Max Dehn Seminar at U Utah

11/30 3:15 pm LCB 215 Dept. of Math.

Arithmeticity of Smooth Maximal Rank Positive Entropy Actions of $\mathbb{R}^k$:

by Alp Uzman.

Plan:

1) Fast Introduction
   → Matsumoto Thm.
2) History of Measure Rigidity.
   → Abstract Measure Rigidity
   → (Algebraic) ~~Measure Rigidity~~
   → Uniform Measure Rigidity
   → Nonuniform Measure Rigidity
3) Background (General Purpose)
   → Relevant Space of Dynamical Systems. $\mathcal{E}^r(\mathbb{R}^k \curvearrowright M)$.
   → Suspensions / Time Crystals (Zotohi-Gurevich Thm?)
   → Oseledets Thm
   ~~Rigorous Statement~~ Problem 4 (and Conjecture 1) from NMR.
4) Rigorous Statement of $\mathbb{R}^k$ Arithmeticity Thm, after $\mathbb{Z}^k$ Arithmeticity Thm.
5) Elements of Proof.; (strategy and challenges).

1) Fast Introduction :

- M compact C^∞ manifold, $k \in \mathbb{Z}_{\geq 2}$, $\alpha_\circ : \mathbb{R}^k \to \mathrm{Diff}(M)$
(k commuting vector fields on M).

Aim: show certain α (MRPES) are suspensions of algebraic $\mathbb{Z}^k$ actions.

→ use tools from nonuniform measure rigidity.

→ nonuniform / measurable version of the following:

Thm (Matsumoto) : Let M be a closed oriented C^∞ manifold, $\alpha_\circ : \mathbb{R}^k \to \mathrm{Diff}^\infty_+(M)$ ($k \in \mathbb{Z}_{\geq 2}$) be a C^∞ action. If

- α is locally free
- $\dim(M) = 2k+1$, $k \in \mathbb{Z}_{\geq 2}$
- α is split Anosov,

then

- $\exists$ affine Cartan action $\gamma_\circ : \mathbb{Z}^k \to \mathrm{Aff}_+(\mathbb{T}^{k+1})$,
- $\exists\, t \in GL(k, \mathbb{R})^\circ$

such that

α is C^∞ isomorphic to $\overset{t\text{- time change}}{\text{the}}$ suspension $\tilde{\gamma}^{\gamma,t}$ of γ.

In particular M is the total space of a C^∞ $\mathbb{T}^{k+1}$-bundle over $\mathbb{T}^k$ with affine monodromy.

Glossary :

- "locally free" = all stabilizers are discrete
- "suspension" = (horizontal) natural flow on mapping torus [more later]
- "κ-time change" = $h_t^{\gamma,\kappa} = h^{\gamma}_{\kappa(t)}$
- "affine Cartan" = $\forall t \in \mathbb{Z}^k \setminus 0$: linear part of γ_t is a hyperbolic matrix in $GL(k+1, \mathbb{Z})$.
- "split Anosov" (~ Cartan according to Katok-Lewis)

$\exists$ Ad^{α}-invariant C^0 splitting $TM = O \oplus \bigoplus_{i=1}^{k+1} E^i$,

where O is tangent to orbits of α and

$\exists$ $\tau_1, \tau_2, \ldots, \tau_{k+1} \in \mathbb{R}^k$, $\forall i$:

$$S(\alpha_{\tau_i}) = E^i$$

$$U(\alpha_{\tau_i}) = S(\alpha_{-\tau_i}) = \bigoplus_{j\neq i} E^j$$

W/r/t some (hence any) C^0 norm on M, α_{τ_i} contracts vectors in E^i exponentially fast and $\alpha_{-\tau_i}$ contracts vectors in $\bigoplus_{j\neq i} E^j$ exponentially fast.

