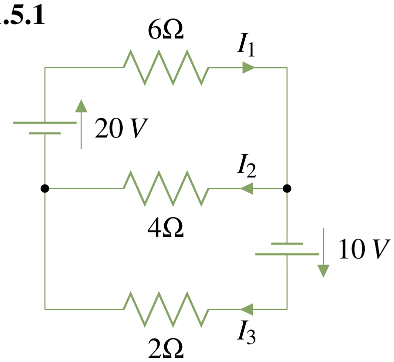
$$I_1 = \tfrac{15}{20} \quad I_4 = \tfrac{28}{20}$$
$$I_2 = \tfrac{-1}{20} \quad I_5 = \tfrac{12}{20}$$
$$I_3 = \tfrac{16}{20} \quad I_6 = \tfrac{27}{20}$$

The fact that I_2 is negative means, of course, that this current is in the opposite direction, with a magnitude of $\frac{1}{20}$ amperes.

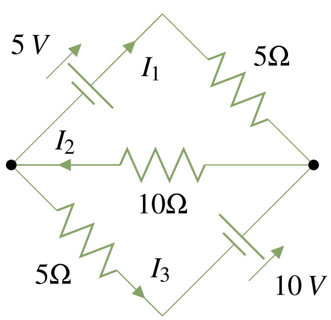
Exercises for 1.5

In Exercises 1 to 4, find the currents in the circuits.

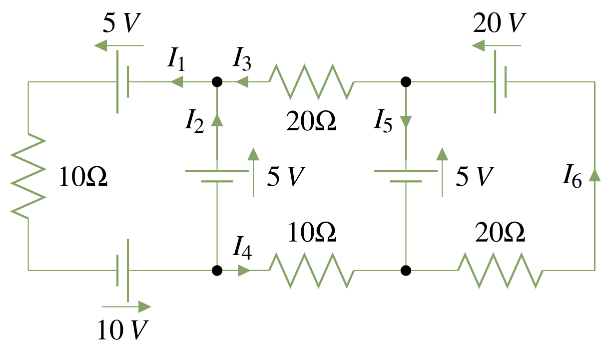
Exercise 1.5.1



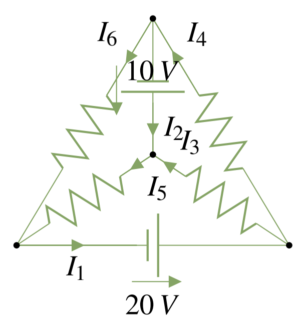
Exercise 1.5.2



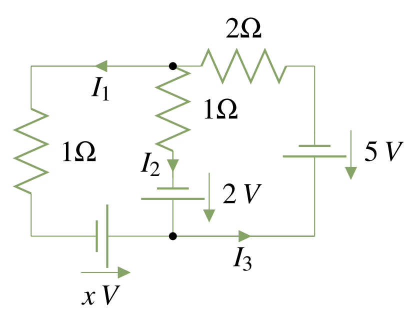
Exercise 1.5.3



Exercise 1.5.4 All resistances are 10Ω.



Exercise 1.5.5
Find the voltage x such that the current $I_1 = 0$.



1.6 An Application to Chemical Reactions

When a chemical reaction takes place a number of molecules combine to produce new molecules. Hence, when hydrogen H_2 and oxygen O_2 molecules combine, the result is water H_2O. We express this as

$$H_2 + O_2 \rightarrow H_2O$$

Individual atoms are neither created nor destroyed, so the number of hydrogen and oxygen atoms going into the reaction must equal the number coming out (in the form of water). In this case the reaction is said to be *balanced*. Note that each hydrogen molecule H_2 consists of two atoms as does each oxygen molecule O_2, while a water molecule H_2O consists of two hydrogen atoms and one oxygen atom. In the above reaction, this requires that twice as many hydrogen molecules enter the reaction; we express this as follows:

$$2H_2 + O_2 \rightarrow 2H_2O$$

This is now balanced because there are 4 hydrogen atoms and 2 oxygen atoms on each side of the reaction.

Example 1.6.1

Balance the following reaction for burning octane C_8H_{18} in oxygen O_2:

$$C_8H_{18} + O_2 \rightarrow CO_2 + H_2O$$

where CO_2 represents carbon dioxide. We must find positive integers x, y, z, and w such that

$$xC_8H_{18} + yO_2 \rightarrow zCO_2 + wH_2O$$

Equating the number of carbon, hydrogen, and oxygen atoms on each side gives $8x = z$, $18x = 2w$ and $2y = 2z + w$, respectively. These can be written as a homogeneous linear system

$$\begin{array}{rcrcrcrcl} 8x & & & - & z & & & = & 0 \\ 18x & & & & & - & 2w & = & 0 \\ & & 2y & - & 2z & - & w & = & 0 \end{array}$$

which can be solved by gaussian elimination. In larger systems this is necessary but, in such a simple situation, it is easier to solve directly. Set $w = t$, so that $x = \frac{1}{9}t$, $z = \frac{8}{9}t$, $2y = \frac{16}{9}t + t = \frac{25}{9}t$. But x, y, z, and w must be positive integers, so the smallest value of t that eliminates fractions is 18. Hence, $x = 2$, $y = 25$, $z = 16$, and $w = 18$, and the balanced reaction is

$$2C_8H_{18} + 25O_2 \rightarrow 16CO_2 + 18H_2O$$

The reader can verify that this is indeed balanced.

It is worth noting that this problem introduces a new element into the theory of linear equations: the insistence that the solution must consist of positive integers.

Exercises for 1.6

In each case balance the chemical reaction.

Exercise 1.6.1 $CH_4 + O_2 \to CO_2 + H_2O$. This is the burning of methane CH_4.

Exercise 1.6.2 $NH_3 + CuO \to N_2 + Cu + H_2O$. Here NH_3 is ammonia, CuO is copper oxide, Cu is copper, and N_2 is nitrogen.

Exercise 1.6.3 $CO_2 + H_2O \to C_6H_{12}O_6 + O_2$. This is called the photosynthesis reaction—$C_6H_{12}O_6$ is glucose.

Exercise 1.6.4 $Pb(N_3)_2 + Cr(MnO_4)_2 \to Cr_2O_3 + MnO_2 + Pb_3O_4 + NO$.

Supplementary Exercises for Chapter 1

Exercise 1.1 We show in Chapter 4 that the graph of an equation $ax+by+cz=d$ is a plane in space when not all of a, b, and c are zero.

a. By examining the possible positions of planes in space, show that three equations in three variables can have zero, one, or infinitely many solutions.

b. Can two equations in three variables have a unique solution? Give reasons for your answer.

Exercise 1.2 Find all solutions to the following systems of linear equations.

a.
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ -3x_1 - 7x_2 + 7x_3 - 5x_4 &= 7 \\ x_1 + 3x_2 - 4x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 + 4x_2 - x_3 + x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 5 \\ x_1 - 6x_2 + 3x_3 \phantom{{}+2x_4} &= 1 \\ x_1 + 14x_2 - 5x_3 + 2x_4 &= 3 \end{aligned}$$

Exercise 1.3 In each case find (if possible) conditions on a, b, and c such that the system has zero, one, or infinitely many solutions.

a.
$$\begin{aligned} x + 2y - 4z &= 4 \\ 3x - y + 13z &= 2 \\ 4x + y + a^2 z &= a+3 \end{aligned}$$

b.
$$\begin{aligned} x + y + 3z &= a \\ ax + y + 5z &= 4 \\ x + ay + 4z &= a \end{aligned}$$

Exercise 1.4 Show that any two rows of a matrix can be interchanged by elementary row transformations of the other two types.

Exercise 1.5 If $ad \neq bc$, show that $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ has reduced row-echelon form $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Exercise 1.6 Find a, b, and c so that the system

$$\begin{aligned} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{aligned}$$

has the solution $x=3$, $y=-1$, $z=2$.

Exercise 1.7 Solve the system

$$\begin{aligned} x + 2y + 2z &= -3 \\ 2x + y + z &= -4 \\ x - y + iz &= i \end{aligned}$$

where $i^2 = -1$. [See Appendix A.]

Exercise 1.8 Show that the *real* system

$$\left\{ \begin{aligned} x + y + z &= 5 \\ 2x - y - z &= 1 \\ -3x + 2y + 2z &= 0 \end{aligned} \right.$$

has a *complex* solution: $x=2$, $y=i$, $z=3-i$ where $i^2=-1$. Explain. What happens when such a real system has a unique solution?

Exercise 1.9 A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the accompanying table.

	Vitamin		
Brand	**A**	**B**	**C**
1	1	2	4
2	1	1	3
3	0	1	1

a. Find all combinations of pills that provide exactly the required amount of vitamins (no partial pills allowed).

b. If brands 1, 2, and 3 cost 3¢, 2¢, and 5¢ per pill, respectively, find the least expensive treatment.

Exercise 1.10 A restaurant owner plans to use x tables seating 4, y tables seating 6, and z tables seating 8, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables, and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, and z.

Exercise 1.11

a. Show that a matrix with two rows and two columns that is in reduced row-echelon form must have one of the following forms:

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & * \\ 0 & 0 \end{array}\right]$$

[*Hint*: The leading 1 in the first row must be in column 1 or 2 or not exist.]

b. List the seven reduced row-echelon forms for matrices with two rows and three columns.

c. List the four reduced row-echelon forms for matrices with three rows and two columns.

Exercise 1.12 An amusement park charges \$7 for adults, \$2 for youths, and \$0.50 for children. If 150 people enter and pay a total of \$100, find the numbers of adults, youths, and children. [*Hint*: These numbers are nonnegative *integers*.]

Exercise 1.13 Solve the following system of equations for x and y.

$$\begin{aligned} x^2 + xy - y^2 &= 1 \\ 2x^2 - xy + 3y^2 &= 13 \\ x^2 + 3xy + 2y^2 &= 0 \end{aligned}$$

[*Hint*: These equations are linear in the new variables $x_1 = x^2$, $x_2 = xy$, and $x_3 = y^2$.]

2. Matrix Algebra

In the study of systems of linear equations in Chapter 1, we found it convenient to manipulate the augmented matrix of the system. Our aim was to reduce it to row-echelon form (using elementary row operations) and hence to write down all solutions to the system. In the present chapter we consider matrices for their own sake. While some of the motivation comes from linear equations, it turns out that matrices can be multiplied and added and so form an algebraic system somewhat analogous to the real numbers. This "matrix algebra" is useful in ways that are quite different from the study of linear equations. For example, the geometrical transformations obtained by rotating the euclidean plane about the origin can be viewed as multiplications by certain 2×2 matrices. These "matrix transformations" are an important tool in geometry and, in turn, the geometry provides a "picture" of the matrices. Furthermore, matrix algebra has many other applications, some of which will be explored in this chapter. This subject is quite old and was first studied systematically in 1858 by Arthur Cayley.[1]

2.1 Matrix Addition, Scalar Multiplication, and Transposition

A rectangular array of numbers is called a **matrix** (the plural is **matrices**), and the numbers are called the **entries** of the matrix. Matrices are usually denoted by uppercase letters: A, B, C, and so on. Hence,

$$A = \left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 5 & 6 \end{array} \right] \quad B = \left[\begin{array}{rr} 1 & -1 \\ 0 & 2 \end{array} \right] \quad C = \left[\begin{array}{r} 1 \\ 3 \\ 2 \end{array} \right]$$

are matrices. Clearly matrices come in various shapes depending on the number of **rows** and **columns**. For example, the matrix A shown has 2 rows and 3 columns. In general, a matrix with m rows and n columns is referred to as an $\boldsymbol{m \times n}$ ***matrix*** or as having **size** $\boldsymbol{m \times n}$. Thus matrices A, B, and C above have sizes 2×3, 2×2, and 3×1, respectively. A matrix of size $1 \times n$ is called a **row matrix**, whereas one of size $m \times 1$ is called a **column matrix**. Matrices of size $n \times n$ for some n are called **square** matrices.

Each entry of a matrix is identified by the row and column in which it lies. The rows are numbered from the top down, and the columns are numbered from left to right. Then the $\boldsymbol{(i,\ j)}$**-entry** of a matrix is

[1] Arthur Cayley (1821-1895) showed his mathematical talent early and graduated from Cambridge in 1842 as senior wrangler. With no employment in mathematics in view, he took legal training and worked as a lawyer while continuing to do mathematics, publishing nearly 300 papers in fourteen years. Finally, in 1863, he accepted the Sadlerian professorship in Cambridge and remained there for the rest of his life, valued for his administrative and teaching skills as well as for his scholarship. His mathematical achievements were of the first rank. In addition to originating matrix theory and the theory of determinants, he did fundamental work in group theory, in higher-dimensional geometry, and in the theory of invariants. He was one of the most prolific mathematicians of all time and produced 966 papers.

the number lying simultaneously in row i and column j. For example,

$$\text{The } (1,\ 2)\text{-entry of } \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array} \right] \text{ is } -1.$$

$$\text{The } (2,\ 3)\text{-entry of } \left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 5 & 6 \end{array} \right] \text{ is } 6.$$

A special notation is commonly used for the entries of a matrix. If A is an $m \times n$ matrix, and if the $(i,\ j)$-entry of A is denoted as a_{ij}, then A is displayed as follows:

$$A = \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array} \right]$$

This is usually denoted simply as $A = \left[a_{ij} \right]$. Thus a_{ij} is the entry in row i and column j of A. For example, a 3×4 matrix in this notation is written

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right]$$

It is worth pointing out a convention regarding rows and columns: *Rows are mentioned before columns.* For example:

- *If a matrix has size $m \times n$, it has m rows and n columns.*
- *If we speak of the $(i,\ j)$-entry of a matrix, it lies in row i and column j.*
- *If an entry is denoted a_{ij}, the first subscript i refers to the row and the second subscript j to the column in which a_{ij} lies.*

Two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ in the plane are equal if and only if[2] they have the same coordinates, that is $x_1 = x_2$ and $y_1 = y_2$. Similarly, two matrices A and B are called **equal** (written $A = B$) if and only if:

1. *They have the same size.*
2. *Corresponding entries are equal.*

If the entries of A and B are written in the form $A = \left[a_{ij} \right]$, $B = \left[b_{ij} \right]$, described earlier, then the second condition takes the following form:

$$A = \left[a_{ij} \right] = \left[b_{ij} \right] \text{ means } a_{ij} = b_{ij} \text{ for all } i \text{ and } j$$

---

[2]If p and q are statements, we say that p implies q if q is true whenever p is true. Then "p if and only if q" means that both p implies q and q implies p. See Appendix B for more on this.

Example 2.1.1

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ discuss the possibility that $A = B$, $B = C$, $A = C$.

Solution. $A = B$ is impossible because A and B are of different sizes: A is 2×2 whereas B is 2×3. Similarly, $B = C$ is impossible. But $A = C$ is possible provided that corresponding entries are equal: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ means $a = 1$, $b = 0$, $c = -1$, and $d = 2$.

Matrix Addition

Definition 2.1 Matrix Addition

If A and B are matrices of the same size, their **sum** *$A + B$ is the matrix formed by adding corresponding entries.*

If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, this takes the form

$$A + B = \left[a_{ij} + b_{ij}\right]$$

Note that addition is *not* defined for matrices of different sizes.

Example 2.1.2

If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}$, compute $A + B$.

Solution.

$$A + B = \begin{bmatrix} 2+1 & 1+1 & 3-1 \\ -1+2 & 2+0 & 0+6 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

Example 2.1.3

Find a, b, and c if $\begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} c & a & b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$.

Solution. Add the matrices on the left side to obtain

$$\begin{bmatrix} a+c & b+a & c+b \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

Because corresponding entries must be equal, this gives three equations: $a + c = 3$, $b + a = 2$, and $c + b = -1$. Solving these yields $a = 3$, $b = -1$, $c = 0$.

If A, B, and C are any matrices *of the same size*, then

$$A+B=B+A \qquad \text{(commutative law)}$$
$$A+(B+C)=(A+B)+C \qquad \text{(associative law)}$$

In fact, if $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then the $(i,\ j)$-entries of $A+B$ and $B+A$ are, respectively, $a_{ij}+b_{ij}$ and $b_{ij}+a_{ij}$. Since these are equal for all i and j, we get

$$A+B=\left[\begin{array}{c} a_{ij}+b_{ij} \end{array}\right]=\left[\begin{array}{c} b_{ij}+a_{ij} \end{array}\right]=B+A$$

The associative law is verified similarly.

The $m \times n$ matrix in which every entry is zero is called the $m \times n$ **zero matrix** and is denoted as 0 (or 0_{mn} if it is important to emphasize the size). Hence,

$$0+X=X$$

holds for all $m \times n$ matrices X. The **negative** of an $m \times n$ matrix A (written $-A$) is defined to be the $m \times n$ matrix obtained by multiplying each entry of A by -1. If $A=\left[a_{ij}\right]$, this becomes $-A=\left[-a_{ij}\right]$. Hence,

$$A+(-A)=0$$

holds for all matrices A where, of course, 0 is the zero matrix of the same size as A.

A closely related notion is that of subtracting matrices. If A and B are two $m \times n$ matrices, their **difference** $A-B$ is defined by

$$A-B=A+(-B)$$

Note that if $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$, then

$$A-B=\left[a_{ij}\right]+\left[-b_{ij}\right]=\left[a_{ij}-b_{ij}\right]$$

is the $m \times n$ matrix formed by *subtracting* corresponding entries.

Example 2.1.4

Let $A=\left[\begin{array}{rrr} 3 & -1 & 0 \\ 1 & 2 & -4 \end{array}\right]$, $B=\left[\begin{array}{rrr} 1 & -1 & 1 \\ -2 & 0 & 6 \end{array}\right]$, $C=\left[\begin{array}{rrr} 1 & 0 & -2 \\ 3 & 1 & 1 \end{array}\right]$. Compute $-A$, $A-B$, and $A+B-C$.

Solution.

$$-A=\left[\begin{array}{rrr} -3 & 1 & 0 \\ -1 & -2 & 4 \end{array}\right]$$
$$A-B=\left[\begin{array}{lcc} 3-1 & -1-(-1) & 0-1 \\ 1-(-2) & 2-0 & -4-6 \end{array}\right]=\left[\begin{array}{rrr} 2 & 0 & -1 \\ 3 & 2 & -10 \end{array}\right]$$
$$A+B-C=\left[\begin{array}{lcc} 3+1-1 & -1-1-0 & 0+1-(-2) \\ 1-2-3 & 2+0-1 & -4+6-1 \end{array}\right]=\left[\begin{array}{rrr} 3 & -2 & 3 \\ -4 & 1 & 1 \end{array}\right]$$

Example 2.1.5

Solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ where X is a matrix.

Solution. We solve a numerical equation $a + x = b$ by subtracting the number a from both sides to obtain $x = b - a$. This also works for matrices. To solve $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ simply subtract the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ from both sides to get

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-3 & 0-2 \\ -1-(-1) & 2-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

The reader should verify that this matrix X does indeed satisfy the original equation.

The solution in Example 2.1.5 solves the single matrix equation $A + X = B$ directly via matrix subtraction: $X = B - A$. This ability to work with matrices as entities lies at the heart of matrix algebra.

It is important to note that the sizes of matrices involved in some calculations are often determined by the context. For example, if

$$A + C = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

then A and C must be the same size (so that $A + C$ makes sense), and that size must be 2×3 (so that the sum is 2×3). For simplicity we shall often omit reference to such facts when they are clear from the context.

Scalar Multiplication

In gaussian elimination, multiplying a row of a matrix by a number k means multiplying *every* entry of that row by k.

Definition 2.2 Matrix Scalar Multiplication

*More generally, if A is any matrix and k is any number, the **scalar multiple** kA is the matrix obtained from A by multiplying each entry of A by k.*

If $A = \left[a_{ij}\right]$, this is

$$kA = \left[ka_{ij}\right]$$

Thus $1A = A$ and $(-1)A = -A$ for any matrix A.

The term *scalar* arises here because the set of numbers from which the entries are drawn is usually referred to as the set of scalars. We have been using real numbers as scalars, but we could equally well have been using complex numbers.

Example 2.1.6

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}$ compute $5A$, $\frac{1}{2}B$, and $3A - 2B$.

Solution.

$$5A = \begin{bmatrix} 15 & -5 & 20 \\ 10 & 0 & 30 \end{bmatrix}, \quad \frac{1}{2}B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \frac{3}{2} & 1 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 9 & -3 & 12 \\ 6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 14 \\ 6 & -6 & 14 \end{bmatrix}$$

If A is any matrix, note that kA is the same size as A for all scalars k. We also have

$$0A = 0 \quad \text{and} \quad k0 = 0$$

because the zero matrix has every entry zero. In other words, $kA = 0$ if either $k = 0$ or $A = 0$. The converse of this statement is also true, as Example 2.1.7 shows.

Example 2.1.7

If $kA = 0$, show that either $k = 0$ or $A = 0$.

Solution. Write $A = \left[a_{ij} \right]$ so that $kA = 0$ means $ka_{ij} = 0$ for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$ for all i and j; that is, $A = 0$.

For future reference, the basic properties of matrix addition and scalar multiplication are listed in Theorem 2.1.1.

Theorem 2.1.1

Let A, B, and C denote arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denote arbitrary real numbers. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. *There is an $m \times n$ matrix 0, such that $0 + A = A$ for each A.*
4. *For each A there is an $m \times n$ matrix, $-A$, such that $A + (-A) = 0$.*
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$.

Proof. Properties 1–4 were given previously. To check Property 5, let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ denote matrices of the same size. Then $A+B = \left[a_{ij}+b_{ij}\right]$, as before, so the $(i,\ j)$-entry of $k(A+B)$ is

$$k(a_{ij}+b_{ij}) = ka_{ij}+kb_{ij}$$

But this is just the $(i,\ j)$-entry of $kA+kB$, and it follows that $k(A+B) = kA+kB$. The other Properties can be similarly verified; the details are left to the reader. □

The Properties in Theorem 2.1.1 enable us to do calculations with matrices in much the same way that numerical calculations are carried out. To begin, Property 2 implies that the sum

$$(A+B)+C = A+(B+C)$$

is the same no matter how it is formed and so is written as $A+B+C$. Similarly, the sum

$$A+B+C+D$$

is independent of how it is formed; for example, it equals both $(A+B)+(C+D)$ and $A+[B+(C+D)]$. Furthermore, property 1 ensures that, for example,

$$B+D+A+C = A+B+C+D$$

In other words, the *order* in which the matrices are added does not matter. A similar remark applies to sums of five (or more) matrices.

Properties 5 and 6 in Theorem 2.1.1 are called **distributive laws** for scalar multiplication, and they extend to sums of more than two terms. For example,

$$k(A+B-C) = kA+kB-kC$$

$$(k+p-m)A = kA+pA-mA$$

Similar observations hold for more than three summands. These facts, together with properties 7 and 8, enable us to simplify expressions by collecting like terms, expanding, and taking common factors in exactly the same way that algebraic expressions involving variables and real numbers are manipulated. The following example illustrates these techniques.

Example 2.1.8

Simplify $2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right]$ where A, B, and C are all matrices of the same size.

Solution. The reduction proceeds as though A, B, and C were variables.

$$\begin{aligned} &2(A+3C)-3(2C-B)-3\left[2(2A+B-4C)-4(A-2C)\right] \\ &\quad = 2A+6C-6C+3B-3\left[4A+2B-8C-4A+8C\right] \\ &\quad = 2A+3B-3\left[2B\right] \\ &\quad = 2A-3B \end{aligned}$$

Transpose of a Matrix

Many results about a matrix A involve the *rows* of A, and the corresponding result for columns is derived in an analogous way, essentially by replacing the word *row* by the word *column* throughout. The following definition is made with such applications in mind.

Definition 2.3 Transpose of a Matrix

If A is an $m \times n$ matrix, the **transpose** *of A, written A^T, is the $n \times m$ matrix whose rows are just the columns of A in the same order.*

In other words, the first row of A^T is the first column of A (that is it consists of the entries of column 1 in order). Similarly the second row of A^T is the second column of A, and so on.

Example 2.1.9

Write down the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution.

$$A^T = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \; B^T = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \text{ and } D^T = D.$$

If $A = \left[a_{ij}\right]$ is a matrix, write $A^T = \left[b_{ij}\right]$. Then b_{ij} is the jth element of the ith row of A^T and so is the jth element of the ith *column* of A. This means $b_{ij} = a_{ji}$, so the definition of A^T can be stated as follows:

$$\text{If } A = \left[a_{ij}\right], \text{ then } A^T = \left[a_{ji}\right]. \tag{2.1}$$

This is useful in verifying the following properties of transposition.

Theorem 2.1.2

Let A and B denote matrices of the same size, and let k denote a scalar.

1. *If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.*
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A+B)^T = A^T + B^T$.

Proof. Property 1 is part of the definition of A^T, and Property 2 follows from (2.1). As to Property 3: If $A = \left[a_{ij}\right]$, then $kA = \left[ka_{ij}\right]$, so (2.1) gives

$$(kA)^T = \left[ka_{ji}\right] = k\left[a_{ji}\right] = kA^T$$

Finally, if $B = \left[b_{ij}\right]$, then $A+B = \left[c_{ij}\right]$ where $c_{ij} = a_{ij} + b_{ij}$ Then (2.1) gives Property 4:

$$(A+B)^T = \left[c_{ij}\right]^T = \left[c_{ji}\right] = \left[a_{ji} + b_{ji}\right] = \left[a_{ji}\right] + \left[b_{ji}\right] = A^T + B^T$$

□

There is another useful way to think of transposition. If $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, the elements a_{11}, a_{22}, a_{33}, $\dots$ are called the **main diagonal** of A. Hence the main diagonal extends down and to the right from the upper left corner of the matrix A; it is shaded in the following examples:

$$\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}\right] \quad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \quad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \quad \left[\begin{array}{c} a_{11} \\ a_{21} \end{array}\right]$$

Thus forming the transpose of a matrix A can be viewed as "flipping" A about its main diagonal, or as "rotating" A through 180° about the line containing the main diagonal. This makes Property 2 in Theorem 2.1.2 transparent.

Example 2.1.10

Solve for A if $\left(2A^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]\right)^T = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]$.

Solution. Using Theorem 2.1.2, the left side of the equation is

$$\left(2A^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]\right)^T = 2\left(A^T\right)^T - 3\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]^T = 2A - 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right]$$

Hence the equation becomes

$$2A - 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right] = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right]$$

Thus $2A = \left[\begin{array}{rr} 2 & 3 \\ -1 & 2 \end{array}\right] + 3\left[\begin{array}{rr} 1 & -1 \\ 2 & 1 \end{array}\right] = \left[\begin{array}{rr} 5 & 0 \\ 5 & 5 \end{array}\right]$, so finally $A = \frac{1}{2}\left[\begin{array}{rr} 5 & 0 \\ 5 & 5 \end{array}\right] = \frac{5}{2}\left[\begin{array}{rr} 1 & 0 \\ 1 & 1 \end{array}\right]$.

Note that Example 2.1.10 can also be solved by first transposing both sides, then solving for A^T, and so obtaining $A = (A^T)^T$. The reader should do this.

The matrix $D = \left[\begin{array}{cc} 1 & 2 \\ 2 & 5 \end{array}\right]$ in Example 2.1.9 has the property that $D = D^T$. Such matrices are important; a matrix A is called **symmetric** if $A = A^T$. A symmetric matrix A is necessarily square (if A is $m \times n$, then A^T is $n \times m$, so $A = A^T$ forces $n = m$). The name comes from the fact that these matrices exhibit a symmetry about the main diagonal. That is, entries that are directly across the main diagonal from each other are equal.

For example, $\left[\begin{array}{ccc} a & b & c \\ b' & d & e \\ c' & e' & f \end{array}\right]$ is symmetric when $b=b'$, $c=c'$, and $e=e'$.

Example 2.1.11

If A and B are symmetric $n\times n$ matrices, show that $A+B$ is symmetric.

Solution. We have $A^T=A$ and $B^T=B$, so, by Theorem 2.1.2, we have $(A+B)^T=A^T+B^T=A+B$. Hence $A+B$ is symmetric.

Example 2.1.12

Suppose a square matrix A satisfies $A=2A^T$. Show that necessarily $A=0$.

Solution. If we iterate the given equation, Theorem 2.1.2 gives

$$A=2A^T=2\left[2A^T\right]^T=2\left[2(A^T)^T\right]=4A$$

Subtracting A from both sides gives $3A=0$, so $A=\frac{1}{3}(0)=0$.

Exercises for 2.1

Exercise 2.1.1 Find a, b, c, and d if

a. $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]=\left[\begin{array}{cc} c-3d & -d \\ 2a+d & a+b \end{array}\right]$

b. $\left[\begin{array}{cc} a-b & b-c \\ c-d & d-a \end{array}\right]=2\left[\begin{array}{rr} 1 & 1 \\ -3 & 1 \end{array}\right]$

c. $3\left[\begin{array}{c} a \\ b \end{array}\right]+2\left[\begin{array}{c} b \\ a \end{array}\right]=\left[\begin{array}{c} 1 \\ 2 \end{array}\right]$

d. $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]=\left[\begin{array}{cc} b & c \\ d & a \end{array}\right]$

Exercise 2.1.2 Compute the following:

a. $\left[\begin{array}{rrr} 3 & 2 & 1 \\ 5 & 1 & 0 \end{array}\right]-5\left[\begin{array}{rrr} 3 & 0 & -2 \\ 1 & -1 & 2 \end{array}\right]$

b. $3\left[\begin{array}{r} 3 \\ -1 \end{array}\right]-5\left[\begin{array}{r} 6 \\ 2 \end{array}\right]+7\left[\begin{array}{r} 1 \\ -1 \end{array}\right]$

c. $\left[\begin{array}{rr} -2 & 1 \\ 3 & 2 \end{array}\right]-4\left[\begin{array}{rr} 1 & -2 \\ 0 & -1 \end{array}\right]+3\left[\begin{array}{rr} 2 & -3 \\ -1 & -2 \end{array}\right]$

d. $\left[\begin{array}{rrr} 3 & -1 & 2 \end{array}\right]-2\left[\begin{array}{rrr} 9 & 3 & 4 \end{array}\right]+\left[\begin{array}{rrr} 3 & 11 & -6 \end{array}\right]$

e. $\left[\begin{array}{rrrr} 1 & -5 & 4 & 0 \\ 2 & 1 & 0 & 6 \end{array}\right]^T$

f. $\left[\begin{array}{rrr} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{array}\right]^T$

g. $\left[\begin{array}{rr} 3 & -1 \\ 2 & 1 \end{array}\right]-2\left[\begin{array}{rr} 1 & -2 \\ 1 & 1 \end{array}\right]^T$

h. $3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Exercise 2.1.3 Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Compute the following (where possible).

a. $3A - 2B$
b. $5C$
c. $3E^T$
d. $B + D$
e. $4A^T - 3C$
f. $(A + C)^T$
g. $2B - 3E$
h. $A - D$
i. $(B - 2E)^T$

Exercise 2.1.4 Find A if:

a. $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$

b. $3A - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Exercise 2.1.5 Find A in terms of B if:

a. $A + B = 3A + 2B$
b. $2A - B = 5(A + 2B)$

Exercise 2.1.6 If X, Y, A, and B are matrices of the same size, solve the following systems of equations to obtain X and Y in terms of A and B.

a. $\begin{aligned} 5X + 3Y &= A \\ 2X + Y &= B \end{aligned}$

b. $\begin{aligned} 4X + 3Y &= A \\ 5X + 4Y &= B \end{aligned}$

Exercise 2.1.7 Find all matrices X and Y such that:

a. $3X - 2Y = \begin{bmatrix} 3 & -1 \end{bmatrix}$
b. $2X - 5Y = \begin{bmatrix} 1 & 2 \end{bmatrix}$

Exercise 2.1.8 Simplify the following expressions where A, B, and C are matrices.

a. $2[9(A - B) + 7(2B - A)]$
$-2[3(2B + A) - 2(A + 3B) - 5(A + B)]$

b. $5[3(A - B + 2C) - 2(3C - B) - A]$
$+2[3(3A - B + C) + 2(B - 2A) - 2C]$

Exercise 2.1.9 If A is any 2×2 matrix, show that:

a. $A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for some numbers a, b, c, and d.

b. $A = p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + q\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + r\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for some numbers p, q, r, and s.

Exercise 2.1.10 Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. If $rA + sB + tC = 0$ for some scalars r, s, and t, show that necessarily $r = s = t = 0$.

Exercise 2.1.11

a. If $Q + A = A$ holds for every $m \times n$ matrix A, show that $Q = 0_{mn}$.

b. If A is an $m \times n$ matrix and $A + A' = 0_{mn}$, show that $A' = -A$.

Exercise 2.1.12 If A denotes an $m \times n$ matrix, show that $A = -A$ if and only if $A = 0$.

Exercise 2.1.13 A square matrix is called a **diagonal** matrix if all the entries off the main diagonal are zero. If A and B are diagonal matrices, show that the following matrices are also diagonal.

a. $A + B$
b. $A - B$
c. kA for any number k

Exercise 2.1.14 In each case determine all s and t such that the given matrix is symmetric:

a. $\begin{bmatrix} 1 & s \\ -2 & t \end{bmatrix}$
b. $\begin{bmatrix} s & t \\ st & 1 \end{bmatrix}$
c. $\begin{bmatrix} s & 2s & st \\ t & -1 & s \\ t & s^2 & s \end{bmatrix}$
d. $\begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}$

Exercise 2.1.15 In each case find the matrix A.

a. $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$

b. $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$

c. $\left(2A - 3\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}\right)^T = 3A^T + \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$

d. $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

Exercise 2.1.16 Let A and B be symmetric (of the same size). Show that each of the following is symmetric.

a. $(A - B)$

b. kA for any scalar k

Exercise 2.1.17 Show that $A + A^T$ and AA^T are symmetric for *any* square matrix A.

Exercise 2.1.18 If A is a square matrix and $A = kA^T$ where $k \neq \pm 1$, show that $A = 0$.

Exercise 2.1.19 In each case either show that the statement is true or give an example showing it is false.

a. If $A + B = A + C$, then B and C have the same size.

b. If $A + B = 0$, then $B = 0$.

c. If the $(3,\ 1)$-entry of A is 5, then the $(1,\ 3)$-entry of A^T is -5.

d. A and A^T have the same main diagonal for every matrix A.

e. If B is symmetric and $A^T = 3B$, then $A = 3B$.

f. If A and B are symmetric, then $kA + mB$ is symmetric for any scalars k and m.

Exercise 2.1.20 A square matrix W is called **skew-symmetric** if $W^T = -W$. Let A be any square matrix.

a. Show that $A - A^T$ is skew-symmetric.

b. Find a symmetric matrix S and a skew-symmetric matrix W such that $A = S + W$.

c. Show that S and W in part (b) are uniquely determined by A.

Exercise 2.1.21 If W is skew-symmetric (Exercise 2.1.20), show that the entries on the main diagonal are zero.

Exercise 2.1.22 Prove the following parts of Theorem 2.1.1.

a. $(k + p)A = kA + pA$

b. $(kp)A = k(pA)$

Exercise 2.1.23 Let A, A_1, A_2, $\dots$, A_n denote matrices of the same size. Use induction on n to verify the following extensions of properties 5 and 6 of Theorem 2.1.1.

a. $k(A_1 + A_2 + \cdots + A_n) = kA_1 + kA_2 + \cdots + kA_n$ for any number k

b. $(k_1 + k_2 + \cdots + k_n)A = k_1A + k_2A + \cdots + k_nA$ for any numbers k_1, k_2, $\dots$, k_n

Exercise 2.1.24 Let A be a square matrix. If $A = pB^T$ and $B = qA^T$ for some matrix B and numbers p and q, show that either $A = 0 = B$ or $pq = 1$.
[*Hint*: Example 2.1.7.]

2.2 Matrix-Vector Multiplication

Up to now we have used matrices to solve systems of linear equations by manipulating the rows of the augmented matrix. In this section we introduce a different way of describing linear systems that makes more use of the coefficient matrix of the system and leads to a useful way of "multiplying" matrices.

Vectors

It is a well-known fact in analytic geometry that two points in the plane with coordinates $(a_1,\ a_2)$ and $(b_1,\ b_2)$ are equal if and only if $a_1 = b_1$ and $a_2 = b_2$. Moreover, a similar condition applies to points $(a_1,\ a_2,\ a_3)$ in space. We extend this idea as follows.

An ordered sequence $(a_1,\ a_2,\ \dots,\ a_n)$ of real numbers is called an **ordered *n*-tuple**. The word "ordered" here reflects our insistence that two ordered n-tuples are equal if and only if corresponding entries are the same. In other words,

$$(a_1,\ a_2,\ \dots,\ a_n) = (b_1,\ b_2,\ \dots,\ b_n) \quad \text{if and only if} \quad a_1 = b_1,\ a_2 = b_2,\ \dots,\ \text{ and } a_n = b_n.$$

Thus the ordered 2-tuples and 3-tuples are just the ordered pairs and triples familiar from geometry.

Definition 2.4 The set $\mathbb{R}^n$ of ordered n-tuples of real numbers

Let $\mathbb{R}$ denote the set of all real numbers. The set of all *ordered n-tuples from $\mathbb{R}$ has a special notation:*

$$\mathbb{R}^n \text{ denotes the set of all ordered } n\text{-tuples of real numbers.}$$

There are two commonly used ways to denote the n-tuples in $\mathbb{R}^n$: As rows $(r_1,\ r_2,\ \dots,\ r_n)$ or columns $\left[\begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_n \end{array}\right]$; the notation we use depends on the context. In any event they are called **vectors** or ***n*-vectors** and will be denoted using bold type such as $\mathbf{x}$ or $\mathbf{v}$. For example, an $m \times n$ matrix A will be written as a row of columns:

$$A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] \text{ where } \mathbf{a}_j \text{ denotes column } j \text{ of } A \text{ for each } j.$$

If $\mathbf{x}$ and $\mathbf{y}$ are two n-vectors in $\mathbb{R}^n$, it is clear that their matrix sum $\mathbf{x}+\mathbf{y}$ is also in $\mathbb{R}^n$ as is the scalar multiple $k\mathbf{x}$ for any real number k. We express this observation by saying that $\mathbb{R}^n$ is **closed** under addition and scalar multiplication. In particular, all the basic properties in Theorem 2.1.1 are true of these n-vectors. These properties are fundamental and will be used frequently below without comment. As for matrices in general, the $n \times 1$ zero matrix is called the **zero *n*-vector** in $\mathbb{R}^n$ and, if $\mathbf{x}$ is an n-vector, the n-vector $-\mathbf{x}$ is called the **negative x**.

Of course, we have already encountered these n-vectors in Section 1.3 as the solutions to systems of linear equations with n variables. In particular we defined the notion of a linear combination of vectors and showed that a linear combination of solutions to a homogeneous system is again a solution. Clearly, a linear combination of n-vectors in $\mathbb{R}^n$ is again in $\mathbb{R}^n$, a fact that we will be using.

Matrix-Vector Multiplication

Given a system of linear equations, the left sides of the equations depend only on the coefficient matrix A and the column $\mathbf{x}$ of variables, and not on the constants. This observation leads to a fundamental idea in linear algebra: We view the left sides of the equations as the "product" $A\mathbf{x}$ of the matrix A and the vector $\mathbf{x}$. This simple change of perspective leads to a completely new way of viewing linear systems—one that is very useful and will occupy our attention throughout this book.

To motivate the definition of the "product" $A\mathbf{x}$, consider first the following system of two equations in three variables:

$$\begin{aligned} ax_1 + \ bx_2 + \ cx_3 &= b_1 \\ a'x_1 + b'x_2 + c'x_3 &= b_1 \end{aligned} \tag{2.2}$$

and let $A = \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ denote the coefficient matrix, the variable matrix, and the constant matrix, respectively. The system (2.2) can be expressed as a single vector equation

$$\begin{bmatrix} ax_1 + \ bx_2 + \ cx_3 \\ a'x_1 + b'x_2 + c'x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

which in turn can be written as follows:

$$x_1 \begin{bmatrix} a \\ a' \end{bmatrix} + x_2 \begin{bmatrix} b \\ b' \end{bmatrix} + x_3 \begin{bmatrix} c \\ c' \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Now observe that the vectors appearing on the left side are just the columns

$$\mathbf{a}_1 = \begin{bmatrix} a \\ a' \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} b \\ b' \end{bmatrix}, \ \text{and } \mathbf{a}_3 = \begin{bmatrix} c \\ c' \end{bmatrix}$$

of the coefficient matrix A. Hence the system (2.2) takes the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \tag{2.3}$$

This shows that the system (2.2) has a solution if and only if the constant matrix $\mathbf{b}$ is a linear combination[3] of the columns of A, and that in this case the entries of the solution are the coefficients x_1, x_2, and x_3 in this linear combination.

Moreover, this holds in general. If A is any $m \times n$ matrix, it is often convenient to view A as a row of columns. That is, if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A, we write

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

and say that $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ is *given in terms of its columns*.

Now consider any system of linear equations with $m \times n$ coefficient matrix A. If $\mathbf{b}$ is the constant matrix of the system, and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the matrix of variables then, exactly as above, the system can

[3]Linear combinations were introduced in Section 1.3 to describe the solutions of homogeneous systems of linear equations. They will be used extensively in what follows.

be written as a single vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{2.4}$$

Example 2.2.1

Write the system $\left\{\begin{array}{rcr} 3x_1 + 2x_2 - 4x_3 &=& 0 \\ x_1 - 3x_2 + \ \ x_3 &=& 3 \\ x_2 - 5x_3 &=& -1 \end{array}\right.$ in the form given in (2.4).

Solution.

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

As mentioned above, we view the left side of (2.4) as the *product* of the matrix A and the vector $\mathbf{x}$. This basic idea is formalized in the following definition:

Definition 2.5 Matrix-Vector Multiplication

*Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n-vector, the **product** $A\mathbf{x}$ is defined to be the m-vector given by:*

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

In other words, if A is $m \times n$ and $\mathbf{x}$ is an n-vector, the product $A\mathbf{x}$ is the linear combination of the columns of A where the coefficients are the entries of $\mathbf{x}$ (in order).

Note that if A is an $m \times n$ matrix, the product $A\mathbf{x}$ is only defined if $\mathbf{x}$ is an n-vector and then the vector $A\mathbf{x}$ is an m-vector because this is true of each column $\mathbf{a}_j$ of A. But in this case the *system* of linear equations with coefficient matrix A and constant vector $\mathbf{b}$ takes the form of a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The following theorem combines Definition 2.5 and equation (2.4) and summarizes the above discussion. Recall that a system of linear equations is said to be *consistent* if it has at least one solution.

Theorem 2.2.1

1. *Every system of linear equations has the form $A\mathbf{x} = \mathbf{b}$ where A is the coefficient matrix, $\mathbf{b}$ is the constant matrix, and $\mathbf{x}$ is the matrix of variables.*

2. *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

3. If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of A and if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then $\mathbf{x}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$ if and only if $x_1, x_2, \dots, x_n$ are a solution of the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

A system of linear equations in the form $A\mathbf{x} = \mathbf{b}$ as in (1) of Theorem 2.2.1 is said to be written in **matrix form**. This is a useful way to view linear systems as we shall see.

Theorem 2.2.1 transforms the problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ into the problem of expressing the constant matrix B as a linear combination of the columns of the coefficient matrix A. Such a change in perspective is very useful because one approach or the other may be better in a particular situation; the importance of the theorem is that there is a choice.

Example 2.2.2

If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, compute $A\mathbf{x}$.

Solution. By Definition 2.5: $A\mathbf{x} = 2\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$.

Example 2.2.3

Given columns $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in $\mathbb{R}^3$, write $2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$ in the form $A\mathbf{x}$ where A is a matrix and $\mathbf{x}$ is a vector.

Solution. Here the column of coefficients is $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 5 \\ 1 \end{bmatrix}$. Hence Definition 2.5 gives

$$A\mathbf{x} = 2\mathbf{a}_1 - 3\mathbf{a}_2 + 5\mathbf{a}_3 + \mathbf{a}_4$$

where $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ is the matrix with $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ as its columns.

Example 2.2.4

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. In each case below, either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a. $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b. $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Solution. By Theorem 2.2.1, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has a solution). So in each case we carry the augmented matrix $[A|\mathbf{b}]$ of the system $A\mathbf{x} = \mathbf{b}$ to reduced form.

a. Here $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 3 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ has no solution in this case. Hence $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

b. Now $\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 3 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ -1 & 1 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, so the system $A\mathbf{x} = \mathbf{b}$ is consistent.

Thus $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in this case. In fact the general solution is $x_1 = 1 - 2s - t$, $x_2 = 2 + s - t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ for *any* choice of s and t. If we take $s = 0$ and $t = 0$, this becomes $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{b}$, whereas taking $s = 1 = t$ gives $-2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = \mathbf{b}$.

Example 2.2.5

Taking A to be the zero matrix, we have $0\mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x}$ by Definition 2.5 because every column of the zero matrix is zero. Similarly, $A\mathbf{0} = \mathbf{0}$ for all matrices A because every entry of the zero vector is zero.

Example 2.2.6

If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^3$.

Solution. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then Definition 2.5 gives

$$I\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}$$

The matrix I in Example 2.2.6 is called the 3×3 **identity matrix**, and we will encounter such matrices again in Example 2.2.11 below. Before proceeding, we develop some algebraic properties of matrix-vector multiplication that are used extensively throughout linear algebra.

Theorem 2.2.2

Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
2. $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ *for all scalars a.*
3. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

Proof. We prove (3); the other verifications are similar and are left as exercises. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ be given in terms of their columns. Since adding two matrices is the same as adding their columns, we have

$$A + B = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \cdots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}$$

If we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ Definition 2.5 gives

$$\begin{aligned} (A + B)\mathbf{x} &= x_1(\mathbf{a}_1 + \mathbf{b}_1) + x_2(\mathbf{a}_2 + \mathbf{b}_2) + \cdots + x_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= (x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n) + (x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) \\ &= A\mathbf{x} + B\mathbf{x} \end{aligned}$$

□

Theorem 2.2.2 allows matrix-vector computations to be carried out much as in ordinary arithmetic. For example, for any $m \times n$ matrices A and B and any n-vectors $\mathbf{x}$ and $\mathbf{y}$, we have:

$$A(2\mathbf{x} - 5\mathbf{y}) = 2A\mathbf{x} - 5A\mathbf{y} \quad \text{and} \quad (3A - 7B)\mathbf{x} = 3A\mathbf{x} - 7B\mathbf{x}$$

We will use such manipulations throughout the book, often without mention.

Linear Equations

Theorem 2.2.2 also gives a useful way to describe the solutions to a system

$$A\mathbf{x} = \mathbf{b}$$

of linear equations. There is a related system

$$A\mathbf{x} = \mathbf{0}$$

called the **associated homogeneous system**, obtained from the original system $A\mathbf{x} = \mathbf{b}$ by replacing all the constants by zeros. Suppose $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$ (that is $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_0 = \mathbf{0}$). Then $\mathbf{x}_1 + \mathbf{x}_0$ is another solution to $A\mathbf{x} = \mathbf{b}$. Indeed, Theorem 2.2.2 gives

$$A(\mathbf{x}_1 + \mathbf{x}_0) = A\mathbf{x}_1 + A\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

This observation has a useful converse.

Theorem 2.2.3

Suppose $\mathbf{x}_1$ is any particular solution to the system $A\mathbf{x} = \mathbf{b}$ of linear equations. Then every solution $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$$

for some solution $\mathbf{x}_0$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. Suppose $\mathbf{x}_2$ is also a solution to $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{x}_2 = \mathbf{b}$. Write $\mathbf{x}_0 = \mathbf{x}_2 - \mathbf{x}_1$. Then $\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$ and, using Theorem 2.2.2, we compute

$$A\mathbf{x}_0 = A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence $\mathbf{x}_0$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. □

Note that gaussian elimination provides one such representation.

Example 2.2.7

Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 6 \\ x_1 \phantom{{}-x_2} - 2x_3 + x_4 &= 4 \end{aligned}$$

Solution. Gaussian elimination gives $x_1 = 4 + 2s - t$, $x_2 = 2 + s + 2t$, $x_3 = s$, and $x_4 = t$ where s and t are arbitrary parameters. Hence the general solution can be written

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4+2s-t \\ 2+s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \left(s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Thus $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution (where $s = 0 = t$), and $\mathbf{x}_0 = s \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ gives *all* solutions to the associated homogeneous system. (To see why this is so, carry out the gaussian elimination again but with all the constants set equal to zero.)

The following useful result is included with no proof.

Theorem 2.2.4

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. Write rank $A = r$.

1. rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ *is either r or $r+1$.*
2. *The system is consistent if and only if* rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r$.
3. *The system is inconsistent if and only if* rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r+1$.

The Dot Product

Definition 2.5 is not always the easiest way to compute a matrix-vector product $A\mathbf{x}$ because it requires that the columns of A be explicitly identified. There is another way to find such a product which uses the matrix A as a whole with no reference to its columns, and hence is useful in practice. The method depends on the following notion.

Definition 2.6 Dot Product in $\mathbb{R}^n$

*If $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are two ordered n-tuples, their **dot product** is defined to be the number*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

To see how this relates to matrix products, let A denote a 3×4 matrix and let $\mathbf{x}$ be a 4-vector. Writing

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

in the notation of Section 2.1, we compute

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix} \end{aligned}$$

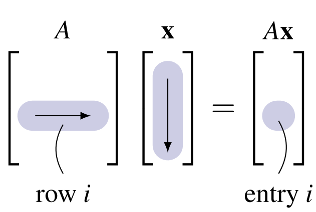
From this we see that each entry of $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$. This computation goes through in general, and we record the result in Theorem 2.2.5.

Theorem 2.2.5: Dot Product Rule

Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.

This result is used extensively throughout linear algebra.

If A is $m \times n$ and $\mathbf{x}$ is an n-vector, the computation of $A\mathbf{x}$ by the dot product rule is simpler than using Definition 2.5 because the computation can be carried out directly with no explicit reference to the columns of A (as in Definition 2.5). The first entry of $A\mathbf{x}$ is the dot product of row 1 of A with $\mathbf{x}$. In hand calculations this is computed by going *across* row one of A, going *down* the column $\mathbf{x}$, multiplying corresponding entries, and adding the results. The other entries of $A\mathbf{x}$ are computed in the same way using the other rows of A with the column $\mathbf{x}$.



In general, compute entry i of $A\mathbf{x}$ as follows (see the diagram):

Go *across* row i of A and *down* column $\mathbf{x}$, multiply corresponding entries, and add the results.

As an illustration, we rework Example 2.2.2 using the dot product rule instead of Definition 2.5.

Example 2.2.8

If $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, compute $A\mathbf{x}$.

Solution. The entries of $A\mathbf{x}$ are the dot products of the rows of A with $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & + & (-1)1 & + & 3\cdot 0 & + & 5(-2) \\ 0\cdot 2 & + & 2\cdot 1 & + & (-3)0 & + & 1(-2) \\ (-3)2 & + & 4\cdot 1 & + & 1\cdot 0 & + & 2(-2) \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -6 \end{bmatrix}$$

Of course, this agrees with the outcome in Example 2.2.2.

Example 2.2.9

Write the following system of linear equations in the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 &= 8 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 &= -2 \\ -x_1 + x_2 - 2x_3 + \quad - 3x_5 &= 0 \end{aligned}$$

Solution. Write $A = \begin{bmatrix} 5 & -1 & 2 & 1 & -3 \\ 1 & 1 & 3 & -5 & 2 \\ -1 & 1 & -2 & 0 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Then the dot product rule gives $A\mathbf{x} = \begin{bmatrix} 5x_1 - x_2 + 2x_3 + x_4 - 3x_5 \\ x_1 + x_2 + 3x_3 - 5x_4 + 2x_5 \\ -x_1 + x_2 - 2x_3 \qquad - 3x_5 \end{bmatrix}$, so the entries of $A\mathbf{x}$ are the left sides of the equations in the linear system. Hence the system becomes $A\mathbf{x} = \mathbf{b}$ because matrices are equal if and only corresponding entries are equal.

Example 2.2.10

If A is the zero $m \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ for each n-vector $\mathbf{x}$.

Solution. For each k, entry k of $A\mathbf{x}$ is the dot product of row k of A with $\mathbf{x}$, and this is zero because row k of A consists of zeros.

Definition 2.7 The Identity Matrix

For each $n > 2$, the ***identity matrix*** *I_n is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.*

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \dots$$

In Example 2.2.6 we showed that $I_3\mathbf{x} = \mathbf{x}$ for each 3-vector $\mathbf{x}$ using Definition 2.5. The following result shows that this holds in general, and is the reason for the name.

Example 2.2.11

For each $n \geq 2$ we have $I_n\mathbf{x} = \mathbf{x}$ for each n-vector $\mathbf{x}$ in $\mathbb{R}^n$.

Solution. We verify the case $n = 4$. Given the 4-vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 + 0 \\ 0 + x_2 + 0 + 0 \\ 0 + 0 + x_3 + 0 \\ 0 + 0 + 0 + x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}$$

In general, $I_n\mathbf{x} = \mathbf{x}$ because entry k of $I_n\mathbf{x}$ is the dot product of row k of I_n with $\mathbf{x}$, and row k of I_n has 1 in position k and zeros elsewhere.

Example 2.2.12

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be any $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{e}_j$ denotes column j of the $n \times n$ identity matrix I_n, then $A\mathbf{e}_j = \mathbf{a}_j$ for each $j = 1, 2, \dots, n$.

Solution. Write $\mathbf{e}_j = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$ where $t_j = 1$, but $t_i = 0$ for all $i \neq j$. Then Theorem 2.2.5 gives

$$A\mathbf{e}_j = t_1\mathbf{a}_1 + \cdots + t_j\mathbf{a}_j + \cdots + t_n\mathbf{a}_n = 0 + \cdots + \mathbf{a}_j + \cdots + 0 = \mathbf{a}_j$$

Example 2.2.12 will be referred to later; for now we use it to prove:

Theorem 2.2.6

Let A and B be $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, then $A = B$.

Proof. Write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ and in terms of their columns. It is enough to show that $\mathbf{a}_k = \mathbf{b}_k$ holds for all k. But we are assuming that $A\mathbf{e}_k = B\mathbf{e}_k$, which gives $\mathbf{a}_k = \mathbf{b}_k$ by Example 2.2.12. □

We have introduced matrix-vector multiplication as a new way to think about systems of linear equations. But it has several other uses as well. It turns out that many geometric operations can be described using matrix multiplication, and we now investigate how this happens. As a bonus, this description provides a geometric "picture" of a matrix by revealing the effect on a vector when it is multiplied by A. This "geometric view" of matrices is a fundamental tool in understanding them.

Transformations

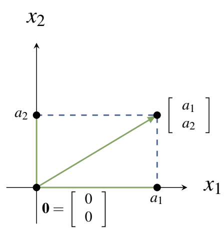


Figure 2.2.1

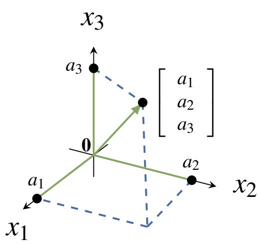


Figure 2.2.2

The set $\mathbb{R}^2$ has a geometrical interpretation as the euclidean plane where a vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ represents the point (a_1, a_2) in the plane (see Figure 2.2.1). In this way we regard $\mathbb{R}^2$ as the set of all points in the plane. Accordingly, we will refer to vectors in $\mathbb{R}^2$ as points, and denote their coordinates as a column rather than a row. To enhance this geometrical interpretation of the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, it is denoted graphically by an arrow from the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the vector as in Figure 2.2.1.

Similarly we identify $\mathbb{R}^3$ with 3-dimensional space by writing a point (a_1, a_2, a_3) as the vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ in $\mathbb{R}^3$, again represented by an arrow[4] from the origin to the point as in Figure 2.2.2. In this way the terms "point" and "vector" mean the same thing in the plane or in space.

We begin by describing a particular geometrical transformation of the plane $\mathbb{R}^2$.

Example 2.2.13

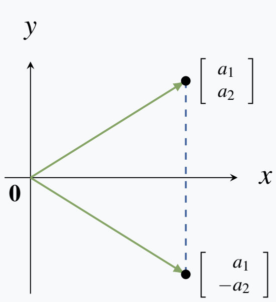


Figure 2.2.3

Consider the transformation of $\mathbb{R}^2$ given by *reflection* in the x axis. This operation carries the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ to its reflection $\begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$ as in Figure 2.2.3. Now observe that

$$\begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

so reflecting $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in the x axis can be achieved by multiplying by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

[4]This "arrow" representation of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used extensively in Chapter 4.

If we write $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, Example 2.2.13 shows that reflection in the x axis carries each vector $\mathbf{x}$ in $\mathbb{R}^2$ to the vector $A\mathbf{x}$ in $\mathbb{R}^2$. It is thus an example of a function

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2$$

As such it is a generalization of the familiar functions $f : \mathbb{R} \to \mathbb{R}$ that carry a *number* x to another real *number* $f(x)$.

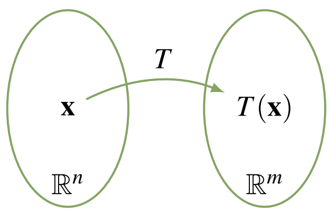


Figure 2.2.4

More generally, functions $T : \mathbb{R}^n \to \mathbb{R}^m$ are called **transformations** from $\mathbb{R}^n$ to $\mathbb{R}^m$. Such a transformation T is a rule that assigns to every vector $\mathbf{x}$ in $\mathbb{R}^n$ a uniquely determined vector $T(\mathbf{x})$ in $\mathbb{R}^m$ called the **image** of $\mathbf{x}$ under T. We denote this state of affairs by writing

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{or} \quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

The transformation T can be visualized as in Figure 2.2.4.

To describe a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we must specify the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ for every $\mathbf{x}$ in $\mathbb{R}^n$. This is referred to as **defining** T, or as specifying the **action** of T. Saying that the action *defines* the transformation means that we regard two transformations $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ as **equal** if they have the **same action**; more formally

$$S = T \quad \text{if and only if} \quad S(\mathbf{x}) = T(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Again, this what we mean by $f = g$ where $f,\ g : \mathbb{R} \to \mathbb{R}$ are ordinary functions.

Functions $f : \mathbb{R} \to \mathbb{R}$ are often described by a formula, examples being $f(x) = x^2 + 1$ and $f(x) = \sin x$. The same is true of transformations; here is an example.

Example 2.2.14

The formula $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$ defines a transformation $\mathbb{R}^4 \to \mathbb{R}^3$.

Example 2.2.13 suggests that matrix multiplication is an important way of defining transformations $\mathbb{R}^n \to \mathbb{R}^m$. If A is any $m \times n$ matrix, multiplication by A gives a transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by} \quad T_A(\mathbf{x}) = A\mathbf{x} \text{ for every } \mathbf{x} \text{ in } \mathbb{R}^n$$

Definition 2.8 Matrix Transformation T_A

*T_A is called the **matrix transformation induced** by A.*

Thus Example 2.2.13 shows that reflection in the x axis is the matrix transformation $\mathbb{R}^2 \to \mathbb{R}^2$ induced by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Also, the transformation $R : \mathbb{R}^4 \to \mathbb{R}^3$ in Example 2.2.13 is the matrix transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$$

Example 2.2.15

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation about the origin through $\frac{\pi}{2}$ radians (that is, 90°)[5]. Show that $R_{\frac{\pi}{2}}$ is induced by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution.

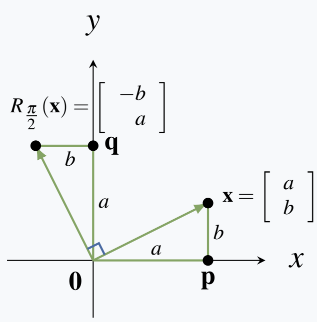


Figure 2.2.5

The effect of $R_{\frac{\pi}{2}}$ is to rotate the vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ counterclockwise through $\frac{\pi}{2}$ to produce the vector $R_{\frac{\pi}{2}}(\mathbf{x})$ shown in Figure 2.2.5. Since triangles $\mathbf{0px}$ and $\mathbf{0q}R_{\frac{\pi}{2}}(\mathbf{x})$ are identical, we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = \begin{bmatrix} -b \\ a \end{bmatrix}$. But $\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In other words, $R_{\frac{\pi}{2}}$ is the matrix transformation induced by A.

If A is the $m \times n$ zero matrix, then A induces the transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{given by} \quad T(\mathbf{x}) = A\mathbf{x} = \mathbf{0} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

This is called the **zero transformation**, and is denoted $T = 0$.

Another important example is the **identity transformation**

$$1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{given by} \quad 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

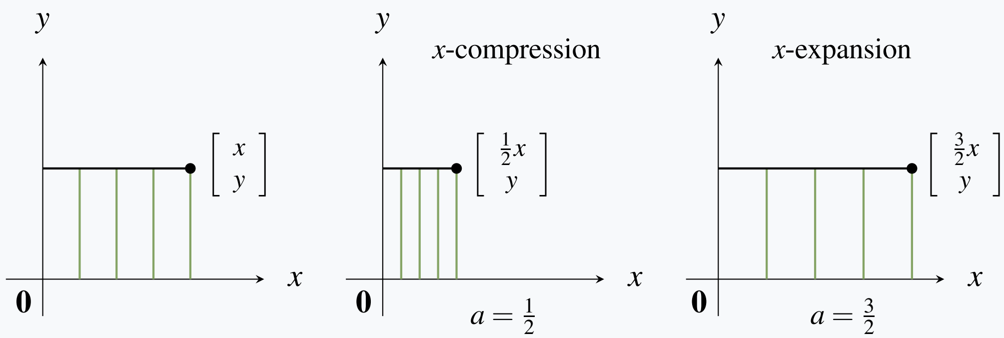
That is, the action of $1_{\mathbb{R}^n}$ on $\mathbf{x}$ is to do nothing to it. If I_n denotes the $n \times n$ identity matrix, we showed in Example 2.2.11 that $I_n\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Hence $1_{\mathbb{R}^n}(\mathbf{x}) = I_n\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$; that is, the identity matrix I_n induces the identity transformation.

Here are two more examples of matrix transformations with a clear geometric description.

[5]*Radian measure* for angles is based on the fact that 360° equals 2π radians. Hence π radians $= 180^\circ$ and $\frac{\pi}{2}$ radians $= 90^\circ$.

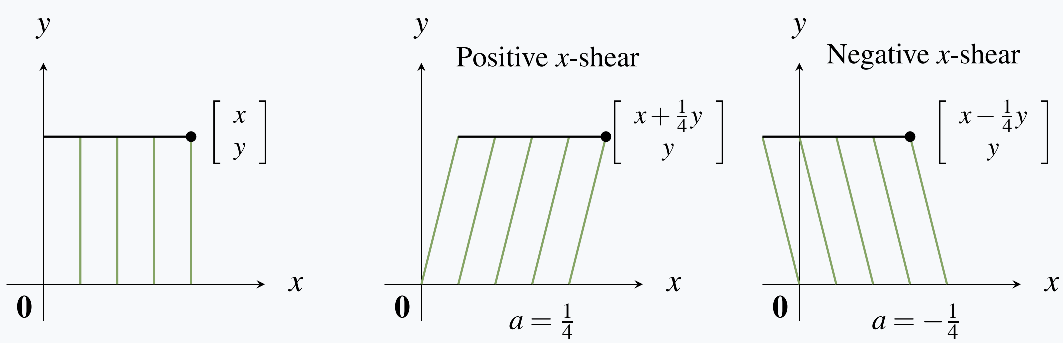
Example 2.2.16

If $a > 0$, the matrix transformation $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is called an ***x*-expansion** of $\mathbb{R}^2$ if $a > 1$, and an ***x*-compression** if $0 < a < 1$. The reason for the names is clear in the diagram below. Similarly, if $b > 0$ the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ gives rise to ***y*-expansions** and ***y*-compressions**.



Example 2.2.17

If a is a number, the matrix transformation $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is called an ***x*-shear** of $\mathbb{R}^2$ (**positive** if $a > 0$ and **negative** if $a < 0$). Its effect is illustrated below when $a = \frac{1}{4}$ and $a = -\frac{1}{4}$.



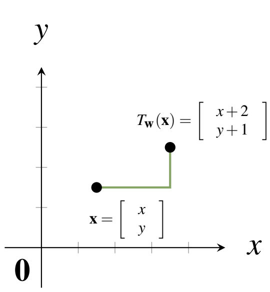


Figure 2.2.6

We hasten to note that there are important geometric transformations that are *not* matrix transformations. For example, if $\mathbf{w}$ is a fixed column in $\mathbb{R}^n$, define the transformation $T_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + \mathbf{w} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then $T_{\mathbf{w}}$ is called **translation** by $\mathbf{w}$. In particular, if $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$, the effect of $T_{\mathbf{w}}$ on $\begin{bmatrix} x \\ y \end{bmatrix}$ is to translate it two units to the right and one unit up (see Figure 2.2.6).

The translation $T_{\mathbf{w}}$ is not a matrix transformation unless $\mathbf{w} = \mathbf{0}$. Indeed, if $T_{\mathbf{w}}$ were induced by a matrix A, then $A\mathbf{x} = T_{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + \mathbf{w}$ would hold for every $\mathbf{x}$ in $\mathbb{R}^n$. In particular, taking $\mathbf{x} = \mathbf{0}$ gives $\mathbf{w} = A\mathbf{0} = \mathbf{0}$.

Exercises for 2.2

Exercise 2.2.1 In each case find a system of equations that is equivalent to the given vector equation. (Do not solve the system.)

a. $x_1 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -3 \end{bmatrix}$

b. $x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

Exercise 2.2.2 In each case find a vector equation that is equivalent to the given system of equations. (Do not solve the equation.)

a. $$\begin{aligned} x_1 - x_2 + 3x_3 &= 5 \\ -3x_1 + x_2 + x_3 &= -6 \\ 5x_1 - 8x_2 &= 9 \end{aligned}$$

b. $$\begin{aligned} x_1 - 2x_2 - x_3 + x_4 &= 5 \\ -x_1 + x_3 - 2x_4 &= -3 \\ 2x_1 - 2x_2 + 7x_3 &= 8 \\ 3x_1 - 4x_2 + 9x_3 - 2x_4 &= 12 \end{aligned}$$

Exercise 2.2.3 In each case compute $A\mathbf{x}$ using: (i) Definition 2.5. (ii) Theorem 2.2.5.

a. $A = \begin{bmatrix} 3 & -2 & 0 \\ 5 & -4 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

c. $A = \begin{bmatrix} -2 & 0 & 5 & 4 \\ 1 & 2 & 0 & 3 \\ -5 & 6 & -7 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

d. $A = \begin{bmatrix} 3 & -4 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ -8 & 7 & -3 & 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Exercise 2.2.4 Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$ be the 3×4 matrix given in terms of its columns $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and $\mathbf{a}_4 = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$. In each case either express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, or show that it is not such a linear combination. Explain what your answer means for the corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations.

a. $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

b. $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Exercise 2.2.5 In each case, express every solution of the system as a sum of a specific solution plus a solution of the associated homogeneous system.

a. $$\begin{aligned} x + y + z &= 2 \\ 2x + y &= 3 \\ x - y - 3z &= 0 \end{aligned}$$

b. $$\begin{aligned} x - y - 4z &= -4 \\ x + 2y + 5z &= 2 \\ x + y + 2z &= 0 \end{aligned}$$

c. $$\begin{aligned} x_1 + x_2 - x_3 - 5x_5 &= 2 \\ x_2 + x_3 - 4x_5 &= -1 \\ x_2 + x_3 + x_4 - x_5 &= -1 \\ 2x_1 - 4x_3 + x_4 + x_5 &= 6 \end{aligned}$$

d. $$\begin{aligned} 2x_1 + x_2 - x_3 - x_4 &= -1 \\ 3x_1 + x_2 + x_3 - 2x_4 &= -2 \\ -x_1 - x_2 + 2x_3 + x_4 &= 2 \\ -2x_1 - x_2 + 2x_4 &= 3 \end{aligned}$$

Exercise 2.2.6 If $\mathbf{x}_0$ and $\mathbf{x}_1$ are solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, use Theorem 2.2.2 to show that $s\mathbf{x}_0 + t\mathbf{x}_1$ is also a solution for any scalars s and t (called a **linear combination** of $\mathbf{x}_0$ and $\mathbf{x}_1$).

Exercise 2.2.7 Assume that $A\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \mathbf{0} = A\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

Show that $\mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find a two-parameter family of solutions to $A\mathbf{x} = \mathbf{b}$.

Exercise 2.2.8 In each case write the system in the form $A\mathbf{x} = \mathbf{b}$, use the gaussian algorithm to solve the system, and express the solution as a particular solution plus a linear combination of basic solutions to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. $$\begin{aligned} x_1 - 2x_2 + x_3 + 4x_4 - x_5 &= 8 \\ -2x_1 + 4x_2 + x_3 - 2x_4 - 4x_5 &= -1 \\ 3x_1 - 6x_2 + 8x_3 + 4x_4 - 13x_5 &= 1 \\ 8x_1 - 16x_2 + 7x_3 + 12x_4 - 6x_5 &= 11 \end{aligned}$$

b. $$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= -4 \\ -3x_1 + 6x_2 - 2x_3 - 3x_4 - 11x_5 &= 11 \\ -2x_1 + 4x_2 - x_3 + x_4 - 8x_5 &= 7 \\ -x_1 + 2x_2 \qquad + 3x_4 - 5x_5 &= 3 \end{aligned}$$

Exercise 2.2.9 Given vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, find a vector $\mathbf{b}$ that is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Justify your answer. [*Hint*: Part (2) of Theorem 2.2.1.]

Exercise 2.2.10 In each case either show that the statement is true, or give an example showing that it is false.

a. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

b. If $A\mathbf{x}$ has a zero entry, then A has a row of zeros.

c. If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \neq \mathbf{0}$, then $A = 0$.

d. Every linear combination of vectors in $\mathbb{R}^n$ can be written in the form $A\mathbf{x}$.

e. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if $\mathbf{b} = 3\mathbf{a}_1 - 2\mathbf{a}_2$, then the system $A\mathbf{x} = \mathbf{b}$ has a solution.

f. If $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns, and if the system $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b} = s\mathbf{a}_1 + t\mathbf{a}_2$ for some s, t.

g. If A is $m \times n$ and $m < n$, then $A\mathbf{x} = \mathbf{b}$ has a solution for every column $\mathbf{b}$.

h. If $A\mathbf{x} = \mathbf{b}$ has a solution for some column $\mathbf{b}$, then it has a solution for every column $\mathbf{b}$.

i. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

j. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ in terms of its columns. If $\mathbf{a}_3 = s\mathbf{a}_1 + t\mathbf{a}_2$, then $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} s \\ t \\ -1 \end{bmatrix}$.

Exercise 2.2.11 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the y axis.

b. T is a reflection in the line $y = x$.

c. T is a reflection in the line $y = -x$.

d. T is a clockwise rotation through $\frac{\pi}{2}$.

Exercise 2.2.12 The **projection** $P : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$. Show that P is induced by a matrix and find the matrix.

Exercise 2.2.13 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation. In each case show that T is induced by a matrix and find the matrix.

a. T is a reflection in the $x - y$ plane.

b. T is a reflection in the $y - z$ plane.

Exercise 2.2.14 Fix $a > 0$ in $\mathbb{R}$, and define $T_a : \mathbb{R}^4 \to \mathbb{R}^4$ by $T_a(\mathbf{x}) = a\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. Show that T is induced by a matrix and find the matrix. [T is called a **dilation** if $a > 1$ and a **contraction** if $a < 1$.]

Exercise 2.2.15 Let A be $m \times n$ and let $\mathbf{x}$ be in $\mathbb{R}^n$. If A has a row of zeros, show that $A\mathbf{x}$ has a zero entry.

Exercise 2.2.16 If a vector $\mathbf{b}$ is a linear combination of the columns of A, show that the system $A\mathbf{x} = \mathbf{b}$ is consistent (that is, it has at least one solution.)

Exercise 2.2.17 If a system $A\mathbf{x} = \mathbf{b}$ is inconsistent (no solution), show that $\mathbf{b}$ is not a linear combination of the columns of A.

Exercise 2.2.18 Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

a. Show that $\mathbf{x}_1 + \mathbf{x}_2$ is a solution to $A\mathbf{x} = \mathbf{0}$.

b. Show that $t\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{0}$ for any scalar t.

Exercise 2.2.19 Suppose $\mathbf{x}_1$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{x}_0$ is any nontrivial solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}_1 + t\mathbf{x}_0$, t a scalar, is an infinite one parameter family of solutions to $A\mathbf{x} = \mathbf{b}$. [*Hint*: Example 2.1.7 Section 2.1.]

Exercise 2.2.20 Let A and B be matrices of the same size. If $\mathbf{x}$ is a solution to both the system $A\mathbf{x} = \mathbf{0}$ and the system $B\mathbf{x} = \mathbf{0}$, show that $\mathbf{x}$ is a solution to the system $(A+B)\mathbf{x} = \mathbf{0}$.

Exercise 2.2.21 If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, show that $A = 0$ is the zero matrix. [*Hint*: Consider $A\mathbf{e}_j$ where $\mathbf{e}_j$ is the jth column of I_n; that is, $\mathbf{e}_j$ is the vector in $\mathbb{R}^n$ with 1 as entry j and every other entry 0.]

Exercise 2.2.22 Prove part (1) of Theorem 2.2.2.

Exercise 2.2.23 Prove part (2) of Theorem 2.2.2.

2.3 Matrix Multiplication

In Section 2.2 matrix-vector products were introduced. If A is an $m \times n$ matrix, the product $A\mathbf{x}$ was defined for any n-column $\mathbf{x}$ in $\mathbb{R}^n$ as follows: If $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right]$ where the $\mathbf{a}_j$ are the columns of A, and if $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$, Definition 2.5 reads

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \tag{2.5}$$

This was motivated as a way of describing systems of linear equations with coefficient matrix A. Indeed every such system has the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is the column of constants.

In this section we extend this matrix-vector multiplication to a way of multiplying matrices in general, and then investigate matrix algebra for its own sake. While it shares several properties of ordinary arithmetic, it will soon become clear that matrix arithmetic is different in a number of ways.

