- # This code examines the variance and covariance of a very large swarm of robots as they move inside a square workplace.

The swarm is large, but the robots are small in comparison, and together cover an area of constant volume A.
 When they are pushed to a side, they flow like water.
We want to know the mean position and variance of this swarm
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It turns out (if the square is infinitely large) we can achieve any covairance. We cannot control the correlation:
correlation (X,Y) is
cor(X,Y) = $\frac{\mathrm{cov}(X, Y)}{\mathrm{sd}(X) * \mathrm{sd}(Y)}$

For a right triangle aligned with the world axis, cor(X,Y) is $\frac{1}{2}$ or $\frac{-1}{2}$.

And covariance is ±Area/18

todo: as a function of α, determine the 8 regions of operation
plot the area for each region
solve for the mean in each region
solve for the variance and covariance for each region
make a plot of mean x,y as afunction of α and A
make a plot of covariance as a function of α and A
make a plot of variance as a function of α and A

Background Math Equations

The centroid (center of mass) = integral over A of x/Area

$$C = \frac{\int x g(x)\, dx}{\int g(x)\, dx},$$ g (x) is the characteristic function (1 inside the region, 0 outside)

For a triangle, whose end points are L = { x_L, y_L}, M = { x_M, y_M}, N = { x_N, y_N}

$$C = \frac{1}{3}(L + M + N) = \left(\frac{1}{3}(x_L + x_M + x_N),\ \frac{1}{3}(y_L + y_M + y_N)\right).$$

The centroid of a non-self-intersecting closed polygon defined by n vertices (x0,y0), (x1,y1), ..., (xn−1,yn−1) is the point (Cx, Cy), where

$$C_x = \frac{1}{6A}\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i\ y_{i+1} - x_{i+1}\ y_i)$$

and where A is the polygon's signed area,

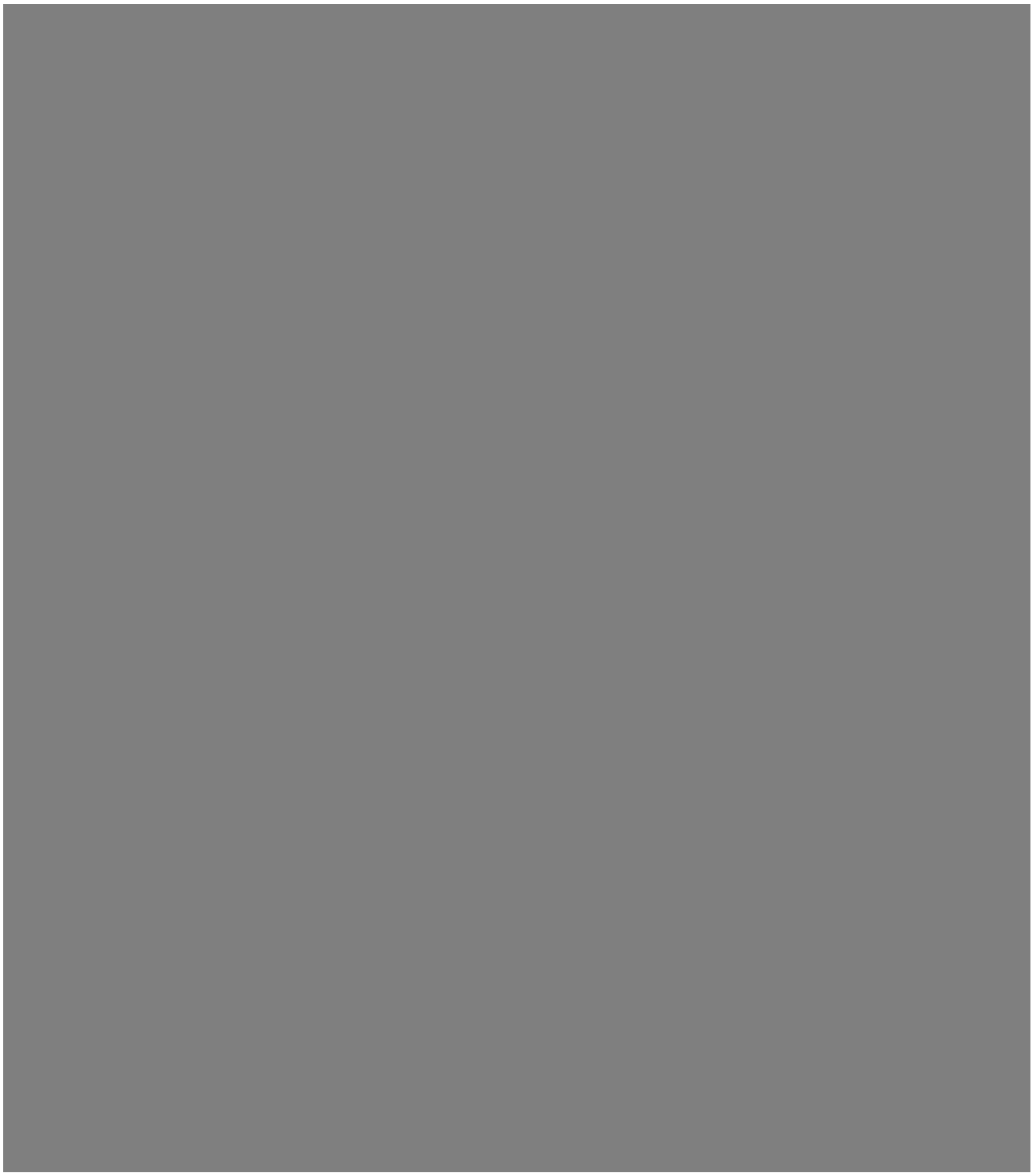
$$A = \frac{1}{2}\sum_{i=0}^{n-1}(x_i\ y_{i+1} - x_{i+1}\ y_i)$$

MonteCarlo Simulation

This simulation shows that the correlation of a swarm of robots in a triangle is always $\pm\frac{1}{2}$.

The covariance is always ±**Area/18**

```
pts = RandomReal[{0, 1}, {100 000, 2}];
Manipulate[Module[{tpts, npts, cov, mean},
  tpts  = Select[ pts, #[[1]] ≤ b - #[[2]] 4 b^2 &];
  (*npts  =Select[ pts,#[[1]]>b-#[[2]]4b^2&];*)
  cov  = Covariance[tpts];
  mean  = Mean[tpts];
  ListPlot[{(*npts,*)tpts}, PlotMarkers → None,
   PlotStyle -> {PointSize[Tiny]},
   PlotRange → {{-.2, 1.2}, {-.2, 1.2}},
   AspectRatio → 1,
   PlotLabel → StringForm["Cov[xy]=``,Cor[XY]=``",
     cov[[1, 2]],
     cov[[1, 2]] / (√(cov[[1, 1]]) √(cov[[2, 2]]))],
   Epilog → {Green, PointSize[Large], Point[mean],
     Opacity[0], EdgeForm[{Thick, Red}],
     Ellipsoid[mean, 6 cov]}]], {b, 0.25, 1}]
```



```
0 = m 1 / (4 b) + b
(*slope-intercept equation of a line for a
  triangle*)
-b 4 b = m
a b = 1 / 4
a = 1 / (4 b)

rise = b, run =  1 / (4 b)

rise * run = 1 / 4
```

```
N[(1/8)/18]
```