$\exists\, C, \lambda > 0$:

$$|T\alpha_{n\tau_i} v^i| \leq Ce^{-\lambda n}|v^i|.$$

"Ad^{α}-invariant"

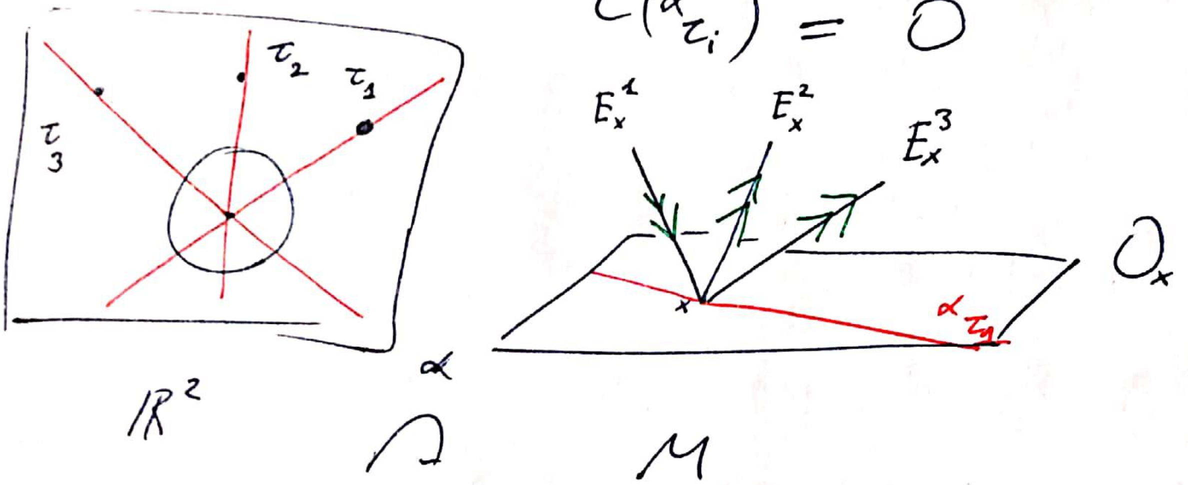
$$T_x\alpha_t\,(E^i_x) = E^i_{\alpha_t(x)}$$

Pictures for Matsumoto:

$k=2$. $\alpha: \mathbb{R}^2 \curvearrowright M^5$

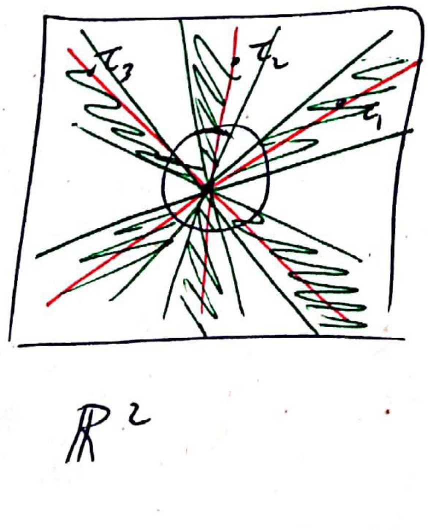
$\exists\ \tau_1, \tau_2, \tau_3 \in \mathbb{R}^2$:

$$S(\alpha_{\tau_i}) = E^i$$
$$U(\alpha_{\tau_i}) = E^j \oplus E^k$$
$$C(\alpha_{\tau_i}) = 0$$



3 hyperbolic directions $\Rightarrow$ dense set of hyperbolic directions.

Obs: structural stability gives 3 cones of hyperbolic directions



Ref.: Brown et al
Entropy, Lyapunov Exponents and Rigidity of Group Actions.

Ex (from Brown et al):

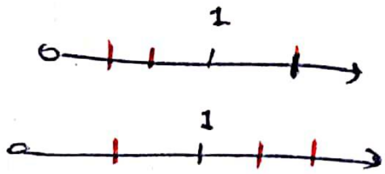
$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$A, B \in SL(3, \mathbb{Z})$

$AB = BA$

A

B



Obs: Linear Cartan $\Rightarrow$ Affine Cartan $\Rightarrow$ split Anosov

* But, this is somewhat misleading, since the methods I use are disjoint from Matsumoto.
(almost %100)

Even the implied notion of "dynamical system" is wrong/different.

2) History of Measure Rigidity:

> Good ref. for (i), (ii): Lindenstrauss - Rigidity of Multiparameter Actions.

(i) Abstract / Algebraic Measure Rigidity [1967–]

Thm (Furstenberg, 1967): Consider $\alpha : \mathbb{Z}^2_{\geq 0} \to \underset{\text{Lie}}{\mathrm{End}}(\mathbb{T})$

$\alpha_{(t_1,t_2)} : x \mapsto 2^{t_1} 3^{t_2} x$. Then $\forall$ closed α-invariant $F \subseteq \mathbb{T}$:

$$F = \mathbb{T} \quad \text{xor} \quad F \subseteq \mathbb{Q}/\mathbb{Z} \text{ and is finite.}$$

* switch from closed invariant sets to invariant measures

Obs: μ invariant $\Rightarrow$ $\mathrm{supp}(\mu)$ invariant

$\mathrm{supp}(\mu) = \left\{ x \in X \;\middle|\; \begin{array}{l} \forall N \in \mathcal{N}(x): \\ \mu(N) > 0 \end{array} \right\}$.