Matrix multiplication is closely related to composition of transformations.

Composition and Matrix Multiplication

Sometimes two transformations "link" together as follows:

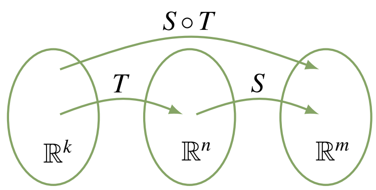
$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

In this case we can apply T first and then apply S, and the result is a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$



The action of $S \circ T$ can be described as "first T then S" (note the order!)[6]. This new transformation is described in the diagram. The reader will have encountered composition of ordinary functions: For example, consider $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$ where $f(x) = x^2$ and $g(x) = x+1$ for all x in $\mathbb{R}$. Then

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] = f(x+1) = (x+1)^2 \\ (g \circ f)(x) &= g[f(x)] = g(x^2) = x^2+1\end{aligned}$$

for all x in $\mathbb{R}$.

Our concern here is with matrix transformations. Suppose that A is an $m \times n$ matrix and B is an $n \times k$ matrix, and let $\mathbb{R}^k \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$ be the matrix transformations induced by B and A respectively, that is:

$$T_B(\mathbf{x}) = B\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k \quad \text{and} \quad T_A(\mathbf{y}) = A\mathbf{y} \text{ for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

Write $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array} \right]$ where $\mathbf{b}_j$ denotes column j of B for each j. Hence each $\mathbf{b}_j$ is an n-vector (B is $n \times k$) so we can form the matrix-vector product $A\mathbf{b}_j$. In particular, we obtain an $m \times k$ matrix

$$\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array} \right]$$

with columns $A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_k$. Now compute $(T_A \circ T_B)(\mathbf{x})$ for any $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \end{array} \right]$ in $\mathbb{R}^k$:

$$\begin{array}{rcll} (T_A \circ T_B)(\mathbf{x}) & = & T_A[T_B(\mathbf{x})] & \text{Definition of } T_A \circ T_B \\ & = & A(B\mathbf{x}) & A \text{ and } B \text{ induce } T_A \text{ and } T_B \\ & = & A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k) & \text{Equation 2.5 above} \\ & = & A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_k\mathbf{b}_k) & \text{Theorem 2.2.2} \\ & = & x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_k(A\mathbf{b}_k) & \text{Theorem 2.2.2} \\ & = & \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array} \right] \mathbf{x} & \text{Equation 2.5 above} \end{array}$$

Because $\mathbf{x}$ was an arbitrary vector in $\mathbb{R}^n$, this shows that $T_A \circ T_B$ is the matrix transformation induced by the matrix $\left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array} \right]$. This motivates the following definition.

[6]When reading the notation $S \circ T$, we read S first and then T even though the action is "first T then S". This annoying state of affairs results because we write $T(\mathbf{x})$ for the effect of the transformation T on $\mathbf{x}$, with T on the left. If we wrote this instead as $(\mathbf{x})T$, the confusion would not occur. However the notation $T(\mathbf{x})$ is well established.

Definition 2.9 Matrix Multiplication

Let A be an $m \times n$ matrix, let B be an $n \times k$ matrix, and write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$ where $\mathbf{b}_j$ is column j of B for each j. The product matrix AB is the $m \times k$ matrix defined as follows:

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}$$

Thus the product matrix AB is given in terms of its columns $A\mathbf{b}_1, A\mathbf{b}_2, \dots, A\mathbf{b}_n$: Column j of AB is the matrix-vector product $A\mathbf{b}_j$ of A and the corresponding column $\mathbf{b}_j$ of B. Note that each such product $A\mathbf{b}_j$ makes sense by Definition 2.5 because A is $m \times n$ and each $\mathbf{b}_j$ is in $\mathbb{R}^n$ (since B has n rows). Note also that if B is a column matrix, this definition reduces to Definition 2.5 for matrix-vector multiplication.

Given matrices A and B, Definition 2.9 and the above computation give

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix} \mathbf{x} = (AB)\mathbf{x}$$

for all $\mathbf{x}$ in $\mathbb{R}^k$. We record this for reference.

Theorem 2.3.1

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Then the product matrix AB is $m \times k$ and satisfies

$$A(B\mathbf{x}) = (AB)\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

Here is an example of how to compute the product AB of two matrices using Definition 2.9.

Example 2.3.1

Compute AB if $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{bmatrix}$.

Solution. The columns of B are $\mathbf{b}_1 = \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}$, so Definition 2.5 gives

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 67 \\ 78 \\ 55 \end{bmatrix} \text{ and } A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \\ 10 \end{bmatrix}$$

Hence Definition 2.9 above gives $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{bmatrix}$.

Example 2.3.2

If A is $m \times n$ and B is $n \times k$, Theorem 2.3.1 gives a simple formula for the composite of the matrix transformations T_A and T_B:

$$T_A \circ T_B = T_{AB}$$

Solution. Given any $\mathbf{x}$ in $\mathbb{R}^k$,

$$\begin{aligned}
(T_A \circ T_B)(\mathbf{x}) &= T_A[T_B(\mathbf{x})] \\
&= A[B\mathbf{x}] \\
&= (AB)\mathbf{x} \\
&= T_{AB}(\mathbf{x})
\end{aligned}$$

While Definition 2.9 is important, there is another way to compute the matrix product AB that gives a way to calculate each individual entry. In Section 2.2 we defined the dot product of two n-tuples to be the sum of the products of corresponding entries. We went on to show (Theorem 2.2.5) that if A is an $m \times n$ matrix and $\mathbf{x}$ is an n-vector, then entry j of the product $A\mathbf{x}$ is the dot product of row j of A with $\mathbf{x}$. This observation was called the "dot product rule" for matrix-vector multiplication, and the next theorem shows that it extends to matrix multiplication in general.

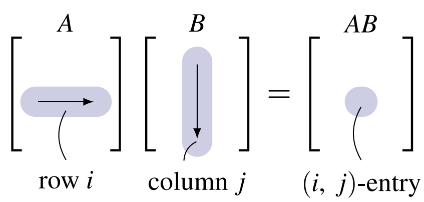
Theorem 2.3.2: Dot Product Rule

Let A and B be matrices of sizes $m \times n$ and $n \times k$, respectively. Then the $(i,\ j)$-entry of AB is the dot product of row i of A with column j of B.

Proof. Write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$ in terms of its columns. Then $A\mathbf{b}_j$ is column j of AB for each j. Hence the $(i,\ j)$-entry of AB is entry i of $A\mathbf{b}_j$, which is the dot product of row i of A with $\mathbf{b}_j$. This proves the theorem. □

Thus to compute the $(i,\ j)$-entry of AB, proceed as follows (see the diagram):

Go *across* row i of A, and *down* column j of B, multiply corresponding entries, and add the results.



Note that this requires that the rows of A must be the same length as the columns of B. The following rule is useful for remembering this and for deciding the size of the product matrix AB.

Compatibility Rule

$$\begin{array}{cc} A & B \\ m \times n & n' \times k \end{array}$$

Let A and B denote matrices. If A is $m \times n$ and B is $n' \times k$, the product AB can be formed if and only if $n = n'$. In this case the size of the product matrix AB is $m \times k$, and we say that AB is **defined**, or that A and B are **compatible** for multiplication.

The diagram provides a useful mnemonic for remembering this. We adopt the following convention:

Convention

Whenever a product of matrices is written, it is tacitly assumed that the sizes of the factors are such that the product is defined.

To illustrate the dot product rule, we recompute the matrix product in Example 2.3.1.

Example 2.3.3

Compute AB if $A = \left[\begin{array}{ccc} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{array}\right]$ and $B = \left[\begin{array}{cc} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{array}\right]$.

Solution. Here A is 3×3 and B is 3×2, so the product matrix AB is defined and will be of size 3×2. Theorem 2.3.2 gives each entry of AB as the dot product of the corresponding row of A with the corresponding column of B_j that is,

$$AB = \left[\begin{array}{ccc} 2 & 3 & 5 \\ 1 & 4 & 7 \\ 0 & 1 & 8 \end{array}\right] \left[\begin{array}{cc} 8 & 9 \\ 7 & 2 \\ 6 & 1 \end{array}\right] = \left[\begin{array}{cc} 2 \cdot 8 + 3 \cdot 7 + 5 \cdot 6 & 2 \cdot 9 + 3 \cdot 2 + 5 \cdot 1 \\ 1 \cdot 8 + 4 \cdot 7 + 7 \cdot 6 & 1 \cdot 9 + 4 \cdot 2 + 7 \cdot 1 \\ 0 \cdot 8 + 1 \cdot 7 + 8 \cdot 6 & 0 \cdot 9 + 1 \cdot 2 + 8 \cdot 1 \end{array}\right] = \left[\begin{array}{cc} 67 & 29 \\ 78 & 24 \\ 55 & 10 \end{array}\right]$$

Of course, this agrees with Example 2.3.1.

Example 2.3.4

Compute the $(1, 3)$- and $(2, 4)$-entries of AB where

$$A = \left[\begin{array}{rrr} 3 & -1 & 2 \\ 0 & 1 & 4 \end{array}\right] \text{ and } B = \left[\begin{array}{rrrr} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{array}\right].$$

Then compute AB.

Solution. The $(1, 3)$-entry of AB is the dot product of row 1 of A and column 3 of B (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\left[\begin{array}{rrr} 3 & -1 & 2 \\ 0 & 1 & 4 \end{array}\right] \left[\begin{array}{rrrr} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{array}\right] \quad (1, 3)\text{-entry} = 3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = 25$$

Similarly, the $(2,\ 4)$-entry of AB involves row 2 of A and column 4 of B.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} \quad (2,\ 4)\text{-entry} = 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$$

Since A is 2×3 and B is 3×4, the product is 2×4.

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Example 2.3.5

If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, compute A^2, AB, BA, and B^2 when they are defined.[7]

Solution. Here, A is a 1×3 matrix and B is a 3×1 matrix, so A^2 and B^2 are not defined. However, the compatibility rule reads

$$\begin{matrix} A & B \\ 1 \times 3 & 3 \times 1 \end{matrix} \quad \text{and} \quad \begin{matrix} B & A \\ 3 \times 1 & 1 \times 3 \end{matrix}$$

so both AB and BA can be formed and these are 1×1 and 3×3 matrices, respectively.

$$AB = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 3 \cdot 6 + 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 2 \\ 6 \cdot 1 & 6 \cdot 3 & 6 \cdot 2 \\ 4 \cdot 1 & 4 \cdot 3 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 15 & 10 \\ 6 & 18 & 12 \\ 4 & 12 & 8 \end{bmatrix}$$

Unlike numerical multiplication, matrix products AB and BA *need not be equal*. In fact they need not even be the same size, as Example 2.3.5 shows. It turns out to be rare that $AB = BA$ (although it is by no means impossible), and A and B are said to **commute** when this happens.

Example 2.3.6

Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA.

[7] As for numbers, we write $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc. Note that A^2 is defined if and only if A is of size $n \times n$ for some n.

Solution. $A^2 = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $A^2 = 0$ can occur even if $A \neq 0$. Next,

$$AB = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 2 & -8 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ -6 & -9 \end{bmatrix}$$

Hence $AB \neq BA$, even though AB and BA are the same size.

Example 2.3.7

If A is any matrix, then $IA = A$ and $AI = A$, and where I denotes an identity matrix of a size so that the multiplications are defined.

Solution. These both follow from the dot product rule as the reader should verify. For a more formal proof, write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ where $\mathbf{a}_j$ is column j of A. Then Definition 2.9 and Example 2.2.11 give

$$IA = \begin{bmatrix} I\mathbf{a}_1 & I\mathbf{a}_2 & \cdots & I\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

If $\mathbf{e}_j$ denotes column j of I, then $A\mathbf{e}_j = \mathbf{a}_j$ for each j by Example 2.2.12. Hence Definition 2.9 gives:

$$AI = A\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = A$$

The following theorem collects several results about matrix multiplication that are used everywhere in linear algebra.

Theorem 2.3.3

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. *$IA = A$ and $AI = A$ where I denotes an identity matrix.*
2. *$A(BC) = (AB)C$.*
3. *$A(B+C) = AB + AC$.*
4. *$(B+C)A = BA + CA$.*
5. *$a(AB) = (aA)B = A(aB)$.*
6. *$(AB)^T = B^T A^T$.*

Proof. Condition (1) is Example 2.3.7; we prove (2), (4), and (6) and leave (3) and (5) as exercises.

1. If $C = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_k \end{bmatrix}$ in terms of its columns, then $BC = \begin{bmatrix} B\mathbf{c}_1 & B\mathbf{c}_2 & \cdots & B\mathbf{c}_k \end{bmatrix}$ by Defini-

tion 2.9, so

$$\begin{aligned} A(BC) &= \left[\begin{array}{cccc} A(B\mathbf{c}_1) & A(B\mathbf{c}_2) & \cdots & A(B\mathbf{c}_k) \end{array} \right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} (AB)\mathbf{c}_1 & (AB)\mathbf{c}_2 & \cdots & (AB)\mathbf{c}_k) \end{array} \right] && \text{Theorem 2.3.1} \\ &= (AB)C && \text{Definition 2.9} \end{aligned}$$

4. We know (Theorem 2.2.2) that $(B+C)\mathbf{x} = B\mathbf{x} + C\mathbf{x}$ holds for every column $\mathbf{x}$. If we write $A = \left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array} \right]$ in terms of its columns, we get

$$\begin{aligned} (B+C)A &= \left[\begin{array}{cccc} (B+C)\mathbf{a}_1 & (B+C)\mathbf{a}_2 & \cdots & (B+C)\mathbf{a}_n \end{array} \right] && \text{Definition 2.9} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 + C\mathbf{a}_1 & B\mathbf{a}_2 + C\mathbf{a}_2 & \cdots & B\mathbf{a}_n + C\mathbf{a}_n \end{array} \right] && \text{Theorem 2.2.2} \\ &= \left[\begin{array}{cccc} B\mathbf{a}_1 & B\mathbf{a}_2 & \cdots & B\mathbf{a}_n \end{array} \right] + \left[\begin{array}{cccc} C\mathbf{a}_1 & C\mathbf{a}_2 & \cdots & C\mathbf{a}_n \end{array} \right] && \text{Adding Columns} \\ &= BA + CA && \text{Definition 2.9} \end{aligned}$$

6. As in Section 2.1, write $A = [a_{ij}]$ and $B = [b_{ij}]$, so that $A^T = [a'_{ij}]$ and $B^T = [b'_{ij}]$ where $a'_{ij} = a_{ji}$ and $b'_{ji} = b_{ij}$ for all i and j. If c_{ij} denotes the $(i,\ j)$-entry of $B^T A^T$, then c_{ij} is the dot product of row i of B^T with column j of A^T. Hence

$$\begin{aligned} c_{ij} = b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{im}a'_{mj} &= b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{mi}a_{jm} \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jm}b_{mi} \end{aligned}$$

But this is the dot product of row j of A with column i of B; that is, the $(j,\ i)$-entry of AB; that is, the $(i,\ j)$-entry of $(AB)^T$. This proves (6). □

Property 2 in Theorem 2.3.3 is called the **associative law** of matrix multiplication. It asserts that the equation $A(BC) = (AB)C$ holds for all matrices (if the products are defined). Hence this product is the same no matter how it is formed, and so is written simply as ABC. This extends: The product $ABCD$ of four matrices can be formed several ways—for example, $(AB)(CD)$, $[A(BC)]D$, and $A[B(CD)]$—but the associative law implies that they are all equal and so are written as $ABCD$. A similar remark applies in general: Matrix products can be written unambiguously with no parentheses.

However, a note of caution about matrix multiplication must be taken: The fact that AB and BA need *not* be equal means that the *order* of the factors is important in a product of matrices. For example $ABCD$ and $ADCB$ may *not* be equal.

Warning

If the order of the factors in a product of matrices is changed, the product matrix may change (or may not be defined). Ignoring this warning is a source of many errors by students of linear algebra!

Properties 3 and 4 in Theorem 2.3.3 are called **distributive laws**. They assert that $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$ hold whenever the sums and products are defined. These rules extend to more than two terms and, together with Property 5, ensure that many manipulations familiar from ordinary algebra extend to matrices. For example

$$\begin{aligned} A(2B-3C+D-5E) &= 2AB-3AC+AD-5AE \\ (A+3C-2D)B &= AB+3CB-2DB \end{aligned}$$

Note again that the warning is in effect: For example $A(B-C)$ need *not* equal $AB-CA$. These rules make possible a lot of simplification of matrix expressions.

Example 2.3.8

Simplify the expression $A(BC-CD)+A(C-B)D-AB(C-D)$.

Solution.

$$\begin{aligned} A(BC-CD)+A(C-B)D-AB(C-D) &= A(BC)-A(CD)+(AC-AB)D-(AB)C+(AB)D \\ &= ABC-ACD+ACD-ABD-ABC+ABD \\ &= 0 \end{aligned}$$

Example 2.3.9 and Example 2.3.10 below show how we can use the properties in Theorem 2.3.2 to deduce other facts about matrix multiplication. Matrices A and B are said to **commute** if $AB=BA$.

Example 2.3.9

Suppose that A, B, and C are $n \times n$ matrices and that both A and B commute with C; that is, $AC=CA$ and $BC=CB$. Show that AB commutes with C.

Solution. Showing that AB commutes with C means verifying that $(AB)C=C(AB)$. The computation uses the associative law several times, as well as the given facts that $AC=CA$ and $BC=CB$.

$$(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$$

Example 2.3.10

Show that $AB=BA$ if and only if $(A-B)(A+B)=A^2-B^2$.

Solution. The following *always* holds:

$$(A-B)(A+B) = A(A+B)-B(A+B) = A^2+AB-BA-B^2 \tag{2.6}$$

Hence if $AB=BA$, then $(A-B)(A+B)=A^2-B^2$ follows. Conversely, if this last equation holds, then equation (2.6) becomes

$$A^2-B^2 = A^2+AB-BA-B^2$$

This gives $0=AB-BA$, and $AB=BA$ follows.

In Section 2.2 we saw (in Theorem 2.2.1) that every system of linear equations has the form

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix, $\mathbf{x}$ is the column of variables, and $\mathbf{b}$ is the constant matrix. Thus the *system* of linear equations becomes a single matrix equation. Matrix multiplication can yield information about such a system.

Example 2.3.11

Consider a system $A\mathbf{x} = \mathbf{b}$ of linear equations where A is an $m \times n$ matrix. Assume that a matrix C exists such that $CA = I_n$. If the system $A\mathbf{x} = \mathbf{b}$ *has* a solution, show that this solution must be $C\mathbf{b}$. Give a condition guaranteeing that $C\mathbf{b}$ *is in fact* a solution.

Solution. Suppose that $\mathbf{x}$ is any solution to the system, so that $A\mathbf{x} = \mathbf{b}$. Multiply both sides of this matrix equation by C to obtain, successively,

$$C(A\mathbf{x}) = C\mathbf{b}, \qquad (CA)\mathbf{x} = C\mathbf{b}, \qquad I_n\mathbf{x} = C\mathbf{b}, \qquad \mathbf{x} = C\mathbf{b}$$

This shows that *if* the system has a solution $\mathbf{x}$, then that solution must be $\mathbf{x} = C\mathbf{b}$, as required. But it does *not* guarantee that the system *has* a solution. However, if we write $\mathbf{x}_1 = C\mathbf{b}$, then

$$A\mathbf{x}_1 = A(C\mathbf{b}) = (AC)\mathbf{b}$$

Thus $\mathbf{x}_1 = C\mathbf{b}$ will be a solution if the condition $AC = I_m$ is satisfied.

The ideas in Example 2.3.11 lead to important information about matrices; this will be pursued in the next section.

Block Multiplication

Definition 2.10 Block Partition of a Matrix

It is often useful to consider matrices whose entries are themselves matrices (called ***blocks****). A matrix viewed in this way is said to be* ***partitioned into blocks****.*

For example, writing a matrix B in the form

$$B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array} \right] \text{ where the } \mathbf{b}_j \text{ are the columns of } B$$

is such a block partition of B. Here is another example.

Consider the matrices

$$A = \left[\begin{array}{rr|rrr} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array} \right] = \left[\begin{array}{cc} I_2 & 0_{23} \\ P & Q \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{rr} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array} \right] = \left[\begin{array}{c} X \\ Y \end{array} \right]$$

where the blocks have been labelled as indicated. This is a natural way to partition A into blocks in view of the blocks I_2 and 0_{23} that occur. This notation is particularly useful when we are multiplying the matrices A and B because the product AB can be computed in block form as follows:

$$AB = \left[\begin{array}{cc} I & 0 \\ P & Q \end{array}\right] \left[\begin{array}{c} X \\ Y \end{array}\right] = \left[\begin{array}{c} IX+0Y \\ PX+QY \end{array}\right] = \left[\begin{array}{c} X \\ PX+QY \end{array}\right] = \left[\begin{array}{rr} 4 & -2 \\ 5 & 6 \\ \hline 30 & 8 \\ 8 & 27 \end{array}\right]$$

This is easily checked to be the product AB, computed in the conventional manner.

In other words, *we can compute the product AB by ordinary matrix multiplication, using blocks as entries*. The only requirement is that the blocks be **compatible**. That is, *the sizes of the blocks must be such that all (matrix) products of blocks that occur make sense*. This means that the number of columns in each block of A must equal the number of rows in the corresponding block of B.

Theorem 2.3.4: Block Multiplication

If matrices A and B are partitioned compatibly into blocks, the product AB can be computed by matrix multiplication using blocks as entries.

We omit the proof.

We have been using two cases of block multiplication. If $B = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right]$ is a matrix where the $\mathbf{b}_j$ are the columns of B, and if the matrix product AB is defined, then we have

$$AB = A\left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{array}\right]$$

This is Definition 2.9 and is a block multiplication where $A = [A]$ has only one block. As another illustration,

$$B\mathbf{x} = \left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_k \end{array}\right] = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_k\mathbf{b}_k$$

where $\mathbf{x}$ is any $k \times 1$ column matrix (this is Definition 2.5).

It is not our intention to pursue block multiplication in detail here. However, we give one more example because it will be used below.

Theorem 2.3.5

Suppose matrices $A = \left[\begin{array}{cc} B & X \\ 0 & C \end{array}\right]$ and $A_1 = \left[\begin{array}{cc} B_1 & X_1 \\ 0 & C_1 \end{array}\right]$ are partitioned as shown where B and B_1 are square matrices of the same size, and C and C_1 are also square of the same size. These are compatible partitionings and block multiplication gives

$$AA_1 = \left[\begin{array}{cc} B & X \\ 0 & C \end{array}\right] \left[\begin{array}{cc} B_1 & X_1 \\ 0 & C_1 \end{array}\right] = \left[\begin{array}{cc} BB_1 & BX_1 + XC_1 \\ 0 & CC_1 \end{array}\right]$$

Example 2.3.12

Obtain a formula for A^k where $A = \left[\begin{array}{cc} I & X \\ 0 & 0 \end{array} \right]$ is square and I is an identity matrix.

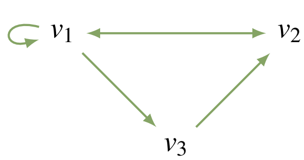
Solution. We have $A^2 = \left[\begin{array}{cc} I & X \\ 0 & 0 \end{array} \right] \left[\begin{array}{cc} I & X \\ 0 & 0 \end{array} \right] = \left[\begin{array}{cc} I^2 & IX + X0 \\ 0 & 0^2 \end{array} \right] = \left[\begin{array}{cc} I & X \\ 0 & 0 \end{array} \right] = A$. Hence $A^3 = AA^2 = AA = A^2 = A$. Continuing in this way, we see that $A^k = A$ for every $k \geq 1$.

Block multiplication has theoretical uses as we shall see. However, it is also useful in computing products of matrices in a computer with limited memory capacity. The matrices are partitioned into blocks in such a way that each product of blocks can be handled. Then the blocks are stored in auxiliary memory and their products are computed one by one.

Directed Graphs

The study of directed graphs illustrates how matrix multiplication arises in ways other than the study of linear equations or matrix transformations.

A **directed graph** consists of a set of points (called **vertices**) connected by arrows (called **edges**). For example, the vertices could represent cities and the edges available flights. If the graph has n vertices $v_1, v_2, \dots, v_n$, the **adjacency** matrix $A = \left[a_{ij} \right]$ is the $n \times n$ matrix whose (i, j)-entry a_{ij} is 1 if there is an edge from v_j to v_i (note the order), and zero otherwise. For example, the adjacency matrix of the directed graph shown is $A = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right]$.



A **path of length** r (or an r-**path**) from vertex j to vertex i is a sequence of r edges leading from v_j to v_i. Thus $v_1 \to v_2 \to v_1 \to v_1 \to v_3$ is a 4-path from v_1 to v_3 in the given graph. The edges are just the paths of length 1, so the (i, j)-entry a_{ij} of the adjacency matrix A is the number of 1-paths from v_j to v_i. This observation has an important extension:

Theorem 2.3.6

If A is the adjacency matrix of a directed graph with n vertices, then the (i, j)-entry of A^r is the number of r-paths $v_j \to v_i$.

As an illustration, consider the adjacency matrix A in the graph shown. Then

$$A = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right], \quad A^2 = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right], \quad \text{and} \quad A^3 = \left[\begin{array}{ccc} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{array} \right]$$

Hence, since the $(2, 1)$-entry of A^2 is 2, there are two 2-paths $v_1 \to v_2$ (in fact they are $v_1 \to v_1 \to v_2$ and $v_1 \to v_3 \to v_2$). Similarly, the $(2, 3)$-entry of A^2 is zero, so there are *no* 2-paths $v_3 \to v_2$, as the reader can verify. The fact that no entry of A^3 is zero shows that it is possible to go from any vertex to any other vertex in exactly three steps.

To see why Theorem 2.3.6 is true, observe that it asserts that

$$\text{the } (i,\ j)\text{-entry of } A^r \text{ equals the number of } r\text{-paths } v_j \to v_i \tag{2.7}$$

holds for each $r \geq 1$. We proceed by induction on r (see Appendix C). The case $r = 1$ is the definition of the adjacency matrix. So assume inductively that (2.7) is true for some $r \geq 1$; we must prove that (2.7) also holds for $r+1$. But every $(r+1)$-path $v_j \to v_i$ is the result of an r-path $v_j \to v_k$ for some k, followed by a 1-path $v_k \to v_i$. Writing $A = \left[a_{ij}\right]$ and $A^r = \left[b_{ij}\right]$, there are b_{kj} paths of the former type (by induction) and a_{ik} of the latter type, and so there are $a_{ik}b_{kj}$ such paths in all. Summing over k, this shows that there are

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad (r+1)\text{-paths } v_j \to v_i$$

But this sum is the dot product of the ith row $\left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right]$ of A with the jth column $\left[\begin{array}{cccc} b_{1j} & b_{2j} & \cdots & b_{nj} \end{array}\right]^T$ of A^r. As such, it is the $(i,\ j)$-entry of the matrix product $A^rA = A^{r+1}$. This shows that (2.7) holds for $r+1$, as required.

Exercises for 2.3

Exercise 2.3.1 Compute the following matrix products.

a. $\left[\begin{array}{rr} 1 & 3 \\ 0 & -2 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ 0 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 2 & 0 & 4 \end{array}\right]\left[\begin{array}{rrr} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{array}\right]$

c. $\left[\begin{array}{rrr} 5 & 0 & -7 \\ 1 & 5 & 9 \end{array}\right]\left[\begin{array}{r} 3 \\ 1 \\ -1 \end{array}\right]$

d. $\left[\begin{array}{rrr} 1 & 3 & -3 \end{array}\right]\left[\begin{array}{rr} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{array}\right]$

e. $\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{rr} 3 & -2 \\ 5 & -7 \\ 9 & 7 \end{array}\right]$

f. $\left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right]\left[\begin{array}{r} 2 \\ 1 \\ -8 \end{array}\right]$

g. $\left[\begin{array}{r} 2 \\ 1 \\ -7 \end{array}\right]\left[\begin{array}{rrr} 1 & -1 & 3 \end{array}\right]$

h. $\left[\begin{array}{rr} 3 & 1 \\ 5 & 2 \end{array}\right]\left[\begin{array}{rr} 2 & -1 \\ -5 & 3 \end{array}\right]$

i. $\left[\begin{array}{rrr} 2 & 3 & 1 \\ 5 & 7 & 4 \end{array}\right]\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]$

j. $\left[\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right]\left[\begin{array}{ccc} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{array}\right]$

Exercise 2.3.2 In each of the following cases, find all possible products A^2, AB, AC, and so on.

a. $A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ -1 & 0 & 0 \end{array}\right]$, $B = \left[\begin{array}{rr} 1 & -2 \\ \frac{1}{2} & 3 \end{array}\right]$, $C = \left[\begin{array}{rr} -1 & 0 \\ 2 & 5 \\ 0 & 5 \end{array}\right]$

b. $A = \left[\begin{array}{rrr} 1 & 2 & 4 \\ 0 & 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{rr} -1 & 6 \\ 1 & 0 \end{array}\right]$, $C = \left[\begin{array}{rr} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{array}\right]$

Exercise 2.3.3 Find a, b, a_1, and b_1 if:

a. $\left[\begin{array}{rr} a & b \\ a_1 & b_1 \end{array}\right]\left[\begin{array}{rr} 3 & -5 \\ -1 & 2 \end{array}\right]=\left[\begin{array}{rr} 1 & -1 \\ 2 & 0 \end{array}\right]$

b. $\left[\begin{array}{rr} 2 & 1 \\ -1 & 2 \end{array}\right]\left[\begin{array}{rr} a & b \\ a_1 & b_1 \end{array}\right]=\left[\begin{array}{rr} 7 & 2 \\ -1 & 4 \end{array}\right]$

Exercise 2.3.4 Verify that $A^2-A-6I=0$ if:

a. $\left[\begin{array}{rr} 3 & -1 \\ 0 & -2 \end{array}\right]$ b. $\left[\begin{array}{rr} 2 & 2 \\ 2 & -1 \end{array}\right]$

Exercise 2.3.5

Given $A=\left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right]$, $B=\left[\begin{array}{rrr} 1 & 0 & -2 \\ 3 & 1 & 0 \end{array}\right]$,
$C=\left[\begin{array}{rr} 1 & 0 \\ 2 & 1 \\ 5 & 8 \end{array}\right]$, and $D=\left[\begin{array}{rrr} 3 & -1 & 2 \\ 1 & 0 & 5 \end{array}\right]$, verify the following facts from Theorem 2.3.1.

a. $A(B-D)=AB-AD$ b. $A(BC)=(AB)C$

c. $(CD)^T=D^TC^T$

Exercise 2.3.6 Let A be a 2×2 matrix.

a. If A commutes with $\left[\begin{array}{rr} 0 & 1 \\ 0 & 0 \end{array}\right]$, show that $A=\left[\begin{array}{rr} a & b \\ 0 & a \end{array}\right]$ for some a and b.

b. If A commutes with $\left[\begin{array}{rr} 0 & 0 \\ 1 & 0 \end{array}\right]$, show that $A=\left[\begin{array}{rr} a & 0 \\ c & a \end{array}\right]$ for some a and c.

c. Show that A commutes with *every* 2×2 matrix if and only if $A=\left[\begin{array}{rr} a & 0 \\ 0 & a \end{array}\right]$ for some a.

Exercise 2.3.7

a. If A^2 can be formed, what can be said about the size of A?

b. If AB and BA can both be formed, describe the sizes of A and B.

c. If ABC can be formed, A is 3×3, and C is 5×5, what size is B?

Exercise 2.3.8

a. Find two 2×2 matrices A such that $A^2=0$.

b. Find three 2×2 matrices A such that (i) $A^2=I$; (ii) $A^2=A$.

c. Find 2×2 matrices A and B such that $AB=0$ but $BA\neq 0$.

Exercise 2.3.9 Write $P=\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$, and let A be $3\times n$ and B be $m\times 3$.

a. Describe PA in terms of the rows of A.

b. Describe BP in terms of the columns of B.

Exercise 2.3.10 Let A, B, and C be as in Exercise 2.3.5. Find the $(3,\ 1)$-entry of CAB using exactly six numerical multiplications.

Exercise 2.3.11 Compute AB, using the indicated block partitioning.

$$A=\left[\begin{array}{rr|rr} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad B=\left[\begin{array}{rr|r} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

Exercise 2.3.12 In each case give formulas for all powers A, A^2, A^3, $\dots$ of A using the block decomposition indicated.

a. $A=\left[\begin{array}{r|rr} 1 & 0 & 0 \\ \hline 1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right]$

b. $A=\left[\begin{array}{rr|rr} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Exercise 2.3.13 Compute the following using block multiplication (all blocks are $k\times k$).

a. $\left[\begin{array}{rr} I & X \\ -Y & I \end{array}\right]\left[\begin{array}{rr} I & 0 \\ Y & I \end{array}\right]$ b. $\left[\begin{array}{rr} I & X \\ 0 & I \end{array}\right]\left[\begin{array}{rr} I & -X \\ 0 & I \end{array}\right]$

c. $\left[\begin{array}{rr} I & X \end{array}\right]\left[\begin{array}{rr} I & X \end{array}\right]^T$ d. $\left[\begin{array}{rr} I & X^T \end{array}\right]\left[\begin{array}{rr} -X & I \end{array}\right]^T$

e. $\left[\begin{array}{rr} I & X \\ 0 & -I \end{array}\right]^n$ any $n\geq 1$

f. $\left[\begin{array}{rr} 0 & X \\ I & 0 \end{array}\right]^n$ any $n\geq 1$

Exercise 2.3.14 Let A denote an $m \times n$ matrix.

a. If $AX = 0$ for every $n \times 1$ matrix X, show that $A = 0$.

b. If $YA = 0$ for every $1 \times m$ matrix Y, show that $A = 0$.

Exercise 2.3.15

a. If $U = \left[\begin{array}{rr} 1 & 2 \\ 0 & -1 \end{array} \right]$, and $AU = 0$, show that $A = 0$.

b. Let U be such that $AU = 0$ implies that $A = 0$. If $PU = QU$, show that $P = Q$.

Exercise 2.3.16 Simplify the following expressions where A, B, and C represent matrices.

a. $A(3B - C) + (A - 2B)C + 2B(C + 2A)$

b. $A(B + C - D) + B(C - A + D) - (A + B)C + (A - B)D$

c. $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$

d. $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

Exercise 2.3.17 If $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$ where $a \neq 0$, show that A factors in the form $A = \left[\begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right] \left[\begin{array}{cc} y & z \\ 0 & w \end{array} \right]$.

Exercise 2.3.18 If A and B commute with C, show that the same is true of:

a. $A + B$

b. kA, k any scalar

Exercise 2.3.19 If A is any matrix, show that both AA^T and A^TA are symmetric.

Exercise 2.3.20 If A and B are symmetric, show that AB is symmetric if and only if $AB = BA$.

Exercise 2.3.21 If A is a 2×2 matrix, show that $A^TA = AA^T$ if and only if A is symmetric or $A = \left[\begin{array}{rr} a & b \\ -b & a \end{array} \right]$ for some a and b.

Exercise 2.3.22

a. Find all symmetric 2×2 matrices A such that $A^2 = 0$.

b. Repeat (a) if A is 3×3.

c. Repeat (a) if A is $n \times n$.

Exercise 2.3.23 Show that there exist no 2×2 matrices A and B such that $AB - BA = I$. [*Hint*: Examine the (1, 1)- and (2, 2)-entries.]

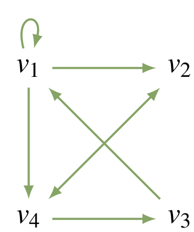
Exercise 2.3.24 Let B be an $n \times n$ matrix. Suppose $AB = 0$ for some nonzero $m \times n$ matrix A. Show that no $n \times n$ matrix C exists such that $BC = I$.

Exercise 2.3.25 An autoparts manufacturer makes fenders, doors, and hoods. Each requires assembly and packaging carried out at factories: Plant 1, Plant 2, and Plant 3. Matrix A below gives the number of hours for assembly and packaging, and matrix B gives the hourly rates at the three plants. Explain the meaning of the (3, 2)-entry in the matrix AB. Which plant is the most economical to operate? Give reasons.

$$\begin{array}{l} \\ \text{Fenders} \\ \text{Doors} \\ \text{Hoods} \end{array} \begin{array}{c} \begin{array}{cc} \text{Assembly} & \text{Packaging} \end{array} \\ \left[\begin{array}{cc} 12 & 2 \\ 21 & 3 \\ 10 & 2 \end{array} \right] \end{array} = A$$

$$\begin{array}{l} \\ \text{Assembly} \\ \text{Packaging} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} \end{array} \\ \left[\begin{array}{ccc} 21 & 18 & 20 \\ 14 & 10 & 13 \end{array} \right] \end{array} = B$$

Exercise 2.3.26 For the directed graph below, find the adjacency matrix A, compute A^3, and determine the number of paths of length 3 from v_1 to v_4 and from v_2 to v_3.



Exercise 2.3.27 In each case either show the statement is true, or give an example showing that it is false.

a. If $A^2 = I$, then $A = I$.

b. If $AJ = A$, then $J = I$.

c. If A is square, then $(A^T)^3 = (A^3)^T$.

d. If A is symmetric, then $I + A$ is symmetric.

e. If $AB = AC$ and $A \neq 0$, then $B = C$.

f. If $A \neq 0$, then $A^2 \neq 0$.

g. If A has a row of zeros, so also does BA for all B.

h. If A commutes with $A+B$, then A commutes with B.

i. If B has a column of zeros, so also does AB.

j. If AB has a column of zeros, so also does B.

k. If A has a row of zeros, so also does AB.

l. If AB has a row of zeros, so also does A.

Exercise 2.3.28

a. If A and B are 2×2 matrices whose rows sum to 1, show that the rows of AB also sum to 1.

b. Repeat part (a) for the case where A and B are $n \times n$.

Exercise 2.3.29 Let A and B be $n \times n$ matrices for which the systems of equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Show that the system $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Exercise 2.3.30 The **trace** of a square matrix A, denoted $\text{tr}\, A$, is the sum of the elements on the main diagonal of A. Show that, if A and B are $n \times n$ matrices:

a. $\text{tr}\,(A+B) = \text{tr}\, A + \text{tr}\, B$.

b. $\text{tr}\,(kA) = k\, \text{tr}\,(A)$ for any number k.

c. $\text{tr}\,(A^T) = \text{tr}\,(A)$.

d. $\text{tr}\,(AB) = \text{tr}\,(BA)$.

e. $\text{tr}\,(AA^T)$ is the sum of the squares of all entries of A.

Exercise 2.3.31 Show that $AB - BA = I$ is impossible.

[*Hint*: See the preceding exercise.]

Exercise 2.3.32 A square matrix P is called an **idempotent** if $P^2 = P$. Show that:

a. 0 and I are idempotents.

b. $\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$, $\left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}\right]$, and $\frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$, are idempotents.

c. If P is an idempotent, so is $I - P$. Show further that $P(I - P) = 0$.

d. If P is an idempotent, so is P^T.

e. If P is an idempotent, so is $Q = P + AP - PAP$ for any square matrix A (of the same size as P).

f. If A is $n \times m$ and B is $m \times n$, and if $AB = I_n$, then BA is an idempotent.

Exercise 2.3.33 Let A and B be $n \times n$ **diagonal matrices** (all entries off the main diagonal are zero).

a. Show that AB is diagonal and $AB = BA$.

b. Formulate a rule for calculating XA if X is $m \times n$.

c. Formulate a rule for calculating AY if Y is $n \times k$.

Exercise 2.3.34 If A and B are $n \times n$ matrices, show that:

a. $AB = BA$ if and only if

$$(A+B)^2 = A^2 + 2AB + B^2$$

b. $AB = BA$ if and only if

$$(A+B)(A-B) = (A-B)(A+B)$$

Exercise 2.3.35 In Theorem 2.3.3, prove

a. part 3;

b. part 5.

2.4 Matrix Inverses

Three basic operations on matrices, addition, multiplication, and subtraction, are analogs for matrices of the same operations for numbers. In this section we introduce the matrix analog of numerical division.

To begin, consider how a numerical equation $ax = b$ is solved when a and b are known numbers. If $a = 0$, there is no solution (unless $b = 0$). But if $a \neq 0$, we can multiply both sides by the inverse $a^{-1} = \frac{1}{a}$ to obtain the solution $x = a^{-1}b$. Of course multiplying by a^{-1} is just dividing by a, and the property of a^{-1} that makes this work is that $a^{-1}a = 1$. Moreover, we saw in Section 2.2 that the role that 1 plays in arithmetic is played in matrix algebra by the identity matrix I. This suggests the following definition.

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix***.[8]

Example 2.4.1

Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. Compute AB and BA.

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence $AB = I = BA$, so B is indeed an inverse of A.

Example 2.4.2

Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse.

Solution. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote an arbitrary 2×2 matrix. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a+3c & b+3d \end{bmatrix}$$

so AB has a row of zeros. Hence AB cannot equal I for any B.

[8] Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B could exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m < n$ there exists a nonzero column $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ (by Theorem 1.3.1), so $\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\mathbf{0}) = \mathbf{0}$, a contradiction. Hence $m \geq n$. Similarly, the condition $AB = I_m$ implies that $n \geq m$. Hence $m = n$ so A is square.

The argument in Example 2.4.2 shows that no zero matrix has an inverse. But Example 2.4.2 also shows that, unlike arithmetic, *it is possible for a nonzero matrix to have no inverse*. However, if a matrix *does* have an inverse, it has only one.

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Proof. Since B and C are both inverses of A, we have $CA = I = AB$. Hence

$$B = IB = (CA)B = C(AB) = CI = C$$

□

If A is an invertible matrix, the (unique) inverse of A is denoted A^{-1}. Hence A^{-1} (when it exists) is a square matrix of the same size as A with the property that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

These equations characterize A^{-1} in the following sense:

> **Inverse Criterion:** *If somehow a matrix B can be found such that $AB = I$ and $BA = I$, then A is invertible and B is the inverse of A; in symbols, $B = A^{-1}$.*

This is a way to verify that the inverse of a matrix exists. Example 2.4.3 and Example 2.4.4 offer illustrations.

Example 2.4.3

If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}.

Solution. We have $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, and so

$$A^3 = A^2A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence $A^3 = I$, as asserted. This can be written as $A^2A = I = AA^2$, so it shows that A^2 is the inverse of A. That is, $A^{-1} = A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

The next example presents a useful formula for the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when it exists. To state it, we define the **determinant** $\det A$ and the **adjugate** $\operatorname{adj} A$ of the matrix A as follows:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \text{and} \quad \operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.4.4

If $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$, show that A has an inverse if and only if $\det A \neq 0$, and in this case

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Solution. For convenience, write $e = \det A = ad - bc$ and $B = \operatorname{adj} A = \left[\begin{array}{rr} d & -b \\ -c & a \end{array} \right]$. Then $AB = eI = BA$ as the reader can verify. So if $e \neq 0$, scalar multiplication by $\frac{1}{e}$ gives

$$A(\tfrac{1}{e}B) = I = (\tfrac{1}{e}B)A$$

Hence A is invertible and $A^{-1} = \frac{1}{e}B$. Thus it remains only to show that if A^{-1} exists, then $e \neq 0$. We prove this by showing that assuming $e = 0$ leads to a contradiction. In fact, if $e = 0$, then $AB = eI = 0$, so left multiplication by A^{-1} gives $A^{-1}AB = A^{-1}0$; that is, $IB = 0$, so $B = 0$. But this implies that a, b, c, and d are *all* zero, so $A = 0$, contrary to the assumption that A^{-1} exists.

As an illustration, if $A = \left[\begin{array}{rr} 2 & 4 \\ -3 & 8 \end{array} \right]$ then $\det A = 2 \cdot 8 - 4 \cdot (-3) = 28 \neq 0$. Hence A is invertible and $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{28} \left[\begin{array}{rr} 8 & -4 \\ 3 & 2 \end{array} \right]$, as the reader is invited to verify.

The determinant and adjugate will be defined in Chapter 3 for any square matrix, and the conclusions in Example 2.4.4 will be proved in full generality.

Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations. Recall that a *system* of linear equations can be written as a *single* matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A and $\mathbf{b}$ are known and $\mathbf{x}$ is to be determined. If A is invertible, we multiply each side of the equation on the left by A^{-1} to get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This gives the solution to the system of equations (the reader should verify that $\mathbf{x} = A^{-1}\mathbf{b}$ really does satisfy $A\mathbf{x} = \mathbf{b}$). Furthermore, the argument shows that if $\mathbf{x}$ is *any* solution, then necessarily $\mathbf{x} = A^{-1}\mathbf{b}$, so the solution is unique. Of course the technique works only when the coefficient matrix A has an inverse. This proves Theorem 2.4.2.

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\mathbf{x} = \mathbf{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 2.4.5

Use Example 2.4.4 to solve the system $\left\{ \begin{array}{rl} 5x_1 - 3x_2 = & -4 \\ 7x_1 + 4x_2 = & 8 \end{array} \right.$.

Solution. In matrix form this is $A\mathbf{x} = \mathbf{b}$ where $A = \left[\begin{array}{rr} 5 & -3 \\ 7 & 4 \end{array} \right]$, $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$, and $\mathbf{b} = \left[\begin{array}{r} -4 \\ 8 \end{array} \right]$. Then $\det A = 5 \cdot 4 - (-3) \cdot 7 = 41$, so A is invertible and $A^{-1} = \frac{1}{41} \left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array} \right]$ by Example 2.4.4. Thus Theorem 2.4.2 gives

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{41} \left[\begin{array}{rr} 4 & 3 \\ -7 & 5 \end{array} \right] \left[\begin{array}{r} -4 \\ 8 \end{array} \right] = \frac{1}{41} \left[\begin{array}{r} 8 \\ 68 \end{array} \right]$$

so the solution is $x_1 = \frac{8}{41}$ and $x_2 = \frac{68}{41}$.

An Inversion Method

If a matrix A is $n \times n$ and invertible, it is desirable to have an efficient technique for finding the inverse. The following procedure will be justified in Section 2.5.

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\left[\begin{array}{cc} A & I \end{array} \right] \to \left[\begin{array}{cc} I & A^{-1} \end{array} \right]$$

where the row operations on A and I are carried out simultaneously.

Example 2.4.6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution. Apply elementary row operations to the double matrix

$$\left[\begin{array}{cc} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Given any $n \times n$ matrix A, Theorem 1.2.1 shows that A can be carried by elementary row operations to a matrix R in reduced row-echelon form. If $R = I$, the matrix A is invertible (this will be proved in the next section), so the algorithm produces A^{-1}. If $R \neq I$, then R has a row of zeros (it is square), so no system of linear equations $A\mathbf{x} = \mathbf{b}$ can have a unique solution. But then A is not invertible by Theorem 2.4.2. Hence, the algorithm is effective in the sense conveyed in Theorem 2.4.3.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Properties of Inverses

The following properties of an invertible matrix are used everywhere.

Example 2.4.7: Cancellation Laws

Let A be an invertible matrix. Show that:

1. If $AB = AC$, then $B = C$.
2. If $BA = CA$, then $B = C$.

Solution. Given the equation $AB = AC$, left multiply both sides by A^{-1} to obtain $A^{-1}AB = A^{-1}AC$. Thus $IB = IC$, that is $B = C$. This proves (1) and the proof of (2) is left to the reader.

Properties (1) and (2) in Example 2.4.7 are described by saying that an invertible matrix can be "left cancelled" and "right cancelled", respectively. Note however that "mixed" cancellation does not hold in general: If A is invertible and $AB = CA$, then B and C may *not* be equal, even if both are 2×2. Here is a specific example:

$$A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right], \quad B = \left[\begin{array}{cc} 0 & 0 \\ 1 & 2 \end{array} \right], C = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$

Sometimes the inverse of a matrix is given by a formula. Example 2.4.4 is one illustration; Example 2.4.8 and Example 2.4.9 provide two more. The idea is the *Inverse Criterion*: If a matrix B can be found such that $AB = I = BA$, then A is invertible and $A^{-1} = B$.

Example 2.4.8

If A is an invertible matrix, show that the transpose A^T is also invertible. Show further that the inverse of A^T is just the transpose of A^{-1}; in symbols, $(A^T)^{-1} = (A^{-1})^T$.

Solution. A^{-1} exists (by assumption). Its transpose $(A^{-1})^T$ is the candidate proposed for the inverse of A^T. Using the inverse criterion, we test it as follows:

$$\begin{array}{c} A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \\ (A^{-1})^T A^T = (AA^{-1})^T = I^T = I \end{array}$$

Hence $(A^{-1})^T$ is indeed the inverse of A^T; that is, $(A^T)^{-1} = (A^{-1})^T$.

Example 2.4.9

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. We are given a candidate for the inverse of AB, namely $B^{-1}A^{-1}$. We test it as follows:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence $B^{-1}A^{-1}$ is the inverse of AB; in symbols, $(AB)^{-1} = B^{-1}A^{-1}$.

We now collect several basic properties of matrix inverses for reference.

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. *I is invertible and $I^{-1} = I$.*
2. *If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.*
3. *If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
4. *If $A_1, A_2, \dots, A_k$ are all invertible, so is their product $A_1A_2\cdots A_k$, and*
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. *If A is invertible, so is A^k for any $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*
7. *If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Proof.

1. This is an immediate consequence of the fact that $I^2 = I$.
2. The equations $AA^{-1} = I = A^{-1}A$ show that A is the inverse of A^{-1}; in symbols, $(A^{-1})^{-1} = A$.
3. This is Example 2.4.9.
4. Use induction on k. If $k = 1$, there is nothing to prove, and if $k = 2$, the result is property 3. If $k > 2$, assume inductively that $(A_1A_2\cdots A_{k-1})^{-1} = A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$. We apply this fact together with property 3 as follows:
$$\begin{aligned}[A_1A_2\cdots A_{k-1}A_k]^{-1} &= [(A_1A_2\cdots A_{k-1})A_k]^{-1} \\ &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1} \\ &= A_k^{-1}\left(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}\right)\end{aligned}$$

So the proof by induction is complete.

5. This is property 4 with $A_1 = A_2 = \cdots = A_k = A$.

6. This is left as Exercise 2.4.29.

7. This is Example 2.4.8. □

The reversal of the order of the inverses in properties 3 and 4 of Theorem 2.4.4 is a consequence of the fact that matrix multiplication is not commutative. Another manifestation of this comes when matrix equations are dealt with. If a matrix equation $B = C$ is given, it can be *left-multiplied* by a matrix A to yield $AB = AC$. Similarly, *right-multiplication* gives $BA = CA$. However, we cannot mix the two: If $B = C$, it need *not* be the case that $AB = CA$ even if A is invertible, for example, $A = \left[\begin{array}{rr} 1 & 1 \\ 0 & 1 \end{array}\right]$, $B = \left[\begin{array}{rr} 0 & 0 \\ 1 & 0 \end{array}\right] = C$.

Part 7 of Theorem 2.4.4 together with the fact that $(A^T)^T = A$ gives

Corollary 2.4.1

A square matrix A is invertible if and only if A^T is invertible.

Example 2.4.10

Find A if $(A^T - 2I)^{-1} = \left[\begin{array}{rr} 2 & 1 \\ -1 & 0 \end{array}\right]$.

Solution. By Theorem 2.4.4(2) and Example 2.4.4, we have

$$(A^T - 2I) = \left[\left(A^T - 2I\right)^{-1}\right]^{-1} = \left[\begin{array}{rr} 2 & 1 \\ -1 & 0 \end{array}\right]^{-1} = \left[\begin{array}{rr} 0 & -1 \\ 1 & 2 \end{array}\right]$$

Hence $A^T = 2I + \left[\begin{array}{rr} 0 & -1 \\ 1 & 2 \end{array}\right] = \left[\begin{array}{rr} 2 & -1 \\ 1 & 4 \end{array}\right]$, so $A = \left[\begin{array}{rr} 2 & 1 \\ -1 & 4 \end{array}\right]$ by Theorem 2.4.4(7).

The following important theorem collects a number of conditions all equivalent[9] to invertibility. It will be referred to frequently below.

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*

3. *A can be carried to the identity matrix I_n by elementary row operations.*

[9] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q"). See Appendix B.

4. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*

5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

Proof. We show that each of these conditions implies the next, and that (5) implies (1).

(1) $\Rightarrow$ (2). If A^{-1} exists, then $A\mathbf{x} = \mathbf{0}$ gives $\mathbf{x} = I_n\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

(2) $\Rightarrow$ (3). Assume that (2) is true. Certainly $A \to R$ by row operations where R is a reduced, row-echelon matrix. It suffices to show that $R = I_n$. Suppose that this is not the case. Then R has a row of zeros (being square). Now consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{0}$. Then $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right] \to \left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ is the reduced form, and $\left[\begin{array}{c|c} R & \mathbf{0} \end{array}\right]$ also has a row of zeros. Since R is square there must be at least one nonleading variable, and hence at least one parameter. Hence the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, contrary to (2). So $R = I_n$ after all.

(3) $\Rightarrow$ (4). Consider the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ of the system $A\mathbf{x} = \mathbf{b}$. Using (3), let $A \to I_n$ by a sequence of row operations. Then these same operations carry $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] \to \left[\begin{array}{c|c} I_n & \mathbf{c} \end{array}\right]$ for some column $\mathbf{c}$. Hence the system $A\mathbf{x} = \mathbf{b}$ has a solution (in fact unique) by gaussian elimination. This proves (4).

(4) $\Rightarrow$ (5). Write $I_n = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right]$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n. For each $j = 1, 2, \dots, n$, the system $A\mathbf{x} = \mathbf{e}_j$ has a solution $\mathbf{c}_j$ by (4), so $A\mathbf{c}_j = \mathbf{e}_j$. Now let $C = \left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right]$ be the $n \times n$ matrix with these matrices $\mathbf{c}_j$ as its columns. Then Definition 2.9 gives (5):

$$AC = A\left[\begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{c}_1 & A\mathbf{c}_2 & \cdots & A\mathbf{c}_n \end{array}\right] = \left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = I_n$$

(5) $\Rightarrow$ (1). Assume that (5) is true so that $AC = I_n$ for some matrix C. Then $C\mathbf{x} = 0$ implies $\mathbf{x} = \mathbf{0}$ (because $\mathbf{x} = I_n\mathbf{x} = AC\mathbf{x} = A\mathbf{0} = \mathbf{0}$). Thus condition (2) holds for the matrix C rather than A. Hence the argument above that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) (with A replaced by C) shows that a matrix C' exists such that $CC' = I_n$. But then

$$A = AI_n = A(CC') = (AC)C' = I_nC' = C'$$

Thus $CA = CC' = I_n$ which, together with $AC = I_n$, shows that C is the inverse of A. This proves (1). □

The proof of (5) $\Rightarrow$ (1) in Theorem 2.4.5 shows that if $AC = I$ for square matrices, then necessarily $CA = I$, and hence that C and A are inverses of each other. We record this important fact for reference.

Corollary 2.4.1

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Here is a quick way to remember Corollary 2.4.1. If A is a square matrix, then

1. If $AC = I$ then $C = A^{-1}$.
2. If $CA = I$ then $C = A^{-1}$.

Observe that Corollary 2.4.1 is false if A and C are not square matrices. For example, we have

$$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right]\left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{array}\right] = I_2 \quad \text{but} \quad \left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right] \neq I_3$$

In fact, it is verified in the footnote on page 80 that if $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, then $m = n$ and A and B are (square) inverses of each other.

An $n \times n$ matrix A has rank n if and only if (3) of Theorem 2.4.5 holds. Hence

Corollary 2.4.2

An $n \times n$ matrix A is invertible if and only if rank $A = n$.

Here is a useful fact about inverses of block matrices.

Example 2.4.11

Let $P = \left[\begin{array}{cc} A & X \\ 0 & B \end{array} \right]$ and $Q = \left[\begin{array}{cc} A & 0 \\ Y & B \end{array} \right]$ be block matrices where A is $m \times m$ and B is $n \times n$ (possibly $m \neq n$).

a. Show that P is invertible if and only if A and B are both invertible. In this case, show that

$$P^{-1} = \left[\begin{array}{cc} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{array} \right]$$

b. Show that Q is invertible if and only if A and B are both invertible. In this case, show that

$$Q^{-1} = \left[\begin{array}{cc} A^{-1} & 0 \\ -B^{-1}YA^{-1} & B^{-1} \end{array} \right]$$

Solution. We do (a.) and leave (b.) for the reader.

a. If A^{-1} and B^{-1} both exist, write $R = \left[\begin{array}{cc} A^{-1} & -A^{-1}XB^{-1} \\ 0 & B^{-1} \end{array} \right]$. Using block multiplication, one verifies that $PR = I_{m+n} = RP$, so P is invertible, and $P^{-1} = R$. Conversely, suppose that P is invertible, and write $P^{-1} = \left[\begin{array}{cc} C & V \\ W & D \end{array} \right]$ in block form, where C is $m \times m$ and D is $n \times n$.

Then the equation $PP^{-1} = I_{n+m}$ becomes

$$\left[\begin{array}{cc} A & X \\ 0 & B \end{array} \right] \left[\begin{array}{cc} C & V \\ W & D \end{array} \right] = \left[\begin{array}{cc} AC + XW & AV + XD \\ BW & BD \end{array} \right] = I_{m+n} = \left[\begin{array}{cc} I_m & 0 \\ 0 & I_n \end{array} \right]$$

using block notation. Equating corresponding blocks, we find

$$AC + XW = I_m, \quad BW = 0, \quad \text{and } BD = I_n$$

Hence B is invertible because $BD = I_n$ (by Corollary 2.4.1), then $W = 0$ because $BW = 0$, and finally, $AC = I_m$ (so A is invertible, again by Corollary 2.4.1).

Inverses of Matrix Transformations

Let $T = T_A : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by the $n \times n$ matrix A. Since A is square, it may very well be invertible, and this leads to the question:

What does it mean geometrically for T that A is invertible?

To answer this, let $T' = T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$ denote the transformation induced by A^{-1}. Then

$$\left.\begin{aligned} T'\left[T(\mathbf{x})\right] &= A^{-1}\left[A\mathbf{x}\right] = I\mathbf{x} = \mathbf{x} \\ T\left[T'(\mathbf{x})\right] &= A\left[A^{-1}\mathbf{x}\right] = I\mathbf{x} = \mathbf{x} \end{aligned}\right\} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2.8}$$

The first of these equations asserts that, if T carries $\mathbf{x}$ to a vector $T(\mathbf{x})$, then T' carries $T(\mathbf{x})$ right back to $\mathbf{x}$; that is T' "reverses" the action of T. Similarly T "reverses" the action of T'. Conditions (2.8) can be stated compactly in terms of composition:

$$T' \circ T = 1_{\mathbb{R}^n} \quad \text{and} \quad T \circ T' = 1_{\mathbb{R}^n} \tag{2.9}$$

When these conditions hold, we say that the matrix transformation T' is an **inverse** of T, and we have shown that if the matrix A of T is invertible, then T has an inverse (induced by A^{-1}).

The converse is also true: If T has an inverse, then its matrix A must be invertible. Indeed, suppose $S : \mathbb{R}^n \to \mathbb{R}^n$ is any inverse of T, so that $S \circ T = 1_{\mathbb{R}_n}$ and $T \circ S = 1_{\mathbb{R}_n}$. It can be shown that S is also a matrix transformation. If B is the matrix of S, we have

$$BA\mathbf{x} = S\left[T(\mathbf{x})\right] = (S \circ T)(\mathbf{x}) = 1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} = I_n\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

It follows by Theorem 2.2.6 that $BA = I_n$, and a similar argument shows that $AB = I_n$. Hence A is invertible with $A^{-1} = B$. Furthermore, the inverse transformation S has matrix A^{-1}, so $S = T'$ using the earlier notation. This proves the following important theorem.

Theorem 2.4.6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by an $n \times n$ matrix A. Then

A is invertible if and only if T has an inverse.

In this case, T has exactly one inverse (which we denote as T^{-1}), and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the transformation induced by the matrix A^{-1}. In other words

$$(T_A)^{-1} = T_{A^{-1}}$$

The geometrical relationship between T and T^{-1} is embodied in equations (2.8) above:

$$T^{-1}\left[T(\mathbf{x})\right] = \mathbf{x} \quad \text{and} \quad T\left[T^{-1}(\mathbf{x})\right] = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

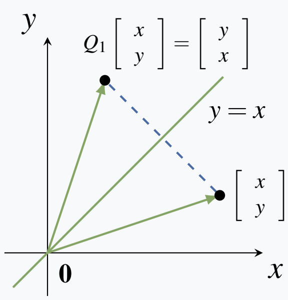
These equations are called the **fundamental identities** relating T and T^{-1}. Loosely speaking, they assert that each of T and T^{-1} "reverses" or "undoes" the action of the other.

This geometric view of the inverse of a linear transformation provides a new way to find the inverse of a matrix A. More precisely, if A is an invertible matrix, we proceed as follows:

1. *Let T be the linear transformation induced by A.*
2. *Obtain the linear transformation T^{-1} which "reverses" the action of T.*
3. *Then A^{-1} is the matrix of T^{-1}.*

Here is an example.

Example 2.4.12



Find the inverse of $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$ by viewing it as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution. If $\mathbf{x} = \left[\begin{array}{c} x \\ y \end{array}\right]$ the vector $A\mathbf{x} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} y \\ x \end{array}\right]$ is the result of reflecting $\mathbf{x}$ in the line $y = x$ (see the diagram). Hence, if $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection in the line $y = x$, then A is the matrix of Q_1. Now observe that Q_1 reverses itself because reflecting a vector $\mathbf{x}$ twice results in $\mathbf{x}$. Consequently $Q_1^{-1} = Q_1$. Since A^{-1} is the matrix of Q_1^{-1} and A is the matrix of Q, it follows that $A^{-1} = A$. Of course this conclusion is clear by simply observing directly that $A^2 = I$, but the geometric method can often work where these other methods may be less straightforward.

Exercises for 2.4

Exercise 2.4.1 In each case, show that the matrices are inverses of each other.

a. $\left[\begin{array}{cc} 3 & 5 \\ 1 & 2 \end{array}\right], \left[\begin{array}{rr} 2 & -5 \\ -1 & 3 \end{array}\right]$

b. $\left[\begin{array}{rr} 3 & 0 \\ 1 & -4 \end{array}\right], \frac{1}{2}\left[\begin{array}{rr} 4 & 0 \\ 1 & -3 \end{array}\right]$

c. $\left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{array}\right], \left[\begin{array}{rrr} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{array}\right]$

d. $\left[\begin{array}{cc} 3 & 0 \\ 0 & 5 \end{array}\right], \left[\begin{array}{cc} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{array}\right]$

Exercise 2.4.2 Find the inverse of each of the following matrices.

a. $\left[\begin{array}{rr} 1 & -1 \\ -1 & 3 \end{array}\right]$

b. $\left[\begin{array}{cc} 4 & 1 \\ 3 & 2 \end{array}\right]$

c. $\left[\begin{array}{rrr} 1 & 0 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrr} 1 & -1 & 2 \\ -5 & 7 & -11 \\ -2 & 3 & -5 \end{array}\right]$

e. $\left[\begin{array}{ccc} 3 & 5 & 0 \\ 3 & 7 & 1 \\ 1 & 2 & 1 \end{array}\right]$

f. $\left[\begin{array}{rrr} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{array}\right]$

g. $\left[\begin{array}{ccc} 2 & 4 & 1 \\ 3 & 3 & 2 \\ 4 & 1 & 4 \end{array}\right]$

h. $\left[\begin{array}{rrr} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{array}\right]$

i. $\left[\begin{array}{rrr} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{array}\right]$

j. $\left[\begin{array}{rrrr} -1 & 4 & 5 & 2 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$

k. $\left[\begin{array}{rrrr} 1 & 0 & 7 & 5 \\ 0 & 1 & 3 & 6 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 1 \end{array}\right]$ l. $\left[\begin{array}{rrrrr} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Exercise 2.4.3 In each case, solve the systems of equations by finding the inverse of the coefficient matrix.

a. $\begin{array}{l} 3x - \ \ y = 5 \\ 2x + 2y = 1 \end{array}$ b. $\begin{array}{l} 2x - 3y = 0 \\ \ \ x - 4y = 1 \end{array}$

c. $\begin{array}{rcl} x + \ \ y + 2z &=& 5 \\ x + \ \ y + \ \ z &=& 0 \\ x + 2y + 4z &=& -2 \end{array}$ d. $\begin{array}{rcl} x + 4y + 2z &=& 1 \\ 2x + 3y + 3z &=& -1 \\ 4x + \ \ y + 4z &=& 0 \end{array}$

Exercise 2.4.4 Given $A^{-1} = \left[\begin{array}{rrr} 1 & -1 & 3 \\ 2 & 0 & 5 \\ -1 & 1 & 0 \end{array}\right]$:

a. Solve the system of equations $A\mathbf{x} = \left[\begin{array}{r} 1 \\ -1 \\ 3 \end{array}\right]$.

b. Find a matrix B such that
$AB = \left[\begin{array}{rrr} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}\right]$.

c. Find a matrix C such that
$CA = \left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 1 & 1 \end{array}\right]$.

Exercise 2.4.5 Find A when

a. $(3A)^{-1} = \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right]$ b. $(2A)^T = \left[\begin{array}{rr} 1 & -1 \\ 2 & 3 \end{array}\right]^{-1}$

c. $(I + 3A)^{-1} = \left[\begin{array}{rr} 2 & 0 \\ 1 & -1 \end{array}\right]$

d. $(I - 2A^T)^{-1} = \left[\begin{array}{rr} 2 & 1 \\ 1 & 1 \end{array}\right]$

e. $\left(A\left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right]\right)^{-1} = \left[\begin{array}{rr} 2 & 3 \\ 1 & 1 \end{array}\right]$

f. $\left(\left[\begin{array}{rr} 1 & 0 \\ 2 & 1 \end{array}\right]A\right)^{-1} = \left[\begin{array}{rr} 1 & 0 \\ 2 & 2 \end{array}\right]$

g. $\left(A^T - 2I\right)^{-1} = 2\left[\begin{array}{rr} 1 & 1 \\ 2 & 3 \end{array}\right]$

h. $\left(A^{-1} - 2I\right)^T = -2\left[\begin{array}{rr} 1 & 1 \\ 1 & 0 \end{array}\right]$

Exercise 2.4.6 Find A when:

a. $A^{-1} = \left[\begin{array}{rrr} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{array}\right]$ b. $A^{-1} = \left[\begin{array}{rrr} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{array}\right]$

Exercise 2.4.7 Given $\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{rrr} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]$ and $\left[\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}\right] = \left[\begin{array}{rrr} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]$, express the variables x_1, x_2, and x_3 in terms of z_1, z_2, and z_3.

Exercise 2.4.8

a. In the system $\begin{array}{l} 3x + 4y = 7 \\ 4x + 5y = 1 \end{array}$, substitute the new variables x' and y' given by $\begin{array}{l} x = -5x' + 4y' \\ y = \ \ \ 4x' - 3y' \end{array}$. Then find x and y.

b. Explain part (a) by writing the equations as $A\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 7 \\ 1 \end{array}\right]$ and $\left[\begin{array}{c} x \\ y \end{array}\right] = B\left[\begin{array}{c} x' \\ y' \end{array}\right]$. What is the relationship between A and B?

Exercise 2.4.9 In each case either prove the assertion or give an example showing that it is false.

a. If $A \neq 0$ is a square matrix, then A is invertible.

b. If A and B are both invertible, then $A + B$ is invertible.

c. If A and B are both invertible, then $(A^{-1}B)^T$ is invertible.

d. If $A^4 = 3I$, then A is invertible.

e. If $A^2 = A$ and $A \neq 0$, then A is invertible.

f. If $AB = B$ for some $B \neq 0$, then A is invertible.

g. If A is invertible and skew symmetric ($A^T = -A$), the same is true of A^{-1}.

h. If A^2 is invertible, then A is invertible.

i. If $AB = I$, then A and B commute.

Exercise 2.4.10

a. If A, B, and C are square matrices and $AB = I$, $I = CA$, show that A is invertible and $B = C = A^{-1}$.

b. If $C^{-1} = A$, find the inverse of C^T in terms of A.

Exercise 2.4.11 Suppose $CA = I_m$, where C is $m \times n$ and A is $n \times m$. Consider the system $A\mathbf{x} = \mathbf{b}$ of n equations in m variables.

a. Show that this system has a unique solution CB if it is consistent.

b. If $C = \left[\begin{array}{rrr} 0 & -5 & 1 \\ 3 & 0 & -1 \end{array}\right]$ and $A = \left[\begin{array}{rr} 2 & -3 \\ 1 & -2 \\ 6 & -10 \end{array}\right]$, find $\mathbf{x}$ (if it exists) when (i) $\mathbf{b} = \left[\begin{array}{r} 1 \\ 0 \\ 3 \end{array}\right]$; and (ii) $\mathbf{b} = \left[\begin{array}{r} 7 \\ 4 \\ 22 \end{array}\right]$.

Exercise 2.4.12 Verify that $A = \left[\begin{array}{rr} 1 & -1 \\ 0 & 2 \end{array}\right]$ satisfies $A^2 - 3A + 2I = 0$, and use this fact to show that $A^{-1} = \frac{1}{2}(3I - A)$.

Exercise 2.4.13 Let $Q = \left[\begin{array}{rrrr} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{array}\right]$. Compute QQ^T and so find Q^{-1} if $Q \neq 0$.

Exercise 2.4.14 Let $U = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$. Show that each of U, $-U$, and $-I_2$ is its own inverse and that the product of any two of these is the third.

Exercise 2.4.15 Consider $A = \left[\begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array}\right]$, $B = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$, $C = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{array}\right]$. Find the inverses by computing (a) A^6; (b) B^4; and (c) C^3.

Exercise 2.4.16 Find the inverse of $\left[\begin{array}{ccc} 1 & 0 & 1 \\ c & 1 & c \\ 3 & c & 2 \end{array}\right]$ in terms of c.

Exercise 2.4.17 If $c \neq 0$, find the inverse of $\left[\begin{array}{rrr} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 2 & c \end{array}\right]$ in terms of c.

Exercise 2.4.18 Show that A has no inverse when:

a. A has a row of zeros.

b. A has a column of zeros.

c. each row of A sums to 0. [*Hint*: Theorem 2.4.5(2).]

d. each column of A sums to 0.

[*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

Exercise 2.4.19 Let A denote a square matrix.

a. Let $YA = 0$ for some matrix $Y \neq 0$. Show that A has no inverse. [*Hint*: Corollary 2.4.1, Theorem 2.4.4.]

b. Use part (a) to show that (i) $\left[\begin{array}{rrr} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right]$; and (ii) $\left[\begin{array}{rrr} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{array}\right]$ have no inverse.

[*Hint*: For part (ii) compare row 3 with the difference between row 1 and row 2.]

Exercise 2.4.20 If A is invertible, show that

a. $A^2 \neq 0$.

b. $A^k \neq 0$ for all $k = 1, 2, \dots$.

Exercise 2.4.21 Suppose $AB = 0$, where A and B are square matrices. Show that:

a. If one of A and B has an inverse, the other is zero.

b. It is impossible for both A and B to have inverses.

c. $(BA)^2 = 0$.

Exercise 2.4.22 Find the inverse of the x-expansion in Example 2.2.16 and describe it geometrically.

Exercise 2.4.23 Find the inverse of the shear transformation in Example 2.2.17 and describe it geometrically.

Exercise 2.4.24 In each case assume that A is a square matrix that satisfies the given condition. Show that A is invertible and find a formula for A^{-1} in terms of A.

a. $A^3 - 3A + 2I = 0$.

b. $A^4 + 2A^3 - A - 4I = 0$.

Exercise 2.4.25 Let A and B denote $n \times n$ matrices.

a. If A and AB are invertible, show that B is invertible using only (2) and (3) of Theorem 2.4.4.

b. If AB is invertible, show that both A and B are invertible using Theorem 2.4.5.

Exercise 2.4.26 In each case find the inverse of the matrix A using Example 2.4.11.

a. $A = \left[\begin{array}{rrr} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{array}\right]$ b. $A = \left[\begin{array}{rrr} 3 & 1 & 0 \\ 5 & 2 & 0 \\ 1 & 3 & -1 \end{array}\right]$

c. $A = \left[\begin{array}{rrrr} 3 & 4 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 1 & 4 \end{array}\right]$

d. $A = \left[\begin{array}{rrrr} 2 & 1 & 5 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

Exercise 2.4.27 If A and B are invertible symmetric matrices such that $AB = BA$, show that A^{-1}, AB, AB^{-1}, and $A^{-1}B^{-1}$ are also invertible and symmetric.

Exercise 2.4.28 Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix.

a. If $A^2 = 0$, verify that $(I - A)^{-1} = I + A$.

b. If $A^3 = 0$, verify that $(I - A)^{-1} = I + A + A^2$.

c. Find the inverse of $\left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]$.

d. If $A^n = 0$, find the formula for $(I - A)^{-1}$.

Exercise 2.4.29 Prove property 6 of Theorem 2.4.4: If A is invertible and $a \neq 0$, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$

Exercise 2.4.30 Let A, B, and C denote $n \times n$ matrices. Using only Theorem 2.4.4, show that:

a. If A, C, and ABC are all invertible, B is invertible.

b. If AB and BA are both invertible, A and B are both invertible.

Exercise 2.4.31 Let A and B denote invertible $n \times n$ matrices.

a. If $A^{-1} = B^{-1}$, does it mean that $A = B$? Explain.

b. Show that $A = B$ if and only if $A^{-1}B = I$.

Exercise 2.4.32 Let A, B, and C be $n \times n$ matrices, with A and B invertible. Show that

a. If A commutes with C, then A^{-1} commutes with C.

b. If A commutes with B, then A^{-1} commutes with B^{-1}.

Exercise 2.4.33 Let A and B be square matrices of the same size.

a. Show that $(AB)^2 = A^2B^2$ if $AB = BA$.

b. If A and B are invertible and $(AB)^2 = A^2B^2$, show that $AB = BA$.

c. If $A = \left[\begin{array}{rr} 1 & 0 \\ 0 & 0 \end{array}\right]$ and $B = \left[\begin{array}{rr} 1 & 1 \\ 0 & 0 \end{array}\right]$, show that $(AB)^2 = A^2B^2$ but $AB \neq BA$.