0.00694444

For α between 1/16 π to 7/16 π, what is the covariance?

$$xm = \text{FullSimplify}\left[ell = 2\sqrt{A\,\text{Csc}[2\,\alpha]};\ \frac{\int_0^{ell\,\text{Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x} x\,dy\right)dx}{A}\right]$$

$$\text{FullSimplify}\left[\frac{0 + 2\,ell\,\text{Cos}[\alpha]}{3}\right]$$

$$ym = \text{FullSimplify}\left[ell = 2\sqrt{A\,\text{Csc}[2\,\alpha]};\ \frac{\int_0^{ell\,\text{Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x} y\,dy\right)dx}{A}\right]$$

$$\text{FullSimplify}\left[\frac{2*0 + ell\,\text{Sin}[\alpha]}{3}\right]$$

$$\frac{4}{3}\,\text{Cos}[\alpha]\,\sqrt{A\,\text{Csc}[2\,\alpha]}$$

$$\frac{4}{3}\,\text{Cos}[\alpha]\,\sqrt{A\,\text{Csc}[2\,\alpha]}$$

$$\frac{2}{3}\,\sqrt{A\,\text{Csc}[2\,\alpha]}\,\text{Sin}[\alpha]$$

$$\frac{2}{3}\,\sqrt{A\,\text{Csc}[2\,\alpha]}\,\text{Sin}[\alpha]$$

```
Clear[a]
```

$$\int_0^{b\,a}\left(\int_0^{x/b^2} y\,dy\right)dx$$

$$\left(\int_0^{b\,a}\left(\int_0^{x/b^2} (y)\,dy\right)dx\right)$$

$$\left(\int_0^{b\,a}\left(\int_0^{x/b^2} (x)\,dy\right)dx\right)$$

$$\frac{a^3}{6\,b}$$

$$\frac{a^3}{6\,b}$$

$$\frac{a^3\,b}{3}$$

Compute covariance for a triangle with area $\frac{1}{2}$ a^2 http : // www.talkstats.com/showthread.php/15046 - Uniform - distribution - on - a - triangle

```
Clear[a, b]
```

$$\text{Simplify}\left[\frac{\int_0^{b\,a}\left(\int_0^{x/b^2}(x\,y)\,dy\right)dx}{\frac{1}{2}\,a^2} - \frac{\left(\int_0^{b\,a}\left(\int_0^{x/b^2}(x)\,dy\right)dx\right)}{\frac{1}{2}\,a^2}\,\frac{\left(\int_0^{b\,a}\left(\int_0^{x/b^2}(y)\,dy\right)dx\right)}{\frac{1}{2}\,a^2}\right]$$

$$\text{Simplify}\left[\frac{\int_0^{b\,a}\left(\int_0^{x/b^2}(x)^2\,dy\right)dx}{\frac{1}{2}\,a^2} - \frac{\left(\int_0^{b\,a}\left(\int_0^{x/b^2}(x)\,dy\right)dx\right)^2}{\frac{1}{2}\,a^2}\right]$$

$$\text{Simplify}\left[\frac{\int_0^{b\,a}\left(\int_0^{x/b^2}(y)^2\,dy\right)dx}{\frac{1}{2}\,a^2} - \frac{\left(\int_0^{b\,a}\left(\int_0^{x/b^2}(y)\,dy\right)dx\right)^2}{\frac{1}{2}\,a^2}\right]$$

$$\frac{a^2}{36}$$

$$\frac{1}{18}\,a^2\,\left(9-4\,a^2\right)\,b^2$$

$$-\frac{a^2\,\left(-3+a^2\right)}{18\,b^2}$$

$$\text{Simplify}\left[\frac{1}{18}\,a^2\,\left(9-4\,a^2\right)\,b^2 * -\frac{a^2\,\left(-3+a^2\right)}{18\,b^2}\right]$$

$$\frac{1}{324}\,a^4\,\left(-3+a^2\right)\,\left(-9+4\,a^2\right)$$

```
Clear[a]
```

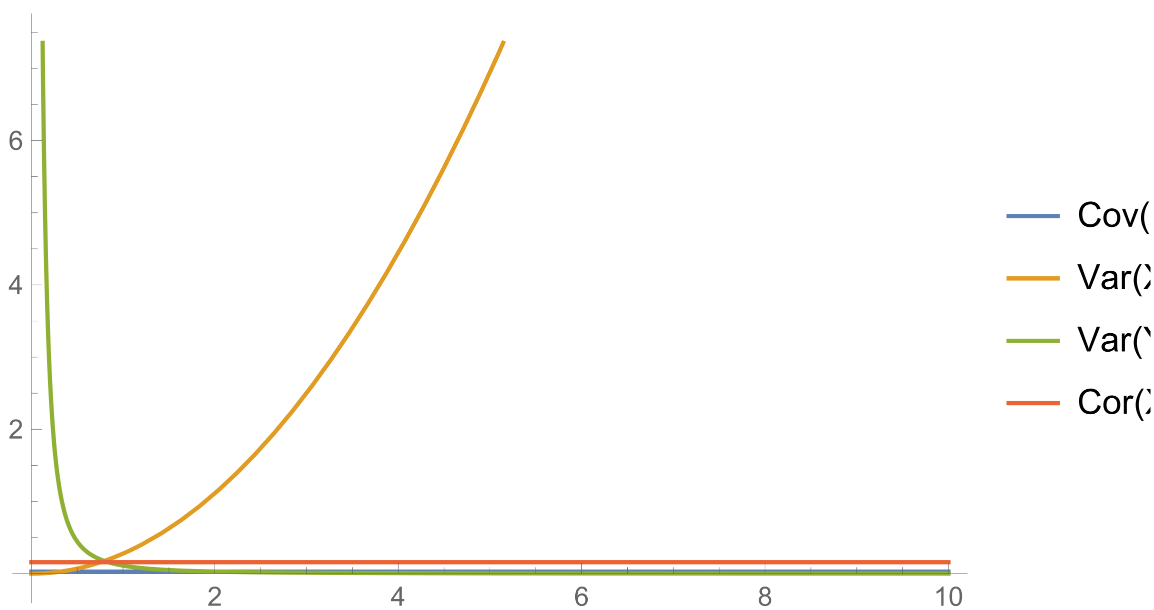
$$\text{FullSimplify}\left[\frac{\frac{a^2}{36}}{\sqrt{\left(\frac{1}{18} a^2 \left(9-4 a^2\right) b^2\right)} \sqrt{\left(\frac{a^2 \left(-3+a^2\right)}{18 b^2}\right)}}\right]$$

$$\frac{a^2}{2 \sqrt{\frac{a^2 \left(-3+a^2\right)}{b^2}} \sqrt{a^2 \left(9-4 a^2\right) b^2}}$$

```
a = 1;
```