Conjecture (Furstenberg): $\forall$ α-invariant and α-ergodic Borel probability measure μ on $\mathbb{T}$:

$$\mu = \mathrm{haar}_{\mathbb{T}} \quad \text{xor}$$

$$\mu \text{ is atomic and } \mathrm{supp}(\mu) \subseteq \mathbb{Q}/\mathbb{Z}.$$

More generally one can take $(t_1, t_2) A x = a^{t_1} b^{t_2} x$ with $a, b \in \mathbb{Z}_{>1}$ multiplicatively independent (no positive integer power of a is a positive integer power of b).

Thm (Lyons-Rudolph-Johnson, 90's): Furstenberg conjecture is true, if $\exists\, t^* \in \mathbb{Z}^2_{\geq 0}$: $\mathrm{ent}_\mu(\alpha_{t^*}) > 0$.

metric / Kolmogorov-Sinai entropy

$\mathrm{topent}(f) = \sup_{\mu \text{ } f\text{-inv.}} \mathrm{ent}_\mu(f)$

* Motto of measure rigidity:
For dynamical systems with multiparameter time, any invariant measure ought to be familiar.

(ii) Uniform Geometric Measure Rigidity. [1998 –]

Thm (Katok-Spatzier, late 90's; Einsiedler-Lindenstrauss early 2000)

Let $k \in \mathbb{Z}_{\geq 2}$, $d \in \mathbb{Z}_{\geq 1}$, $\alpha_\circ : \mathbb{Z}^k \to \underline{\mathrm{Aut}}_{\underline{\mathrm{Lie}}}(\mathbb{T}^d)$.

If α is strongly irreducible and faithful, then

$\forall$ ergodic $\mu \in \mathrm{Prob}(\mathbb{T}^d, \alpha)$:

if $\exists\, t^* \in \mathbb{Z}^k$: $\mathrm{ent}_\mu(\alpha_{t^*}) > 0$,

then $\mu = \mathrm{haar}_{\mathbb{T}^d}$.

($\Leftrightarrow$)

Glossary:

"α is irreducible": $\forall$ α-invariant closed subgroups $H \leq \mathbb{T}^d$: H is finite.

"α is strongly irreducible" "totally": $\forall$ finite index subgroup $T \overset{i}{\hookrightarrow} \mathbb{Z}^k$: $\alpha \circ i$ is irred.

This is to avoid examples like

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \curvearrowright \mathbb{T}^4, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$\downarrow$$

$A \curvearrowright \mathbb{T}^2$ rank-one factor [$\leadsto$ Vinhage].

Conj. (Katok - Spatzier): Any Anosov action of $\mathbb{R}^{k_1} \oplus \mathbb{Z}^{k_2}$ $k_1 + k_2 \notin \mathbb{Z}_{\geq 2}$ by diffeomorphisms is virtually smoothly isomorphic to an algebraic action, provided that it has "no "rank-one factors"".

* New perspective: Lyapunov geometry
  - → Lyapunov foliations
  - → Conditionals along Lyapunov leaves
  - → Lyapunov hyperplanes

(iii) Nonuniform Geometric Measure Rigidity [2007–]

- Adaptation of Lyapunov geometry to the nonuniformly hyperbolic (Pesin theoretical) setting.
  (objects are rougher, things work so, … Hölder continuity only on certain regions)

<u>Thm (Kalinin–Katok–RH 2011)</u>: Let M be a compact C^∞ manifold, $\alpha_0 : \mathbb{R}^k \to \mathrm{Diff}^r(M^{2k+1})$ or $\alpha_0 : \mathbb{Z}^k \to \mathrm{Diff}^r(M^{k+1})$ ($k \in \mathbb{Z}_{\geq 2}$, $r = (1, \theta)$, $\theta \in]0, 1]$) be a C^r action. Then $\forall$ ergodic $\mu \in \mathrm{Prob}(M, \alpha)$:

(A) if $\exists\, t^* \in \mathbb{Z}^k : \mathrm{ent}_\mu(\alpha_{t^*}) > 0$ and

(μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes $K^1, K^2, \ldots, K^{k+1} \leq \mathbb{R}^k$ and they are in general position,

then $\mu \ll \mathrm{leb}_M$.

