Research Statement

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1 Introduction

I am primarily interested in the question of time. Stated this way, this is a notoriously difficult philosophical question that is very broad, and as such almost nonsensical. In my opinion, however, a good way to attempt answering this question is to impose on it a mathematical framework[1]; the mathematical discipline that is closest in spirit to what one gets by imposing a mathematical framework onto the question of time is dynamics, and dynamics is my specialization as a mathematician. In this statement I would like to outline my more specific research interests; both my current studies as well as studies I would like to take up in the near future. Currently I am primarily interested in the measure rigidity of abelian group actions. I am also interested in entropy theories (in the sense of Kolmogorov-Sinai) for smooth actions of groups more general than $\mathbb{Z}$ or $\mathbb{R}$ and I will mention some related work in progress. For future directions I will highlight a result on nonstationary normal forms and an extension of that branch of nonuniform hyperbolicity called Pesin theory which has close connections with measure rigidity

2 Current Studies

Arithmeticity Let M be a smooth compact manifold and G be a Lie group (including discrete groups). We'll call a pair (μ, α) a G system on M if μ is a Borel probability measure on M and α is a C^r action of G on M for some $r \in \mathbb{R}_{>1}$ preserving μ. In [Uzm22] I adapt the machinery developed in [KRH16] for $\mathbb{Z}^k$ systems to $\mathbb{R}^k$ systems for $k \in \mathbb{Z}_{\geq 2}$, and I use this machinery to show that any $\mathbb{R}^k$ system that satisfies the maximal rank positive entropy hypotheses is measure theoretically isomorphic to a constant linear time change of the suspension of a $\mathbb{Z}^k$ system on a space crystal T^{k+1} of dimension $k+1$ each of whose time-t maps is an affine automorphism with linear part a hyperbolic element

[1] Indeed, generally speaking mathematics is a good way to start doing philosophy. It should be noted that I am a mathematician who is more than keen on at least some kinds of philosophy; this has certain ramifications regarding my view of mathematics and how I conduct research, although for the rest of this statement this will not be relevant.

in $\mathrm{GL}(k+1,\mathbb{Z})$. Here a "space crystal" is either the torus $\mathbb{T}^{k+1} \cong \mathbb{R}^{k+1}/\mathbb{Z}^{k+1}$ or the $\pm$-infratorus $\mathbb{T}^{k+1}_{\pm} \cong \mathbb{R}^{k+1}/\mathbb{Z}^{k+1} \rtimes \{\pm I\}$. I am using the phrase "space crystal" to distinguish the torus $\mathbb{T}^k$ over which the suspension fibers; I call this torus the "time crystal" of the system[2].

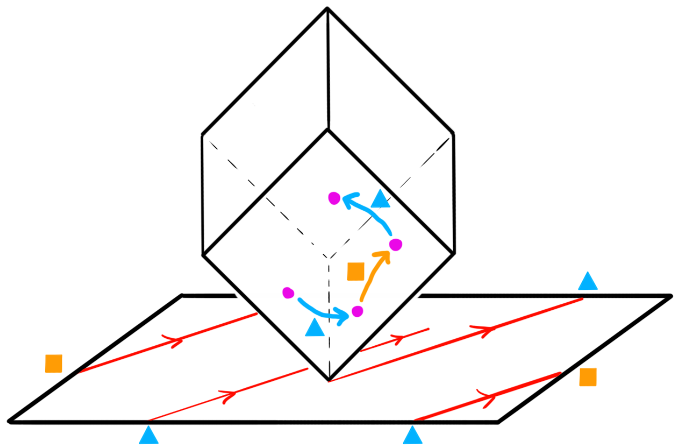

Figure 1: In the case of $k = 2$, we have that the time crystal of (μ, α) is $\mathbb{T}^2$ and the space crystal of (μ, α) is T^3. The space crystal carries a $\mathbb{Z}^2$ system with generators ▲ and ■. We think of the space crystal sliding along the time crystal (this is the translation action $\mathbb{T}^2 \curvearrowright \mathbb{T}^2$). The discrete time progresses on T^3 according to which sides of the time crystal are hit as the space crystal slides.