$$\text{Plot}\left[\left\{\frac{a^2}{36}, \frac{1}{18} a^2 \left(9-4 a^2\right) b^2, -\frac{a^2 \left(-3+a^2\right)}{18 b^2}, \frac{\frac{a^2}{36}}{\sqrt{\left(\frac{1}{18} a^2 \left(9-4 a^2\right) b^2\right)} \sqrt{\left(-\frac{a^2 \left(-3+a^2\right)}{18 b^2}\right)}}\right\}, \{b, 0, 10\},\right.$$

```
PlotLegends → {"Cov(X,Y)", "Var(X)", "Var(Y)",
   "Cor(X,Y)"}]
```



```
(*compute covariances.  Why is the covariance
 constant?  That doesn't make sense*)
```

$$\text{Simplify}\left[\text{ell} = 2\sqrt{\text{A Csc}[2\,\alpha]};\ \frac{\int_0^{\text{ell Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x}\left(x - \frac{4}{3}\text{Cos}[\alpha]\sqrt{\text{A Csc}[2\,\alpha]}\right)^2 dy\right)dx}{A}\right]$$

$$\text{Simplify}\left[\text{ell} = 2\sqrt{\text{A Csc}[2\,\alpha]};\ \frac{\int_0^{\text{ell Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x}(x)^2\,dy\right)dx}{A} - \text{xm}^2\right]$$

$$\text{Simplify}\left[\text{ell} = 2\sqrt{\text{A Csc}[2\,\alpha]};\ \frac{\int_0^{\text{ell Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x}(x\,y)\,dy\right)dx}{A} - \text{xm ym}\right]$$

$$\text{Simplify}\left[\text{ell} = 2\sqrt{\text{A Csc}[2\,\alpha]};\ \frac{1}{A}\left(\int_0^{\text{ell Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x}\left(x - \frac{4}{3}\text{Cos}[\alpha]\sqrt{\text{A Csc}[2\,\alpha]}\right)\left(y - \frac{2}{3}\sqrt{\text{A Csc}[2\,\alpha]}\,\text{Sin}[\alpha]\right)dy\right)dx\right)\right]$$

$$\text{Simplify}\left[\text{ell} = 2\sqrt{\text{A Csc}[2\,\alpha]};\ \frac{\int_0^{\text{ell Cos}[\alpha]}\left(\int_0^{\text{Tan}[\alpha]\,x}(y)^2\,dy\right)dx}{A} - \text{ym}^2\right]$$

$$\frac{1}{9}\,\text{A Cot}[\alpha]$$

$$\frac{1}{9}\,\text{A Cot}[\alpha]$$

$$\frac{A}{18}$$

$$\frac{A}{18}$$

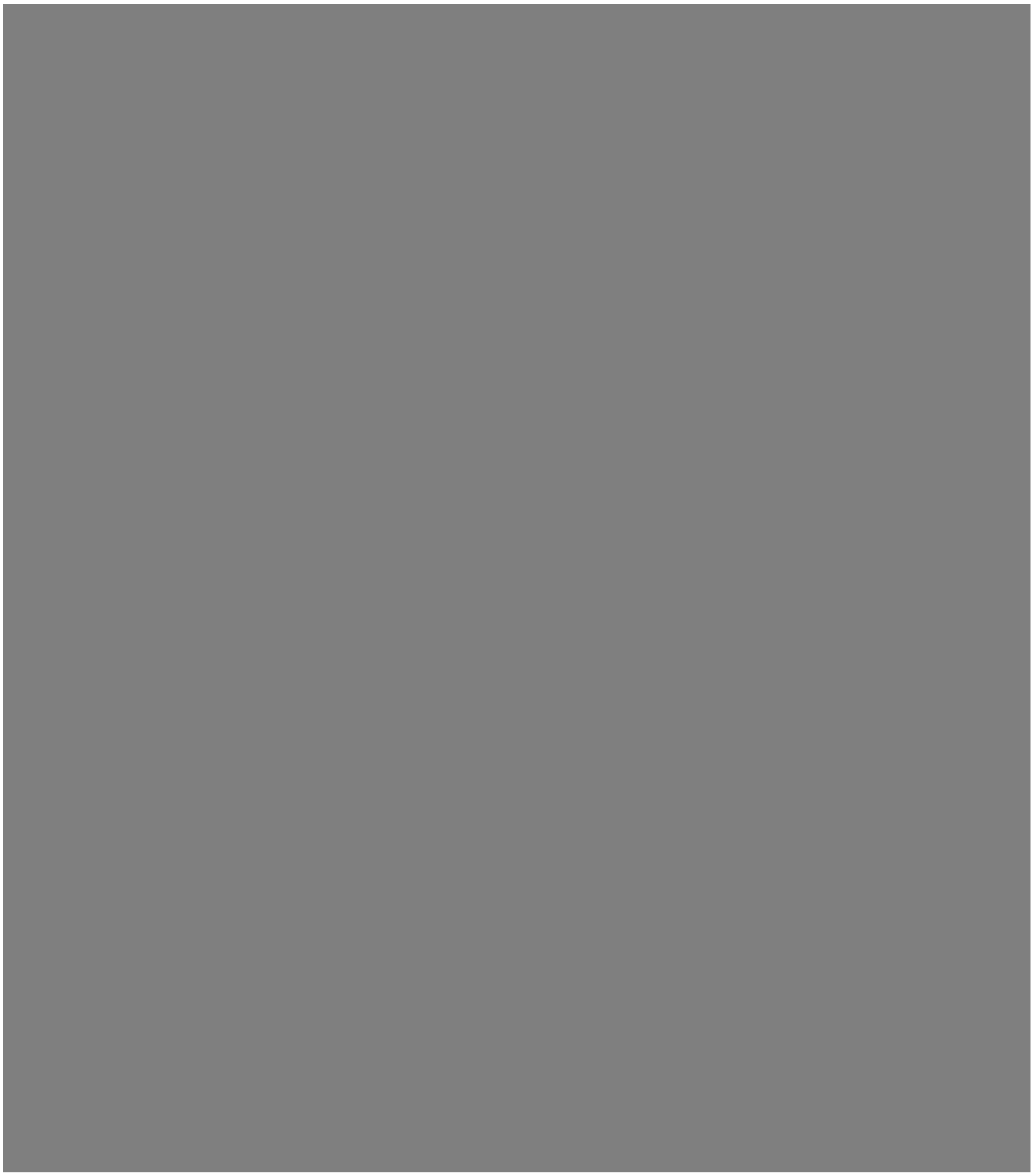
$$\frac{1}{9}\, A \operatorname{Tan}[\alpha]$$

```
Simplify[A^2/2 - 8/9 A Cos[α] Csc[2 α] Sin[α]]
```