["maximal rank positive entropy" paradigm].

Glossary:

"Lyapunov hyperplanes": <u>next section</u>.

"gen. pos.": $\dim\left(K^{i_1} \cap K^{i_2} \cap \dots \cap K^{i_p}\right) = \max\{k-p, 0\}$.

"$\mu \ll \mathrm{leb}_M$": In local charts μ is given by a density that is bounded away from ∞.

<u>Thm (Kushnirenko 1965, Abramov 1959, Hu 1993)</u>:

$$E_{(\mu,\alpha)} : \mathbb{R}^k \longrightarrow \mathbb{R}_{\geq 0} \quad \text{is a seminorm.}$$
$$t \longmapsto \mathrm{ent}_\mu(\alpha_t)$$

Call it the <u>entropy gauge</u> of (μ, α).

<u>Prop</u>: The (*) hypotheses of KKRH Thm is equivalent to:

$E_{(\mu,\alpha)}$ is a norm and

(μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes. $\rightarrow$ $\boxed{\text{This leads to Fried entropy.}}$

3) Background (General Purpose):

(i) Relevant space of Dynamical systems $\nearrow$ implicit in Mañé, Viana, Ruelle, ...

Def: Smooth ergodic theory of $\mathbb{R}^k$ actions on M (with no potential):

$$\mathcal{E}^r(\mathbb{R}^k \curvearrowright M) = \left\{ (\mu, \alpha) \;\middle|\; \begin{array}{l} \alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M) \ C^r \text{ action} \\ \mu \in \mathrm{Prob}(M) \ \alpha\text{-invariant} \end{array} \right\}$$

$$\begin{array}{ccc} & \mathcal{E}^r(\mathbb{R}^k \curvearrowright M) & \\ \swarrow & & \searrow \\ \mathrm{Prob}(M) & & \mathcal{A}^r(\mathbb{R}^k \curvearrowright M) \end{array}$$

* elements of $\mathcal{E}^r(\mathbb{R}^k \curvearrowright M)$: $\mathbb{R}^k$-systems on M.
* "action" actually means "system".

* $\mathcal{E}^r(\mathbb{R}^k \curvearrowright M) \subseteq \mathrm{Prob}(M) \times C^r(\mathbb{R}^k \times M; M)$

$\uparrow$ vague topology $\qquad \uparrow$ compact-open C^r topology.

* $\mathcal{E}^r(\mathbb{R}^k \curvearrowright M)$ carries ~~a~~ conjugation actions.

Def: {SET with potential: $\mathcal{E}^{r,\square}(\mathbb{R}^k \curvearrowright M; H)$

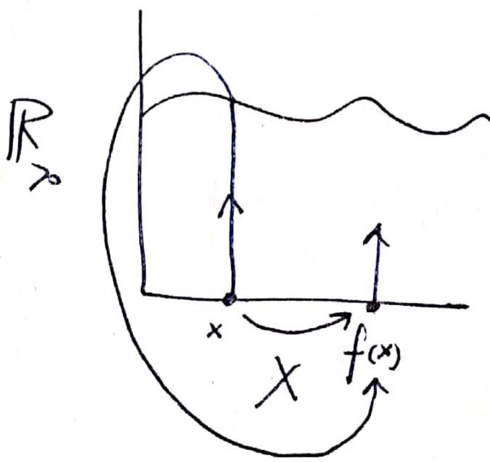
$$= \left\{ (\mu, \alpha, u) \;\middle|\; \begin{array}{l} \alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M) \ C^r \text{ action} \\ \mu \in \mathrm{Prob}(M) \ \alpha\text{-invariant} \\ u : \mathbb{R}^k \times M \longrightarrow H \ \alpha\text{ 1-cocycle} \\ \quad \text{with } \square \text{ regularity} \end{array} \right\}$$

* Also carries a transfer action in the third coordinate.

(ii) Suspensions / Time Crystals.

Classical: Produce an $\mathbb{R}$-system from a $\mathbb{Z}$-system on a space with 1 extra dimension.

