Research Statement

Alp Uzman

Last Revised: 2025-10-14 14:17:12-06:00

I am primarily interested in the question of time. Stated this way, this is a notoriously difficult philosophical question that is very broad. In my opinion, however, a good way to attempt answering this question is to impose on it a mathematical framework[1]; the mathematical discipline that is closest in spirit to what one gets by imposing a mathematical framework onto the question of time is dynamics, and dynamics is my specialization as a mathematician.

One can approach dynamics from a pure, applied or interdisciplinary perspective. One can approach mathematics at large from a dynamical perspective also, due to the insight of the interchangeability of "symmetry" and "time" [2]. As such, I have broad mathematical interests, within and without dynamics. With this dynamical perspective, I am committed to designing research projects that are both mathematically substantial and accessible to students.

Summary of Research

My work so far has focused in the smooth dynamics of abelian Lie groups. In simple terms, I study, using methods from calculus, closed systems that evolve with respect to a predetermined law of time evolution, where time has multiple independent dimensions. Mathematically, a closed system is a "manifold", time is a "group", and the law of time evolution is a "group action".

In more technical terms, previously in [Uzm23], building on [KKRH11, KRH16], I showed that any sufficiently smooth action of $\mathbb{R}^k$ on a $2k+1$ dimensional manifold is the suspension of an algebraic action of $\mathbb{Z}^k$ up to a measure theoretical change of coordinates, provided that the original action satisfies certain positive entropy assumptions. My ongoing work involves on the one hand the development of entropy theory for smooth actions of nilpotent Lie groups[3] ([Uzm25], building on [Fri83, KKRH14]) and on the other hand, joint with Prof. Kurt Vinhage, the study of the effects of time changes of Anosov abelian actions ([UV25a], building on [Vin22]) as well as detection of suspension structures for Anosov abelian actions of $\mathbb{R}^k$ on manifolds of arbitrary dimension ([UV25b], building on [Pla72, SV19]).

I will not provide more detailed and rigorous descriptions of the above statements as well as definitions of the technical terms; here it suffices to note that this specific field of dynamics requires a strong background in many fields, including functional analysis, measure theory, information theory, Lie theory and differential geometry. As such, I presume what I specialize in as such is appropriate only for advanced undergraduates[4]. Still, the breadth of the background allows me to operate as a generalist and connect with students and colleagues across mathematical disciplines.

Undergraduate Research and Mentoring

I can clarify what I mean by "operating as a generalist" by focusing on a specific class I designed and taught. In Fall 2024, I taught a class focusing on undergraduate research in mathematics, MATH

[1]Indeed, generally speaking mathematics is a good way to start doing philosophy. It should be noted that I am a mathematician who is more than keen on at least some kinds of philosophy; this has certain ramifications regarding my view of mathematics and how I conduct research, although for the rest of this statement this will not be relevant.

[2]https://math.stackexchange.com/q/626927/169085

[3]Nilpotent Lie groups are roughly groups that are made up of finitely many abelian Lie groups.

[4]This semester I am supervising two such undergraduates studying foundational papers [Fri83, Hu93, KKRH14] in the smooth ergodic theory of abelian Lie group actions.

4800. This is a class that is offered quite regularly at the U of Utah, and from the topic to grading policy, everything is designed by the instructor that teaches the class that semester. I had a total of nine students, and the students were graded partly based on weekly reports and partly based on a final report and an accompanying presentation. The students were allowed to work and report on anything they found interesting each week, insofar as their studies were related to the main topic I chose for the class, namely "fractal geometry and dynamics". The students were allowed to experiment with different areas as they were deciding on what to write their final report, and once they had an idea as to what their final reports were going to be on, they were allowed to report on partial progress in their weekly reports. One constraint I had imposed was that no two students were allowed to submit the same material for weekly reports. This constraint was to encourage the students to find their own path, and indeed I had chosen the topic for the course so that there were sufficiently many diverse paths students could decide to follow. By the end of the semester we had reports ranging from continued fractions and dimension theory to Brownian motion and wildfire dynamics[5].

Two directions I identify as particularly well-suited for undergraduate involvement are ergodic optimization and analytical dimension theory. Both of these directions are active areas of interest to the broader mathematical community and simultaneously well-aligned with my research and background.