This in particular solves Problem 4 from the landmark paper [KKRH11], which can be considered as a suspension rigidity problem[3]. On the other hand, my result is also a foundational result in the geometric approach to measure rigidity, in that it supports the idea that taking the suspension of a discrete system is not only categorically but also a dynamically natural operation; taking suspensions is crucial for Lyapunov geometry, that is, the geometry of the Lyapunov hyperplanes and Weyl chambers.

Let me now state part of my $\mathbb{R}^k$ arithmeticity result precisely and briefly mention the new complications that are not dealt with in the proof from the earlier $\mathbb{Z}^k$ arithmeticity result [KRH16].

Theorem 1 ([Uzm22])**:** Let (μ, α) be an $\mathbb{R}^k$ system on M. If (μ, α) satisfies the maximal rank positive entropy hypotheses, that is,

[2] I borrow this nomenclature from contemporary physics (see [Wil12, YN18, YCLZ20]), although the way I use the phrase "time crystal" does not match with the use of the term in the physics literature. The relation between the two uses is roughly the same relation between Pugh's and Anosov's closing lemmas (see [Pug11]). Note that in classical ergodic theory typically the discrete system that is being suspended is thought of as the "horizontal" space, especially if there is a cocycle (i.e. roof function). In the study of actions of groups more complicated than $\mathbb{Z}$ or $\mathbb{R}$ the perspective is typically flipped, so that the suspended system is "vertical". In any suspension with a cocycle, existence of a time crystal signifies the triviality (interpreted appropriately) of the cocycle, so that time crystals are obstructions to the suspension system having the K property, and existence of a space crystal signifies algebraicity of the suspended system; the Totoki-Gurevich Theorem (see [Tot70, Gur69] and [Uzm] for further references) is important for this discussion, and it also makes the case that even for classical ergodic theory, and even when there is a cocycle it is better to think of the suspended system as vertical.

[3] One of the most famous suspension rigidity problems is the Verjovsky conjecture for codimension-one Anosov flows (see [Ver74, BBGRH21]), although as far as I know the lines of attack that were used to tackle this problem so far are different from the methods I'm using for my arithmeticity result.

- $k \in \mathbb{Z}_{\geq 2}$ and $\dim(M) = 2k+1$,
- (μ, α) is locally free and ergodic,
- The system (μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes, and
- For any $t \in \mathbb{R}^k \setminus 0$ the time-t map α_t has positive metric entropy w/r/t μ,

then there is

- an affine Cartan action $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{Aff}(T^{k+1})$, and
- a $\kappa \in \mathrm{GL}(k, \mathbb{R})^\circ$

such that (μ, α) is measure theoretically isomorphic to the κ time change of the suspension of $(\mathrm{haar}_{T^{k+1}}, \gamma)$. Moreover the measure theoretical isomorphism has certain conditional smoothness properties, e.g. off a closed subset of arbitrarily small μ measure the regularity of the isomorphism is arbitrarily close to the regularity of the original system.

⌟

The main idea of the proof is to use nonstationary linearizations of Lyapunov æ-foliations (which are all 1 dimensional in this case) and holonomies defined in terms of them to extend the system (μ, α) measure theoretically to the standard action of a subgroup of $\mathrm{Aff}(\mathbb{R}^{2k+1})$ with diagonal linear part. Identifying the symmetries of this extension (the "synthetic homoclinic group") and quotienting them out produce the crystals. There are two distinctions with the $\mathbb{Z}^k$ case: first is that in the $\mathbb{Z}^k$ case by a classical result of Pesin the action is virtually[4] by Bernoulli diffeomorphisms; in the $\mathbb{R}^k$ case this "weak mixing reduction" available (the same work of Pesin gives only an dichotomy for flows). The second is that in order for the holonomies to produce the covering map, no matter in which order the Lyapunov leaves are used they have to be consistent, however for $\mathbb{R}^k$ actions a priori they may bleed into the orbit directions. Accordingly, in the $\mathbb{R}^k$ case the synthetic homoclinic group has to be shown to have a k-dimensional trivial part[5].

