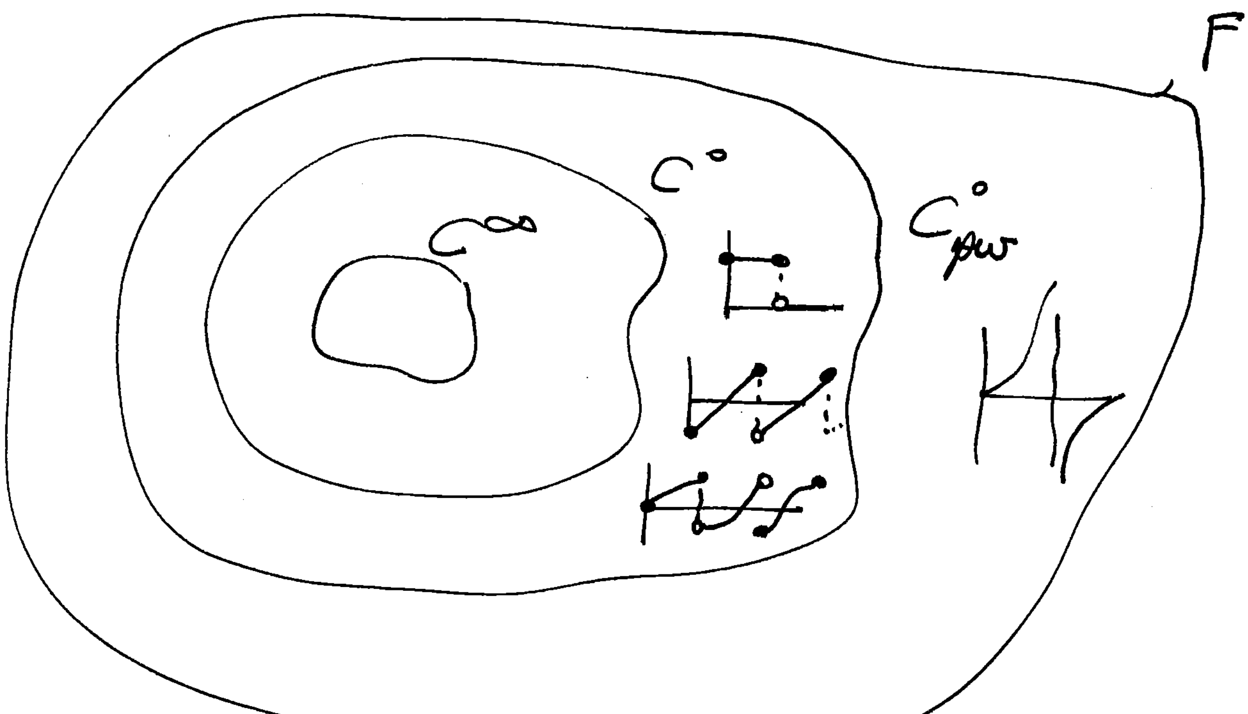
The Laplace Transform.

- Let $f : [0,\infty[\to \mathbb{R}$ be a function. Its <u>Laplace transform</u> is:

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$$

We want to interpret $\mathcal{L}$ as a function (of functions), so we need to determine when $\mathcal{L}(f)$ makes sense.

- $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is <u>piecewise continuous</u> if on any closed interval $[a,b]$ it has only finitely many jump discontinuities. Denote by $C^0_{pw}([0,\infty[, \mathbb{R})$ the linear space of all piecewise continuous functions.



$$C^0_{pw}([0,\infty[, \mathbb{R}) \xrightarrow{\cong} \{ f \in C^0_{pw}(\mathbb{R}, \mathbb{R}) \mid t < 0 \Rightarrow f(t) = 0 \}$$

