Supplement to Shaping a Swarm With a Shared Control Input Using Boundary Walls and Wall Friction

Paper-ID 66

***Abstract*—Describes videos. Also includes algorithms and equations too lengthy for main paper, but potentially useful for the community.**

I. Video Attachments

Five videos animate the key algorithms in this paper.

A. *CircleSwarmAnalytical.mov*

The video *CircleSwarmAnalytical.mov* shows the stable configuration of a swarm under a constant global input. Animated plots show mean, variance, covariance, and correlation for a swarm in a circular workspace.

B. *SquareSwarmAnalytical.mov*

The video *SquareSwarmAnalytical.mov* shows the stable configuration of a swarm under a constant global input. Animated plots show mean, variance, covariance, and correlation for a swarm in a square workspace.

C. *TwoRobotPosition.mp4*

Animates Algs. 1, 2, 3 in Mathematica to show how two robots can be arbitrarily positioned in a square workspace. In this video the desired initial and ending positions of the two robots are manipulated, and the path that the robots should follow is drawn. The video ends with an extreme case where the robots must exchange positions.

D. *nRobotPostionPacman.mp4*

An implementation of Alg. 4 in MATLAB that illustrates how the two robots positioning algorithm is extendable to n robots. In this video all robots gets the same input, but by exploiting wall friction each robot reaches its goal.

E. *AutomaticCovControl.mp4*

A closed-loop controller that steers a swarm of particles to a desired covariance, implemented with a box2D simulator. In this video the green ellipse is the desired covariance ellipse, the red ellipse is the current covariance ellipse of the swarm and the red dot is the mean position of the robots. Robots follow the algorithm to achieve the desired values for σ_{goalxy}, σ_x^2 and σ_y^2.

II. Calculations for modeling swarm as fluid in a simple planar workspace

Two workspaces are used, a square and a circular workspace.

A. *Square Workspace*

This section provides formulas for the mean, variance, covariance and correlation of a very large swarm of robots as they move inside a square workplace under the influence of gravity pointing in the direction β. The swarm is large, but the robots are small in comparison, and together cover an area of constant volume A. Under a global input such as gravity, they flow like water, moving to a side of the workplace and forming a polygonal shape. The workspace is

The range of possible angles for the global input angle β is [0,2π). In this range of angles, the swarm assumes eight different polygonal shapes. The shapes alternate between triangles and trapezoids when the area A<1/2, and alternate between squares with one corner removed and trapezoids when A>1/2.

Two representative formulas are attached, the outline of the swarm shapes in (II) and $\bar{x}(\beta, A)$ in (I).

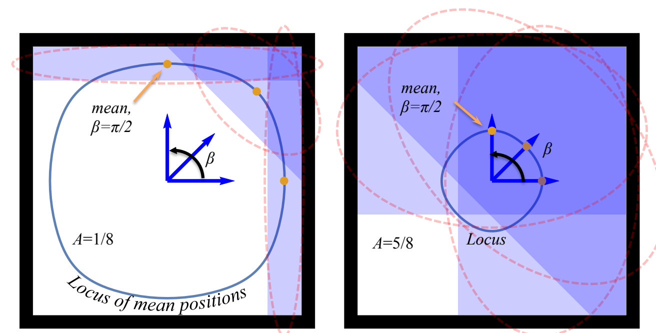

Fig. 1. A swarm in a square workspace under a constant global input assumes either a triangular or a trapezoidal shape if $A < 1/2$. If $A > 1/2$ the swarm is either a squares with one corner removed or a trapezoidal shape.