$f: X \circlearrowleft \qquad u: X \to \mathbb{R}_{>0}$ "roof function" (cocycle).



$$\rightsquigarrow \phi_\bullet^{f,u} : \mathbb{R} \curvearrowright \mathbb{R} \otimes_{f,u} X$$

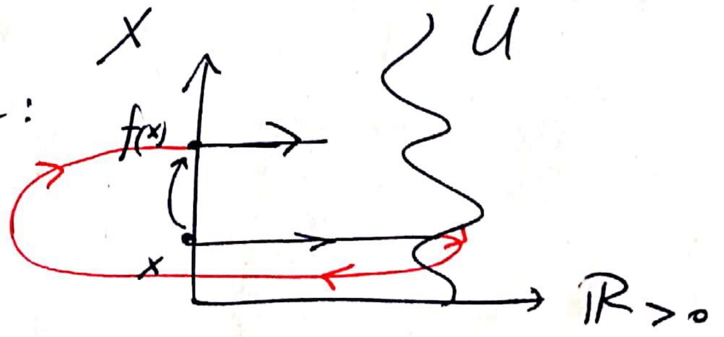
$$\phi_t^{f,u}[r,x] = [t+r, x].$$

$$\mu \in \mathrm{Prob}(X,f)$$

$$\rightsquigarrow \mathrm{haar}_{\mathbb{T}} \otimes_{f,u} \mu \quad (A \times B) = \frac{\mathrm{haar}_{\mathbb{T}}(A)\,\mu(B)}{\mathbb{E}_\mu(u)}$$

$$(\mu, f) \in \mathcal{E}^r(\mathbb{Z} \curvearrowright X) \rightsquigarrow \left(\mathrm{haar}_{\mathbb{T}} \otimes_{f,u} \mu, \phi^{f,u}\right) \in \mathcal{E}^r(\mathbb{R} \curvearrowright \mathbb{R} \otimes_{f,u} X).$$

- $u \equiv 1 \Rightarrow \mathbb{R} \otimes_f X$ is the mapping torus of f.
- Flip the perspective:



• There is a factor circle $\mathbb{R} \otimes_{f,u} X \longrightarrow \mathbb{T}$.

The factor circle carries a foliation that is the projection of ~~[illegible]~~ orbits of $\mathfrak{h}^{f,u}$, but this foliation does <u>not</u> necessarily have a canonically defined time parameter (i.e., it may fail to be a factor system of the suspension system $\mathfrak{h}^{f,u}$)

When it does have a canonically defined time parameter, call the factor circle the <u>time crystal</u> of $\mathfrak{h}^{f,u}$.

• Time crystals are obstructions to K-property.

<u>Thm</u> (Totoki-Gurevich 1970's):

$f : X^{\mathbb{Z}} \circlearrowleft$ shift. $\mu = \nu^{\otimes \mathbb{Z}}$,

(μ, f) Bernoulli system. $U : X^{\mathbb{Z}} \to \mathbb{R}$ measurable, only depends on x_0. Then

(i) If U is lattice distributed, then $\mathfrak{h}^{f,u}$ is not weak mixing
(time crystal case)

(ii) If U is not lattice distributed, then $\mathfrak{h}^{f,u}$ is K.

("lat. dist." $= \exists \ell \in \mathbb{Z}_{\geq 1} : \vec{U}(\mu)(\ell \mathbb{Z}) = 1$)

• Suspensions for $\mathbb{Z}^k$ systems.

$$(\nu,\beta) \in \mathcal{E}^r(\mathbb{Z}^k \curvearrowright N) \rightsquigarrow \left(\mathrm{haar}_{\mathbb{T}^k} \otimes_\beta \nu, \hbar^\beta\right) \in \mathcal{E}^r\left(\mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\beta N\right)$$

$$\gamma^\beta_\bullet : \mathbb{Z}^k \curvearrowright (\mathbb{R}^k \times N)$$

$$\gamma^\beta_t(r,x) = (r-t, \beta_t(x)).$$

$$\mathbb{R}^k \otimes_\beta N = (\mathbb{R}^k \times N)/\gamma^\beta$$

$$\hbar^\beta_\bullet : \mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\beta N$$

$$\hbar^\beta_t[r,x] = [t+r, x].$$

More generally if one also has a cocycle over β: $u : \mathbb{Z}^k \times N \to \mathbb{R}^k$