Two Follow-up Problems: the Anosov Case and the Rank One Case There are two follow-up problems to my arithmeticity result, both due to Prof. F. Rodríguez Hertz[6]. The first problem is to upgrade the measure theoretical isomorphism to C^r everywhere, under the extra assumption that α is Anosov. Let us call the measure theoretical isomorphism promised in the arithmeticity result the arithmeticity isomorphism.

Problem 1 (FRH)**:** Let (μ, α) be an $\mathbb{R}^k$ system on M. If (μ, α) satisfies the maximal rank positive entropy hypotheses, and in addition

- There is a $t^\dagger \in \mathbb{R}^k$ such that $\alpha_{t^\dagger}$ is uniformly hyperbolic normal to the orbit foliation,

[4] i.e., up to finite index subgroup.

[5] See my poster available at `https://alpuzman.github.io/research/uzman_simons2022posterkrha.pdf` for further details.

[6] (personal communication)

then the arithmeticity isomorphism is everywhere defined and is a C^r diffeomorphism.

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Provisionally, a solution to this problem requires extra ingredients from the landmark papers [RH07, RHW14] that champion the idea of "finding many Anosovs from one". This in particular recovers Matsumoto's result [Mat09], of which my arithmeticity result can be considered as a nonuniform analog.

The second follow-up problem is in rank one. Now we don't have Lyapunov geometry, and we don't have measure rigidity; still if one assumes absolute continuity of the invariant measure, the measurable affine extension and the synthetic homoclinic group are still available. I will state this problem vaguely, as the way I noted it seems too good to be true:

Problem 2 (FRH)**:** Let (ν, g) be a $\mathbb{Z}$ system on a surface and (μ, φ) be a $\mathbb{R}$ system on a 3-manifold.

- ν and μ are of Lebesgue class,
- The derivatives are $\log^+$-integrable, and
- Both systems have positive entropy.

Then (ν, g) and (μ, φ) are rigid. Find precise instantiations of "rigid" appropriate to this context.

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There are many other questions to be explored regarding the arithmeticity machinery.

Entropy Gauges and Fried Entropy Let us now switch back to the general case of G systems. Given any G system (μ, α), denote by $\mathfrak{e}_{(\mu,\alpha)} : G \to \mathbb{R}_{\geq 0}$ that map which takes a time t and sends it to the metric entropy of the diffeomorphism α_t w/r/t μ. We call $\mathfrak{e}_{(\mu,\alpha)}$ the entropy gauge associated to the system (μ, α), as indeed in the case of $G = \mathbb{R}^k$ by a classical result of Abramov (see [Abr59]) and the thesis work of Hu ([Hu93b, Hu93a]) $\mathfrak{e}_{(\mu,\alpha)}$ is a seminorm on $\mathbb{R}^k$. This observation is interesting in and of itself, a more important reason why one would consider such entropy gauges is to develop an entropy theory for smooth actions of groups more complicated than $\mathbb{Z}$ or $\mathbb{R}$. Indeed, either the definition of entropy is to be changed so that the whole system has finite entropy, in which case time-t maps of the action will have infinite entropy, or else one needs to be content with the entropy of the whole system being zero[7]. Neither of these approaches is satisfactory, in that the first case violates the Kushnirenko hypothesis ("classical systems have finite entropy"), and the second case dismisses the whole endeavor as trivial. For an alternative, following [Fri83, KKRH14] let us fix a (*not necessarily translation invariant*) distance function d on G compatible with its topology and a Haar measure haar_G, and define the Fried entropy of (μ, α) by

[7] As far as I know this was first observed by Conze in [Con73].

$$\mathrm{Friedent}(\mu,\alpha) = \mathrm{Friedent}_{(d,\mathrm{haar}_G)}(\mu,\alpha) = \frac{\mathrm{haar}_G(\{t \in G \mid d(t,e_G) \leq 1\})}{\mathrm{haar}_G(\{t \in G \mid \mathfrak{e}_{(\mu,\alpha)}(t) \leq 1\})}.$$

In the aforementioned papers, it is proven that for G compactly generated abelian and d the ℓ^1 distance, Friedent is multiplicative w/r/t taking products of systems, for $G = \mathbb{Z}$ or $\mathbb{R}$ recovers the standard definition of metric entropy, and satisfies the Kushnirenko and Abramov hypotheses[8].