$$\frac{1}{18}\, A\, (-8 + 9 A)$$

```
Manipulate[Module[{A = 1/8, ell , mean, cov},
  ell  = 2 Sqrt[A Csc[2 α]] (*= Sqrt[(2A/(Sin[α]Cos[α]))] *);
  mean = {(0 + 2 ell Cos[α])/3, (2*0 + ell Sin[α])/3};
  cov  = {{1/9 A^2 Cot[α], A^2/18}, {A^2/18, 1/9 A^2 Tan[α]}};
  Plot[{If[x < ell Cos[α], Tan[α] x, 0], A/(2 x)},
   {x, 0, 2}, AspectRatio → 1,
   Filling → {1 -> Axis, None},
   PlotRange → {{0, 2}, {0, 2}},

   Epilog → {Red, PointSize[Large], Point[mean],
     Opacity[0], EdgeForm[{Thick, Red}],
     Ellipsoid[mean, Sqrt[cov]],
     Ellipsoid[mean, 2 Sqrt[cov]]
    },
   PlotLabel → StringForm["Area = ``",
     1/2 ell Cos[α] * Tan[α] ell Cos[α]]]],
 {α, 1/64 π , 15/32 π}]
```



$$\text{Simplify}\left[\int_0^1 \left(\int_0^x (x - xm)\,(y - ym)\,dy\right) dx\right]$$

$$\text{Simplify}\left[\int_0^1 \left(\int_0^x y\,y\,dy\right) dx\right]$$

$$\text{Simplify}\left[\int_0^1 \left(\int_0^x x\,x\,dy\right) dx\right]$$

$$\frac{1}{24}\,(3 - 8\,ym + 4\,xm\,(-1 + 3\,ym))$$

$$\frac{1}{12}$$

$$\frac{1}{4}$$

$$\text{Simplify}\left[\int_{cx}^1 \left(\int_0^{\frac{cy}{1-cx}x-\frac{cy}{1-cx}cx} dy\right) dx\right]$$

$$\text{Simplify}\left[\int_{cx}^1 \left(\int_0^{\frac{cy}{1-cx}x-\frac{cy}{1-cx}cx} x\,dy\right) dx\right]$$

$$\text{Simplify}\left[\int_{cx}^{1}\left(\int_{0}^{\frac{cy}{1-cx}x-\frac{cy}{1-cx}cx}\left(x-\frac{cx+2*1}{3}\right)^{2}dy\right)dx\right]$$

$$\text{Simplify}\left[\int_{cx}^{1}\left(\int_{0}^{\frac{cy}{1-cx}x-\frac{cy}{1-cx}cx}\left(x-\frac{cx+2*1}{3}\right)\left(y-\frac{cy}{3}\right)dy\right)dx\right]$$

$$\text{Simplify}\left[\int_{cx}^{1}\left(\int_{0}^{\frac{cy}{1-cx}x-\frac{cy}{1-cx}cx}\left(y-\frac{cy}{3}\right)^{2}dy\right)dx\right]$$

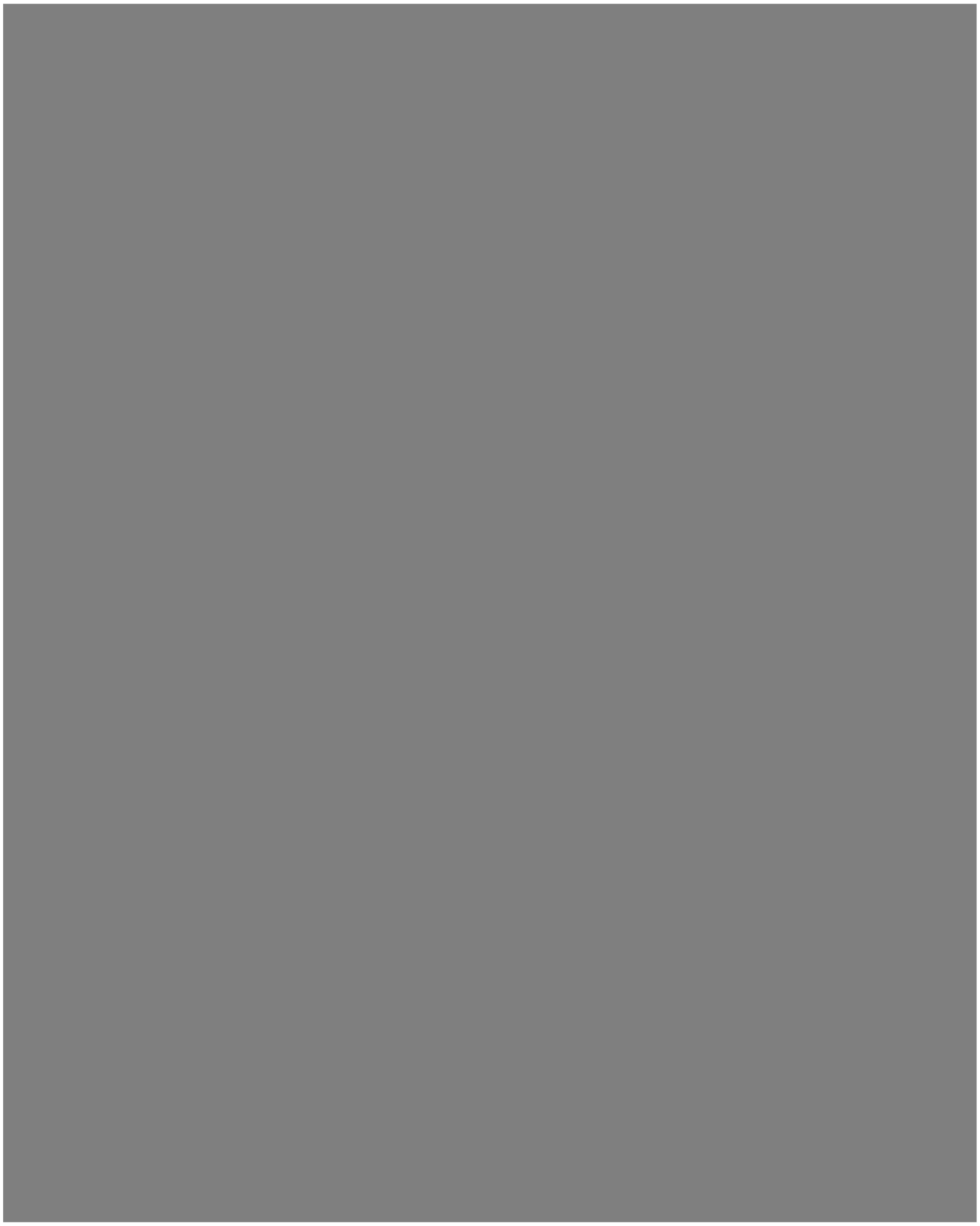
$$-\frac{1}{36}(-1+cx)^{3}\,cy$$

$$\frac{1}{72}(-1+cx)^{2}\,cy^{2}$$

$$-\frac{1}{36}(-1+cx)\,cy^{3}$$

```
Manipulate[Module[{mean, cov, cxy, bwid = 0.05, A},
  A = (1 - cx) cy/2;
  mean = {(cx + 2 * 1)/3, cy/3};
  cxy = 1/72 (-1 + cx)^2 cy^2;
  cov = {{-1/36 (-1 + cx)^3 cy, cxy},
     {cxy, -1/36 (-1 + cx) cy^3}};
  Plot[cy/(1 - cx) x - cy/(1 - cx) cx, {x, 0, 1},
   PlotRange -> {{-bwid, 2 + bwid}, {-bwid, 2 + bwid}},
   AspectRatio -> 1,
   PlotLabel -> StringForm["Area = ``,mean=``, cov=``",
     A, mean, cov],
   Prolog -> {Darker[Red],
     Rectangle[{0, 0} - bwid, {1, 1} + bwid], White,
     Rectangle[{0, 0}, {1, 1}]},
   Epilog -> {Red, PointSize[Large], Point[mean],
     Opacity[0], EdgeForm[{Thick, Red}],
     Ellipsoid[mean, Sqrt[cov]],
     Ellipsoid[mean, 2 Sqrt[cov]]}


  ]],
 {cx, 0, 0.99}, {cy, 0.01, 1}]
```