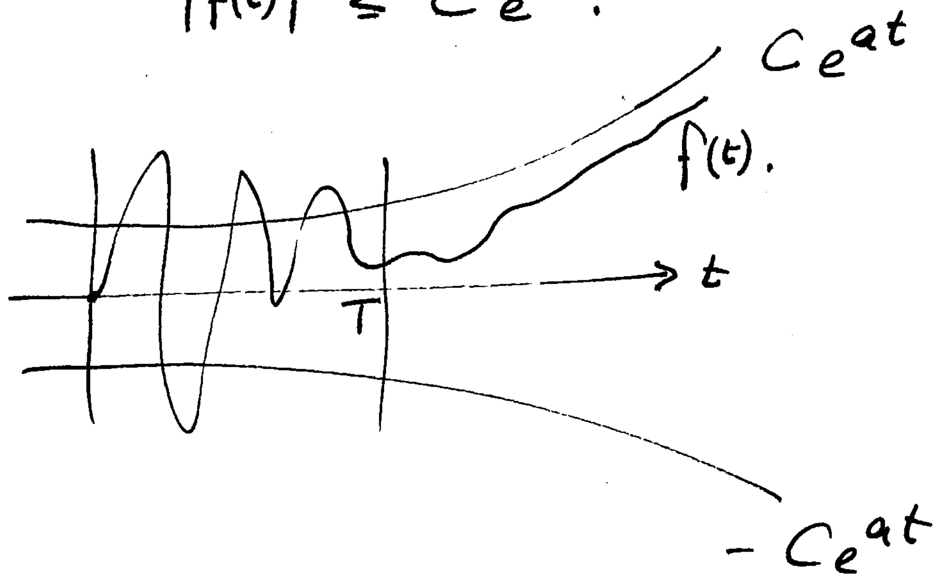
$$\varphi \longmapsto \varphi u_0 = \begin{cases} \varphi, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}.$$

$$\psi|_{[0,\infty[} \longleftarrow\!\mapsto \psi$$

• Let $a \in \mathbb{R}$, $f : [0, \infty[\longrightarrow \mathbb{R}$. Then $f(t) = \underset{t\to\infty}{O}(e^{at})$ ("f is <u>big oh of e^{at}</u> as $t \to \infty$.") if

there are $C > 0$, $T > 0$, for all $t \geq T$:

$$|f(t)| \leq C e^{at}.$$

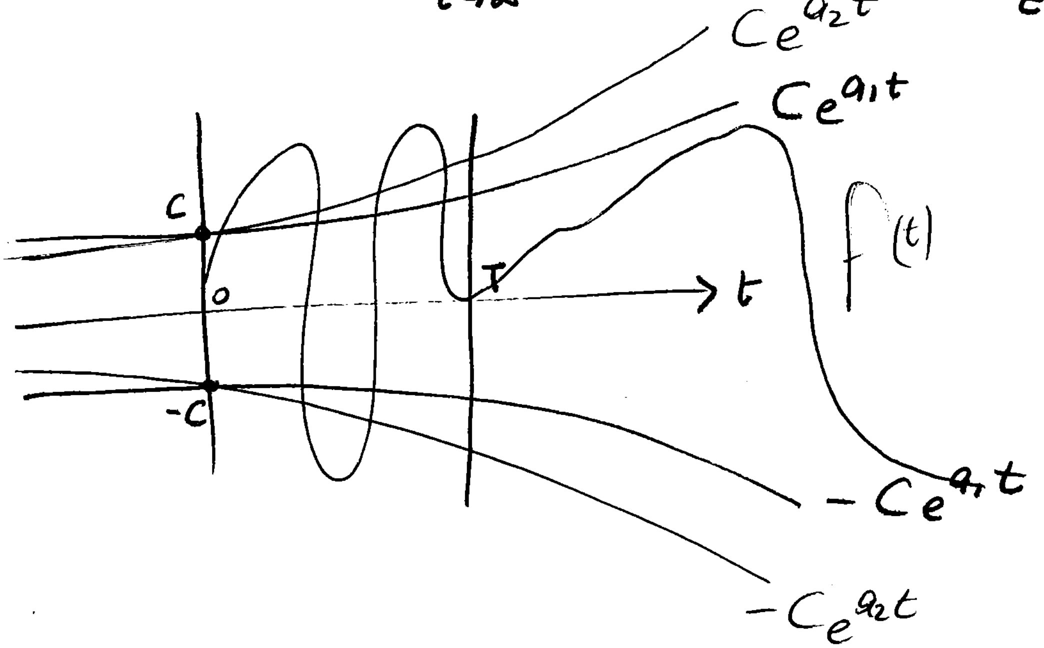


eg. bounded functions ($\sin$, $\cos$, χ_I, $u_c = \chi_{[c,\infty[}$, ...) are $\underset{t\to\infty}{O}(1)$ $(a = 0)$

exponentials & polynomials are $\underset{t\to\infty}{O}(e^{at})$ $(a > 0)$.