Exercise 2.4.34 Let A and B be $n \times n$ matrices for which AB is invertible. Show that A and B are both invertible.

Exercise 2.4.35 Consider $A = \left[\begin{array}{rrr} 1 & 3 & -1 \\ 2 & 1 & 5 \\ 1 & -7 & 13 \end{array}\right]$,

$B = \left[\begin{array}{rrr} 1 & 1 & 2 \\ 3 & 0 & -3 \\ -2 & 5 & 17 \end{array}\right]$.

a. Show that A is not invertible by finding a nonzero 1×3 matrix Y such that $YA = 0$.

[*Hint*: Row 3 of A equals 2(row 2) $-$ 3(row 1).]

b. Show that B is not invertible.

[*Hint*: Column 3 = 3(column 2) − column 1.]

Exercise 2.4.36 Show that a square matrix A is invertible if and only if it can be left-cancelled: $AB = AC$ implies $B = C$.

Exercise 2.4.37 If $U^2 = I$, show that $I+U$ is not invertible unless $U = I$.

Exercise 2.4.38

a. If J is the 4×4 matrix with every entry 1, show that $I - \frac{1}{2}J$ is self-inverse and symmetric.

b. If X is $n \times m$ and satisfies $X^T X = I_m$, show that $I_n - 2XX^T$ is self-inverse and symmetric.

Exercise 2.4.39 An $n \times n$ matrix P is called an idempotent if $P^2 = P$. Show that:

a. I is the only invertible idempotent.

b. P is an idempotent if and only if $I - 2P$ is self-inverse.

c. U is self-inverse if and only if $U = I - 2P$ for some idempotent P.

d. $I - aP$ is invertible for any $a \neq 1$, and that $(I - aP)^{-1} = I + \left(\frac{a}{1-a}\right)^P$.

Exercise 2.4.40 If $A^2 = kA$, where $k \neq 0$, show that A is invertible if and only if $A = kI$.

Exercise 2.4.41 Let A and B denote $n \times n$ invertible matrices.

a. Show that $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$.

b. If $A+B$ is also invertible, show that $A^{-1} + B^{-1}$ is invertible and find a formula for $(A^{-1} + B^{-1})^{-1}$.

Exercise 2.4.42 Let A and B be $n \times n$ matrices, and let I be the $n \times n$ identity matrix.

a. Verify that $A(I+BA) = (I+AB)A$ and that $(I+BA)B = B(I+AB)$.

b. If $I+AB$ is invertible, verify that $I+BA$ is also invertible and that $(I+BA)^{-1} = I - B(I+AB)^{-1}A$.

2.5 Elementary Matrices

It is now clear that elementary row operations are important in linear algebra: They are essential in solving linear systems (using the gaussian algorithm) and in inverting a matrix (using the matrix inversion algorithm). It turns out that they can be performed by left multiplying by certain invertible matrices. These matrices are the subject of this section.

Definition 2.12 Elementary Matrices

An $n \times n$ matrix E is called an ***elementary matrix*** *if it can be obtained from the identity matrix I_n by a single elementary row operation (called the operation* ***corresponding*** *to E). We say that E is of type I, II, or III if the operation is of that type (see Definition 1.2).*

Hence

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

are elementary of types I, II, and III, respectively, obtained from the 2×2 identity matrix by interchanging rows 1 and 2, multiplying row 2 by 9, and adding 5 times row 2 to row 1.

Suppose now that the matrix $A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$ is left multiplied by the above elementary matrices E_1, E_2, and E_3. The results are:

$$E_1A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} p & q & r \\ a & b & c \end{bmatrix}$$
$$E_2A = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a & b & c \\ 9p & 9q & 9r \end{bmatrix}$$
$$E_3A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+5p & b+5q & c+5r \\ p & q & r \end{bmatrix}$$

In each case, left multiplying A by the elementary matrix has the *same* effect as doing the corresponding row operation to A. This works in general.

Lemma 2.5.1: [10]

If an elementary row operation is performed on an $m \times n$ matrix A, the result is EA where E is the elementary matrix obtained by performing the same operation on the $m \times m$ identity matrix.

Proof. We prove it for operations of type III; the proofs for types I and II are left as exercises. Let E be the elementary matrix corresponding to the operation that adds k times row p to row $q \neq p$. The proof depends on the fact that each row of EA is equal to the corresponding row of E times A. Let $K_1, K_2, \dots, K_m$ denote the rows of I_m. Then row i of E is K_i if $i \neq q$, while row q of E is $K_q + kK_p$. Hence:

$$\begin{aligned} \text{If } i \neq q \text{ then row } i \text{ of } EA = \quad & K_iA = (\text{row } i \text{ of } A). \\ \text{Row } q \text{ of } EA = (K_q + kK_p)A = \quad & K_qA + k(K_pA) \\ = \quad & (\text{row } q \text{ of } A) \text{ plus } k \text{ (row } p \text{ of } A). \end{aligned}$$

Thus EA is the result of adding k times row p of A to row q, as required. □

The effect of an elementary row operation can be reversed by another such operation (called its inverse) which is also elementary of the same type (see the discussion following (Example 1.1.3). It follows that each elementary matrix E is invertible. In fact, if a row operation on I produces E, then the inverse operation carries E back to I. If F is the elementary matrix corresponding to the inverse operation, this means $FE = I$ (by Lemma 2.5.1). Thus $F = E^{-1}$ and we have proved

Lemma 2.5.2

Every elementary matrix E is invertible, and E^{-1} is also a elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$, $k \neq 0$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q, $q \neq p$

[10] A *lemma* is an auxiliary theorem used in the proof of other theorems.

Note that elementary matrices of type I are self-inverse.

Example 2.5.1

Find the inverse of each of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. E_1, E_2, and E_3 are of type I, II, and III respectively, so the table gives

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}, \quad \text{and} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Inverses and Elementary Matrices

Suppose that an $m \times n$ matrix A is carried to a matrix B (written $A \to B$) by a series of k elementary row operations. Let $E_1, E_2, \dots, E_k$ denote the corresponding elementary matrices. By Lemma 2.5.1, the reduction becomes

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = B$$

In other words,

$$A \to UA = B \quad \text{where } U = E_kE_{k-1}\cdots E_2E_1$$

The matrix $U = E_kE_{k-1}\cdots E_2E_1$ is invertible, being a product of invertible matrices by Lemma 2.5.2. Moreover, U can be computed without finding the E_i as follows: If the above series of operations carrying $A \to B$ is performed on I_m in place of A, the result is $I_m \to UI_m = U$. Hence this series of operations carries the block matrix $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} B & U \end{bmatrix}$. This, together with the above discussion, proves

Theorem 2.5.1

Suppose A is $m \times n$ and $A \to B$ by elementary row operations.

1. *$B = UA$ where U is an $m \times m$ invertible matrix.*
2. *U can be computed by $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} B & U \end{bmatrix}$ using the operations carrying $A \to B$.*
3. *$U = E_kE_{k-1}\cdots E_2E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying A to B.*

Example 2.5.2

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, express the reduced row-echelon form R of A as $R = UA$ where U is invertible.

Solution. Reduce the double matrix $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} R & U \end{bmatrix}$ as follows:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

Hence $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Now suppose that A is invertible. We know that $A \to I$ by Theorem 2.4.5, so taking $B = I$ in Theorem 2.5.1 gives $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & U \end{bmatrix}$ where $I = UA$. Thus $U = A^{-1}$, so we have $\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & A^{-1} \end{bmatrix}$. This is the matrix inversion algorithm in Section 2.4. However, more is true: Theorem 2.5.1 gives $A^{-1} = U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the row operations carrying $A \to I$. Hence

$$A = \left(A^{-1}\right)^{-1} = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} \tag{2.10}$$

By Lemma 2.5.2, this shows that every invertible matrix A is a product of elementary matrices. Since elementary matrices are invertible (again by Lemma 2.5.2), this proves the following important characterization of invertible matrices.

Theorem 2.5.2

A square matrix is invertible if and only if it is a product of elementary matrices.

It follows from Theorem 2.5.1 that $A \to B$ by row operations if and only if $B = UA$ for some invertible matrix B. In this case we say that A and B are **row-equivalent**. (See Exercise 2.5.17.)

Example 2.5.3

Express $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$ as a product of elementary matrices.

Solution. Using Lemma 2.5.1, the reduction of $A \to I$ is as follows:

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \to E_1 A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \to E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \to E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where the corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Hence $(E_3\ E_2\ E_1)A = I$, so:

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Smith Normal Form

Let A be an $m \times n$ matrix of rank r, and let R be the reduced row-echelon form of A. Theorem 2.5.1 shows that $R = UA$ where U is invertible, and that U can be found from $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$.

The matrix R has r leading ones (since $\operatorname{rank} A = r$) so, as R is reduced, the $n \times m$ matrix R^T contains each row of I_r in the first r columns. Thus row operations will carry $R^T \to \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}$. Hence Theorem 2.5.1 (again) shows that $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n \times m} = U_1R^T$ where U_1 is an $n \times n$ invertible matrix. Writing $V = U_1^T$, we obtain

$$UAV = RV = RU_1^T = \left(U_1R^T\right)^T = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}\right)^T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

Moreover, the matrix $U_1 = V^T$ can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n \times m} & V^T \end{array}\right]$. This proves

Theorem 2.5.3

Let A be an $m \times n$ matrix of rank r. There exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

Moreover, if R is the reduced row-echelon form of A, then:

1. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$;*
2. *V can be computed by $\left[\begin{array}{cc} R^T & I_n \end{array}\right] \to \left[\begin{array}{cc} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n \times m} & V^T \end{array}\right]$.*

If A is an $m \times n$ matrix of rank r, the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the **Smith normal form**[11] of A. Whereas the reduced row-echelon form of A is the "nicest" matrix to which A can be carried by row operations, the Smith canonical form is the "nicest" matrix to which A can be carried by *row and column* operations. This is because doing row operations to R^T amounts to doing *column* operations to R and then transposing.

[11] Named after Henry John Stephen Smith (1826–83).

Example 2.5.4

Given $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$, find invertible matrices U and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \text{rank } A$.

Solution. The matrix U and the reduced row-echelon form R of A are computed by the row reduction $\left[\begin{array}{cc} A & I_3 \end{array}\right] \to \left[\begin{array}{cc} R & U \end{array}\right]$:

$$\left[\begin{array}{rrrr|rrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrr} 1 & -1 & 0 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Hence

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

In particular, $r = \text{rank } R = 2$. Now row-reduce $\left[\begin{array}{cc} R^T & I_4 \end{array}\right] \to \left[\begin{array}{cc} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} & V^T \end{array}\right]$:

$$\left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{array}\right]$$

whence

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & -1 \end{bmatrix} \quad \text{so} \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $UAV = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ as is easily verified.

Uniqueness of the Reduced Row-echelon Form

In this short subsection, Theorem 2.5.1 is used to prove the following important theorem.

Theorem 2.5.4

If a matrix A is carried to reduced row-echelon matrices R and S by row operations, then $R = S$.

Proof. Observe first that $UR = S$ for some invertible matrix U (by Theorem 2.5.1 there exist invertible matrices P and Q such that $R = PA$ and $S = QA$; take $U = QP^{-1}$). We show that $R = S$ by induction on the number m of rows of R and S. The case $m = 1$ is left to the reader. If R_j and S_j denote column j in R and S respectively, the fact that $UR = S$ gives

$$UR_j = S_j \quad \text{for each } j \tag{2.11}$$

Since U is invertible, this shows that R and S have the same zero columns. Hence, by passing to the matrices obtained by deleting the zero columns from R and S, we may assume that R and S have no zero columns.

But then the first column of R and S is the first column of I_m because R and S are row-echelon, so (2.11) shows that the first column of U is column 1 of I_m. Now write U, R, and S in block form as follows.

$$U = \left[\begin{array}{cc} 1 & X \\ 0 & V \end{array}\right], \quad R = \left[\begin{array}{cc} 1 & X \\ 0 & R' \end{array}\right], \quad \text{and} \quad S = \left[\begin{array}{cc} 1 & Z \\ 0 & S' \end{array}\right]$$

Since $UR = S$, block multiplication gives $VR' = S'$ so, since V is invertible (U is invertible) and both R' and S' are reduced row-echelon, we obtain $R' = S'$ by induction. Hence R and S have the same number (say r) of leading 1s, and so both have m–r zero rows.

In fact, R and S have leading ones in the same columns, say r of them. Applying (2.11) to these columns shows that the first r columns of U are the first r columns of I_m. Hence we can write U, R, and S in block form as follows:

$$U = \left[\begin{array}{cc} I_r & M \\ 0 & W \end{array}\right], \quad R = \left[\begin{array}{cc} R_1 & R_2 \\ 0 & 0 \end{array}\right], \quad \text{and} \quad S = \left[\begin{array}{cc} S_1 & S_2 \\ 0 & 0 \end{array}\right]$$

where R_1 and S_1 are $r \times r$. Then block multiplication gives $UR = R$; that is, $S = R$. This completes the proof. □

Exercises for 2.5

Exercise 2.5.1 For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.

a. $E = \left[\begin{array}{ccc} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

b. $E = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$

c. $E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array}\right]$

d. $E = \left[\begin{array}{rcc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

e. $E = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$

f. $E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{array}\right]$

Exercise 2.5.2 In each case find an elementary matrix E such that $B = EA$.

a. $A = \left[\begin{array}{cr} 2 & 1 \\ 3 & -1 \end{array}\right]$, $B = \left[\begin{array}{cr} 2 & 1 \\ 1 & -2 \end{array}\right]$

b. $A = \left[\begin{array}{rc} -1 & 2 \\ 0 & 1 \end{array}\right]$, $B = \left[\begin{array}{cr} 1 & -2 \\ 0 & 1 \end{array}\right]$

c. $A = \left[\begin{array}{rc} 1 & 1 \\ -1 & 2 \end{array}\right]$, $B = \left[\begin{array}{rc} -1 & 2 \\ 1 & 1 \end{array}\right]$

d. $A = \left[\begin{array}{cc} 4 & 1 \\ 3 & 2 \end{array}\right]$, $B = \left[\begin{array}{cr} 1 & -1 \\ 3 & 2 \end{array}\right]$

e. $A = \left[\begin{array}{rr} -1 & 1 \\ 1 & -1 \end{array}\right]$, $B = \left[\begin{array}{rc} -1 & 1 \\ -1 & 1 \end{array}\right]$

f. $A=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, B=\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

Exercise 2.5.3 Let $A=\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

a. Find elementary matrices E_1 and E_2 such that $C=E_2E_1A$.

b. Show that there is *no* elementary matrix E such that $C=EA$.

Exercise 2.5.4 If E is elementary, show that A and EA differ in at most two rows.

Exercise 2.5.5

a. Is I an elementary matrix? Explain.

b. Is 0 an elementary matrix? Explain.

Exercise 2.5.6 In each case find an invertible matrix U such that $UA=R$ is in reduced row-echelon form, and express U as a product of elementary matrices.

a. $A=\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$

b. $A=\begin{bmatrix} 1 & 2 & 1 \\ 5 & 12 & -1 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 1 & 1 & 2 \\ 1 & -3 & 3 & 2 \end{bmatrix}$

d. $A=\begin{bmatrix} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 \end{bmatrix}$

Exercise 2.5.7 In each case find an invertible matrix U such that $UA=B$, and express U as a product of elementary matrices.

a. $A=\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}, B=\begin{bmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \end{bmatrix}$

b. $A=\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B=\begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

Exercise 2.5.8 In each case factor A as a product of elementary matrices.

a. $A=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

b. $A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

d. $A=\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ -2 & 2 & 15 \end{bmatrix}$

Exercise 2.5.9 Let E be an elementary matrix.

a. Show that E^T is also elementary of the same type.

b. Show that $E^T=E$ if E is of type I or II.

Exercise 2.5.10 Show that every matrix A can be factored as $A=UR$ where U is invertible and R is in reduced row-echelon form.

Exercise 2.5.11 If $A=\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ and $B=\begin{bmatrix} 5 & 2 \\ -5 & -3 \end{bmatrix}$ find an elementary matrix F such that $AF=B$.

[*Hint*: See Exercise 2.5.9.]

Exercise 2.5.12 In each case find invertible U and V such that $UAV=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r=\operatorname{rank} A$.

a. $A=\begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 4 \end{bmatrix}$

b. $A=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

c. $A=\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 0 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}$

d. $A=\begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$

Exercise 2.5.13 Prove Lemma 2.5.1 for elementary matrices of:

a. type I;

b. type II.

Exercise 2.5.14 While trying to invert A, $\begin{bmatrix} A & I \end{bmatrix}$ is carried to $\begin{bmatrix} P & Q \end{bmatrix}$ by row operations. Show that $P=QA$.

Exercise 2.5.15 If A and B are $n\times n$ matrices and AB is a product of elementary matrices, show that the same is true of A.

Exercise 2.5.16 If U is invertible, show that the reduced row-echelon form of a matrix $\left[\begin{array}{cc} U & A \end{array} \right]$ is $\left[\begin{array}{cc} I & U^{-1}A \end{array} \right]$.

Exercise 2.5.17 Two matrices A and B are called **row-equivalent** (written $A \overset{r}{\sim} B$) if there is a sequence of elementary row operations carrying A to B.

a. Show that $A \overset{r}{\sim} B$ if and only if $A = UB$ for some invertible matrix U.

b. Show that:

   i. $A \overset{r}{\sim} A$ for all matrices A.

   ii. If $A \overset{r}{\sim} B$, then $B \overset{r}{\sim} A$

   iii. If $A \overset{r}{\sim} B$ and $B \overset{r}{\sim} C$, then $A \overset{r}{\sim} C$.

c. Show that, if A and B are both row-equivalent to some third matrix, then $A \overset{r}{\sim} B$.

d. Show that $\left[\begin{array}{rrrr} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{array} \right]$ and $\left[\begin{array}{rrrr} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{array} \right]$ are row-equivalent. [*Hint*: Consider (c) and Theorem 1.2.1.]

Exercise 2.5.18 If U and V are invertible $n \times n$ matrices, show that $U \overset{r}{\sim} V$. (See Exercise 2.5.17.)

Exercise 2.5.19 (See Exercise 2.5.17.) Find all matrices that are row-equivalent to:

a. $\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$

b. $\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$

c. $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$

d. $\left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array} \right]$

Exercise 2.5.20 Let A and B be $m \times n$ and $n \times m$ matrices, respectively. If $m > n$, show that AB is not invertible. [*Hint*: Use Theorem 1.3.1 to find $\mathbf{x} \neq \mathbf{0}$ with $B\mathbf{x} = \mathbf{0}$.]

Exercise 2.5.21 Define an *elementary column operation* on a matrix to be one of the following: (I) Interchange two columns. (II) Multiply a column by a nonzero scalar. (III) Add a multiple of a column to another column. Show that:

a. If an elementary column operation is done to an $m \times n$ matrix A, the result is AF, where F is an $n \times n$ elementary matrix.

b. Given any $m \times n$ matrix A, there exist $m \times m$ elementary matrices $E_1, \dots, E_k$ and $n \times n$ elementary matrices $F_1, \dots, F_p$ such that, in block form,

$$E_k \cdots E_1 A F_1 \cdots F_p = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right]$$

Exercise 2.5.22 Suppose B is obtained from A by:

a. interchanging rows i and j;

b. multiplying row i by $k \neq 0$;

c. adding k times row i to row j $(i \neq j)$.

In each case describe how to obtain B^{-1} from A^{-1}. [*Hint*: See part (a) of the preceding exercise.]

Exercise 2.5.23 Two $m \times n$ matrices A and B are called **equivalent** (written $A \overset{e}{\sim} B$) if there exist invertible matrices U and V (sizes $m \times m$ and $n \times n$) such that $A = UBV$.

a. Prove the following the properties of equivalence.

   i. $A \overset{e}{\sim} A$ for all $m \times n$ matrices A.

   ii. If $A \overset{e}{\sim} B$, then $B \overset{e}{\sim} A$.

   iii. If $A \overset{e}{\sim} B$ and $B \overset{e}{\sim} C$, then $A \overset{e}{\sim} C$.

b. Prove that two $m \times n$ matrices are equivalent if they have the same rank. [*Hint*: Use part (a) and Theorem 2.5.3.]

2.6 Linear Transformations

If A is an $m \times n$ matrix, recall that the transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

is called the *matrix transformation induced* by A. In Section 2.2, we saw that many important geometric transformations were in fact matrix transformations. These transformations can be characterized in a different way. The new idea is that of a linear transformation, one of the basic notions in linear algebra. We define these transformations in this section, and show that they are really just the matrix transformations looked at in another way. Having these two ways to view them turns out to be useful because, in a given situation, one perspective or the other may be preferable.

Linear Transformations

Definition 2.13 Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

*A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if it satisfies the following two conditions for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a:*

T1 $\quad T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

T2 $\quad T(a\mathbf{x}) = aT(\mathbf{x})$

Of course, $\mathbf{x}+\mathbf{y}$ and $a\mathbf{x}$ here are computed in $\mathbb{R}^n$, while $T(\mathbf{x})+T(\mathbf{y})$ and $aT(\mathbf{x})$ are in $\mathbb{R}^m$. We say that T *preserves addition* if T1 holds, and that T *preserves scalar multiplication* if T2 holds. Moreover, taking $a = 0$ and $a = -1$ in T2 gives

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(-\mathbf{x}) = -T(\mathbf{x}) \quad \text{for all } \mathbf{x}$$

Hence T preserves the zero vector and the negative of a vector. Even more is true.

Recall that a vector $\mathbf{y}$ in $\mathbb{R}^n$ is called a **linear combination** of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ if $\mathbf{y}$ has the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$$

for some scalars $a_1, a_2, \dots, a_k$. Conditions T1 and T2 combine to show that every linear transformation T *preserves linear combinations* in the sense of the following theorem. This result is used repeatedly in linear algebra.

Theorem 2.6.1: Linearity Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \dots$

$$T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) = a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) + \cdots + a_kT(\mathbf{x}_k)$$

for all scalars a_i and all vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

Proof. If $k=1$, it reads $T(a_1\mathbf{x}_1) = a_1T(\mathbf{x}_1)$ which is Condition T1. If $k=2$, we have

$$\begin{aligned} T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) &= T(a_1\mathbf{x}_1) + T(a_2\mathbf{x}_2) && \text{by Condition T1} \\ &= a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) && \text{by Condition T2} \end{aligned}$$

If $k=3$, we use the case $k=2$ to obtain

$$\begin{aligned} T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3) &= T[(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + a_3\mathbf{x}_3] && \text{collect terms} \\ &= T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2) + T(a_3\mathbf{x}_3) && \text{by Condition T1} \\ &= [a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2)] + T(a_3\mathbf{x}_3) && \text{by the case } k=2 \\ &= [a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2)] + a_3T(\mathbf{x}_3) && \text{by Condition T2} \end{aligned}$$

The proof for any k is similar, using the previous case $k-1$ and Conditions T1 and T2. □

The method of proof in Theorem 2.6.1 is called *mathematical induction* (Appendix C).

Theorem 2.6.1 shows that if T is a linear transformation and $T(\mathbf{x}_1), T(\mathbf{x}_2), \dots, T(\mathbf{x}_k)$ are all known, then $T(\mathbf{y})$ can be easily computed for any linear combination $\mathbf{y}$ of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$. This is a very useful property of linear transformations, and is illustrated in the next example.

Example 2.6.1

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $T\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 2 \\ -3 \end{array}\right]$ and $T\left[\begin{array}{r} 1 \\ -2 \end{array}\right] = \left[\begin{array}{c} 5 \\ 1 \end{array}\right]$, find $T\left[\begin{array}{c} 4 \\ 3 \end{array}\right]$.

Solution. Write $\mathbf{z} = \left[\begin{array}{c} 4 \\ 3 \end{array}\right]$, $\mathbf{x} = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$, and $\mathbf{y} = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$ for convenience. Then we know $T(\mathbf{x})$ and $T(\mathbf{y})$ and we want $T(\mathbf{z})$, so it is enough by Theorem 2.6.1 to express $\mathbf{z}$ as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. That is, we want to find numbers a and b such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. Equating entries gives two equations $4 = a+b$ and $3 = a-2b$. The solution is, $a = \frac{11}{3}$ and $b = \frac{1}{3}$, so $\mathbf{z} = \frac{11}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$. Thus Theorem 2.6.1 gives

$$T(\mathbf{z}) = \tfrac{11}{3}T(\mathbf{x}) + \tfrac{1}{3}T(\mathbf{y}) = \tfrac{11}{3}\left[\begin{array}{r} 2 \\ -3 \end{array}\right] + \tfrac{1}{3}\left[\begin{array}{c} 5 \\ 1 \end{array}\right] = \tfrac{1}{3}\left[\begin{array}{r} 27 \\ -32 \end{array}\right]$$

This is what we wanted.

Example 2.6.2

If A is $m \times n$, the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$, is a linear transformation.

Solution. We have $T_A(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, so Theorem 2.2.2 gives

$$T_A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T_A(\mathbf{x}) + T_A(\mathbf{y})$$

and

$$T_A(a\mathbf{x}) = A(a\mathbf{x}) = a(A\mathbf{x}) = aT_A(\mathbf{x})$$

hold for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. Hence T_A satisfies T1 and T2, and so is linear.

The remarkable thing is that the *converse* of Example 2.6.2 is true: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is actually a matrix transformation. To see why, we define the **standard basis** of $\mathbb{R}^n$ to be the set of columns

$$\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$$

of the identity matrix I_n. Then each $\mathbf{e}_i$ is in $\mathbb{R}^n$ and every vector $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ in $\mathbb{R}^n$ is a linear combination of the $\mathbf{e}_i$. In fact:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

as the reader can verify. Hence Theorem 2.6.1 shows that

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Now observe that each $T(\mathbf{e}_i)$ is a column in $\mathbb{R}^m$, so

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

is an $m \times n$ matrix. Hence we can apply Definition 2.5 to get

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n) = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] = A\mathbf{x}$$

Since this holds for every $\mathbf{x}$ in $\mathbb{R}^n$, it shows that T is the matrix transformation induced by A, and so proves most of the following theorem.

Theorem 2.6.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. *T is linear if and only if it is a matrix transformation.*
2. *In this case $T = T_A$ is the matrix transformation induced by a unique $m \times n$ matrix A, given in terms of its columns by*

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{array}\right]$$

where $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Proof. It remains to verify that the matrix A is unique. Suppose that T is induced by another matrix B. Then $T(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. But $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$, so $B\mathbf{x} = A\mathbf{x}$ for every $\mathbf{x}$. Hence $A = B$ by Theorem 2.2.6. $\square$

Hence we can speak of *the* matrix of a linear transformation. Because of Theorem 2.6.2 we may (and shall) use the phrases "linear transformation" and "matrix transformation" interchangeably.

Example 2.6.3

Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. Show that T is a linear transformation and use Theorem 2.6.2 to find its matrix.

Solution. Write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, so that $\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}$. Hence

$$T(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\mathbf{x}) + T(\mathbf{y})$$

Similarly, the reader can verify that $T(a\mathbf{x}) = aT(\mathbf{x})$ for all a in $\mathbb{R}$, so T is a linear transformation. Now the standard basis of $\mathbb{R}^3$ is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so, by Theorem 2.6.2, the matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Of course, the fact that $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ shows directly that T is a matrix transformation (hence linear) and reveals the matrix.

To illustrate how Theorem 2.6.2 is used, we rederive the matrices of the transformations in Examples 2.2.13 and 2.2.15.

Example 2.6.4

Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the x axis (as in Example 2.2.13) and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\frac{\pi}{2}$ about the origin (as in Example 2.2.15). Use Theorem 2.6.2 to find the matrices of Q_0 and $R_{\frac{\pi}{2}}$.

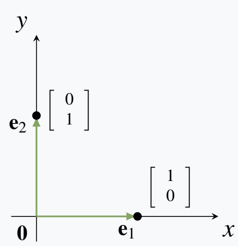


Figure 2.6.1

Solution. Observe that Q_0 and $R_{\frac{\pi}{2}}$ are linear by Example 2.6.2 (they are matrix transformations), so Theorem 2.6.2 applies to them. The standard basis of $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ points along the positive x axis, and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ points along the positive y axis (see Figure 2.6.1).

The reflection of $\mathbf{e}_1$ in the x axis is $\mathbf{e}_1$ itself because $\mathbf{e}_1$ points along the x axis, and the reflection of $\mathbf{e}_2$ in the x axis is $-\mathbf{e}_2$ because $\mathbf{e}_2$ is perpendicular to the x axis. In other words, $Q_0(\mathbf{e}_1) = \mathbf{e}_1$ and $Q_0(\mathbf{e}_2) = -\mathbf{e}_2$. Hence Theorem 2.6.2 shows that the matrix of Q_0 is

$$\left[\begin{array}{cc} Q_0(\mathbf{e}_1) & Q_0(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_1 & -\mathbf{e}_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$$

which agrees with Example 2.2.13.

Similarly, rotating $\mathbf{e}_1$ through $\frac{\pi}{2}$ counterclockwise about the origin produces $\mathbf{e}_2$, and rotating $\mathbf{e}_2$ through $\frac{\pi}{2}$ counterclockwise about the origin gives $-\mathbf{e}_1$. That is, $R_{\frac{\pi}{2}}(\mathbf{e}_1) = \mathbf{e}_2$ and $R_{\frac{\pi}{2}}(\mathbf{e}_2) = -\mathbf{e}_2$. Hence, again by Theorem 2.6.2, the matrix of $R_{\frac{\pi}{2}}$ is

$$\left[\begin{array}{cc} R_{\frac{\pi}{2}}(\mathbf{e}_1) & R_{\frac{\pi}{2}}(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_2 & -\mathbf{e}_1 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$$

agreeing with Example 2.2.15.

Example 2.6.5

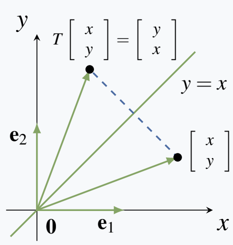


Figure 2.6.2

Let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = x$. Show that Q_1 is a matrix transformation, find its matrix, and use it to illustrate Theorem 2.6.2.

Solution. Figure 2.6.2 shows that $Q_1 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} y \\ x \end{array}\right]$. Hence $Q_1 \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} y \\ x \end{array}\right]$, so Q_1 is the matrix transformation induced by the matrix $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$. Hence Q_1 is linear (by Example 2.6.2) and so Theorem 2.6.2 applies. If $\mathbf{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\mathbf{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ are the standard basis of $\mathbb{R}^2$, then it is clear geometrically that $Q_1(\mathbf{e}_1) = \mathbf{e}_2$ and $Q_1(\mathbf{e}_2) = \mathbf{e}_1$. Thus (by Theorem 2.6.2) the matrix of Q_1 is $\left[\begin{array}{cc} Q_1(\mathbf{e}_1) & Q_1(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_2 & \mathbf{e}_1 \end{array}\right] = A$ as before.

Recall that, given two "linked" transformations

$$\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$$

we can apply T first and then apply S, and so obtain a new transformation

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m$$

called the **composite** of S and T, defined by

$$(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

If S and T are linear, the action of $S \circ T$ can be computed by multiplying their matrices.

Theorem 2.6.3

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and let A and B be the matrices of S and T respectively. Then $S \circ T$ is linear with matrix AB.

Proof. $(S \circ T)(\mathbf{x}) = S[T(\mathbf{x})] = A[B\mathbf{x}] = (AB)\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^k$. □

Theorem 2.6.3 shows that the action of the composite $S \circ T$ is determined by the matrices of S and T. But it also provides a very useful interpretation of matrix multiplication. If A and B are matrices, the product matrix AB induces the transformation resulting from first applying B and then applying A. Thus the study of matrices can cast light on geometrical transformations and vice-versa. Here is an example.

Example 2.6.6

Show that reflection in the x axis followed by rotation through $\frac{\pi}{2}$ is reflection in the line $y = x$.

Solution. The composite in question is $R_{\frac{\pi}{2}} \circ Q_0$ where Q_0 is reflection in the x axis and $R_{\frac{\pi}{2}}$ is rotation through $\frac{\pi}{2}$. By Example 2.6.4, $R_{\frac{\pi}{2}}$ has matrix $A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$ and Q_0 has matrix $B = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$. Hence Theorem 2.6.3 shows that the matrix of $R_{\frac{\pi}{2}} \circ Q_0$ is $AB = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$, which is the matrix of reflection in the line $y = x$ by Example 2.6.3.

This conclusion can also be seen geometrically. Let $\mathbf{x}$ be a typical point in $\mathbb{R}^2$, and assume that $\mathbf{x}$ makes an angle α with the positive x axis. The effect of first applying Q_0 and then applying $R_{\frac{\pi}{2}}$ is shown in Figure 2.6.3. The fact that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ makes the angle α with the positive y axis shows that $R_{\frac{\pi}{2}}[Q_0(\mathbf{x})]$ is the reflection of $\mathbf{x}$ in the line $y = x$.

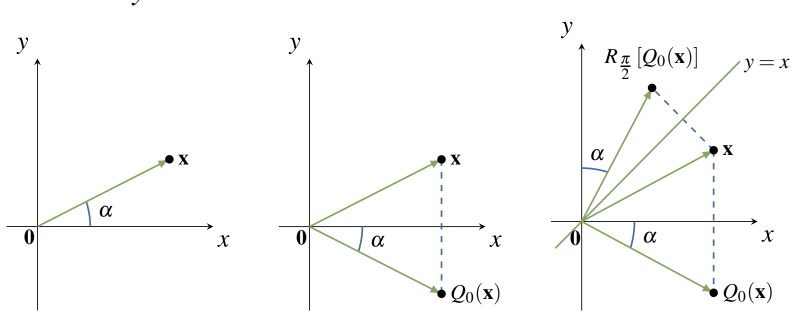


Figure 2.6.3

In Theorem 2.6.3, we saw that the matrix of the composite of two linear transformations is the product of their matrices (in fact, matrix products were defined so that this is the case). We are going to apply this fact to rotations, reflections, and projections in the plane. Before proceeding, we pause to present useful geometrical descriptions of vector addition and scalar multiplication in the plane, and to give a short review of angles and the trigonometric functions.

Some Geometry

As we have seen, it is convenient to view a vector $\mathbf{x}$ in $\mathbb{R}^2$ as an arrow from the origin to the point $\mathbf{x}$ (see Section 2.2). This enables us to visualize what sums and scalar multiples mean geometrically. For example consider $\mathbf{x} = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$ in $\mathbb{R}^2$. Then $2\mathbf{x} = \left[\begin{array}{c} 2 \\ 4 \end{array}\right]$, $\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array}\right]$ and $-\frac{1}{2}\mathbf{x} = \left[\begin{array}{c} -\frac{1}{2} \\ -1 \end{array}\right]$, and these are shown as arrows in Figure 2.6.4.

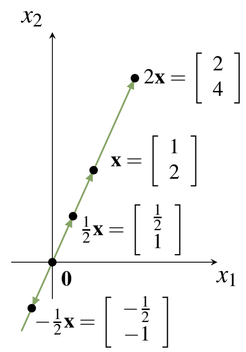


Figure 2.6.4

Observe that the arrow for $2\mathbf{x}$ is twice as long as the arrow for $\mathbf{x}$ and in the same direction, and that the arrows for $\frac{1}{2}\mathbf{x}$ is also in the same direction as the arrow for $\mathbf{x}$, but only half as long. On the other hand, the arrow for $-\frac{1}{2}\mathbf{x}$ is half as long as the arrow for $\mathbf{x}$, but in the *opposite* direction. More generally, we have the following geometrical description of scalar multiplication in $\mathbb{R}^2$:

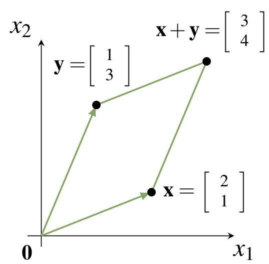


Figure 2.6.5

Scalar Multiple Law

Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$. The arrow for $k\mathbf{x}$ is $|k|$ times[12] as long as the arrow for $\mathbf{x}$, and is in the same direction as the arrow for $\mathbf{x}$ if $k > 0$, and in the opposite direction if $k < 0$.

Now consider two vectors $\mathbf{x} = \left[\begin{array}{c} 2 \\ 1 \end{array}\right]$ and $\mathbf{y} = \left[\begin{array}{c} 1 \\ 3 \end{array}\right]$ in $\mathbb{R}^2$. They are plotted in Figure 2.6.5 along with their sum $\mathbf{x}+\mathbf{y} = \left[\begin{array}{c} 3 \\ 4 \end{array}\right]$. It is a routine matter to verify that the four points $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}+\mathbf{y}$ form the vertices of a **parallelogram**–that is opposite sides are parallel and of the same length. (The reader should verify that the side from $\mathbf{0}$ to $\mathbf{x}$ has slope of $\frac{1}{2}$, as does the side from $\mathbf{y}$ to $\mathbf{x}+\mathbf{y}$, so these sides are parallel.) We state this as follows:

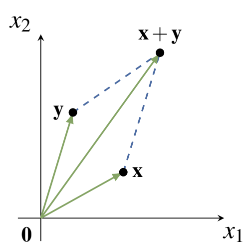


Figure 2.6.6

Parallelogram Law

Consider vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$. If the arrows for $\mathbf{x}$ and $\mathbf{y}$ are drawn (see Figure 2.6.6), the arrow for $\mathbf{x}+\mathbf{y}$ corresponds to the fourth vertex of the parallelogram determined by the points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{0}$.

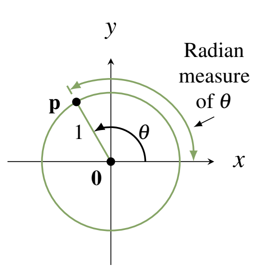


Figure 2.6.7

We will have more to say about this in Chapter 4.

Before proceeding we turn to a brief review of angles and the trigonometric functions. Recall that an angle θ is said to be in **standard position** if it is measured counterclockwise from the positive x axis (as in Figure 2.6.7). Then θ uniquely determines a point $\mathbf{p}$ on the **unit circle**

[12]If k is a real number, $|k|$ denotes the **absolute value** of k; that is, $|k| = k$ if $k \geq 0$ and $|k| = -k$ if $k < 0$.

(radius 1, centre at the origin). The **radian** measure of θ is the length of the arc on the unit circle from the positive x axis to $\mathbf{p}$. Thus $360^\circ = 2\pi$ radians, $180^\circ = \pi$, $90^\circ = \frac{\pi}{2}$, and so on.

The point $\mathbf{p}$ in Figure 2.6.7 is also closely linked to the trigonometric functions **cosine** and **sine**, written $\cos\theta$ and $\sin\theta$ respectively. In fact these functions are *defined* to be the x and y coordinates of $\mathbf{p}$; that is $\mathbf{p} = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$. This defines $\cos\theta$ and $\sin\theta$ for the arbitrary angle θ (possibly negative), and agrees with the usual values when θ is an acute angle $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$ as the reader should verify. For more discussion of this, see Appendix A.

Rotations

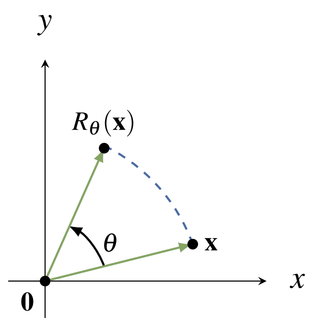


Figure 2.6.8

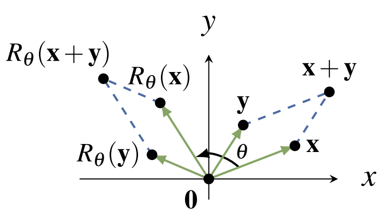


Figure 2.6.9

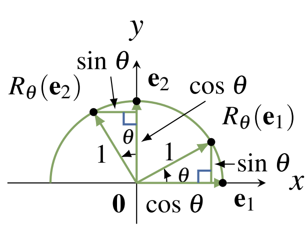


Figure 2.6.10

We can now describe rotations in the plane. Given an angle θ, let

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denote counterclockwise rotation of $\mathbb{R}^2$ about the origin through the angle θ. The action of R_θ is depicted in Figure 2.6.8. We have already looked at $R_{\frac{\pi}{2}}$ (in Example 2.2.15) and found it to be a matrix transformation. It turns out that R_θ is a matrix transformation for *every* angle θ (with a simple formula for the matrix), but it is not clear how to find the matrix. Our approach is to first establish the (somewhat surprising) fact that R_θ is *linear*, and then obtain the matrix from Theorem 2.6.2.

Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^2$. Then $\mathbf{x}+\mathbf{y}$ is the diagonal of the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$ as in Figure 2.6.9.

The effect of R_θ is to rotate the *entire* parallelogram to obtain the new parallelogram determined by $R_\theta(\mathbf{x})$ and $R_\theta(\mathbf{y})$, with diagonal $R_\theta(\mathbf{x}+\mathbf{y})$. But this diagonal is $R_\theta(\mathbf{x}) + R_\theta(\mathbf{y})$ by the parallelogram law (applied to the new parallelogram). It follows that

$$R_\theta(\mathbf{x}+\mathbf{y}) = R_\theta(\mathbf{x}) + R_\theta(\mathbf{y})$$

A similar argument shows that $R_\theta(a\mathbf{x}) = aR_\theta(\mathbf{x})$ for any scalar a, so $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is indeed a linear transformation.

With linearity established we can find the matrix of R_θ. Let $\mathbf{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\mathbf{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ denote the standard basis of $\mathbb{R}^2$. By Figure 2.6.10 we see that

$$R_\theta(\mathbf{e}_1) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right] \quad \text{and} \quad R_\theta(\mathbf{e}_2) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right]$$

Hence Theorem 2.6.2 shows that R_θ is induced by the matrix

$$\left[\begin{array}{cc} R_\theta(\mathbf{e}_1) & R_\theta(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

We record this as

Theorem 2.6.4

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with matrix $\left[\begin{array}{rr} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$.

For example, $R_{\frac{\pi}{2}}$ and R_π have matrices $\left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$ and $\left[\begin{array}{rr} -1 & 0 \\ 0 & -1 \end{array}\right]$, respectively, by Theorem 2.6.4. The first of these confirms the result in Example 2.2.15. The second shows that rotating a vector $\mathbf{x} = \left[\begin{array}{c} x \\ y \end{array}\right]$ through the angle π results in $R_\pi(\mathbf{x}) = \left[\begin{array}{rr} -1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} -x \\ -y \end{array}\right] = -\mathbf{x}$. Thus applying R_π is the same as negating $\mathbf{x}$, a fact that is evident without Theorem 2.6.4.

Example 2.6.7

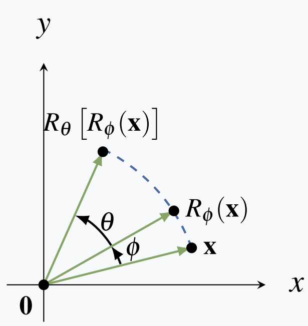


Figure 2.6.11

Let θ and ϕ be angles. By finding the matrix of the composite $R_\theta \circ R_\phi$, obtain expressions for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

Solution. Consider the transformations $\mathbb{R}^2 \xrightarrow{R_\phi} \mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2$. Their composite $R_\theta \circ R_\phi$ is the transformation that first rotates the plane through ϕ and then rotates it through θ, and so is the rotation through the angle $\theta+\phi$ (see Figure 2.6.11).
In other words

$$R_{\theta+\phi} = R_\theta \circ R_\phi$$

Theorem 2.6.3 shows that the corresponding equation holds for the matrices of these transformations, so Theorem 2.6.4 gives:

$$\left[\begin{array}{cc} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{array}\right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]\left[\begin{array}{cc} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array}\right]$$

If we perform the matrix multiplication on the right, and then compare first column entries, we obtain

$$\begin{aligned} \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi \end{aligned}$$

These are the two basic identities from which most of trigonometry can be derived.

Reflections

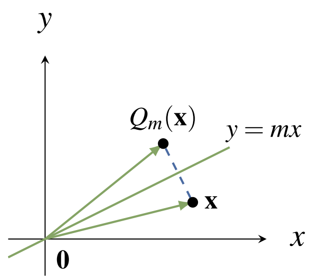


Figure 2.6.12

The line through the origin with slope m has equation $y = mx$, and we let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$.

This transformation is described geometrically in Figure 2.6.12. In words, $Q_m(\mathbf{x})$ is the "mirror image" of $\mathbf{x}$ in the line $y = mx$. If $m = 0$ then Q_0 is reflection in the x axis, so we already know Q_0 is linear. While we could show directly that Q_m is linear (with an argument like that for R_θ), we prefer to do it another way that is instructive and derives the matrix of Q_m directly without using Theorem 2.6.2.

Let θ denote the angle between the positive x axis and the line $y = mx$. The key observation is that the transformation Q_m can be accomplished in three steps: First rotate through $-\theta$ (so our line coincides with the x axis), then reflect in the x axis, and finally rotate back through θ. In other words:

$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$$

Since $R_{-\theta}$, Q_0, and R_θ are all linear, this (with Theorem 2.6.3) shows that Q_m is linear and that its matrix is the product of the matrices of R_θ, Q_0, and $R_{-\theta}$. If we write $c = \cos\theta$ and $s = \sin\theta$ for simplicity, then the matrices of R_θ, $R_{-\theta}$, and Q_0 are

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right], \quad \left[\begin{array}{rr} c & s \\ -s & c \end{array}\right], \quad \text{and} \quad \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] \text{ respectively.}^{13}$$

Hence, by Theorem 2.6.3, the matrix of $Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$ is

$$\left[\begin{array}{rr} c & -s \\ s & c \end{array}\right] \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] \left[\begin{array}{rr} c & s \\ -s & c \end{array}\right] = \left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$$

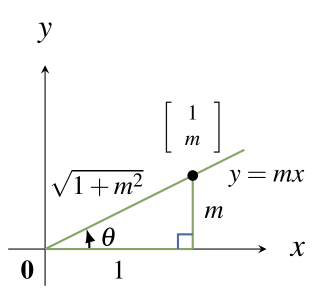


Figure 2.6.13

We can obtain this matrix in terms of m alone. Figure 2.6.13 shows that

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

so the matrix $\left[\begin{array}{cc} c^2 - s^2 & 2sc \\ 2sc & s^2 - c^2 \end{array}\right]$ of Q_m becomes $\frac{1}{1+m^2} \left[\begin{array}{cc} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{array}\right]$.

Theorem 2.6.5

Let Q_m denote reflection in the line $y = mx$. Then Q_m is a linear transformation with matrix $\frac{1}{1+m^2} \left[\begin{array}{cc} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{array}\right]$.

[13] The matrix of $R_{-\theta}$ comes from the matrix of R_θ using the fact that, for all angles θ, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$.

Note that if $m = 0$, the matrix in Theorem 2.6.5 becomes $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, as expected. Of course this analysis fails for reflection in the y axis because vertical lines have no slope. However it is an easy exercise to verify directly that reflection in the y axis is indeed linear with matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.[14]

Example 2.6.8

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through $-\frac{\pi}{2}$ followed by reflection in the y axis. Show that T is a reflection in a line through the origin and find the line.

Solution. The matrix of $R_{-\frac{\pi}{2}}$ is $\begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the matrix of reflection in the y axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence the matrix of T is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and this is reflection in the line $y = -x$ (take $m = -1$ in Theorem 2.6.5).

Projections

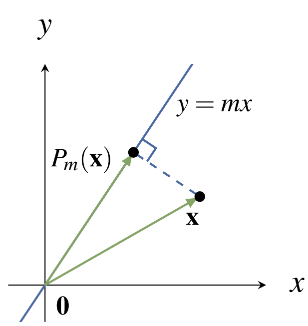


Figure 2.6.14

The method in the proof of Theorem 2.6.5 works more generally. Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote projection on the line $y = mx$. This transformation is described geometrically in Figure 2.6.14.

If $m = 0$, then $P_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, so P_0 is linear with matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence the argument above for Q_m goes through for P_m. First observe that

$$P_m = R_\theta \circ P_0 \circ R_{-\theta}$$

as before. So, P_m is linear with matrix

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

where $c = \cos\theta = \frac{1}{\sqrt{1+m^2}}$ and $s = \sin\theta = \frac{m}{\sqrt{1+m^2}}$.

[14] Note that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \lim_{m\to\infty} \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

This gives:

Theorem 2.6.6

Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ be projection on the line $y = mx$. Then P_m is a linear transformation with matrix $\frac{1}{1+m^2}\left[\begin{array}{cc} 1 & m \\ m & m^2 \end{array}\right]$.

Again, if $m = 0$, then the matrix in Theorem 2.6.6 reduces to $\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$ as expected. As the y axis has no slope, the analysis fails for projection on the y axis, but this transformation is indeed linear with matrix $\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$ as is easily verified directly.

Note that the formula for the matrix of Q_m in Theorem 2.6.5 can be derived from the above formula for the matrix of P_m. Using Figure 2.6.12, observe that $Q_m(\mathbf{x}) = \mathbf{x} + 2[P_m(\mathbf{x}) - \mathbf{x}]$ so $Q_m(x) = 2P_m(\mathbf{x}) - \mathbf{x}$. Substituting the matrices for $P_m(\mathbf{x})$ and $1_{\mathbb{R}^2}(\mathbf{x})$ gives the desired formula.

Example 2.6.9

Given $\mathbf{x}$ in $\mathbb{R}^2$, write $\mathbf{y} = P_m(\mathbf{x})$. The fact that $\mathbf{y}$ lies on the line $y = mx$ means that $P_m(\mathbf{y}) = \mathbf{y}$. But then

$$(P_m \circ P_m)(\mathbf{x}) = P_m(\mathbf{y}) = \mathbf{y} = P_m(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^2, \quad \text{that is, } P_m \circ P_m = P_m.$$

In particular, if we write the matrix of P_m as $A = \frac{1}{1+m^2}\left[\begin{array}{cc} 1 & m \\ m & m^2 \end{array}\right]$, then $A^2 = A$. The reader should verify this directly.

Exercises for 2.6

Exercise 2.6.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

a. Find $T\left[\begin{array}{r} 8 \\ 3 \\ 7 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \end{array}\right]$ and $T\left[\begin{array}{r} 2 \\ 1 \\ 3 \end{array}\right] = \left[\begin{array}{r} -1 \\ 0 \end{array}\right]$.

b. Find $T\left[\begin{array}{r} 5 \\ 6 \\ -13 \end{array}\right]$ if $T\left[\begin{array}{r} 3 \\ 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} 3 \\ 5 \end{array}\right]$ and $T\left[\begin{array}{r} 2 \\ 0 \\ 5 \end{array}\right] = \left[\begin{array}{r} -1 \\ 2 \end{array}\right]$.

Exercise 2.6.2 Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation.

a. Find $T\left[\begin{array}{r} 1 \\ 3 \\ -2 \\ -3 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 1 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \\ -1 \end{array}\right]$ and $T\left[\begin{array}{r} 0 \\ -1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 5 \\ 0 \\ 1 \end{array}\right]$.

b. Find $T\left[\begin{array}{r} 5 \\ -1 \\ 2 \\ -4 \end{array}\right]$ if $T\left[\begin{array}{r} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 5 \\ 1 \\ -3 \end{array}\right]$

and $T\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 2.6.3 In each case assume that the transformation T is linear, and use Theorem 2.6.2 to obtain the matrix A of T.

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = -x$.

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = -\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$.

c. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is clockwise rotation through $\frac{\pi}{4}$.

d. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation through $\frac{\pi}{4}$.

Exercise 2.6.4 In each case use Theorem 2.6.2 to obtain the matrix A of the transformation T. You may assume that T is linear in each case.

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $x - z$ plane.

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the $y - z$ plane.

Exercise 2.6.5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a. If $\mathbf{x}$ is in $\mathbb{R}^n$, we say that $\mathbf{x}$ is in the *kernel* of T if $T(\mathbf{x}) = \mathbf{0}$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both in the kernel of T, show that $a\mathbf{x}_1 + b\mathbf{x}_2$ is also in the kernel of T for all scalars a and b.

b. If $\mathbf{y}$ is in $\mathbb{R}^n$, we say that $\mathbf{y}$ is in the *image* of T if $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{y}_1$ and $\mathbf{y}_2$ are both in the image of T, show that $a\mathbf{y}_1 + b\mathbf{y}_2$ is also in the image of T for all scalars a and b.

Exercise 2.6.6 Use Theorem 2.6.2 to find the matrix of the **identity transformation** $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 2.6.7 In each case show that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not a linear transformation.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ 0 \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 \end{bmatrix}$

Exercise 2.6.8 In each case show that T is either reflection in a line or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x+y \\ -x+y \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} 8x+6y \\ 6x-8y \end{bmatrix}$

Exercise 2.6.9 Express reflection in the line $y = -x$ as the composition of a rotation followed by reflection in the line $y = x$.

Exercise 2.6.10 Find the matrix of $T : \mathbb{R}^3 \to \mathbb{R}^3$ in each case:

a. T is rotation through θ about the x axis (from the y axis to the z axis).

b. T is rotation through θ about the y axis (from the x axis to the z axis).

Exercise 2.6.11 Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line making an angle θ with the positive x axis.

a. Show that the matrix of T_θ is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ for all θ.

b. Show that $T_\theta \circ R_{2\phi} = T_{\theta-\phi}$ for all θ and ϕ.

Exercise 2.6.12 In each case find a rotation or reflection that equals the given transformation.

a. Reflection in the y axis followed by rotation through $\frac{\pi}{2}$.

b. Rotation through π followed by reflection in the x axis.

c. Rotation through $\frac{\pi}{2}$ followed by reflection in the line $y = x$.

d. Reflection in the x axis followed by rotation through $\frac{\pi}{2}$.

e. Reflection in the line $y = x$ followed by reflection in the x axis.

f. Reflection in the x axis followed by reflection in the line $y = x$.

Exercise 2.6.13 Let R and S be matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ induced by matrices A and B respectively. In each case, show that T is a matrix transformation and describe its matrix in terms of A and B.

a. $T(\mathbf{x}) = R(\mathbf{x}) + S(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

b. $T(\mathbf{x}) = aR(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$ (where a is a fixed real number).

Exercise 2.6.14 Show that the following hold for all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$:

a. $T(\mathbf{0}) = \mathbf{0}$

b. $T(-\mathbf{x}) = -T(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{R}^n$

Exercise 2.6.15 The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^n$ is called the **zero transformation**.

a. Show that the zero transformation is linear and find its matrix.

b. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $T(\mathbf{e}_i) = \mathbf{0}$ for each i, show that T is the zero transformation. [*Hint*: Theorem 2.6.1.]

Exercise 2.6.16 Write the elements of $\mathbb{R}^n$ and $\mathbb{R}^m$ as rows. If A is an $m \times n$ matrix, define $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\mathbf{y}) = \mathbf{y}A$ for all rows $\mathbf{y}$ in $\mathbb{R}^m$. Show that:

a. T is a linear transformation.

b. the rows of A are $T(\mathbf{f}_1), T(\mathbf{f}_2), \dots, T(\mathbf{f}_m)$ where $\mathbf{f}_i$ denotes row i of I_m. [*Hint*: Show that $\mathbf{f}_i A$ is row i of A.]

Exercise 2.6.17 Let $S : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations with matrices A and B respectively.

a. Show that $B^2 = B$ if and only if $T^2 = T$ (where T^2 means $T \circ T$).

b. Show that $B^2 = I$ if and only if $T^2 = 1_{\mathbb{R}^n}$.

c. Show that $AB = BA$ if and only if $S \circ T = T \circ S$.

[*Hint*: Theorem 2.6.3.]

Exercise 2.6.18 Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the x axis, let $Q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$, let $Q_{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = -x$, and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation through $\frac{\pi}{2}$.

a. Show that $Q_1 \circ R_{\frac{\pi}{2}} = Q_0$.

b. Show that $Q_1 \circ Q_0 = R_{\frac{\pi}{2}}$.

c. Show that $R_{\frac{\pi}{2}} \circ Q_0 = Q_1$.

d. Show that $Q_0 \circ R_{\frac{\pi}{2}} = Q_{-1}$.

Exercise 2.6.19 For any slope m, show that:

a. $Q_m \circ P_m = P_m$

b. $P_m \circ Q_m = P_m$

Exercise 2.6.20 Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$. Show that T is a linear transformation and find its matrix.

Exercise 2.6.21 Given c in $\mathbb{R}$, define $T_c : \mathbb{R}^n \to \mathbb{R}$ by $T_c(\mathbf{x}) = c\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that T_c is a linear transformation and find its matrix.

Exercise 2.6.22 Given vectors $\mathbf{w}$ and $\mathbf{x}$ in $\mathbb{R}^n$, denote their dot product by $\mathbf{w} \cdot \mathbf{x}$.

a. Given $\mathbf{w}$ in $\mathbb{R}^n$, define $T_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}$ by $T_{\mathbf{w}}(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Show that $T_{\mathbf{w}}$ is a linear transformation.

b. Show that *every* linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ is given as in (a); that is $T = T_{\mathbf{w}}$ for some $\mathbf{w}$ in $\mathbb{R}^n$.

Exercise 2.6.23 If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$. [*Hint*: By Definition 2.5, find a matrix A such that $A\mathbf{x} = \mathbf{y}$.]

Exercise 2.6.24 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k$ be two linear transformations. Show directly that $S \circ T$ is linear. That is:

a. Show that $(S \circ T)(\mathbf{x} + \mathbf{y}) = (S \circ T)\mathbf{x} + (S \circ T)\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

b. Show that $(S \circ T)(a\mathbf{x}) = a[(S \circ T)\mathbf{x}]$ for all $\mathbf{x}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$.

Exercise 2.6.25 Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k \xrightarrow{R} \mathbb{R}^k$ be linear. Show that $R \circ (S \circ T) = (R \circ S) \circ T$ by showing directly that $[R \circ (S \circ T)](\mathbf{x}) = [(R \circ S) \circ T)](\mathbf{x})$ holds for each vector $\mathbf{x}$ in $\mathbb{R}^n$.

2.7 LU-Factorization[15]

The solution to a system $A\mathbf{x} = \mathbf{b}$ of linear equations can be solved quickly if A can be factored as $A = LU$ where L and U are of a particularly nice form. In this section we show that gaussian elimination can be used to find such factorizations.

Triangular Matrices

As for square matrices, if $A = \left[a_{ij}\right]$ is an $m \times n$ matrix, the elements a_{11}, a_{22}, a_{33}, $\dots$ form the **main diagonal** of A. Then A is called **upper triangular** if every entry below and to the left of the main diagonal is zero. Every row-echelon matrix is upper triangular, as are the matrices

$$\left[\begin{array}{rrrr} 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{rrrrr} 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

By analogy, a matrix A is called **lower triangular** if its transpose is upper triangular, that is if each entry above and to the right of the main diagonal is zero. A matrix is called **triangular** if it is upper or lower triangular.

Example 2.7.1

Solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &= 3 \\ 5x_3 + x_4 + x_5 &= 8 \\ 2x_5 &= 6 \end{aligned}$$

where the coefficient matrix is upper triangular.

Solution. As in gaussian elimination, let the "non-leading" variables be parameters: $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows. The last equation gives

$$x_5 = \frac{6}{2} = 3$$

Substitution into the second last equation gives

$$x_3 = 1 - \frac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \frac{2}{5}t$$

The method used in Example 2.7.1 is called **back substitution** because later variables are substituted into earlier equations. It works because the coefficient matrix is upper triangular. Similarly, if the coeffi-

[15]This section is not used later and so may be omitted with no loss of continuity.

cient matrix is lower triangular the system can be solved by **forward substitution** where earlier variables are substituted into later equations. As observed in Section 1.2, these procedures are more numerically efficient than gaussian elimination.

Now consider a system $A\mathbf{x} = \mathbf{b}$ where A can be factored as $A = LU$ where L is lower triangular and U is upper triangular. Then the system $A\mathbf{x} = \mathbf{b}$ can be solved in two stages as follows:

1. *First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by forward substitution.*
2. *Then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by back substitution.*

Then $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$ because $A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}$. Moreover, every solution $\mathbf{x}$ arises this way (take $\mathbf{y} = U\mathbf{x}$). Furthermore the method adapts easily for use in a computer.

This focuses attention on efficiently obtaining such factorizations $A = LU$. The following result will be needed; the proof is straightforward and is left as Exercises 2.7.7 and 2.7.8.

Lemma 2.7.1

Let A and B denote matrices.

1. *If A and B are both lower (upper) triangular, the same is true of AB.*
2. *If A is $n \times n$ and lower (upper) triangular, then A is invertible if and only if every main diagonal entry is nonzero. In this case A^{-1} is also lower (upper) triangular.*

LU-Factorization

Let A be an $m \times n$ matrix. Then A can be carried to a row-echelon matrix U (that is, upper triangular). As in Section 2.5, the reduction is

$$A \to E_1A \to E_2E_1A \to E_3E_2E_1A \to \cdots \to E_kE_{k-1}\cdots E_2E_1A = U$$

where $E_1, E_2, \dots, E_k$ are elementary matrices corresponding to the row operations used. Hence

$$A = LU$$

where $L = (E_kE_{k-1}\cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k^{-1}$. If we do not insist that U is reduced then, except for row interchanges, none of these row operations involve adding a row to a row *above* it. Thus, if no row interchanges are used, all the E_i are *lower* triangular, and so L is lower triangular (and invertible) by Lemma 2.7.1. This proves the following theorem. For convenience, let us say that A can be **lower reduced** if it can be carried to row-echelon form using no row interchanges.

Theorem 2.7.1

If A can be lower reduced to a row-echelon matrix U, then

$$A = LU$$

where L is lower triangular and invertible and U is upper triangular and row-echelon.

Definition 2.14 LU-factorization

A factorization $A = LU$ as in Theorem 2.7.1 is called an ***LU-factorization*** *of A.*

Such a factorization may not exist (Exercise 2.7.4) because A cannot be carried to row-echelon form using no row interchange. A procedure for dealing with this situation will be outlined later. However, if an LU-factorization $A = LU$ does exist, then the gaussian algorithm gives U and also leads to a procedure for finding L. Example 2.7.2 provides an illustration. For convenience, the first nonzero column from the left in a matrix A is called the **leading column** of A.

Example 2.7.2

Find an LU-factorization of $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$.

Solution. We lower reduce A to row-echelon form as follows:

$$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The circled columns are determined as follows: The first is the leading column of A, and is used (by lower reduction) to create the first leading 1 and create zeros below it. This completes the work on row 1, and we repeat the procedure on the matrix consisting of the remaining rows. Thus the second circled column is the leading column of this smaller matrix, which we use to create the second leading 1 and the zeros below it. As the remaining row is zero here, we are finished. Then $A = LU$ where

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$

This matrix L is obtained from I_3 by replacing the bottom of the first two columns by the circled columns in the reduction. Note that the rank of A is 2 here, and this is the number of circled columns.

The calculation in Example 2.7.2 works in general. There is no need to calculate the elementary matrices E_i, and the method is suitable for use in a computer because the circled columns can be stored in memory as they are created. The procedure can be formally stated as follows:

LU-Algorithm

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. *If $A = 0$, take $L = I_m$ and $U = 0$.*
2. *If $A \neq 0$, write $A_1 = A$ and let $\mathbf{c}_1$ be the leading column of A_1. Use $\mathbf{c}_1$ to create the first leading 1 and create zeros below it (using lower reduction). When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.*
3. *If $A_2 \neq 0$, let $\mathbf{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.*
4. *Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.*
5. *Create L by placing $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.*

A proof of the LU-algorithm is given at the end of this section.

LU-factorization is particularly important if, as often happens in business and industry, a series of equations $A\mathbf{x} = B_1, A\mathbf{x} = B_2, \dots, A\mathbf{x} = B_k$, must be solved, each with the same coefficient matrix A. It is very efficient to solve the first system by gaussian elimination, simultaneously creating an LU-factorization of A, and then using the factorization to solve the remaining systems by forward and back substitution.

Example 2.7.3

Find an LU-factorization for $A = \left[\begin{array}{rrrrr} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{array}\right]$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} \boxed{\begin{matrix} 5 \\ -3 \\ -2 \\ 1 \end{matrix}} & \begin{matrix} -5 & 10 & 0 & 5 \\ 3 & 2 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 10 & 2 & 5 \end{matrix} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

If U denotes this row-echelon matrix, then $A = LU$, where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

The next example deals with a case where no row of zeros is present in U (in fact, A is invertible).

Example 2.7.4

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \boxed{5} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Hence $A = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

There are matrices (for example $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) that have no LU-factorization and so require at least one row interchange when being carried to row-echelon form via the gaussian algorithm. However, it turns out that, if all the row interchanges encountered in the algorithm are carried out first, the resulting matrix requires no interchanges and so has an LU-factorization. Here is the precise result.

Theorem 2.7.2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the gaussian algorithm. Let $P_1, P_2, \dots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. *PA is the matrix obtained from A by doing these interchanges (in order) to A.*
2. *PA has an LU-factorization.*

The proof is given at the end of this section.

A matrix P that is the product of elementary matrices corresponding to row interchanges is called a **permutation matrix**. Such a matrix is obtained from the identity matrix by arranging the rows in a different order, so it has exactly one 1 in each row and each column, and has zeros elsewhere. We regard the identity matrix as a permutation matrix. The elementary permutation matrices are those obtained from I by a single row interchange, and every permutation matrix is a product of elementary ones.

Example 2.7.5

If $A = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix}$, find a permutation matrix P such that PA has an LU-factorization, and then find the factorization.

Solution. Apply the gaussian algorithm to A:

$$A \xrightarrow{*} \begin{bmatrix} -1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 10 \\ 0 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{*} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Two row interchanges were needed (marked with $*$), first rows 1 and 2 and then rows 2 and 3. Hence, as in Theorem 2.7.2,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we do these interchanges (in order) to A, the result is PA. Now apply the LU-algorithm to PA:

$$PA = \begin{bmatrix} \boxed{-1} & -1 & 1 & 2 \\ 2 & 1 & -3 & 6 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & \boxed{-1} & -1 & 10 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & \boxed{-1} & 2 \\ 0 & 0 & -2 & 14 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & \boxed{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

Hence, $PA = LU$, where $L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 10 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Theorem 2.7.2 provides an important general factorization theorem for matrices. If A is any $m \times n$ matrix, it asserts that there exists a permutation matrix P and an LU-factorization $PA = LU$. Moreover, it shows that either $P = I$ or $P = P_s \cdots P_2 P_1$, where $P_1, P_2, \dots, P_s$ are the elementary permutation matrices arising in the reduction of A to row-echelon form. Now observe that $P_i^{-1} = P_i$ for each i (they are elementary row interchanges). Thus, $P^{-1} = P_1 P_2 \cdots P_s$, so the matrix A can be factored as

$$A = P^{-1} LU$$

where P^{-1} is a permutation matrix, L is lower triangular and invertible, and U is a row-echelon matrix. This is called a **PLU-factorization** of A.

The LU-factorization in Theorem 2.7.1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

However, it is necessary here that the row-echelon matrix has a row of zeros. Recall that the rank of a matrix A is the number of nonzero rows in any row-echelon matrix U to which A can be carried by row operations. Thus, if A is $m \times n$, the matrix U has no row of zeros if and only if A has rank m.

Theorem 2.7.3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m (that is, U has no row of zeros), then L and U are uniquely determined by A.

Proof. Suppose $A = MV$ is another LU-factorization of A, so M is lower triangular and invertible and V is row-echelon. Hence $LU = MV$, and we must show that $L = M$ and $U = V$. We write $N = M^{-1}L$. Then N

is lower triangular and invertible (Lemma 2.7.1) and $NU = V$, so it suffices to prove that $N = I$. If N is $m \times m$, we use induction on m. The case $m = 1$ is left to the reader. If $m > 1$, observe first that column 1 of V is N times column 1 of U. Thus if either column is zero, so is the other (N is invertible). Hence, we can assume (by deleting zero columns) that the $(1,\ 1)$-entry is 1 in both U and V.

Now we write $N = \left[\begin{array}{cc} a & 0 \\ X & N_1 \end{array} \right]$, $U = \left[\begin{array}{cc} 1 & Y \\ 0 & U_1 \end{array} \right]$, and $V = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array} \right]$ in block form. Then $NU = V$ becomes $\left[\begin{array}{cc} a & aY \\ X & XY + N_1U_1 \end{array} \right] = \left[\begin{array}{cc} 1 & Z \\ 0 & V_1 \end{array} \right]$. Hence $a = 1$, $Y = Z$, $X = 0$, and $N_1U_1 = V_1$. But $N_1U_1 = V_1$ implies $N_1 = I$ by induction, whence $N = I$. □

If A is an $m \times m$ invertible matrix, then A has rank m by Theorem 2.4.5. Hence, we get the following important special case of Theorem 2.7.3.

Corollary 2.7.1

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

Of course, in this case U is an upper triangular matrix with 1s along the main diagonal.

Proofs of Theorems

Proof of the LU-Algorithm. If $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ are columns of lengths $m,\ m-1,\ \dots,\ m-r+1$, respectively, write $L^{(m)}(\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r)$ for the lower triangular $m \times m$ matrix obtained from I_m by placing $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ at the bottom of the first r columns of I_m.

Proceed by induction on n. If $A = 0$ or $n = 1$, it is left to the reader. If $n > 1$, let $\mathbf{c}_1$ denote the leading column of A and let $\mathbf{k}_1$ denote the first column of the $m \times m$ identity matrix. There exist elementary matrices $E_1,\ \dots,\ E_k$ such that, in block form,

$$(E_k \cdots E_2E_1)A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline A_1 \end{array} \end{array} \right] \quad \text{where } (E_k \cdots E_2E_1)\mathbf{c}_1 = \mathbf{k}_1$$

Moreover, each E_j can be taken to be lower triangular (by assumption). Write

$$G = (E_k \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_k^{-1}$$

Then G is lower triangular, and $G\mathbf{k}_1 = \mathbf{c}_1$. Also, each E_j (and so each E_j^{-1}) is the result of either multiplying row 1 of I_m by a constant or adding a multiple of row 1 to another row. Hence,

$$G = (E_1^{-1}E_2^{-1} \cdots E_k^{-1})I_m = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array} \right]$$

in block form. Now, by induction, let $A_1 = L_1U_1$ be an LU-factorization of A_1, where $L_1 = L^{(m-1)}[\mathbf{c}_2,\ \dots,\ \mathbf{c}_r]$ and U_1 is row-echelon. Then block multiplication gives

$$G^{-1}A = \left[\begin{array}{c|c|c} 0 & \mathbf{k}_1 & \begin{array}{c} X_1 \\ \hline L_1U_1 \end{array} \end{array} \right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array} \right] \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array} \right]$$

Hence $A = LU$, where $U = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & U_1 \end{array}\right]$ is row-echelon and

$$L = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline I_{m-1} \end{array} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{c}_1 & \begin{array}{c} 0 \\ \hline L \end{array} \end{array}\right] = L^{(m)}\left[\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r\right]$$

This completes the proof. □

Proof of Theorem 2.7.2. Let A be a nonzero $m \times n$ matrix and let $\mathbf{k}_j$ denote column j of I_m. There is a permutation matrix P_1 (where either P_1 is elementary or $P_1 = I_m$) such that the first nonzero column $\mathbf{c}_1$ of P_1A has a nonzero entry on top. Hence, as in the LU-algorithm,

$$L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1 \end{array}\right]$$

in block form. Then let P_2 be a permutation matrix (either elementary or I_m) such that

$$P_2 \cdot L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & A_1' \end{array}\right]$$

and the first nonzero column $\mathbf{c}_2$ of A_1' has a nonzero entry on top. Thus,

$$L^{(m)}\left[\mathbf{k}_1,\ \mathbf{c}_2\right]^{-1} \cdot P_2 \cdot L^{(m)}\left[\mathbf{c}_1\right]^{-1} \cdot P_1 \cdot A = \left[\begin{array}{c|c|c} 0 & 1 & X_1 \\ \hline 0 & 0 & \begin{array}{c|c|c} 0 & 1 & X_2 \\ \hline 0 & 0 & A_2 \end{array} \end{array}\right]$$

in block form. Continue to obtain elementary permutation matrices $P_1,\ P_2,\ \dots,\ P_r$ and columns $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_r$ of lengths $m,\ m-1,\ \dots,$ such that

$$(L_rP_rL_{r-1}P_{r-1}\cdots L_2P_2L_1P_1)A = U$$

where U is a row-echelon matrix and $L_j = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j\right]^{-1}$ for each j, where the notation means the first $j-1$ columns are those of I_m. It is not hard to verify that each L_j has the form $L_j = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j'\right]$ where $\mathbf{c}_j'$ is a column of length $m-j+1$. We now claim that each permutation matrix P_k can be "moved past" each matrix L_j to the right of it, in the sense that

$$P_kL_j = L_j'P_k$$

where $L_j' = L^{(m)}\left[\mathbf{k}_1,\ \dots,\ \mathbf{k}_{j-1},\ \mathbf{c}_j''\right]$ for some column $\mathbf{c}_j''$ of length $m-j+1$. Given that this is true, we obtain a factorization of the form

$$(L_rL_{r-1}'\cdots L_2'L_1')(P_rP_{r-1}\cdots P_2P_1)A = U$$

If we write $P = P_rP_{r-1}\cdots P_2P_1$, this shows that PA has an LU-factorization because $L_rL_{r-1}'\cdots L_2'L_1'$ is lower triangular and invertible. All that remains is to prove the following rather technical result. □

Lemma 2.7.2

Let P_k result from interchanging row k of I_m with a row below it. If $j < k$, let c_j be a column of length $m - j + 1$. Then there is another column $\mathbf{c}'_j$ of length $m - j + 1$ such that

$$P_k \cdot L^{(m)} \left[\mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{c}_j\right] = L^{(m)} \left[\mathbf{k}_1, \ldots, \mathbf{k}_{j-1}, \mathbf{c}'_j\right] \cdot P_k$$

The proof is left as Exercise 2.7.11.