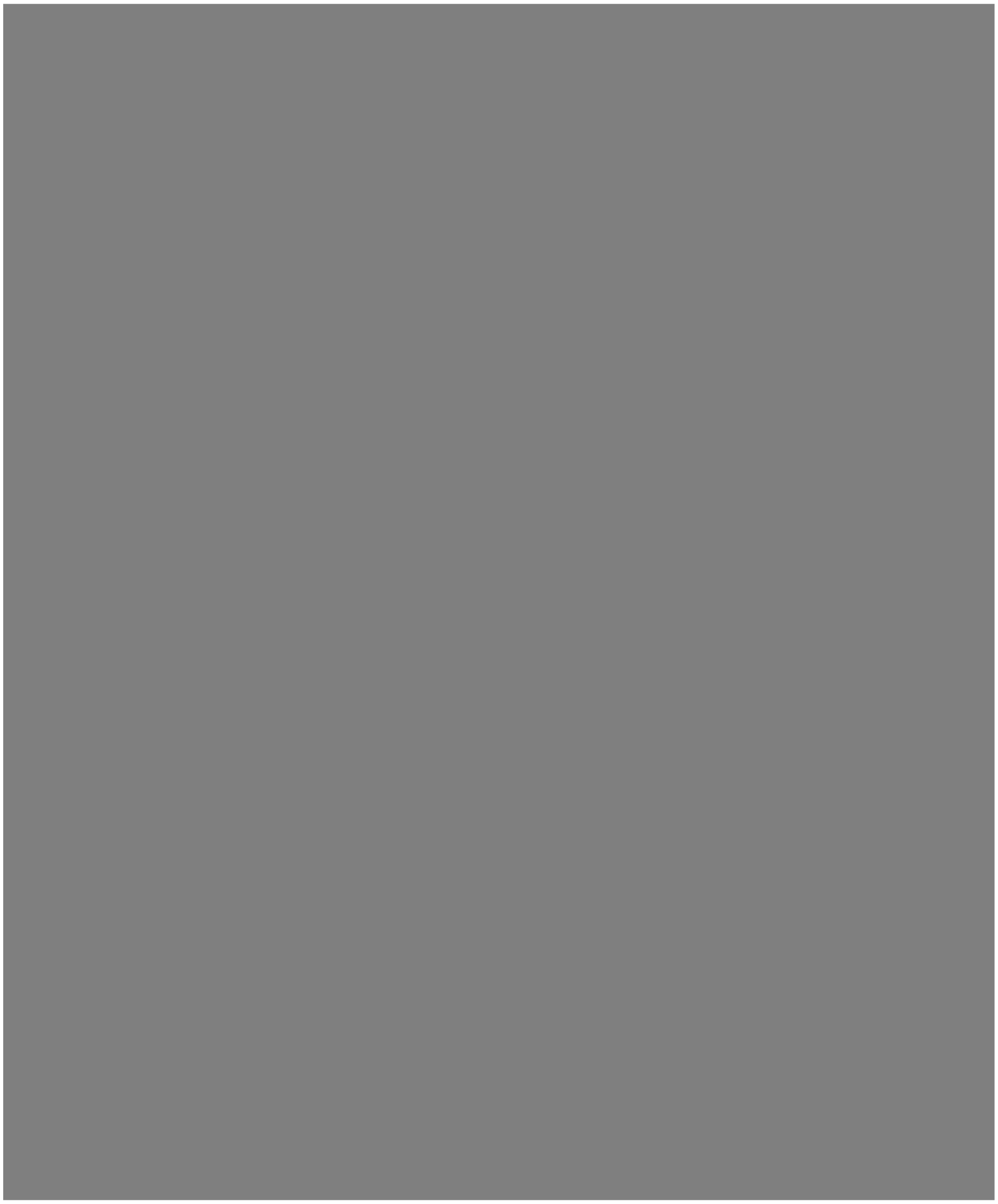
```
Manipulate[Module[{mean, cov, c, bwid = 0.05, A},
  A = Piecewise[{{1 + (-1/2 + b) Cot[α], b + Tan[α] ≥ 1},
                 {1/2 (1 + b Cot[α])^2 Tan[α], True}}];

  mean = Piecewise[{{{-1/12
        (-2 - 3 b + 3 b^2 + (4 - 3 b + 3 b^2) Cos[2 α])
        Csc[α]^2, 1/6 (3 + (-2 + 3 b) Cot[α])}, b + ⋯},
      {{1/6 (2 + 3 b Cot[α] - b^3 Cot[α]^3) Tan[α],
        1/6 (1 + b Cot[α])^3 Tan[α]^2}, Tru⋯}}]

  c = Piecewise[{{{-1/12
        (-2 - 3 b + 3 b^2 + (4 - 3 b + 3 b^2) Cos[2 α])
        Csc[α]^2, 1/6 (3 + (-2 + 3 b) Cot[α])}, b + Tan[⋯},
      {{1/6 (2 + 3 b Cot[α] - b^3 Cot[α]^3) Tan[α],
        1/6 (1 + b Cot[α])^3 Tan[α]^2,
        Cot[α]/12 + 1/3 (1 + (-1 + b) Cot[α])}, True}}]

  cov = {{c[[1]], c[[2]]}, {c[[2]], c[[3]]}};
  RegionPlot[y - Tan[α] x < b, {x, 0, 1}, {y, 0, 1},
   PlotLabel → StringForm["Area = ``,``", A, mean],
   PlotRange → {{-bwid, 1 + bwid}, {-bwid, 1 + bwid}},
   Prolog → {Opacity[0], EdgeForm[{Thick, Red}],
     Point[mean]},
   Epilog → {Darker[Red],
     Rectangle[{0, 0} - bwid, {1, 1} + bwid], White,
     Rectangle[{0, 0}, {1, 1}]}

  ]],
 {α, 0.01, π/2}, {b, -1, 1}]
```