Ergodic Optimization In ergodic optimization[6] one fixes a compact metric space X and a compactly generated abelian Lie group A[7] and considers the space of all triples (μ, α, ϕ), where μ is a Borel probability measure on X, $\alpha : A \times X \to X$ is a continuous group action that preserves the measure μ, and $\phi : X \to \mathbb{R}$ is an observable. There is a natural notion of equivalence defined for such triples, and to any such triple one can associate certain dynamical quantities that are invariant under equivalence (for instance topological pressure or measure theoretical entropy). Broadly speaking, in ergodic optimization one fixes either one (or more) dynamical quantities, or one (or more) of μ, α, or ϕ, and optimizes the free variables accordingly. In another variant, one considers quadruples $(\mu, \alpha, \kappa, \phi)$, where $\kappa : A \times X \to A$ is an additional time change variable[8]; varying κ to obtain special properties typically is referred to as "synchronization"[9]. As far as I am aware, ergodic optimization with time changes is mostly unexplored, even in the $A = \mathbb{Z}$ or $\mathbb{R}$ case. While some metric space topology, measure theory and convex analysis is needed, ergodic optimization requires relatively little overhead, especially if $A = \mathbb{Z}$ or $\mathbb{R}$ and hence is approachable by undergraduate students.

Analytical Dimension Theory In analytical dimension theory again one fixes a metric space X and associates to sufficiently well-behaved subsets A of X, a number $\dim(A)$ in such a way that for X a vector space and A a vector subspace, $\dim(A)$ is the dimension of A as one is familiar with from linear algebra. In principle analytical notions of dimension are not axiomatic, although there are well-known recipes that result in meaningful notions of dimension, for instance Minkowski or Hausdorff dimensions, as well as intermediate dimensions[10] that interpolate between these two. The dimension of a subset quantifies how complicated the set is, and as such dimension theory is very useful in diverse fields. One particular use I am interested in is the applications of dimension theory in dynamical systems. Say $f : X \to X$ is a continuous function, and A is an "attractor" of f. This roughly means that A is invariant under f, and understanding A gives us important information about the dynamics. Hence we are interested in the dimensions of attractors of dynamical

[5] The presentation schedule is available at https://github.com/AlpUzman/MATH_4800_001_FALL_2024/blob/main/MATH_4800_001_FALL_2024_PSCHE_RED.pdf, the names of the students have been redacted due to privacy concerns. An unredacted version, as well as the written reports are available upon request.

[6] See e.g. [Jen06] for a survey.

[7] Thus A is of the form $\mathbb{Z}^{k_1} \times \mathbb{R}^{k_2}$ modulo a compact subgroup. The case $A = \mathbb{Z}$ corresponds to iterating a homeomorphism $f : X \to X$ and the case $A = \mathbb{R}$ corresponds to a continuous flow $\varphi : \mathbb{R} \times X \to X$.

[8] One considers for $x \in X$ the trajectory of x under α, except with the reparameterization given by κ: at time $t \in A$, x moves under α according to time $\kappa(t, x)$.

[9] For instance as in [Par86].

[10] See e.g. [Fal21] for a survey.

systems. As there are many different notions of dimension, computing the dimensions of an attractor A relative to as many notions of dimension gives valuable information; even the validation that under different notions of dimension, the number $\dim(A)$ does not change is valuable information! Analytical dimension theory again requires some background in measure theory, but there are many attractors or more general sets whose dimension (exactly or approximately) needs to be calculated; it seems this is appropriate for undergraduates. Another interesting study appropriate for undergraduates would be the investigation of one of the recipes to define meaningful notions of dimension: namely first define a family of measures $\{\mu_s\}_{s\in S}$ (or outer measures, or outer contents), and then define for A a μ_s-measurable subset for any $s \in S$, the dimension $\dim(A)$ to be to be the subset $S' \subseteq S$ of "phase transitions"[11] of $\mu_s(A)$ as a function of s. Then given any group G of symmetries of X, one can infimize $\dim(A)$ over G to obtain a G-invariant notion of dimension[12]. It is well-known that many dimension-like quantities used in fractal geometry and thermodynamic formalism can be obtained this way[13], but the general axiomatic study of this method is far from complete.