Exercises for 2.7

Exercise 2.7.1 Find an LU-factorization of the following matrices.

a. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 3 & 9 & -3 & 3 & 1 \\ -1 & -3 & 1 & -3 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 2 & 4 & 2 \\ 1 & -1 & 3 \\ -1 & 7 & -7 \end{array}\right]$

c. $\left[\begin{array}{rrrrr} 2 & 6 & -2 & 0 & 2 \\ 1 & 5 & -1 & 2 & 5 \\ 3 & 7 & -3 & -2 & 5 \\ -1 & -1 & 1 & 2 & 3 \end{array}\right]$

d. $\left[\begin{array}{rrrrr} -1 & -3 & 1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 & 1 \\ 0 & -2 & -4 & -2 & 0 \end{array}\right]$

e. $\left[\begin{array}{rrrrrr} 2 & 2 & 4 & 6 & 0 & 2 \\ 1 & -1 & 2 & 1 & 3 & 1 \\ -2 & 2 & -4 & -1 & 1 & 6 \\ 0 & 2 & 0 & 3 & 4 & 8 \\ -2 & 4 & -4 & 1 & -2 & 6 \end{array}\right]$

f. $\left[\begin{array}{rrrrr} 2 & 2 & -2 & 4 & 2 \\ 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & -2 & 6 & 3 \\ 1 & 3 & -2 & 2 & 1 \end{array}\right]$

Exercise 2.7.2 Find a permutation matrix P and an LU-factorization of PA if A is:

a. $\left[\begin{array}{rrr} 0 & 0 & 2 \\ 0 & -1 & 4 \\ 3 & 5 & 1 \end{array}\right]$

b. $\left[\begin{array}{rrr} 0 & -1 & 2 \\ 0 & 0 & 4 \\ -1 & 2 & 1 \end{array}\right]$

c. $\left[\begin{array}{rrrrr} 0 & -1 & 2 & 1 & 3 \\ -1 & 1 & 3 & 1 & 4 \\ 1 & -1 & -3 & 6 & 2 \\ 2 & -2 & -4 & 1 & 0 \end{array}\right]$

d. $\left[\begin{array}{rrrr} -1 & -2 & 3 & 0 \\ 2 & 4 & -6 & 5 \\ 1 & 1 & -1 & 3 \\ 2 & 5 & -10 & 1 \end{array}\right]$

Exercise 2.7.3 In each case use the given LU-decomposition of A to solve the system $A\mathbf{x} = \mathbf{b}$ by finding $\mathbf{y}$ such that $L\mathbf{y} = \mathbf{b}$, and then $\mathbf{x}$ such that $U\mathbf{x} = \mathbf{y}$:

a. $A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 3 \end{array}\right] \left[\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$;

$\mathbf{b} = \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right]$

b. $A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 1 \end{array}\right] \left[\begin{array}{rrrr} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

$\mathbf{b} = \left[\begin{array}{r} -2 \\ -1 \\ 1 \end{array}\right]$

c. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix};$

$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$

$\mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 4 \\ 5 \end{bmatrix}$

Exercise 2.7.4 Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = LU$ is impossible where L is lower triangular and U is upper triangular.

Exercise 2.7.5 Show that we can accomplish any row interchange by using only row operations of other types.

Exercise 2.7.6

a. Let L and L_1 be invertible lower triangular matrices, and let U and U_1 be invertible upper triangular matrices. Show that $LU = L_1U_1$ if and only if there exists an invertible diagonal matrix D such that $L_1 = LD$ and $U_1 = D^{-1}U$. [*Hint*: Scrutinize $L^{-1}L_1 = UU_1^{-1}$.]

b. Use part (a) to prove Theorem 2.7.3 in the case that A is invertible.

Exercise 2.7.7 Prove Lemma 2.7.1(1). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.8 Prove Lemma 2.7.1(2). [*Hint*: Use block multiplication and induction.]

Exercise 2.7.9 A triangular matrix is called **unit triangular** if it is square and every main diagonal element is a 1.

a. If A can be carried by the gaussian algorithm to row-echelon form using no row interchanges, show that $A = LU$ where L is unit lower triangular and U is upper triangular.

b. Show that the factorization in (a.) is unique.

Exercise 2.7.10 Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ be columns of lengths $m, m-1, \dots, m-r+1$. If $\mathbf{k}_j$ denotes column j of I_m, show that $L^{(m)}[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r] = L^{(m)}[\mathbf{c}_1] L^{(m)}[\mathbf{k}_1, \mathbf{c}_2] L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \mathbf{c}_3] \cdots$
$L^{(m)}[\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_{r-1}, \mathbf{c}_r]$. The notation is as in the proof of Theorem 2.7.2. [*Hint*: Use induction on m and block multiplication.]

Exercise 2.7.11 Prove Lemma 2.7.2. [*Hint*: $P_k^{-1} = P_k$. Write $P_k = \begin{bmatrix} I_k & 0 \\ 0 & P_0 \end{bmatrix}$ in block form where P_0 is an $(m-k) \times (m-k)$ permutation matrix.]

2.8 An Application to Input-Output Economic Models[16]

In 1973 Wassily Leontief was awarded the Nobel prize in economics for his work on mathematical models.[17] Roughly speaking, an economic system in this model consists of several industries, each of which produces a product and each of which uses some of the production of the other industries. The following example is typical.

[16]The applications in this section and the next are independent and may be taken in any order.

[17]See W. W. Leontief, "The world economy of the year 2000," *Scientific American,* Sept. 1980.

Example 2.8.1

A primitive society has three basic needs: food, shelter, and clothing. There are thus three industries in the society—the farming, housing, and garment industries—that produce these commodities. Each of these industries consumes a certain proportion of the total output of each commodity according to the following table.

		OUTPUT		
		Farming	**Housing**	**Garment**
	Farming	0.4	0.2	0.3
CONSUMPTION	**Housing**	0.2	0.6	0.4
	Garment	0.4	0.2	0.3

Find the annual prices that each industry must charge for its income to equal its expenditures.

Solution. Let p_1, p_2, and p_3 be the prices charged per year by the farming, housing, and garment industries, respectively, for their total output. To see how these prices are determined, consider the farming industry. It receives p_1 for its production in any year. But it *consumes* products from all these industries in the following amounts (from row 1 of the table): 40% of the food, 20% of the housing, and 30% of the clothing. Hence, the expenditures of the farming industry are $0.4p_1 + 0.2p_2 + 0.3p_3$, so

$$0.4p_1 + 0.2p_2 + 0.3p_3 = p_1$$

A similar analysis of the other two industries leads to the following system of equations.

$$\begin{aligned} 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_1 \\ 0.2p_1 + 0.6p_2 + 0.4p_3 &= p_2 \\ 0.4p_1 + 0.2p_2 + 0.3p_3 &= p_3 \end{aligned}$$

This has the matrix form $E\mathbf{p} = \mathbf{p}$, where

$$E = \left[\begin{array}{ccc} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.3 \end{array} \right] \quad \text{and} \quad \mathbf{p} = \left[\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \right]$$

The equations can be written as the homogeneous system

$$(I - E)\mathbf{p} = \mathbf{0}$$

where I is the 3×3 identity matrix, and the solutions are

$$\mathbf{p} = \left[\begin{array}{c} 2t \\ 3t \\ 2t \end{array} \right]$$

where t is a parameter. Thus, the pricing must be such that the total output of the farming industry has the same value as the total output of the garment industry, whereas the total value of the housing industry must be $\frac{3}{2}$ as much.

In general, suppose an economy has n industries, each of which uses some (possibly none) of the production of every industry. We assume first that the economy is **closed** (that is, no product is exported or imported) and that all product is used. Given two industries i and j, let e_{ij} denote the proportion of the total annual output of industry j that is consumed by industry i. Then $E = \left[e_{ij}\right]$ is called the **input-output** matrix for the economy. Clearly,

$$0 \leq e_{ij} \leq 1 \quad \text{for all } i \text{ and } j \tag{2.12}$$

Moreover, all the output from industry j is used by *some* industry (the model is closed), so

$$e_{1j} + e_{2j} + \cdots + e_{ij} = 1 \quad \text{for each } j \tag{2.13}$$

This condition asserts that each column of E sums to 1. Matrices satisfying conditions (2.12) and (2.13) are called **stochastic matrices**.

As in Example 2.8.1, let p_i denote the price of the total annual production of industry i. Then p_i is the annual revenue of industry i. On the other hand, industry i spends $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$ annually for the product it uses ($e_{ij}p_j$ is the cost for product from industry j). The closed economic system is said to be in **equilibrium** if the annual expenditure equals the annual revenue for each industry—that is, if

$$e_{1j}p_1 + e_{2j}p_2 + \cdots + e_{ij}p_n = p_i \quad \text{for each } i = 1,\ 2,\ \dots,\ n$$

If we write $\mathbf{p} = \left[\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \right]$, these equations can be written as the matrix equation

$$E\mathbf{p} = \mathbf{p}$$

This is called the **equilibrium condition**, and the solutions $\mathbf{p}$ are called **equilibrium price structures**. The equilibrium condition can be written as

$$(I - E)\mathbf{p} = \mathbf{0}$$

which is a system of homogeneous equations for $\mathbf{p}$. Moreover, there is always a nontrivial solution $\mathbf{p}$. Indeed, the column sums of $I - E$ are all 0 (because E is stochastic), so the row-echelon form of $I - E$ has a row of zeros. In fact, more is true:

Theorem 2.8.1

Let E be any $n \times n$ stochastic matrix. Then there is a nonzero $n \times 1$ vector $\mathbf{p}$ with nonnegative entries such that $E\mathbf{p} = \mathbf{p}$. If all the entries of E are positive, the matrix $\mathbf{p}$ can be chosen with all entries positive.

Theorem 2.8.1 guarantees the existence of an equilibrium price structure for any closed input-output system of the type discussed here. The proof is beyond the scope of this book.[18]

[18]The interested reader is referred to P. Lancaster's *Theory of Matrices* (New York: Academic Press, 1969) or to E. Seneta's *Non-negative Matrices* (New York: Wiley, 1973).

Example 2.8.2

Find the equilibrium price structures for four industries if the input-output matrix is

$$E = \begin{bmatrix} 0.6 & 0.2 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0 \\ 0.1 & 0.3 & 0.5 & 0.2 \\ 0 & 0.1 & 0.2 & 0.7 \end{bmatrix}$$

Find the prices if the total value of business is \$1000.

Solution. If $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$ is the equilibrium price structure, then the equilibrium condition reads $E\mathbf{p} = \mathbf{p}$. When we write this as $(I - E)\mathbf{p} = \mathbf{0}$, the methods of Chapter 1 yield the following family of solutions:

$$\mathbf{p} = \begin{bmatrix} 44t \\ 39t \\ 51t \\ 47t \end{bmatrix}$$

where t is a parameter. If we insist that $p_1 + p_2 + p_3 + p_4 = 1000$, then $t = 5.525$. Hence

$$\mathbf{p} = \begin{bmatrix} 243.09 \\ 215.47 \\ 281.76 \\ 259.67 \end{bmatrix}$$

to five figures.

The Open Model

We now assume that there is a demand for products in the **open sector** of the economy, which is the part of the economy other than the producing industries (for example, consumers). Let d_i denote the total value of the demand for product i in the open sector. If p_i and e_{ij} are as before, the value of the annual demand for product i by the producing industries themselves is $e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n$, so the total annual revenue p_i of industry i breaks down as follows:

$$p_i = (e_{i1}p_1 + e_{i2}p_2 + \cdots + e_{in}p_n) + d_i \quad \text{for each } i = 1,\ 2,\ \dots,\ n$$

The column $\mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ is called the **demand matrix**, and this gives a matrix equation

$$\mathbf{p} = E\mathbf{p} + \mathbf{d}$$

or

$$(I-E)\mathbf{p}=\mathbf{d} \tag{2.14}$$

This is a system of linear equations for $\mathbf{p}$, and we ask for a solution $\mathbf{p}$ with every entry nonnegative. Note that every entry of E is between 0 and 1, but the column sums of E need not equal 1 as in the closed model.

Before proceeding, it is convenient to introduce a useful notation. If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are matrices of the same size, we write $A>B$ if $a_{ij}>b_{ij}$ for all i and j, and we write $A\geq B$ if $a_{ij}\geq b_{ij}$ for all i and j. Thus $P\geq 0$ means that every entry of P is nonnegative. Note that $A\geq 0$ and $B\geq 0$ implies that $AB\geq 0$.

Now, given a demand matrix $\mathbf{d}\geq \mathbf{0}$, we look for a production matrix $\mathbf{p}\geq \mathbf{0}$ satisfying equation (2.14). This certainly exists if $I-E$ is invertible and $(I-E)^{-1}\geq 0$. On the other hand, the fact that $\mathbf{d}\geq \mathbf{0}$ means any solution $\mathbf{p}$ to equation (2.14) satisfies $\mathbf{p}\geq E\mathbf{p}$. Hence, the following theorem is not too surprising.

Theorem 2.8.2

Let $E\geq 0$ be a square matrix. Then $I-E$ is invertible and $(I-E)^{-1}\geq 0$ if and only if there exists a column $\mathbf{p}>\mathbf{0}$ such that $\mathbf{p}>E\mathbf{p}$.

Heuristic Proof.

If $(I-E)^{-1}\geq 0$, the existence of $\mathbf{p}>\mathbf{0}$ with $\mathbf{p}>E\mathbf{p}$ is left as Exercise 2.8.11. Conversely, suppose such a column $\mathbf{p}$ exists. Observe that

$$(I-E)(I+E+E^2+\cdots+E^{k-1})=I-E^k$$

holds for all $k\geq 2$. If we can show that every entry of E^k approaches 0 as k becomes large then, intuitively, the infinite matrix sum

$$U=I+E+E^2+\cdots$$

exists and $(I-E)U=I$. Since $U\geq 0$, this does it. To show that E^k approaches 0, it suffices to show that $EP<\mu P$ for some number μ with $0<\mu<1$ (then $E^kP<\mu^kP$ for all $k\geq 1$ by induction). The existence of μ is left as Exercise 2.8.12. □

The condition $\mathbf{p}>E\mathbf{p}$ in Theorem 2.8.2 has a simple economic interpretation. If $\mathbf{p}$ is a production matrix, entry i of $E\mathbf{p}$ is the total value of all product used by industry i in a year. Hence, the condition $\mathbf{p}>E\mathbf{p}$ means that, for each i, the value of product produced by industry i exceeds the value of the product it uses. In other words, each industry runs at a profit.

Example 2.8.3

If $E=\left[\begin{array}{ccc}0.6&0.2&0.3\\0.1&0.4&0.2\\0.2&0.5&0.1\end{array}\right]$, show that $I-E$ is invertible and $(I-E)^{-1}\geq 0$.

Solution. Use $\mathbf{p}=(3,\ 2,\ 2)^T$ in Theorem 2.8.2.

If $\mathbf{p}_0=(1,\ 1,\ 1)^T$, the entries of $E\mathbf{p}_0$ are the row sums of E. Hence $\mathbf{p}_0>E\mathbf{p}_0$ holds if the row sums of E are all less than 1. This proves the first of the following useful facts (the second is Exercise 2.8.10).

Corollary 2.8.1

Let $E \geq 0$ be a square matrix. In each case, $I - E$ is invertible and $(I - E)^{-1} \geq 0$:

1. *All row sums of E are less than 1.*
2. *All column sums of E are less than 1.*

Exercises for 2.8

Exercise 2.8.1 Find the possible equilibrium price structures when the input-output matrices are:

a. $\left[\begin{array}{ccc} 0.1 & 0.2 & 0.3 \\ 0.6 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.4 \end{array} \right]$ b. $\left[\begin{array}{ccc} 0.5 & 0 & 0.5 \\ 0.1 & 0.9 & 0.2 \\ 0.4 & 0.1 & 0.3 \end{array} \right]$

c. $\left[\begin{array}{cccc} 0.3 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.1 & 0 \\ 0.3 & 0.3 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.6 & 0.7 \end{array} \right]$

d. $\left[\begin{array}{cccc} 0.5 & 0 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0 & 0.1 \\ 0.1 & 0.2 & 0.8 & 0.2 \\ 0.2 & 0.1 & 0.1 & 0.6 \end{array} \right]$

Exercise 2.8.2 Three industries A, B, and C are such that all the output of A is used by B, all the output of B is used by C, and all the output of C is used by A. Find the possible equilibrium price structures.

Exercise 2.8.3 Find the possible equilibrium price structures for three industries where the input-output matrix is $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]$. Discuss why there are two parameters here.

Exercise 2.8.4 Prove Theorem 2.8.1 for a 2×2 stochastic matrix E by first writing it in the form $E = \left[\begin{array}{cc} a & b \\ 1-a & 1-b \end{array} \right]$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

Exercise 2.8.5 If E is an $n \times n$ stochastic matrix and $\mathbf{c}$ is an $n \times 1$ matrix, show that the sum of the entries of $\mathbf{c}$ equals the sum of the entries of the $n \times 1$ matrix $E\mathbf{c}$.

Exercise 2.8.6 Let $W = \left[\begin{array}{ccccc} 1 & 1 & 1 & \cdots & 1 \end{array} \right]$. Let E and F denote $n \times n$ matrices with nonnegative entries.

a. Show that E is a stochastic matrix if and only if $WE = W$.

b. Use part (a.) to deduce that, if E and F are both stochastic matrices, then EF is also stochastic.

Exercise 2.8.7 Find a 2×2 matrix E with entries between 0 and 1 such that:

a. $I - E$ has no inverse.

b. $I - E$ has an inverse but not all entries of $(I - E)^{-1}$ are nonnegative.

Exercise 2.8.8 If E is a 2×2 matrix with entries between 0 and 1, show that $I - E$ is invertible and $(I - E)^{-1} \geq 0$ if and only if $\text{tr } E < 1 + \det E$. Here, if $E = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$, then $\text{tr } E = a + d$ and $\det E = ad - bc$.

Exercise 2.8.9 In each case show that $I - E$ is invertible and $(I - E)^{-1} \geq 0$.

a. $\left[\begin{array}{ccc} 0.6 & 0.5 & 0.1 \\ 0.1 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{array} \right]$ b. $\left[\begin{array}{ccc} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.1 & 0.4 \end{array} \right]$

c. $\left[\begin{array}{ccc} 0.6 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.5 & 0.1 \end{array} \right]$ d. $\left[\begin{array}{ccc} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.1 & 0.2 \\ 0.3 & 0.3 & 0.2 \end{array} \right]$

Exercise 2.8.10 Prove that (1) implies (2) in the Corollary to Theorem 2.8.2.

Exercise 2.8.11 If $(I - E)^{-1} \geq 0$, find $\mathbf{p} > 0$ such that $\mathbf{p} > E\mathbf{p}$.

Exercise 2.8.12 If $E\mathbf{p} < \mathbf{p}$ where $E \geq 0$ and $\mathbf{p} > 0$, find a number μ such that $E\mathbf{p} < \mu\mathbf{p}$ and $0 < \mu < 1$.

[*Hint*: If $E\mathbf{p} = (q_1, \dots, q_n)^T$ and $\mathbf{p} = (p_1, \dots, p_n)^T$, take any number μ where $\max \left\{ \frac{q_1}{p_1}, \dots, \frac{q_n}{p_n} \right\} < \mu < 1$.]

2.9 An Application to Markov Chains

Many natural phenomena progress through various stages and can be in a variety of states at each stage. For example, the weather in a given city progresses day by day and, on any given day, may be sunny or rainy. Here the states are "sun" and "rain," and the weather progresses from one state to another in daily stages. Another example might be a football team: The stages of its evolution are the games it plays, and the possible states are "win," "draw," and "loss."

The general setup is as follows: A real conceptual "system" is run generating a sequence of outcomes. The system evolves through a series of "stages," and at any stage it can be in any one of a finite number of "states." At any given stage, the state to which it will go at the next stage depends on the past and present history of the system—that is, on the sequence of states it has occupied to date.

Definition 2.15 Markov Chain

A ***Markov chain*** *is such an evolving system wherein the state to which it will go next depends only on its present state and does not depend on the earlier history of the system.*[19]

Even in the case of a Markov chain, the state the system will occupy at any stage is determined only in terms of probabilities. In other words, chance plays a role. For example, if a football team wins a particular game, we do not know whether it will win, draw, or lose the next game. On the other hand, we may know that the team tends to persist in winning streaks; for example, if it wins one game it may win the next game $\frac{1}{2}$ of the time, lose $\frac{4}{10}$ of the time, and draw $\frac{1}{10}$ of the time. These fractions are called the **probabilities** of these various possibilities. Similarly, if the team loses, it may lose the next game with probability $\frac{1}{2}$ (that is, half the time), win with probability $\frac{1}{4}$, and draw with probability $\frac{1}{4}$. The probabilities of the various outcomes after a drawn game will also be known.

We shall treat probabilities informally here: *The probability that a given event will occur is the long-run proportion of the time that the event does indeed occur*. Hence, all probabilities are numbers between 0 and 1. A probability of 0 means the event is impossible and never occurs; events with probability 1 are certain to occur.

If a Markov chain is in a particular state, the probabilities that it goes to the various states at the next stage of its evolution are called the **transition probabilities** for the chain, and they are assumed to be known quantities. To motivate the general conditions that follow, consider the following simple example. Here the system is a man, the stages are his successive lunches, and the states are the two restaurants he chooses.

Example 2.9.1

A man always eats lunch at one of two restaurants, A and B. He never eats at A twice in a row. However, if he eats at B, he is three times as likely to eat at B next time as at A. Initially, he is equally likely to eat at either restaurant.

a. What is the probability that he eats at A on the third day after the initial one?

[19] The name honours Andrei Andreyevich Markov (1856–1922) who was a professor at the university in St. Petersburg, Russia.

b. What proportion of his lunches does he eat at A?

Solution. The table of transition probabilities follows. The A column indicates that if he eats at A on one day, he never eats there again on the next day and so is certain to go to B.

		Present Lunch	
		A	B
Next	A	0	0.25
Lunch	B	1	0.75

The B column shows that, if he eats at B on one day, he will eat there on the next day $\frac{3}{4}$ of the time and switches to A only $\frac{1}{4}$ of the time.
The restaurant he visits on a given day is not determined. The most that we can expect is to know the probability that he will visit A or B on that day.

Let $\mathbf{s}_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \end{bmatrix}$ denote the *state vector* for day m. Here $s_1^{(m)}$ denotes the probability that he eats at A on day m, and $s_2^{(m)}$ is the probability that he eats at B on day m. It is convenient to let $\mathbf{s}_0$ correspond to the initial day. Because he is equally likely to eat at A or B on that initial day, $s_1^{(0)} = 0.5$ and $s_2^{(0)} = 0.5$, so $\mathbf{s}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. Now let

$$P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$$

denote the *transition matrix*. We claim that the relationship

$$\mathbf{s}_{m+1} = P\mathbf{s}_m$$

holds for all integers $m \geq 0$. This will be derived later; for now, we use it as follows to successively compute $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$.

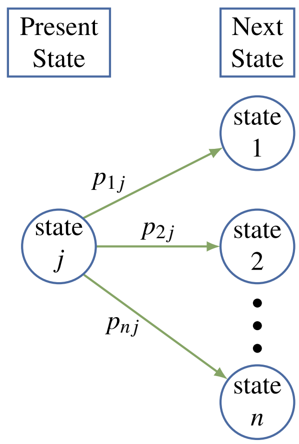
$$\begin{aligned}
\mathbf{s}_1 = P\mathbf{s}_0 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix} \\
\mathbf{s}_2 = P\mathbf{s}_1 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix} = \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix} \\
\mathbf{s}_3 = P\mathbf{s}_2 &= \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix} = \begin{bmatrix} 0.1953125 \\ 0.8046875 \end{bmatrix}
\end{aligned}$$

Hence, the probability that his third lunch (after the initial one) is at A is approximately 0.195, whereas the probability that it is at B is 0.805. If we carry these calculations on, the next state vectors are (to five figures):

$$\mathbf{s}_4 = \begin{bmatrix} 0.20117 \\ 0.79883 \end{bmatrix} \quad \mathbf{s}_5 = \begin{bmatrix} 0.19971 \\ 0.80029 \end{bmatrix}$$

$$\mathbf{s}_6 = \begin{bmatrix} 0.20007 \\ 0.79993 \end{bmatrix} \quad \mathbf{s}_7 = \begin{bmatrix} 0.19998 \\ 0.80002 \end{bmatrix}$$

Moreover, as m increases the entries of $\mathbf{s}_m$ get closer and closer to the corresponding entries of $\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. Hence, in the long run, he eats 20% of his lunches at A and 80% at B.



Example 2.9.1 incorporates most of the essential features of all Markov chains. The general model is as follows: The system evolves through various stages and at each stage can be in exactly one of n distinct states. It progresses through a sequence of states as time goes on. If a Markov chain is in state j at a particular stage of its development, the probability p_{ij} that it goes to state i at the next stage is called the **transition probability**. The $n \times n$ matrix $P = \left[p_{ij}\right]$ is called the **transition matrix** for the Markov chain. The situation is depicted graphically in the diagram.

We make one important assumption about the transition matrix $P = \left[p_{ij}\right]$: It does *not* depend on which stage the process is in. This assumption means that the transition probabilities are *independent of time*—that is, they do not change as time goes on. It is this assumption that distinguishes Markov chains in the literature of this subject.

Example 2.9.2

Suppose the transition matrix of a three-state Markov chain is

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{array}{c} \text{Present state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 0.3 & 0.1 & 0.6 \\ 0.5 & 0.9 & 0.2 \\ 0.2 & 0.0 & 0.2 \end{array}\right] \end{array} \begin{array}{l} \\ \\ 1 \\ 2 \;\text{Next state} \\ 3 \end{array}$$

If, for example, the system is in state 2, then column 2 lists the probabilities of where it goes next. Thus, the probability is $p_{12} = 0.1$ that it goes from state 2 to state 1, and the probability is $p_{22} = 0.9$ that it goes from state 2 to state 2. The fact that $p_{32} = 0$ means that it is impossible for it to go from state 2 to state 3 at the next stage.

Consider the jth column of the transition matrix P.

$$\begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{bmatrix}$$

If the system is in state j at some stage of its evolution, the transition probabilities $p_{1j}, p_{2j}, \dots, p_{nj}$ represent the fraction of the time that the system will move to state 1, state 2, ..., state n, respectively, at the next stage. We assume that it has to go to *some* state at each transition, so the sum of these probabilities is 1:

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1 \quad \text{for each } j$$

Thus, the columns of P all sum to 1 and the entries of P lie between 0 and 1. Hence P is called a **stochastic matrix**.

As in Example 2.9.1, we introduce the following notation: Let $s_i^{(m)}$ denote the probability that the system is in state i after m transitions. The $n \times 1$ matrices

$$\mathbf{s}_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \\ \vdots \\ s_n^{(m)} \end{bmatrix} \qquad m = 0,\ 1,\ 2,\ \dots$$

are called the **state vectors** for the Markov chain. Note that the sum of the entries of $\mathbf{s}_m$ must equal 1 because the system must be in *some* state after m transitions. The matrix $\mathbf{s}_0$ is called the **initial state vector** for the Markov chain and is given as part of the data of the particular chain. For example, if the chain has only two states, then an initial vector $\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means that it started in state 1. If it started in state 2, the initial vector would be $\mathbf{s}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $\mathbf{s}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, it is equally likely that the system started in state 1 or in state 2.

Theorem 2.9.1

Let P be the transition matrix for an n-state Markov chain. If $\mathbf{s}_m$ is the state vector at stage m, then

$$\mathbf{s}_{m+1} = P\mathbf{s}_m$$

for each $m = 0,\ 1,\ 2,\ \dots$.

Heuristic Proof. Suppose that the Markov chain has been run N times, each time starting with the same initial state vector. Recall that p_{ij} is the proportion of the time the system goes from state j at some stage to state i at the next stage, whereas $s_i^{(m)}$ is the proportion of the time it is in state i at stage m. Hence

$$s_i^{m+1}N$$

is (approximately) the number of times the system is in state i at stage $m+1$. We are going to calculate this number another way. The system got to state i at stage $m+1$ through *some* other state (say state j) at stage m. The number of times it was *in* state j at that stage is (approximately) $s_j^{(m)}N$, so the number of times it got to state i via state j is $p_{ij}(s_j^{(m)}N)$. Summing over j gives the number of times the system is in state i (at stage $m+1$). This is the number we calculated before, so

$$s_i^{(m+1)}N = p_{i1}s_1^{(m)}N + p_{i2}s_2^{(m)}N + \cdots + p_{in}s_n^{(m)}N$$

Dividing by N gives $s_i^{(m+1)} = p_{i1}s_1^{(m)} + p_{i2}s_2^{(m)} + \cdots + p_{in}s_n^{(m)}$ for each i, and this can be expressed as the matrix equation $\mathbf{s}_{m+1} = P\mathbf{s}_m$. □

If the initial probability vector $\mathbf{s}_0$ and the transition matrix P are given, Theorem 2.9.1 gives $\mathbf{s}_1,\ \mathbf{s}_2,\ \mathbf{s}_3,\ \dots,$ one after the other, as follows:

$$\begin{aligned} \mathbf{s}_1 &= P\mathbf{s}_0 \\ \mathbf{s}_2 &= P\mathbf{s}_1 \\ \mathbf{s}_3 &= P\mathbf{s}_2 \\ &\vdots \end{aligned}$$

Hence, the state vector $\mathbf{s}_m$ is completely determined for each $m = 0,\ 1,\ 2,\ \dots$ by P and $\mathbf{s}_0$.

Example 2.9.3

A wolf pack always hunts in one of three regions R_1, R_2, and R_3. Its hunting habits are as follows:

1. If it hunts in some region one day, it is as likely as not to hunt there again the next day.
2. If it hunts in R_1, it never hunts in R_2 the next day.
3. If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other regions the next day.

If the pack hunts in R_1 on Monday, find the probability that it hunts there on Thursday.

Solution. The stages of this process are the successive days; the states are the three regions. The transition matrix P is determined as follows (see the table): The first habit asserts that $p_{11} = p_{22} = p_{33} = \frac{1}{2}$. Now column 1 displays what happens when the pack starts in R_1: It never goes to state 2, so $p_{21} = 0$ and, because the column must sum to 1, $p_{31} = \frac{1}{2}$. Column 2 describes what happens if it starts in R_2: $p_{22} = \frac{1}{2}$ and p_{12} and p_{32} are equal (by habit 3), so $p_{12} = p_{32} = \frac{1}{2}$ because the column sum must equal 1. Column 3 is filled in a similar way.

	R_1	R_2	R_3
R_1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
R_2	0	$\frac{1}{2}$	$\frac{1}{4}$
R_3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$

Now let Monday be the initial stage. Then $\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the pack hunts in R_1 on that day.

Then $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ describe Tuesday, Wednesday, and Thursday, respectively, and we compute them using Theorem 2.9.1.

$$\mathbf{s}_1 = P\mathbf{s}_0 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad \mathbf{s}_2 = P\mathbf{s}_1 = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{4}{8} \end{bmatrix} \qquad \mathbf{s}_3 = P\mathbf{s}_2 = \begin{bmatrix} \frac{11}{32} \\ \frac{6}{32} \\ \frac{15}{32} \end{bmatrix}$$

Hence, the probability that the pack hunts in Region R_1 on Thursday is $\frac{11}{32}$.

Steady State Vector

Another phenomenon that was observed in Example 2.9.1 can be expressed in general terms. The state vectors $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \dots$ were calculated in that example and were found to "approach" $\mathbf{s} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$. This means that the first component of $\mathbf{s}_m$ becomes and remains very close to 0.2 as m becomes large, whereas the second component gets close to 0.8 as m increases. When this is the case, we say that $\mathbf{s}_m$ **converges** to $\mathbf{s}$. For large m, then, there is very little error in taking $\mathbf{s}_m = \mathbf{s}$, so the long-term probability that the system is in state 1 is 0.2, whereas the probability that it is in state 2 is 0.8. In Example 2.9.1, enough state vectors were computed for the limiting vector $\mathbf{s}$ to be apparent. However, there is a better way to do this that works in most cases.

Suppose P is the transition matrix of a Markov chain, and assume that the state vectors $\mathbf{s}_m$ converge to a limiting vector $\mathbf{s}$. Then $\mathbf{s}_m$ is very close to $\mathbf{s}$ for sufficiently large m, so $\mathbf{s}_{m+1}$ is also very close to $\mathbf{s}$. Thus, the equation $\mathbf{s}_{m+1} = P\mathbf{s}_m$ from Theorem 2.9.1 is closely approximated by

$$\mathbf{s} = P\mathbf{s}$$

so it is not surprising that $\mathbf{s}$ should be a solution to this matrix equation. Moreover, it is easily solved because it can be written as a system of homogeneous linear equations

$$(I - P)\mathbf{s} = \mathbf{0}$$

with the entries of $\mathbf{s}$ as variables.

In Example 2.9.1, where $P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$, the general solution to $(I - P)\mathbf{s} = \mathbf{0}$ is $\mathbf{s} = \begin{bmatrix} t \\ 4t \end{bmatrix}$, where t is a parameter. But if we insist that the entries of S sum to 1 (as must be true of all state vectors), we find $t = 0.2$ and so $\mathbf{s} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ as before.

All this is predicated on the existence of a limiting vector for the sequence of state vectors of the Markov chain, and such a vector may not always exist. However, it does exist in one commonly occurring situation. A stochastic matrix P is called **regular** if some power P^m of P has every entry greater than zero. The matrix $P = \begin{bmatrix} 0 & 0.25 \\ 1 & 0.75 \end{bmatrix}$ of Example 2.9.1 is regular (in this case, each entry of P^2 is positive), and the general theorem is as follows:

Theorem 2.9.2

Let P be the transition matrix of a Markov chain and assume that P is regular. Then there is a unique column matrix $\mathbf{s}$ satisfying the following conditions:

1. $P\mathbf{s} = \mathbf{s}$.
2. *The entries of $\mathbf{s}$ are positive and sum to 1.*

Moreover, condition 1 can be written as

$$(I - P)\mathbf{s} = \mathbf{0}$$

and so gives a homogeneous system of linear equations for $\mathbf{s}$*. Finally, the sequence of state vectors* $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \dots$ *converges to* $\mathbf{s}$ *in the sense that if* m *is large enough, each entry of* $\mathbf{s}_m$ *is closely approximated by the corresponding entry of* $\mathbf{s}$*.*

This theorem will not be proved here.[20]

If P is the regular transition matrix of a Markov chain, the column $\mathbf{s}$ satisfying conditions 1 and 2 of Theorem 2.9.2 is called the **steady-state vector** for the Markov chain. The entries of $\mathbf{s}$ are the long-term probabilities that the chain will be in each of the various states.

Example 2.9.4

A man eats one of three soups—beef, chicken, and vegetable—each day. He never eats the same soup two days in a row. If he eats beef soup on a certain day, he is equally likely to eat each of the others the next day; if he does not eat beef soup, he is twice as likely to eat it the next day as the alternative.

a. If he has beef soup one day, what is the probability that he has it again two days later?

b. What are the long-run probabilities that he eats each of the three soups?

Solution. The states here are B, C, and V, the three soups. The transition matrix P is given in the table. (Recall that, for each state, the corresponding column lists the probabilities for the next state.)

	B	C	V
B	0	$\frac{2}{3}$	$\frac{2}{3}$
C	$\frac{1}{2}$	0	$\frac{1}{3}$
V	$\frac{1}{2}$	$\frac{1}{3}$	0

If he has beef soup initially, then the initial state vector is

$$\mathbf{s}_0 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

Then two days later the state vector is $\mathbf{s}_2$. If P is the transition matrix, then

$$\mathbf{s}_1 = P\mathbf{s}_0 = \frac{1}{2}\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right], \quad \mathbf{s}_2 = P\mathbf{s}_1 = \frac{1}{6}\left[\begin{array}{c} 4 \\ 1 \\ 1 \end{array}\right]$$

so he eats beef soup two days later with probability $\frac{2}{3}$. This answers (a.) and also shows that he eats chicken and vegetable soup each with probability $\frac{1}{6}$.

[20]The interested reader can find an elementary proof in J. Kemeny, H. Mirkil, J. Snell, and G. Thompson, *Finite Mathematical Structures* (Englewood Cliffs, N.J.: Prentice-Hall, 1958).

To find the long-run probabilities, we must find the steady-state vector $\mathbf{s}$. Theorem 2.9.2 applies because P is regular (P^2 has positive entries), so $\mathbf{s}$ satisfies $P\mathbf{s} = \mathbf{s}$. That is, $(I - P)\mathbf{s} = \mathbf{0}$ where

$$I - P = \frac{1}{6}\left[\begin{array}{rrr} 6 & -4 & -4 \\ -3 & 6 & -2 \\ -3 & -2 & 6 \end{array}\right]$$

The solution is $\mathbf{s} = \left[\begin{array}{c} 4t \\ 3t \\ 3t \end{array}\right]$, where t is a parameter, and we use $\mathbf{s} = \left[\begin{array}{c} 0.4 \\ 0.3 \\ 0.3 \end{array}\right]$ because the entries of $\mathbf{s}$ must sum to 1. Hence, in the long run, he eats beef soup 40% of the time and eats chicken soup and vegetable soup each 30% of the time.

Exercises for 2.9

Exercise 2.9.1 Which of the following stochastic matrices is regular?

a. $\left[\begin{array}{ccc} 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{array}\right]$ b. $\left[\begin{array}{ccc} \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{4} & 1 & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{3} \end{array}\right]$

Exercise 2.9.2 In each case find the steady-state vector and, assuming that it starts in state 1, find the probability that it is in state 2 after 3 transitions.

a. $\left[\begin{array}{cc} 0.5 & 0.3 \\ 0.5 & 0.7 \end{array}\right]$ b. $\left[\begin{array}{cc} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$ d. $\left[\begin{array}{ccc} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{array}\right]$

e. $\left[\begin{array}{ccc} 0.8 & 0.0 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.4 & 0.7 \end{array}\right]$ f. $\left[\begin{array}{ccc} 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.6 & 0.6 & 0.1 \end{array}\right]$

Exercise 2.9.3 A fox hunts in three territories A, B, and C. He never hunts in the same territory on two successive days. If he hunts in A, then he hunts in C the next day. If he hunts in B or C, he is twice as likely to hunt in A the next day as in the other territory.

a. What proportion of his time does he spend in A, in B, and in C?

b. If he hunts in A on Monday (C on Monday), what is the probability that he will hunt in B on Thursday?

Exercise 2.9.4 Assume that there are three social classes—upper, middle, and lower—and that social mobility behaves as follows:

1. Of the children of upper-class parents, 70% remain upper-class, whereas 10% become middle-class and 20% become lower-class.
2. Of the children of middle-class parents, 80% remain middle-class, whereas the others are evenly split between the upper class and the lower class.
3. For the children of lower-class parents, 60% remain lower-class, whereas 30% become middle-class and 10% upper-class.

a. Find the probability that the grandchild of lower-class parents becomes upper-class.

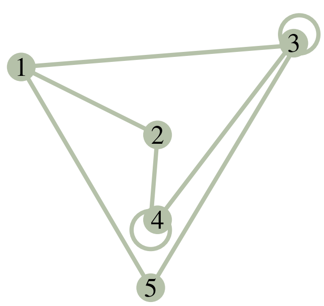
b. Find the long-term breakdown of society into classes.

Exercise 2.9.5 The prime minister says she will call an election. This gossip is passed from person to person with a probability $p \neq 0$ that the information is passed incorrectly at any stage. Assume that when a person hears the gossip he or she passes it to one person who does not know. Find the long-term probability that a person will hear that there is going to be an election.

Exercise 2.9.6 John makes it to work on time one Monday out of four. On other work days his behaviour is as follows: If he is late one day, he is twice as likely to come to work on time the next day as to be late. If he is on time one day, he is as likely to be late as not the next day. Find the probability of his being late and that of his being on time Wednesdays.

Exercise 2.9.7 Suppose you have 1¢ and match coins with a friend. At each match you either win or lose 1¢ with equal probability. If you go broke or ever get 4¢, you quit. Assume your friend never quits. If the states are 0, 1, 2, 3, and 4 representing your wealth, show that the corresponding transition matrix P is not regular. Find the probability that you will go broke after 3 matches.

Exercise 2.9.8 A mouse is put into a maze of compartments, as in the diagram. Assume that he always leaves any compartment he enters and that he is equally likely to take any tunnel entry.



a. If he starts in compartment 1, find the probability that he is in compartment 1 again after 3 moves.

b. Find the compartment in which he spends most of his time if he is left for a long time.

Exercise 2.9.9 If a stochastic matrix has a 1 on its main diagonal, show that it cannot be regular. Assume it is not 1×1.

Exercise 2.9.10 If $\mathbf{s}_m$ is the stage-m state vector for a Markov chain, show that $\mathbf{s}_{m+k} = P^k \mathbf{s}_m$ holds for all $m \geq 1$ and $k \geq 1$ (where P is the transition matrix).

Exercise 2.9.11 A stochastic matrix is **doubly stochastic** if all the row sums also equal 1. Find the steady-state vector for a doubly stochastic matrix.

Exercise 2.9.12 Consider the 2×2 stochastic matrix

$$P = \left[\begin{array}{cc} 1-p & q \\ p & 1-q \end{array} \right],$$

where $0 < p < 1$ and $0 < q < 1$.

a. Show that $\frac{1}{p+q} \left[\begin{array}{c} q \\ p \end{array} \right]$ is the steady-state vector for P.

b. Show that P^m converges to the matrix $\frac{1}{p+q} \left[\begin{array}{cc} q & q \\ p & p \end{array} \right]$ by first verifying inductively that $P^m = \frac{1}{p+q} \left[\begin{array}{cc} q & q \\ p & p \end{array} \right] + \frac{(1-p-q)^m}{p+q} \left[\begin{array}{cc} p & -q \\ -p & q \end{array} \right]$ for $m = 1, 2, \dots$. (It can be shown that the sequence of powers $P, P^2, P^3, \dots$ of any regular transition matrix converges to the matrix each of whose columns equals the steady-state vector for P.)

Supplementary Exercises for Chapter 2

Exercise 2.1 Solve for the matrix X if:

a. $PXQ = R$;

b. $XP = S$;

where $P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$,
$R = \begin{bmatrix} -1 & 1 & -4 \\ -4 & 0 & -6 \\ 6 & 6 & -6 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 6 \\ 3 & 1 \end{bmatrix}$

Exercise 2.2 Consider

$$p(X) = X^3 - 5X^2 + 11X - 4I.$$

a. If $p(U) = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ compute $p(U^T)$.

b. If $p(U) = 0$ where U is $n \times n$, find U^{-1} in terms of U.

Exercise 2.3 Show that, if a (possibly nonhomogeneous) system of equations is consistent and has more variables than equations, then it must have infinitely many solutions. [*Hint*: Use Theorem 2.2.2 and Theorem 1.3.1.]

Exercise 2.4 Assume that a system $A\mathbf{x} = \mathbf{b}$ of linear equations has at least two distinct solutions $\mathbf{y}$ and $\mathbf{z}$.

a. Show that $\mathbf{x}_k = \mathbf{y} + k(\mathbf{y} - \mathbf{z})$ is a solution for every k.

b. Show that $\mathbf{x}_k = \mathbf{x}_m$ implies $k = m$. [*Hint*: See Example 2.1.7.]

c. Deduce that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Exercise 2.5

a. Let A be a 3×3 matrix with all entries on and below the main diagonal zero. Show that $A^3 = 0$.

b. Generalize to the $n \times n$ case and prove your answer.

Exercise 2.6 Let I_{pq} denote the $n \times n$ matrix with (p, q)-entry equal to 1 and all other entries 0. Show that:

a. $I_n = I_{11} + I_{22} + \cdots + I_{nn}$.

b. $I_{pq}I_{rs} = \begin{cases} I_{ps} & \text{if } q = r \\ 0 & \text{if } q \neq r \end{cases}$.

c. If $A = [a_{ij}]$ is $n \times n$, then $A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}I_{ij}$.

d. If $A = [a_{ij}]$, then $I_{pq}AI_{rs} = a_{qr}I_{ps}$ for all p, q, r, and s.

Exercise 2.7 A matrix of the form aI_n, where a is a number, is called an $n \times n$ **scalar matrix**.

a. Show that each $n \times n$ scalar matrix commutes with every $n \times n$ matrix.

b. Show that A is a scalar matrix if it commutes with every $n \times n$ matrix. [*Hint*: See part (d.) of Exercise 2.6.]

Exercise 2.8 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, and D are all $n \times n$ and each commutes with all the others. If $M^2 = 0$, show that $(A + D)^3 = 0$. [*Hint*: First show that $A^2 = -BC = D^2$ and that

$$B(A + D) = 0 = C(A + D).]$$

Exercise 2.9 If A is 2×2, show that $A^{-1} = A^T$ if and only if $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some θ or
$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for some θ.

[*Hint*: If $a^2 + b^2 = 1$, then $a = \cos\theta$, $b = \sin\theta$ for some θ. Use

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.]$$

Exercise 2.10

a. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A^2 = I$.

b. What is wrong with the following argument? If $A^2 = I$, then $A^2 - I = 0$, so $(A - I)(A + I) = 0$, whence $A = I$ or $A = -I$.

Exercise 2.11 Let E and F be elementary matrices obtained from the identity matrix by adding multiples of row k to rows p and q. If $k \neq p$ and $k \neq q$, show that $EF = FE$.

Exercise 2.12 If A is a 2×2 real matrix, $A^2 = A$ and $A^T = A$, show that either A is one of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, or $A = \begin{bmatrix} a & b \\ b & 1-a \end{bmatrix}$ where $a^2 + b^2 = a$, $-\frac{1}{2} \leq b \leq \frac{1}{2}$ and $b \neq 0$.

Exercise 2.13 Show that the following are equivalent for matrices P, Q:

1. P, Q, and $P + Q$ are all invertible and

$$(P+Q)^{-1} = P^{-1} + Q^{-1}$$

2. P is invertible and $Q = PG$ where $G^2 + G + I = 0$.

3. Determinants and Diagonalization

With each square matrix we can calculate a number, called the determinant of the matrix, which tells us whether or not the matrix is invertible. In fact, determinants can be used to give a formula for the inverse of a matrix. They also arise in calculating certain numbers (called eigenvalues) associated with the matrix. These eigenvalues are essential to a technique called diagonalization that is used in many applications where it is desired to predict the future behaviour of a system. For example, we use it to predict whether a species will become extinct.

Determinants were first studied by Leibnitz in 1696, and the term "determinant" was first used in 1801 by Gauss is his *Disquisitiones Arithmeticae*. Determinants are much older than matrices (which were introduced by Cayley in 1878) and were used extensively in the eighteenth and nineteenth centuries, primarily because of their significance in geometry (see Section 4.4). Although they are somewhat less important today, determinants still play a role in the theory and application of matrix algebra.

3.1 The Cofactor Expansion

In Section 2.4 we defined the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as follows:[1]

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and showed (in Example 2.4.4) that A has an inverse if and only if $\det A \neq 0$. One objective of this chapter is to do this for *any* square matrix A. There is no difficulty for 1×1 matrices: If $A = [a]$, we define $\det A = \det\,[a] = a$ and note that A is invertible if and only if $a \neq 0$.

If A is 3×3 and invertible, we look for a suitable definition of $\det A$ by trying to carry A to the identity matrix by row operations. The first column is not zero (A is invertible); suppose the (1, 1)-entry a is not zero. Then row operations give

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & ae-bd & af-cd \\ 0 & ah-bg & ai-cg \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & u & af-cd \\ 0 & v & ai-cg \end{bmatrix}$$

where $u = ae - bd$ and $v = ah - bg$. Since A is invertible, one of u and v is nonzero (by Example 2.4.11); suppose that $u \neq 0$. Then the reduction proceeds

$$A \rightarrow \begin{bmatrix} a & b & c \\ 0 & u & af-cd \\ 0 & v & ai-cg \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & u & af-cd \\ 0 & uv & u(ai-cg) \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ 0 & u & af-cd \\ 0 & 0 & w \end{bmatrix}$$

[1]Determinants are commonly written $|A| = \det A$ using vertical bars. We will use both notations.

where $w = u(ai - cg) - v(af - cd) = a(aei + bfg + cdh - ceg - afh - bdi)$. We define

$$\det A = aei + bfg + cdh - ceg - afh - bdi \tag{3.1}$$

and observe that $\det A \neq 0$ because $a \det A = w \neq 0$ (is invertible).

To motivate the definition below, collect the terms in Equation 3.1 involving the entries a, b, and c in row 1 of A:

$$\begin{aligned}
\det A = \left|\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}\right| &= aei + bfg + cdh - ceg - afh - bdi \\
&= a(ei - fh) - b(di - fg) + c(dh - eg) \\
&= a\left|\begin{array}{cc} e & f \\ h & i \end{array}\right| - b\left|\begin{array}{cc} d & f \\ g & i \end{array}\right| + c\left|\begin{array}{cc} d & e \\ g & h \end{array}\right|
\end{aligned}$$

This last expression can be described as follows: To compute the determinant of a 3×3 matrix A, multiply each entry in row 1 by a sign times the determinant of the 2×2 matrix obtained by deleting the row and column of that entry, and add the results. The signs alternate down row 1, starting with $+$. It is this observation that we generalize below.

Example 3.1.1

$$\begin{aligned}
\det \begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix} &= 2\left|\begin{array}{cc} 0 & 6 \\ 5 & 0 \end{array}\right| - 3\left|\begin{array}{cc} -4 & 6 \\ 1 & 0 \end{array}\right| + 7\left|\begin{array}{cc} -4 & 0 \\ 1 & 5 \end{array}\right| \\
&= 2(-30) - 3(-6) + 7(-20) \\
&= -182
\end{aligned}$$

This suggests an inductive method of defining the determinant of any square matrix in terms of determinants of matrices one size smaller. The idea is to define determinants of 3×3 matrices in terms of determinants of 2×2 matrices, then we do 4×4 matrices in terms of 3×3 matrices, and so on.

To describe this, we need some terminology.

Definition 3.1 Cofactors of a Matrix

Assume that determinants of $(n-1) \times (n-1)$ matrices have been defined. Given the $n \times n$ matrix A, let

A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

*Then the $(i,\ j)$-**cofactor** $c_{ij}(A)$ is the scalar defined by*

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij})$$

*Here $(-1)^{i+j}$ is called the **sign** of the $(i,\ j)$-position.*

The sign of a position is clearly 1 or -1, and the following diagram is useful for remembering it:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Note that the signs alternate along each row and column with $+$ in the upper left corner.

Example 3.1.2

Find the cofactors of positions $(1, 2)$, $(3, 1)$, and $(2, 3)$ in the following matrix.

$$A = \begin{bmatrix} 3 & -1 & 6 \\ 5 & 2 & 7 \\ 8 & 9 & 4 \end{bmatrix}$$

Solution. Here A_{12} is the matrix $\begin{bmatrix} 5 & 7 \\ 8 & 4 \end{bmatrix}$ that remains when row 1 and column 2 are deleted. The sign of position $(1, 2)$ is $(-1)^{1+2} = -1$ (this is also the $(1, 2)$-entry in the sign diagram), so the $(1, 2)$-cofactor is

$$c_{12}(A) = (-1)^{1+2} \begin{vmatrix} 5 & 7 \\ 8 & 4 \end{vmatrix} = (-1)(5 \cdot 4 - 7 \cdot 8) = (-1)(-36) = 36$$

Turning to position $(3, 1)$, we find

$$c_{31}(A) = (-1)^{3+1} A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 6 \\ 2 & 7 \end{vmatrix} = (+1)(-7-12) = -19$$

Finally, the $(2, 3)$-cofactor is

$$c_{23}(A) = (-1)^{2+3} A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -1 \\ 8 & 9 \end{vmatrix} = (-1)(27+8) = -35$$

Clearly other cofactors can be found—there are nine in all, one for each position in the matrix.

We can now define $\det A$ for any square matrix A

Definition 3.2 Cofactor expansion of a Matrix

Assume that determinants of $(n-1) \times (n-1)$ matrices have been defined. If $A = \left[a_{ij}\right]$ is $n \times n$ define

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

This is called the ***cofactor expansion*** *of $\det A$ along row 1.*

It asserts that $\det A$ can be computed by multiplying the entries of row 1 by the corresponding cofactors, and adding the results. The astonishing thing is that $\det A$ can be computed by taking the cofactor expansion along *any row or column*: Simply multiply each entry of that row or column by the corresponding cofactor and add.

Theorem 3.1.1: Cofactor Expansion Theorem[2]

The determinant of an $n \times n$ matrix A can be computed by using the cofactor expansion along any row or column of A. That is $\det A$ can be computed by multiplying each entry of the row or column by the corresponding cofactor and adding the results.

The proof will be given in Section 3.6.

Example 3.1.3

Compute the determinant of $A = \left[\begin{array}{rrr} 3 & 4 & 5 \\ 1 & 7 & 2 \\ 9 & 8 & -6 \end{array} \right]$.

Solution. The cofactor expansion along the first row is as follows:

$$\begin{aligned} \det A &= 3c_{11}(A) + 4c_{12}(A) + 5c_{13}(A) \\ &= 3\left| \begin{array}{rr} 7 & 2 \\ 8 & -6 \end{array} \right| - 4\left| \begin{array}{rr} 1 & 2 \\ 9 & -6 \end{array} \right| + 3\left| \begin{array}{rr} 1 & 7 \\ 9 & 8 \end{array} \right| \\ &= 3(-58) - 4(-24) + 5(-55) \\ &= -353 \end{aligned}$$

Note that the signs alternate along the row (indeed along *any* row or column). Now we compute $\det A$ by expanding along the first column.

$$\begin{aligned} \det A &= 3c_{11}(A) + 1c_{21}(A) + 9c_{31}(A) \\ &= 3\left| \begin{array}{rr} 7 & 2 \\ 8 & -6 \end{array} \right| - \left| \begin{array}{rr} 4 & 5 \\ 8 & -6 \end{array} \right| + 9\left| \begin{array}{rr} 4 & 5 \\ 7 & 2 \end{array} \right| \\ &= 3(-58) - (-64) + 9(-27) \\ &= -353 \end{aligned}$$

The reader is invited to verify that $\det A$ can be computed by expanding along any other row or column.

The fact that the cofactor expansion along *any row or column* of a matrix A always gives the same result (the determinant of A) is remarkable, to say the least. The choice of a particular row or column can simplify the calculation.

[2]The cofactor expansion is due to Pierre Simon de Laplace (1749–1827), who discovered it in 1772 as part of a study of linear differential equations. Laplace is primarily remembered for his work in astronomy and applied mathematics.

Example 3.1.4

Compute $\det A$ where $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ -6 & 3 & 1 & 0 \end{bmatrix}$.

Solution. The first choice we must make is which row or column to use in the cofactor expansion. The expansion involves multiplying entries by cofactors, so the work is minimized when the row or column contains as many zero entries as possible. Row 1 is a best choice in this matrix (column 4 would do as well), and the expansion is

$$\begin{aligned} \det A &= 3c_{11}(A) + 0c_{12}(A) + 0c_{13}(A) + 0c_{14}(A) \\ &= 3\begin{vmatrix} 1 & 2 & 0 \\ 6 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix} \end{aligned}$$

This is the first stage of the calculation, and we have succeeded in expressing the determinant of the 4×4 matrix A in terms of the determinant of a 3×3 matrix. The next stage involves this 3×3 matrix. Again, we can use any row or column for the cofactor expansion. The third column is preferred (with two zeros), so

$$\begin{aligned} \det A &= 3\left(0\begin{vmatrix} 6 & 0 \\ 3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix}\right) \\ &= 3[0 + 1(-5) + 0] \\ &= -15 \end{aligned}$$

This completes the calculation.

Computing the determinant of a matrix A can be tedious. For example, if A is a 4×4 matrix, the cofactor expansion along any row or column involves calculating four cofactors, each of which involves the determinant of a 3×3 matrix. And if A is 5×5, the expansion involves five determinants of 4×4 matrices! There is a clear need for some techniques to cut down the work.[3]

The motivation for the method is the observation (see Example 3.1.4) that calculating a determinant is simplified a great deal when a row or column consists mostly of zeros. (In fact, when a row or column consists *entirely* of zeros, the determinant is zero—simply expand along that row or column.)

Recall next that one method of *creating* zeros in a matrix is to apply elementary row operations to it. Hence, a natural question to ask is what effect such a row operation has on the determinant of the matrix. It turns out that the effect is easy to determine and that elementary *column* operations can be used in the same way. These observations lead to a technique for evaluating determinants that greatly reduces the

[3] If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ we can calculate $\det A$ by considering $\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}$ obtained from A by adjoining columns 1 and 2 on the right. Then $\det A = aei + bfg + cdh - ceg - afh - bdi$, where the positive terms aei, bfg, and cdh are the products down and to the right starting at a, b, and c, and the negative terms ceg, afh, and bdi are the products down and to the left starting at c, a, and b. **Warning**: This rule does **not** apply to $n \times n$ matrices where $n > 3$ or $n = 2$.

labour involved. The necessary information is given in Theorem 3.1.2.

Theorem 3.1.2

Let A denote an $n \times n$ matrix.

1. *If A has a row or column of zeros,* $\det A = 0$.
2. *If two distinct rows (or columns) of A are interchanged, the determinant of the resulting matrix is* $-\det A$.
3. *If a row (or column) of A is multiplied by a constant u, the determinant of the resulting matrix is* $u(\det A)$.
4. *If two distinct rows (or columns) of A are identical,* $\det A = 0$.
5. *If a multiple of one row of A is added to a different row (or if a multiple of a column is added to a different column), the determinant of the resulting matrix is* $\det A$.

Proof. We prove properties 2, 4, and 5 and leave the rest as exercises.

Property 2. If A is $n \times n$, this follows by induction on n. If $n = 2$, the verification is left to the reader. If $n > 2$ and two rows are interchanged, let B denote the resulting matrix. Expand $\det A$ and $\det B$ along a row *other than* the two that were interchanged. The entries in this row are the same for both A and B, but the cofactors in B are the negatives of those in A (by induction) because the corresponding $(n-1) \times (n-1)$ matrices have two rows interchanged. Hence, $\det B = -\det A$, as required. A similar argument works if two columns are interchanged.

Property 4. If two rows of A are equal, let B be the matrix obtained by interchanging them. Then $B = A$, so $\det B = detA$. But $\det B = -\det A$ by property 2, so $\det A = \det B = 0$. Again, the same argument works for columns.

Property 5. Let B be obtained from $A = \left[a_{ij}\right]$ by adding u times row p to row q. Then row q of B is

$$(a_{q1} + ua_{p1},\ a_{q2} + ua_{p2},\ \dots,\ a_{qn} + ua_{pn})$$

The cofactors of these elements in B are the same as in A (they do not involve row q): in symbols, $c_{qj}(B) = c_{qj}(A)$ for each j. Hence, expanding B along row q gives

$$\begin{aligned}\det A &= (a_{q1} + ua_{p1})c_{q1}(A) + (a_{q2} + ua_{p2})c_{q2}(A) + \cdots + (a_{qn} + ua_{pn})c_{qn}(A) \\ &= [a_{q1}c_{q1}(A) + a_{q2}c_{q2}(A) + \cdots + a_{qn}c_{qn}(A)] + u[a_{p1}c_{q1}(A) + a_{p2}c_{q2}(A) + \cdots + a_{pn}c_{qn}(A)] \\ &= \det A + u \det C\end{aligned}$$

where C is the matrix obtained from A by replacing row q by row p (and both expansions are along row q). Because rows p and q of C are equal, $\det C = 0$ by property 4. Hence, $\det B = detA$, as required. As before, a similar proof holds for columns. □

To illustrate Theorem 3.1.2, consider the following determinants.

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$ (because the last row consists of zeros)

$$\begin{vmatrix} 3 & -1 & 5 \\ 2 & 8 & 7 \\ 1 & 2 & -1 \end{vmatrix} = -\begin{vmatrix} 5 & -1 & 3 \\ 7 & 8 & 2 \\ -1 & 2 & 1 \end{vmatrix}$$ (because two columns are interchanged)

$$\begin{vmatrix} 8 & 1 & 2 \\ 3 & 0 & 9 \\ 1 & 2 & -1 \end{vmatrix} = 3\begin{vmatrix} 8 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$ (because the second row of the matrix on the left is 3 times the second row of the matrix on the right)

$$\begin{vmatrix} 2 & 1 & 2 \\ 4 & 0 & 4 \\ 1 & 3 & 1 \end{vmatrix} = 0$$ (because two columns are identical)

$$\begin{vmatrix} 2 & 5 & 2 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 20 \\ -1 & 2 & 9 \\ 3 & 1 & 1 \end{vmatrix}$$ (because twice the second row of the matrix on the left was added to the first row)

The following four examples illustrate how Theorem 3.1.2 is used to evaluate determinants.

Example 3.1.5

Evaluate $\det A$ when $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{bmatrix}$.

Solution. The matrix does have zero entries, so expansion along (say) the second row would involve somewhat less work. However, a column operation can be used to get a zero in position $(2, 3)$—namely, add column 1 to column 3. Because this does not change the value of the determinant, we obtain

$$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 4 \\ 1 & 0 & 0 \\ 2 & 1 & 8 \end{vmatrix} = -\begin{vmatrix} -1 & 4 \\ 1 & 8 \end{vmatrix} = 12$$

where we expanded the second 3×3 matrix along row 2.

Example 3.1.6

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 6$, evaluate $\det A$ where $A = \begin{bmatrix} a+x & b+y & c+z \\ 3x & 3y & 3z \\ -p & -q & -r \end{bmatrix}$.

Solution. First take common factors out of rows 2 and 3.

$$\det A = 3(-1)\det \begin{bmatrix} a+x & b+y & c+z \\ x & y & z \\ p & q & r \end{bmatrix}$$

Now subtract the second row from the first and interchange the last two rows.

$$\det A = -3\det \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} = 3\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = 3\cdot 6 = 18$$

The determinant of a matrix is a sum of products of its entries. In particular, if these entries are polynomials in x, then the determinant itself is a polynomial in x. It is often of interest to determine which values of x make the determinant zero, so it is very useful if the determinant is given in factored form. Theorem 3.1.2 can help.

Example 3.1.7

Find the values of x for which $\det A = 0$, where $A = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$.

Solution. To evaluate $\det A$, first subtract x times row 1 from rows 2 and 3.

$$\det A = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x \\ 0 & 1-x^2 & x-x^2 \\ 0 & x-x^2 & 1-x^2 \end{vmatrix} = \begin{vmatrix} 1-x^2 & x-x^2 \\ x-x^2 & 1-x^2 \end{vmatrix}$$

At this stage we could simply evaluate the determinant (the result is $2x^3-3x^2+1$). But then we would have to factor this polynomial to find the values of x that make it zero. However, this factorization can be obtained directly by first factoring each entry in the determinant and taking a common factor of $(1-x)$ from each row.

$$\begin{aligned} \det A = \begin{vmatrix} (1-x)(1+x) & x(1-x) \\ x(1-x) & (1-x)(1+x) \end{vmatrix} &= (1-x)^2 \begin{vmatrix} 1+x & x \\ x & 1+x \end{vmatrix} \\ &= (1-x)^2(2x+1) \end{aligned}$$

Hence, $\det A = 0$ means $(1-x)^2(2x+1) = 0$, that is $x = 1$ or $x = -\frac{1}{2}$.

Example 3.1.8

If a_1, a_2, and a_3 are given show that

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = (a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$$

Solution. Begin by subtracting row 1 from rows 2 and 3, and then expand along column 1:

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix} = \begin{bmatrix} a_2 - a_1 & a_2^2 - a_1^2 \\ a_3 - a_1 & a_3^2 - a_1^2 \end{bmatrix}$$

Now $(a_2 - a_1)$ and $(a_3 - a_1)$ are common factors in rows 1 and 2, respectively, so

$$\begin{aligned} \det \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix} &= (a_2 - a_1)(a_3 - a_1) \det \begin{bmatrix} 1 & a_2 + a_1 \\ 1 & a_3 + a_1 \end{bmatrix} \\ &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \end{aligned}$$

The matrix in Example 3.1.8 is called a Vandermonde matrix, and the formula for its determinant can be generalized to the $n \times n$ case (see Theorem 3.2.7).

If A is an $n \times n$ matrix, forming uA means multiplying *every* row of A by u. Applying property 3 of Theorem 3.1.2, we can take the common factor u out of each row and so obtain the following useful result.

Theorem 3.1.3

If A is an $n \times n$ matrix, then $\det(uA) = u^n \det A$ for any number u.

The next example displays a type of matrix whose determinant is easy to compute.

Example 3.1.9

Evaluate $\det A$ if $A = \begin{bmatrix} a & 0 & 0 & 0 \\ u & b & 0 & 0 \\ v & w & c & 0 \\ x & y & z & d \end{bmatrix}$.

Solution. Expand along row 1 to get $\det A = a \begin{vmatrix} b & 0 & 0 \\ w & c & 0 \\ y & z & d \end{vmatrix}$. Now expand this along the top row to get $\det A = ab \begin{vmatrix} c & 0 \\ z & d \end{vmatrix} = abcd$, the product of the main diagonal entries.

A square matrix is called a **lower triangular matrix** if all entries above the main diagonal are zero (as in Example 3.1.9). Similarly, an **upper triangular matrix** is one for which all entries below the main diagonal are zero. A **triangular matrix** is one that is either upper or lower triangular. Theorem 3.1.4 gives an easy rule for calculating the determinant of any triangular matrix. The proof is like the solution to Example 3.1.9.

Theorem 3.1.4

If A is a square triangular matrix, then $\det A$ is the product of the entries on the main diagonal.

Theorem 3.1.4 is useful in computer calculations because it is a routine matter to carry a matrix to triangular form using row operations.

Block matrices such as those in the next theorem arise frequently in practice, and the theorem gives an easy method for computing their determinants. This dovetails with Example 2.4.11.

Theorem 3.1.5

Consider matrices $\left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right]$ *and* $\left[\begin{array}{cc} A & 0 \\ Y & B \end{array}\right]$ *in block form, where A and B are square matrices. Then*

$$\det \left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right] = \det A \det B \textit{ and } \det \left[\begin{array}{cc} A & 0 \\ Y & B \end{array}\right] = \det A \det B$$

Proof. Write $T = \det \left[\begin{array}{cc} A & X \\ 0 & B \end{array}\right]$ and proceed by induction on k where A is $k \times k$. If $k = 1$, it is the cofactor expansion along column 1. In general let $S_i(T)$ denote the matrix obtained from T by deleting row i and column 1. Then the cofactor expansion of $\det T$ along the first column is

$$\det T = a_{11} \det(S_1(T)) - a_{21} \det(S_2(T)) + \cdots \pm a_{k1} \det(S_k(T)) \tag{3.2}$$

where $a_{11}, a_{21}, \cdots, a_{k1}$ are the entries in the first column of A. But $S_i(T) = \left[\begin{array}{cc} S_i(A) & X_i \\ 0 & B \end{array}\right]$ for each $i = 1, 2, \cdots, k$, so $\det(S_i(T)) = \det(S_i(A)) \cdot \det B$ by induction. Hence, Equation 3.2 becomes

$$\begin{aligned} \det T &= \{a_{11} \det(S_1(T)) - a_{21} \det(S_2(T)) + \cdots \pm a_{k1} \det(S_k(T))\} \det B \\ &= \{\det A\} \det B \end{aligned}$$

as required. The lower triangular case is similar. □

Example 3.1.10

$$\det \left[\begin{array}{rrrr} 2 & 3 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \end{array}\right] = - \left|\begin{array}{rrrr} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{array}\right| = - \left|\begin{array}{rr} 2 & 1 \\ 1 & -1 \end{array}\right| \left|\begin{array}{rr} 1 & 1 \\ 4 & 1 \end{array}\right| = -(-3)(-3) = -9$$

The next result shows that $\det A$ is a linear transformation when regarded as a function of a fixed column of A. The proof is Exercise 3.1.21.

Theorem 3.1.6

Given columns $\mathbf{c}_1, \cdots, \mathbf{c}_{j-1}, \mathbf{c}_{j+1}, \cdots, \mathbf{c}_n$ in $\mathbb{R}^n$, define $T : \mathbb{R}^n \to \mathbb{R}$ by

$$T(\mathbf{x}) = \det \left[\begin{array}{ccccccc} \mathbf{c}_1 & \cdots & \mathbf{c}_{j-1} & \mathbf{x} & \mathbf{c}_{j+1} & \cdots & \mathbf{c}_n \end{array} \right] \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then, for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all a in $\mathbb{R}$,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \text{and} \quad T(a\mathbf{x}) = aT(\mathbf{x})$$

Exercises for 3.1

Exercise 3.1.1 Compute the determinants of the following matrices.

a. $\left[\begin{array}{rr} 2 & -1 \\ 3 & 2 \end{array} \right]$

b. $\left[\begin{array}{rr} 6 & 9 \\ 8 & 12 \end{array} \right]$

c. $\left[\begin{array}{cc} a^2 & ab \\ ab & b^2 \end{array} \right]$

d. $\left[\begin{array}{cc} a+1 & a \\ a & a-1 \end{array} \right]$

e. $\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$

f. $\left[\begin{array}{rrr} 2 & 0 & -3 \\ 1 & 2 & 5 \\ 0 & 3 & 0 \end{array} \right]$

g. $\left[\begin{array}{rrr} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right]$

h. $\left[\begin{array}{ccc} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{array} \right]$

i. $\left[\begin{array}{ccc} 1 & b & c \\ b & c & 1 \\ c & 1 & b \end{array} \right]$

j. $\left[\begin{array}{ccc} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{array} \right]$

k. $\left[\begin{array}{rrrr} 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \end{array} \right]$

l. $\left[\begin{array}{rrrr} 1 & 0 & 3 & 1 \\ 2 & 2 & 6 & 0 \\ -1 & 0 & -3 & 1 \\ 4 & 1 & 12 & 0 \end{array} \right]$

m. $\left[\begin{array}{rrrr} 3 & 1 & -5 & 2 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 1 & 1 & 2 & -1 \end{array} \right]$

n. $\left[\begin{array}{rrrr} 4 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \end{array} \right]$

o. $\left[\begin{array}{rrrr} 1 & -1 & 5 & 5 \\ 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & 1 & 2 & -1 \end{array} \right]$

p. $\left[\begin{array}{cccc} 0 & 0 & 0 & a \\ 0 & 0 & b & p \\ 0 & c & q & k \\ d & s & t & u \end{array} \right]$

Exercise 3.1.2 Show that $\det A = 0$ if A has a row or column consisting of zeros.

Exercise 3.1.3 Show that the sign of the position in the last row and the last column of A is always $+1$.

Exercise 3.1.4 Show that $\det I = 1$ for any identity matrix I.

Exercise 3.1.5 Evaluate the determinant of each matrix by reducing it to upper triangular form.

a. $\left[\begin{array}{rrr} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -1 & 3 \end{array} \right]$

b. $\left[\begin{array}{rrr} -1 & 3 & 1 \\ 2 & 5 & 3 \\ 1 & -2 & 1 \end{array} \right]$

c. $\left[\begin{array}{rrrr} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{array} \right]$

d. $\left[\begin{array}{rrrr} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{array} \right]$

Exercise 3.1.6 Evaluate by cursory inspection:

a. $\det \left[\begin{array}{ccc} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{array} \right]$

b. $\det \begin{bmatrix} a & b & c \\ a+b & 2b & c+b \\ 2 & 2 & 2 \end{bmatrix}$

Exercise 3.1.7 If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$ compute:

a. $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$

b. $\det \begin{bmatrix} -2a & -2b & -2c \\ 2p+x & 2q+y & 2r+z \\ 3x & 3y & 3z \end{bmatrix}$

Exercise 3.1.8 Show that:

a. $\det \begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = 2 \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

b. $\det \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix} = 9 \det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$

Exercise 3.1.9 In each case either prove the statement or give an example showing that it is false:

a. $\det(A+B) = \det A + \det B$.

b. If $\det A = 0$, then A has two equal rows.

c. If A is 2×2, then $\det(A^T) = \det A$.

d. If R is the reduced row-echelon form of A, then $\det A = \det R$.

e. If A is 2×2, then $\det(7A) = 49 \det A$.

f. $\det(A^T) = -\det A$.

g. $\det(-A) = -\det A$.

h. If $\det A = \det B$ where A and B are the same size, then A = B.

Exercise 3.1.10 Compute the determinant of each matrix, using Theorem 3.1.5.

a. $\begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ -1 & 3 & 1 & 4 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 \end{bmatrix}$

Exercise 3.1.11 If $\det A = 2$, $\det B = -1$, and $\det C = 3$, find:

a. $\det \begin{bmatrix} A & X & Y \\ 0 & B & Z \\ 0 & 0 & C \end{bmatrix}$

b. $\det \begin{bmatrix} A & 0 & 0 \\ X & B & 0 \\ Y & Z & C \end{bmatrix}$

c. $\det \begin{bmatrix} A & X & Y \\ 0 & B & 0 \\ 0 & Z & C \end{bmatrix}$

d. $\det \begin{bmatrix} A & X & 0 \\ 0 & B & 0 \\ Y & Z & C \end{bmatrix}$

Exercise 3.1.12 If A has three columns with only the top two entries nonzero, show that $\det A = 0$.

Exercise 3.1.13

a. Find $\det A$ if A is 3×3 and $\det(2A) = 6$.

b. Under what conditions is $\det(-A) = \det A$?

Exercise 3.1.14 Evaluate by first adding all other rows to the first row.

a. $\det \begin{bmatrix} x-1 & 2 & 3 \\ 2 & -3 & x-2 \\ -2 & x & -2 \end{bmatrix}$

b. $\det \begin{bmatrix} x-1 & -3 & 1 \\ 2 & -1 & x-1 \\ -3 & x+2 & -2 \end{bmatrix}$

Exercise 3.1.15

a. Find b if $\det \begin{bmatrix} 5 & -1 & x \\ 2 & 6 & y \\ -5 & 4 & z \end{bmatrix} = ax + by + cz$.

b. Find c if $\det \left[\begin{array}{rrr} 2 & x & -1 \\ 1 & y & 3 \\ -3 & z & 4 \end{array} \right] = ax+by+cz$.