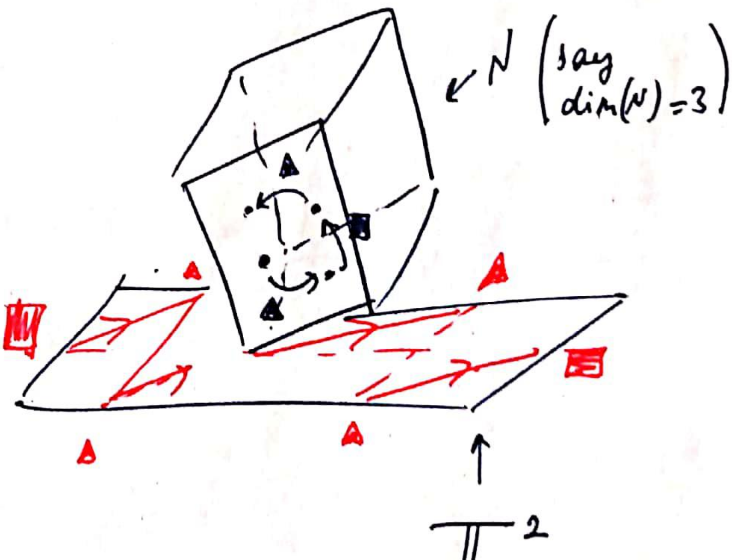
$0 = u(s,x) - u(t+s,x) + u(t,\beta_s(x))$, then

$$\gamma^{\beta,u}_t(r,x) = (r - u(t,x), \beta_t(x)).$$

$k=2$.

$$\begin{tikzcd}
\mathbb{R}^k \otimes_\beta N \arrow[d] & \mathbb{R}^k \times N \arrow[l] \arrow[d] \\
\mathbb{T}^k & \mathbb{R}^k \arrow[l]
\end{tikzcd}$$

time crystal.



▲, ■ : generators of β.

Obs: If $(\nu, \beta) \in \mathcal{E}^r(\mathbb{Z}^k \curvearrowright N)$ satisfies the hypotheses of KKRH Thm, then so does its suspension $\in \mathcal{E}^r(\mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\beta N)$.

Obs: $\text{haar}_{\mathbb{T}^k} \otimes_\beta \nu \sim \text{leb}_{\mathbb{R}^k \otimes_\beta N}$

$$\Longrightarrow \nu \sim \text{leb}_N .$$

Problem 4 (: KKRH): Are there any $\mathbb{R}^k$ systems satisfying the hypotheses of KKRH Thm that are not (constant time changes) of suspensions of $\mathbb{Z}^k$ systems satisfying the hypotheses of KKRH Thm?

My answer: No (and then some)

(iii) Oseledets for Smooth Actions.

Thm: Let M be a compact C^∞ manifold, $k \in \mathbb{Z}_{\geq 1}$, $(\mu, \alpha) \in \mathcal{E}^1(\mathbb{R}^k \curvearrowright M)$ be locally free and ergodic. Then $k \leq \dim(M)$. If $k = \dim(M)$, then the only Lyapunov exponent of (μ, α) is 0 μ-a.e. Otherwise (i.e. $k \lneq \dim(M)$), there is a Borel measurable subset $Osel =_\mu M$ such that

$\boxed{EX1\,1}$ $\exists!\ l \in \{1, 2, \dots, \dim(M) - k\}$,

$\exists!\ \delta^1, \delta^2, \dots, \delta^l \in \{1, 2, \dots, \dim(M)\}$:

$$\sum_{i=1}^{l} \delta^i = \dim(M) - k,$$

$\boxed{EX1\,2}$ $\exists!$ (up to reordering) linear operator

$$X = (X^1, X^2, \dots, X^{\dim(M)-k}, \underbrace{0, 0, \dots, 0}_{k}) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{\dim(M)})$$

$$= (\underbrace{x^1, x^1, \dots, x^1}_{\delta^1 \text{ many}}, \underbrace{x^2, \dots, x^2}_{\delta^2 \text{ many}}, \dots, \underbrace{x^l, \dots, x^l}_{\delta^l \text{ many}}, \underbrace{0, \dots, 0}_{k\text{-many}}).$$

with $i \neq j \Rightarrow x^i \neq x^j$ (but not necessarily $x^i \neq 0$),

$\boxed{\text{EX13}}$ $\forall x \in \text{Osel}$ $\exists!$ splitting

$$T_x M = O_x \oplus \bigoplus_{i=1}^{\ell} L^i_x$$

with $\dim (L^i_x) = \delta^i$,

such that

$\boxed{\text{ASYM 1}}$ $\forall\, C^0$ norm on M:

$\forall x \in \text{Osel}, \forall i$:

$$\lim_{|t| \to \infty} \sup_{v^i \in L^i \setminus 0} \frac{\log |T_x \alpha_t (v^i)| - \chi^i(t)}{|t|} = 0$$

$\boxed{\text{ASYM 2}}$ $\forall\, C^0$ density on M:

$$\forall x \in \text{Osel} : \lim_{|t| \to \infty} \frac{\log \operatorname{Jac}_x(\alpha_t) - \sum_{i=1}^{\ell} \delta^i \chi^i(t)}{|t|} = 0$$

$\boxed{\text{ASYM3}}$ $\forall$ C^0 Riemannian metric on M:

$\forall x \in \mathrm{Osel}$, $\forall I \subseteq \{1, 2, \ldots, \ell\}$

$$\lim_{|t|\to 0} \frac{\log \left\| \mathrm{proj}\left(T_x \alpha_t \left(\bigoplus_{i\in I} L_x^i\right)\right) - \mathrm{proj}\left(O_{\alpha_t(x)}\right) \right\|}{|t|} = 0$$

$$\lim_{|t|\to 0} \frac{\log \left\| \mathrm{proj}\left(T_x \alpha_t \left(\bigoplus_{i\in I} L_x^i\right)\right) - \mathrm{proj}\left(T_x \alpha_t \left(\bigoplus_{i\notin I} L_x^i\right)\right) \right\|}{|t|} = 0$$

$\boxed{\text{INV4}}$ $TM =_{\mu} O \oplus \bigoplus_{i=1}^{\ell} L^i$

is a measurable ae-splitting that is Ad^{α}-invariant.

Definitions,

χ^i : Lyapunov exponent

X : Lyapunov operator

L^i : Lyapunov $\mathfrak{X}$-subbundle.

$TM = \underset{\Gamma}{O} \oplus \bigoplus_i L^i$: (fine) Lyapunov $\mathfrak{X}$-splitting

$K^i = \ker(\chi^i)$: Lyapunov hyperplane (when $\chi^i \neq 0$.)

δ^i : dynamical multiplicity.

$S(\alpha_t) = \bigoplus_{\chi^i(t)<0} L^i$: stable $\mathfrak{X}$-subbundle of time-t map.

$U(\alpha_t) = \bigoplus_{\chi^i(t)>0} L^i$: unstable $\mathfrak{X}$-subbundle of time-t map.

$C(\alpha_t) = \bigoplus_{\chi^i(t)=0} L^i \geq O$: center $\mathfrak{X}$-subbundle of time-t map.

Rem: $\boxed{ASYM\ 1}$ says: α stretches/shrinks v^i like $e^{\chi^i(t)}$

$\boxed{ASYM\ 2}$ says: α distorts volumes ~~[illegible]~~ (geometric) like $e^{\sum_i \delta^i \chi^i(t)}$

$\boxed{ASYM\ 3}$ says: the splitting becomes orthogonal exponentially

4) Rigorous Statement of $\mathbb{R}^k$ Arithmeticity:

Thm: Let M be a compact C^∞ manifold, $r \in (1, \infty)$, $\theta \in]0, 1]$, $(\mu, \alpha) \in \mathcal{E}^r(\mathbb{R}^k \curvearrowright M)$. If

- α is locally free and ergodic
- $\dim(M) = 2k+1$, $k \in \mathbb{Z}_{\geq 2}$
- (μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes, and
- $\forall t \in \mathbb{R}^k \setminus 0$: $\boxminus_{(\mu,\alpha)}(t) > 0$, then
- $\exists$ affine Cartan $\gamma : \mathbb{Z}^k \to \mathrm{Aff}(\mathbb{T}^{k+1})$ where $\mathbb{T}^{k+1} = \mathbb{T}^{k+1}$ or $\mathbb{T}^{k+1}/\{\pm I\}$
- $\exists\, K \in GL(k, \mathbb{R})$ such that

(μ, α) is measure theoretically isomorphic to ~~$\mathrm{haar}_{\mathbb{T}^k} \otimes_\gamma \mu, h$~~

$$\left(\mathrm{haar}_{\mathbb{T}^k} \otimes_\gamma \mathrm{haar}_{\mathbb{T}^{k+1}},\ h^{\gamma, K}\right)$$

Further, the arithmeticity isomorphism

$$\Phi_{(\mu,\alpha)} : (\mu, \alpha) \xrightarrow{\cong} \left(\mathrm{haar}_{\mathbb{T}^k} \otimes_\gamma \mathrm{haar}_{\mathbb{T}^{k+1}},\ t_h^{\gamma,k}\right)$$

is C^r on on $f^{\mathcal{B}}$

and $\Phi_{(\mu,\alpha)}$ is C^r in the Whitney sense.