In [Uzm23] I investigate the global dependency of Friedent on the system, given G and M, and also extend and strengthen the above result to a uniqueness result for nilpotent systems. A provisional theorem is as follows[9]:

Theorem 2 ([Uzm23]; Provisional)**:** Let G be a Carnot group and M be a smooth compact manifold. Denote by d_{CC} the subriemannian distance function on G. Then $\mathrm{Friedent}_{(d_{CC},\mathrm{haar}_G)}$ is the unique $\mathbb{R}_{\geq 0}$-valued function on the space of G systems on M that is multiplicative along the steps of the grading, at the linear step coincides with the Fried entropy of the abelian case, and satisfies the Abramov hypothesis.

⌟

It would also be interesting to consider the Fried entropy given by the Mitchell measure (which is the Hausdorff measure associated to d_{CC} of appropriate dimension, see the next section). The paper [HSW18] by Hu et al. is also relevant to this discussion; for instance there they show that for G the 3 dimensional discrete Heisenberg group, the entropy of the time-t map for t in the central direction is always zero; signifying the necessity for slow entropy compatible with the grading of G[10].

3 Future Studies

Let me finally highlight one measure rigidity problem that I find very interesting.

Subriemannian Measure Rigidity The problem in question is a big open problem in subriemannian geometry. Let M be a smooth manifold and H be a sufficiently smooth CC structure on M. Briefly[11], this means that H is the image of a C^r bundle map $E \to TM$, where $E \to M$ is a C^∞ vector bundle, and taking iterated brackets of vector fields with values in H one can get all vector fields on M ("Hörmander condition"):

$$\begin{aligned}
&\exists r_\bullet : M \to \mathbb{Z}_{\geq 0}, \forall x \in M : \\
&0 = H^0_x \leq H^1_x = H_x \leq \cdots \leq H^i_x = \mathrm{Span}_x(H^{i-1} + [H, H^{i-1}]) \leq \cdots \leq H^{r_x}_x = T_x M.
\end{aligned}$$

[8]The latter means that switching to a finite index subgroup should multiplicatively contribute the index to entropy.

[9]We refer to [Mon02] for definitions related to CC geometry.

[10]This is exactly their Question 1.3..

[11]We again refer to the aforementioned book by Montgomery or the book [BR96]

Call a point $x \in M$ H-regular if $r_\bullet$ as well as the signature of the flag $H^\bullet$ above are constant on a neighborhood of x. $H = H^1$ marks the admissible directions in M, it is also the directions of linear growth. H^2 / H^1 next marks the directions of quadratic growth etc. Mitchell showed in [Mit85] a formula analogous to Pesin entropy formula. Namely, if d^H_{CC} is the subriemannian distance function defined by H, then for any H-regular point $x \in M$:

$$\mathrm{Hausdim}^{d^H_{CC}}(M) = \sum_{i=1}^{q_x} i\left(\dim(H^i_x) - \dim(H^{i-1}_x)\right).$$

Problem 3: Develop the analogy between Pesin entropy formula and Mitchell dimension formula. Establish an analog of Ledrappier-Young theory in the context of CC geometry. ⌟

There is an alternative, differential geometric volume measure that can be defined on any CC manifold, called the Popp measure. Now we can state the problem:

Problem 4: Is the Mitchell measure a locally constant multiple of the Popp measure? ⌟

When M is a Carnot group with its standard CC structure, both of these measures are Haar, so in the homogeneous case the problem is trivial. The problem is solved in [ABB12] when every point is H-regular and both $r_\bullet$ and the signature of the flag $H^\bullet$ is constant; in this case it is shown that in fact the Mitchell measure is a constant multiple of the Popp measure. In full generality as far as I know the problem is open, and even if it weren't open I think a Ledrappier-Young theoretical approach to this question would be a fruitful endeavor, both for CC geometry and for dynamics; see also [BR13, Don14][12].

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[12] The questions at the intersection of dynamics and CC geometry seem very interesting in general; see [Sim16] for a sampler.

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