Exercise 3.1.16 Find the real numbers x and y such that $\det A = 0$ if:

a. $A = \left[\begin{array}{ccc} 0 & x & y \\ y & 0 & x \\ x & y & 0 \end{array} \right]$

b. $A = \left[\begin{array}{rrr} 1 & x & x \\ -x & -2 & x \\ -x & -x & -3 \end{array} \right]$

c. $A = \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{array} \right]$

d. $A = \left[\begin{array}{cccc} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ y & 0 & 0 & x \end{array} \right]$

Exercise 3.1.17 Show that

$$\det \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{array} \right] = -3x^2$$

Exercise 3.1.18 Show that

$$\det \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \\ a & 1 & x & x^2 \\ p & b & 1 & x \\ q & r & c & 1 \end{array} \right] = (1-ax)(1-bx)(1-cx).$$

Exercise 3.1.19

Given the polynomial $p(x) = a+bx+cx^2+dx^3+x^4$, the matrix $C = \left[\begin{array}{rrrr} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{array} \right]$ is called the **companion matrix** of $p(x)$. Show that $\det(xI-C) = p(x)$.

Exercise 3.1.20 Show that

$$\det \left[\begin{array}{ccc} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{array} \right]$$
$$= (a+b+c+3x)[(ab+ac+bc)-(a^2+b^2+c^2)]$$

Exercise 3.1.21 . Prove Theorem 3.1.6. [*Hint*: Expand the determinant along column j.]

Exercise 3.1.22 Show that

$$\det \left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & a_2 & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & * & * \\ a_n & * & \cdots & * & * \end{array} \right] = (-1)^k a_1 a_2 \cdots a_n$$

where either $n = 2k$ or $n = 2k+1$, and $*$-entries are arbitrary.

Exercise 3.1.23 By expanding along the first column, show that:

$$\det \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array} \right] = 1+(-1)^{n+1}$$

if the matrix is $n \times n$, $n \geq 2$.

Exercise 3.1.24 Form matrix B from a matrix A by writing the columns of A in reverse order. Express $\det B$ in terms of $\det A$.

Exercise 3.1.25 Prove property 3 of Theorem 3.1.2 by expanding along the row (or column) in question.

Exercise 3.1.26 Show that the line through two distinct points (x_1, y_1) and (x_2, y_2) in the plane has equation

$$\det \left[\begin{array}{ccc} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{array} \right] = 0$$

Exercise 3.1.27 Let A be an $n \times n$ matrix. Given a polynomial $p(x) = a_0 + a_1 x + \cdots + a_m x^m$, we write $p(A) = a_0 I + a_1 A + \cdots + a_m A^m$.

For example, if $p(x) = 2-3x+5x^2$, then $p(A) = 2I-3A+5A^2$. The *characteristic polynomial* of A is defined to be $c_A(x) = \det[xI-A]$, and the Cayley-Hamilton theorem asserts that $c_A(A) = 0$ for any matrix A.

a. Verify the theorem for

i. $A = \left[\begin{array}{rr} 3 & 2 \\ 1 & -1 \end{array} \right]$

ii. $A = \left[\begin{array}{rrr} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 8 & 2 & 2 \end{array} \right]$

b. Prove the theorem for $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$

3.2 Determinants and Matrix Inverses

In this section, several theorems about determinants are derived. One consequence of these theorems is that a square matrix A is invertible if and only if $\det A \neq 0$. Moreover, determinants are used to give a formula for A^{-1} which, in turn, yields a formula (called Cramer's rule) for the solution of any system of linear equations with an invertible coefficient matrix.

We begin with a remarkable theorem (due to Cauchy in 1812) about the determinant of a product of matrices. The proof is given at the end of this section.

Theorem 3.2.1: Product Theorem

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.

The complexity of matrix multiplication makes the product theorem quite unexpected. Here is an example where it reveals an important numerical identity.

Example 3.2.1

If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$ then $AB = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix}$.

Hence $\det A \det B = \det(AB)$ gives the identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

Theorem 3.2.1 extends easily to $\det(ABC) = \det A \det B \det C$. In fact, induction gives

$$\det(A_1A_2\cdots A_{k-1}A_k) = \det A_1 \det A_2 \cdots \det A_{k-1} \det A_k$$

for any square matrices $A_1, \dots, A_k$ of the same size. In particular, if each $A_i = A$, we obtain

$$\det(A^k) = (detA)^k, \text{ for any } k \geq 1$$

We can now give the invertibility condition.

Theorem 3.2.2

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. *When this is the case,* $\det(A^{-1}) = \frac{1}{\det A}$

Proof. If A is invertible, then $AA^{-1} = I$; so the product theorem gives

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$$

Hence, $\det A \neq 0$ and also $\det A^{-1} = \frac{1}{\det A}$.

Conversely, if $\det A \neq 0$, we show that A can be carried to I by elementary row operations (and invoke Theorem 2.4.5). Certainly, A can be carried to its reduced row-echelon form R, so $R = E_k \cdots E_2 E_1 A$ where the E_i are elementary matrices (Theorem 2.5.1). Hence the product theorem gives

$$\det R = \det E_k \cdots \det E_2 \det E_1 \det A$$

Since $\det E \neq 0$ for all elementary matrices E, this shows $\det R \neq 0$. In particular, R has no row of zeros, so $R = I$ because R is square and reduced row-echelon. This is what we wanted. □

Example 3.2.2

For which values of c does $A = \left[\begin{array}{rrr} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array}\right]$ have an inverse?

Solution. Compute $\det A$ by first adding c times column 1 to column 3 and then expanding along row 1.

$$\det A = \det \left[\begin{array}{rrr} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array}\right] = \det \left[\begin{array}{rrc} 1 & 0 & 0 \\ -1 & 3 & 1-c \\ 0 & 2c & -4 \end{array}\right] = 2(c+2)(c-3)$$

Hence, $\det A = 0$ if $c = -2$ or $c = 3$, and A has an inverse if $c \neq -2$ and $c \neq 3$.

Example 3.2.3

If a product $A_1 A_2 \cdots A_k$ of square matrices is invertible, show that each A_i is invertible.

Solution. We have $\det A_1 \det A_2 \cdots \det A_k = \det(A_1 A_2 \cdots A_k)$ by the product theorem, and $\det(A_1 A_2 \cdots A_k) \neq 0$ by Theorem 3.2.2 because $A_1 A_2 \cdots A_k$ is invertible. Hence

$$\det A_1 \det A_2 \cdots \det A_k \neq 0$$

so $\det A_i \neq 0$ for each i. This shows that each A_i is invertible, again by Theorem 3.2.2.

Theorem 3.2.3

If A is any square matrix, $\det A^T = \det A$.

Proof. Consider first the case of an elementary matrix E. If E is of type I or II, then $E^T = E$; so certainly $\det E^T = \det E$. If E is of type III, then E^T is also of type III; so $\det E^T = 1 = \det E$ by Theorem 3.1.2. Hence, $\det E^T = \det E$ for every elementary matrix E.

Now let A be any square matrix. If A is not invertible, then neither is A^T; so $\det A^T = 0 = \det A$ by Theorem 3.2.2. On the other hand, if A is invertible, then $A = E_k \cdots E_2 E_1$, where the E_i are elementary matrices (Theorem 2.5.2). Hence, $A^T = E_1^T E_2^T \cdots E_k^T$ so the product theorem gives

$$\begin{aligned}\det A^T = \det E_1^T \det E_2^T \cdots \det E_k^T &= \det E_1 \det E_2 \cdots \det E_k \\ &= \det E_k \cdots \det E_2 \det E_1 \\ &= \det A\end{aligned}$$

This completes the proof. □

Example 3.2.4

If $\det A = 2$ and $\det B = 5$, calculate $\det (A^3 B^{-1} A^T B^2)$.

Solution. We use several of the facts just derived.

$$\begin{aligned}\det (A^3 B^{-1} A^T B^2) &= \det (A^3) \det (B^{-1}) \det (A^T) \det (B^2) \\ &= (\det A)^3 \frac{1}{\det B} \det A (\det B)^2 \\ &= 2^3 \cdot \frac{1}{5} \cdot 2 \cdot 5^2 \\ &= 80\end{aligned}$$

Example 3.2.5

A square matrix is called **orthogonal** if $A^{-1} = A^T$. What are the possible values of $\det A$ if A is orthogonal?

Solution. If A is orthogonal, we have $I = AA^T$. Take determinants to obtain

$$1 = \det I = \det (AA^T) = \det A \det A^T = (\det A)^2$$

Since $\det A$ is a number, this means $\det A = \pm 1$.

Hence Theorems 2.6.4 and 2.6.5 imply that rotation about the origin and reflection about a line through the origin in $\mathbb{R}^2$ have orthogonal matrices with determinants 1 and -1 respectively. In fact they are the *only* such transformations of $\mathbb{R}^2$. We have more to say about this in Section 8.2.

Adjugates

In Section 2.4 we defined the adjugate of a 2×2 matrix $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ to be $\text{adj}\,(A) = \left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right]$. Then we verified that $A(\text{adj}\, A) = (\det A)I = (\text{adj}\, A)A$ and hence that, if $\det A \neq 0$, $A^{-1} = \frac{1}{\det A} \text{adj}\, A$. We are now able to define the adjugate of an arbitrary square matrix and to show that this formula for the inverse remains valid (when the inverse exists).

Recall that the (i, j)-cofactor $c_{ij}(A)$ of a square matrix A is a number defined for each position (i, j) in the matrix. If A is a square matrix, the **cofactor matrix of** A is defined to be the matrix $\left[c_{ij}(A)\right]$ whose (i, j)-entry is the (i, j)-cofactor of A.

Definition 3.3 Adjugate of a Matrix

The **adjugate**[4] *of* A, *denoted* $\text{adj}(A)$, *is the transpose of this cofactor matrix; in symbols,*

$$\text{adj}(A) = \left[c_{ij}(A)\right]^T$$

This agrees with the earlier definition for a 2×2 matrix A as the reader can verify.

Example 3.2.6

Compute the adjugate of $A = \left[\begin{array}{rrr} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{array}\right]$ and calculate $A(\text{adj } A)$ and $(\text{adj } A)A$.

Solution. We first find the cofactor matrix.

$$\left[\begin{array}{ccc} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{array}\right] = \left[\begin{array}{ccc} \left|\begin{array}{rr} 1 & 5 \\ -6 & 7 \end{array}\right| & -\left|\begin{array}{rr} 0 & 5 \\ -2 & 7 \end{array}\right| & \left|\begin{array}{rr} 0 & 1 \\ -2 & -6 \end{array}\right| \\ -\left|\begin{array}{rr} 3 & -2 \\ -6 & 7 \end{array}\right| & \left|\begin{array}{rr} 1 & -2 \\ -2 & 7 \end{array}\right| & -\left|\begin{array}{rr} 1 & 3 \\ -2 & -6 \end{array}\right| \\ \left|\begin{array}{rr} 3 & -2 \\ 1 & 5 \end{array}\right| & -\left|\begin{array}{rr} 1 & -2 \\ 0 & 5 \end{array}\right| & \left|\begin{array}{rr} 1 & 3 \\ 0 & 1 \end{array}\right| \end{array}\right]$$

$$= \left[\begin{array}{rrr} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{array}\right]$$

Then the adjugate of A is the transpose of this cofactor matrix.

$$\text{adj } A = \left[\begin{array}{rrr} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{array}\right]^T = \left[\begin{array}{rrr} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{array}\right]$$

The computation of $A(\text{adj } A)$ gives

$$A(\text{adj } A) = \left[\begin{array}{rrr} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{array}\right] \left[\begin{array}{rrr} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array}\right] = 3I$$

and the reader can verify that also $(\text{adj } A)A = 3I$. Hence, analogy with the 2×2 case would indicate that $\det A = 3$; this is, in fact, the case.

The relationship $A(\text{adj } A) = (\det A)I$ holds for any square matrix A. To see why this is so, consider

[4]This is also called the classical adjoint of A, but the term "adjoint" has another meaning.

the general 3×3 case. Writing $c_{ij}(A) = c_{ij}$ for short, we have

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

If $A = \left[a_{ij}\right]$ in the usual notation, we are to verify that $A(\text{adj } A) = (\det A)I$. That is,

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

Consider the $(1,\ 1)$-entry in the product. It is given by $a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$, and this is just the cofactor expansion of $\det A$ along the first row of A. Similarly, the $(2,\ 2)$-entry and the $(3,\ 3)$-entry are the cofactor expansions of $\det A$ along rows 2 and 3, respectively.

So it remains to be seen why the off-diagonal elements in the matrix product $A(\text{adj } A)$ are all zero. Consider the $(1,\ 2)$-entry of the product. It is given by $a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23}$. This *looks* like the cofactor expansion of the determinant of *some* matrix. To see which, observe that c_{21}, c_{22}, and c_{23} are all computed by *deleting* row 2 of A (and one of the columns), so they remain the same if row 2 of A is changed. In particular, if row 2 of A is replaced by row 1, we obtain

$$a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0$$

where the expansion is along row 2 and where the determinant is zero because two rows are identical. A similar argument shows that the other off-diagonal entries are zero.

This argument works in general and yields the first part of Theorem 3.2.4. The second assertion follows from the first by multiplying through by the scalar $\frac{1}{\det A}$.

Theorem 3.2.4: Adjugate Formula

If A is any square matrix, then

$$A(\text{adj } A) = (\det A)I = (\text{adj } A)A$$

In particular, if $\det A \neq 0$, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$

It is important to note that this theorem is *not* an efficient way to find the inverse of the matrix A. For example, if A were 10×10, the calculation of $\text{adj } A$ would require computing $10^2 = 100$ determinants of 9×9 matrices! On the other hand, the matrix inversion algorithm would find A^{-1} with about the same effort as finding $\det A$. Clearly, Theorem 3.2.4 is not a *practical* result: its virtue is that it gives a formula for A^{-1} that is useful for *theoretical* purposes.

Example 3.2.7

Find the $(2,\ 3)$-entry of A^{-1} if $A = \left[\begin{array}{rrr} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{array} \right]$.

Solution. First compute

$$\det A = \left| \begin{array}{rrr} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{array} \right| = \left| \begin{array}{rrr} 2 & 1 & 7 \\ 5 & -7 & 11 \\ 3 & 0 & 0 \end{array} \right| = 3 \left| \begin{array}{rr} 1 & 7 \\ -7 & 11 \end{array} \right| = 180$$

Since $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{180} \left[c_{ij}(A) \right]^T$, the $(2,\ 3)$-entry of A^{-1} is the $(3,\ 2)$-entry of the matrix $\frac{1}{180} \left[c_{ij}(A) \right]$; that is, it equals $\frac{1}{180} c_{32}(A) = \frac{1}{180} \left(- \left| \begin{array}{rr} 2 & 3 \\ 5 & 1 \end{array} \right| \right) = \frac{13}{180}$.

Example 3.2.8

If A is $n \times n$, $n \geq 2$, show that $\det (\operatorname{adj} A) = (\det A)^{n-1}$.

Solution. Write $d = \det A$; we must show that $\det (\operatorname{adj} A) = d^{n-1}$. We have $A(\operatorname{adj} A) = dI$ by Theorem 3.2.4, so taking determinants gives $d \det (\operatorname{adj} A) = d^n$. Hence we are done if $d \neq 0$. Assume $d = 0$; we must show that $\det (\operatorname{adj} A) = 0$, that is, $\operatorname{adj} A$ is not invertible. If $A \neq 0$, this follows from $A(\operatorname{adj} A) = dI = 0$; if $A = 0$, it follows because then $\operatorname{adj} A = 0$.

Cramer's Rule

Theorem 3.2.4 has a nice application to linear equations. Suppose

$$A\mathbf{x} = \mathbf{b}$$

is a system of n equations in n variables $x_1,\ x_2,\ \dots,\ x_n$. Here A is the $n \times n$ coefficient matrix, and $\mathbf{x}$ and $\mathbf{b}$ are the columns

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \text{ and } \mathbf{b} = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

of variables and constants, respectively. If $\det A \neq 0$, we left multiply by A^{-1} to obtain the solution $\mathbf{x} = A^{-1}\mathbf{b}$. When we use the adjugate formula, this becomes

$$\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = \frac{1}{\det A} (\operatorname{adj} A)\mathbf{b}$$

$$= \frac{1}{\det A} \begin{bmatrix} c_{11}(A) & c_{21}(A) & \cdots & c_{n1}(A) \\ c_{12}(A) & c_{22}(A) & \cdots & c_{n2}(A) \\ \vdots & \vdots & & \vdots \\ c_{1n}(A) & c_{2n}(A) & \cdots & c_{nn}(A) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Hence, the variables $x_1, x_2, \dots, x_n$ are given by

$$\begin{aligned} x_1 &= \tfrac{1}{\det A}\left[b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A)\right] \\ x_2 &= \tfrac{1}{\det A}\left[b_1c_{12}(A) + b_2c_{22}(A) + \cdots + b_nc_{n2}(A)\right] \\ &\ \ \vdots \qquad\qquad\qquad \vdots \\ x_n &= \tfrac{1}{\det A}\left[b_1c_{1n}(A) + b_2c_{2n}(A) + \cdots + b_nc_{nn}(A)\right] \end{aligned}$$

Now the quantity $b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A)$ occurring in the formula for x_1 looks like the cofactor expansion of the determinant of a matrix. The cofactors involved are $c_{11}(A), c_{21}(A), \dots, c_{n1}(A)$, corresponding to the first column of A. If A_1 is obtained from A by replacing the first column of A by $\mathbf{b}$, then $c_{i1}(A_1) = c_{i1}(A)$ for each i because column 1 is deleted when computing them. Hence, expanding $\det(A_1)$ by the first column gives

$$\begin{aligned} \det A_1 &= b_1c_{11}(A_1) + b_2c_{21}(A_1) + \cdots + b_nc_{n1}(A_1) \\ &= b_1c_{11}(A) + b_2c_{21}(A) + \cdots + b_nc_{n1}(A) \\ &= (\det A)x_1 \end{aligned}$$

Hence, $x_1 = \frac{\det A_1}{\det A}$ and similar results hold for the other variables.

Theorem 3.2.5: Cramer's Rule[5]

If A is an invertible $n \times n$ matrix, the solution to the system

$$A\mathbf{x} = \mathbf{b}$$

of n equations in the variables $x_1, x_2, \dots, x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing column k by $\mathbf{b}$.

Example 3.2.9

Find x_1, given the following system of equations.

$$\begin{aligned} 5x_1 + x_2 - x_3 &= 4 \\ 9x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

[5]Gabriel Cramer (1704–1752) was a Swiss mathematician who wrote an introductory work on algebraic curves. He popularized the rule that bears his name, but the idea was known earlier.

Solution. Compute the determinants of the coefficient matrix A and the matrix A_1 obtained from it by replacing the first column by the column of constants.

$$\det A = \det \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix} = -16$$

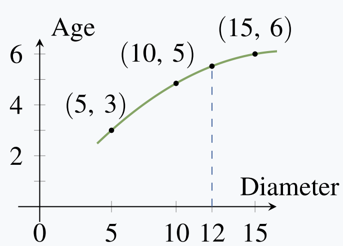
$$\det A_1 = \det \begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix} = 12$$

Hence, $x_1 = \frac{\det A_1}{\det A} = -\frac{3}{4}$ by Cramer's rule.

Cramer's rule is *not* an efficient way to solve linear systems or invert matrices. True, it enabled us to calculate x_1 here without computing x_2 or x_3. Although this might seem an advantage, the truth of the matter is that, for large systems of equations, the number of computations needed to find *all* the variables by the gaussian algorithm is comparable to the number required to find *one* of the determinants involved in Cramer's rule. Furthermore, the algorithm works when the matrix of the system is not invertible and even when the coefficient matrix is not square. Like the adjugate formula, then, Cramer's rule is *not* a practical numerical technique; its virtue is theoretical.

Polynomial Interpolation

Example 3.2.10



A forester wants to estimate the age (in years) of a tree by measuring the diameter of the trunk (in cm). She obtains the following data:

	Tree 1	Tree 2	Tree 3
Trunk Diameter	5	10	15
Age	3	5	6

Estimate the age of a tree with a trunk diameter of 12 cm.

Solution.
The forester decides to "fit" a quadratic polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2$$

to the data, that is choose the coefficients r_0, r_1, and r_2 so that $p(5) = 3$, $p(10) = 5$, and $p(15) = 6$, and then use $p(12)$ as the estimate. These conditions give three linear equations:

$$\begin{aligned} r_0 + 5r_1 + 25r_2 &= 3 \\ r_0 + 10r_1 + 100r_2 &= 5 \\ r_0 + 15r_1 + 225r_2 &= 6 \end{aligned}$$

The (unique) solution is $r_0 = 0$, $r_1 = \frac{7}{10}$, and $r_2 = -\frac{1}{50}$, so

$$p(x) = \frac{7}{10}x - \frac{1}{50}x^2 = \frac{1}{50}x(35 - x)$$

Hence the estimate is $p(12) = 5.52$.

As in Example 3.2.10, it often happens that two variables x and y are related but the actual functional form $y = f(x)$ of the relationship is unknown. Suppose that for certain values $x_1, x_2, \dots, x_n$ of x the corresponding values $y_1, y_2, \dots, y_n$ are known (say from experimental measurements). One way to estimate the value of y corresponding to some other value a of x is to find a polynomial[6]

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1}x^{n-1}$$

that "fits" the data, that is $p(x_i) = y_i$ holds for each $i = 1, 2, \dots, n$. Then the estimate for y is $p(a)$. As we will see, such a polynomial always exists if the x_i are distinct.

The conditions that $p(x_i) = y_i$ are

$$\begin{array}{ccccccccccc}
r_0 & + & r_1 x_1 & + & r_2 x_1^2 & + & \cdots & + & r_{n-1}x_1^{n-1} & = & y_1 \\
r_0 & + & r_1 x_2 & + & r_2 x_2^2 & + & \cdots & + & r_{n-1}x_2^{n-1} & = & y_2 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
r_0 & + & r_1 x_n & + & r_2 x_n^2 & + & \cdots & + & r_{n-1}x_n^{n-1} & = & y_n
\end{array}$$

In matrix form, this is

$$\left[\begin{array}{ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{array}\right] \left[\begin{array}{c} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{array}\right] = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \tag{3.3}$$

It can be shown (see Theorem 3.2.7) that the determinant of the coefficient matrix equals the product of all terms $(x_i - x_j)$ with $i > j$ and so is nonzero (because the x_i are distinct). Hence the equations have a unique solution $r_0, r_1, \dots, r_{n-1}$. This proves

Theorem 3.2.6

Let n data pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be given, and assume that the x_i are distinct. Then there exists a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1}x^{n-1}$$

such that $p(x_i) = y_i$ for each $i = 1, 2, \dots, n$.

The polynomial in Theorem 3.2.6 is called the **interpolating polynomial** for the data.

[6]A **polynomial** is an expression of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where the a_i are numbers and x is a variable. If $a_n \neq 0$, the integer n is called the degree of the polynomial, and a_n is called the leading coefficient. See Appendix D.

We conclude by evaluating the determinant of the coefficient matrix in Equation 3.3. If $a_1,\ a_2,\ \dots,\ a_n$ are numbers, the determinant

$$\det \left[\begin{array}{ccccc} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{array} \right]$$

is called a **Vandermonde determinant**.[7] There is a simple formula for this determinant. If $n = 2$, it equals $(a_2 - a_1)$; if $n = 3$, it is $(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$ by Example 3.1.8. The general result is the product

$$\prod_{1 \leq j < i \leq n} (a_i - a_j)$$

of all factors $(a_i - a_j)$ where $1 \leq j < i \leq n$. For example, if $n = 4$, it is

$$(a_4 - a_3)(a_4 - a_2)(a_4 - a_1)(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

Theorem 3.2.7

Let $a_1,\ a_2,\ \dots,\ a_n$ be numbers where $n \geq 2$. Then the corresponding Vandermonde determinant is given by

$$\det \left[\begin{array}{ccccc} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{array} \right] = \prod_{1 \leq j < i \leq n} (a_i - a_j)$$

Proof. We may assume that the a_i are distinct; otherwise both sides are zero. We proceed by induction on $n \geq 2$; we have it for $n = 2,\ 3$. So assume it holds for $n - 1$. The trick is to replace a_n by a variable x, and consider the determinant

$$p(x) = \det \left[\begin{array}{ccccc} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{array} \right]$$

Then $p(x)$ is a polynomial of degree at most $n - 1$ (expand along the last row), and $p(a_i) = 0$ for each $i = 1,\ 2,\ \dots,\ n - 1$ because in each case there are two identical rows in the determinant. In particular, $p(a_1) = 0$, so we have $p(x) = (x - a_1)p_1(x)$ by the factor theorem (see Appendix D). Since $a_2 \neq a_1$, we obtain $p_1(a_2) = 0$, and so $p_1(x) = (x - a_2)p_2(x)$. Thus $p(x) = (x - a_1)(x - a_2)p_2(x)$. As the a_i are distinct, this process continues to obtain

$$p(x) = (x - a_1)(x - a_2) \cdots (x - a_{n-1})d \tag{3.4}$$

[7]Alexandre Théophile Vandermonde (1735–1796) was a French mathematician who made contributions to the theory of equations.

where d is the coefficient of x^{n-1} in $p(x)$. By the cofactor expansion of $p(x)$ along the last row we get

$$d = (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{bmatrix}$$

Because $(-1)^{n+n} = 1$ the induction hypothesis shows that d is the product of all factors $(a_i - a_j)$ where $1 \leq j < i \leq n-1$. The result now follows from Equation 3.4 by substituting a_n for x in $p(x)$. □

Proof of Theorem 3.2.1. If A and B are $n \times n$ matrices we must show that

$$\det(AB) = \det A \det B \tag{3.5}$$

Recall that if E is an elementary matrix obtained by doing one row operation to I_n, then doing that operation to a matrix C (Lemma 2.5.1) results in EC. By looking at the three types of elementary matrices separately, Theorem 3.1.2 shows that

$$\det(EC) = \det E \det C \quad \text{for any matrix } C \tag{3.6}$$

Thus if $E_1, E_2, \ldots, E_k$ are all elementary matrices, it follows by induction that

$$\det(E_k \cdots E_2 E_1 C) = \det E_k \cdots \det E_2 \det E_1 \det C \text{ for any matrix } C \tag{3.7}$$

Lemma. If A has no inverse, then $\det A = 0$.

Proof. Let $A \to R$ where R is reduced row-echelon, say $E_n \cdots E_2 E_1 A = R$. Then R has a row of zeros by Part (4) of Theorem 2.4.5, and hence $\det R = 0$. But then Equation 3.7 gives $\det A = 0$ because $\det E \neq 0$ for any elementary matrix E. This proves the Lemma.

Now we can prove Equation 3.5 by considering two cases.

Case 1. A has no inverse. Then AB also has no inverse (otherwise $A[B(AB)^{-1}] = I$) so A is invertible by Corollary 2.4.2 to Theorem 2.4.5. Hence the above Lemma (twice) gives

$$\det(AB) = 0 = 0 \det B = \det A \det B$$

proving Equation 3.5 in this case.

Case 2. A has an inverse. Then A is a product of elementary matrices by Theorem 2.5.2, say $A = E_1 E_2 \cdots E_k$. Then Equation 3.7 with $C = I$ gives

$$\det A = \det(E_1 E_2 \cdots E_k) = \det E_1 \det E_2 \cdots \det E_k$$

But then Equation 3.7 with $C = B$ gives

$$\det(AB) = \det[(E_1 E_2 \cdots E_k)B] = \det E_1 \det E_2 \cdots \det E_k \det B = \det A \det B$$

and Equation 3.5 holds in this case too. □

Exercises for 3.2

Exercise 3.2.1 Find the adjugate of each of the following matrices.

a. $\begin{bmatrix} 5 & 1 & 3 \\ -1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

d. $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Exercise 3.2.2 Use determinants to find which real values of c make each of the following matrices invertible.

a. $\begin{bmatrix} 1 & 0 & 3 \\ 3 & -4 & c \\ 2 & 5 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 0 & c & -c \\ -1 & 2 & 1 \\ c & -c & c \end{bmatrix}$

c. $\begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$

d. $\begin{bmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 2 & c & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & c & -1 \\ c & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$

Exercise 3.2.3 Let A, B, and C denote $n \times n$ matrices and assume that $\det A = -1$, $\det B = 2$, and $\det C = 3$. Evaluate:

a. $\det(A^3BC^TB^{-1})$

b. $\det(B^2C^{-1}AB^{-1}C^T)$

Exercise 3.2.4 Let A and B be invertible $n \times n$ matrices. Evaluate:

a. $\det(B^{-1}AB)$

b. $\det(A^{-1}B^{-1}AB)$

Exercise 3.2.5 If A is 3×3 and $\det(2A^{-1}) = -4$ and $\det(A^3(B^{-1})^T) = -4$, find $\det A$ and $\det B$.

Exercise 3.2.6 Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$ and assume that $\det A = 3$. Compute:

a. $\det(2B^{-1})$ where $B = \begin{bmatrix} 4u & 2a & -p \\ 4v & 2b & -q \\ 4w & 2c & -r \end{bmatrix}$

b. $\det(2C^{-1})$ where $C = \begin{bmatrix} 2p & -a+u & 3u \\ 2q & -b+v & 3v \\ 2r & -c+w & 3w \end{bmatrix}$

Exercise 3.2.7 If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -2$ calculate:

a. $\det \begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}$

b. $\det \begin{bmatrix} 2b & 0 & 4d \\ 1 & 2 & -2 \\ a+1 & 2 & 2(c-1) \end{bmatrix}$

c. $\det(3A^{-1})$ where $A = \begin{bmatrix} 3c & a+c \\ 3d & b+d \end{bmatrix}$

Exercise 3.2.8 Solve each of the following by Cramer's rule:

a. $\begin{aligned} 2x + y &= 1 \\ 3x + 7y &= -2 \end{aligned}$

b. $\begin{aligned} 3x + 4y &= 9 \\ 2x - y &= -1 \end{aligned}$

c. $\begin{aligned} 5x + y - z &= -7 \\ 2x - y - 2z &= 6 \\ 3x \qquad + 2z &= -7 \end{aligned}$

d. $\begin{aligned} 4x - y + 3z &= 1 \\ 6x + 2y - z &= 0 \\ 3x + 3y + 2z &= -1 \end{aligned}$

Exercise 3.2.9 Use Theorem 3.2.4 to find the (2, 3)-entry of A^{-1} if:

a. $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 4 & 7 \end{bmatrix}$

Exercise 3.2.10 Explain what can be said about $\det A$ if:

a. $A^2 = A$

b. $A^2 = I$

c. $A^3 = A$

d. $PA = P$ and P is invertible

e. $A^2 = uA$ and A is $n \times n$

f. $A = -A^T$ and A is $n \times n$

g. $A^2 + I = 0$ and A is $n \times n$

Exercise 3.2.11 Let A be $n \times n$. Show that $uA = (uI)A$, and use this with Theorem 3.2.1 to deduce the result in Theorem 3.1.3: $\det(uA) = u^n \det A$.

Exercise 3.2.12 If A and B are $n \times n$ matrices, if $AB = -BA$, and if n is odd, show that either A or B has no inverse.

Exercise 3.2.13 Show that $\det AB = \det BA$ holds for any two $n \times n$ matrices A and B.

Exercise 3.2.14 If $A^k = 0$ for some $k \geq 1$, show that A is not invertible.

Exercise 3.2.15 If $A^{-1} = A^T$, describe the cofactor matrix of A in terms of A.

Exercise 3.2.16 Show that no 3×3 matrix A exists such that $A^2 + I = 0$. Find a 2×2 matrix A with this property.

Exercise 3.2.17 Show that $\det(A + B^T) = \det(A^T + B)$ for any $n \times n$ matrices A and B.

Exercise 3.2.18 Let A and B be invertible $n \times n$ matrices. Show that $\det A = \det B$ if and only if $A = UB$ where U is a matrix with $\det U = 1$.

Exercise 3.2.19 For each of the matrices in Exercise 2, find the inverse for those values of c for which it exists.

Exercise 3.2.20 In each case either prove the statement or give an example showing that it is false:

a. If $\operatorname{adj} A$ exists, then A is invertible.

b. If A is invertible and $\operatorname{adj} A = A^{-1}$, then $\det A = 1$.

c. $\det(AB) = \det(B^T A)$.

d. If $\det A \neq 0$ and $AB = AC$, then $B = C$.

e. If $A^T = -A$, then $\det A = -1$.

f. If $\operatorname{adj} A = 0$, then $A = 0$.

g. If A is invertible, then $\operatorname{adj} A$ is invertible.

h. If A has a row of zeros, so also does $\operatorname{adj} A$.

i. $\det(A^T A) > 0$ for all square matrices A.

j. $\det(I + A) = 1 + \det A$.

k. If AB is invertible, then A and B are invertible.

l. If $\det A = 1$, then $\operatorname{adj} A = A$.

m. If A is invertible and $\det A = d$, then $\operatorname{adj} A = dA^{-1}$.

Exercise 3.2.21 If A is 2×2 and $\det A = 0$, show that one column of A is a scalar multiple of the other. [*Hint*: Definition 2.5 and Part (2) of Theorem 2.4.5.]

Exercise 3.2.22 Find a polynomial $p(x)$ of degree 2 such that:

a. $p(0) = 2$, $p(1) = 3$, $p(3) = 8$

b. $p(0) = 5$, $p(1) = 3$, $p(2) = 5$

Exercise 3.2.23 Find a polynomial $p(x)$ of degree 3 such that:

a. $p(0) = p(1) = 1$, $p(-1) = 4$, $p(2) = -5$

b. $p(0) = p(1) = 1$, $p(-1) = 2$, $p(-2) = -3$

Exercise 3.2.24 Given the following data pairs, find the interpolating polynomial of degree 3 and estimate the value of y corresponding to $x = 1.5$.

a. $(0,\ 1)$, $(1,\ 2)$, $(2,\ 5)$, $(3,\ 10)$

b. $(0,\ 1)$, $(1,\ 1.49)$, $(2,\ -0.42)$, $(3,\ -11.33)$

c. $(0,\ 2)$, $(1,\ 2.03)$, $(2,\ -0.40)$, $(-1,\ 0.89)$

Exercise 3.2.25 If $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ show that $\det A = 1 + a^2 + b^2 + c^2$. Hence, find A^{-1} for any a, b, and c.

Exercise 3.2.26

a. Show that $A = \begin{bmatrix} a & p & q \\ 0 & b & r \\ 0 & 0 & c \end{bmatrix}$ has an inverse if and only if $abc \neq 0$, and find A^{-1} in that case.

b. Show that if an upper triangular matrix is invertible, the inverse is also upper triangular.

Exercise 3.2.27 Let A be a matrix each of whose entries are integers. Show that each of the following conditions implies the other.

1. A is invertible and A^{-1} has integer entries.

2. $\det A = 1$ or -1.

Exercise 3.2.28 If $A^{-1} = \left[\begin{array}{rrr} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{array} \right]$ find $\mathrm{adj}\, A$.

Exercise 3.2.29 If A is 3×3 and $\det A = 2$, find $\det(A^{-1} + 4\,\mathrm{adj}\, A)$.

Exercise 3.2.30 Show that $\det \left[\begin{array}{cc} 0 & A \\ B & X \end{array} \right] = \det A \det B$ when A and B are 2×2. What if A and B are 3×3?

[*Hint*: Block multiply by $\left[\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right]$.]

Exercise 3.2.31 Let A be $n \times n$, $n \geq 2$, and assume one column of A consists of zeros. Find the possible values of $\mathrm{rank}\,(\mathrm{adj}\, A)$.

Exercise 3.2.32 If A is 3×3 and invertible, compute $\det(-A^2(\mathrm{adj}\, A)^{-1})$.

Exercise 3.2.33 Show that $\mathrm{adj}\,(uA) = u^{n-1}\,\mathrm{adj}\, A$ for all $n \times n$ matrices A.

Exercise 3.2.34 Let A and B denote invertible $n \times n$ matrices. Show that:

a. $\mathrm{adj}\,(\mathrm{adj}\, A) = (\det A)^{n-2}A$ (here $n \geq 2$) [*Hint*: See Example 3.2.8.]

b. $\mathrm{adj}\,(A^{-1}) = (\mathrm{adj}\, A)^{-1}$

c. $\mathrm{adj}\,(A^T) = (\mathrm{adj}\, A)^T$

d. $\mathrm{adj}\,(AB) = (\mathrm{adj}\, B)(\mathrm{adj}\, A)$ [*Hint*: Show that $AB\,\mathrm{adj}\,(AB) = AB\,\mathrm{adj}\, B\,\mathrm{adj}\, A$.]

3.3 Diagonalization and Eigenvalues

The world is filled with examples of systems that evolve in time—the weather in a region, the economy of a nation, the diversity of an ecosystem, etc. Describing such systems is difficult in general and various methods have been developed in special cases. In this section we describe one such method, called *diagonalization,* which is one of the most important techniques in linear algebra. A very fertile example of this procedure is in modelling the growth of the population of an animal species. This has attracted more attention in recent years with the ever increasing awareness that many species are endangered. To motivate the technique, we begin by setting up a simple model of a bird population in which we make assumptions about survival and reproduction rates.

Example 3.3.1

Consider the evolution of the population of a species of birds. Because the number of males and females are nearly equal, we count only females. We assume that each female remains a juvenile for one year and then becomes an adult, and that only adults have offspring. We make three assumptions about reproduction and survival rates:

1. The number of juvenile females hatched in any year is twice the number of adult females alive the year before (we say the **reproduction rate** is 2).
2. Half of the adult females in any year survive to the next year (the **adult survival rate** is $\frac{1}{2}$).
3. One quarter of the juvenile females in any year survive into adulthood (the **juvenile survival rate** is $\frac{1}{4}$).

If there were 100 adult females and 40 juvenile females alive initially, compute the population of females k years later.

Solution. Let a_k and j_k denote, respectively, the number of adult and juvenile females after k years, so that the total female population is the sum $a_k + j_k$. Assumption 1 shows that $j_{k+1} = 2a_k$, while assumptions 2 and 3 show that $a_{k+1} = \frac{1}{2}a_k + \frac{1}{4}j_k$. Hence the numbers a_k and j_k in successive years are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $\mathbf{v}_k = \left[\begin{array}{c} a_k \\ j_k \end{array}\right]$ and $A = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{array}\right]$ these equations take the matrix form

$$\mathbf{v}_{k+1} = A\mathbf{v}_k, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Taking $k = 0$ gives $\mathbf{v}_1 = A\mathbf{v}_0$, then taking $k = 1$ gives $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and taking $k = 2$ gives $\mathbf{v}_3 = A\mathbf{v}_2 = A^3\mathbf{v}_0$. Continuing in this way, we get

$$\mathbf{v}_k = A^k\mathbf{v}_0, \quad \text{for each } k = 0,\ 1,\ 2,\ \dots$$

Since $\mathbf{v}_0 = \left[\begin{array}{c} a_0 \\ j_0 \end{array}\right] = \left[\begin{array}{c} 100 \\ 40 \end{array}\right]$ is known, finding the population profile $\mathbf{v}_k$ amounts to computing A^k for all $k \geq 0$. We will complete this calculation in Example 3.3.12 after some new techniques have been developed.

Let A be a fixed $n \times n$ matrix. A sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots$ of column vectors in $\mathbb{R}^n$ is called a **linear dynamical system**[8] if $\mathbf{v}_0$ is known and the other $\mathbf{v}_k$ are determined (as in Example 3.3.1) by the conditions

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for each } k = 0,\ 1,\ 2,\ \dots$$

These conditions are called a **matrix recurrence** for the vectors $\mathbf{v}_k$. As in Example 3.3.1, they imply that

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for all } k \geq 0$$

so finding the columns $\mathbf{v}_k$ amounts to calculating A^k for $k \geq 0$.

Direct computation of the powers A^k of a square matrix A can be time-consuming, so we adopt an indirect method that is commonly used. The idea is to first **diagonalize** the matrix A, that is, to find an invertible matrix P such that

$$P^{-1}AP = D \text{ is a diagonal matrix} \tag{3.8}$$

This works because the powers D^k of the diagonal matrix D are easy to compute, and Equation 3.8 enables us to compute powers A^k of the matrix A in terms of powers D^k of D. Indeed, we can solve Equation 3.8 for A to get $A = PDP^{-1}$. Squaring this gives

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

Using this we can compute A^3 as follows:

$$A^3 = AA^2 = (PDP^{-1})(PD^2P^{-1}) = PD^3P^{-1}$$

[8] More precisely, this is *a linear discrete* dynamical system. Many models regard $\mathbf{v}_t$ as a continuous function of the time t, and replace our condition between $\mathbf{b}_{k+1}$ and $A\mathbf{v}_k$ with a differential relationship viewed as functions of time.

Continuing in this way we obtain Theorem 3.3.1 (even if D is not diagonal).

Theorem 3.3.1

If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$ for each $k = 1, 2, \ldots$.

Hence computing A^k comes down to finding an invertible matrix P as in equation Equation 3.8. To do this it is necessary to first compute certain numbers (called eigenvalues) associated with the matrix A.

Eigenvalues and Eigenvectors

Definition 3.4 Eigenvalues and Eigenvectors of a Matrix

If A is an $n \times n$ matrix, a number λ is called an ***eigenvalue*** *of A if*

$$A\mathbf{x} = \lambda\mathbf{x} \text{ for some column } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

In this case, $\mathbf{x}$ is called an ***eigenvector*** *of A corresponding to the eigenvalue λ, or a λ-**eigenvector** for short.*

Example 3.3.2

If $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ then $A\mathbf{x} = 4\mathbf{x}$ so $\lambda = 4$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.

The matrix A in Example 3.3.2 has another eigenvalue in addition to $\lambda = 4$. To find it, we develop a general procedure for *any* $n \times n$ matrix A.

By definition a number λ is an eigenvalue of the $n \times n$ matrix A if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some column $\mathbf{x} \neq \mathbf{0}$. This is equivalent to asking that the homogeneous system

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$. By Theorem 2.4.5 this happens if and only if the matrix $\lambda I - A$ is not invertible and this, in turn, holds if and only if the determinant of the coefficient matrix is zero:

$$\det(\lambda I - A) = 0$$

This last condition prompts the following definition:

Definition 3.5 Characteristic Polynomial of a Matrix

If A is an $n \times n$ matrix, the ***characteristic polynomial*** *$c_A(x)$ of A is defined by*

$$c_A(x) = \det(xI - A)$$

Note that $c_A(x)$ is indeed a polynomial in the variable x, and it has degree n when A is an $n \times n$ matrix (this is illustrated in the examples below). The above discussion shows that a number λ is an eigenvalue of A if and only if $c_A(\lambda) = 0$, that is if and only if λ is a **root** of the characteristic polynomial $c_A(x)$. We record these observations in

Theorem 3.3.2

Let A be an $n \times n$ matrix.

1. *The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

In practice, solving the equations in part 2 of Theorem 3.3.2 is a routine application of gaussian elimination, but finding the eigenvalues can be difficult, often requiring computers (see Section 8.5). For now, the examples and exercises will be constructed so that the roots of the characteristic polynomials are relatively easy to find (usually integers). However, the reader should not be misled by this into thinking that eigenvalues are so easily obtained for the matrices that occur in practical applications!

Example 3.3.3

Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ discussed in Example 3.3.2, and then find all the eigenvalues and their eigenvectors.

Solution. Since $xI - A = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix}$ we get

$$c_A(x) = \det \begin{bmatrix} x-3 & -5 \\ -1 & x+1 \end{bmatrix} = x^2 - 2x - 8 = (x-4)(x+2)$$

Hence, the roots of $c_A(x)$ are $\lambda_1 = 4$ and $\lambda_2 = -2$, so these are the eigenvalues of A. Note that $\lambda_1 = 4$ was the eigenvalue mentioned in Example 3.3.2, but we have found a new one: $\lambda_2 = -2$. To find the eigenvectors corresponding to $\lambda_2 = -2$, observe that in this case

$$(\lambda_2 I - A)\mathbf{x} = \begin{bmatrix} \lambda_2 - 3 & -5 \\ -1 & \lambda_2 + 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$$

so the general solution to $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where t is an arbitrary real number.

Hence, the eigenvectors $\mathbf{x}$ corresponding to λ_2 are $\mathbf{x} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t \neq 0$ is arbitrary. Similarly, $\lambda_1 = 4$ gives rise to the eigenvectors $\mathbf{x} = t\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $t \neq 0$ which includes the observation in Example 3.3.2.

Note that a square matrix A has *many* eigenvectors associated with any given eigenvalue λ. In fact *every* nonzero solution $\mathbf{x}$ of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is an eigenvector. Recall that these solutions are all linear combinations of certain basic solutions determined by the gaussian algorithm (see Theorem 1.3.2). Observe that any nonzero multiple of an eigenvector is again an eigenvector,[9] and such multiples are often more convenient.[10] Any set of nonzero multiples of the basic solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$ will be called a set of **basic eigenvectors** corresponding to λ.

Example 3.3.4

Find the characteristic polynomial, eigenvalues, and basic eigenvectors for

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

Solution. Here the characteristic polynomial is given by

$$c_A(x) = \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x-2 & 1 \\ -1 & -3 & x+2 \end{bmatrix} = (x-2)(x-1)(x+1)$$

so the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$. To find all eigenvectors for $\lambda_1 = 2$, compute

$$\lambda_1 I - A = \begin{bmatrix} \lambda_1 - 2 & 0 & 0 \\ -1 & \lambda_1 - 2 & 1 \\ -1 & -3 & \lambda_1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -3 & 4 \end{bmatrix}$$

We want the (nonzero) solutions to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$. The augmented matrix becomes

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

using row operations. Hence, the general solution $\mathbf{x}$ to $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ where t is arbitrary, so we can use $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the basic eigenvector corresponding to $\lambda_1 = 2$. As the reader can verify, the gaussian algorithm gives basic eigenvectors $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = 1$ and $\lambda_3 = -1$, respectively. Note that to eliminate fractions, we could instead use $3\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ as the basic λ_3-eigenvector.

[9] In fact, any nonzero linear combination of λ-eigenvectors is again a λ-eigenvector.

[10] Allowing nonzero multiples helps eliminate round-off error when the eigenvectors involve fractions.

Example 3.3.5

If A is a square matrix, show that A and A^T have the same characteristic polynomial, and hence the same eigenvalues.

Solution. We use the fact that $xI - A^T = (xI - A)^T$. Then

$$c_{A^T}(x) = \det \left(xI - A^T\right) = \det \left[(xI - A)^T\right] = \det(xI - A) = c_A(x)$$

by Theorem 3.2.3. Hence $c_{A^T}(x)$ and $c_A(x)$ have the same roots, and so A^T and A have the same eigenvalues (by Theorem 3.3.2).

The eigenvalues of a matrix need not be distinct. For example, if $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$ the characteristic polynomial is $(x-1)^2$ so the eigenvalue 1 occurs twice. Furthermore, eigenvalues are usually not computed as the roots of the characteristic polynomial. There are iterative, numerical methods (for example the QR-algorithm in Section 8.5) that are much more efficient for large matrices.

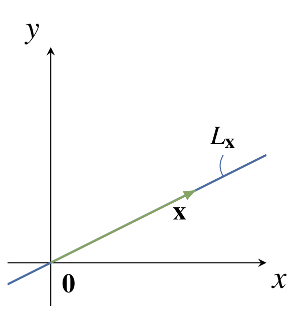
A-Invariance

If A is a 2×2 matrix, we can describe the eigenvectors of A geometrically using the following concept. A line L through the origin in $\mathbb{R}^2$ is called A-**invariant** if $A\mathbf{x}$ is in L whenever $\mathbf{x}$ is in L. If we think of A as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, this asks that A carries L into itself, that is the image $A\mathbf{x}$ of each vector $\mathbf{x}$ in L is again in L.

Example 3.3.6

The x axis $L = \left\{ \left[\begin{array}{c} x \\ 0 \end{array}\right] \mid x \text{ in } \mathbb{R} \right\}$ is A-invariant for any matrix of the form

$$A = \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right] \text{ because } \left[\begin{array}{cc} a & b \\ 0 & c \end{array}\right]\left[\begin{array}{c} x \\ 0 \end{array}\right] = \left[\begin{array}{c} ax \\ 0 \end{array}\right] \text{ is } L \text{ for all } \mathbf{x} = \left[\begin{array}{c} x \\ 0 \end{array}\right] \text{ in } L$$



To see the connection with eigenvectors, let $\mathbf{x} \neq \mathbf{0}$ be any nonzero vector in $\mathbb{R}^2$ and let $L_{\mathbf{x}}$ denote the unique line through the origin containing $\mathbf{x}$ (see the diagram). By the definition of scalar multiplication in Section 2.6, we see that $L_{\mathbf{x}}$ consists of all scalar multiples of $\mathbf{x}$, that is

$$L_{\mathbf{x}} = \mathbb{R}\mathbf{x} = \{t\mathbf{x} \mid t \text{ in } \mathbb{R}\}$$

Now suppose that $\mathbf{x}$ is an eigenvector of A, say $A\mathbf{x} = \lambda\mathbf{x}$ for some λ in $\mathbb{R}$. Then if $t\mathbf{x}$ is in $L_{\mathbf{x}}$ then

$$A(t\mathbf{x}) = t(A\mathbf{x}) = t(\lambda\mathbf{x}) = (t\lambda)\mathbf{x} \text{ is again in } L_{\mathbf{x}}$$

That is, $L_{\mathbf{x}}$ is A-invariant. On the other hand, if $L_{\mathbf{x}}$ is A-invariant then $A\mathbf{x}$ is in $L_{\mathbf{x}}$ (since $\mathbf{x}$ is in $L_{\mathbf{x}}$). Hence $A\mathbf{x} = t\mathbf{x}$ for some t in $\mathbb{R}$, so $\mathbf{x}$ is an eigenvector for A (with eigenvalue t). This proves:

Theorem 3.3.3

Let A be a 2×2 matrix, let $\mathbf{x} \neq \mathbf{0}$ be a vector in $\mathbb{R}^2$, and let $L_\mathbf{x}$ be the line through the origin in $\mathbb{R}^2$ containing $\mathbf{x}$. Then

$$\mathbf{x} \text{ is an eigenvector of } A \quad \text{if and only if} \quad L_\mathbf{x} \text{ is } A\text{-invariant}$$

Example 3.3.7

1. If θ is not a multiple of π, show that $A = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]$ has no real eigenvalue.
2. If m is real show that $B = \frac{1}{1+m^2} \left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array} \right]$ has a 1 as an eigenvalue.

Solution.

1. A induces rotation about the origin through the angle θ (Theorem 2.6.4). Since θ is not a multiple of π, this shows that no line through the origin is A-invariant. Hence A has no eigenvector by Theorem 3.3.3, and so has no eigenvalue.
2. B induces reflection Q_m in the line through the origin with slope m by Theorem 2.6.5. If $\mathbf{x}$ is any nonzero point on this line then it is clear that $Q_m\mathbf{x} = \mathbf{x}$, that is $Q_m\mathbf{x} = 1\mathbf{x}$. Hence 1 is an eigenvalue (with eigenvector $\mathbf{x}$).

If $\theta = \frac{\pi}{2}$ in Example 3.3.7, then $A = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$ so $c_A(x) = x^2 + 1$. This polynomial has no root in $\mathbb{R}$, so A has no (real) eigenvalue, and hence no eigenvector. In fact its eigenvalues are the complex numbers i and $-i$, with corresponding eigenvectors $\left[\begin{array}{c} 1 \\ -i \end{array} \right]$ and $\left[\begin{array}{c} 1 \\ i \end{array} \right]$ In other words, A *has* eigenvalues and eigenvectors, just not real ones.

Note that *every* polynomial has complex roots,[11] so every matrix has complex eigenvalues. While these eigenvalues may very well be real, this suggests that we really should be doing linear algebra over the complex numbers. Indeed, everything we have done (gaussian elimination, matrix algebra, determinants, etc.) works if all the scalars are complex.

[11]This is called the *Fundamental Theorem of Algebra* and was first proved by Gauss in his doctoral dissertation.

Diagonalization

An $n \times n$ matrix D is called a **diagonal matrix** if all its entries off the main diagonal are zero, that is if D has the form

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{diag}\,(\lambda_1,\, \lambda_2,\, \cdots,\, \lambda_n)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are numbers. Calculations with diagonal matrices are very easy. Indeed, if $D = \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $E = \text{diag}\,(\mu_1, \mu_2, \dots, \mu_n)$ are two diagonal matrices, their product DE and sum $D+E$ are again diagonal, and are obtained by doing the same operations to corresponding diagonal elements:

$$\begin{aligned} DE &= \text{diag}\,(\lambda_1\mu_1,\, \lambda_2\mu_2,\, \dots,\, \lambda_n\mu_n) \\ D+E &= \text{diag}\,(\lambda_1+\mu_1,\, \lambda_2+\mu_2,\, \dots,\, \lambda_n+\mu_n) \end{aligned}$$

Because of the simplicity of these formulas, and with an eye on Theorem 3.3.1 and the discussion preceding it, we make another definition:

Definition 3.6 Diagonalizable Matrices

An $n \times n$ matrix A is called ***diagonalizable*** *if*

$$P^{-1}AP \text{ is diagonal for some invertible } n \times n \text{ matrix } P$$

Here the invertible matrix P is called a ***diagonalizing matrix*** *for A.*

To discover when such a matrix P exists, we let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ denote the columns of P and look for ways to determine when such $\mathbf{x}_i$ exist and how to compute them. To this end, write P in terms of its columns as follows:

$$P = [\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n]$$

Observe that $P^{-1}AP = D$ for some diagonal matrix D holds if and only if

$$AP = PD$$

If we write $D = \text{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n)$, where the λ_i are numbers to be determined, the equation $AP = PD$ becomes

$$A\,[\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n] = [\mathbf{x}_1,\, \mathbf{x}_2,\, \cdots,\, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

By the definition of matrix multiplication, each side simplifies as follows

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix}$$

Comparing columns shows that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for each i, so

$$P^{-1}AP = D \quad \text{if and only if } A\mathbf{x}_i = \lambda_i\mathbf{x}_i \text{ for each } i$$

In other words, $P^{-1}AP = D$ holds if and only if the diagonal entries of D are eigenvalues of A and the columns of P are corresponding eigenvectors. This proves the following fundamental result.

Theorem 3.3.4

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that the matrix $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array} \right]$ is invertible.*
2. *When this is the case, $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

Example 3.3.8

Diagonalize the matrix $A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{array} \right]$ in Example 3.3.4.

Solution. By Example 3.3.4, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -1$, with corresponding basic eigenvectors $\mathbf{x}_1 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right]$, $\mathbf{x}_2 = \left[\begin{array}{r} 0 \\ 1 \\ 1 \end{array} \right]$, and $\mathbf{x}_3 = \left[\begin{array}{r} 0 \\ 1 \\ 3 \end{array} \right]$ respectively. Since the matrix $P = \left[\begin{array}{ccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{array} \right] = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{array} \right]$ is invertible, Theorem 3.3.4 guarantees that

$$P^{-1}AP = \left[\begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] = D$$

The reader can verify this directly—easier to check $AP = PD$.

In Example 3.3.8, suppose we let $Q = \left[\begin{array}{ccc} \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{x}_3 \end{array} \right]$ be the matrix formed from the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ of A, but in a *different order* than that used to form P. Then $Q^{-1}AQ = \operatorname{diag}(\lambda_2, \lambda_1, \lambda_3)$ is diagonal by Theorem 3.3.4, but the eigenvalues are in the *new* order. Hence we can choose the diagonalizing matrix P so that the eigenvalues λ_i appear in any order we want along the main diagonal of D.

In every example above each eigenvalue has had only one basic eigenvector. Here is a diagonalizable matrix where this is not the case.

Example 3.3.9

Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution. To compute the characteristic polynomial of A first add rows 2 and 3 of $xI - A$ to row 1:

$$\begin{aligned} c_A(x) &= \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} = \det \begin{bmatrix} x-2 & x-2 & x-2 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} \\ &= \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x+1 & 0 \\ -1 & 0 & x+1 \end{bmatrix} = (x-2)(x+1)^2 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, with λ_2 repeated twice (we say that λ_2 has *multiplicity* two). However, A *is* diagonalizable. For $\lambda_1 = 2$, the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the reader can verify, so a basic λ_1-eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Turning to the repeated eigenvalue $\lambda_2 = -1$, we must solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$. By gaussian elimination, the general solution is $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where s and t are arbitrary. Hence the gaussian algorithm produces *two* basic λ_2-eigenvectors $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ If we take $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we find that P is invertible. Hence $P^{-1}AP = \operatorname{diag}(2, -1, -1)$ by Theorem 3.3.4.

Example 3.3.9 typifies every diagonalizable matrix. To describe the general case, we need some terminology.

Definition 3.7 Multiplicity of an Eigenvalue

An eigenvalue λ of a square matrix A is said to have **multiplicity** *m if it occurs m times as a root of the characteristic polynomial $c_A(x)$.*

For example, the eigenvalue $\lambda_2 = -1$ in Example 3.3.9 has multiplicity 2. In that example the gaussian algorithm yields two basic λ_2-eigenvectors, the same number as the multiplicity. This works in general.

Theorem 3.3.5

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors; that is, if and only if the general solution of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has exactly m parameters.

One case of Theorem 3.3.5 deserves mention.

Theorem 3.3.6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The proofs of Theorem 3.3.5 and Theorem 3.3.6 require more advanced techniques and are given in Chapter 5. The following procedure summarizes the method.

Diagonalization Algorithm

To diagonalize an $n \times n$ matrix A:

Step 1. Find the distinct eigenvalues λ of A.

Step 2. Compute a set of basic eigenvectors corresponding to each of these eigenvalues λ as basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Step 3. The matrix A is diagonalizable if and only if there are n basic eigenvectors in all.

Step 4. If A is diagonalizable, the $n \times n$ matrix P with these basic eigenvectors as its columns is a diagonalizing matrix for A, that is, P is invertible and $P^{-1}AP$ is diagonal.

The diagonalization algorithm is valid even if the eigenvalues are nonreal complex numbers. In this case the eigenvectors will also have complex entries, but we will not pursue this here.

Example 3.3.10

Show that $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$ is not diagonalizable.

Solution 1. The characteristic polynomial is $c_A(x) = (x-1)^2$, so A has only one eigenvalue $\lambda_1 = 1$ of multiplicity 2. But the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $t \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$, so there is only one parameter, and so only one basic eigenvector $\left[\begin{array}{c} 1 \\ 2 \end{array} \right]$. Hence A is not diagonalizable.

Solution 2. We have $c_A(x) = (x-1)^2$ so the only eigenvalue of A is $\lambda = 1$. Hence, if A were diagonalizable, Theorem 3.3.4 would give $P^{-1}AP = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I$ for some invertible matrix P. But then $A = PIP^{-1} = I$, which is not the case. So A cannot be diagonalizable.

Diagonalizable matrices share many properties of their eigenvalues. The following example illustrates why.

Example 3.3.11

If $\lambda^3 = 5\lambda$ for every eigenvalue of the diagonalizable matrix A, show that $A^3 = 5A$.

Solution. Let $P^{-1}AP = D = \text{diag}\,(\lambda_1, \dots, \lambda_n)$. Because $\lambda_i^3 = 5\lambda_i$ for each i, we obtain

$$D^3 = \text{diag}\,(\lambda_1^3, \dots, \lambda_n^3) = \text{diag}\,(5\lambda_1, \dots, 5\lambda_n) = 5D$$

Hence $A^3 = (PDP^{-1})^3 = PD^3P^{-1} = P(5D)P^{-1} = 5(PDP^{-1}) = 5A$ using Theorem 3.3.1. This is what we wanted.

If $p(x)$ is any polynomial and $p(\lambda) = 0$ for every eigenvalue of the diagonalizable matrix A, an argument similar to that in Example 3.3.11 shows that $p(A) = 0$. Thus Example 3.3.11 deals with the case $p(x) = x^3 - 5x$. In general, $p(A)$ is called the *evaluation* of the polynomial $p(x)$ at the matrix A. For example, if $p(x) = 2x^3 - 3x + 5$, then $p(A) = 2A^3 - 3A + 5I$—note the use of the identity matrix.

In particular, if $c_A(x)$ denotes the characteristic polynomial of A, we certainly have $c_A(\lambda) = 0$ for each eigenvalue λ of A (Theorem 3.3.2). Hence $c_A(A) = 0$ for every diagonalizable matrix A. This is, in fact, true for *any* square matrix, diagonalizable or not, and the general result is called the Cayley-Hamilton theorem. It is proved in Section 8.7 and again in Section 11.1.

Linear Dynamical Systems

We began Section 3.3 with an example from ecology which models the evolution of the population of a species of birds as time goes on. As promised, we now complete the example—Example 3.3.12 below.

The bird population was described by computing the female population profile $\mathbf{v}_k = \left[\begin{array}{c} a_k \\ j_k \end{array} \right]$ of the species, where a_k and j_k represent the number of adult and juvenile females present k years after the initial values a_0 and j_0 were observed. The model assumes that these numbers are related by the following equations:

$$\begin{aligned} a_{k+1} &= \tfrac{1}{2}a_k + \tfrac{1}{4}j_k \\ j_{k+1} &= 2a_k \end{aligned}$$

If we write $A = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{array} \right]$ the columns $\mathbf{v}_k$ satisfy $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k = 0, 1, 2, \dots$.

Hence $\mathbf{v}_k = A^k\mathbf{v}_0$ for each $k = 1, 2, \dots$. We can now use our diagonalization techniques to determine the population profile $\mathbf{v}_k$ for all values of k in terms of the initial values.

Example 3.3.12

Assuming that the initial values were $a_0 = 100$ adult females and $j_0 = 40$ juvenile females, compute a_k and j_k for $k = 1, 2, \dots$.

Solution. The characteristic polynomial of the matrix $A = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{array} \right]$ is $c_A(x) = x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$ and gaussian elimination gives corresponding basic eigenvectors $\left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right]$ and $\left[\begin{array}{c} -\frac{1}{4} \\ 1 \end{array} \right]$. For convenience, we can use multiples $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 2 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} -1 \\ 4 \end{array} \right]$ respectively. Hence a diagonalizing matrix is $P = \left[\begin{array}{rr} 1 & -1 \\ 2 & 4 \end{array} \right]$ and we obtain

$$P^{-1}AP = D \text{ where } D = \left[\begin{array}{cc} 1 & 0 \\ 0 & -\frac{1}{2} \end{array} \right]$$

This gives $A = PDP^{-1}$ so, for each $k \geq 0$, we can compute A^k explicitly:

$$\begin{aligned} A^k = PD^kP^{-1} &= \left[\begin{array}{rr} 1 & -1 \\ 2 & 4 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{array} \right] \frac{1}{6} \left[\begin{array}{rr} 4 & 1 \\ -2 & 4 \end{array} \right] \\ &= \frac{1}{6} \left[\begin{array}{cc} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{array} \right] \end{aligned}$$

Hence we obtain

$$\begin{aligned} \left[\begin{array}{c} a_k \\ j_k \end{array} \right] = \mathbf{v}_k = A^k\mathbf{v}_0 &= \frac{1}{6} \left[\begin{array}{cc} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{array} \right] \left[\begin{array}{r} 100 \\ 40 \end{array} \right] \\ &= \frac{1}{6} \left[\begin{array}{c} 440+160(-\frac{1}{2})^k \\ 880-640(-\frac{1}{2})^k \end{array} \right] \end{aligned}$$

Equating top and bottom entries, we obtain exact formulas for a_k and j_k:

$$a_k = \frac{220}{3} + \frac{80}{3}\left(-\frac{1}{2}\right)^k \text{ and } j_k = \frac{440}{3} + \frac{320}{3}\left(-\frac{1}{2}\right)^k \text{ for } k = 1,\ 2,\ \cdots$$

In practice, the exact values of a_k and j_k are not usually required. What is needed is a measure of how these numbers behave for large values of k. This is easy to obtain here. Since $(-\frac{1}{2})^k$ is nearly zero for large k, we have the following approximate values

$$a_k \approx \frac{220}{3} \text{ and } j_k \approx \frac{440}{3} \text{ if } k \text{ is large}$$

Hence, in the long term, the female population stabilizes with approximately twice as many juveniles as adults.

Definition 3.8 Linear Dynamical System

If A is an $n \times n$ matrix, a sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots$ of columns in $\mathbb{R}^n$ is called a ***linear dynamical system*** *if $\mathbf{v}_0$ is specified and $\mathbf{v}_1,\ \mathbf{v}_2,\ \dots$ are given by the matrix recurrence $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k \geq 0$. We call A the* ***migration*** *matrix of the system.*

We have $\mathbf{v}_1 = A\mathbf{v}_0$, then $\mathbf{v}_2 = A\mathbf{v}_1 = A^2\mathbf{v}_0$, and continuing we find

$$\mathbf{v}_k = A^k\mathbf{v}_0 \text{ for each } k = 1,\ 2,\ \cdots \tag{3.9}$$

Hence the columns $\mathbf{v}_k$ are determined by the powers A^k of the matrix A and, as we have seen, these powers can be efficiently computed if A is diagonalizable. In fact Equation 3.9 can be used to give a nice "formula" for the columns $\mathbf{v}_k$ in this case.

Assume that A is diagonalizable with eigenvalues $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_n$ and corresponding basic eigenvectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_n$. If $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array} \right]$ is a diagonalizing matrix with the $\mathbf{x}_i$ as columns, then P is invertible and

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n)$$

by Theorem 3.3.4. Hence $A = PDP^{-1}$ so Equation 3.9 and Theorem 3.3.1 give

$$\mathbf{v}_k = A^k\mathbf{v}_0 = (PDP^{-1})^k\mathbf{v}_0 = (PD^kP^{-1})\mathbf{v}_0 = PD^k(P^{-1}\mathbf{v}_0)$$

for each $k = 1,\ 2,\ \dots$. For convenience, we denote the column $P^{-1}\mathbf{v}_0$ arising here as follows:

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

Then matrix multiplication gives

$$\begin{aligned}
\mathbf{v}_k &= PD^k(P^{-1}\mathbf{v}_0) \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] \left[\begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right] \\
&= \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{array} \right] \left[\begin{array}{c} b_1\lambda_1^k \\ b_2\lambda_2^k \\ \vdots \\ b_3\lambda_n^k \end{array} \right] \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n
\end{aligned} \tag{3.10}$$

for each $k \geq 0$. This is a useful **exact formula** for the columns $\mathbf{v}_k$. Note that, in particular,

$$\mathbf{v}_0 = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n$$

However, such an exact formula for $\mathbf{v}_k$ is often not required in practice; all that is needed is to *estimate* $\mathbf{v}_k$ for large values of k (as was done in Example 3.3.12). This can be easily done if A has a largest eigenvalue. An eigenvalue λ of a matrix A is called a **dominant eigenvalue** of A if it has multiplicity 1 and

$$|\lambda| > |\mu| \text{ for all eigenvalues } \mu \neq \lambda$$

where $|\lambda|$ denotes the absolute value of the number λ. For example, $\lambda_1 = 1$ is dominant in Example 3.3.12.

Returning to the above discussion, suppose that A has a dominant eigenvalue. By choosing the order in which the columns $\mathbf{x}_i$ are placed in P, we may assume that λ_1 is dominant among the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A (see the discussion following Example 3.3.8). Now recall the exact expression for $\mathbf{v}_k$ in Equation 3.10 above:

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n$$

Take λ_1^k out as a common factor in this equation to get

$$\mathbf{v}_k = \lambda_1^k\left[b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right]$$

for each $k \geq 0$. Since λ_1 is dominant, we have $|\lambda_i| < |\lambda_1|$ for each $i \geq 2$, so each of the numbers $(\lambda_i/\lambda_1)^k$ become small in absolute value as k increases. Hence $\mathbf{v}_k$ is approximately equal to the first term $\lambda_1^k b_1\mathbf{x}_1$, and we write this as $\mathbf{v}_k \approx \lambda_1^k b_1\mathbf{x}_1$. These observations are summarized in the following theorem (together with the above exact formula for $\mathbf{v}_k$).

Theorem 3.3.7

Consider the dynamical system $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ with matrix recurrence

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for } k \geq 0$$

where A and $\mathbf{v}_0$ are given. Assume that A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding basic eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, and let $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array}\right]$ be the diagonalizing matrix. Then an exact formula for $\mathbf{v}_k$ is

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \text{ for each } k \geq 0$$

where the coefficients b_i come from

$$\boldsymbol{b} = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right]$$

Moreover, if A has dominant[12]eigenvalue λ_1, then $\mathbf{v}_k$ is approximated by

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 \text{ for sufficiently large } k.$$

[12]Similar results can be found in other situations. If for example, eigenvalues λ_1 and λ_2 (possibly equal) satisfy $|\lambda_1| = |\lambda_2| > |\lambda_i|$ for all $i > 2$, then we obtain $\mathbf{v}_k \approx b_1\lambda_1^k x_1 + b_2\lambda_2^k x_2$ for large k.

Example 3.3.13

Returning to Example 3.3.12, we see that $\lambda_1 = 1$ is the dominant eigenvalue, with eigenvector $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$. Here $P = \left[\begin{array}{rr} 1 & -1 \\ 2 & 4 \end{array}\right]$ and $\mathbf{v}_0 = \left[\begin{array}{c} 100 \\ 40 \end{array}\right]$ so $P^{-1}\mathbf{v}_0 = \frac{1}{3}\left[\begin{array}{r} 220 \\ -80 \end{array}\right]$. Hence $b_1 = \frac{220}{3}$ in the notation of Theorem 3.3.7, so

$$\left[\begin{array}{c} a_k \\ j_k \end{array}\right] = \mathbf{v}_k \approx b_1\lambda_1^k\mathbf{x}_1 = \frac{220}{3}1^k\left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

where k is large. Hence $a_k \approx \frac{220}{3}$ and $j_k \approx \frac{440}{3}$ as in Example 3.3.12.

This next example uses Theorem 3.3.7 to solve a "linear recurrence." See also Section 3.4.

Example 3.3.14

Suppose a sequence $x_0, x_1, x_2, \dots$ is determined by insisting that

$$x_0 = 1, \; x_1 = -1, \;\text{ and } x_{k+2} = 2x_k - x_{k+1} \text{ for every } k \geq 0$$

Find a formula for x_k in terms of k.

Solution. Using the linear recurrence $x_{k+2} = 2x_k - x_{k+1}$ repeatedly gives

$$x_2 = 2x_0 - x_1 = 3, \quad x_3 = 2x_1 - x_2 = -5, \quad x_4 = 11, \quad x_5 = -21, \dots$$

so the x_i are determined but no pattern is apparent. The idea is to find $\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right]$ for each k instead, and then retrieve x_k as the top component of $\mathbf{v}_k$. The reason this works is that the linear recurrence guarantees that these $\mathbf{v}_k$ are a dynamical system:

$$\mathbf{v}_{k+1} = \left[\begin{array}{c} x_{k+1} \\ x_{k+2} \end{array}\right] = \left[\begin{array}{c} x_{k+1} \\ 2x_k - x_{k+1} \end{array}\right] = A\mathbf{v}_k \text{ where } A = \left[\begin{array}{rr} 0 & 1 \\ 2 & -1 \end{array}\right]$$

The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 1$ with eigenvectors $\mathbf{x}_1 = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$, so the diagonalizing matrix is $P = \left[\begin{array}{rr} 1 & 1 \\ -2 & 1 \end{array}\right]$.

Moreover, $\mathbf{b} = P_0^{-1}\mathbf{v}_0 = \frac{1}{3}\left[\begin{array}{c} 2 \\ 1 \end{array}\right]$ so the exact formula for $\mathbf{v}_k$ is

$$\left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \frac{2}{3}(-2)^k\left[\begin{array}{r} 1 \\ -2 \end{array}\right] + \frac{1}{3}1^k\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Equating top entries gives the desired formula for x_k:

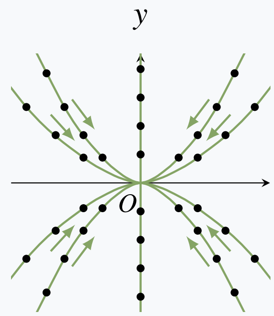
$$x_k = \frac{1}{3}\left[2(-2)^k + 1\right] \text{ for all } k = 0, 1, 2, \dots$$

The reader should check this for the first few values of k.

Graphical Description of Dynamical Systems

If a dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ is given, the sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ is called the **trajectory** of the system starting at $\mathbf{v}_0$. It is instructive to obtain a graphical plot of the system by writing $\mathbf{v}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and plotting the successive values as points in the plane, identifying $\mathbf{v}_k$ with the point (x_k, y_k) in the plane. We give several examples which illustrate properties of dynamical systems. For ease of calculation we assume that the matrix A is simple, usually diagonal.

Example 3.3.15

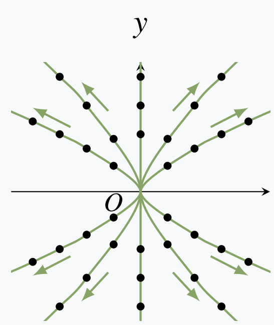


Let $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ Then the eigenvalues are $\frac{1}{2}$ and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The exact formula is

$$\mathbf{v}_k = b_1 \left(\tfrac{1}{2}\right)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \left(\tfrac{1}{3}\right)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $k = 0, 1, 2, \dots$ by Theorem 3.3.7, where the coefficients b_1 and b_2 depend on the initial point $\mathbf{v}_0$. Several trajectories are plotted in the diagram and, for each choice of $\mathbf{v}_0$, the trajectories converge toward the origin because both eigenvalues are less than 1 in absolute value. For this reason, the origin is called an **attractor** for the system.

Example 3.3.16

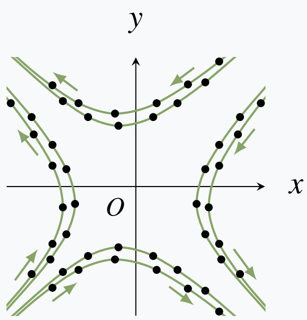


Let $A = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{4}{3} \end{bmatrix}$. Here the eigenvalues are $\frac{3}{2}$ and $\frac{4}{3}$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as before. The exact formula is

$$\mathbf{v}_k = b_1 \left(\tfrac{3}{2}\right)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \left(\tfrac{4}{3}\right)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $k = 0, 1, 2, \dots$. Since both eigenvalues are greater than 1 in absolute value, the trajectories diverge away from the origin for every choice of initial point V_0. For this reason, the origin is called a **repellor** for the system.