Calculate the Area for triangle beyond line

y=Tan[α]x + b

$$\text{Simplify}\left[\text{Assuming}\left[b < 0\ \&\&\ \alpha > 0\ \&\&\ \alpha < \pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{1}\int_{0}^{\text{Tan}[\alpha]\,x+b} dy\,dx\right]\right]$$

$$\frac{1}{2}\,(1 + b\,\text{Cot}[\alpha])^2\,\text{Tan}[\alpha]$$

$$\text{Assuming}\left[b < 0\ \&\&\ \alpha > 0\ \&\&\ \alpha < \pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{(1-b)/\text{Tan}[\alpha]}\int_{0}^{\text{Tan}[\alpha]\,x+b} dy\,dx + \int_{(1-b)/\text{Tan}[\alpha]}^{1}\int_{0}^{1} dy\,dx\right]$$

$$1 + \frac{\text{Cot}[\alpha]}{2} - (1-b)\,\text{Cot}[\alpha]$$

$$\text{Simplify}\left[\text{If}\left[\text{Tan}[\alpha] + b < 1,\ b + b^2\,\text{Cot}[\alpha] - \frac{1}{2}b^2\,\text{Csc}[\alpha]\,\text{Sec}[\alpha] + \frac{\text{Tan}[\alpha]}{2} + \frac{1}{2}b^2\,\text{Tan}[\alpha],\ 1 + \frac{\text{Cot}[\alpha]}{2} - (1-b)\,\text{Cot}[\alpha]\right]\right]$$

$$\begin{cases} 1 + \left(-\frac{1}{2} + b\right)\text{Cot}[\alpha] & b + \text{Tan}[\alpha] \geq 1 \\ \frac{1}{2}\,(1 + b\,\text{Cot}[\alpha])^2\,\text{Tan}[\alpha] & \text{True} \end{cases}$$

Calculate the Mean for triangle beyond

line

Tan[α]x + b

$$\text{Simplify}\left[\text{Assuming}\left[b<0\ \&\&\ \alpha>0\ \&\&\ \alpha<\pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{1}\int_{0}^{\text{Tan}[\alpha]\,x+b} x\,dy\,dx\right]\right]$$

$$\text{Simplify}\left[\text{Assuming}\left[b<0\ \&\&\ \alpha>0\ \&\&\ \alpha<\pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{1}\int_{0}^{\text{Tan}[\alpha]\,x+b} y\,dy\,dx\right]\right]$$

$$\frac{1}{6}\left(2+3\,b\,\text{Cot}[\alpha]-b^3\,\text{Cot}[\alpha]^3\right)\text{Tan}[\alpha]$$

$$\frac{1}{6}\,(1+b\,\text{Cot}[\alpha])^3\,\text{Tan}[\alpha]^2$$

$$\text{Assuming}\left[b<0\ \&\&\ \alpha>0\ \&\&\ \alpha<\pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{(1-b)/\text{Tan}[\alpha]}\int_{0}^{\text{Tan}[\alpha]\,x+b} x\,dy\,dx+\int_{(1-b)/\text{Tan}[\alpha]}^{1}\int_{0}^{1} x\,dy\,dx\right]$$

$$\text{Assuming}\left[b<0\ \&\&\ \alpha>0\ \&\&\ \alpha<\pi/2,\ \int_{-b/\text{Tan}[\alpha]}^{(1-b)/\text{Tan}[\alpha]}\int_{0}^{\text{Tan}[\alpha]\,x+b} y\,dy\,dx+\int_{(1-b)/\text{Tan}[\alpha]}^{1}\int_{0}^{1} y\,dy\,dx\right]$$

$$\frac{\text{Cot}[\alpha]^2}{3}-\frac{1}{2}\,b\,\text{Cot}[\alpha]^2+\frac{1}{2}\left(2+(-2+b)\,b-(-1+b)^2\,\text{Csc}[\alpha]^2\right)$$

$$\frac{\text{Cot}[\alpha]}{6}+\frac{1}{2}\,(1+(-1+b)\,\text{Cot}[\alpha])$$

```
Simplify[If[Tan[α] + b < 1,
  {1/6 (2 + 3 b Cot[α] - b^3 Cot[α]^3) Tan[α],
   1/6 (1 + b Cot[α])^3 Tan[α]^2}
  ,
  {Cot[α]^2/3 - 1/2 b Cot[α]^2 +
    1/2 (2 + (-2 + b) b - (-1 + b)^2 Csc[α]^2),
   Cot[α]/6 + 1/2 (1 + (-1 + b) Cot[α])}]]
```

$$\begin{cases} \left\{-\frac{1}{12}\left(-2-3\,b+3\,b^{2}+\left(4-3\,b+3\,b^{2}\right)\operatorname{Cos}[2\,\alpha]\right)\operatorname{Csc}[\alpha]^{2},\ \frac{1}{6}\,(3+(-2+3\,b)\operatorname{Cot}[\alpha])\right\} & b+\operatorname{Tan}[\alpha]\geq 1 \\ \left\{\frac{1}{6}\left(2+3\,b\operatorname{Cot}[\alpha]-b^{3}\operatorname{Cot}[\alpha]^{3}\right)\operatorname{Tan}[\alpha],\ \frac{1}{6}\,(1+b\operatorname{Cot}[\alpha])^{3}\operatorname{Tan}[\alpha]^{2}\right\} & \text{True} \end{cases}$$

Calculate the Variance for triangle beyond line

Tan[α]x + b

$$\text{Simplify}\Big[\text{Assuming}\Big[b < 0 \;\&\&\; \alpha > 0 \;\&\&\; \alpha < \pi/2,\; \int_{-b/\text{Tan}[\alpha]}^{1} \Big(x^2 \int_0^{\text{Tan}[\alpha]\,x+b} dy\Big)\, dx\Big]\Big]$$

$$\text{Simplify}\Big[\text{Assuming}\Big[b < 0 \;\&\&\; \alpha > 0 \;\&\&\; \alpha < \pi/2,\; \int_{-b/\text{Tan}[\alpha]}^{1} \int_0^{\text{Tan}[\alpha]\,x+b} y^2\, dy\, dx\Big]\Big]$$

$$\text{Simplify}\Big[\text{Assuming}\Big[b < 0 \;\&\&\; \alpha > 0 \;\&\&\; \alpha < \pi/2,\; \int_{-b/\text{Tan}[\alpha]}^{1} \Big(x \int_0^{\text{Tan}[\alpha]\,x+b} y\, dy\Big)\, dx\Big]\Big]$$

$$\frac{1}{12}\left(3 + 4\, b\, \text{Cot}[\alpha] + b^4\, \text{Cot}[\alpha]^4\right) \text{Tan}[\alpha]$$

$$\frac{1}{12}\left(1 + b\, \text{Cot}[\alpha]\right)^4 \text{Tan}[\alpha]^3$$

$$-\frac{1}{6}\, \text{Csc}[2\,\alpha]^2\, (b\, \text{Cos}[\alpha] - 3\, \text{Sin}[\alpha])\, (b\, \text{Cos}[\alpha] + \text{Sin}[\alpha])^3$$

```
Assuming[b < 0 && α > 0 && α < π / 2,
 ∫_{-b/Tan[α]}^{(1-b)/Tan[α]} ∫_0^{Tan[α] x + b} x^2 dy dx + ∫_{(1-b)/Tan[α]}^1 ∫_0^1 x^2 dy dx]
Assuming[b < 0 && α > 0 && α < π / 2,
 ∫_{-b/Tan[α]}^{(1-b)/Tan[α]} ∫_0^{Tan[α] x + b} x y dy dx + ∫_{(1-b)/Tan[α]}^1 ∫_0^1 x y dy dx]
Assuming[b < 0 && α > 0 && α < π / 2,
 ∫_{-b/Tan[α]}^{(1-b)/Tan[α]} ∫_0^{Tan[α] x + b} y^2 dy dx + ∫_{(1-b)/Tan[α]}^1 ∫_0^1 y^2 dy dx]
```