<u>SW</u>: e^{t^2}, $e^{e^t} \neq \underset{t\to\infty}{O}(e^{at})$ for any $a \in \mathbb{R}$.

$a_1 < a_2 \Rightarrow$ if $f(t) = \underset{t\to\infty}{O}(e^{a_1 t})$, then $f(t) = \underset{t\to\infty}{O}(e^{a_2 t})$.



•

$\forall a \in \mathbb{R}$: $dom(\mathcal{L})_a := \left\{ f \in C^0_{pw}([0,\infty[, \mathbb{R}) \;\middle|\; f(t) = \underset{t\to\infty}{O}(e^{at}) \right\}$.

linear space of Laplaceable functions of exponential type a.

Thm: Let $a \in \mathbb{R}$, $f \in dom(\mathcal{L})_a$. Then

$$\mathcal{L}(f) :]a,\infty[\longrightarrow \mathbb{R}$$
$$s \longmapsto \int_0^\infty e^{-st} f(t)\,dt$$

is a well-defined function.

$\left(\begin{array}{l} \forall = \text{"for all"} / \text{"for any"} \\ \exists = \text{"there exists"} \\ \in = \text{"element of"} \end{array}\right)$

Pf: Fix $s > a$.

$$|\mathcal{L}(f)(s)| = \left| \int_0^\infty e^{-st} f(t)\,dt \right| \leq \int_0^\infty \left| e^{-st} f(t) \right| dt$$

$$= \underbrace{\int_0^T e^{-st} |f(t)|\,dt}_{=:\, M < \infty} + \int_T^\infty e^{-st} |f(t)|\,dt$$

$$\leq M + C \int_T^\infty e^{(a-s)t}\,dt = M + C \lim_{B \to \infty} \int_T^B e^{(a-s)t}\,dt$$

$$= M + C \lim_{B \to \infty} \left[\frac{e^{(a-s)t}}{a-s} \right] \Bigg|_{t=T}^{B} = M + C \lim_{B \to \infty} \frac{\overbrace{e^{(a-s)B}}^{s>a,\ \to 0} - e^{(a-s)T}}{a-s}$$

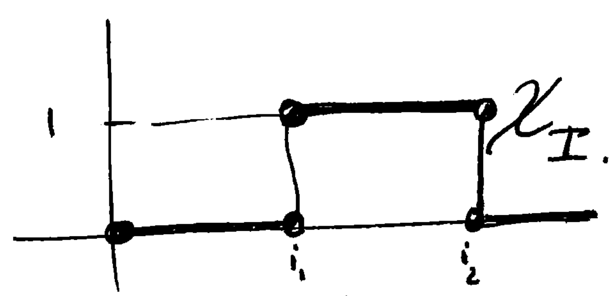
$$= M + C \, \frac{-e^{(a-s)T}}{a-s} < \infty. \quad \checkmark$$

Ex: • $I = [i_1, i_2] \subseteq \mathbb{R}_{\geq 0}$. $\chi_I : [0, \infty[\longrightarrow \mathbb{R}$

$$t \longmapsto \left\{ \begin{array}{ll} 1, & \text{if } i_1 \overset{\leq}{<} t \overset{\leq}{<} i_2 \\ 0, & \text{ow.} \end{array} \right\}$$

$\Rightarrow \mathcal{L}(\chi_I) :]0, \infty[\longrightarrow \mathbb{R}$.

is well-defined.



$s > 0$. $\mathcal{L}(\chi_I)(s)$

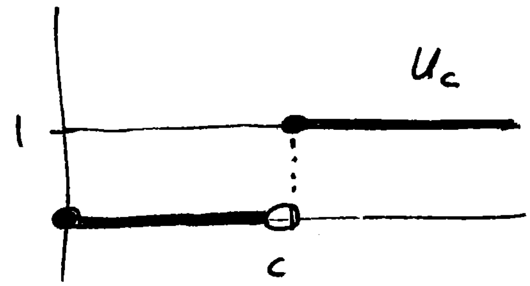
$$= \int_0^\infty e^{-st} \chi_I(t)\,dt = \int_{i_1}^{i_2} e^{-st}\,dt = \left[\frac{e^{-st}}{-s} \right]_{t=i_1}^{i_2}$$

$$= \frac{e^{-si_2}}{-s} - \frac{e^{-si_1}}{-s} = \frac{e^{-si_1} - e^{-si_2}}{s}.$$