Lean Finally, I am keen to incorporate formal verification using the Lean proof assistant into my research program. I am aware of only the basic syntax and ideas of Lean, although I would be very interested in the opportunity to develop a curriculum or research program that incorporates formalization. Indeed, recently there has been plenty of activities in both the education[14] and research[15] directions. A concrete proposal in the research direction would be the formalization of the results in Bowen's classic book [Bow75]. This short book was first published in 1975, and in 2008 it was TeX'ed by J.-R. Chazottes ([Bow08]), with a second edition appearing in 2017. There are still errors in this latest addition (as I have discovered during my PhD), and a formalized account would ultimately be greatly appreciated by the dynamics community. This project could also serve as a source of undergraduate research.

References

[Bow75] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, Vol. 470, Springer-Verlag, Berlin-New York, 1975. MR 0442989

[Bow08] ______, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, revised ed., Lecture Notes in Mathematics, vol. 470, Springer-Verlag, Berlin, 2008, With a preface by David Ruelle, Edited by Jean-René Chazottes. MR 2423393

[Edg04] Gerald A. Edgar (ed.), *Classics on fractals*, Studies in Nonlinearity, Westview Press. Advanced Book Program, Boulder, CO, 2004. MR 2049443

[Fal21] Kenneth J. Falconer, *Intermediate dimensions: a survey*, Thermodynamic formalism, Lecture Notes in Math., vol. 2290, Springer, Cham, [2021] ©2021, pp. 469–493. MR 4436834

[Fri83] David Fried, *Entropy for smooth abelian actions*, Proc. Amer. Math. Soc. **87** (1983), no. 1, 111–116. MR 677244

[Gou22] Sébastien Gouëzel, *A formalization of the change of variables formula for integrals in mathlib*, Intelligent computer mathematics, Lecture Notes in Comput. Sci., vol. 13467, Springer, Cham, [2022] ©2022, pp. 3–18. MR 4570911

[11] In the case of Hausdorff dimension, $S = [0, \infty]$, and S' is a singleton.

[12] For instance, for X compact, infimizing Hausdorff dimension over G the group of homeomorphisms X produces the Lebesgue covering dimension by the Pontrjagin-Schrinelmann Theorem [Edg04].

[13] See e.g. [Pes97].

[14] See e.g. https://www.youtube.com/live/alByz_LoANE?si=2YUOLe_0u7VTFpb3, https://terrytao.wordpress.com/2025/05/31/a-lean-companion-to-analysis-i/.

[15] See e.g. [Gou22].

[Hu93] Hu Yi Hu, *Some ergodic properties of commuting diffeomorphisms*, Ergodic Theory Dynam. Systems **13** (1993), no. 1, 73–100. MR 1213080

[Jen06] Oliver Jenkinson, *Ergodic optimization*, Discrete Contin. Dyn. Syst. **15** (2006), no. 1, 197–224. MR 2191393

[KKRH11] Boris Kalinin, Anatole Katok, and Federico Rodriguez Hertz, *Nonuniform measure rigidity*, Ann. of Math. (2) **174** (2011), no. 1, 361–400. MR 2811602

[KKRH14] Anatole Katok, Svetlana Katok, and Federico Rodriguez Hertz, *The Fried average entropy and slow entropy for actions of higher rank abelian groups*, Geom. Funct. Anal. **24** (2014), no. 4, 1204–1228. MR 3248484

[KRH16] Anatole Katok and Federico Rodriguez Hertz, *Arithmeticity and topology of smooth actions of higher rank abelian groups*, J. Mod. Dyn. **10** (2016), 135–172. MR 3503686

[Par86] William Parry, *Synchronisation of canonical measures for hyperbolic attractors*, Comm. Math. Phys. **106** (1986), no. 2, 267–275. MR 855312

[Pes97] Yakov B. Pesin, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1997, Contemporary views and applications. MR 1489237

[Pla72] Joseph F. Plante, *Anosov flows*, Amer. J. Math. **94** (1972), 729–754. MR 377930

[SV19] Ralf Spatzier and Kurt Vinhage, *Cartan actions of higher rank abelian groups and their classification*, 2019.

[UV25a] Alp Uzman and Kurt Vinhage, *Fragility of rank one factors*, In preparation, 2025.

[UV25b] ———, *Higher rank Plante alternative*, In preparation, 2025.

[Uzm23] Alp Uzman, *Arithmeticity for smooth maximal rank positive entropy actions of* $\mathbb{R}^k$, submitted to Journal of Modern Dynamics, under review, 2023.

[Uzm25] ———, *Fried average entropy for actions of nilpotent groups*, In preparation, 2025.

[Vin22] Kurt Vinhage, *Instability for rank one factors of product actions*, 2022.