Example 3.3.17



Let $A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. Now the eigenvalues are $\frac{3}{2}$ and $\frac{1}{2}$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The exact formula is

$$\mathbf{v}_k = b_1 \left(\tfrac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b_2 \left(\tfrac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

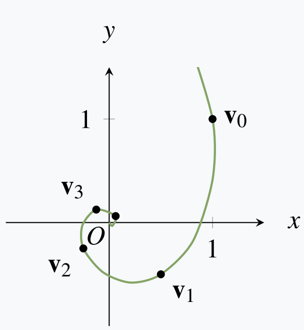
for $k = 0, 1, 2, \dots$. In this case $\frac{3}{2}$ is the dominant eigenvalue so, if $b_1 \neq 0$, we have $\mathbf{v}_k \approx b_1 \left(\frac{3}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for large k and $\mathbf{v}_k$ is approaching the line $y = -x$.

However, if $b_1 = 0$, then $\mathbf{v}_k = b_2 \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so approaches the origin along the line $y = x$. In general the trajectories appear as in the diagram, and the origin is called a **saddle point** for the dynamical system in this case.

Example 3.3.18

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$. Now the characteristic polynomial is $c_A(x) = x^2 + \frac{1}{4}$, so the eigenvalues are the complex numbers $\frac{i}{2}$ and $-\frac{i}{2}$ where $i^2 = -1$. Hence A is not diagonalizable as a real matrix. However, the trajectories are not difficult to describe. If we start with $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then the trajectory begins as

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{8} \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} \frac{1}{32} \\ -\frac{1}{32} \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{64} \end{bmatrix}, \dots$$



The first five of these points are plotted in the diagram. Here each trajectory spirals in toward the origin, so the origin is an attractor. Note that the two (complex) eigenvalues have absolute value less than 1 here. If they had absolute value greater than 1, the trajectories would spiral out from the origin.

Google PageRank

Dominant eigenvalues are useful to the Google search engine for finding information on the Web. If an information query comes in from a client, Google has a sophisticated method of establishing the "relevance" of each site to that query. When the relevant sites have been determined, they are placed in order of importance using a ranking of *all* sites called the PageRank. The relevant sites with the highest PageRank are the ones presented to the client. It is the construction of the PageRank that is our interest here.

The Web contains many links from one site to another. Google interprets a link from site j to site i as a "vote" for the importance of site i. Hence if site i has more links to it than does site j, then i is regarded as more "important" and assigned a higher PageRank. One way to look at this is to view the sites as vertices in a huge directed graph (see Section 2.2). Then if site j links to site i there is an edge from j to i, and hence the (i, j)-entry is a 1 in the associated adjacency matrix (called the *connectivity* matrix in this context). Thus a large number of 1s in row i of this matrix is a measure of the PageRank of site i.[13]

However this does not take into account the PageRank of the sites that link to i. Intuitively, the higher the rank of these sites, the higher the rank of site i. One approach is to compute a dominant eigenvector $\mathbf{x}$ for the connectivity matrix. In most cases the entries of $\mathbf{x}$ can be chosen to be positive with sum 1. Each site corresponds to an entry of $\mathbf{x}$, so the sum of the entries of sites linking to a given site i is a measure of the rank of site i. In fact, Google chooses the PageRank of a site so that it is proportional to this sum.[14]

Exercises for 3.3

Exercise 3.3.1 In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$

f. $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

g. $A = \begin{bmatrix} 3 & 1 & 1 \\ -4 & -2 & -5 \\ 2 & 2 & 5 \end{bmatrix}$

h. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

i. $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \lambda \neq \mu$

Exercise 3.3.2 Consider a linear dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for $k \geq 0$. In each case approximate $\mathbf{v}_k$ using Theorem 3.3.7.

a. $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}, \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

[13] For more on PageRank, visit https://en.wikipedia.org/wiki/PageRank.

[14] See the articles "Searching the web with eigenvectors" by Herbert S. Wilf, UMAP Journal 23(2), 2002, pages 101–103, and "The worlds largest matrix computation: Google's PageRank is an eigenvector of a matrix of order 2.7 billion" by Cleve Moler, Matlab News and Notes, October 2002, pages 12–13.

d. $A = \left[\begin{array}{rrr} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{array}\right]$, $\mathbf{v}_0 = \left[\begin{array}{r} 2 \\ 0 \\ 1 \end{array}\right]$

Exercise 3.3.3 Show that A has $\lambda = 0$ as an eigenvalue if and only if A is not invertible.

Exercise 3.3.4 Let A denote an $n \times n$ matrix and put $A_1 = A - \alpha I$, α in $\mathbb{R}$. Show that λ is an eigenvalue of A if and only if $\lambda - \alpha$ is an eigenvalue of A_1. (Hence, the eigenvalues of A_1 are just those of A "shifted" by α.) How do the eigenvectors compare?

Exercise 3.3.5 Show that the eigenvalues of $\left[\begin{array}{rr} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$ are $e^{i\theta}$ and $e^{-i\theta}$. (See Appendix A)

Exercise 3.3.6 Find the characteristic polynomial of the $n \times n$ identity matrix I. Show that I has exactly one eigenvalue and find the eigenvectors.

Exercise 3.3.7 Given $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ show that:

a. $c_A(x) = x^2 - \text{tr } Ax + \det A$, where $\text{tr } A = a + d$ is called the **trace** of A.

b. The eigenvalues are $\frac{1}{2}\left[(a+d) \pm \sqrt{(a-b)^2 + 4bc}\right]$.

Exercise 3.3.8 In each case, find $P^{-1}AP$ and then compute A^n.

a. $A = \left[\begin{array}{rr} 6 & -5 \\ 2 & -1 \end{array}\right]$, $P = \left[\begin{array}{rr} 1 & 5 \\ 1 & 2 \end{array}\right]$

b. $A = \left[\begin{array}{rr} -7 & -12 \\ 6 & -10 \end{array}\right]$, $P = \left[\begin{array}{rr} -3 & 4 \\ 2 & -3 \end{array}\right]$

[*Hint*: $(PDP^{-1})^n = PD^nP^{-1}$ for each $n = 1, 2, \dots$.]

Exercise 3.3.9

a. If $A = \left[\begin{array}{rr} 1 & 3 \\ 0 & 2 \end{array}\right]$ and $B = \left[\begin{array}{rr} 2 & 0 \\ 0 & 1 \end{array}\right]$ verify that A and B are diagonalizable, but AB is not.

b. If $D = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$ find a diagonalizable matrix A such that $D + A$ is not diagonalizable.

Exercise 3.3.10 If A is an $n \times n$ matrix, show that A is diagonalizable if and only if A^T is diagonalizable.

Exercise 3.3.11 If A is diagonalizable, show that each of the following is also diagonalizable.

a. A^n, $n \geq 1$

b. kA, k any scalar.

c. $p(A)$, $p(x)$ any polynomial (Theorem 3.3.1)

d. $U^{-1}AU$ for any invertible matrix U.

e. $kI + A$ for any scalar k.

Exercise 3.3.12 Give an example of two diagonalizable matrices A and B whose sum $A + B$ is not diagonalizable.

Exercise 3.3.13 If A is diagonalizable and 1 and -1 are the only eigenvalues, show that $A^{-1} = A$.

Exercise 3.3.14 If A is diagonalizable and 0 and 1 are the only eigenvalues, show that $A^2 = A$.

Exercise 3.3.15 If A is diagonalizable and $\lambda \geq 0$ for each eigenvalue of A, show that $A = B^2$ for some matrix B.

Exercise 3.3.16 If $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, show that $AB = BA$. [*Hint*: Diagonal matrices commute.]

Exercise 3.3.17 A square matrix A is called **nilpotent** if $A^n = 0$ for some $n \geq 1$. Find all nilpotent diagonalizable matrices. [*Hint*: Theorem 3.3.1.]

Exercise 3.3.18 Let A be any $n \times n$ matrix and $r \neq 0$ a real number.

a. Show that the eigenvalues of rA are precisely the numbers $r\lambda$, where λ is an eigenvalue of A.

b. Show that $c_{rA}(x) = r^n c_A\left(\frac{x}{r}\right)$.

Exercise 3.3.19

a. If all rows of A have the same sum s, show that s is an eigenvalue.

b. If all columns of A have the same sum s, show that s is an eigenvalue.

Exercise 3.3.20 Let A be an invertible $n \times n$ matrix.

a. Show that the eigenvalues of A are nonzero.

b. Show that the eigenvalues of A^{-1} are precisely the numbers $1/\lambda$, where λ is an eigenvalue of A.

c. Show that $c_{A^{-1}}(x) = \frac{(-x)^n}{\det A} c_A\left(\frac{1}{x}\right)$.

Exercise 3.3.21 Suppose λ is an eigenvalue of a square matrix A with eigenvector $\mathbf{x} \neq \mathbf{0}$.

a. Show that λ^2 is an eigenvalue of A^2 (with the same $\mathbf{x}$).

b. Show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.

c. Show that $p(\lambda)$ is an eigenvalue of $p(A)$ for any nonzero polynomial $p(x)$.

Exercise 3.3.22 If A is an $n \times n$ matrix, show that $c_{A^2}(x^2) = (-1)^n c_A(x) c_A(-x)$.

Exercise 3.3.23 An $n \times n$ matrix A is called nilpotent if $A^m = 0$ for some $m \geq 1$.

a. Show that every triangular matrix with zeros on the main diagonal is nilpotent.

b. If A is nilpotent, show that $\lambda = 0$ is the only eigenvalue (even complex) of A.

c. Deduce that $c_A(x) = x^n$, if A is $n \times n$ and nilpotent.

Exercise 3.3.24 Let A be diagonalizable with real eigenvalues and assume that $A^m = I$ for some $m \geq 1$.

a. Show that $A^2 = I$.

b. If m is odd, show that $A = I$.

[*Hint*: Theorem A.3]

Exercise 3.3.25 Let $A^2 = I$, and assume that $A \neq I$ and $A \neq -I$.

a. Show that the only eigenvalues of A are $\lambda = 1$ and $\lambda = -1$.

b. Show that A is diagonalizable. [*Hint*: Verify that $A(A+I) = A+I$ and $A(A-I) = -(A-I)$, and then look at nonzero columns of $A+I$ and of $A-I$.]

c. If $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = mx$ where $m \neq 0$, use (b) to show that the matrix of Q_m is diagonalizable for each m.

d. Now prove (c) geometrically using Theorem 3.3.3.

Exercise 3.3.26 Let $A = \left[\begin{array}{rrr} 2 & 3 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{array}\right]$ and $B = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 0 \end{array}\right]$. Show that $c_A(x) = c_B(x) = (x+1)^2(x-2)$, but A is diagonalizable and B is not.

Exercise 3.3.27

a. Show that the only diagonalizable matrix A that has only one eigenvalue λ is the scalar matrix $A = \lambda I$.

b. Is $\left[\begin{array}{rr} 3 & -2 \\ 2 & -1 \end{array}\right]$ diagonalizable?

Exercise 3.3.28 Characterize the diagonalizable $n \times n$ matrices A such that $A^2 - 3A + 2I = 0$ in terms of their eigenvalues. [*Hint*: Theorem 3.3.1.]

Exercise 3.3.29 Let $A = \left[\begin{array}{cc} B & 0 \\ 0 & C \end{array}\right]$ where B and C are square matrices.

a. If B and C are diagonalizable via Q and R (that is, $Q^{-1}BQ$ and $R^{-1}CR$ are diagonal), show that A is diagonalizable via $\left[\begin{array}{cc} Q & 0 \\ 0 & R \end{array}\right]$

b. Use (a) to diagonalize A if $B = \left[\begin{array}{rr} 5 & 3 \\ 3 & 5 \end{array}\right]$ and $C = \left[\begin{array}{rr} 7 & -1 \\ -1 & 7 \end{array}\right]$.

Exercise 3.3.30 Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B and C are square matrices.

a. Show that $c_A(x) = c_B(x)c_C(x)$.

b. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of B and C, respectively, show that $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix}$ are eigenvectors of A, and show how every eigenvector of A arises from such eigenvectors.

Exercise 3.3.31 Referring to the model in Example 3.3.1, determine if the population stabilizes, becomes extinct, or becomes large in each case. Denote the adult and juvenile survival rates as A and J, and the reproduction rate as R.

	R	A	J
$a.$	2	$\frac{1}{2}$	$\frac{1}{2}$
$b.$	3	$\frac{1}{4}$	$\frac{1}{4}$
$c.$	2	$\frac{1}{4}$	$\frac{1}{3}$
$d.$	3	$\frac{3}{5}$	$\frac{1}{5}$

Exercise 3.3.32 In the model of Example 3.3.1, does the final outcome depend on the initial population of adult and juvenile females? Support your answer.

Exercise 3.3.33 In Example 3.3.1, keep the same reproduction rate of 2 and the same adult survival rate of $\frac{1}{2}$, but suppose that the juvenile survival rate is ρ. Determine which values of ρ cause the population to become extinct or to become large.

Exercise 3.3.34 In Example 3.3.1, let the juvenile survival rate be $\frac{2}{5}$ and let the reproduction rate be 2. What values of the adult survival rate α will ensure that the population stabilizes?

3.4 An Application to Linear Recurrences

It often happens that a problem can be solved by finding a sequence of numbers $x_0, x_1, x_2, \dots$ where the first few are known, and subsequent numbers are given in terms of earlier ones. Here is a combinatorial example where the object is to count the number of ways to do something.

Example 3.4.1

An urban planner wants to determine the number x_k of ways that a row of k parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find the first few values of x_k.

Solution. Clearly, $x_0 = 1$ and $x_1 = 1$, while $x_2 = 2$ since there can be two cars or one truck. We have $x_3 = 3$ (the 3 configurations are *ccc*, *cT*, and *Tc*) and $x_4 = 5$ (*cccc, ccT, cTc, Tcc,* and *TT*). The key to this method is to find a way to express each subsequent x_k in terms of earlier values. In this case we claim that

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0 \tag{3.11}$$

Indeed, every way to fill $k+2$ spaces falls into one of two categories: Either a car is parked in the first space (and the remaining $k+1$ spaces are filled in x_{k+1} ways), or a truck is parked in the first two spaces (with the other k spaces filled in x_k ways). Hence, there are $x_{k+1} + x_k$ ways to fill the $k+2$ spaces. This is Equation 3.11.

The recurrence in Equation 3.11 determines x_k for every $k \geq 2$ since x_0 and x_1 are given. In fact, the first few values are

$$\begin{array}{rclcl} x_0 & = & 1 & & \\ x_1 & = & 1 & & \\ x_2 & = & x_0 + x_1 & = & 2 \\ x_3 & = & x_1 + x_2 & = & 3 \\ x_4 & = & x_2 + x_3 & = & 5 \\ x_5 & = & x_3 + x_4 & = & 8 \\ \vdots & & \vdots & & \vdots \end{array}$$

Clearly, we can find x_k for any value of k, but one wishes for a "formula" for x_k as a function of k. It turns out that such a formula can be found using diagonalization. We will return to this example later.

A sequence $x_0, x_1, x_2, \dots$ of numbers is said to be given **recursively** if each number in the sequence is completely determined by those that come before it. Such sequences arise frequently in mathematics and computer science, and also occur in other parts of science. The formula $x_{k+2} = x_{k+1} + x_k$ in Example 3.4.1 is an example of a **linear recurrence relation** of length 2 because x_{k+2} is the sum of the two preceding terms x_{k+1} and x_k; in general, the **length** is m if x_{k+m} is a sum of multiples of $x_k, x_{k+1}, \dots, x_{k+m-1}$.

The simplest linear recursive sequences are of length 1, that is x_{k+1} is a fixed multiple of x_k for each k, say $x_{k+1} = ax_k$. If x_0 is specified, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and $x_3 = ax_2 = a^3x_0, \dots$. Continuing, we obtain $x_k = a^k x_0$ for each $k \geq 0$, which is an explicit formula for x_k as a function of k (when x_0 is given).

Such formulas are not always so easy to find for all choices of the initial values. Here is an example where diagonalization helps.

Example 3.4.2

Suppose the numbers $x_0, x_1, x_2, \dots$ are given by the linear recurrence relation

$$x_{k+2} = x_{k+1} + 6x_k \text{ for } k \geq 0$$

where x_0 and x_1 are specified. Find a formula for x_k when $x_0 = 1$ and $x_1 = 3$, and also when $x_0 = 1$ and $x_1 = 1$.

Solution. If $x_0 = 1$ and $x_1 = 3$, then

$$x_2 = x_1 + 6x_0 = 9, \quad x_3 = x_2 + 6x_1 = 27, \quad x_4 = x_3 + 6x_2 = 81$$

and it is apparent that

$$x_k = 3^k \text{ for } k = 0,\ 1,\ 2,\ 3, \text{ and } 4$$

This formula holds for all k because it is true for $k = 0$ and $k = 1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ for each k as is readily checked.
However, if we begin instead with $x_0 = 1$ and $x_1 = 1$, the sequence continues

$$x_2 = 7, \quad x_3 = 13, \quad x_4 = 55, \quad x_5 = 133, \quad \dots$$

In this case, the sequence is uniquely determined but no formula is apparent. Nonetheless, a simple device transforms the recurrence into a matrix recurrence to which our diagonalization techniques apply.

The idea is to compute the sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ of columns instead of the numbers $x_0, x_1, x_2, \dots,$ where

$$\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right] \text{ for each } k \geq 0$$

Then $\mathbf{v}_0 = \left[\begin{array}{c} x_0 \\ x_1 \end{array} \right] = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$ is specified, and the numerical recurrence $x_{k+2} = x_{k+1} + 6x_k$ transforms into a matrix recurrence as follows:

$$\mathbf{v}_{k+1} = \left[\begin{array}{c} x_{k+1} \\ x_{k+2} \end{array} \right] = \left[\begin{array}{c} x_{k+1} \\ 6x_k + x_{k+1} \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ 6 & 1 \end{array} \right] \left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right] = A\mathbf{v}_k$$

where $A = \left[\begin{array}{cc} 0 & 1 \\ 6 & 1 \end{array} \right]$. Thus these columns $\mathbf{v}_k$ are a linear dynamical system, so Theorem 3.3.7 applies provided the matrix A is diagonalizable.

We have $c_A(x) = (x-3)(x+2)$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 3 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} -1 \\ 2 \end{array} \right]$ as the reader can check. Since $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \right] = \left[\begin{array}{cc} 1 & -1 \\ 3 & 2 \end{array} \right]$ is invertible, it is a diagonalizing matrix for A. The coefficients b_i in Theorem 3.3.7 are given by $\left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} \frac{3}{5} \\ \frac{-2}{5} \end{array} \right]$, so that the theorem gives

$$\left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right] = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \tfrac{3}{5}3^k \left[\begin{array}{c} 1 \\ 3 \end{array} \right] + \tfrac{-2}{5}(-2)^k \left[\begin{array}{c} -1 \\ 2 \end{array} \right]$$

Equating top entries yields

$$x_k = \tfrac{1}{5}\left[3^{k+1} - (-2)^{k+1}\right] \text{ for } k \geq 0$$

This gives $x_0 = 1 = x_1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ as is easily verified. Hence, it is the desired formula for the x_k.

Returning to Example 3.4.1, these methods give an exact formula and a good approximation for the numbers x_k in that problem.

Example 3.4.3

In Example 3.4.1, an urban planner wants to determine x_k, the number of ways that a row of k parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find a formula for x_k and estimate it for large k.

Solution. We saw in Example 3.4.1 that the numbers x_k satisfy a linear recurrence

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0$$

If we write $\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right]$ as before, this recurrence becomes a matrix recurrence for the $\mathbf{v}_k$:

$$\mathbf{v}_{k+1} = \left[\begin{array}{c} x_{k+1} \\ x_{k+2} \end{array} \right] = \left[\begin{array}{c} x_{k+1} \\ x_k + x_{k+1} \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right] \left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right] = A\mathbf{v}_k$$

for all $k \geq 0$ where $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right]$. Moreover, A is diagonalizable here. The characteristic polynomial is $c_A(x) = x^2 - x - 1$ with roots $\frac{1}{2}\left[1 \pm \sqrt{5}\right]$ by the quadratic formula, so A has eigenvalues

$$\lambda_1 = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \text{ and } \lambda_2 = \tfrac{1}{2}\left(1 - \sqrt{5}\right)$$

Corresponding eigenvectors are $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ \lambda_1 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ \lambda_2 \end{array} \right]$ respectively as the reader can verify.
As the matrix $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ \lambda_1 & \lambda_2 \end{array} \right]$ is invertible, it is a diagonalizing matrix for A. We compute the coefficients b_1 and b_2 (in Theorem 3.3.7) as follows:

$$\left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] = P^{-1}\mathbf{v}_0 = \frac{1}{-\sqrt{5}} \left[\begin{array}{cc} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \end{array} \right] = \frac{1}{\sqrt{5}} \left[\begin{array}{c} \lambda_1 \\ -\lambda_2 \end{array} \right]$$

where we used the fact that $\lambda_1 + \lambda_2 = 1$. Thus Theorem 3.3.7 gives

$$\left[\begin{array}{c} x_k \\ x_{k+1} \end{array} \right] = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \frac{\lambda_1}{\sqrt{5}}\lambda_1^k \left[\begin{array}{c} 1 \\ \lambda_1 \end{array} \right] - \frac{\lambda_2}{\sqrt{5}}\lambda_2^k \left[\begin{array}{c} 1 \\ \lambda_2 \end{array} \right]$$

Comparing top entries gives an exact formula for the numbers x_k:

$$x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right] \text{ for } k \geq 0$$

Finally, observe that λ_1 is dominant here (in fact, $\lambda_1 = 1.618$ and $\lambda_2 = -0.618$ to three decimal places) so λ_2^{k+1} is negligible compared with λ_1^{k+1} is large. Thus,

$$x_k \approx \frac{1}{\sqrt{5}}\lambda_1^{k+1} \text{ for each } k \geq 0.$$

This is a good approximation, even for as small a value as $k = 12$. Indeed, repeated use of the recurrence $x_{k+2} = x_k + x_{k+1}$ gives the exact value $x_{12} = 233$, while the approximation is $x_{12} \approx \frac{(1.618)^{13}}{\sqrt{5}} = 232.94$.

The sequence $x_0, x_1, x_2, \dots$ in Example 3.4.3 was first discussed in 1202 by Leonardo Pisano of Pisa, also known as Fibonacci,[15] and is now called the **Fibonacci sequence**. It is completely determined by the conditions $x_0 = 1$, $x_1 = 1$ and the recurrence $x_{k+2} = x_k + x_{k+1}$ for each $k \geq 0$. These numbers have

[15] Fibonacci was born in Italy. As a young man he travelled to India where he encountered the "Fibonacci" sequence. He returned to Italy and published this in his book *Liber Abaci* in 1202. In the book he is the first to bring the Hindu decimal system for representing numbers to Europe.

been studied for centuries and have many interesting properties (there is even a journal, the *Fibonacci Quarterly*, devoted exclusively to them). For example, biologists have discovered that the arrangement of leaves around the stems of some plants follow a Fibonacci pattern. The formula $x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right]$ in Example 3.4.3 is called the **Binet formula**. It is remarkable in that the x_k are integers but λ_1 and λ_2 are not. This phenomenon can occur even if the eigenvalues λ_i are nonreal complex numbers.

We conclude with an example showing that *nonlinear* recurrences can be very complicated.

Example 3.4.4

Suppose a sequence $x_0, x_1, x_2, \dots$ satisfies the following recurrence:

$$x_{k+1} = \begin{cases} \frac{1}{2}x_k & \text{if } x_k \text{ is even} \\ 3x_k + 1 & \text{if } x_k \text{ is odd} \end{cases}$$

If $x_0 = 1$, the sequence is $1, 4, 2, 1, 4, 2, 1, \dots$ and so continues to cycle indefinitely. The same thing happens if $x_0 = 7$. Then the sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \dots$$

and it again cycles. However, it is not known whether every choice of x_0 will lead eventually to 1. It is quite possible that, for some x_0, the sequence will continue to produce different values indefinitely, or will repeat a value and cycle without reaching 1. No one knows for sure.

Exercises for 3.4

Exercise 3.4.1 Solve the following linear recurrences.

a. $x_{k+2} = 3x_k + 2x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

b. $x_{k+2} = 2x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 2$.

c. $x_{k+2} = 2x_k + x_{k+1}$, where $x_0 = 0$ and $x_1 = 1$.

d. $x_{k+2} = 6x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

Exercise 3.4.2 Solve the following linear recurrences.

a. $x_{k+3} = 6x_{k+2} - 11x_{k+1} + 6x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

b. $x_{k+3} = -2x_{k+2} + x_{k+1} + 2x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

[*Hint*: Use $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$.]

Exercise 3.4.3 In Example 3.4.1 suppose buses are also allowed to park, and let x_k denote the number of ways a row of k parking spaces can be filled with cars, trucks, and buses.

a. If trucks and buses take up 2 and 3 spaces respectively, show that $x_{k+3} = x_k + x_{k+1} + x_{k+2}$ for each k, and use this recurrence to compute x_{10}. [*Hint*: The eigenvalues are of little use.]

b. If buses take up 4 spaces, find a recurrence for the x_k and compute x_{10}.

Exercise 3.4.4 A man must climb a flight of k steps. He always takes one or two steps at a time. Thus he can climb 3 steps in the following ways: 1, 1, 1; 1, 2; or 2, 1. Find s_k, the number of ways he can climb the flight of k steps. [*Hint*: Fibonacci.]

Exercise 3.4.5 How many "words" of k letters can be made from the letters $\{a, b\}$ if there are no adjacent a's?

Exercise 3.4.6 How many sequences of k flips of a coin are there with no HH?

Exercise 3.4.7 Find x_k, the number of ways to make a stack of k poker chips if only red, blue, and gold chips are used and no two gold chips are adjacent. [*Hint*: Show that $x_{k+2} = 2x_{k+1} + 2x_k$ by considering how many stacks have a red, blue, or gold chip on top.]

Exercise 3.4.8 A nuclear reactor contains α- and β-particles. In every second each α-particle splits into three β-particles, and each β-particle splits into an α-particle and two β-particles. If there is a single α-particle in the reactor at time $t = 0$, how many α-particles are there at $t = 20$ seconds? [*Hint*: Let x_k and y_k denote the number of α- and β-particles at time $t = k$ seconds. Find x_{k+1} and y_{k+1} in terms of x_k and y_k.]

Exercise 3.4.9 The annual yield of wheat in a certain country has been found to equal the average of the yield in the previous two years. If the yields in 1990 and 1991 were 10 and 12 million tons respectively, find a formula for the yield k years after 1990. What is the long-term average yield?

Exercise 3.4.10 Find the general solution to the recurrence $x_{k+1} = rx_k + c$ where r and c are constants. [*Hint*: Consider the cases $r = 1$ and $r \neq 1$ separately. If $r \neq 1$, you will need the identity $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$ for $n \geq 1$.]

Exercise 3.4.11 Consider the length 3 recurrence $x_{k+3} = ax_k + bx_{k+1} + cx_{k+2}$.

a. If $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ show that $\mathbf{v}_{k+1} = A\mathbf{v}_k$.

b. If λ is any eigenvalue of A, show that $\mathbf{x} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$ is a λ-eigenvector.

[*Hint*: Show directly that $A\mathbf{x} = \lambda\mathbf{x}$.]

c. Generalize (a) and (b) to a recurrence

$$x_{k+4} = ax_k + bx_{k+1} + cx_{k+2} + dx_{k+3}$$

of length 4.

Exercise 3.4.12 Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c$$

where c may not be zero.

a. If $a + b \neq 1$ show that p can be found such that, if we set $y_k = x_k + p$, then $y_{k+2} = ay_{k+1} + by_k$. [Hence, the sequence x_k can be found provided y_k can be found by the methods of this section (or otherwise).]

b. Use (a) to solve $x_{k+2} = x_{k+1} + 6x_k + 5$ where $x_0 = 1$ and $x_1 = 1$.

Exercise 3.4.13 Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c(k) \tag{3.12}$$

where $c(k)$ is a function of k, and consider the related recurrence

$$x_{k+2} = ax_{k+1} + bx_k \tag{3.13}$$

Suppose that $x_k = p_k$ is a particular solution of Equation 3.12.

a. If q_k is any solution of Equation 3.13, show that $q_k + p_k$ is a solution of Equation 3.12.

b. Show that every solution of Equation 3.12 arises as in (a) as the sum of a solution of Equation 3.13 plus the particular solution p_k of Equation 3.12.

3.5 An Application to Systems of Differential Equations

A function f of a real variable is said to be **differentiable** if its derivative exists and, in this case, we let f' denote the derivative. If f and g are differentiable functions, a system

$$\begin{aligned} f' &= 3f + 5g \\ g' &= -f + 2g \end{aligned}$$

is called a *system of first order differential equations,* or a *differential system* for short. Solving many practical problems often comes down to finding sets of functions that satisfy such a system (often involving more than two functions). In this section we show how diagonalization can help. Of course an acquaintance with calculus is required.

The Exponential Function

The simplest differential system is the following single equation:

$$f' = af \text{ where } a \text{ is constant} \tag{3.14}$$

It is easily verified that $f(x) = e^{ax}$ is one solution; in fact, Equation 3.14 is simple enough for us to find *all* solutions. Suppose that f is any solution, so that $f'(x) = af(x)$ for all x. Consider the new function g given by $g(x) = f(x)e^{-ax}$. Then the product rule of differentiation gives

$$\begin{aligned} g'(x) &= f(x)\left[-ae^{-ax}\right] + f'(x)e^{-ax} \\ &= -af(x)e^{-ax} + [af(x)]\,e^{-ax} \\ &= 0 \end{aligned}$$

for all x. Hence the function $g(x)$ has zero derivative and so must be a constant, say $g(x) = c$. Thus $c = g(x) = f(x)e^{-ax}$, that is

$$f(x) = ce^{ax}$$

In other words, every solution $f(x)$ of Equation 3.14 is just a scalar multiple of e^{ax}. Since every such scalar multiple is easily seen to be a solution of Equation 3.14, we have proved

Theorem 3.5.1

The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\} = \mathbb{R}e^{ax}$.

Remarkably, this result together with diagonalization enables us to solve a wide variety of differential systems.

Example 3.5.1

Assume that the number $n(t)$ of bacteria in a culture at time t has the property that the rate of change of n is proportional to n itself. If there are n_0 bacteria present when $t = 0$, find the number at time t.

Solution. Let k denote the proportionality constant. The rate of change of $n(t)$ is its time-derivative $n'(t)$, so the given relationship is $n'(t) = kn(t)$. Thus Theorem 3.5.1 shows that all solutions n are given by $n(t) = ce^{kt}$, where c is a constant. In this case, the constant c is determined by the requirement that there be n_0 bacteria present when $t = 0$. Hence $n_0 = n(0) = ce^{k0} = c$, so

$$n(t) = n_0 e^{kt}$$

gives the number at time t. Of course the constant k depends on the strain of bacteria.

The condition that $n(0) = n_0$ in Example 3.5.1 is called an **initial condition** or a **boundary condition** and serves to select one solution from the available solutions.

General Differential Systems

Solving a variety of problems, particularly in science and engineering, comes down to solving a system of linear differential equations. Diagonalization enters into this as follows. The general problem is to find differentiable functions $f_1, f_2, \dots, f_n$ that satisfy a system of equations of the form

$$\begin{array}{l} f_1' = a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n \\ f_2' = a_{21}f_1 + a_{22}f_2 + \cdots + a_{2n}f_n \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ f_n' = a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn}f_n \end{array}$$

where the a_{ij} are constants. This is called a **linear system of differential equations** or simply a **differential system**. The first step is to put it in matrix form. Write

$$\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right] \quad \mathbf{f}' = \left[\begin{array}{c} f_1' \\ f_2' \\ \vdots \\ f_n' \end{array} \right] \quad A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right]$$

Then the system can be written compactly using matrix multiplication:

$$\mathbf{f}' = A\mathbf{f}$$

Hence, given the matrix A, the problem is to find a column $\mathbf{f}$ of differentiable functions that satisfies this condition. This can be done if A is diagonalizable. Here is an example.

Example 3.5.2

Find a solution to the system

$$\begin{aligned} f_1' &= f_1 + 3f_2 \\ f_2' &= 2f_1 + 2f_2 \end{aligned}$$

that satisfies $f_1(0) = 0$, $f_2(0) = 5$.

Solution. This is $\mathbf{f}' = A\mathbf{f}$, where $\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \end{array}\right]$ and $A = \left[\begin{array}{cc} 1 & 3 \\ 2 & 2 \end{array}\right]$. The reader can verify that $c_A(x) = (x-4)(x+1)$, and that $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$ and $\mathbf{x}_2 = \left[\begin{array}{r} 3 \\ -2 \end{array}\right]$ are eigenvectors corresponding to the eigenvalues 4 and -1, respectively. Hence the diagonalization algorithm gives $P^{-1}AP = \left[\begin{array}{rr} 4 & 0 \\ 0 & -1 \end{array}\right]$, where $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 3 \\ 1 & -2 \end{array}\right]$. Now consider new functions g_1 and g_2 given by $\mathbf{f} = P\mathbf{g}$ (equivalently, $\mathbf{g} = P^{-1}\mathbf{f}$), where $\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \end{array}\right]$ Then

$$\left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} g_1 \\ g_2 \end{array}\right] \qquad \text{that is,} \quad \begin{array}{l} f_1 = g_1 + 3g_2 \\ f_2 = g_1 - 2g_2 \end{array}$$

Hence $f_1' = g_1' + 3g_2'$ and $f_2' = g_1' - 2g_2'$ so that

$$\mathbf{f}' = \left[\begin{array}{c} f_1' \\ f_2' \end{array}\right] = \left[\begin{array}{rr} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} g_1' \\ g_2' \end{array}\right] = P\mathbf{g}'$$

If this is substituted in $\mathbf{f}' = A\mathbf{f}$, the result is $P\mathbf{g}' = AP\mathbf{g}$, whence

$$\mathbf{g}' = P^{-1}AP\mathbf{g}$$

But this means that

$$\left[\begin{array}{c} g_1' \\ g_2' \end{array}\right] = \left[\begin{array}{rr} 4 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} g_1 \\ g_2 \end{array}\right], \qquad \text{so} \quad \begin{array}{l} g_1' = 4g_1 \\ g_2' = -g_2 \end{array}$$

Hence Theorem 3.5.1 gives $g_1(x) = ce^{4x}$, $g_2(x) = de^{-x}$, where c and d are constants. Finally, then,

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right] = P\left[\begin{array}{c} g_1(x) \\ g_2(x) \end{array}\right] = \left[\begin{array}{rr} 1 & 3 \\ 1 & -2 \end{array}\right]\left[\begin{array}{c} ce^{4x} \\ de^{-x} \end{array}\right] = \left[\begin{array}{c} ce^{4x} + 3de^{-x} \\ ce^{4x} - 2de^{-x} \end{array}\right]$$

so the *general solution* is

$$\begin{aligned} f_1(x) &= ce^{4x} + 3de^{-x} \\ f_2(x) &= ce^{4x} - 2de^{-x} \end{aligned} \qquad c \text{ and } d \text{ constants}$$

It is worth observing that this can be written in matrix form as

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array}\right] = c\left[\begin{array}{c} 1 \\ 1 \end{array}\right]e^{4x} + d\left[\begin{array}{r} 3 \\ -2 \end{array}\right]e^{-x}$$

That is,

$$\mathbf{f}(x) = c\mathbf{x}_1 e^{4x} + d\mathbf{x}_2 e^{-x}$$

This form of the solution works more generally, as will be shown.
Finally, the requirement that $f_1(0) = 0$ and $f_2(0) = 5$ in this example determines the constants c and d:

$$\begin{aligned} 0 &= f_1(0) = ce^0 + 3de^0 = c + 3d \\ 5 &= f_2(0) = ce^0 - 2de^0 = c - 2d \end{aligned}$$

These equations give $c = 3$ and $d = -1$, so

$$\begin{aligned} f_1(x) &= 3e^{4x} - 3e^{-x} \\ f_2(x) &= 3e^{4x} + 2e^{-x} \end{aligned}$$

satisfy all the requirements.

The technique in this example works in general.

Theorem 3.5.2

Consider a linear system

$$\mathbf{f}' = A\mathbf{f}$$

of differential equations, where A is an $n \times n$ diagonalizable matrix. Let $P^{-1}AP$ be diagonal, where P is given in terms of its columns

$$P = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$

and $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ are eigenvectors of A. If $\mathbf{x}_i$ corresponds to the eigenvalue λ_i for each i, then every solution $\mathbf{f}$ of $\mathbf{f}' = A\mathbf{f}$ has the form

$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Proof. By Theorem 3.3.4, the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{bmatrix}$ is invertible and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

As in Example 3.5.2, write $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ and define $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$ by $\mathbf{g} = P^{-1}\mathbf{f}$; equivalently, $\mathbf{f} = P\mathbf{g}$. If $P = \begin{bmatrix} p_{ij} \end{bmatrix}$, this gives

$$f_i = p_{i1}g_1 + p_{i2}g_2 + \cdots + p_{in}g_n$$

Since the p_{ij} are constants, differentiation preserves this relationship:

$$f_i' = p_{i1}g_1' + p_{i2}g_2' + \cdots + p_{in}g_n'$$

so $\mathbf{f}' = P\mathbf{g}'$. Substituting this into $\mathbf{f}' = A\mathbf{f}$ gives $P\mathbf{g}' = AP\mathbf{g}$. But then left multiplication by P^{-1} gives $\mathbf{g}' = P^{-1}AP\mathbf{g}$, so the original system of equations $\mathbf{f}' = A\mathbf{f}$ for $\mathbf{f}$ becomes much simpler in terms of $\mathbf{g}$:

$$\left[\begin{array}{c} g_1' \\ g_2' \\ \vdots \\ g_n' \end{array}\right] = \left[\begin{array}{cccc} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{array}\right] \left[\begin{array}{c} g_1 \\ g_2 \\ \vdots \\ g_n \end{array}\right]$$

Hence $g_i' = \lambda_i g_i$ holds for each i, and Theorem 3.5.1 implies that the only solutions are

$$g_i(x) = c_i e^{\lambda_i x} \quad c_i \text{ some constant}$$

Then the relationship $\mathbf{f} = P\mathbf{g}$ gives the functions $f_1, f_2, \dots, f_n$ as follows:

$$\mathbf{f}(x) = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] \left[\begin{array}{c} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{array}\right] = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

This is what we wanted. □

The theorem shows that *every* solution to $\mathbf{f}' = A\mathbf{f}$ is a linear combination

$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

where the coefficients c_i are arbitrary. Hence this is called the **general solution** to the system of differential equations. In most cases the solution functions $f_i(x)$ are required to satisfy boundary conditions, often of the form $f_i(a) = b_i$, where $a, b_1, \dots, b_n$ are prescribed numbers. These conditions determine the constants c_i. The following example illustrates this and displays a situation where one eigenvalue has multiplicity greater than 1.

Example 3.5.3

Find the general solution to the system

$$\begin{aligned} f_1' &= 5f_1 + 8f_2 + 16f_3 \\ f_2' &= 4f_1 + f_2 + 8f_3 \\ f_3' &= -4f_1 - 4f_2 - 11f_3 \end{aligned}$$

Then find a solution satisfying the boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$.

Solution. The system has the form $\mathbf{f}' = A\mathbf{f}$, where $A = \left[\begin{array}{rrr} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{array}\right]$. In this case $c_A(x) = (x+3)^2(x-1)$ and eigenvectors corresponding to the eigenvalues -3, -3, and 1 are, respectively,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Hence, by Theorem 3.5.2, the general solution is

$$\mathbf{f}(x) = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} e^{x}, \qquad c_i \text{ constants.}$$

The boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$ determine the constants c_i.

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{f}(0) &= c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

The solution is $c_1 = -3$, $c_2 = 5$, $c_3 = 4$, so the required specific solution is

$$\begin{aligned} f_1(x) &= -7e^{-3x} + 8e^{x} \\ f_2(x) &= -3e^{-3x} + 4e^{x} \\ f_3(x) &= 5e^{-3x} - 4e^{x} \end{aligned}$$

Exercises for 3.5

Exercise 3.5.1 Use Theorem 3.5.1 to find the general solution to each of the following systems. Then find a specific solution satisfying the given boundary condition.

a. $f_1' = 2f_1 + 4f_2, \; f_1(0) = 0$
$f_2' = 3f_1 + 3f_2, \; f_2(0) = 1$

b. $f_1' = -f_1 + 5f_2, \; f_1(0) = 1$
$f_2' = f_1 + 3f_2, \; f_2(0) = -1$

c. $f_1' = 4f_2 + 4f_3$
$f_2' = f_1 + f_2 - 2f_3$
$f_3' = -f_1 + f_2 + 4f_3$

$f_1(0) = f_2(0) = f_3(0) = 1$

d. $f_1' = 2f_1 + f_2 + 2f_3$
$f_2' = 2f_1 + 2f_2 - 2f_3$
$f_3' = 3f_1 + f_2 + f_3$

$f_1(0) = f_2(0) = f_3(0) = 1$

Exercise 3.5.2 Show that the solution to $f' = af$ satisfying $f(x_0) = k$ is $f(x) = ke^{a(x-x_0)}$.

Exercise 3.5.3 A radioactive element decays at a rate proportional to the amount present. Suppose an initial mass of 10 g decays to 8 g in 3 hours.

a. Find the mass t hours later.

b. Find the half-life of the element—the time taken to decay to half its mass.

Exercise 3.5.4 The population $N(t)$ of a region at time t increases at a rate proportional to the population. If the population doubles every 5 years and is 3 million initially, find $N(t)$.

Exercise 3.5.5 Let A be an invertible diagonalizable $n \times n$ matrix and let $\mathbf{b}$ be an n-column of constant functions. We can solve the system $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ as follows:

a. If $\mathbf{g}$ satisfies $\mathbf{g}' = A\mathbf{g}$ (using Theorem 3.5.2), show that $\mathbf{f} = \mathbf{g} - A^{-1}\mathbf{b}$ is a solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$.

b. Show that every solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ arises as in (a) for some solution $\mathbf{g}$ to $\mathbf{g}' = A\mathbf{g}$.

Exercise 3.5.6 Denote the second derivative of f by $f'' = (f')'$. Consider the second order differential equation

$$f'' - a_1 f' - a_2 f = 0, \quad a_1 \text{ and } a_2 \text{ real numbers} \quad (3.15)$$

a. If f is a solution to Equation 3.15 let $f_1 = f$ and $f_2 = f' - a_1 f$. Show that
$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 \end{cases},$$
that is $\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

b. Conversely, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a solution to the system in (a), show that f_1 is a solution to Equation 3.15.

Exercise 3.5.7 Writing $f''' = (f'')'$, consider the third order differential equation

$$f''' - a_1 f'' - a_2 f' - a_3 f = 0$$

where a_1, a_2, and a_3 are real numbers. Let $f_1 = f$, $f_2 = f' - a_1 f$ and $f_3 = f'' - a_1 f' - a_2 f''$.

a. Show that $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is a solution to the system
$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 + f_3, \\ f_3' = a_3 f_1 \end{cases}$$
that is $\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

b. Show further that if $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is any solution to this system, then $f = f_1$ is a solution to Equation 3.15.

Remark. A similar construction casts every linear differential equation of order n (with constant coefficients) as an $n \times n$ linear system of first order equations. However, the matrix need not be diagonalizable, so other methods have been developed.

3.6 Proof of the Cofactor Expansion Theorem

Recall that our definition of the term *determinant* is inductive: The determinant of any 1×1 matrix is defined first; then it is used to define the determinants of 2×2 matrices. Then that is used for the 3×3 case, and so on. The case of a 1×1 matrix $[a]$ poses no problem. We simply define

$$\det [a] = a$$

as in Section 3.1. Given an $n \times n$ matrix A, define A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j. Now assume that the determinant of any $(n-1) \times (n-1)$ matrix has been defined. Then the determinant of A is *defined* to be

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n+1} a_{n1} \det A_{n1} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1} \end{aligned}$$

where summation notation has been introduced for convenience.[16] Observe that, in the terminology of Section 3.1, this is just the cofactor expansion of $\det A$ along the first column, and that $(-1)^{i+j} \det A_{ij}$ is the $(i,\ j)$-cofactor (previously denoted as $c_{ij}(A)$).[17] To illustrate the definition, consider the 2×2 matrix $A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right]$. Then the definition gives

$$\det \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] = a_{11} \det \left[a_{22} \right] - a_{21} \det \left[a_{12} \right] = a_{11}a_{22} - a_{21}a_{12}$$

and this is the same as the definition in Section 3.1.

Of course, the task now is to use this definition to *prove* that the cofactor expansion along *any* row or column yields $\det A$ (this is Theorem 3.1.1). The proof proceeds by first establishing the properties of determinants stated in Theorem 3.1.2 but for *rows* only (see Lemma 3.6.2). This being done, the full proof of Theorem 3.1.1 is not difficult. The proof of Lemma 3.6.2 requires the following preliminary result.

Lemma 3.6.1

Let A, B, and C be $n \times n$ matrices that are identical except that the pth row of A is the sum of the pth rows of B and C. Then

$$\det A = \det B + \det C$$

Proof. We proceed by induction on n, the cases $n = 1$ and $n = 2$ being easily checked. Consider a_{i1} and A_{i1}:

Case 1: If $i \neq p$,

$$a_{i1} = b_{i1} = c_{i1} \quad \text{and} \quad \det A_{i1} = \det B_{i1} = \det C_{i1}$$

by induction because A_{i1}, B_{i1}, C_{i1} are identical except that one row of A_{i1} is the sum of the corresponding rows of B_{i1} and C_{i1}.

Case 2: If $i = p$,

$$a_{p1} = b_{p1} + c_{p1} \quad \text{and} \quad A_{p1} = B_{p1} = C_{p1}$$

Now write out the defining sum for $\det A$, splitting off the pth term for special attention.

$$\begin{aligned} \det A &= \sum_{i \neq p} a_{i1}(-1)^{i+1} \det A_{i1} + a_{p1}(-1)^{p+1} \det A_{p1} \\ &= \sum_{i \neq p} a_{i1}(-1)^{i+1} \left[\det B_{i1} + \det B_{i1} \right] + (b_{p1} + c_{p1})(-1)^{p+1} \det A_{p1} \end{aligned}$$

where $\det A_{i1} = \det B_{i1} + \det C_{i1}$ by induction. But the terms here involving B_{i1} and b_{p1} add up to $\det B$ because $a_{i1} = b_{i1}$ if $i \neq p$ and $A_{p1} = B_{p1}$. Similarly, the terms involving C_{i1} and c_{p1} add up to $\det C$. Hence $\det A = \det B + \det C$, as required. □

[16]Summation notation is a convenient shorthand way to write sums of similar expressions. For example $a_1 + a_2 + a_3 + a_4 = \sum_{i=1}^{4} a_i$, $a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8 = \sum_{k=5}^{8} a_kb_k$, and $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{j=1}^{5} j^2$.

[17]Note that we used the expansion along *row* 1 at the beginning of Section 3.1. The column 1 expansion definition is more convenient here.

Lemma 3.6.2

Let $A = \left[a_{ij} \right]$ denote an $n \times n$ matrix.

1. *If $B = \left[b_{ij} \right]$ is formed from A by multiplying a row of A by a number u, then $\det B = u \det A$.*
2. *If A contains a row of zeros, then $\det A = 0$.*
3. *If $B = \left[b_{ij} \right]$ is formed by interchanging two rows of A, then $\det B = -\det A$.*
4. *If A contains two identical rows, then $\det A = 0$.*
5. *If $B = \left[b_{ij} \right]$ is formed by adding a multiple of one row of A to a different row, then $\det B = \det A$.*

Proof. For later reference the defining sums for $\det A$ and $\det B$ are as follows:

$$\det A = \sum_{i=1}^{n} a_{i1}(-1)^{i+1} \det A_{i1} \tag{3.16}$$

$$\det B = \sum_{i=1}^{n} b_{i1}(-1)^{i+1} \det B_{i1} \tag{3.17}$$

Property 1. The proof is by induction on n, the cases $n = 1$ and $n = 2$ being easily verified. Consider the ith term in the sum 3.17 for $\det B$ where B is the result of multiplying row p of A by u.

a. If $i \neq p$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = u \det A_{i1}$ by induction because B_{i1} comes from A_{i1} by multiplying a row by u.

b. If $i = p$, then $b_{p1} = ua_{p1}$ and $B_{p1} = A_{p1}$.

In either case, each term in Equation 3.17 is u times the corresponding term in Equation 3.16, so it is clear that $\det B = u \det A$.

Property 2. This is clear by property 1 because the row of zeros has a common factor $u = 0$.

Property 3. Observe first that it suffices to prove property 3 for interchanges of adjacent rows. (Rows p and q $(q > p)$ can be interchanged by carrying out $2(q-p)-1$ adjacent changes, which results in an *odd* number of sign changes in the determinant.) So suppose that rows p and $p+1$ of A are interchanged to obtain B. Again consider the ith term in Equation 3.17.

a. If $i \neq p$ and $i \neq p+1$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = -\det A_{i1}$ by induction because B_{i1} results from interchanging adjacent rows in A_{i1}. Hence the ith term in Equation 3.17 is the negative of the ith term in Equation 3.16. Hence $\det B = -\det A$ in this case.

b. If $i = p$ or $i = p+1$, then $b_{p1} = a_{p+1,\,1}$ and $B_{p1} = A_{p+1,\,1}$, whereas $b_{p+1,\,1} = a_{p1}$ and $B_{p+1,\,1} = A_{p1}$. Hence terms p and $p+1$ in Equation 3.17 are

$$b_{p1}(-1)^{p+1} \det B_{p1} = -a_{p+1,\,1}(-1)^{(p+1)+1} \det(A_{p+1,\,1})$$

$$b_{p+1,\,1}(-1)^{(p+1)+1} \det B_{p+1,\,1} = -a_{p1}(-1)^{p+1} \det(A_{p1})$$

This means that terms p and $p+1$ in Equation 3.17 are the same as these terms in Equation 3.16, except that the order is reversed and the signs are changed. Thus the sum 3.17 is the negative of the sum 3.16; that is, $\det B = -\det A$.

Property 4. If rows p and q in A are identical, let B be obtained from A by interchanging these rows. Then $B = A$ so $\det A = \det B$. But $\det B = -\det A$ by property 3 so $\det A = -\det A$. This implies that $\det A = 0$.

Property 5. Suppose B results from adding u times row q of A to row p. Then Lemma 3.6.1 applies to B to show that $\det B = \det A + \det C$, where C is obtained from A by replacing row p by u times row q. It now follows from properties 1 and 4 that $\det C = 0$ so $\det B = \det A$, as asserted. □

These facts are enough to enable us to prove Theorem 3.1.1. For convenience, it is restated here in the notation of the foregoing lemmas. The only difference between the notations is that the $(i,\ j)$-cofactor of an $n \times n$ matrix A was denoted earlier by

$$c_{ij}(A) = (-1)^{i+j} \det A_{ij}$$

Theorem 3.6.1

If $A = \left[a_{ij}\right]$ is an $n \times n$ matrix, then

1. $\det A = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$ *(cofactor expansion along column j).*
2. $\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$ *(cofactor expansion along row i).*

Here A_{ij} denotes the matrix obtained from A by deleting row i and column j.

Proof. Lemma 3.6.2 establishes the truth of Theorem 3.1.2 for *rows*. With this information, the arguments in Section 3.2 proceed exactly as written to establish that $\det A = \det A^T$ holds for any $n \times n$ matrix A. Now suppose B is obtained from A by interchanging two columns. Then B^T is obtained from A^T by interchanging two rows so, by property 3 of Lemma 3.6.2,

$$\det B = \det B^T = -\det A^T = -\det A$$

Hence property 3 of Lemma 3.6.2 holds for *columns* too.

This enables us to prove the cofactor expansion for columns. Given an $n \times n$ matrix $A = \left[a_{ij}\right]$, let $B = \left[b_{ij}\right]$ be obtained by moving column j to the left side, using $j-1$ interchanges of adjacent columns. Then $\det B = (-1)^{j-1} \det A$ and, because $B_{i1} = A_{ij}$ and $b_{i1} = a_{ij}$ for all i, we obtain

$$\begin{aligned} \det A &= (-1)^{j-1} \det B = (-1)^{j-1} \sum_{i=1}^{n} b_{i1}(-1)^{i+1} \det B_{i1} \\ &= \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} \end{aligned}$$

This is the cofactor expansion of $\det A$ along column j.

Finally, to prove the row expansion, write $B = A^T$. Then $B_{ij} = (A_{ij}^T)$ and $b_{ij} = a_{ji}$ for all i and j. Expanding $\det B$ along column j gives

$$\begin{aligned}\det A = \det A^T = \det B &= \sum_{i=1}^{n} b_{ij}(-1)^{i+j} \det B_{ij} \\ &= \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det \left[(A_{ji}^T)\right] = \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det A_{ji}\end{aligned}$$

This is the required expansion of $\det A$ along row j. □

Exercises for 3.6

Exercise 3.6.1 Prove Lemma 3.6.1 for columns.

Exercise 3.6.2 Verify that interchanging rows p and q $(q > p)$ can be accomplished using $2(q-p)-1$ adjacent interchanges.

Exercise 3.6.3 If u is a number and A is an $n \times n$ matrix, prove that $\det(uA) = u^n \det A$ by induction on n, using only the definition of $\det A$.

Supplementary Exercises for Chapter 3

Exercise 3.1 Show that

$$\det \left[\begin{array}{ccc} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{array} \right] = (1+x^3) \det \left[\begin{array}{ccc} a & b & c \\ p & q & r \\ u & v & w \end{array} \right]$$

Exercise 3.2

a. Show that $(A_{ij})^T = (A^T)_{ji}$ for all i, j, and all square matrices A.

b. Use (a) to prove that $\det A^T = \det A$. [*Hint*: Induction on n where A is $n \times n$.]

Exercise 3.3 Show that $\det \left[\begin{array}{cc} 0 & I_n \\ I_m & 0 \end{array} \right] = (-1)^{nm}$ for all $n \geq 1$ and $m \geq 1$.

Exercise 3.4 Show that

$$\det \left[\begin{array}{ccc} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{array} \right] = (b-a)(c-a)(c-b)(a+b+c)$$

Exercise 3.5 Let $A = \left[\begin{array}{c} R_1 \\ R_2 \end{array} \right]$ be a 2×2 matrix with rows R_1 and R_2. If $\det A = 5$, find $\det B$ where

$$B = \left[\begin{array}{c} 3R_1 + 2R_3 \\ 2R_1 + 5R_2 \end{array} \right]$$

Exercise 3.6 Let $A = \left[\begin{array}{rr} 3 & -4 \\ 2 & -3 \end{array} \right]$ and let $\mathbf{v}_k = A^k \mathbf{v}_0$ for each $k \geq 0$.

a. Show that A has no dominant eigenvalue.

b. Find $\mathbf{v}_k$ if $\mathbf{v}_0$ equals:

i. $\left[\begin{array}{c} 1 \\ 1 \end{array} \right]$

ii. $\left[\begin{array}{c} 2 \\ 1 \end{array} \right]$

iii. $\left[\begin{array}{c} x \\ y \end{array} \right] \neq \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$ or $\left[\begin{array}{c} 2 \\ 1 \end{array} \right]$

4. Vector Geometry

4.1 Vectors and Lines

In this chapter we study the geometry of 3-dimensional space. We view a point in 3-space as an arrow from the origin to that point. Doing so provides a "picture" of the point that is truly worth a thousand words. We used this idea earlier, in Section 2.6, to describe rotations, reflections, and projections of the plane $\mathbb{R}^2$. We now apply the same techniques to 3-space to examine similar transformations of $\mathbb{R}^3$. Moreover, the method enables us to completely describe all lines and planes in space.

Vectors in $\mathbb{R}^3$

Introduce a coordinate system in 3-dimensional space in the usual way. First choose a point O called the *origin*, then choose three mutually perpendicular lines through O, called the x, y, and z *axes*, and establish a number scale on each axis with zero at the origin. Given a point P in 3-space we associate three numbers x, y, and z with P, as described in Figure 4.1.1. These numbers are called the *coordinates* of P, and we denote the point as $(x,\ y,\ z)$, or $P(x,\ y,\ z)$ to emphasize the label P. The result is called a *cartesian*[1] coordinate system for 3-space, and the resulting description of 3-space is called *cartesian geometry*.

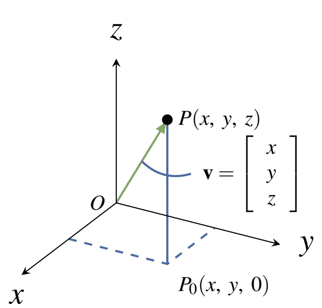


Figure 4.1.1

As in the plane, we introduce vectors by identifying each point $P(x,\ y,\ z)$ with the vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, represented by the **arrow** from the origin to P as in Figure 4.1.1. Informally, we say that the point P *has vector* $\mathbf{v}$, and that vector $\mathbf{v}$ *has point* P. In this way 3-space is identified with $\mathbb{R}^3$, and this identification will be made throughout this chapter, often without comment. In particular, the terms "vector" and "point" are interchangeable.[2] The resulting description of 3-space is called **vector geometry**. Note that the origin is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[1]Named after René Descartes who introduced the idea in 1637.

[2]Recall that we defined $\mathbb{R}^n$ as the set of all ordered n-tuples of real numbers, and reserved the right to denote them as rows or as columns.

Length and Direction

We are going to discuss two fundamental geometric properties of vectors in $\mathbb{R}^3$: length and direction. First, if $\mathbf{v}$ is a vector with point P, the **length** $\|\mathbf{v}\|$ of vector $\mathbf{v}$ is defined to be the distance from the origin to P, that is the length of the arrow representing $\mathbf{v}$. The following properties of length will be used frequently.

Theorem 4.1.1

Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be a vector.*

1. $\|\mathbf{v}\| = \sqrt{x^2+y^2+z^2}$.[3]
2. $\mathbf{v} = \mathbf{0}$ *if and only if* $\|\mathbf{v}\| = 0$
3. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$ *for all scalars* a.[4]

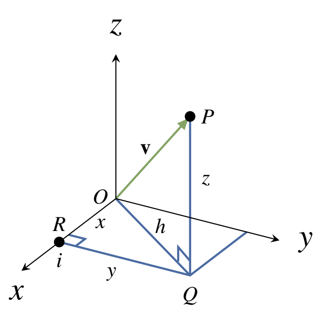


Figure 4.1.2

Proof. Let $\mathbf{v}$ have point $P(x,\ y,\ z)$.

1. In Figure 4.1.2, $\|\mathbf{v}\|$ is the hypotenuse of the right triangle OQP, and so $\|\mathbf{v}\|^2 = h^2+z^2$ by Pythagoras' theorem.[5] But h is the hypotenuse of the right triangle ORQ, so $h^2 = x^2+y^2$. Now (1) follows by eliminating h^2 and taking positive square roots.
2. If $\|\mathbf{v}\| = 0$, then $x^2+y^2+z^2 = 0$ by (1). Because squares of real numbers are nonnegative, it follows that $x = y = z = 0$, and hence that $\mathbf{v} = \mathbf{0}$. The converse is because $\|\mathbf{0}\| = 0$.
3. We have $a\mathbf{v} = \begin{bmatrix} ax & ay & az \end{bmatrix}^T$ so (1) gives

$$\|a\mathbf{v}\|^2 = (ax)^2+(ay)^2+(az)^2 = a^2\|\mathbf{v}\|^2$$

Hence $\|a\mathbf{v}\| = \sqrt{a^2}\|\mathbf{v}\|$, and we are done because $\sqrt{a^2} = |a|$ for any real number a. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.1 also holds.

[3] When we write $\sqrt{p}$ we mean the positive square root of p.

[4] Recall that the absolute value $|a|$ of a real number is defined by $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$.

[5] Pythagoras' theorem states that if a and b are sides of right triangle with hypotenuse c, then $a^2+b^2 = c^2$. A proof is given at the end of this section.

Example 4.1.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ then $\|\mathbf{v}\| = \sqrt{4+1+9} = \sqrt{14}$. Similarly if $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ in 2-space then $\|\mathbf{v}\| = \sqrt{9+16} = 5$.

When we view two nonzero vectors as arrows emanating from the origin, it is clear geometrically what we mean by saying that they have the same or opposite **direction**. This leads to a fundamental new description of vectors.

Theorem 4.1.2

Let $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ be vectors in $\mathbb{R}^3$. Then $\mathbf{v} = \mathbf{w}$ as matrices if and only if $\mathbf{v}$ and $\mathbf{w}$ have the same direction and the same length.[6]

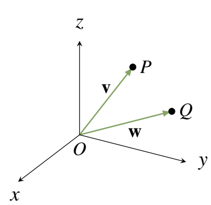


Figure 4.1.3

Proof. If $\mathbf{v} = \mathbf{w}$, they clearly have the same direction and length. Conversely, let $\mathbf{v}$ and $\mathbf{w}$ be vectors with points $P(x,\ y,\ z)$ and $Q(x_1,\ y_1,\ z_1)$ respectively. If $\mathbf{v}$ and $\mathbf{w}$ have the same length and direction then, geometrically, P and Q must be the same point (see Figure 4.1.3). Hence $x = x_1$, $y = y_1$, and $z = z_1$, that is

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \mathbf{w}. \qquad \square$$

A characterization of a vector in terms of its length and direction only is called an **intrinsic** description of the vector. The point to note is that such a description does *not* depend on the choice of coordinate system in $\mathbb{R}^3$. Such descriptions are important in applications because physical laws are often stated in terms of vectors, and these laws cannot depend on the particular coordinate system used to describe the situation.

Geometric Vectors

If A and B are distinct points in space, the arrow from A to B has length and direction.

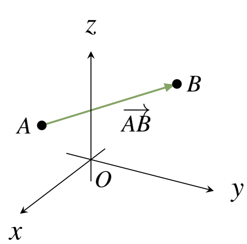


Figure 4.1.4

[6]It is Theorem 4.1.2 that gives vectors their power in science and engineering because many physical quantities are determined by their length and magnitude (and are called **vector quantities**). For example, saying that an airplane is flying at 200 km/h does not describe where it is going; the direction must also be specified. The speed and direction comprise the **velocity** of the airplane, a vector quantity.

Hence:

Definition 4.1 Geometric Vectors

Suppose that A and B are any two points in $\mathbb{R}^3$. In Figure 4.1.4 the line segment from A to B is denoted $\overrightarrow{AB}$ and is called the **geometric vector** *from A to B. Point A is called the* **tail** *of $\overrightarrow{AB}$, B is called the* **tip** *of $\overrightarrow{AB}$, and the* **length** *of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.*

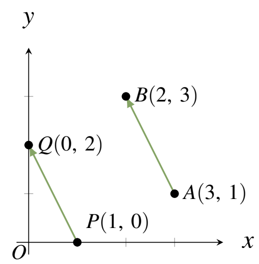


Figure 4.1.5

Note that if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ with point P then $\mathbf{v} = \overrightarrow{OP}$ is itself a geometric vector where O is the origin. Referring to $\overrightarrow{AB}$ as a "vector" seems justified by Theorem 4.1.2 because it has a direction (from A to B) and a length $\|\overrightarrow{AB}\|$. However there appears to be a problem because two geometric vectors can have the same length and direction even if the tips and tails are different. For example $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in Figure 4.1.5 have the same length $\sqrt{5}$ and the same direction (1 unit left and 2 units up) so, by Theorem 4.1.2, they are the same vector! The best way to understand this apparent paradox is to see $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ as different *representations* of the same[7] underlying vector $\left[\begin{array}{r} -1 \\ 2 \end{array}\right]$. Once it is clarified, this phenomenon is a great benefit because, thanks to Theorem 4.1.2, it means that the same geometric vector can be positioned anywhere in space; what is important is the length and direction, not the location of the tip and tail. This ability to move geometric vectors about is very useful as we shall soon see.

The Parallelogram Law

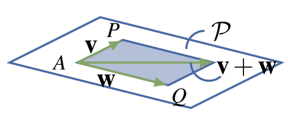


Figure 4.1.6

We now give an intrinsic description of the sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, that is a description that depends only on the lengths and directions of $\mathbf{v}$ and $\mathbf{w}$ and not on the choice of coordinate system. Using Theorem 4.1.2 we can think of these vectors as having a common tail A. If their tips are P and Q respectively, then they both lie in a plane $\mathcal{P}$ containing A, P, and Q, as shown in Figure 4.1.6. The vectors $\mathbf{v}$ and $\mathbf{w}$ create a parallelogram[8] in $\mathcal{P}$, shaded in Figure 4.1.6, called the parallelogram **determined** by $\mathbf{v}$ and $\mathbf{w}$.

If we now choose a coordinate system in the plane $\mathcal{P}$ with A as origin, then the parallelogram law in the plane (Section 2.6) shows that their sum $\mathbf{v}+\mathbf{w}$ is the diagonal of the parallelogram they determine with tail A. This is an intrinsic description of the sum $\mathbf{v}+\mathbf{w}$ because it makes no reference to coordinates. This discussion proves:

[7]Fractions provide another example of quantities that can be the same but *look* different. For example $\frac{6}{9}$ and $\frac{14}{21}$ certainly appear different, but they are equal fractions—both equal $\frac{2}{3}$ in "lowest terms".

[8]Recall that a parallelogram is a four-sided figure whose opposite sides are parallel and of equal length.

The Parallelogram Law

In the parallelogram determined by two vectors $\mathbf{v}$ *and* $\mathbf{w}$*, the vector* $\mathbf{v}+\mathbf{w}$ *is the diagonal with the same tail as* $\mathbf{v}$ *and* $\mathbf{w}$*.*

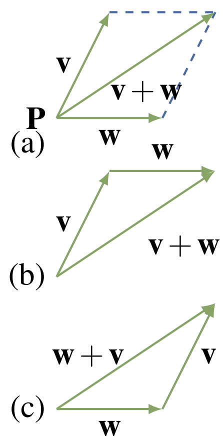


Figure 4.1.7

Because a vector can be positioned with its tail at any point, the parallelogram law leads to another way to view vector addition. In Figure 4.1.7(a) the sum $\mathbf{v}+\mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ is shown as given by the parallelogram law. If $\mathbf{w}$ is moved so its tail coincides with the tip of $\mathbf{v}$ (Figure 4.1.7(b)) then the sum $\mathbf{v}+\mathbf{w}$ is seen as "first $\mathbf{v}$ and then $\mathbf{w}$. Similarly, moving the tail of $\mathbf{v}$ to the tip of $\mathbf{w}$ shows in Figure 4.1.7(c) that $\mathbf{v}+\mathbf{w}$ is "first $\mathbf{w}$ and then $\mathbf{v}$." This will be referred to as the **tip-to-tail rule**, and it gives a graphic illustration of why $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

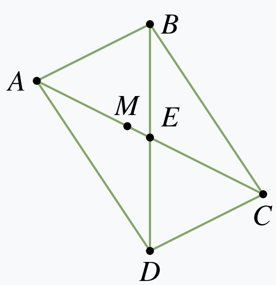
Since $\overrightarrow{AB}$ denotes the vector from a point A to a point B, the tip-to-tail rule takes the easily remembered form

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

for any points A, B, and C. The next example uses this to derive a theorem in geometry without using coordinates.

Example 4.1.2

Show that the diagonals of a parallelogram bisect each other.



Solution. Let the parallelogram have vertices A, B, C, and D, as shown; let E denote the intersection of the two diagonals; and let M denote the midpoint of diagonal AC. We must show that $M=E$ and that this is the midpoint of diagonal BD. This is accomplished by showing that $\overrightarrow{BM}=\overrightarrow{MD}$. (Then the fact that these vectors have the same direction means that $M=E$, and the fact that they have the same length means that $M=E$ is the midpoint of BD.) Now $\overrightarrow{AM}=\overrightarrow{MC}$ because M is the midpoint of AC, and $\overrightarrow{BA}=\overrightarrow{CD}$ because the figure is a parallelogram. Hence

$$\overrightarrow{BM}=\overrightarrow{BA}+\overrightarrow{AM}=\overrightarrow{CD}+\overrightarrow{MC}=\overrightarrow{MC}+\overrightarrow{CD}=\overrightarrow{MD}$$

where the first and last equalities use the tip-to-tail rule of vector addition.

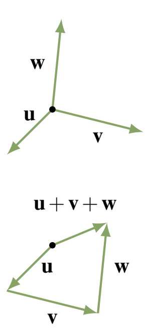


Figure 4.1.8

One reason for the importance of the tip-to-tail rule is that it means two or more vectors can be added by placing them tip-to-tail in sequence. This gives a useful "picture" of the sum of several vectors, and is illustrated for three vectors in Figure 4.1.8 where $\mathbf{u}+\mathbf{v}+\mathbf{w}$ is viewed as first $\mathbf{u}$, then $\mathbf{v}$, then $\mathbf{w}$.

There is a simple geometrical way to visualize the (matrix) **difference** $\mathbf{v}-\mathbf{w}$ of two vectors. If $\mathbf{v}$ and $\mathbf{w}$ are positioned so that they have a common tail A (see Figure 4.1.9), and if B and C are their respective tips, then the tip-to-tail rule gives $\mathbf{w} + \overrightarrow{CB} = \mathbf{v}$. Hence $\mathbf{v} - \mathbf{w} = \overrightarrow{CB}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$. Thus both $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ appear as diagonals in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ (see Figure 4.1.9). We record this for reference.

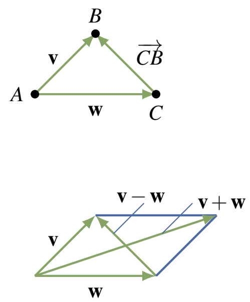


Figure 4.1.9

Theorem 4.1.3

If $\mathbf{v}$ and $\mathbf{w}$ have a common tail, then $\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

One of the most useful applications of vector subtraction is that it gives a simple formula for the vector from one point to another, and for the distance between the points.

Theorem 4.1.4

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then:

1. $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$.

2. *The distance between P_1 and P_2 is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

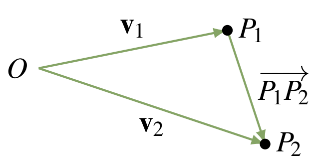


Figure 4.1.10

Proof. If O is the origin, write

$$\mathbf{v}_1 = \overrightarrow{OP}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \overrightarrow{OP}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

as in Figure 4.1.10.

Then Theorem 4.1.3 gives $\overrightarrow{P_1P_2} = \mathbf{v}_2 - \mathbf{v}_1$, and (1) follows. But the distance between P_1 and P_2 is $\|\overrightarrow{P_1P_2}\|$, so (2) follows from (1) and Theorem 4.1.1. □

Of course the $\mathbb{R}^2$-version of Theorem 4.1.4 is also valid: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in $\mathbb{R}^2$, then $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$, and the distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 4.1.3

The distance between $P_1(2, -1, 3)$ and $P_2(1, 1, 4)$ is $\sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{6}$, and the vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

As for the parallelogram law, the intrinsic rule for finding the length and direction of a scalar multiple of a vector in $\mathbb{R}^3$ follows easily from the same situation in $\mathbb{R}^2$.

Scalar Multiple Law

If a is a real number and $\mathbf{v} \neq \mathbf{0}$ is a vector then:

1. *The length of $a\mathbf{v}$ is $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.*
2. *If[9] $a\mathbf{v} \neq \mathbf{0}$, the direction of $a\mathbf{v}$ is* $\begin{cases} \text{the same as } \mathbf{v} \text{ if } a > 0, \\ \text{opposite to } \mathbf{v} \text{ if } a < 0. \end{cases}$

Proof.

1. This is part of Theorem 4.1.1.
2. Let O denote the origin in $\mathbb{R}^3$, let $\mathbf{v}$ have point P, and choose any plane containing O and P. If we set up a coordinate system in this plane with O as origin, then $\mathbf{v} = \overrightarrow{OP}$ so the result in (2) follows from the scalar multiple law in the plane (Section 2.6). □

Figure 4.1.11 gives several examples of scalar multiples of a vector $\mathbf{v}$.

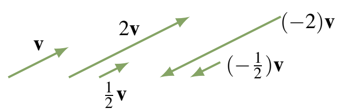


Figure 4.1.11

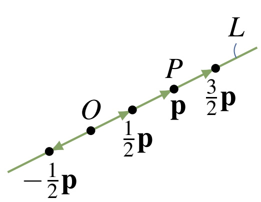


Figure 4.1.12

Consider a line L through the origin, let P be any point on L other than the origin O, and let $\mathbf{p} = \overrightarrow{OP}$. If $t \neq 0$, then $t\mathbf{p}$ is a point on L because it has direction the same or opposite as that of $\mathbf{p}$. Moreover $t > 0$ or $t < 0$ according as the point $t\mathbf{p}$ lies on the same or opposite side of the origin as P. This is illustrated in Figure 4.1.12.

A vector $\mathbf{u}$ is called a **unit vector** if $\|\mathbf{u}\| = 1$. Then $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors, called the **coordinate** vectors. We discuss them in more detail in Section 4.2.

Example 4.1.4

If $\mathbf{v} \neq \mathbf{0}$ show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unique unit vector in the same direction as $\mathbf{v}$.

Solution. The vectors in the same direction as $\mathbf{v}$ are the scalar multiples $a\mathbf{v}$ where $a > 0$. But $\|a\mathbf{v}\| = |a|\|\mathbf{v}\| = a\|\mathbf{v}\|$ when $a > 0$, so $a\mathbf{v}$ is a unit vector if and only if $a = \frac{1}{\|\mathbf{v}\|}$.

The next example shows how to find the coordinates of a point on the line segment between two given points. The technique is important and will be used again below.

[9] Since the zero vector has no direction, we deal only with the case $a\mathbf{v} \neq \mathbf{0}$.