• $\forall c \geqslant 0: \quad u_c := \chi_{[c,\infty[} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$

$$t \longmapsto \begin{cases} 1, & \text{if } c \leq t \\ 0, & \text{if } 0 \leq t < c \end{cases}$$

(Heaviside function / unit step function)



$$u_c = \chi_{[c,\infty[} = \lim_{B\to\infty} \chi_{[c,B]}$$

$$\Rightarrow \mathcal{L}(u_c)(s) = \mathcal{L}\left(\lim_{B\to\infty} \chi_{[c,B]}\right)(s)$$

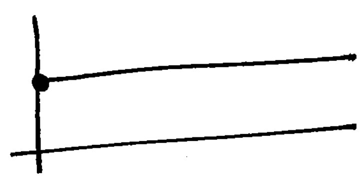
$$= \lim_{B\to\infty} \mathcal{L}\left(\chi_{[c,B]}\right)(s) = \lim_{B\to\infty} \frac{e^{-sc} - e^{-sB}}{s} = \frac{e^{-sc}}{s}.$$

continuity of $\mathcal{L}$.

$\mathcal{L}(u_c) :]0,\infty[\longrightarrow \mathbb{R}$. $\boxed{\#12}$

(SW: check by direct computation)

• $c := 0$. $\quad u_0 = 1$



$u_0 = 1.$

$$\Rightarrow \qquad \Rightarrow \mathcal{L}(1)(s) = \mathcal{L}(u_0)(s) = \frac{1}{s}. \quad \boxed{\#1}$$

Ex: $c \in \mathbb{R}$. $\quad \varphi : [0,\infty[\longrightarrow \mathbb{R}$
$\qquad t \longmapsto e^{ct}$.

$\varphi(t) = \underset{t\to\infty}{O}\left(e^{ct}\right)$

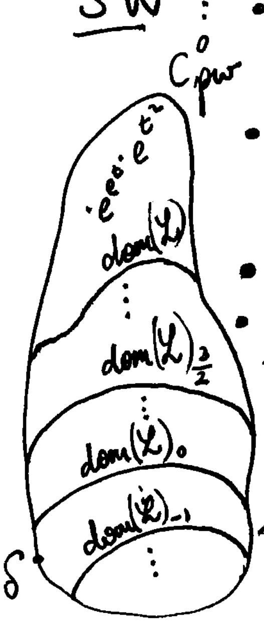
$\Rightarrow \varphi \in \mathrm{dom}(\mathcal{L})_c$.

$\Rightarrow \mathcal{L}(\varphi) :]c,\infty[\longrightarrow \mathbb{R}$ is well-def.

$s > c$.

$$\mathcal{L}(\varphi)(s) = \mathcal{L}\{e^{ct}\}(s) = \int_0^\infty e^{-st} e^{ct}\, dt = \int_0^\infty e^{(c-s)t} dt$$

$$= \lim_{B\to\infty} \int_0^B e^{(c-s)t} dt = \lim_{B\to\infty} \left[\frac{e^{(c-s)t}}{c-s}\right]\Bigg|_{t=0}^{B} = \frac{1}{s-c} \quad \boxed{\#2}$$



<u>SW</u> : • Laplace transform of $\sin(\omega t)$ & $\cos(\omega t)$? $\boxed{\#5, \#6}$

• $\forall a \in \mathbb{R}$: $\mathcal{L} : \mathrm{dom}(\mathcal{L})_a \to F(]a,\infty[, \mathbb{R})$ is linear.

• Let $\mathrm{dom}(\mathcal{L})$ be (the weak closure of) $\bigcup_{a \in \mathbb{R}} \mathrm{dom}(\mathcal{L})_a$. Then $\mathcal{L}$ is linear on it.

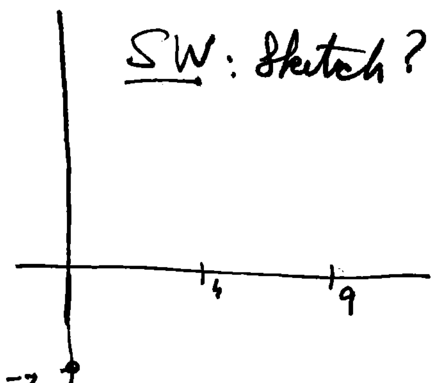
• $\chi_{[a,b[} = u_a - u_b$. , $\chi_{]a,b[} = \lim_{n\to\infty} u_{a+\frac{1}{n}} - u_b$,

$\chi_{[a,b]} = u_a - \lim_{n\to\infty} u_{b+\frac{1}{n}}$.

