Supplement to Shaping a Swarm With a Shared Control Input Using Boundary Walls and Wall Friction

Paper-ID 66

***Abstract*—Includes algorithms and equations too lengthy for main paper, but potentially useful for the community.**

I. Calculations for modeling swarm as fluid in a simple planar workspace

Two workspaces are used, a square and a circular workspace.

A. Square Workspace

This section provides formulas for the mean, variance, covariance and correlation of a very large swarm of robots as they move inside a square workplace under the influence of gravity pointing in the direction β. The swarm is large, but the robots are small in comparison, and together cover an area of constant volume A. Under a global input such as gravity, they flow like water, moving to a side of the workplace and forming a polygonal shape. The workspace is

The range of possible angles for the global input angle β is [0,2π). In this range of angles, the swarm assumes eight different polygonal shapes. The shapes alternate between triangles and trapezoids when the area A<1/2, and alternate between squares with one corner removed and trapezoids when A>1/2.

Two representative formulas are attached, the outline of the swarm shapes in (II) and $\bar{x}(\beta, A)$ in (I).

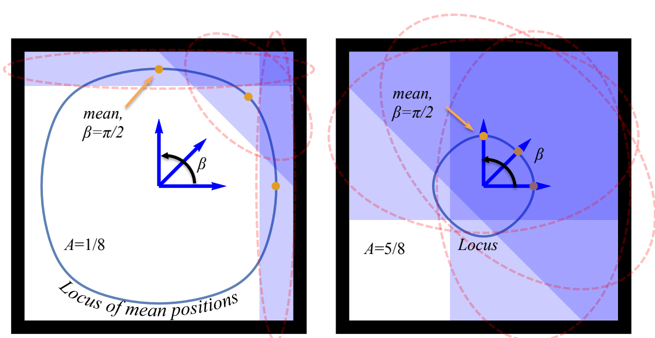

Fig. 1. A swarm in a

B. Circle Workspace

The area under a chord of a circle is the area of a sector less the area of the triangle originating at the circle center: provement of area formula: $A = S(sector) - S(triangle) = 1/2LR - 1/2C(1-h)$, thus

$$A = (1/2)\left[LR - c(R-h)\right] \tag{3}$$

where L is arc length, c is chord length, R is radius and h is height. Solving for L and C gives

$$L = 2\cos^{-1}(1-h) \tag{4}$$

$$C = 2\sqrt{h(2-h)} \tag{5}$$

Therefor the area under a chord is

$$\cos^{-1}(1-h) - (1-h)\sqrt{(2-h)h} \tag{6}$$

The variance of x and y are:

$$\sigma_x^2(h) = \frac{64(h-2)^3h^3}{144\left(\sqrt{-(h-2)h}(h-1)+\arccos(1-h)\right)^2} + \frac{9\left(\sqrt{-(h-2)h}(h-1)+\arccos(1-h)\right)\left(\sin\left(4\arcsin(1-h)\right)+4\arccos(1-h)\right)}{144\left(\sqrt{-(h-2)h}(h-1)+\arccos(1-h)\right)^2} \tag{7}$$

$$\sigma_y^2(h) = \frac{12\arccos(1-h) - 8\sin\left(2\arccos(1-h)\right) + \sin\left(4\arccos(1-h)\right)}{48\left(\sqrt{-(h-2)h}(h-1)+\arccos(1-h)\right)} \tag{8}$$

II. Algorithm for generating desired y spacing between two robots using wall friction

$$
\bar{x}(\beta, A) = \begin{aligned}
A \leq \frac{1}{2} :& \begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \leq 2\pi \\
1 - \frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) < \beta \leq \frac{\pi}{2} - \tan^{-1}(2A) \\
\frac{\cot(\beta)}{12A} + \frac{1}{2} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{\pi}{2} \\
\frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}(2A) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \pi \\
\frac{1}{3}\sqrt{2}\sqrt{A\tan(\beta)} & \tan^{-1}(2A) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}(2A) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \leq \tan^{-1}(2A) + \frac{3\pi}{2} \\
1 - \frac{1}{3}\sqrt{2}\sqrt{-A\tan(\beta)} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 :& \begin{cases}
-\frac{\tan^2(\beta)}{24A} - \frac{A}{2} + 1 & 0 \leq \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq 2\pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{6A+\cot(\beta)}{12A} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} \\
\frac{-2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} < \beta \leq \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{\tan^2(\beta)}{24A} + \frac{A}{2} & \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi \\
\frac{2\sqrt{2}\sqrt{(1-A)\tan(\beta)}(1-A)+6A-3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi < \beta \leq \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\frac{1}{2} - \frac{\cot(\beta)}{12A} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \leq \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} \\
\frac{2\sqrt{2}\sqrt{(A-1)\tan(\beta)}(A-1)+3}{6A} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} < \beta \leq 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right)
\end{cases} \\
A = 1 :& \frac{1}{2}
\end{aligned} \tag{1}
$$