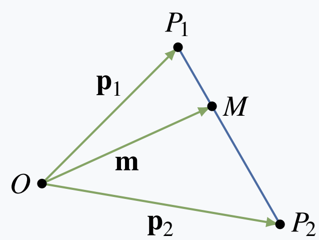
Example 4.1.5

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors of two points P_1 and P_2. If M is the point one third the way from P_1 to P_2, show that the vector $\mathbf{m}$ of M is given by

$$\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$$

Conclude that if $P_1 = P_1(x_1,\ y_1,\ z_1)$ and $P_2 = P_2(x_2,\ y_2,\ z_2)$, then M has coordinates

$$M = M\left(\frac{2}{3}x_1 + \frac{1}{3}x_2,\ \frac{2}{3}y_1 + \frac{1}{3}y_2,\ \frac{2}{3}z_1 + \frac{1}{3}z_2\right)$$



Solution. The vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{m}$ are shown in the diagram. We have $\overrightarrow{P_1M} = \frac{1}{3}\overrightarrow{P_1P_2}$ because $\overrightarrow{P_1M}$ is in the same direction as $\overrightarrow{P_1P_2}$ and $\frac{1}{3}$ as long. By Theorem 4.1.3 we have $\overrightarrow{P_1P_2} = \mathbf{p}_2 - \mathbf{p}_1$, so tip-to-tail addition gives

$$\mathbf{m} = \mathbf{p}_1 + \overrightarrow{P_1M} = \mathbf{p}_1 + \frac{1}{3}(\mathbf{p}_2 - \mathbf{p}_1) = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$$

as required. For the coordinates, we have $\mathbf{p}_1 = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$ and $\mathbf{p}_2 = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$, so

$$\mathbf{m} = \frac{2}{3}\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right] + \frac{1}{3}\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right] = \left[\begin{array}{c} \frac{2}{3}x_1 + \frac{1}{3}x_2 \\ \frac{2}{3}y_1 + \frac{1}{3}y_2 \\ \frac{2}{3}z_1 + \frac{1}{3}z_2 \end{array}\right]$$

by matrix addition. The last statement follows.

Note that in Example 4.1.5 $\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$ is a "weighted average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ with more weight on $\mathbf{p}_1$ because $\mathbf{m}$ is closer to $\mathbf{p}_1$.

The point M halfway between points P_1 and P_2 is called the **midpoint** between these points. In the same way, the vector $\mathbf{m}$ of M is

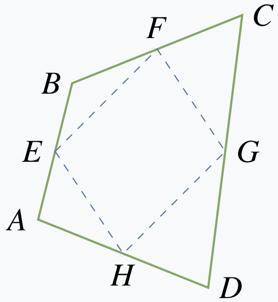
$$\mathbf{m} = \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2 = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$$

as the reader can verify, so $\mathbf{m}$ is the "average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ in this case.

Example 4.1.6

Show that the midpoints of the four sides of any quadrilateral are the vertices of a parallelogram. Here a quadrilateral is any figure with four vertices and straight sides.

Solution. Suppose that the vertices of the quadrilateral are A, B, C, and D (in that order) and that E, F, G, and H are the midpoints of the sides as shown in the diagram. It suffices to show $\overrightarrow{EF} = \overrightarrow{HG}$ (because then sides EF and HG are parallel and of equal length).



Now the fact that E is the midpoint of AB means that $\overrightarrow{EB} = \frac{1}{2}\overrightarrow{AB}$. Similarly, $\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$, so

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AC}$$

A similar argument shows that $\overrightarrow{HG} = \frac{1}{2}\overrightarrow{AC}$ too, so $\overrightarrow{EF} = \overrightarrow{HG}$ as required.

Definition 4.2 Parallel Vectors in $\mathbb{R}^3$

Two nonzero vectors are called ***parallel*** *if they have the same or opposite direction.*

Many geometrical propositions involve this notion, so the following theorem will be referred to repeatedly.

Theorem 4.1.5

Two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel if and only if one is a scalar multiple of the other.*

Proof. If one of them is a scalar multiple of the other, they are parallel by the scalar multiple law.

Conversely, assume that $\mathbf{v}$ and $\mathbf{w}$ are parallel and write $d = \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|}$ for convenience. Then $\mathbf{v}$ and $\mathbf{w}$ have the same or opposite direction. If they have the same direction we show that $\mathbf{v} = d\mathbf{w}$ by showing that $\mathbf{v}$ and $d\mathbf{w}$ have the same length and direction. In fact, $\|d\mathbf{w}\| = |d|\|\mathbf{w}\| = \|\mathbf{v}\|$ by Theorem 4.1.1; as to the direction, $d\mathbf{w}$ and $\mathbf{w}$ have the same direction because $d > 0$, and this is the direction of $\mathbf{v}$ by assumption. Hence $\mathbf{v} = d\mathbf{w}$ in this case by Theorem 4.1.2. In the other case, $\mathbf{v}$ and $\mathbf{w}$ have opposite direction and a similar argument shows that $\mathbf{v} = -d\mathbf{w}$. We leave the details to the reader. □

Example 4.1.7

Given points $P(2,\ -1,\ 4)$, $Q(3,\ -1,\ 3)$, $A(0,\ 2,\ 1)$, and $B(1,\ 3,\ 0)$, determine if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

Solution. By Theorem 4.1.3, $\overrightarrow{PQ} = (1,\ 0,\ -1)$ and $\overrightarrow{AB} = (1,\ 1,\ -1)$. If $\overrightarrow{PQ} = t\overrightarrow{AB}$ then $(1,\ 0,\ -1) = (t,\ t,\ -t)$, so $1 = t$ and $0 = t$, which is impossible. Hence $\overrightarrow{PQ}$ is *not* a scalar multiple of $\overrightarrow{AB}$, so these vectors are not parallel by Theorem 4.1.5.

Lines in Space

These vector techniques can be used to give a very simple way of describing straight lines in space. In order to do this, we first need a way to specify the orientation of such a line, much as the slope does in the plane.

Definition 4.3 Direction Vector of a Line

*With this in mind, we call a nonzero vector $\mathbf{d} \neq \mathbf{0}$ a **direction vector** for the line if it is parallel to $\overrightarrow{AB}$ for some pair of distinct points A and B on the line.*

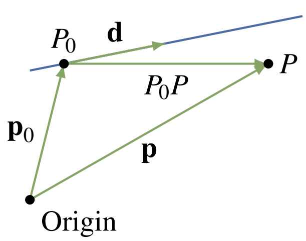


Figure 4.1.13

Of course it is then parallel to $\overrightarrow{CD}$ for *any* distinct points C and D on the line. In particular, any nonzero scalar multiple of $\mathbf{d}$ will also serve as a direction vector of the line.

We use the fact that there is exactly one line that passes through a particular point $P_0(x_0,\ y_0,\ z_0)$ and has a given direction vector $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right]$. We want to describe this line by giving a condition on x, y, and z that the point $P(x,\ y,\ z)$ lies on this line. Let $\mathbf{p}_0 = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right]$ and $\mathbf{p} = \left[\begin{array}{c} x \\ y \\ z \end{array} \right]$ denote the vectors of P_0 and P, respectively (see Figure 4.1.13). Then

$$\mathbf{p} = \mathbf{p}_0 + \overrightarrow{P_0P}$$

Hence P lies on the line if and only if $\overrightarrow{P_0P}$ is parallel to $\mathbf{d}$—that is, if and only if $\overrightarrow{P_0P} = t\mathbf{d}$ for some scalar t by Theorem 4.1.5. Thus $\mathbf{p}$ is the vector of a point on the line if and only if $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$ for some scalar t. This discussion is summed up as follows.

Vector Equation of a Line

The line parallel to $\mathbf{d} \neq \mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \quad t \text{ any scalar}$$

In other words, the point P with vector $\mathbf{p}$ is on this line if and only if a real number t exists such that $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$.

In component form the vector equation becomes

$$\left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right] + t \left[\begin{array}{c} a \\ b \\ c \end{array} \right]$$

Equating components gives a different description of the line.

Parametric Equations of a Line

The line through $P_0(x_0, y_0, z_0)$ *with direction vector* $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *is given by*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \quad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x, y, z)$ *is on this line if and only if a real number* t *exists such that* $x = x_0 + ta$, $y = y_0 + tb$, *and* $z = z_0 + tc$.

Example 4.1.8

Find the equations of the line through the points $P_0(2, 0, 1)$ and $P_1(4, -1, 1)$.

Solution. Let $\mathbf{d} = \overrightarrow{P_0P_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ denote the vector from P_0 to P_1. Then $\mathbf{d}$ is parallel to the line (P_0 and P_1 are *on* the line), so $\mathbf{d}$ serves as a direction vector for the line. Using P_0 as the point on the line leads to the parametric equations

$$\begin{aligned} x &= 2 + 2t \\ y &= -t \qquad t \text{ a parameter} \\ z &= 1 \end{aligned}$$

Note that if P_1 is used (rather than P_0), the equations are

$$\begin{aligned} x &= 4 + 2s \\ y &= -1 - s \quad s \text{ a parameter} \\ z &= 1 \end{aligned}$$

These are different from the preceding equations, but this is merely the result of a change of parameter. In fact, $s = t - 1$.

Example 4.1.9

Find the equations of the line through $P_0(3, -1, 2)$ parallel to the line with equations

$$\begin{aligned} x &= -1 + 2t \\ y &= 1 + t \\ z &= -3 + 4t \end{aligned}$$

Solution. The coefficients of t give a direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ of the given line. Because the line we seek is parallel to this line, $\mathbf{d}$ also serves as a direction vector for the new line. It passes through P_0, so the parametric equations are

$$\begin{aligned} x &= 3 + 2t \\ y &= -1 + t \\ z &= 2 + 4t \end{aligned}$$

Example 4.1.10

Determine whether the following lines intersect and, if so, find the point of intersection.

$$\begin{array}{ll} x = 1 - 3t & x = -1 + s \\ y = 2 + 5t & y = 3 - 4s \\ z = 1 + t & z = 1 - s \end{array}$$

Solution. Suppose $P(x,\ y,\ z)$ with vector $\mathbf{p}$ lies on both lines. Then

$$\begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + s \\ 3 - 4s \\ 1 - s \end{bmatrix} \text{ for some } t \text{ and } s,$$

where the first (second) equation is because P lies on the first (second) line. Hence the lines intersect if and only if the three equations

$$\begin{aligned} 1 - 3t &= -1 + s \\ 2 + 5t &= 3 - 4s \\ 1 + t &= 1 - s \end{aligned}$$

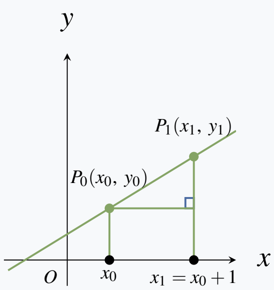
have a solution. In this case, $t = 1$ and $s = -1$ satisfy all three equations, so the lines *do* intersect and the point of intersection is

$$\mathbf{p} = \begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

using $t = 1$. Of course, this point can also be found from $\mathbf{p} = \begin{bmatrix} -1 + s \\ 3 - 4s \\ 1 - s \end{bmatrix}$ using $s = -1$.

Example 4.1.11

Show that the line through $P_0(x_0, y_0)$ with slope m has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and equation $y - y_0 = m(x - x_0)$. This equation is called the *point-slope* formula.



Solution. Let $P_1(x_1, y_1)$ be the point on the line one unit to the right of P_0 (see the diagram). Hence $x_1 = x_0 + 1$. Then $\mathbf{d} = \overrightarrow{P_0P_1}$ serves as direction vector of the line, and $\mathbf{d} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 - y_0 \end{bmatrix}$. But the slope m can be computed as follows:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{1} = y_1 - y_0$$

Hence $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and the parametric equations are $x = x_0 + t$, $y = y_0 + mt$. Eliminating t gives $y - y_0 = mt = m(x - x_0)$, as asserted.

Note that the vertical line through $P_0(x_0, y_0)$ has a direction vector $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that is *not* of the form $\begin{bmatrix} 1 \\ m \end{bmatrix}$ for any m. This result confirms that the notion of slope makes no sense in this case. However, the vector method gives parametric equations for the line:

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \end{aligned}$$

Because y is arbitrary here (t is arbitrary), this is usually written simply as $x = x_0$.

Pythagoras' Theorem

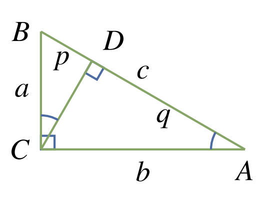


Figure 4.1.14

The Pythagorean theorem was known earlier, but Pythagoras (c. 550 B.C.) is credited with giving the first rigorous, logical, deductive proof of the result. The proof we give depends on a basic property of similar triangles: ratios of corresponding sides are equal.

Theorem 4.1.6: Pythagoras' Theorem

Given a right-angled triangle with hypotenuse c and sides a and b, then $a^2 + b^2 = c^2$.

Proof. Let A, B, and C be the vertices of the triangle as in Figure 4.1.14. Draw a perpendicular line from C to the point D on the hypotenuse, and let p and q be the lengths of BD and DA respectively. Then DBC

and CBA are similar triangles so $\frac{p}{a} = \frac{a}{c}$. This means $a^2 = pc$. In the same way, the similarity of DCA and CBA gives $\frac{q}{b} = \frac{b}{c}$, whence $b^2 = qc$. But then

$$a^2 + b^2 = pc + qc = (p+q)c = c^2$$

because $p+q = c$. This proves Pythagoras' theorem[10]. □

Exercises for 4.1

Exercise 4.1.1 Compute $\|\mathbf{v}\|$ if $\mathbf{v}$ equals:

a. $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ d. $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

e. $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ f. $-3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Exercise 4.1.2 Find a unit vector in the direction of:

a. $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$ b. $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Exercise 4.1.3

a. Find a unit vector in the direction from $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

b. If $\mathbf{u} \neq \mathbf{0}$, for which values of a is $a\mathbf{u}$ a unit vector?

Exercise 4.1.4 Find the distance between the following pairs of points.

a. $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ d. $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

Exercise 4.1.5 Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

Exercise 4.1.6 Let A, B, and C denote the three vertices of a triangle.

a. If E is the midpoint of side BC, show that

$$\overrightarrow{AE} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

b. If F is the midpoint of side AC, show that

$$\overrightarrow{FE} = \frac{1}{2}\overrightarrow{AB}$$

Exercise 4.1.7 Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel in each of the following cases.

a. $\mathbf{u} = \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$

[10] There is an intuitive geometrical proof of Pythagoras' theorem in Example B.3.

Exercise 4.1.8 Let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of points P and Q, respectively, and let R be the point whose vector is $\mathbf{p}+\mathbf{q}$. Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

a. $\overrightarrow{QP}$

b. $\overrightarrow{QR}$

c. $\overrightarrow{RP}$

d. $\overrightarrow{RO}$ where O is the origin

Exercise 4.1.9 In each case, find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$.

a. $P(1,\ -1,\ 3)$, $Q(3,\ 1,\ 0)$

b. $P(2,\ 0,\ 1)$, $Q(1,\ -1,\ 6)$

c. $P(1,\ 0,\ 1)$, $Q(1,\ 0,\ -3)$

d. $P(1,\ -1,\ 2)$, $Q(1,\ -1,\ 2)$

e. $P(1,\ 0,\ -3)$, $Q(-1,\ 0,\ 3)$

f. $P(3,\ -1,\ 6)$, $Q(1,\ 1,\ 4)$

Exercise 4.1.10 In each case, find a point Q such that $\overrightarrow{PQ}$ has (i) the same direction as $\mathbf{v}$; (ii) the opposite direction to $\mathbf{v}$.

a. $P(-1,\ 2,\ 2)$, $\mathbf{v}=\left[\begin{array}{r}1\\3\\1\end{array}\right]$

b. $P(3,\ 0,\ -1)$, $\mathbf{v}=\left[\begin{array}{r}2\\-1\\3\end{array}\right]$

Exercise 4.1.11 Let $\mathbf{u}=\left[\begin{array}{r}3\\-1\\0\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}4\\0\\1\end{array}\right]$, and $\mathbf{w}=\left[\begin{array}{r}-1\\1\\5\end{array}\right]$. In each case, find $\mathbf{x}$ such that:

a. $3(2\mathbf{u}+\mathbf{x})+\mathbf{w}=2\mathbf{x}-\mathbf{v}$

b. $2(3\mathbf{v}-\mathbf{x})=5\mathbf{w}+\mathbf{u}-3\mathbf{x}$

Exercise 4.1.12 Let $\mathbf{u}=\left[\begin{array}{r}1\\1\\2\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}0\\1\\2\end{array}\right]$, and $\mathbf{w}=\left[\begin{array}{r}1\\0\\-1\end{array}\right]$. In each case, find numbers a, b, and c such that $\mathbf{x}=a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$.

a. $\mathbf{x}=\left[\begin{array}{r}2\\-1\\6\end{array}\right]$

b. $\mathbf{x}=\left[\begin{array}{r}1\\3\\0\end{array}\right]$

Exercise 4.1.13 Let $\mathbf{u}=\left[\begin{array}{r}3\\-1\\0\end{array}\right]$, $\mathbf{v}=\left[\begin{array}{r}4\\0\\1\end{array}\right]$, and $\mathbf{z}=\left[\begin{array}{r}1\\1\\1\end{array}\right]$. In each case, show that there are no numbers a, b, and c such that:

a. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z}=\left[\begin{array}{r}1\\2\\1\end{array}\right]$

b. $a\mathbf{u}+b\mathbf{v}+c\mathbf{z}=\left[\begin{array}{r}5\\6\\-1\end{array}\right]$

Exercise 4.1.14 Given $P_1(2,\ 1,\ -2)$ and $P_2(1,\ -2,\ 0)$. Find the coordinates of the point P:

a. $\frac{1}{5}$ the way from P_1 to P_2

b. $\frac{1}{4}$ the way from P_2 to P_1

Exercise 4.1.15 Find the two points trisecting the segment between $P(2,\ 3,\ 5)$ and $Q(8,\ -6,\ 2)$.

Exercise 4.1.16 Let $P_1(x_1,\ y_1,\ z_1)$ and $P_2(x_2,\ y_2,\ z_2)$ be two points with vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If r and s are positive integers, show that the point P lying $\frac{r}{r+s}$ the way from P_1 to P_2 has vector

$$\mathbf{p}=\left(\tfrac{s}{r+s}\right)\mathbf{p}_1+\left(\tfrac{r}{r+s}\right)\mathbf{p}_2$$

Exercise 4.1.17 In each case, find the point Q:

a. $\overrightarrow{PQ}=\left[\begin{array}{r}2\\0\\-3\end{array}\right]$ and $P=P(2,\ -3,\ 1)$

b. $\overrightarrow{PQ}=\left[\begin{array}{r}-1\\4\\7\end{array}\right]$ and $P=P(1,\ 3,\ -4)$

Exercise 4.1.18 Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. In each case find $\mathbf{x}$:

a. $2\mathbf{u} - \|\mathbf{v}\|\mathbf{v} = \frac{3}{2}(\mathbf{u} - 2\mathbf{x})$

b. $3\mathbf{u} + 7\mathbf{v} = \|\mathbf{u}\|^2(2\mathbf{x} + \mathbf{v})$

Exercise 4.1.19 Find all vectors $\mathbf{u}$ that are parallel to $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and satisfy $\|\mathbf{u}\| = 3\|\mathbf{v}\|$.

Exercise 4.1.20 Let P, Q, and R be the vertices of a parallelogram with adjacent sides PQ and PR. In each case, find the other vertex S.

a. $P(3,\ -1,\ -1)$, $Q(1,\ -2,\ 0)$, $R(1,\ -1,\ 2)$

b. $P(2,\ 0,\ -1)$, $Q(-2,\ 4,\ 1)$, $R(3,\ -1,\ 0)$

Exercise 4.1.21 In each case either prove the statement or give an example showing that it is false.

a. The zero vector $\mathbf{0}$ is the only vector of length 0.

b. If $\|\mathbf{v} - \mathbf{w}\| = 0$, then $\mathbf{v} = \mathbf{w}$.

c. If $\mathbf{v} = -\mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.

d. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \mathbf{w}$.

e. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \pm\mathbf{w}$.

f. If $\mathbf{v} = t\mathbf{w}$ for some scalar t, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction.

g. If $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ are nonzero, and $\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ parallel, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.

h. $\|-5\mathbf{v}\| = -5\|\mathbf{v}\|$, for all $\mathbf{v}$.

i. If $\|\mathbf{v}\| = \|2\mathbf{v}\|$, then $\mathbf{v} = \mathbf{0}$.

j. $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$, for all $\mathbf{v}$ and $\mathbf{w}$.

Exercise 4.1.22 Find the vector and parametric equations of the following lines.

a. The line parallel to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and passing through $P(1,\ -1,\ 3)$.

b. The line passing through $P(3,\ -1,\ 4)$ and $Q(1,\ 0,\ -1)$.

c. The line passing through $P(3,\ -1,\ 4)$ and $Q(3,\ -1,\ 5)$.

d. The line parallel to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and passing through $P(1,\ 1,\ 1)$.

e. The line passing through $P(1,\ 0,\ -3)$ and parallel to the line with parametric equations $x = -1 + 2t$, $y = 2 - t$, and $z = 3 + 3t$.

f. The line passing through $P(2,\ -1,\ 1)$ and parallel to the line with parametric equations $x = 2 - t$, $y = 1$, and $z = t$.

g. The lines through $P(1,\ 0,\ 1)$ that meet the line with vector equation $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ at points at distance 3 from $P_0(1,\ 2,\ 0)$.

Exercise 4.1.23 In each case, verify that the points P and Q lie on the line.

a. $\begin{aligned} x &= 3 - 4t \\ y &= 2 + t \\ z &= 1 - t \end{aligned}$ $\quad P(-1,\ 3,\ 0)$, $Q(11,\ 0,\ 3)$

b. $\begin{aligned} x &= 4 - t \\ y &= 3 \\ z &= 1 - 2t \end{aligned}$ $\quad P(2,\ 3,\ -3)$, $Q(-1,\ 3,\ -9)$

Exercise 4.1.24 Find the point of intersection (if any) of the following pairs of lines.

a. $\begin{aligned} x &= 3 + t \\ y &= 1 - 2t \\ z &= 3 + 3t \end{aligned} \quad \begin{aligned} x &= 4 + 2s \\ y &= 6 + 3s \\ z &= 1 + s \end{aligned}$

b. $\begin{aligned} x &= 1 - t \\ y &= 2 + 2t \\ z &= -1 + 3t \end{aligned} \quad \begin{aligned} x &= 2s \\ y &= 1 + s \\ z &= 3 \end{aligned}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

Exercise 4.1.25 Show that if a line passes through the origin, the vectors of points on the line are all scalar multiples of some fixed nonzero vector.

Exercise 4.1.26 Show that every line parallel to the z axis has parametric equations $x = x_0$, $y = y_0$, $z = t$ for some fixed numbers x_0 and y_0.

Exercise 4.1.27 Let $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector where a, b, and c are *all* nonzero. Show that the equations of the line through $P_0(x_0, y_0, z_0)$ with direction vector $\mathbf{d}$ can be written in the form

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This is called the **symmetric form** of the equations.

Exercise 4.1.28 A parallelogram has sides AB, BC, CD, and DA. Given $A(1, -1, 2)$, $C(2, 1, 0)$, and the midpoint $M(1, 0, -3)$ of AB, find $\overrightarrow{BD}$.

Exercise 4.1.29 Find all points C on the line through $A(1, -1, 2)$ and $B = (2, 0, 1)$ such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$.

Exercise 4.1.30 Let A, B, C, D, E, and F be the vertices of a regular hexagon, taken in order. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

Exercise 4.1.31

a. Let P_1, P_2, P_3, P_4, P_5, and P_6 be six points equally spaced on a circle with centre C. Show that

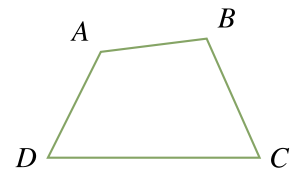
$$\overrightarrow{CP}_1 + \overrightarrow{CP}_2 + \overrightarrow{CP}_3 + \overrightarrow{CP}_4 + \overrightarrow{CP}_5 + \overrightarrow{CP}_6 = \mathbf{0}$$

b. Show that the conclusion in part (a) holds for any *even* set of points evenly spaced on the circle.

c. Show that the conclusion in part (a) holds for *three* points.

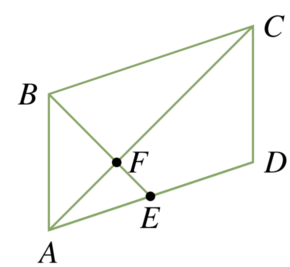
d. Do you think it works for *any* finite set of points evenly spaced around the circle?

Exercise 4.1.32 Consider a quadrilateral with vertices A, B, C, and D in order (as shown in the diagram).



If the diagonals AC and BD bisect each other, show that the quadrilateral is a parallelogram. (This is the converse of Example 4.1.2.) [*Hint*: Let E be the intersection of the diagonals. Show that $\overrightarrow{AB} = \overrightarrow{DC}$ by writing $\overrightarrow{AB} = \overrightarrow{AE} + \overrightarrow{EB}$.]

Exercise 4.1.33 Consider the parallelogram $ABCD$ (see diagram), and let E be the midpoint of side AD.



Show that BE and AC trisect each other; that is, show that the intersection point is one-third of the way from E to B and from A to C. [*Hint*: If F is one-third of the way from A to C, show that $2\overrightarrow{EF} = \overrightarrow{FB}$ and argue as in Example 4.1.2.]

Exercise 4.1.34 The line from a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. If the vertices of a triangle have vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, show that the point on each median that is $\frac{1}{3}$ the way from the midpoint to the vertex has vector $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$. Conclude that the point C with vector $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies on all three medians. This point C is called the **centroid** of the triangle.

Exercise 4.1.35 Given four noncoplanar points in space, the figure with these points as vertices is called a **tetrahedron**. The line from a vertex through the centroid (see previous exercise) of the triangle formed by the remaining vertices is called a **median** of the tetrahedron. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ are the vectors of the four vertices, show that the point on a median one-fourth the way from the centroid to the vertex has vector $\frac{1}{4}(\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{x})$. Conclude that the four medians are concurrent.

4.2 Projections and Planes

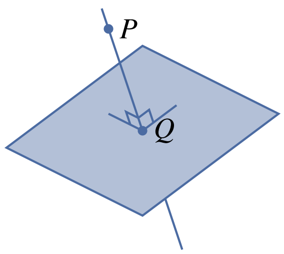


Figure 4.2.1

Any student of geometry soon realizes that the notion of perpendicular lines is fundamental. As an illustration, suppose a point P and a plane are given and it is desired to find the point Q that lies in the plane and is closest to P, as shown in Figure 4.2.1. Clearly, what is required is to find the line through P that is perpendicular to the plane and then to obtain Q as the point of intersection of this line with the plane. Finding the line *perpendicular* to the plane requires a way to determine when two vectors are perpendicular. This can be done using the idea of the dot product of two vectors.

The Dot Product and Angles

Definition 4.4 Dot Product in $\mathbb{R}^3$

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* **dot product** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Because $\mathbf{v} \cdot \mathbf{w}$ is a number, it is sometimes called the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$.[11]

Example 4.2.1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-1) = -5$.

The next theorem lists several basic properties of the dot product.

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*

2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.

3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.

4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

[11]Similarly, if $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2$.

5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.

6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

Proof. (1), (2), and (3) are easily verified, and (4) comes from Theorem 4.1.1. The rest are properties of matrix arithmetic (because $\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{w}$), and are left to the reader. □

The properties in Theorem 4.2.1 enable us to do calculations like

$$3\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}) = 6(\mathbf{u} \cdot \mathbf{v}) - 9(\mathbf{u} \cdot \mathbf{w}) + 12(\mathbf{u} \cdot \mathbf{z})$$

and such computations will be used without comment below. Here is an example.

Example 4.2.2

Verify that $\|\mathbf{v} - 3\mathbf{w}\|^2 = 1$ when $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 1$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

Solution. We apply Theorem 4.2.1 several times:

$$\begin{aligned} \|\mathbf{v} - 3\mathbf{w}\|^2 &= (\mathbf{v} - 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) \\ &= \mathbf{v} \cdot (\mathbf{v} - 3\mathbf{w}) - 3\mathbf{w} \cdot (\mathbf{v} - 3\mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - 3(\mathbf{v} \cdot \mathbf{w}) - 3(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\ &= \|\mathbf{v}\|^2 - 6(\mathbf{v} \cdot \mathbf{w}) + 9\|\mathbf{w}\|^2 \\ &= 4 - 12 + 9 = 1 \end{aligned}$$

There is an intrinsic description of the dot product of two nonzero vectors in $\mathbb{R}^3$. To understand it we require the following result from trigonometry.

Law of Cosines

If a triangle has sides a, b, and c, and if θ is the interior angle opposite c then

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

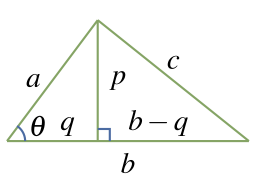


Figure 4.2.2

Proof. We prove it when is θ acute, that is $0 \leq \theta < \frac{\pi}{2}$; the obtuse case is similar. In Figure 4.2.2 we have $p = a\sin\theta$ and $q = a\cos\theta$. Hence Pythagoras' theorem gives

$$\begin{aligned} c^2 = p^2 + (b-q)^2 &= a^2\sin^2\theta + (b - a\cos\theta)^2 \\ &= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta \end{aligned}$$

The law of cosines follows because $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. □

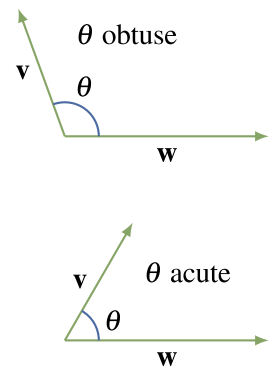


Figure 4.2.3

Note that the law of cosines reduces to Pythagoras' theorem if θ is a right angle (because $\cos \frac{\pi}{2} = 0$).

Now let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors positioned with a common tail as in Figure 4.2.3. Then they determine a unique angle θ in the range

$$0 \leq \theta \leq \pi$$

This angle θ will be called the **angle between** $\mathbf{v}$ and $\mathbf{w}$. Figure 4.2.3 illustrates when θ is acute (less than $\frac{\pi}{2}$) and obtuse (greater than $\frac{\pi}{2}$). Clearly $\mathbf{v}$ and $\mathbf{w}$ are parallel if θ is either 0 or π. Note that we do not define the angle between $\mathbf{v}$ and $\mathbf{w}$ if one of these vectors is $\mathbf{0}$.

The next result gives an easy way to compute the angle between two nonzero vectors using the dot product.

Theorem 4.2.2

Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

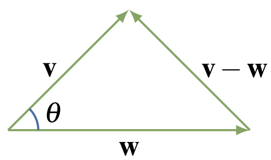


Figure 4.2.4

Proof. We calculate $\|\mathbf{v} - \mathbf{w}\|^2$ in two ways. First apply the law of cosines to the triangle in Figure 4.2.4 to obtain:

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

On the other hand, we use Theorem 4.2.1:

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 - 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \end{aligned}$$

Comparing these we see that $-2\|\mathbf{v}\| \|\mathbf{w}\| \cos \theta = -2(\mathbf{v} \cdot \mathbf{w})$, and the result follows. □

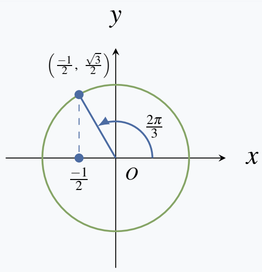
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, Theorem 4.2.2 gives an intrinsic description of $\mathbf{v} \cdot \mathbf{w}$ because $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and the angle θ between $\mathbf{v}$ and $\mathbf{w}$ do not depend on the choice of coordinate system. Moreover, since $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ are nonzero ($\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors), it gives a formula for the cosine of the angle θ:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \tag{4.1}$$

Since $0 \leq \theta \leq \pi$, this can be used to find θ.

Example 4.2.3

Compute the angle between $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.



Solution. Compute $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-2+1-2}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$. Now recall that $\cos\theta$ and $\sin\theta$ are defined so that $(\cos\theta, \sin\theta)$ is the point on the unit circle determined by the angle θ (drawn counterclockwise, starting from the positive x axis). In the present case, we know that $\cos\theta = -\frac{1}{2}$ and that $0 \leq \theta \leq \pi$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, it follows that $\theta = \frac{2\pi}{3}$ (see the diagram).

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, equation (4.1) shows that $\cos\theta$ has the same sign as $\mathbf{v}\cdot\mathbf{w}$, so

$$\begin{array}{lll}
\mathbf{v}\cdot\mathbf{w} > 0 & \text{if and only if} & \theta \text{ is acute } (0 \leq \theta < \frac{\pi}{2}) \\
\mathbf{v}\cdot\mathbf{w} < 0 & \text{if and only if} & \theta \text{ is obtuse } (\frac{\pi}{2} < \theta \leq 0) \\
\mathbf{v}\cdot\mathbf{w} = 0 & \text{if and only if} & \theta = \frac{\pi}{2}
\end{array}$$

In this last case, the (nonzero) vectors are perpendicular. The following terminology is used in linear algebra:

Definition 4.5 Orthogonal Vectors in $\mathbb{R}^3$

*Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **orthogonal** if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or the angle between them is $\frac{\pi}{2}$.*

Since $\mathbf{v}\cdot\mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 4.2.3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example 4.2.4

Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

Solution. The vectors along the sides of the triangle are

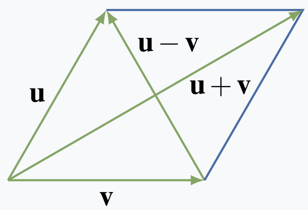
$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \ \text{and } \overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Evidently $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 2 - 2 + 0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides PQ and QR are perpendicular—that is, the angle at Q is a right angle.

Example 4.2.5 demonstrates how the dot product can be used to verify geometrical theorems involving perpendicular lines.

Example 4.2.5

A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.



Solution. Let $\mathbf{u}$ and $\mathbf{v}$ denote vectors along two adjacent sides of a rhombus, as shown in the diagram. Then the diagonals are $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$, and we compute

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v})-\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2-\|\mathbf{v}\|^2\\ &= 0\end{aligned}$$

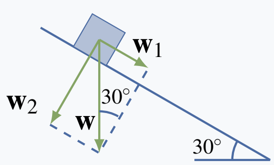
because $\|\mathbf{u}\| = \|\mathbf{v}\|$ (it is a rhombus). Hence $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are orthogonal.

Projections

In applications of vectors, it is frequently useful to write a vector as the sum of two orthogonal vectors. Here is an example.

Example 4.2.6

Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?



Solution. Let $\mathbf{w}$ denote the weight (force due to gravity) exerted on the block. Then $\|\mathbf{w}\| = 10$ kilograms and the direction of $\mathbf{w}$ is vertically down as in the diagram. The idea is to write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is parallel to the inclined surface and $\mathbf{w}_2$ is perpendicular to the surface. Since there is no friction, the force required is $-\mathbf{w}_1$ because the force $\mathbf{w}_2$ has no effect parallel to the surface. As the angle between $\mathbf{w}$ and $\mathbf{w}_2$ is 30° in the diagram, we have $\frac{\|\mathbf{w}_1\|}{\|\mathbf{w}\|} = \sin 30° = \frac{1}{2}$. Hence $\|\mathbf{w}_1\| = \frac{1}{2}\|\mathbf{w}\| = \frac{1}{2}10 = 5$. Thus the required force has a magnitude of 5 kilograms weight directed up the surface.

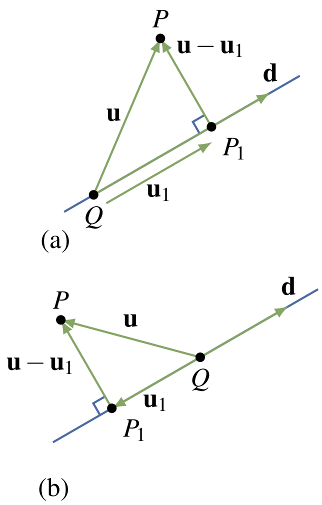


Figure 4.2.5

If a nonzero vector $\mathbf{d}$ is specified, the key idea in Example 4.2.6 is to be able to write an arbitrary vector $\mathbf{u}$ as a sum of two vectors,

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$. Suppose that $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ emanate from a common tail Q (see Figure 4.2.5). Let P be the tip of $\mathbf{u}$, and let P_1 denote the foot of the perpendicular from P to the line through Q parallel to $\mathbf{d}$.

Then $\mathbf{u}_1 = \overrightarrow{QP}_1$ has the required properties:

1. $\mathbf{u}_1$ *is parallel to* $\mathbf{d}$.
2. $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ *is orthogonal to* $\mathbf{d}$.
3. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 4.6 Projection in $\mathbb{R}^3$

*The vector $\mathbf{u}_1 = \overrightarrow{QP}_1$ in Figure 4.2.5 is called **the projection** of $\mathbf{u}$ on $\mathbf{d}$. It is denoted*

$$\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$$

In Figure 4.2.5(a) the vector $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$ (Figure 4.2.5(b)). Note that the projection $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4.2.4

Let $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ be vectors.

1. *The projection of $\mathbf{u}$ on $\mathbf{d}$ is given by* $\operatorname{proj}_{\mathbf{d}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2} \mathbf{d}$.
2. *The vector $\mathbf{u} - \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is orthogonal to $\mathbf{d}$.*

Proof. The vector $\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$ is parallel to $\mathbf{d}$ and so has the form $\mathbf{u}_1 = t\mathbf{d}$ for some scalar t. The requirement that $\mathbf{u} - \mathbf{u}_1$ and $\mathbf{d}$ are orthogonal determines t. In fact, it means that $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{d} = 0$ by Theorem 4.2.3. If $\mathbf{u}_1 = t\mathbf{d}$ is substituted here, the condition is

$$0 = (\mathbf{u} - t\mathbf{d}) \cdot \mathbf{d} = \mathbf{u} \cdot \mathbf{d} - t(\mathbf{d} \cdot \mathbf{d}) = \mathbf{u} \cdot \mathbf{d} - t\|\mathbf{d}\|^2$$

It follows that $t = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$, where the assumption that $\mathbf{d} \neq \mathbf{0}$ guarantees that $\|\mathbf{d}\|^2 \neq 0$. □

Example 4.2.7

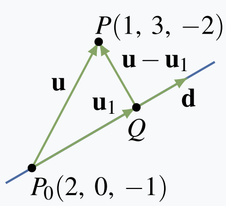
Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution. The projection $\mathbf{u}_1$ of $\mathbf{u}$ on $\mathbf{d}$ is

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{2+3+3}{1^2+(-1)^2+3^2}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \frac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$, and this is orthogonal to $\mathbf{d}$ by Theorem 4.2.4 (alternatively, observe that $\mathbf{d}\cdot\mathbf{u}_2 = 0$). Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, we are done.

Example 4.2.8



Find the shortest distance (see diagram) from the point $P(1,\ 3,\ -2)$ to the line through $P_0(2,\ 0,\ -1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

Solution. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ denote the vector from P_0 to P, and let $\mathbf{u}_1$ denote the projection of $\mathbf{u}$ on $\mathbf{d}$. Thus

$$\mathbf{u}_1 = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{-1-3+0}{1^2+(-1)^2+0^2}\mathbf{d} = -2\mathbf{d} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

by Theorem 4.2.4. We see geometrically that the point Q on the line is closest to P, so the distance is

$$\|\overrightarrow{QP}\| = \|\mathbf{u} - \mathbf{u}_1\| = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \sqrt{3}$$

To find the coordinates of Q, let $\mathbf{p}_0$ and $\mathbf{q}$ denote the vectors of P_0 and Q, respectively. Then $\mathbf{p}_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \mathbf{p}_0 + \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Hence $Q(0,\ 2,\ -1)$ is the required point. It can be checked that the distance from Q to P is $\sqrt{3}$, as expected.

Planes

It is evident geometrically that among all planes that are perpendicular to a given straight line there is exactly one containing any given point. This fact can be used to give a very simple description of a plane. To do this, it is necessary to introduce the following notion:

Definition 4.7 Normal Vector in a Plane

A nonzero vector $\mathbf{n}$ *is called a* ***normal*** *for a plane if it is orthogonal to every vector in the plane.*

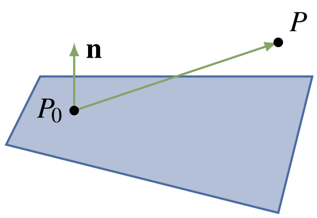


Figure 4.2.6

For example, the coordinate vector $\mathbf{k}$ is a normal for the x-y plane.

Given a point $P_0 = P_0(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$, shaded in Figure 4.2.6. A point $P = P(x, y, z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$—that is, if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. Because $\overrightarrow{P_0P} = \left[\begin{array}{c} x - x_0 \\ y - y_0 \\ z - z_0 \end{array} \right]$ this gives the following result:

Scalar Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ *with normal* $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right] \neq \mathbf{0}$ *as a normal vector is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, a point $P(x, y, z)$ *is on this plane if and only if* x, y, *and* z *satisfy this equation.*

Example 4.2.9

Find an equation of the plane through $P_0(1, -1, 3)$ with $\mathbf{n} = \left[\begin{array}{r} 3 \\ -1 \\ 2 \end{array} \right]$ as normal.

Solution. Here the general scalar equation becomes

$$3(x - 1) - (y + 1) + 2(z - 3) = 0$$

This simplifies to $3x - y + 2z = 10$.

If we write $d = ax_0 + by_0 + cz_0$, the scalar equation shows that every plane with normal $\mathbf{n} = \left[\begin{array}{c} a \\ b \\ c \end{array} \right]$ has a linear equation of the form

$$ax + by + cz = d \tag{4.2}$$

for some constant d. Conversely, the graph of this equation is a plane with $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a, b, and c are not all zero).

Example 4.2.10

Find an equation of the plane through $P_0(3,\ -1,\ 2)$ that is parallel to the plane with equation $2x - 3y = 6$.

Solution. The plane with equation $2x - 3y = 6$ has normal $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x - 3y = d$ for some d by Equation 4.2. Insisting that $P_0(3,\ -1,\ 2)$ lies on the plane determines d; that is, $d = 2 \cdot 3 - 3(-1) = 9$. Hence, the equation is $2x - 3y = 9$.

Consider points $P_0(x_0,\ y_0,\ z_0)$ and $P(x,\ y,\ z)$ with vectors $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given a nonzero vector $\mathbf{n}$, the scalar equation of the plane through $P_0(x_0,\ y_0,\ z_0)$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ takes the vector form:

Vector Equation of a Plane

The plane with normal $\mathbf{n} \neq \mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$$

In other words, the point with vector $\mathbf{p}$ is on the plane if and only if $\mathbf{p}$ satisfies this condition.

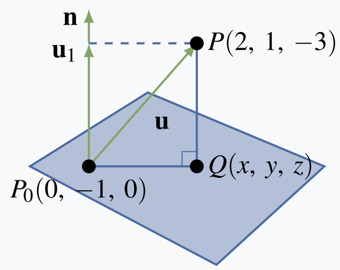
Moreover, Equation 4.2 translates as follows:

Every plane with normal $\mathbf{n}$ has vector equation $\mathbf{n} \cdot \mathbf{p} = d$ for some number d.

This is useful in the second solution of Example 4.2.11.

Example 4.2.11

Find the shortest distance from the point $P(2,\ 1,\ -3)$ to the plane with equation $3x - y + 4z = 1$. Also find the point Q on this plane closest to P.



Solution 1. The plane in question has normal $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$.

Choose any point P_0 on the plane—say $P_0(0,\ -1,\ 0)$—and let $Q(x,\ y,\ z)$ be the point on the plane closest to P (see the diagram).

The vector from P_0 to P is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Now erect $\mathbf{n}$ with its tail at P_0. Then $\overrightarrow{QP} = \mathbf{u}_1$ and $\mathbf{u}_1$ is the projection of $\mathbf{u}$ on $\mathbf{n}$:

$$\mathbf{u}_1 = \frac{\mathbf{n}\cdot\mathbf{u}}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-8}{26}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{-4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Hence the distance is $\|\overrightarrow{QP}\| = \|\mathbf{u}_1\| = \frac{4\sqrt{26}}{13}$. To calculate the point Q, let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p}_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ be the vectors of Q and P_0. Then

$$\mathbf{q} = \mathbf{p}_0 + \mathbf{u} - \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{38}{13} \\ \frac{9}{13} \\ \frac{-23}{13} \end{bmatrix}$$

This gives the coordinates of $Q(\frac{38}{13},\ \frac{9}{13},\ \frac{-23}{13})$.

Solution 2. Let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ be the vectors of Q and P. Then Q is on the line through P with direction vector $\mathbf{n}$, so $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ for some scalar t. In addition, Q lies on the plane, so $\mathbf{n}\cdot\mathbf{q} = 1$. This determines t:

$$1 = \mathbf{n}\cdot\mathbf{q} = \mathbf{n}\cdot(\mathbf{p} + t\mathbf{n}) = \mathbf{n}\cdot\mathbf{p} + t\|\mathbf{n}\|^2 = -7 + t(26)$$

This gives $t = \frac{8}{26} = \frac{4}{13}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{q} = \mathbf{p} + t\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{13}\begin{bmatrix} 38 \\ 9 \\ -23 \end{bmatrix}$$

as before. This determines Q (in the diagram), and the reader can verify that the required distance is $\|\overrightarrow{QP}\| = \frac{4}{13}\sqrt{26}$, as before.

The Cross Product

If P, Q, and R are three distinct points in $\mathbb{R}^3$ that are not all on some line, it is clear geometrically that there is a unique plane containing all three. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in this plane, so finding a normal amounts to finding a nonzero vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product provides a systematic way to do this.

Definition 4.8 Cross Product

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* ***cross product*** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

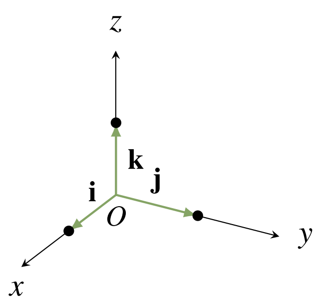


Figure 4.2.7

(Because it is a vector, $\mathbf{v}_1 \times \mathbf{v}_2$ is often called the **vector product**.) There is an easy way to remember this definition using the **coordinate vectors**:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

They are vectors of length 1 pointing along the positive x, y, and z axes, respectively, as in Figure 4.2.7. The reason for the name is that any vector can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

With this, the cross product can be described as follows:

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k}$$

where the determinant is expanded along the first column.

Example 4.2.12

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, then

$$\begin{aligned}
\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & -1 & 3 \\ \mathbf{k} & 4 & 7 \end{bmatrix} &= \begin{vmatrix} -1 & 3 \\ 4 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\
&= -19\mathbf{i} - 10\mathbf{j} + 7\mathbf{k} \\
&= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix}
\end{aligned}$$

Observe that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ in Example 4.2.12. This holds in general as can be verified directly by computing $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$, and is recorded as the first part of the following theorem. It will follow from a more general result which, together with the second part, will be proved in Section 4.3 where a more detailed study of the cross product will be undertaken.

Theorem 4.2.5

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.

1. *$\mathbf{v} \times \mathbf{w}$ is a vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.*
2. *If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.*

It is interesting to contrast Theorem 4.2.5(2) with the assertion (in Theorem 4.2.3) that

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if and only if } \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

Example 4.2.13

Find the equation of the plane through $P(1,\ 3,\ -2)$, $Q(1,\ 1,\ 5)$, and $R(2,\ -2,\ 3)$.

Solution. The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \mathbf{i} & 0 & 1 \\ \mathbf{j} & -2 & -5 \\ \mathbf{k} & 7 & 5 \end{bmatrix} = 25\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

is a normal for the plane (being orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$). Hence the plane has equation

$$25x + 7y + 2z = d \quad \text{for some number } d.$$

Since $P(1,\ 3,\ -2)$ lies in the plane we have $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$. Hence $d = 42$ and the equation is $25x + 7y + 2z = 42$. Incidentally, the same equation is obtained (verify) if $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$, are used as the vectors in the plane.

Example 4.2.14

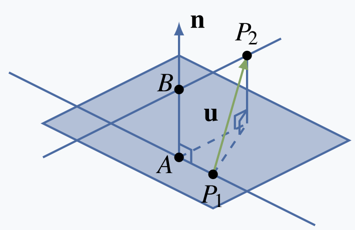
Find the shortest distance between the nonparallel lines

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] + t \left[\begin{array}{c} 2 \\ 0 \\ 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 3 \\ 1 \\ 0 \end{array}\right] + s \left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right]$$

Then find the points A and B on the lines that are closest together.

Solution. Direction vectors for the two lines are $\mathbf{d}_1 = \left[\begin{array}{c} 2 \\ 0 \\ 1 \end{array}\right]$ and $\mathbf{d}_2 = \left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right]$, so

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \det \left[\begin{array}{ccr} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 1 & -1 \end{array}\right] = \left[\begin{array}{r} -1 \\ 3 \\ 2 \end{array}\right]$$



is perpendicular to both lines. Consider the plane shaded in the diagram containing the first line with $\mathbf{n}$ as normal. This plane contains $P_1(1,\ 0,\ -1)$ and is parallel to the second line. Because $P_2(3,\ 1,\ 0)$ is on the second line, the distance in question is just the shortest distance between $P_2(3,\ 1,\ 0)$ and this plane. The vector $\mathbf{u}$ from P_1 to P_2 is $\mathbf{u} = \overrightarrow{P_1P_2} = \left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right]$ and so, as in Example 4.2.11, the distance is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$.

$$\text{distance } = \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} \right\| = \frac{|\mathbf{u} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

Note that it is necessary that $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ be nonzero for this calculation to be possible. As is shown later (Theorem 4.3.4), this is guaranteed by the fact that $\mathbf{d}_1$ and $\mathbf{d}_2$ are *not* parallel.
The points A and B have coordinates $A(1 + 2t,\ 0,\ t - 1)$ and $B(3 + s,\ 1 + s,\ -s)$ for some s and t, so $\overrightarrow{AB} = \left[\begin{array}{c} 2 + s - 2t \\ 1 + s \\ 1 - s - t \end{array}\right]$. This vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$, and the conditions $\overrightarrow{AB} \cdot \mathbf{d}_1 = 0$ and $\overrightarrow{AB} \cdot \mathbf{d}_2 = 0$ give equations $5t - s = 5$ and $t - 3s = 2$. The solution is $s = \frac{-5}{14}$ and $t = \frac{13}{14}$, so the points are $A(\frac{40}{14},\ 0,\ \frac{-1}{14})$ and $B(\frac{37}{14},\ \frac{9}{14},\ \frac{5}{14})$. We have $\|\overrightarrow{AB}\| = \frac{3\sqrt{14}}{14}$, as before.

Exercises for 4.2

Exercise 4.2.1 Compute $\mathbf{u} \cdot \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \mathbf{u}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ -5 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{v} = \mathbf{0}$

Exercise 4.2.2 Find the angle between the following pairs of vectors.

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

e. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

f. $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5\sqrt{2} \\ -7 \\ -1 \end{bmatrix}$

Exercise 4.2.3 Find all real numbers x such that:

a. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

b. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

Exercise 4.2.4 Find all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both:

a. $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

b. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

c. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

d. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Exercise 4.2.5 Find two orthogonal vectors that are both orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Exercise 4.2.6 Consider the triangle with vertices $P(2, 0, -3)$, $Q(5, -2, 1)$, and $R(7, 5, 3)$.

a. Show that it is a right-angled triangle.

b. Find the lengths of the three sides and verify the Pythagorean theorem.

Exercise 4.2.7 Show that the triangle with vertices $A(4, -7, 9)$, $B(6, 4, 4)$, and $C(7, 10, -6)$ is not a right-angled triangle.

Exercise 4.2.8 Find the three internal angles of the triangle with vertices:

a. $A(3, 1, -2)$, $B(3, 0, -1)$, and $C(5, 2, -1)$

b. $A(3, 1, -2)$, $B(5, 2, -1)$, and $C(4, 3, -3)$

Exercise 4.2.9 Show that the line through $P_0(3, 1, 4)$ and $P_1(2, 1, 3)$ is perpendicular to the line through $P_2(1, -1, 2)$ and $P_3(0, 5, 3)$.

Exercise 4.2.10 In each case, compute the projection of $\mathbf{u}$ on $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

Exercise 4.2.11 In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

a. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix}$

Exercise 4.2.12 Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

a. $P(3, 2-1)$
line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

b. $P(1, -1, 3)$
line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

Exercise 4.2.13 Compute $\mathbf{u} \times \mathbf{v}$ where:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

c. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

d. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

Exercise 4.2.14 Find an equation of each of the following planes.

a. Passing through $A(2, 1, 3)$, $B(3, -1, 5)$, and $C(1, 2, -3)$.

b. Passing through $A(1, -1, 6)$, $B(0, 0, 1)$, and $C(4, 7, -11)$.

c. Passing through $P(2, -3, 5)$ and parallel to the plane with equation $3x - 2y - z = 0$.

d. Passing through $P(3, 0, -1)$ and parallel to the plane with equation $2x - y + z = 3$.

e. Containing $P(3, 0, -1)$ and the line
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

f. Containing $P(2, 1, 0)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

g. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

i. Each point of which is equidistant from $P(2, -1, 3)$ and $Q(1, 1, -1)$.

j. Each point of which is equidistant from $P(0, 1, -1)$ and $Q(2, -1, -3)$.

Exercise 4.2.15 In each case, find a vector equation of the line.

a. Passing through $P(3, -1, 4)$ and perpendicular to the plane $3x - 2y - z = 0$.

b. Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x + y = 1$.

c. Passing through $P(0, 0, 0)$ and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

d. Passing through $P(1, 1, -1)$, and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

e. Passing through $P(2, 1, -1)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and perpendicular to that line.

f. Passing through $P(1, 1, 2)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and perpendicular to that line.

Exercise 4.2.16 In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

a. $P(2, 3, 0)$; plane with equation $5x + y + z = 1$.

b. $P(3, 1, -1)$; plane with equation $2x + y - z = 6$.

Exercise 4.2.17

a. Does the line through $P(1, 2, -3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x - y - z = 3$? Explain.

b. Does the plane through $P(4, 0, 5)$, $Q(2, 2, 1)$, and $R(1, -1, 2)$ pass through the origin? Explain.

Exercise 4.2.18 Show that every plane containing $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain $R(-1, 6, -5)$.

Exercise 4.2.19 Find the equations of the line of intersection of the following planes.

a. $2x - 3y + 2z = 5$ and $x + 2y - z = 4$.

b. $3x + y - 2z = 1$ and $x + y + z = 5$.

Exercise 4.2.20 In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

a. $x - 3y + 2z = 4$

b. $2x - y - z = 5$

c. $3x - y + z = 8$

d. $-x - 4y - 3z = 6$

Exercise 4.2.21 Find the equation of *all* planes:

a. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

b. Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

c. Containing the origin.

d. Containing $P(3, 2, -4)$.

e. Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

f. Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

g. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

h. Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

Exercise 4.2.22 If a plane contains two distinct points P_1 and P_2, show that it contains every point on the line through P_1 and P_2.

Exercise 4.2.23 Find the shortest distance between the following pairs of parallel lines.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Exercise 4.2.24 Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$;
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Exercise 4.2.25 Show that two lines in the plane with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. [*Hint*: Example 4.1.11.]

Exercise 4.2.26

a. Show that, of the four diagonals of a cube, no pair is perpendicular.

b. Show that each diagonal is perpendicular to the face diagonals it does not meet.

Exercise 4.2.27 Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, find the angle between a diagonal and one of the longest sides.

Exercise 4.2.28 Consider a rectangular solid with sides of lengths a, b, and c. Show that it has two orthogonal diagonals if and only if the sum of two of a^2, b^2, and c^2 equals the third.

Exercise 4.2.29 Let A, B, and $C(2, -1, 1)$ be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\left[\begin{array}{r} 2 \\ 0 \\ -1 \end{array}\right]$, and angle $C = 90^\circ$. Find the equation of the line through B and C.

Exercise 4.2.30 If the diagonals of a parallelogram have equal length, show that the parallelogram is a rectangle.

Exercise 4.2.31 Given $\mathbf{v} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ in component form, show that the projections of $\mathbf{v}$ on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$, respectively.

Exercise 4.2.32

a. Can $\mathbf{u} \cdot \mathbf{v} = -7$ if $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 2$? Defend your answer.

b. Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \left[\begin{array}{r} 2 \\ -1 \\ 2 \end{array}\right]$, $\|\mathbf{v}\| = 6$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

Exercise 4.2.33 Show $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

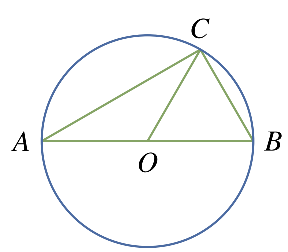
Exercise 4.2.34

a. Show $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

b. What does this say about parallelograms?

Exercise 4.2.35 Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus. [*Hint*: Example 4.2.5.]

Exercise 4.2.36 Let A and B be the end points of a diameter of a circle (see the diagram). If C is any point on the circle, show that AC and BC are perpendicular. [*Hint*: Express $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$ and $\overrightarrow{BC}$ in terms of $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$, where O is the centre.]



Exercise 4.2.37 Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Exercise 4.2.38 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be pairwise orthogonal vectors.

a. Show that $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

b. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all the same length, show that they all make the same angle with $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

Exercise 4.2.39

a. Show that $\mathbf{n} = \left[\begin{array}{c} a \\ b \end{array}\right]$ is orthogonal to every vector along the line $ax + by + c = 0$.

b. Show that the shortest distance from $P_0(x_0,\ y_0)$ to the line is $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.

[*Hint*: If P_1 is on the line, project $\mathbf{u} = \overrightarrow{P_1P_0}$ on $\mathbf{n}$.]

Exercise 4.2.40 Assume $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors that are not parallel. Show that $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ is a nonzero vector that bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

Exercise 4.2.41 Let α, β, and γ be the angles a vector $\mathbf{v} \neq \mathbf{0}$ makes with the positive x, y, and z axes, respectively. Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

a. If $\mathbf{v} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$, show that $\cos\alpha = \frac{a}{\|\mathbf{v}\|}$, $\cos\beta = \frac{b}{\|\mathbf{v}\|}$, and $\cos\gamma = \frac{c}{\|\mathbf{v}\|}$.

b. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Exercise 4.2.42 Let $\mathbf{v} \neq \mathbf{0}$ be any nonzero vector and suppose that a vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ is parallel to $\mathbf{v}$ and $\mathbf{q}$ is orthogonal to $\mathbf{v}$. Show that $\mathbf{p}$ must equal the projection of $\mathbf{u}$ on $\mathbf{v}$. [*Hint*: Argue as in the proof of Theorem 4.2.4.]

Exercise 4.2.43 Let $\mathbf{v} \neq \mathbf{0}$ be a nonzero vector and let $a \neq 0$ be a scalar. If $\mathbf{u}$ is any vector, show that the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u}$ on $a\mathbf{v}$.

Exercise 4.2.44

a. Show that the **Cauchy-Schwarz inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: $|\cos\theta| \leq 1$ for all angles θ.]

b. Show that $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

[*Hint*: When is $\cos\theta = \pm 1$?]

c. Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

d. Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$ for all x, y, and z.

e. Show that $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ holds for all x, y, and z.

Exercise 4.2.45 Prove that the **triangle inequality** $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: Consider the triangle with $\mathbf{u}$ and $\mathbf{v}$ as two sides.]

4.3 More on the Cross Product

The cross product $\mathbf{v} \times \mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$ and $\mathbf{w} = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v} \times \mathbf{w} = \det \left[\begin{array}{ccc} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{array}\right] = \left|\begin{array}{cc} y_1 & y_2 \\ z_1 & z_2 \end{array}\right| \mathbf{i} - \left|\begin{array}{cc} x_1 & x_2 \\ z_1 & z_2 \end{array}\right| \mathbf{j} + \left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| \mathbf{k} \tag{4.3}$$

Here $\mathbf{i} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$, $\mathbf{j} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$, and $\mathbf{k} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 4.2.5 that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 4.3.1

If $\mathbf{u} = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right]$, $\mathbf{v} = \left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]$, *and* $\mathbf{w} = \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$, *then* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \left[\begin{array}{ccc} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{array}\right]$.

Proof. Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ and then adding. Using equation (4.3), the result is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = x_0 \left(\left|\begin{array}{cc} y_1 & y_2 \\ z_1 & z_2 \end{array}\right| \right) + y_0 \left(- \left|\begin{array}{cc} x_1 & x_2 \\ z_1 & z_2 \end{array}\right| \right) + z_0 \left(\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right| \right) = \det \left[\begin{array}{ccc} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{array}\right]$$

where the last determinant is expanded along column 1. □

The result in Theorem 4.3.1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \left[\begin{array}{ccc} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}\right]$$

where $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denotes the matrix with **u**, **v**, and **w** as its columns. Now it is clear that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both **v** and **w** because the determinant of a matrix is zero if two columns are identical.

Because of (4.3) and Theorem 4.3.1, several of the following properties of the cross product follow from properties of determinants (they can also be verified directly).

Theorem 4.3.2

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote arbitrary vectors in* $\mathbb{R}^3$.

1. $\mathbf{u} \times \mathbf{v}$ *is a vector.*
2. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. $\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$.
4. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
5. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
6. $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$ *for any scalar* k.
7. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.
8. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$.

Proof. (1) is clear; (2) follows from Theorem 4.3.1; and (3) and (4) follow because the determinant of a matrix is zero if one column is zero or if two columns are identical. If two columns are interchanged, the determinant changes sign, and this proves (5). The proofs of (6), (7), and (8) are left as Exercise 4.3.15. □

We now come to a fundamental relationship between the dot and cross products.

Theorem 4.3.3: Lagrange Identity[12]

If $\mathbf{u}$ *and* $\mathbf{v}$ *are any two vectors in* $\mathbb{R}^3$, *then*

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

Proof. Given **u** and **v**, introduce a coordinate system and write $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in component form. Then all the terms in the identity can be computed in terms of the components. The detailed proof is left as Exercise 4.3.14. □

An expression for the magnitude of the vector $\mathbf{u} \times \mathbf{v}$ can be easily obtained from the Lagrange identity. If θ is the angle between **u** and **v**, substituting $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$ into the Lagrange identity gives

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta$$

[12] Joseph Louis Lagrange (1736–1813) was born in Italy and spent his early years in Turin. At the age of 19 he solved a famous problem by inventing an entirely new method, known today as the calculus of variations, and went on to become one of the greatest mathematicians of all time. His work brought a new level of rigour to analysis and his *Mécanique Analytique* is a masterpiece in which he introduced methods still in use. In 1766 he was appointed to the Berlin Academy by Frederik the Great who asserted that the "greatest mathematician in Europe" should be at the court of the "greatest king in Europe." After the death of Frederick, Lagrange went to Paris at the invitation of Louis XVI. He remained there throughout the revolution and was made a count by Napoleon.

using the fact that $1-\cos^2\theta = \sin^2\theta$. But $\sin\theta$ is nonnegative on the range $0 \le \theta \le \pi$, so taking the positive square root of both sides gives

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

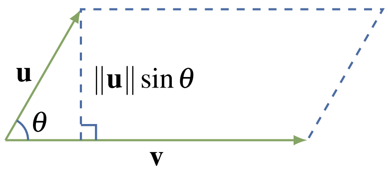


Figure 4.3.1

This expression for $\|\mathbf{u}\times\mathbf{v}\|$ makes no reference to a coordinate system and, moreover, it has a nice geometrical interpretation. The parallelogram determined by the vectors $\mathbf{u}$ and $\mathbf{v}$ has base length $\|\mathbf{v}\|$ and altitude $\|\mathbf{u}\|\sin\theta$ (see Figure 4.3.1). Hence the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is

$$(\|\mathbf{u}\|\sin\theta)\|\mathbf{v}\| = \|\mathbf{u}\times\mathbf{v}\|$$

This proves the first part of Theorem 4.3.4.

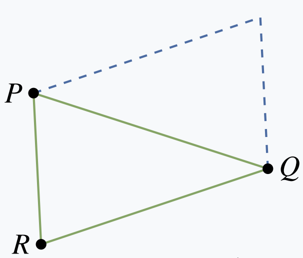
Theorem 4.3.4

If $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

1. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta =$ *the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.*
2. *$\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.*

Proof of (2). By (1), $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if the area of the parallelogram is zero. By Figure 4.3.1 the area vanishes if and only if $\mathbf{u}$ and $\mathbf{v}$ have the same or opposite direction—that is, if and only if they are parallel. □

Example 4.3.1



Find the area of the triangle with vertices $P(2,\ 1,\ 0)$, $Q(3,\ -1,\ 1)$, and $R(1,\ 0,\ 1)$.

Solution. We have $\overrightarrow{RP} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\overrightarrow{RQ} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. The area of the triangle is half the area of the parallelogram (see the diagram), and so equals $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\|$. We have

$$\overrightarrow{RP}\times\overrightarrow{RQ} = \det\begin{bmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{RP}\times\overrightarrow{RQ}\| = \frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$.

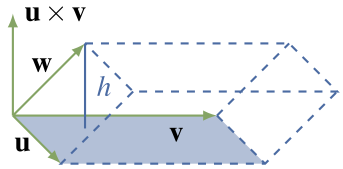


Figure 4.3.2

If three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 4.3.2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, so it has area $A = \|\mathbf{u}\times\mathbf{v}\|$ by Theorem 4.3.4. The height of the solid is the length h of the projection of $\mathbf{w}$ on $\mathbf{u}\times\mathbf{v}$. Hence

$$h = \left|\frac{\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})}{\|\mathbf{u}\times\mathbf{v}\|^2}\right| \|\mathbf{u}\times\mathbf{v}\| = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|} = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|$. This proves

Theorem 4.3.5

The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 4.3.2) is given by $|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|$.

Example 4.3.2

Find the volume of the parallelepiped determined by the vectors

$$\mathbf{w} = \left[\begin{array}{r} 1 \\ 2 \\ -1 \end{array}\right], \ \mathbf{u} = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right], \ \mathbf{v} = \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array}\right]$$

Solution. By Theorem 4.3.1, $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \det\left[\begin{array}{rrr} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] = -3$. Hence the volume is $|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})| = |-3| = 3$ by Theorem 4.3.5.

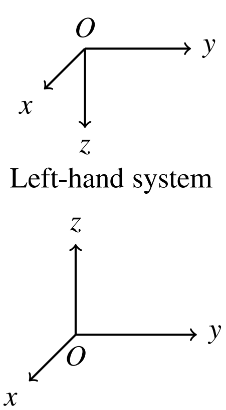


Figure 4.3.3

We can now give an intrinsic description of the cross product $\mathbf{u}\times\mathbf{v}$. Its magnitude $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ is coordinate-free. If $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$, its direction is very nearly determined by the fact that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and so points along the line normal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It remains only to decide which of the two possible directions is correct.

Before this can be done, the basic issue of how coordinates are assigned must be clarified. When coordinate axes are chosen in space, the procedure is as follows: An origin is selected, two perpendicular lines (the x and y axes) are chosen through the origin, and a positive direction on each of these axes is selected quite arbitrarily. Then the line through the origin normal to this x-y plane is called the z axis, but there is a choice of which direction on this axis is the positive one. The two possibilities are shown in Figure 4.3.3, and it is a standard convention that cartesian coordinates are always **right-hand coordinate systems**. The reason for this terminology is that, in such a system, if the z axis is grasped in the right hand with the thumb pointing in the positive z direction, then the fingers curl around from the positive x axis to the positive y axis (through a right angle).

Suppose now that $\mathbf{u}$ and $\mathbf{v}$ are given and that θ is the angle between them (so $0 \leq \theta \leq \pi$). Then the direction of $\|\mathbf{u} \times \mathbf{v}\|$ is given by the right-hand rule.

Right-hand Rule

If the vector $\mathbf{u} \times \mathbf{v}$ is grasped in the right hand and the fingers curl around from $\mathbf{u}$ to $\mathbf{v}$ through the angle θ, the thumb points in the direction for $\mathbf{u} \times \mathbf{v}$.

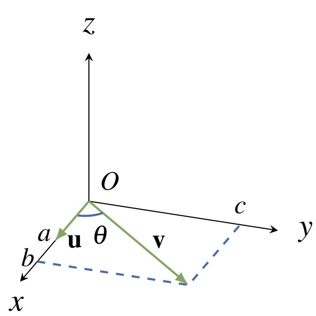


Figure 4.3.4

To indicate why this is true, introduce coordinates in $\mathbb{R}^3$ as follows: Let $\mathbf{u}$ and $\mathbf{v}$ have a common tail O, choose the origin at O, choose the x axis so that $\mathbf{u}$ points in the positive x direction, and then choose the y axis so that $\mathbf{v}$ is in the x-y plane and the positive y axis is on the same side of the x axis as $\mathbf{v}$. Then, in this system, $\mathbf{u}$ and $\mathbf{v}$ have component form $\mathbf{u} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$ where $a > 0$ and $c > 0$. The situation is depicted in Figure 4.3.4. The right-hand rule asserts that $\mathbf{u} \times \mathbf{v}$ should point in the positive z direction. But our definition of $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & a & b \\ \mathbf{j} & 0 & c \\ \mathbf{k} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ac \end{bmatrix} = (ac)\mathbf{k}$$

and $(ac)\mathbf{k}$ has the positive z direction because $ac > 0$.

Exercises for 4.3

Exercise 4.3.1 If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the coordinate vectors, verify that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Exercise 4.3.2 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ need not equal $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ by calculating both when

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise 4.3.3 Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ if:

a. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

b. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Exercise 4.3.4 Find the area of the triangle with the following vertices.

a. $A(3, -1, 2)$, $B(1, 1, 0)$, and $C(1, 2, -1)$

b. $A(3, 0, 1)$, $B(5, 1, 0)$, and $C(7, 2, -1)$

c. $A(1, 1, -1)$, $B(2, 0, 1)$, and $C(1, -1, 3)$

d. $A(3, -1, 1)$, $B(4, 1, 0)$, and $C(2, -3, 0)$

Exercise 4.3.5 Find the volume of the parallelepiped determined by **w**, **u**, and **v** when:

a. $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

b. $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Exercise 4.3.6 Let P_0 be a point with vector $\mathbf{p}_0$, and let $ax+by+cz=d$ be the equation of a plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

a. Show that the point on the plane closest to P_0 has vector $\mathbf{p}$ given by

$$\mathbf{p} = \mathbf{p}_0 + \frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}.$$

[*Hint*: $\mathbf{p} = \mathbf{p}_0 + t\mathbf{n}$ for some t, and $\mathbf{p} \cdot \mathbf{n} = \mathbf{d}$.]

b. Show that the shortest distance from P_0 to the plane is $\frac{|d-(\mathbf{p}_0 \cdot \mathbf{n})|}{\|\mathbf{n}\|}$.

c. Let P_0' denote the reflection of P_0 in the plane—that is, the point on the opposite side of the plane such that the line through P_0 and P_0' is perpendicular to the plane.

Show that $\mathbf{p}_0 + 2\frac{d-(\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}$ is the vector of P_0'.

Exercise 4.3.7 Simplify $(a\mathbf{u}+b\mathbf{v}) \times (c\mathbf{u}+d\mathbf{v})$.

Exercise 4.3.8 Show that the shortest distance from a point P to the line through P_0 with direction vector $\mathbf{d}$ is $\frac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}$.

Exercise 4.3.9 Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero, nonorthogonal vectors. If θ is the angle between them, show that $\tan\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\mathbf{u} \cdot \mathbf{v}}$.

Exercise 4.3.10 Show that points A, B, and C are all on one line if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = 0$

Exercise 4.3.11 Show that points A, B, C, and D are all on one plane if and only if $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = 0$

Exercise 4.3.12 Use Theorem 4.3.5 to confirm that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, the (rectangular) parallelepiped they determine has volume $\|\mathbf{u}\|\|\mathbf{v}\|\|\mathbf{w}\|$.

Exercise 4.3.13 Show that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|^2$.

Exercise 4.3.14 Complete the proof of Theorem 4.3.3.

Exercise 4.3.15 Prove the following properties in Theorem 4.3.2.

a. Property 6

b. Property 7

c. Property 8

Exercise 4.3.16

a. Show that $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{w} \times \mathbf{u})$ holds for all vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$.

b. Show that $\mathbf{v} - \mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u})$ are orthogonal.

Exercise 4.3.17 Show $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \times \mathbf{v})\mathbf{w}$. [*Hint*: First do it for $\mathbf{u} = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; then write $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and use Theorem 4.3.2.]

Exercise 4.3.18 Prove the **Jacobi identity**:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

[*Hint*: The preceding exercise.]

Exercise 4.3.19 Show that

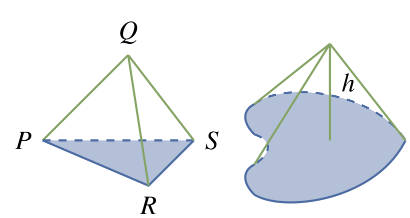
$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = \det \begin{bmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{u} \cdot \mathbf{z} \\ \mathbf{v} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{z} \end{bmatrix}$$

[*Hint*: Exercises 4.3.16 and 4.3.17.]

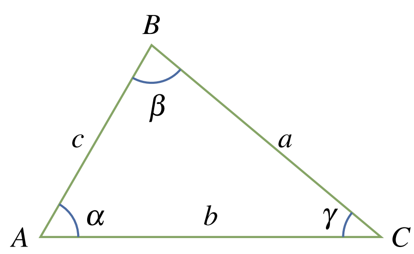
Exercise 4.3.20 Let P, Q, R, and S be four points, not all on one plane, as in the diagram. Show that the volume of the pyramid they determine is

$$\frac{1}{6}|\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS})|.$$

[*Hint*: The volume of a cone with base area A and height h as in the diagram below right is $\frac{1}{3}Ah$.]