<u>Ex</u> : $\varphi : [0,\infty[\to \mathbb{R}$

$$t \longmapsto \begin{cases} 3t^2 - 2, & t < 4 \\ e^{5t} + t, & 4 \le t < 9 \\ \cos(2t), & 9 \le t \end{cases}.$$

<u>SW</u>: Sketch?



$$\varphi(t) = (3t^2-2)\underbrace{(u_0 - u_4)}_{=\chi_{[0,4[}} + (e^{5t}+t)\underbrace{(u_4 - u_9)}_{=\chi_{[4,9[}} + \cos(2t)\underbrace{u_9}_{=\chi_{[9,\infty[}}$$

$$= (3t^2-2) + (e^{5t}+t-3t^2+2)u_4 + (\cos(2t) - e^{5t} - t)u_9.$$

<u>SW</u> : $\varphi : [0,\infty[\to \mathbb{R}$

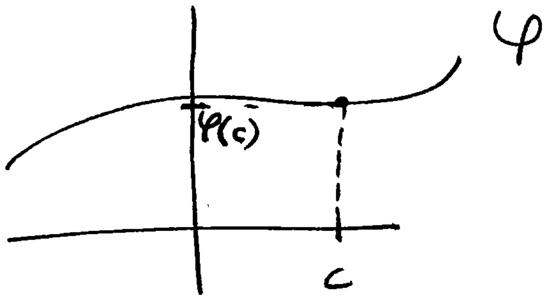
$$t \longmapsto \begin{cases} \varphi_1(t), & 0 \le t < a_1 \\ \varphi_2(t), & a_1 \le t < a_2 \\ \vdots & \\ \varphi_k(t), & a_{k-1} \le t \end{cases}$$

(Re)Write φ using unit step functions.

$\bullet\ \forall c \in \mathbb{R}:\ \delta_c : F(\mathbb{R}_{\geq 0}, \mathbb{R}) \longrightarrow \mathbb{R}$

$\qquad\qquad\qquad\qquad \varphi \longmapsto \varphi(c)$

(<u>Dirac delta</u> at c / <u>unit impulse</u> at c / <u>point mass</u> at c)



<u>SW</u>: δ_c is linear.

Later: Another notation:

$$\varphi(c) = \delta_c(\varphi) = \int_0^\infty \varphi(t)\ \delta(t-c)\,dt$$

$$\Rightarrow \mathcal{L}(\delta_c)(s) = e^{-sc}$$

$$\overset{\shortparallel}{\mathcal{L}\{\delta(t-c)\}(s)}$$

$\boxed{\text{\#17, to be repeated and done properly}}$

$\bullet\ \mathcal{L}\{f(t)\,\delta(t-c)\}(s) = f(c)\,e^{-sc}$

if f is continuous at c.

<u>Ex</u>: $\bullet\ \delta_0(\sin) = \sin(0) = 0.$

$\bullet\ \delta_0(\cos) = \cos(0) = 1.$

$\bullet\ \delta_{\frac{\pi}{4}}(\cos) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$

$\bullet\ \mathcal{L}\{\delta(t-2)\,t^3 e^{-t}\cos^3(\pi t)\} = \int_0^\infty e^{-st}\, t^3 e^{-t} \cos^3(\pi t)\,\delta(t-2)\,dt$

$$= \delta_2\left(e^{-st}\,t^3 e^{-t}\cos^3(\pi t)\right) = e^{-2s}\cdot 8 \cdot e^{-2}\cdot \cos^3(2\pi)$$

$$= 8e^{-2}e^{-2s}$$

$$= 8e^{-2(s+1)}.$$

• $\mathcal{M} : F(\mathbb{R}_{\geq 0}, \mathbb{R}) \longrightarrow F(\mathbb{R}_{\geq 0}, \mathbb{R})$ "multiply by the monomial t"

$$\varphi \longmapsto [\mathcal{M}(\varphi) : t \mapsto t\,\varphi(t)]$$

<u>SW</u> : $\mathcal{M}: \mathrm{dom}(\mathcal{L}) \to \mathrm{dom}(\mathcal{L})$ is a linear operator.