$$\frac{\mathrm{Cot}[\alpha]^3}{4} - \frac{2}{3} b\,\mathrm{Cot}[\alpha]^3 + \frac{1}{2} b^2\,\mathrm{Cot}[\alpha]^3 + \frac{1}{6}\left(1 + (-1+b)\,\mathrm{Cot}[\alpha]\right)\mathrm{Csc}[\alpha]^2 \left(2 + (-2+b)\,b + (-2+b)\,b\,\mathrm{Cos}[2\,\alpha] - (-1+b)\,\mathrm{Sin}[2\,\alpha]\right)$$

$$\frac{1}{24}(3 - 4\,b)\,\mathrm{Cot}[\alpha]^2 + \frac{1}{4}\left(2 + (-2+b)\,b - (-1+b)^2\,\mathrm{Csc}[\alpha]^2\right)$$

$$\frac{\mathrm{Cot}[\alpha]}{12} + \frac{1}{3}\left(1 + (-1+b)\,\mathrm{Cot}[\alpha]\right)$$

```
Simplify[If[Tan[α] + b < 1,
  {1/12 (3 + 4 b Cot[α] + b^4 Cot[α]^4) Tan[α],
   1/12 (1 + b Cot[α])^4 Tan[α]^3,
   -1/6 Csc[2 α]^2 (b Cos[α] - 3 Sin[α]) (b Cos[α] + Sin[α])^3}
  , {Cot[α]^3/4 - 2/3 b Cot[α]^3 + 1/2 b^2 Cot[α]^3 +
     1/6 (1 + (-1 + b) Cot[α]) Csc[α]^2
      (2 + (-2 + b) b + (-2 + b) b Cos[2 α] - (-1 + b) Sin[2 α]),
   1/24 (3 - 4 b) Cot[α]^2 +
     1/4 (2 + (-2 + b) b - (-1 + b)^2 Csc[α]^2), }]]
```

$$
\begin{cases}
\left\{-\frac{1}{12}\left(-2-3b+3b^2+\left(4-3b+3b^2\right)\mathrm{Cos}[2\alpha]\right)\mathrm{Csc}[\alpha]^2,\ \frac{1}{6}\left(3+(-2+3b)\,\mathrm{Cot}[\alpha]\right)\right\} & b+\mathrm{Tan}[\alpha]\geq 1\\
\left\{\frac{1}{6}\left(2+3b\,\mathrm{Cot}[\alpha]-b^3\,\mathrm{Cot}[\alpha]^3\right)\mathrm{Tan}[\alpha],\ \frac{1}{6}\left(1+b\,\mathrm{Cot}[\alpha]\right)^3\mathrm{Tan}[\alpha]^2,\ \frac{\mathrm{Cot}[\alpha]}{12}+\frac{1}{3}\left(1+(-1+b)\,\mathrm{Cot}[\alpha]\right)\right\} & \mathrm{True}
\end{cases}
$$

Line is y = Tan[θ]x+ $1 - A - \frac{\mathrm{Tan}[\theta]}{2}$

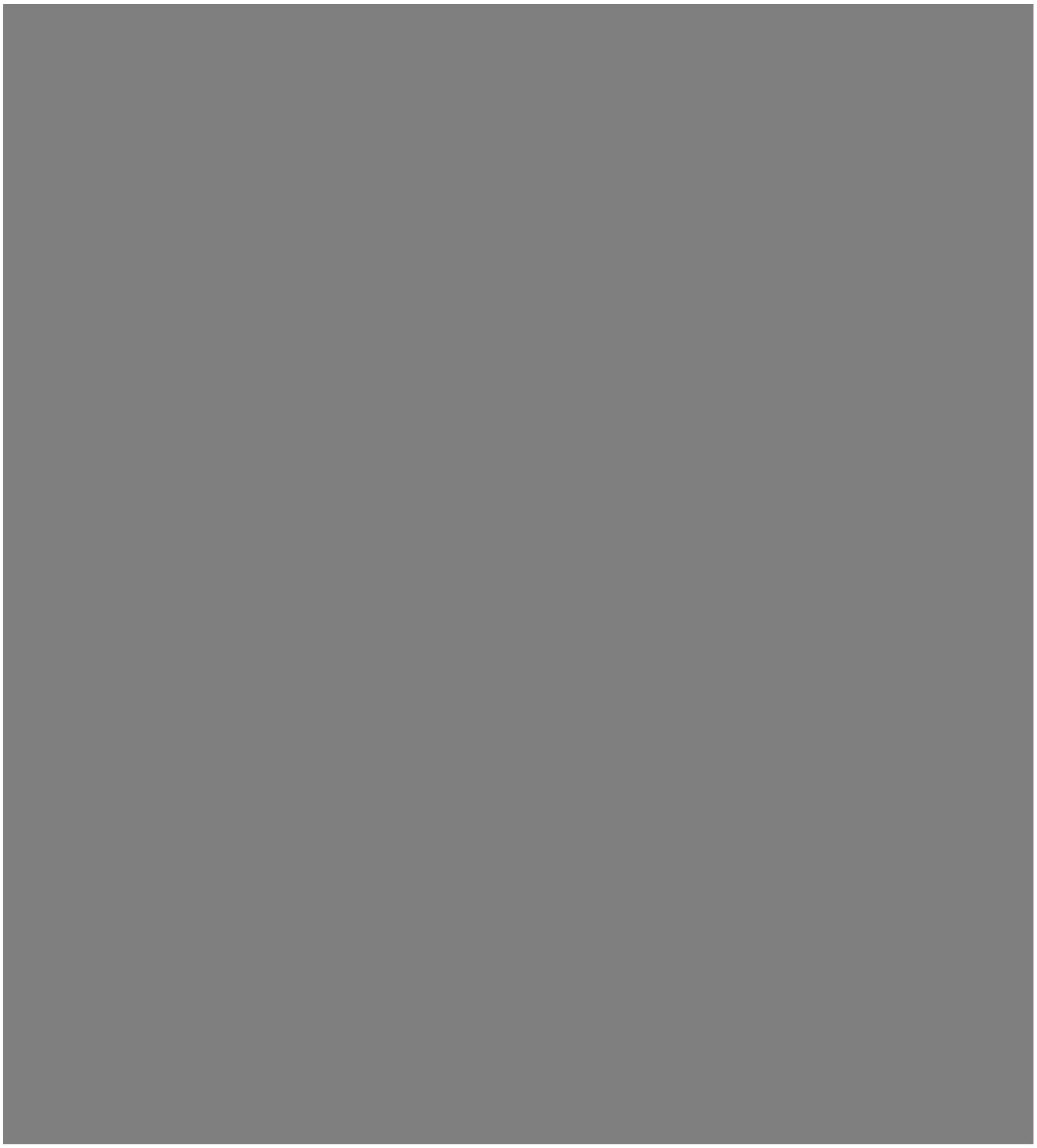
Set::setraw : Cannot assign to raw object 1. ≫

```
(*Plot a constant area place denoted by θ*)
Manipulate[
```

```
Module[{L = 1, θ = ArcTan[-pDir〚2〛, pDir〚1〛]},

 Graphics[{ White, Rectangle[1.05 {1, 1}, 1.05 {1, 1}],
   Darker[Red], Rectangle[-1.025 {1, 1}, 1.025 {1, 1}],
   White, Rectangle[{-1, -1}, {1, 1}],
   Blue, Line[{{0, 0}, pDir}],
   Red, Line[{pDir + {-pDir〚2〛, pDir〚1〛},
     pDir + {pDir[[2]], -pDir[[1]]}}],
   Orange,
   Line[{{0, 1 - A - Tan[θ]/2},
     {1, Tan[θ] + 1 - A - Tan[θ]/2}}],
   Line[{{0, 1 - A - Tan[θ]/2},
     {-1, -Tan[θ] + 1 - A - Tan[θ]/2}}],
   Green,
   Line[{{0, 1 - A / 2 - Tan[θ]/2},
     {1, Tan[θ] + 1 - A / 2 - Tan[θ]/2}}],
   Line[{{0, 1 - A / 2 - Tan[θ]/2},
     {-1, -Tan[θ] + 1 - A / 2 - Tan[θ]/2}}]


  }, PlotRange → {1.025 {-1, 1}, 1.025 {-1, 1}}
 ]
], {{pDir, {2 / 3, 1 / 4}}, {-1, -1}, {1, 1}, Locator},
{{A, 1 / 2}, 1 / 100, 1}
]
```