Exercise 4.3.21 Consider a triangle with vertices A, B, and C, as in the diagram below. Let α, β, and γ denote the angles at A, B, and C, respectively, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Write $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{CA}$.



a. Deduce that $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{0}$.

b. Show that $\mathbf{u}\times\mathbf{v} = \mathbf{w}\times\mathbf{u} = \mathbf{v}\times\mathbf{w}$. [*Hint*: Compute $\mathbf{u}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$ and $\mathbf{v}\times(\mathbf{u}+\mathbf{v}+\mathbf{w})$.]

c. Deduce the **law of sines**:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Exercise 4.3.22 Show that the (shortest) distance between two planes $\mathbf{n}\cdot\mathbf{p} = d_1$ and $\mathbf{n}\cdot\mathbf{p} = d_2$ with $\mathbf{n}$ as normal is $\frac{|d_2-d_1|}{\|\mathbf{n}\|}$.

Exercise 4.3.23 Let A and B be points other than the origin, and let $\mathbf{a}$ and $\mathbf{b}$ be their vectors. If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, show that the plane through A, B, and the origin is given by

$$\left\{P(x,\ y,\ z) \mid \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = s\mathbf{a}+t\mathbf{b} \text{ for some } s \text{ and } t\right\}$$

Exercise 4.3.24 Let A be a 2×3 matrix of rank 2 with rows $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that

$$P = \{XA \mid X = [xy];x,\ y \text{ arbitrary}\}$$

is the plane through the origin with normal $\mathbf{r}_1\times\mathbf{r}_2$.

Exercise 4.3.25 Given the cube with vertices $P(x,\ y,\ z)$, where each of x, y, and z is either 0 or 2, consider the plane perpendicular to the diagonal through $P(0,\ 0,\ 0)$ and $P(2,\ 2,\ 2)$ and bisecting it.

a. Show that the plane meets six of the edges of the cube and bisects them.

b. Show that the six points in (a) are the vertices of a regular hexagon.

4.4 Linear Operators on $\mathbb{R}^3$

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. In this case we showed (in Theorem 2.6.2) that there exists an $m\times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and we say that T is the **matrix transformation induced** by A.

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

*is called a **linear operator** on $\mathbb{R}^n$.*

In Section 2.6 we investigated three important linear operators on $\mathbb{R}^2$: rotations about the origin, reflections in a line through the origin, and projections on this line.

In this section we investigate the analogous operators on $\mathbb{R}^3$: Rotations about a line through the origin, reflections in a plane through the origin, and projections onto a plane or line through the origin in $\mathbb{R}^3$. In every case we show that the operator is linear, and we find the matrices of all the reflections and projections.

To do this we must prove that these reflections, projections, and rotations are actually *linear* operators on $\mathbb{R}^3$. In the case of reflections and rotations, it is convenient to examine a more general situation. A transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be **distance preserving** if the distance between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the distance between $\mathbf{v}$ and $\mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$; that is,

$$\|T(\mathbf{v}) - T(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\| \text{ for all } \mathbf{v} \text{ and } \mathbf{w} \text{ in } \mathbb{R}^3 \tag{4.4}$$

Clearly reflections and rotations are distance preserving, and both carry $\mathbf{0}$ to $\mathbf{0}$, so the following theorem shows that they are both linear.

Theorem 4.4.1

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is distance preserving, and if $T(\mathbf{0}) = \mathbf{0}$, then T is linear.

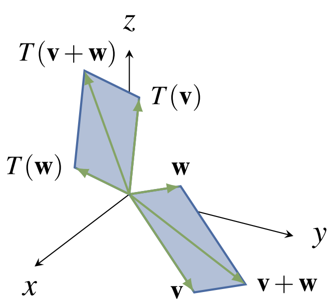


Figure 4.4.1

Proof. Since $T(\mathbf{0}) = \mathbf{0}$, taking $\mathbf{w} = \mathbf{0}$ in (4.4) shows that $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v}$ in $\mathbb{R}^3$, that is T preserves length. Also, $\|T(\mathbf{v}) - T(\mathbf{w})\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$ by (4.4). Since $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|^2$ always holds, it follows that $T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$. Hence (by Theorem 4.2.2) the angle between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$ for all (nonzero) vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

With this we can show that T is linear. Given nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. By the preceding paragraph, the effect of T is to carry this *entire parallelogram* to the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$, with diagonal $T(\mathbf{v} + \mathbf{w})$. But this diagonal is $T(\mathbf{v}) + T(\mathbf{w})$ by the parallelogram law (see Figure 4.4.1).

In other words, $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$. A similar argument shows that $T(a\mathbf{v}) = aT(\mathbf{v})$ for all scalars a, proving that T is indeed linear. □

Distance-preserving linear operators are called **isometries**, and we return to them in Section 10.4.

Reflections and Projections

In Section 2.6 we studied the reflection $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ in the line $y = mx$ and projection $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ on the same line. We found (in Theorems 2.6.5 and 2.6.6) that they are both linear and

$$Q_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \quad \text{and} \quad P_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

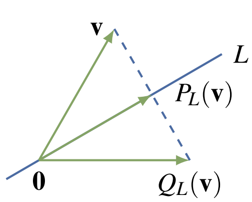


Figure 4.4.2

We now look at the analogues in $\mathbb{R}^3$.

Let L denote a line through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_L(\mathbf{v})$ of $\mathbf{v}$ in L and the projection $P_L(\mathbf{v})$ of $\mathbf{v}$ on L are defined in Figure 4.4.2. In the same figure, we see that

$$P_L(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_L(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_L(\mathbf{v}) + \mathbf{v}] \tag{4.5}$$

so the fact that Q_L is linear (by Theorem 4.4.1) shows that P_L is also linear.[13]

However, Theorem 4.2.4 gives us the matrix of P_L directly. In fact, if $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \neq \mathbf{0}$ is a direction vector for L, and we write $\mathbf{v} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$, then

$$P_L(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{ax+by+cz}{a^2+b^2+c^2}\left[\begin{array}{c} a \\ b \\ c \end{array}\right] = \frac{1}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$$

as the reader can verify. Note that this shows directly that P_L is a matrix transformation and so gives another proof that it is linear.

Theorem 4.4.2

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $\mathbf{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right] \neq \mathbf{0}$. Then P_L and Q_L are both linear and

$$P_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{array}\right]$$

$$Q_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{array}\right]$$

Proof. It remains to find the matrix of Q_L. But (4.5) implies that $Q_L(\mathbf{v}) = 2P_L(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, so if $\mathbf{v} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ we obtain (with some matrix arithmetic):

$$\begin{aligned} Q_L(\mathbf{v}) &= \left\{\frac{2}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{array}\right] - \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right\}\left[\begin{array}{c} x \\ y \\ z \end{array}\right] \\ &= \frac{1}{a^2+b^2+c^2}\left[\begin{array}{ccc} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right] \end{aligned}$$

as required. □

[13]Note that Theorem 4.4.1 does *not* apply to P_L since it does not preserve distance.

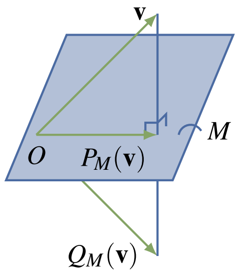


Figure 4.4.3

In $\mathbb{R}^3$ we can reflect in planes as well as lines. Let M denote a plane through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_M(\mathbf{v})$ of $\mathbf{v}$ in M and the projection $P_M(\mathbf{v})$ of $\mathbf{v}$ on M are defined in Figure 4.4.3. As above, we have

$$P_M(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_M(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_M(\mathbf{v}) + \mathbf{v}]$$

so the fact that Q_M is linear (again by Theorem 4.4.1) shows that P_M is also linear.

Again we can obtain the matrix directly. If $\mathbf{n}$ is a normal for the plane M, then Figure 4.4.3 shows that

$$P_M(\mathbf{v}) = \mathbf{v} - \text{proj}_{\mathbf{n}}\, \mathbf{v} = \mathbf{v} - \tfrac{\mathbf{v}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \text{ for all vectors } \mathbf{v}.$$

If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, a computation like the above gives

$$\begin{aligned} P_M(\mathbf{v}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \tfrac{ax+by+cz}{a^2+b^2+c^2} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & b^2+c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

This proves the first part of

Theorem 4.4.3

Let M denote the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_M and Q_M are both linear and

$$P_M \text{ has matrix } \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}$$

$$Q_M \text{ has matrix } \tfrac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

Proof. It remains to compute the matrix of Q_M. Since $Q_M(\mathbf{v}) = 2P_M(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, the computation is similar to the above and is left as an exercise for the reader. □

Rotations

In Section 2.6 we studied the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise about the origin through the angle θ. Moreover, we showed in Theorem 2.6.4 that R_θ is linear and has matrix $\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$. One extension of this is given in the following example.

Example 4.4.1

Let $R_{z,\ \theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation of $\mathbb{R}^3$ about the z axis through an angle θ from the positive x axis toward the positive y axis. Show that $R_{z,\ \theta}$ is linear and find its matrix.

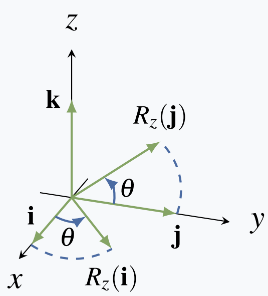


Figure 4.4.4

Solution. First R is distance preserving and so is linear by Theorem 4.4.1. Hence we apply Theorem 2.6.2 to obtain the matrix of $R_{z,\ \theta}$.

Let $\mathbf{i} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$, $\mathbf{j} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$, and $\mathbf{k} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$ denote the standard basis of $\mathbb{R}^3$; we must find $R_{z,\ \theta}(\mathbf{i})$, $R_{z,\ \theta}(\mathbf{j})$, and $R_{z,\ \theta}(\mathbf{k})$. Clearly $R_{z,\ \theta}(\mathbf{k}) = \mathbf{k}$. The effect of $R_{z,\ \theta}$ on the x-y plane is to rotate it counterclockwise through the angle θ. Hence Figure 4.4.4 gives

$$R_{z,\ \theta}(\mathbf{i}) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \\ 0 \end{array}\right], \quad R_{z,\ \theta}(\mathbf{j}) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \\ 0 \end{array}\right]$$

so, by Theorem 2.6.2, $R_{z,\ \theta}$ has matrix

$$\left[\begin{array}{ccc} R_{z,\ \theta}(\mathbf{i}) & R_{z,\ \theta}(\mathbf{j}) & R_{z,\ \theta}(\mathbf{k}) \end{array}\right] = \left[\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Example 4.4.1 begs to be generalized. Given a line L through the origin in $\mathbb{R}^3$, every rotation about L through a fixed angle is clearly distance preserving, and so is a linear operator by Theorem 4.4.1. However, giving a precise description of the matrix of this rotation is not easy and will have to wait until more techniques are available.

Transformations of Areas and Volumes

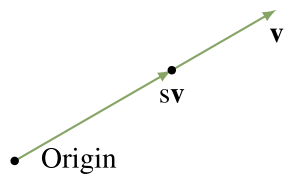


Figure 4.4.5

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Each vector in the same direction as $\mathbf{v}$ whose length is a fraction s of the length of $\mathbf{v}$ has the form $s\mathbf{v}$ (see Figure 4.4.5).

With this, scrutiny of Figure 4.4.6 shows that a vector $\mathbf{u}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ if and only if it has the form $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ where $0 \leq s \leq 1$ and $0 \leq t \leq 1$. But then, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, we have

$$T(s\mathbf{v} + t\mathbf{w}) = T(s\mathbf{v}) + T(t\mathbf{w}) = sT(\mathbf{v}) + tT(\mathbf{w})$$

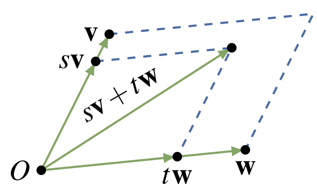


Figure 4.4.6

Hence $T(s\mathbf{v} + t\mathbf{w})$ is in the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$. Conversely, every vector in this parallelogram has the form $T(s\mathbf{v} + t\mathbf{w})$ where $s\mathbf{v} + t\mathbf{w}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. For this reason, the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$ is called the **image** of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. We record this discussion as:

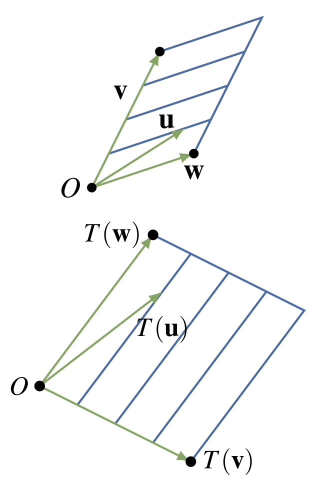


Figure 4.4.7

Theorem 4.4.4

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$) is a linear operator, the image of the parallelogram determined by vectors $\mathbf{v}$ and $\mathbf{w}$ is the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$.

This result is illustrated in Figure 4.4.7, and was used in Examples 2.2.15 and 2.2.16 to reveal the effect of expansion and shear transformations.

We now describe the effect of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ on the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ (see the discussion preceding Theorem 4.3.5). If T has matrix A, Theorem 4.4.4 shows that this parallelepiped is carried to the parallelepiped determined by $T(\mathbf{u}) = A\mathbf{u}$, $T(\mathbf{v}) = A\mathbf{v}$, and $T(\mathbf{w}) = A\mathbf{w}$. In particular, we want to discover how the volume changes, and it turns out to be closely related to the determinant of the matrix A.

Theorem 4.4.5

Let $\mathrm{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ denote the volume of the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, and let $\mathrm{area}(\mathbf{p}, \mathbf{q})$ denote the area of the parallelogram determined by two vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^2$. Then:

1. *If A is a 3×3 matrix, then $\mathrm{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |\det(A)| \cdot \mathrm{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.*
2. *If A is a 2×2 matrix, then $\mathrm{area}(A\mathbf{p}, A\mathbf{q}) = |\det(A)| \cdot \mathrm{area}(\mathbf{p}, \mathbf{q})$.*

Proof.

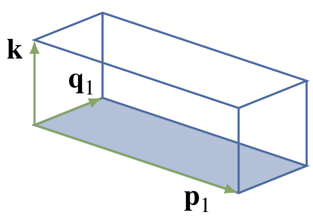
1. Let $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ denote the 3×3 matrix with columns $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then

$$\text{vol}\,(A\mathbf{u},\, A\mathbf{v},\, A\mathbf{w}) = |A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w})|$$

by Theorem 4.3.5. Now apply Theorem 4.3.1 twice to get

$$\begin{aligned} A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w}) = \det \begin{bmatrix} A\mathbf{u} & A\mathbf{v} & A\mathbf{w} \end{bmatrix} &= \det\,(A \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}) \\ &= \det\,(A) \det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \\ &= \det\,(A)(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})) \end{aligned}$$

where we used Definition 2.9 and the product theorem for determinants. Finally (1) follows from Theorem 4.3.5 by taking absolute values.



2. Given $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, $\mathbf{p}_1 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. By the diagram, area $(\mathbf{p},\, \mathbf{q}) = \text{vol}\,(\mathbf{p}_1,\, \mathbf{q}_1,\, \mathbf{k})$ where $\mathbf{k}$ is the (length 1) coordinate vector along the z axis. If A is a 2×2 matrix, write $A_1 = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ in block form, and observe that $(A\mathbf{v})_1 = (A_1\mathbf{v}_1)$ for all $\mathbf{v}$ in $\mathbb{R}^2$ and $A_1\mathbf{k} = \mathbf{k}$. Hence part (1) of this theorem shows

$$\begin{aligned} \text{area}\,(A\mathbf{p},\, A\mathbf{q}) &= \text{vol}\,(A_1\mathbf{p}_1,\, A_1\mathbf{q}_1,\, A_1\mathbf{k}) \\ &= |\det\,(A_1)|\ \text{vol}\,(\mathbf{p}_1,\, \mathbf{q}_1,\, \mathbf{k}) \\ &= |\det\,(A)|\ \text{area}\,(\mathbf{p},\, \mathbf{q}) \end{aligned}$$

as required. □

Define the **unit square** and **unit cube** to be the square and cube corresponding to the coordinate vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then Theorem 4.4.5 gives a geometrical meaning to the determinant of a matrix A:

- If A is a 2×2 matrix, then $|\det\,(A)|$ is the area of the image of the unit square under multiplication by A;
- If A is a 3×3 matrix, then $|\det\,(A)|$ is the volume of the image of the unit cube under multiplication by A.

These results, together with the importance of areas and volumes in geometry, were among the reasons for the initial development of determinants.

Exercises for 4.4

Exercise 4.4.1 In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle.

a. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-y \\ y-x \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -x-y \\ x-y \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$

e. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$

f. $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-\sqrt{3}y \\ \sqrt{3}x+y \end{bmatrix}$

Exercise 4.4.2 Determine the effect of the following transformations.

a. Rotation through $\frac{\pi}{2}$, followed by projection on the y axis, followed by reflection in the line $y=x$.

b. Projection on the line $y=x$ followed by projection on the line $y=-x$.

c. Projection on the x axis followed by reflection in the line $y=x$.

Exercise 4.4.3 In each case solve the problem by finding the matrix of the operator.

a. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x-5y+2z=0$.

b. Find the projection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ on the plane with equation $2x-y+4z=0$.

c. Find the reflection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x-y+3z=0$.

d. Find the reflection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ in the plane with equation $2x+y-5z=0$.

e. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

f. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

g. Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.

h. Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

Exercise 4.4.4

a. Find the rotation of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{4}$.

b. Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

Exercise 4.4.5 Find the matrix of the rotation in $\mathbb{R}^3$ about the x axis through the angle θ (from the positive y axis to the positive z axis).

Exercise 4.4.6 Find the matrix of the rotation about the y axis through the angle θ (from the positive x axis to the positive z axis).

Exercise 4.4.7 If A is 3×3, show that the image of the line in $\mathbb{R}^3$ through $\mathbf{p}_0$ with direction vector $\mathbf{d}$ is the line through $A\mathbf{p}_0$ with direction vector $A\mathbf{d}$, assuming that $A\mathbf{d} \neq \mathbf{0}$. What happens if $A\mathbf{d} = \mathbf{0}$?

Exercise 4.4.8 If A is 3×3 and invertible, show that the image of the plane through the origin with normal $\mathbf{n}$ is the plane through the origin with normal $\mathbf{n}_1 = B\mathbf{n}$ where $B = (A^{-1})^T$. [*Hint*: Use the fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$ to show that $\mathbf{n}_1 \cdot (A\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$ for each $\mathbf{p}$ in $\mathbb{R}^3$.]

Exercise 4.4.9 Let L be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \neq 0$.

a. If P_L denotes projection on L, show that P_L has matrix $\frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

b. If Q_L denotes reflection in L, show that Q_L has matrix $\frac{1}{a^2+b^2}\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$.

Exercise 4.4.10 Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$, let L be the line through the origin with direction vector $\mathbf{n}$, and let M be the plane through the origin with normal $\mathbf{n}$. Show that $P_L(\mathbf{v}) = Q_L(\mathbf{v}) + P_M(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^3$. [In this case, we say that $P_L = Q_L + P_M$.]

Exercise 4.4.11 If M is the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that Q_M has matrix

$$\frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

4.5 An Application to Computer Graphics

Computer graphics deals with images displayed on a computer screen, and so arises in a variety of applications, ranging from word processors, to *Star Wars* animations, to video games, to wire-frame images of an airplane. These images consist of a number of points on the screen, together with instructions on how to fill in areas bounded by lines and curves. Often curves are approximated by a set of short straight-line segments, so that the curve is specified by a series of points on the screen at the end of these segments. Matrix transformations are important here because matrix images of straight line segments are again line segments.[14] Note that a colour image requires that three images are sent, one to each of the red, green, and blue phosphorus dots on the screen, in varying intensities.

Consider displaying the letter A. In reality, it is depicted on the screen, as in Figure 4.5.1, by specifying the coordinates of the 11 corners and filling in the interior.

For simplicity, we will disregard the thickness of the letter, so we require only five coordinates as in Figure 4.5.2.

[14]If $\mathbf{v}_0$ and $\mathbf{v}_1$ are vectors, the vector from $\mathbf{v}_0$ to $\mathbf{v}_1$ is $\mathbf{d} = \mathbf{v}_1 - \mathbf{v}_0$. So a vector $\mathbf{v}$ lies on the line segment between $\mathbf{v}_0$ and $\mathbf{v}_1$ if and only if $\mathbf{v} = \mathbf{v}_0 + t\mathbf{d}$ for some number t in the range $0 \leq t \leq 1$. Thus the image of this segment is the set of vectors $A\mathbf{v} = A\mathbf{v}_0 + tA\mathbf{d}$ with $0 \leq t \leq 1$, that is the image is the segment between $A\mathbf{v}_0$ and $A\mathbf{v}_1$.

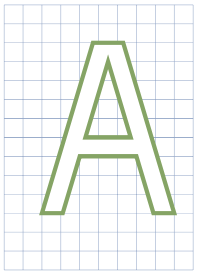


Figure 4.5.1

This simplified letter can then be stored as a data matrix

$$\begin{array}{rc} \textit{Vertex} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ D = & \left[\begin{array}{ccccc} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right] \end{array}$$

where the columns are the coordinates of the vertices in order. Then if we want to transform the letter by a 2×2 matrix A, we left-multiply this data matrix by A (the effect is to multiply each column by A and so transform each vertex).

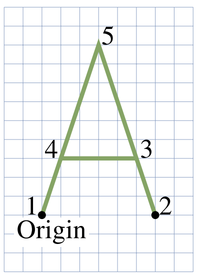


Figure 4.5.2

For example, we can slant the letter to the right by multiplying by an x-shear matrix $A = \left[\begin{array}{cc} 1 & 0.2 \\ 0 & 1 \end{array} \right]$—see Section 2.2. The result is the letter with data matrix

$$A = \left[\begin{array}{cc} 1 & 0.2 \\ 0 & 1 \end{array} \right] \left[\begin{array}{ccccc} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right] = \left[\begin{array}{ccccc} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right]$$

which is shown in Figure 4.5.3.

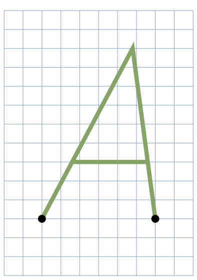


Figure 4.5.3

If we want to make this slanted matrix narrower, we can now apply an x-scale matrix $B = \left[\begin{array}{cc} 0.8 & 0 \\ 0 & 1 \end{array} \right]$ that shrinks the x-coordinate by 0.8. The result is the composite transformation

$$\begin{aligned} BAD &= \left[\begin{array}{cc} 0.8 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 0.2 \\ 0 & 1 \end{array} \right] \left[\begin{array}{ccccc} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right] \\ &= \left[\begin{array}{ccccc} 0 & 4.8 & 4.48 & 1.28 & 3.84 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right] \end{aligned}$$

which is drawn in Figure 4.5.4.

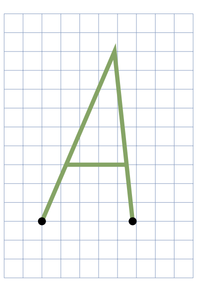


Figure 4.5.4

On the other hand, we can rotate the letter about the origin through $\frac{\pi}{6}$ (or 30°) by multiplying by the matrix $R_{\frac{\pi}{2}} = \left[\begin{array}{cc} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{array} \right] = \left[\begin{array}{cc} 0.866 & -0.5 \\ 0.5 & 0.866 \end{array} \right]$. This gives

$$\begin{aligned} R_{\frac{\pi}{2}} &= \left[\begin{array}{cc} 0.866 & -0.5 \\ 0.5 & 0.866 \end{array} \right] \left[\begin{array}{ccccc} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array} \right] \\ &= \left[\begin{array}{ccccc} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{array} \right] \end{aligned}$$

and is plotted in Figure 4.5.5.

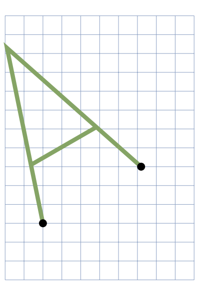


Figure 4.5.5

This poses a problem: How do we rotate at a point other than the origin? It turns out that we can do this when we have solved another more basic problem. It is clearly important to be able to translate a screen image by a fixed vector $\mathbf{w}$, that is apply the transformation $T_w : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_w(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ for all $\mathbf{v}$ in $\mathbb{R}^2$. The problem is that these translations are not matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$ because they do not carry $\mathbf{0}$ to $\mathbf{0}$ (unless $\mathbf{w} = \mathbf{0}$). However, there is a clever way around this.

The idea is to represent a point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a 3×1 column $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, called the **homogeneous coordinates** of $\mathbf{v}$. Then translation by $\mathbf{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+p \\ y+q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\mathbf{w}}(\mathbf{v}) \\ 1 \end{bmatrix}$$

Thus, by using homogeneous coordinates we can implement the translation T_w in the top two coordinates. On the other hand, the matrix transformation induced by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also given by a 3×3 matrix:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{v} \\ 1 \end{bmatrix}$$

So everything can be accomplished at the expense of using 3×3 matrices and homogeneous coordinates.

Example 4.5.1

Rotate the letter A in Figure 4.5.2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Solution. Using homogeneous coordinates for the vertices of the letter results in a data matrix with three rows:

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

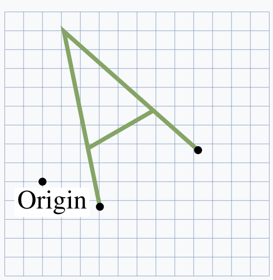


Figure 4.5.6

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point $\mathbf{w}$ moves to the origin, then rotate this translated letter, and then translate it by $\mathbf{w}$ back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.036 & 8.232 & 5.866 & 2.402 & 1.134 \\ -1.33 & 1.67 & 3.768 & 1.768 & 7.964 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This is plotted in Figure 4.5.6.

This discussion merely touches the surface of computer graphics, and the reader is referred to specialized books on the subject. Realistic graphic rendering requires an enormous number of matrix calculations. In fact, matrix multiplication algorithms are now embedded in microchip circuits, and can perform over 100 million matrix multiplications per second. This is particularly important in the field of three-dimensional graphics where the homogeneous coordinates have four components and 4×4 matrices are required.

Exercises for 4.5

Exercise 4.5.1 Consider the letter A described in Figure 4.5.2. Find the data matrix for the letter obtained by:

a. Rotating the letter through $\frac{\pi}{4}$ about the origin.

b. Rotating the letter through $\frac{\pi}{4}$ about the point $\left[\begin{array}{c} 1 \\ 2 \end{array}\right]$.

Exercise 4.5.2 Find the matrix for turning the letter A in Figure 4.5.2 upside-down in place.

Exercise 4.5.3 Find the 3×3 matrix for reflecting in the line $y = mx + b$. Use $\left[\begin{array}{c} 1 \\ m \end{array}\right]$ as direction vector for the line.

Exercise 4.5.4 Find the 3×3 matrix for rotating through the angle θ about the point $P(a, b)$.

Exercise 4.5.5 Find the reflection of the point P in the line $y = 1 + 2x$ in $\mathbb{R}^2$ if:

a. $P = P(1, 1)$

b. $P = P(1, 4)$

c. What about $P = P(1, 3)$? Explain. [*Hint*: Example 4.5.1 and Section 4.4.]

Supplementary Exercises for Chapter 4

Exercise 4.1 Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, and $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$, show that $a = a_1$ and $b = b_1$.

Exercise 4.2 Consider a triangle with vertices A, B, and C. Let E and F be the midpoints of sides AB and AC, respectively, and let the medians EC and FB meet at O. Write $\overrightarrow{EO} = s\overrightarrow{EC}$ and $\overrightarrow{FO} = t\overrightarrow{FB}$, where s and t are scalars. Show that $s = t = \frac{1}{3}$ by expressing $\overrightarrow{AO}$ two ways in the form $a\overrightarrow{EO} + b\overrightarrow{AC}$, and applying Exercise 4.1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).

Exercise 4.3 A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?

Exercise 4.4 A wind is blowing from the south at 75 knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.

Exercise 4.5 An airplane pilot flies at 300 km/h in a direction 30° south of east. The wind is blowing from the south at 150 km/h.

a. Find the resulting direction and speed of the airplane.

b. Find the speed of the airplane if the wind is from the west (at 150 km/h).

Exercise 4.6 A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector $\mathbf{v} = (x, y)$ that she must achieve, assuming the x and y axes point east and north, respectively, and find her resulting speed.

Exercise 4.7 A boat goes 12 knots heading north. The current is 5 knots from the west. In what direction does the boat actually move and at what speed?

Exercise 4.8 Show that the distance from a point A (with vector $\mathbf{a}$) to the plane with vector equation $\mathbf{n} \cdot \mathbf{p} = d$ is $\frac{1}{\|n\|}|\mathbf{n} \cdot \mathbf{a} - d|$.

Exercise 4.9 If two distinct points lie in a plane, show that the line through these points is contained in the plane.

Exercise 4.10 The line through a vertex of a triangle, perpendicular to the opposite side, is called an **altitude** of the triangle. Show that the three altitudes of any triangle are concurrent. (The intersection of the altitudes is called the **orthocentre** of the triangle.) [*Hint*: If P is the intersection of two of the altitudes, show that the line through P and the remaining vertex is perpendicular to the remaining side.]

5. Vector Space $\mathbb{R}^n$

5.1 Subspaces and Spanning

In Section 2.2 we introduced the set $\mathbb{R}^n$ of all n-tuples (called *vectors*), and began our investigation of the matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ given by matrix multiplication by an $m \times n$ matrix. Particular attention was paid to the euclidean plane $\mathbb{R}^2$ where certain simple geometric transformations were seen to be matrix transformations. Then in Section 2.6 we introduced linear transformations, showed that they are all matrix transformations, and found the matrices of rotations and reflections in $\mathbb{R}^2$. We returned to this in Section 4.4 where we showed that projections, reflections, and rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$ were all linear, and where we related areas and volumes to determinants.

In this chapter we investigate $\mathbb{R}^n$ in full generality, and introduce some of the most important concepts and methods in linear algebra. The n-tuples in $\mathbb{R}^n$ will continue to be denoted $\mathbf{x}$, $\mathbf{y}$, and so on, and will be written as rows or columns depending on the context.

Subspaces of $\mathbb{R}^n$

Definition 5.1 Subspace of $\mathbb{R}^n$

A set[1] U *of vectors in* $\mathbb{R}^n$ *is called a* **subspace** *of* $\mathbb{R}^n$ *if it satisfies the following properties:*

S1. The zero vector $\mathbf{0} \in U$.

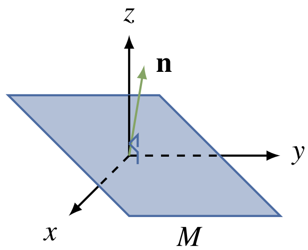
S2. If $\mathbf{x} \in U$ *and* $\mathbf{y} \in U$*, then* $\mathbf{x} + \mathbf{y} \in U$.

S3. If $\mathbf{x} \in U$*, then* $a\mathbf{x} \in U$ *for every real number* a.

We say that the subset U is **closed under addition** if S2 holds, and that U is **closed under scalar multiplication** if S3 holds.

Clearly $\mathbb{R}^n$ is a subspace of itself, and this chapter is about these subspaces and their properties. The set $U = \{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for each a in $\mathbb{R}$; it is called the **zero subspace**. Any subspace of $\mathbb{R}^n$ other than $\{\mathbf{0}\}$ or $\mathbb{R}^n$ is called a **proper** subspace.

[1] We use the language of sets. Informally, a **set** X is a collection of objects, called the **elements** of the set. The fact that x is an element of X is denoted $x \in X$. Two sets X and Y are called equal (written $X = Y$) if they have the same elements. If every element of X is in the set Y, we say that X is a **subset** of Y, and write $X \subseteq Y$. Hence $X \subseteq Y$ and $Y \subseteq X$ both hold if and only if $X = Y$.



We saw in Section 4.2 that every plane M through the origin in $\mathbb{R}^3$ has equation $ax+by+cz=0$ where a, b, and c are not all zero. Here $\mathbf{n}=\left[\begin{array}{c} a \\ b \\ c \end{array}\right]$ is a normal for the plane and

$$M=\{\mathbf{v} \text{ in } \mathbb{R}^3 \mid \mathbf{n}\cdot\mathbf{v}=0\}$$

where $\mathbf{v}=\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ and $\mathbf{n}\cdot\mathbf{v}$ denotes the dot product introduced in Section 2.2 (see the diagram).[2] Then M is a subspace of $\mathbb{R}^3$. Indeed we show that M satisfies S1, S2, and S3 as follows:

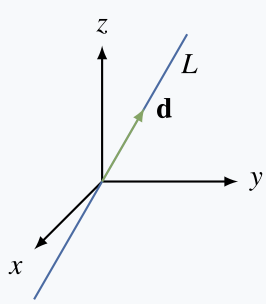
S1. $\mathbf{0}\in M$ *because* $\mathbf{n}\cdot\mathbf{0}=0$*;*

S2. If $\mathbf{v}\in M$ *and* $\mathbf{v}_1\in M$ *, then* $\mathbf{n}\cdot(\mathbf{v}+\mathbf{v}_1)=\mathbf{n}\cdot\mathbf{v}+\mathbf{n}\cdot\mathbf{v}_1=0+0=0$ *, so* $\mathbf{v}+\mathbf{v}_1\in M$*;*

S3. If $\mathbf{v}\in M$ *, then* $\mathbf{n}\cdot(a\mathbf{v})=a(\mathbf{n}\cdot\mathbf{v})=a(0)=0$ *, so* $a\mathbf{v}\in M$*.*

This proves the first part of

Example 5.1.1



Planes and lines through the origin in $\mathbb{R}^3$ are all subspaces of $\mathbb{R}^3$.

Solution. We dealt with planes above. If L is a line through the origin with direction vector $\mathbf{d}$, then $L=\{t\mathbf{d} \mid t\in\mathbb{R}\}$ (see the diagram). We leave it as an exercise to verify that L satisfies S1, S2, and S3.

Example 5.1.1 shows that lines through the origin in $\mathbb{R}^2$ are subspaces; in fact, they are the *only* proper subspaces of $\mathbb{R}^2$ (Exercise 5.1.24). Indeed, we shall see in Example 5.2.14 that lines and planes through the origin in $\mathbb{R}^3$ are the only proper subspaces of $\mathbb{R}^3$. Thus the geometry of lines and planes through the origin is captured by the subspace concept. (Note that *every* line or plane is just a translation of one of these.)

Subspaces can also be used to describe important features of an $m\times n$ matrix A. The **null space** of A, denoted $\operatorname{null} A$, and the **image space** of A, denoted $\operatorname{im} A$, are defined by

$$\operatorname{null} A=\{\mathbf{x}\in\mathbb{R}^n \mid A\mathbf{x}=\mathbf{0}\} \quad \text{and} \quad \operatorname{im} A=\{A\mathbf{x} \mid \mathbf{x}\in\mathbb{R}^n\}$$

In the language of Chapter 2, $\operatorname{null} A$ consists of all solutions $\mathbf{x}$ in $\mathbb{R}^n$ of the homogeneous system $A\mathbf{x}=\mathbf{0}$, and $\operatorname{im} A$ is the set of all vectors $\mathbf{y}$ in $\mathbb{R}^m$ such that $A\mathbf{x}=\mathbf{y}$ *has* a solution $\mathbf{x}$. Note that $\mathbf{x}$ is in $\operatorname{null} A$ if it

[2] We are using set notation here. In general $\{q \mid p\}$ means the set of all objects q with property p.

satisfies the *condition* $A\mathbf{x} = \mathbf{0}$, while $\operatorname{im} A$ consists of vectors of the *form* $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. These two ways to describe subsets occur frequently.

Example 5.1.2

If A is an $m \times n$ matrix, then:

1. $\operatorname{null} A$ is a subspace of $\mathbb{R}^n$.
2. $\operatorname{im} A$ is a subspace of $\mathbb{R}^m$.

Solution.

1. The zero vector $\mathbf{0} \in \mathbb{R}^n$ lies in $\operatorname{null} A$ because $A\mathbf{0} = \mathbf{0}$.[3] If $\mathbf{x}$ and $\mathbf{x}_1$ are in $\operatorname{null} A$, then $\mathbf{x} + \mathbf{x}_1$ and $a\mathbf{x}$ are in $\operatorname{null} A$ because they satisfy the required condition:

$$A(\mathbf{x} + \mathbf{x}_1) = A\mathbf{x} + A\mathbf{x}_1 = \mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad A(a\mathbf{x}) = a(A\mathbf{x}) = a\mathbf{0} = \mathbf{0}$$

   Hence $\operatorname{null} A$ satisfies S1, S2, and S3, and so is a subspace of $\mathbb{R}^n$.

2. The zero vector $\mathbf{0} \in \mathbb{R}^m$ lies in $\operatorname{im} A$ because $\mathbf{0} = A\mathbf{0}$. Suppose that $\mathbf{y}$ and $\mathbf{y}_1$ are in $\operatorname{im} A$, say $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y}_1 = A\mathbf{x}_1$ where $\mathbf{x}$ and $\mathbf{x}_1$ are in $\mathbb{R}^n$. Then

$$\mathbf{y} + \mathbf{y}_1 = A\mathbf{x} + A\mathbf{x}_1 = A(\mathbf{x} + \mathbf{x}_1) \quad \text{and} \quad a\mathbf{y} = a(A\mathbf{x}) = A(a\mathbf{x})$$

   show that $\mathbf{y} + \mathbf{y}_1$ and $a\mathbf{y}$ are both in $\operatorname{im} A$ (they have the required form). Hence $\operatorname{im} A$ is a subspace of $\mathbb{R}^m$.

There are other important subspaces associated with a matrix A that clarify basic properties of A. If A is an $n \times n$ matrix and λ is any number, let

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

A vector $\mathbf{x}$ is in $E_\lambda(A)$ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, so Example 5.1.2 gives:

Example 5.1.3

$E_\lambda(A) = \operatorname{null}(\lambda I - A)$ is a subspace of $\mathbb{R}^n$ for each $n \times n$ matrix A and number λ.

$E_\lambda(A)$ is called the **eigenspace** of A corresponding to λ. The reason for the name is that, in the terminology of Section 3.3, λ is an **eigenvalue** of A if $E_\lambda(A) \neq \{\mathbf{0}\}$. In this case the nonzero vectors in $E_\lambda(A)$ are called the **eigenvectors** of A corresponding to λ.

The reader should not get the impression that *every* subset of $\mathbb{R}^n$ is a subspace. For example:

$$U_1 = \left\{ \left[\begin{array}{c} x \\ y \end{array} \right] \middle| x \geq 0 \right\} \text{ satisfies S1 and S2, but not S3;}$$

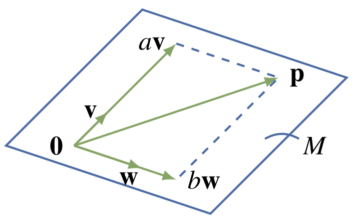
[3] We are using $\mathbf{0}$ to represent the zero vector in both $\mathbb{R}^m$ and $\mathbb{R}^n$. This abuse of notation is common and causes no confusion once everybody knows what is going on.

$$U_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x^2 = y^2 \right\} \text{ satisfies S1 and S3, but not S2;}$$

Hence neither U_1 nor U_2 is a subspace of $\mathbb{R}^2$. (However, see Exercise 5.1.20.)

Spanning Sets

Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. The plane M through the origin containing these vectors is described in Section 4.2 by saying that $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a *normal* for M, and that M consists of all vectors $\mathbf{p}$ such that $\mathbf{n} \cdot \mathbf{p} = 0$.[4] While this is a very useful way to look at planes, there is another approach that is at least as useful in $\mathbb{R}^3$ and, more importantly, works for all subspaces of $\mathbb{R}^n$ for any $n \geq 1$.



The idea is as follows: Observe that, by the diagram, a vector $\mathbf{p}$ is in M if and only if it has the form

$$\mathbf{p} = a\mathbf{v} + b\mathbf{w}$$

for certain real numbers a and b (we say that $\mathbf{p}$ is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$). Hence we can describe M as

$$M = \{a\mathbf{x} + b\mathbf{w} \mid a, b \in \mathbb{R}\}.$$[5]

and we say that $\{\mathbf{v}, \mathbf{w}\}$ is a *spanning set* for M. It is this notion of a spanning set that provides a way to describe all subspaces of $\mathbb{R}^n$.

As in Section 1.3, given vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ in $\mathbb{R}^n$, a vector of the form

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \quad \text{where the } t_i \text{ are scalars}$$

is called a **linear combination** of the $\mathbf{x}_i$, and t_i is called the **coefficient** of $\mathbf{x}_i$ in the linear combination.

Definition 5.2 Linear Combinations and Span in $\mathbb{R}^n$

*The set of all such linear combinations is called the **span** of the $\mathbf{x}_i$ and is denoted*

$$\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \mid t_i \text{ in } \mathbb{R}\}$$

*If $V = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, we say that V is **spanned** by the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$, and that the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ **span** the space V.*

Here are two examples:

$$\operatorname{span}\{\mathbf{x}\} = \{t\mathbf{x} \mid t \in \mathbb{R}\}$$

which we write as $\operatorname{span}\{\mathbf{x}\} = \mathbb{R}\mathbf{x}$ for simplicity.

$$\operatorname{span}\{\mathbf{x}, \mathbf{y}\} = \{r\mathbf{x} + s\mathbf{y} \mid r, s \in \mathbb{R}\}$$

[4]The vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is nonzero because $\mathbf{v}$ and $\mathbf{w}$ are not parallel.

[5]In particular, this implies that any vector $\mathbf{p}$ orthogonal to $\mathbf{v} \times \mathbf{w}$ must be a linear combination $\mathbf{p} = a\mathbf{v} + b\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ for some a and b. Can you prove this directly?

In particular, the above discussion shows that, if $\mathbf{v}$ and $\mathbf{w}$ are two nonzero, nonparallel vectors in $\mathbb{R}^3$, then

$$M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$$

is the plane in $\mathbb{R}^3$ containing $\mathbf{v}$ and $\mathbf{w}$. Moreover, if $\mathbf{d}$ is any nonzero vector in $\mathbb{R}^3$ (or $\mathbb{R}^2$), then

$$L = \text{span}\,\{\mathbf{v}\} = \{t\mathbf{d} \mid t \in \mathbb{R}\} = \mathbb{R}\mathbf{d}$$

is the line with direction vector $\mathbf{d}$. Hence lines and planes can both be described in terms of spanning sets.

Example 5.1.4

Let $\mathbf{x} = (2,\ -1,\ 2,\ 1)$ and $\mathbf{y} = (3,\ 4,\ -1,\ 1)$ in $\mathbb{R}^4$. Determine whether $\mathbf{p} = (0,\ -11,\ 8,\ 1)$ or $\mathbf{q} = (2,\ 3,\ 1,\ 2)$ are in $U = \text{span}\,\{\mathbf{x},\ \mathbf{y}\}$.

Solution. The vector $\mathbf{p}$ is in U if and only if $\mathbf{p} = s\mathbf{x} + t\mathbf{y}$ for scalars s and t. Equating components gives equations

$$2s + 3t = 0, \quad -s + 4t = -11, \quad 2s - t = 8, \quad \text{and} \quad s + t = 1$$

This linear system has solution $s = 3$ and $t = -2$, so $\mathbf{p}$ is in U. On the other hand, asking that $\mathbf{q} = s\mathbf{x} + t\mathbf{y}$ leads to equations

$$2s + 3t = 2, \quad -s + 4t = 3, \quad 2s - t = 1, \quad \text{and} \quad s + t = 2$$

and this system has *no* solution. So $\mathbf{q}$ does *not* lie in U.

Theorem 5.1.1: Span Theorem

Let $U = \text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ in $\mathbb{R}^n$. Then:

1. *U is a subspace of $\mathbb{R}^n$ containing each $\mathbf{x}_i$.*
2. *If W is a subspace of $\mathbb{R}^n$ and each $\mathbf{x}_i \in W$, then $U \subseteq W$.*

Proof.

1. The zero vector $\mathbf{0}$ is in U because $\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$ is a linear combination of the $\mathbf{x}_i$. If $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ and $\mathbf{y} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$ are in U, then $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in U because
$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (t_1 + s_1)\mathbf{x}_1 + (t_2 + s_2)\mathbf{x}_2 + \cdots + (t_k + s_k)\mathbf{x}_k, \text{ and} \\ a\mathbf{x} &= (at_1)\mathbf{x}_1 + (at_2)\mathbf{x}_2 + \cdots + (at_k)\mathbf{x}_k \end{aligned}$$
Finally each $\mathbf{x}_i$ is in U (for example, $\mathbf{x}_2 = 0\mathbf{x}_1 + 1\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$) so S1, S2, and S3 are satisfied for U, proving (1).

2. Let $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ where the t_i are scalars and each $\mathbf{x}_i \in W$. Then each $t_i\mathbf{x}_i \in W$ because W satisfies S3. But then $\mathbf{x} \in W$ because W satisfies S2 (verify). This proves (2). □

Condition (2) in Theorem 5.1.1 can be expressed by saying that $\text{span}\,\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ is the *smallest* subspace of $\mathbb{R}^n$ that contains each $\mathbf{x}_i$. This is useful for showing that two subspaces U and W are equal, since this amounts to showing that both $U \subseteq W$ and $W \subseteq U$. Here is an example of how it is used.

Example 5.1.5

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, show that $\operatorname{span}\{\mathbf{x},\ \mathbf{y}\} = \operatorname{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$.

Solution. Since both $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are in $\operatorname{span}\{\mathbf{x},\ \mathbf{y}\}$, Theorem 5.1.1 gives

$$\operatorname{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\} \subseteq \operatorname{span}\{\mathbf{x},\ \mathbf{y}\}$$

But $\mathbf{x} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) + \frac{1}{2}(\mathbf{x}-\mathbf{y})$ and $\mathbf{y} = \frac{1}{2}(\mathbf{x}+\mathbf{y}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})$ are both in $\operatorname{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$, so

$$\operatorname{span}\{\mathbf{x},\ \mathbf{y}\} \subseteq \operatorname{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$$

again by Theorem 5.1.1. Thus $\operatorname{span}\{\mathbf{x},\ \mathbf{y}\} = \operatorname{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{x}-\mathbf{y}\}$, as desired.

It turns out that many important subspaces are best described by giving a spanning set. Here are three examples, beginning with an important spanning set for $\mathbb{R}^n$ itself. Column j of the $n \times n$ identity matrix I_n is denoted $\mathbf{e}_j$ and called the jth **coordinate vector** in $\mathbb{R}^n$, and the set $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ is called the **standard basis** of $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$ is any vector in $\mathbb{R}^n$, then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, as the reader can verify. This proves:

Example 5.1.6

$\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n\}$ where $\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_n$ are the columns of I_n.

If A is an $m \times n$ matrix A, the next two examples show that it is a routine matter to find spanning sets for $\operatorname{null} A$ and $\operatorname{im} A$.

Example 5.1.7

Given an $m \times n$ matrix A, let $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ denote the basic solutions to the system $A\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm. Then

$$\operatorname{null} A = \operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$$

Solution. If $\mathbf{x} \in \operatorname{null} A$, then $A\mathbf{x} = \mathbf{0}$ so Theorem 1.3.2 shows that $\mathbf{x}$ is a linear combination of the basic solutions; that is, $\operatorname{null} A \subseteq \operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$. On the other hand, if $\mathbf{x}$ is in $\operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$, then $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ for scalars t_i, so

$$A\mathbf{x} = t_1A\mathbf{x}_1 + t_2A\mathbf{x}_2 + \cdots + t_kA\mathbf{x}_k = t_1\mathbf{0} + t_2\mathbf{0} + \cdots + t_k\mathbf{0} = \mathbf{0}$$

This shows that $\mathbf{x} \in \operatorname{null} A$, and hence that $\operatorname{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} \subseteq \operatorname{null} A$. Thus we have equality.

Example 5.1.8

Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the columns of the $m \times n$ matrix A. Then

$$\operatorname{im} A = \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$$

Solution. If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, observe that

$$\left[\begin{array}{cccc} A\mathbf{e}_1 & A\mathbf{e}_2 & \cdots & A\mathbf{e}_n \end{array}\right] = A\left[\begin{array}{cccc} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = AI_n = A = \left[\begin{array}{ccc} \mathbf{c}_1 & \mathbf{c}_2 & \cdots \mathbf{c}_n \end{array}\right].$$

Hence $\mathbf{c}_i = A\mathbf{e}_i$ is in $\operatorname{im} A$ for each i, so $\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\} \subseteq \operatorname{im} A$.

Conversely, let $\mathbf{y}$ be in $\operatorname{im} A$, say $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$, then Definition 2.5 gives

$$\mathbf{y} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \text{ is in } \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$$

This shows that $\operatorname{im} A \subseteq \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$, and the result follows.

Exercises for 5.1

We often write vectors in $\mathbb{R}^n$ as rows.

Exercise 5.1.1 In each case determine whether U is a subspace of $\mathbb{R}^3$. Support your answer.

a. $U = \{(1, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

b. $U = \{(0, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

c. $U = \{(r, s, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}, -r + 3s + 2t = 0\}$.

d. $U = \{(r, 3s, r-2) \mid r \text{ and } s \text{ in } \mathbb{R}\}$.

e. $U = \{(r, 0, s) \mid r^2 + s^2 = 0, r \text{ and } s \text{ in } \mathbb{R}\}$.

f. $U = \{(2r, -s^2, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}\}$.

Exercise 5.1.2 In each case determine if $\mathbf{x}$ lies in $U = \operatorname{span}\{\mathbf{y}, \mathbf{z}\}$. If $\mathbf{x}$ is in U, write it as a linear combination of $\mathbf{y}$ and $\mathbf{z}$; if $\mathbf{x}$ is not in U, show why not.

a. $\mathbf{x} = (2, -1, 0, 1)$, $\mathbf{y} = (1, 0, 0, 1)$, and $\mathbf{z} = (0, 1, 0, 1)$.

b. $\mathbf{x} = (1, 2, 15, 11)$, $\mathbf{y} = (2, -1, 0, 2)$, and $\mathbf{z} = (1, -1, -3, 1)$.

c. $\mathbf{x} = (8, 3, -13, 20)$, $\mathbf{y} = (2, 1, -3, 5)$, and $\mathbf{z} = (-1, 0, 2, -3)$.

d. $\mathbf{x} = (2, 5, 8, 3)$, $\mathbf{y} = (2, -1, 0, 5)$, and $\mathbf{z} = (-1, 2, 2, -3)$.

Exercise 5.1.3 In each case determine if the given vectors span $\mathbb{R}^4$. Support your answer.

a. $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$.

b. $\{(1, 3, -5, 0), (-2, 1, 0, 0), (0, 2, 1, -1), (1, -4, 5, 0)\}$.

Exercise 5.1.4 Is it possible that $\{(1,\ 2,\ 0),\ (2,\ 0,\ 3)\}$ can span the subspace $U = \{(r,\ s,\ 0) \mid r \text{ and } s \text{ in } \mathbb{R}\}$? Defend your answer.

Exercise 5.1.5 Give a spanning set for the zero subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$.

Exercise 5.1.6 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Defend your answer.

Exercise 5.1.7 If $U = \text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x}+t\mathbf{z},\ \mathbf{y},\ \mathbf{z}\}$ for every t in $\mathbb{R}$.

Exercise 5.1.8 If $U = \text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x}+\mathbf{y},\ \mathbf{y}+\mathbf{z},\ \mathbf{z}+\mathbf{x}\}$.

Exercise 5.1.9 If $a \neq 0$ is a scalar, show that $\text{span}\{a\mathbf{x}\} = \text{span}\{\mathbf{x}\}$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 5.1.10 If $a_1,\ a_2,\ \dots,\ a_k$ are nonzero scalars, show that $\text{span}\{a_1\mathbf{x}_1,\ a_2\mathbf{x}_2,\ \dots,\ a_k\mathbf{x}_k\} = \text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ for any vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

Exercise 5.1.11 If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, determine all subspaces of $\text{span}\{\mathbf{x}\}$.

Exercise 5.1.12 Suppose that $U = \text{span}\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$ where each $\mathbf{x}_i$ is in $\mathbb{R}^n$. If A is an $m \times n$ matrix and $A\mathbf{x}_i = \mathbf{0}$ for each i, show that $A\mathbf{y} = \mathbf{0}$ for every vector $\mathbf{y}$ in U.

Exercise 5.1.13 If A is an $m \times n$ matrix, show that, for each invertible $m \times m$ matrix U, $\text{null}(A) = \text{null}(UA)$.

Exercise 5.1.14 If A is an $m \times n$ matrix, show that, for each invertible $n \times n$ matrix V, $\text{im}(A) = \text{im}(AV)$.

Exercise 5.1.15 Let U be a subspace of $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.

a. If $a\mathbf{x}$ is in U where $a \neq 0$ is a number, show that $\mathbf{x}$ is in U.

b. If $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$ are in U where $\mathbf{y}$ is a vector in $\mathbb{R}^n$, show that $\mathbf{x}$ is in U.

Exercise 5.1.16 In each case either show that the statement is true or give an example showing that it is false.

a. If $U \neq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $\mathbf{x}+\mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

b. If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U for all r in $\mathbb{R}$, then $\mathbf{x}$ is in U.

c. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is also in U.

d. If $\mathbf{x}$ is in U and $U = \text{span}\{\mathbf{y},\ \mathbf{z}\}$, then $U = \text{span}\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$.

e. The empty set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

f. $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is in $\text{span}\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right],\ \left[\begin{array}{c} 2 \\ 0 \end{array}\right]\right\}$.

Exercise 5.1.17

a. If A and B are $m \times n$ matrices, show that $U = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = B\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.

b. What if A is $m \times n$, B is $k \times n$, and $m \neq k$?

Exercise 5.1.18 Suppose that $\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k$ are vectors in $\mathbb{R}^n$. If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $\text{span}\{\mathbf{x}_1\, \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\} = \text{span}\{\mathbf{y}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_k\}$.

Exercise 5.1.19 If $U \neq \{\mathbf{0}\}$ is a subspace of $\mathbb{R}$, show that $U = \mathbb{R}$.

Exercise 5.1.20 Let U be a nonempty subset of $\mathbb{R}^n$. Show that U is a subspace if and only if S2 and S3 hold.

Exercise 5.1.21 If S and T are nonempty sets of vectors in $\mathbb{R}^n$, and if $S \subseteq T$, show that $\text{span}\{S\} \subseteq \text{span}\{T\}$.

Exercise 5.1.22 Let U and W be subspaces of $\mathbb{R}^n$. Define their **intersection** $U \cap W$ and their **sum** $U + W$ as follows:

$U \cap W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ belongs to both } U \text{ and } W\}$.

$U + W = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}$ is a sum of a vector in U and a vector in $W\}$.

a. Show that $U \cap W$ is a subspace of $\mathbb{R}^n$.

b. Show that $U + W$ is a subspace of $\mathbb{R}^n$.

Exercise 5.1.23 Let P denote an invertible $n \times n$ matrix. If λ is a number, show that

$$E_\lambda(PAP^{-1}) = \{P\mathbf{x} \mid \mathbf{x} \text{ is in } E_\lambda(A)\}$$

for each $n \times n$ matrix A.

Exercise 5.1.24 Show that every proper subspace U of $\mathbb{R}^2$ is a line through the origin. [*Hint*: If $\mathbf{d}$ is a nonzero vector in U, let $L = \mathbb{R}\mathbf{d} = \{r\mathbf{d} \mid r \text{ in } \mathbb{R}\}$ denote the line with direction vector $\mathbf{d}$. If $\mathbf{u}$ is in U but not in L, argue geometrically that every vector $\mathbf{v}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{d}$.]

5.2 Independence and Dimension

Some spanning sets are better than others. If $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a subspace of $\mathbb{R}^n$, then every vector in U can be written as a linear combination of the $\mathbf{x}_i$ in at least one way. Our interest here is in spanning sets where each vector in U has a *exactly one* representation as a linear combination of these vectors.

Linear Independence

Given $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ in $\mathbb{R}^n$, suppose that two linear combinations are equal:

$$r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$$

We are looking for a condition on the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors that guarantees that this representation is *unique*; that is, $r_i = s_i$ for each i. Taking all terms to the left side gives

$$(r_1 - s_1)\mathbf{x}_1 + (r_2 - s_2)\mathbf{x}_2 + \cdots + (r_k - s_k)\mathbf{x}_k = \mathbf{0}$$

so the required condition is that this equation forces all the coefficients $r_i - s_i$ to be zero.

Definition 5.3 Linear Independence in $\mathbb{R}^n$

*With this in mind, we call a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors **linearly independent** (or simply **independent**) if it satisfies the following condition:*

$$\text{If } t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0} \text{ then } t_1 = t_2 = \cdots = t_k = 0$$

We record the result of the above discussion for reference.

Theorem 5.2.1

*If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is an independent set of vectors in $\mathbb{R}^n$, then every vector in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ has a **unique** representation as a linear combination of the $\mathbf{x}_i$.*

It is useful to state the definition of independence in different language. Let us say that a linear combination **vanishes** if it equals the zero vector, and call a linear combination **trivial** if every coefficient is zero. Then the definition of independence can be compactly stated as follows:

> A set of vectors is independent if and only if the only linear combination that vanishes is the trivial one.

Hence we have a procedure for checking that a set of vectors is independent:

Independence Test

To verify that a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *of vectors in* $\mathbb{R}^n$ *is independent, proceed as follows:*

1. *Set a linear combination equal to zero:* $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$.
2. *Show that* $t_i = 0$ *for each* i *(that is, the linear combination is trivial).*

Of course, if some nontrivial linear combination vanishes, the vectors are not independent.

Example 5.2.1

Determine whether $\{(1, 0, -2, 5), (2, 1, 0, -1), (1, 1, 2, 1)\}$ is independent in $\mathbb{R}^4$.

Solution. Suppose a linear combination vanishes:

$$r(1, 0, -2, 5) + s(2, 1, 0, -1) + t(1, 1, 2, 1) = (0, 0, 0, 0)$$

Equating corresponding entries gives a system of four equations:

$$r + 2s + t = 0, \; s + t = 0, \; -2r + 2t = 0, \; \text{and } 5r - s + t = 0$$

The only solution is the trivial one $r = s = t = 0$ (verify), so these vectors are independent by the independence test.

Example 5.2.2

Show that the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ is independent.

Solution. The components of $t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \cdots + t_n\mathbf{e}_n$ are $t_1, t_2, \dots, t_n$ (see the discussion preceding Example 5.1.6) So the linear combination vanishes if and only if each $t_i = 0$. Hence the independence test applies.

Example 5.2.3

If $\{\mathbf{x}, \mathbf{y}\}$ is independent, show that $\{2\mathbf{x} + 3\mathbf{y}, \mathbf{x} - 5\mathbf{y}\}$ is also independent.

Solution. If $s(2\mathbf{x} + 3\mathbf{y}) + t(\mathbf{x} - 5\mathbf{y}) = \mathbf{0}$, collect terms to get $(2s + t)\mathbf{x} + (3s - 5t)\mathbf{y} = \mathbf{0}$. Since $\{\mathbf{x}, \mathbf{y}\}$ is independent this combination must be trivial; that is, $2s + t = 0$ and $3s - 5t = 0$. These equations have only the trivial solution $s = t = 0$, as required.

Example 5.2.4

Show that the zero vector in $\mathbb{R}^n$ does not belong to any independent set.

Solution. No set $\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors is independent because we have a vanishing, nontrivial linear combination $1 \cdot \mathbf{0} + 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k = \mathbf{0}$.

Example 5.2.5

Given $\mathbf{x}$ in $\mathbb{R}^n$, show that $\{\mathbf{x}\}$ is independent if and only if $\mathbf{x} \neq \mathbf{0}$.

Solution. A vanishing linear combination from $\{\mathbf{x}\}$ takes the form $t\mathbf{x} = \mathbf{0}$, t in $\mathbb{R}$. This implies that $t = 0$ because $\mathbf{x} \neq \mathbf{0}$.

The next example will be needed later.

Example 5.2.6

Show that the nonzero rows of a row-echelon matrix R are independent.

Solution. We illustrate the case with 3 leading 1s; the general case is analogous. Suppose R has the form $R = \left[\begin{array}{cccccc} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$ where $*$ indicates a nonspecified number. Let R_1, R_2, and R_3 denote the nonzero rows of R. If $t_1R_1 + t_2R_2 + t_3R_3 = 0$ we show that $t_1 = 0$, then $t_2 = 0$, and finally $t_3 = 0$. The condition $t_1R_1 + t_2R_2 + t_3R_3 = 0$ becomes

$$(0, t_1, *, *, *, *) + (0, 0, 0, t_2, *, *) + (0, 0, 0, 0, t_3, *) = (0, 0, 0, 0, 0, 0)$$

Equating second entries show that $t_1 = 0$, so the condition becomes $t_2R_2 + t_3R_3 = 0$. Now the same argument shows that $t_2 = 0$. Finally, this gives $t_3R_3 = 0$ and we obtain $t_3 = 0$.

A set of vectors in $\mathbb{R}^n$ is called **linearly dependent** (or simply **dependent**) if it is *not* linearly independent, equivalently if some nontrivial linear combination vanishes.

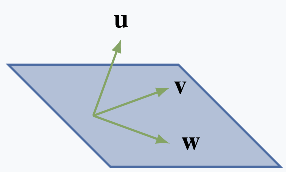
Example 5.2.7

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^3$, show that $\{\mathbf{v}, \mathbf{w}\}$ is dependent if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

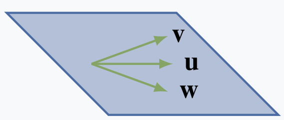
Solution. If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then one is a scalar multiple of the other (Theorem 4.1.4), say $\mathbf{v} = a\mathbf{w}$ for some scalar a. Then the nontrivial linear combination $\mathbf{v} - a\mathbf{w} = \mathbf{0}$ vanishes, so $\{\mathbf{v}, \mathbf{w}\}$ is dependent.
Conversely, if $\{\mathbf{v}, \mathbf{w}\}$ is dependent, let $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ be nontrivial, say $s \neq 0$. Then $\mathbf{v} = -\frac{t}{s}\mathbf{w}$ so $\mathbf{v}$ and $\mathbf{w}$ are parallel (by Theorem 4.1.4). A similar argument works if $t \neq 0$.

With this we can give a geometric description of what it means for a set $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ in $\mathbb{R}^3$ to be independent. Note that this requirement means that $\{\mathbf{v},\ \mathbf{w}\}$ is also independent ($a\mathbf{v}+b\mathbf{w}=\mathbf{0}$ means that $0\mathbf{u}+a\mathbf{v}+b\mathbf{w}=\mathbf{0}$), so $M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$ is the plane containing $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{0}$ (see the discussion preceding Example 5.1.4). So we assume that $\{\mathbf{v},\ \mathbf{w}\}$ is independent in the following example.

Example 5.2.8



$\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ independent



$\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ not independent

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$ where $\{\mathbf{v},\ \mathbf{w}\}$ independent. Show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent if and only if $\mathbf{u}$ is not in the plane $M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$. This is illustrated in the diagrams.

Solution. If $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, suppose $\mathbf{u}$ is in the plane $M = \text{span}\,\{\mathbf{v},\ \mathbf{w}\}$, say $\mathbf{u} = a\mathbf{v}+b\mathbf{w}$, where a and b are in $\mathbb{R}$. Then $1\mathbf{u}-a\mathbf{v}-b\mathbf{w}=\mathbf{0}$, contradicting the independence of $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$.

On the other hand, suppose that $\mathbf{u}$ is not in M; we must show that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent. If $r\mathbf{u}+s\mathbf{v}+t\mathbf{w}=\mathbf{0}$ where r, s, and t are in $\mathbb{R}^3$, then $r=0$ since otherwise $\mathbf{u} = -\frac{s}{r}\mathbf{v}+\frac{-t}{r}\mathbf{w}$ is in M. But then $s\mathbf{v}+t\mathbf{w}=\mathbf{0}$, so $s=t=0$ by our assumption. This shows that $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is independent, as required.

By the inverse theorem, the following conditions are equivalent for an $n\times n$ matrix A:

1. *A is invertible.*
2. *If $A\mathbf{x}=\mathbf{0}$ where $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x}=\mathbf{0}$.*
3. *$A\mathbf{x}=\mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*

While condition 1 makes no sense if A is not square, conditions 2 and 3 are meaningful for any matrix A and, in fact, are related to independence and spanning. Indeed, if $\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n$ are the columns of A, and if we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$$

by Definition 2.5. Hence the definitions of independence and spanning show, respectively, that condition 2 is equivalent to the independence of $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ and condition 3 is equivalent to the requirement that $\text{span}\,\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\} = \mathbb{R}^m$. This discussion is summarized in the following theorem:

Theorem 5.2.2

If A is an $m\times n$ matrix, let $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ denote the columns of A.

1. *$\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ is independent in $\mathbb{R}^m$ if and only if $A\mathbf{x}=\mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x}=\mathbf{0}$.*
2. *$\mathbb{R}^m = \text{span}\,\{\mathbf{c}_1,\ \mathbf{c}_2,\ \dots,\ \mathbf{c}_n\}$ if and only if $A\mathbf{x}=\mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^m$.*

For a *square* matrix A, Theorem 5.2.2 characterizes the invertibility of A in terms of the spanning and independence of its columns (see the discussion preceding Theorem 5.2.2). It is important to be able to discuss these notions for *rows*. If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are $1 \times n$ rows, we define span $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ to be the set of all linear combinations of the $\mathbf{x}_i$ (as matrices), and we say that $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is linearly independent if the only vanishing linear combination is the trivial one (that is, if $\{\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_k^T\}$ is independent in $\mathbb{R}^n$, as the reader can verify).[6]

Theorem 5.2.3

The following are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*
2. *The columns of A are linearly independent.*
3. *The columns of A span $\mathbb{R}^n$.*
4. *The rows of A are linearly independent.*
5. *The rows of A span the set of all $1 \times n$ rows.*

Proof. Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of A.

(1) $\Leftrightarrow$ (2). By Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$; this holds if and only if $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent by Theorem 5.2.2.

(1) $\Leftrightarrow$ (3). Again by Theorem 2.4.5, A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every column B in $\mathbb{R}^n$; this holds if and only if span $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} = \mathbb{R}^n$ by Theorem 5.2.2.

(1) $\Leftrightarrow$ (4). The matrix A is invertible if and only if A^T is invertible (by Corollary 2.4.1 to Theorem 2.4.4); this in turn holds if and only if A^T has independent columns (by (1) $\Leftrightarrow$ (2)); finally, this last statement holds if and only if A has independent rows (because the rows of A are the transposes of the columns of A^T).

(1) $\Leftrightarrow$ (5). The proof is similar to (1) $\Leftrightarrow$ (4). □

Example 5.2.9

Show that $S = \{(2, -2, 5), (-3, 1, 1), (2, 7, -4)\}$ is independent in $\mathbb{R}^3$.

Solution. Consider the matrix $A = \begin{bmatrix} 2 & -2 & 5 \\ -3 & 1 & 1 \\ 2 & 7 & -4 \end{bmatrix}$ with the vectors in S as its rows. A routine computation shows that $\det A = -117 \neq 0$, so A is invertible. Hence S is independent by Theorem 5.2.3. Note that Theorem 5.2.3 also shows that $\mathbb{R}^3 = \text{span } S$.

[6] It is best to view columns and rows as just two different *notations* for ordered n-tuples. This discussion will become redundant in Chapter 6 where we define the general notion of a vector space.

Dimension

It is common geometrical language to say that $\mathbb{R}^3$ is 3-dimensional, that planes are 2-dimensional and that lines are 1-dimensional. The next theorem is a basic tool for clarifying this idea of "dimension". Its importance is difficult to exaggerate.

Theorem 5.2.4: Fundamental Theorem

Let U be a subspace of $\mathbb{R}^n$. If U is spanned by m vectors, and if U contains k linearly independent vectors, then $k \leq m$.

This proof is given in Theorem 6.3.2 in much greater generality.

Definition 5.4 Basis of $\mathbb{R}^n$

*If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ of vectors in U is called a **basis** of U if it satisfies the following two conditions:*

1. $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ *is linearly independent.*
2. $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

The most remarkable result about bases[7] is:

Theorem 5.2.5: Invariance Theorem

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Proof. We have $k \leq m$ by the fundamental theorem because $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ spans U, and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ is independent. Similarly, by interchanging $\mathbf{x}$'s and $\mathbf{y}$'s we get $m \leq k$. Hence $m = k$. $\square$

The invariance theorem guarantees that there is no ambiguity in the following definition:

Definition 5.5 Dimension of a Subspace of $\mathbb{R}^n$

*If U is a subspace of $\mathbb{R}^n$ and $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is any basis of U, the number, m, of vectors in the basis is called the **dimension** of U, denoted*

$$\dim U = m$$

The importance of the invariance theorem is that the dimension of U can be determined by counting the number of vectors in *any* basis.[8]

[7] The plural of "basis" is "bases".

[8] We will show in Theorem 5.2.6 that every subspace of $\mathbb{R}^n$ does indeed *have* a basis.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$, that is the set of columns of the identity matrix. Then $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ by Example 5.1.6, and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is independent by Example 5.2.2. Hence it is indeed a basis of $\mathbb{R}^n$ in the present terminology, and we have

Example 5.2.10

$\dim(\mathbb{R}^n) = n$ and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis.

This agrees with our geometric sense that $\mathbb{R}^2$ is two-dimensional and $\mathbb{R}^3$ is three-dimensional. It also says that $\mathbb{R}^1 = \mathbb{R}$ is one-dimensional, and $\{1\}$ is a basis. Returning to subspaces of $\mathbb{R}^n$, we define

$$\dim\{\mathbf{0}\} = 0$$

This amounts to saying $\{\mathbf{0}\}$ has a basis containing *no* vectors. This makes sense because $\mathbf{0}$ cannot belong to *any* independent set (Example 5.2.4).

Example 5.2.11

Let $U = \left\{ \left[\begin{array}{c} r \\ s \\ r \end{array} \right] \mid r, s \text{ in } \mathbb{R} \right\}$. Show that U is a subspace of $\mathbb{R}^3$, find a basis, and calculate $\dim U$.

Solution. Clearly, $\left[\begin{array}{c} r \\ s \\ r \end{array} \right] = r\mathbf{u} + s\mathbf{v}$ where $\mathbf{u} = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right]$ and $\mathbf{v} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]$. It follows that $U = \text{span}\{\mathbf{u}, \mathbf{v}\}$, and hence that U is a subspace of $\mathbb{R}^3$. Moreover, if $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$, then $\left[\begin{array}{c} r \\ s \\ r \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$ so $r = s = 0$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is independent, and so a **basis** of U. This means $\dim U = 2$.

Example 5.2.12

Let $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be a basis of $\mathbb{R}^n$. If A is an invertible $n \times n$ matrix, then $D = \{A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n\}$ is also a basis of $\mathbb{R}^n$.

Solution. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then $A^{-1}\mathbf{x}$ is in $\mathbb{R}^n$ so, since B is a basis, we have $A^{-1}\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n$ for t_i in $\mathbb{R}$. Left multiplication by A gives $\mathbf{x} = t_1(A\mathbf{x}_1) + t_2(A\mathbf{x}_2) + \cdots + t_n(A\mathbf{x}_n)$, and it follows that D spans $\mathbb{R}^n$. To show independence, let $s_1(A\mathbf{x}_1) + s_2(A\mathbf{x}_2) + \cdots + s_n(A\mathbf{x}_n) = \mathbf{0}$, where the s_i are in $\mathbb{R}$. Then $A(s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n) = \mathbf{0}$ so left multiplication by A^{-1} gives $s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n = \mathbf{0}$. Now the independence of B shows that each $s_i = 0$, and so proves the independence of D. Hence D is a basis of $\mathbb{R}^n$.

While we have found bases in many subspaces of $\mathbb{R}^n$, we have not yet shown that *every* subspace *has* a basis. This is part of the next theorem, the proof of which is deferred to Section 6.4 (Theorem 6.4.1) where it will be proved in more generality.

Theorem 5.2.6

Let $U \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{R}^n$. Then:

1. *U has a basis and $\dim U \leq n$.*
2. *Any independent set in U can be enlarged (by adding vectors from the standard basis) to a basis of U.*
3. *Any spanning set for U can be cut down (by deleting vectors) to a basis of U.*

Example 5.2.13

Find a basis of $\mathbb{R}^4$ containing $S = \{\mathbf{u},\ \mathbf{v}\}$ where $\mathbf{u} = (0,\ 1,\ 2,\ 3)$ and $\mathbf{v} = (2,\ -1,\ 0,\ 1)$.

Solution. By Theorem 5.2.6 we can find such a basis by adding vectors from the standard basis of $\mathbb{R}^4$ to S. If we try $\mathbf{e}_1 = (1,\ 0,\ 0,\ 0)$, we find easily that $\{\mathbf{e}_1,\ \mathbf{u},\ \mathbf{v}\}$ is independent. Now add another vector from the standard basis, say $\mathbf{e}_2$.
Again we find that $B = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{u},\ \mathbf{v}\}$ is independent. Since B has $4 = \dim \mathbb{R}^4$ vectors, then B must span $\mathbb{R}^4$ by Theorem 5.2.7 below (or simply verify it directly). Hence B is a basis of $\mathbb{R}^4$.

Theorem 5.2.6 has a number of useful consequences. Here is the first.

Theorem 5.2.7

Let U be a subspace of $\mathbb{R}^n$ where $\dim U = m$ and let $B = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_m\}$ be a set of m vectors in U. Then B is independent if and only if B spans U.

Proof. Suppose B is independent. If B does not span U then, by Theorem 5.2.6, B can be enlarged to a basis of U containing more than m vectors. This contradicts the invariance theorem because $\dim U = m$, so B spans U. Conversely, if B spans U but is not independent, then B can be cut down to a basis of U containing fewer than m vectors, again a contradiction. So B is independent, as required. □

As we saw in Example 5.2.13, Theorem 5.2.7 is a "labour-saving" result. It asserts that, given a subspace U of dimension m and a set B of exactly m vectors in U, to prove that B is a basis of U it suffices to show either that B spans U or that B is independent. It is not necessary to verify both properties.

Theorem 5.2.8

Let $U \subseteq W$ be subspaces of $\mathbb{R}^n$. Then:

1. *$\dim U \leq \dim W$.*
2. *If $\dim U = \dim W$, then $U = W$.*