B. *Circle Workspace*

The area under a chord of a circle is the area of a sector less the area of the triangle originating at the circle center: $A = S(sector) - S(triangle) = 1/2LR - 1/2C(1-h)$, thus

$$A = (1/2)\left[LR - c(R-h)\right] \tag{3}$$

where L is arc length, c is chord length, R is radius and h is height. Solving for L and C gives

$$L = 2\cos^{-1}(1-h) \tag{4}$$
$$C = 2\sqrt{h(2-h)} \tag{5}$$

Therefore the area under a chord is

$$\cos^{-1}(1-h) - (1-h)\sqrt{(2-h)h} \tag{6}$$

$$
\begin{aligned}
\bar{x}(\beta, A) = A \leq \frac{1}{2} :& \begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \leq 2\pi \\
1 - \frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) < \beta \leq \frac{\pi}{2} - \tan^{-1}(2A) \\
\frac{\cot(\beta)}{12A} + \frac{1}{2} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{\pi}{2} \\
\frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}(2A) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \pi \\
\frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}(2A) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{3\pi}{2} \\
1 - \frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 :& \begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq 2\pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{6A+\cot(\beta)}{12A} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} \\
\frac{-2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(1-A)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} \\
\frac{2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right)
\end{cases} \\
A = 1 :& \frac{1}{2}
\end{aligned} \tag{1}
$$

TABLE I

$\bar{x}$ IN A UNIT-SQUARE WORKSPACE

For a circular workspace, with $\beta = 0$, the variance of x and y are:

$$
\sigma_x^2(h) = \frac{64(h-2)^3h^3}{144\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)^2} + \frac{9\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)\left(\sin\left(4\arcsin(1-h)\right) + 4\arccos(1-h)\right)}{144\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)^2} \tag{7}
$$

$$
\sigma_y^2(h) = \frac{12\arccos(1-h) - 8\sin\left(2\arccos(1-h)\right) + \sin\left(4\arccos(1-h)\right)}{48\left(\sqrt{-(h-2)h}(h-1) + \arccos(1-h)\right)} \tag{8}
$$

For $\beta = 0$, $\sigma_{xy} = 0$. These values can be rotated to calculate $\sigma_x^2(\beta, h), \sigma_y^2(\beta, h)$, and $\sigma_{xy}(\beta, h)$.

REFERENCES

$$
\text{RobotRegion}(\beta, A) = \begin{cases}
A \le \frac{1}{2} : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -A - \frac{\tan(\beta)}{2} + 1 & 1 \\ -A + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & 0 \le \beta \le \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \le 2\pi \\
\begin{pmatrix} 1 & 1 \\ 1 - \sqrt{2}\sqrt{A\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) < \beta \le \frac{\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -A + \frac{\cot(\beta)}{2} + 1 \\ 1 & -A - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \frac{\pi}{2} \\
\begin{pmatrix} 0 & 1 \\ \sqrt{2}\sqrt{-A\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \le \pi - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ A - \frac{\tan(\beta)}{2} & 1 \\ A + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & \pi - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \pi \\
\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\sqrt{A\cot(\beta)} \\ \sqrt{2}\sqrt{A\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}(2A) + \pi < \beta \le \frac{3\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & A - \frac{\cot(\beta)}{2} \\ 0 & A + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \frac{3\pi}{2} \\
\begin{pmatrix} 1 & 0 \\ 1 - \sqrt{2}\sqrt{-A\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \le 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ (1-A) - \frac{\tan(\beta)}{2} & 1 \\ (1-A) + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & 0 \le \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le 2\pi \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & \sqrt{2}\sqrt{(1-A)\cot(\beta)} \\ \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & (1-A) - \frac{\cot(\beta)}{2} \\ 0 & (1-A) + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 - \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} < \beta \le \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -(1-A) - \frac{\tan(\beta)}{2} + 1 & 1 \\ -(1-A) + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 - \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi < \beta \le \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -(1-A) + \frac{\cot(\beta)}{2} + 1 \\ 1 & -(1-A) - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} < \beta \le 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right)
\end{cases}, \\
A = 1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}
\end{cases} \tag{2}
$$

TABLE II
ROBOTREGIONS IN A UNIT-SQUARE WORKSPACE