Preprint Imminent!

__Cor:__ Problem 4 is a No.

__Cor:__ $T^{x}\alpha$ (derivative cocycle along L^x)
is cohomologous to constant (Conj 1 from KKRH)

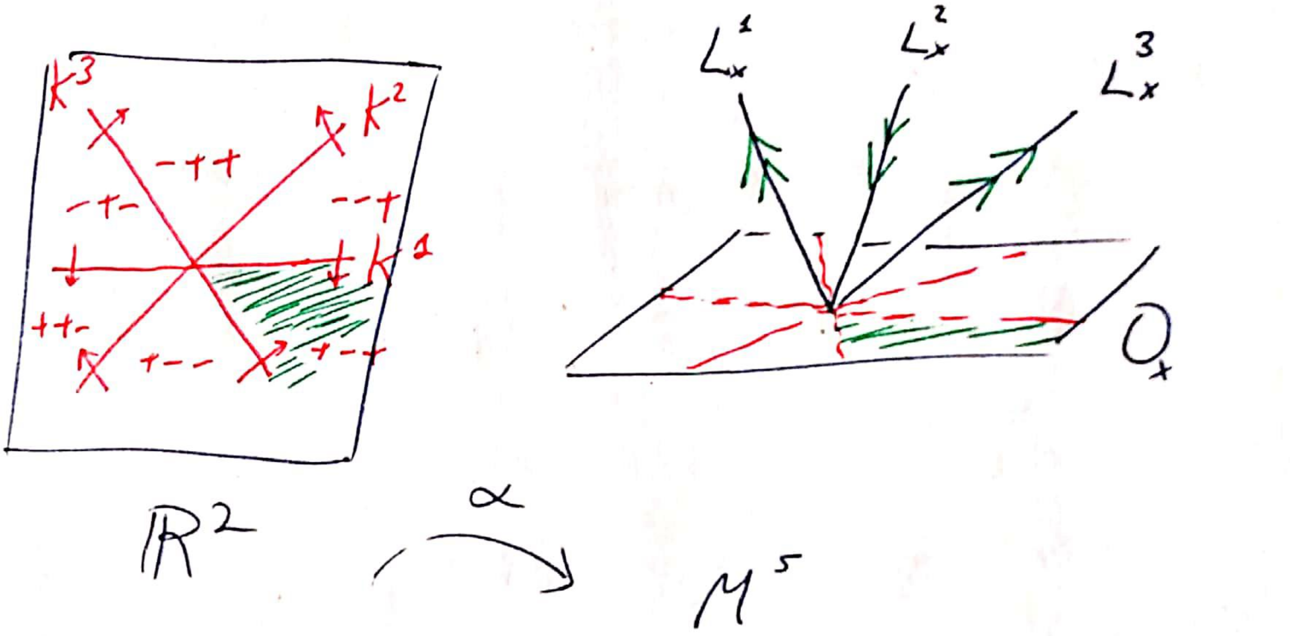
__Cor:__ Lyapunov exponents are logarithms of algebraic numbers.

__Cor:__ Obstructions to K property
((μ,α) is K iff $\forall t \in \mathbb{R}^k \setminus 0$: (μ, α_t) is K).

__Thm (Next to be proven)__ : If in addition $\exists t^* \in \mathbb{R}^k$
α_{t^*} is Anosov, recover Matsumoto.

Pictures for Arithmeticity :

$k = 2, \quad \alpha_0 : \mathbb{R}^2 \curvearrowright M^5 \qquad TM =_\mu O \oplus L^1 \oplus L^2 \oplus L^3$



5) Strategy and Challenges:

+ Adapt the machinery Katok – FRH built for $\mathbb{Z}^k$ ~~actions~~ systems to $\mathbb{R}^k$ systems.

  • They prove a $\mathbb{Z}^k$ arithmeticity result (the time crystal is a finite group).

Challenges: • In $\mathbb{Z}^k$ case, can factor out the time crystal at the first step (after Perry 1980's) "weak mixing reduction"

  • In $\mathbb{R}^k$ case one needs to control the ~~##~~ spill-over of holonomies along the orbit directions.