TABLE I

$\bar{x}$ IN A UNIT-SQUARE WORKSPACE

$$
\text{RobotRegion}(\beta, A) = \begin{cases}
A \le \frac{1}{2} : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -A - \frac{\tan(\beta)}{2} + 1 & 1 \\ -A + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & 0 \le \beta \le \tan^{-1}(2A) \vee 2\pi - \tan^{-1}(2A) < \beta \le 2\pi \\
\begin{pmatrix} 1 & 1 \\ 1 - \sqrt{2}\sqrt{A\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) < \beta \le \frac{\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -A + \frac{\cot(\beta)}{2} + 1 \\ 1 & -A - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \frac{\pi}{2} \\
\begin{pmatrix} 0 & 1 \\ \sqrt{2}\sqrt{-A\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{\pi}{2} < \beta \le \pi - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ A - \frac{\tan(\beta)}{2} & 1 \\ A + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & \pi - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \pi \\
\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\sqrt{A\cot(\beta)} \\ \sqrt{2}\sqrt{A\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}(2A) + \pi < \beta \le \frac{3\pi}{2} - \tan^{-1}(2A) \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & A - \frac{\cot(\beta)}{2} \\ 0 & A + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}(2A) < \beta \le \tan^{-1}(2A) + \frac{3\pi}{2} \\
\begin{pmatrix} 1 & 0 \\ 1 - \sqrt{2}\sqrt{-A\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-A\cot(\beta)} \end{pmatrix} & \tan^{-1}(2A) + \frac{3\pi}{2} < \beta \le 2\pi - \tan^{-1}(2A)
\end{cases} \\
\frac{1}{2} < A < 1 : \begin{cases}
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ (1-A) - \frac{\tan(\beta)}{2} & 1 \\ (1-A) + \frac{\tan(\beta)}{2} & 0 \end{pmatrix} & 0 \le \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) \vee 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le 2\pi \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & \sqrt{2}\sqrt{(1-A)\cot(\beta)} \\ \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 0 \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & (1-A) - \frac{\cot(\beta)}{2} \\ 0 & (1-A) + \frac{\cot(\beta)}{2} \end{pmatrix} & \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 - \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 0 \\ 1 & \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{\pi}{2} < \beta \le \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -(1-A) - \frac{\tan(\beta)}{2} + 1 & 1 \\ -(1-A) + \frac{\tan(\beta)}{2} + 1 & 0 \end{pmatrix} & \pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 - \sqrt{2}\sqrt{(1-A)\tan(\beta)} & 1 \\ 1 & 1 - \sqrt{2}\sqrt{(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \pi < \beta \le \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) \\
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -(1-A) + \frac{\cot(\beta)}{2} + 1 \\ 1 & -(1-A) - \frac{\cot(\beta)}{2} + 1 \end{pmatrix} & \frac{3\pi}{2} - \tan^{-1}\left(\frac{1}{2}, 1-A\right) < \beta \le \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} \\
\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ \sqrt{2}\sqrt{-(1-A)\tan(\beta)} & 1 \\ 0 & 1 - \sqrt{2}\sqrt{-(1-A)\cot(\beta)} \end{pmatrix} & \tan^{-1}\left(\frac{1}{2}, 1-A\right) + \frac{3\pi}{2} < \beta \le 2\pi - \tan^{-1}\left(\frac{1}{2}, 1-A\right)
\end{cases} \\
A = 1 : \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}
\end{cases}, \tag{2}
$$

TABLE II

ROBOTREGIONS IN A UNIT-SQUARE WORKSPACE