Try a different Way

```
A = 1 / 8
N[Tan[2 A] ]
```

$$N\left[\text{Tan}\left[\frac{1}{2\,A}\right]\right]$$

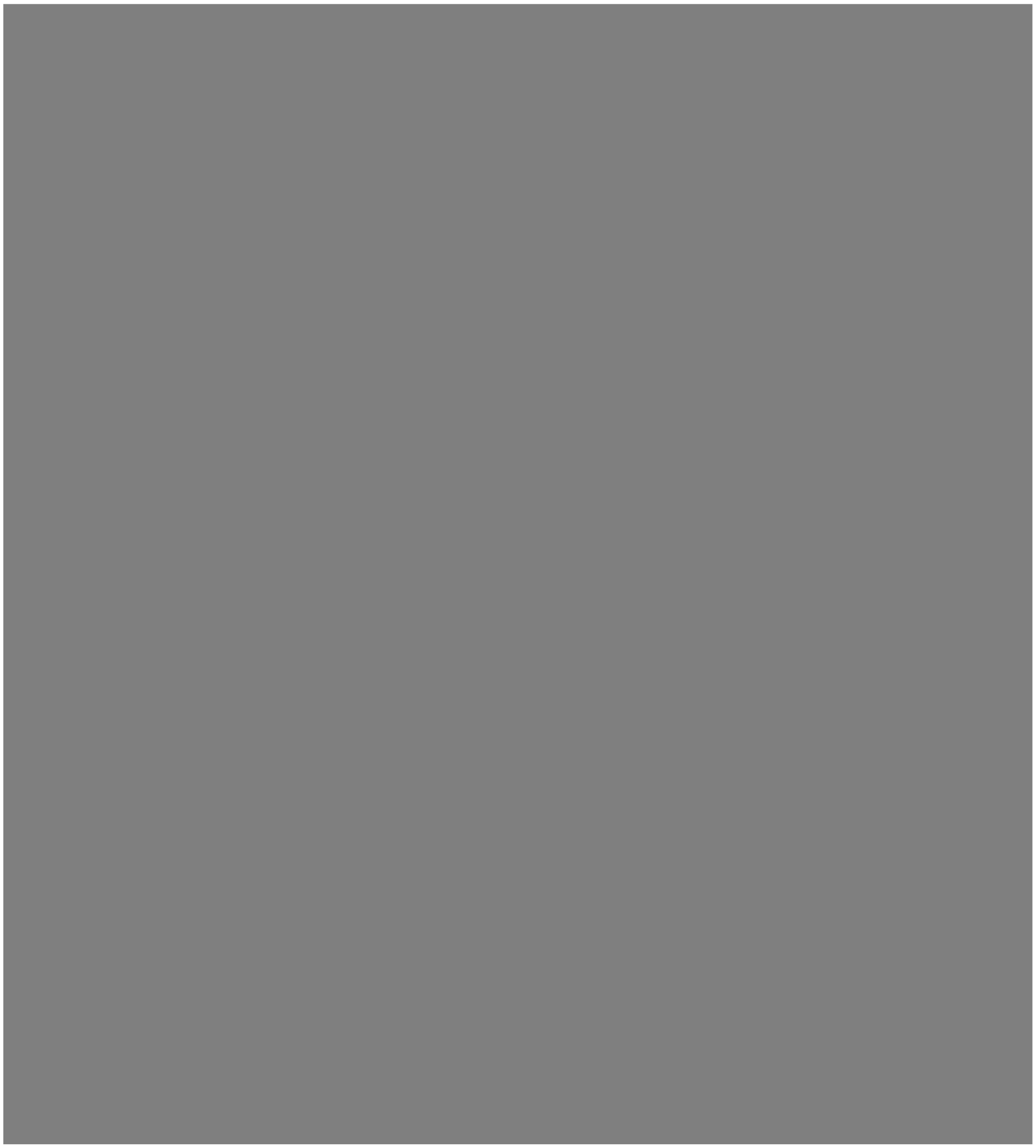
$$\frac{1}{8}$$

0.255342

1.15782

```
(*Plot a constant area place denoted by θ*)
Manipulate[
 Module[{L = 1 / 2, bw = 1 / 20, α = π/2 + θ, el, A = 1 / 8},
  el = √((2 A)/(Cos[α] Sin[α]));
  Graphics[
   { White, Rectangle[1.05 {L, L}, 1.05 {L, L}],
    Blue, HalfPlane[{{0, 0}, {Sin[θ], -Cos[θ]}},
     {Cos[θ], Sin[θ]}],
    Green,
    HalfPlane[{{L - el Cos[α], -L}, {L, -L + el Sin[α]}},
     {Cos[θ], Sin[θ]}],
    (*borders*)
    Darker[Red],
     Rectangle[{- (L + bw), - (L + bw)}, {(L + bw), - (L)}],
     Rectangle[{- (L + bw), - (L + bw)}, {-L, (L + bw)}],
     Rectangle[{(L), (L + bw)}, {-L, L}],
    Rectangle[{(L), - (L)}, {L + bw, L + bw}],
    Text[StringForm["alpha = ``", α], {0, 1 / 4}]

   }, PlotRange → {1.025 {-L, L}, 1.025 {-L, L}}
  ]
 ], {θ, -π / 2, 3 / 2 π}
]
```



```
Cos[α + π / 2]
```

```
-Sin[α]
```