<u>Thm</u> : Let $\varphi \in \mathrm{dom}(\mathcal{L})_a$ be differentiable and $\dot{\varphi} \in \mathrm{dom}(\mathcal{L})_a$. Then

$\mathcal{L}(\dot{\varphi}) :]a, \infty[\longrightarrow \mathbb{R}$ is well-defined and

for $s > a$, $\mathcal{L}(\dot{\varphi})(s) = s\,\mathcal{L}(\varphi)(s) - \varphi(0)$. $\boxed{\#18}$

Pf: Let $s > a$.

$$\mathcal{L}(\dot{\varphi})(s) = \int_0^\infty e^{-st}\,\dot{\varphi}(t)\,dt = \lim_{B\to\infty} \int_0^B e^{-st}\,\dot{\varphi}(t)\,dt$$

$\boxed{\begin{array}{l} u = e^{-st} \\ du = -se^{-st}dt \\ dv = \dot{\varphi}\,dt \\ v = \varphi \end{array}}$

$$= \lim_{B\to\infty} \left(\left[e^{-st}\varphi(t)\right]\Big|_{t=0}^{B} + s\int_0^B e^{-st}\varphi(t)\,dt \right)$$

$$= \lim_{B\to\infty} \left(\underbrace{e^{-sB}\varphi(B)} - \varphi(0) + s\int_0^B e^{-st}\varphi(t)\,dt \right)$$

$$\left(\begin{array}{l} |e^{-sB}\varphi(B)| \leq e^{-sB}|\varphi(B)| \\ \leq C e^{-sB} e^{aB} = Ce^{(a-s)B} \to 0 \end{array} \right)$$

$$= s\int_0^\infty e^{-st}\varphi(t)\,dt - \varphi(0) = s\,\mathcal{L}(\varphi)(s) - \varphi(0). \checkmark$$

$$\Rightarrow \boxed{\mathcal{L} \circ \partial_t = \mathcal{M} \circ \mathcal{L} - \delta_0.}$$

SW: • Meditate on this:

$$\mathcal{L}\,\partial_t^2 = \mathcal{L}\,\partial_t\partial_t = (\mathcal{M}\mathcal{L} - \delta_0)\,\partial_t = \mathcal{M}\mathcal{L}\,\partial_t - \delta_0\,\partial_t$$

$$= \mathcal{M}(\mathcal{M}\mathcal{L} - \delta_0) - \delta_0\,\partial_t = \mathcal{M}^2\mathcal{L} - \mathcal{M}\,\delta_0 - \delta_0\,\partial_t.$$

$$\Rightarrow \boxed{\mathcal{L}\circ\partial_t^2 = \mathcal{M}^2\circ\mathcal{L} - \mathcal{M}\circ\delta_0 - \delta_0\circ\partial_t}$$

i.e., for all φ & s, $\mathcal{L}(\ddot{\varphi})(s) = s^2\,\mathcal{L}(\varphi)(s) - s\,\varphi(0) - \dot{\varphi}(0)$
whenever it makes sense.

• Generalize to higher order derivatives.

Thm: Let $\varphi \in \mathrm{dom}(\mathcal{L})_a$. Then $\mathcal{L}(\varphi) :]a,\infty[\longrightarrow \mathbb{R}$
is differentiable and $-\partial_s\,\mathcal{L}(\varphi)(s) = \mathcal{L}\{t\,\varphi(t)\}(s)$.

$\boxed{\#19}$

Pf: Let $s > a$.

$$\partial_s\,\mathcal{L}(\varphi)(s) = \lim_{h\to 0} \frac{\mathcal{L}(\varphi)(s+h) - \mathcal{L}(\varphi)(s)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_0^\infty e^{-(s+h)t}\varphi(t)\,dt - \int_0^\infty e^{-st}\varphi(t)\,dt\right)$$

$$= \lim_{h\to 0}\int_0^\infty \frac{e^{-ht}-1}{h}\,e^{-st}\varphi(t)\,dt \underset{\uparrow}{=} \int_0^\infty \underbrace{\left(\lim_{h\to 0}\frac{e^{-ht}-1}{h}\right)}\,e^{-st}\varphi(t)\,dt$$

(SW: Why?) $\left(\overset{0}{\underset{0}{=}} \lim_{h\to 0}\frac{-t\,e^{-ht}}{1} = -t\right)$

$$= \int_0^\infty e^{-st}\big(-t\,\varphi(t)\big)\,dt = \mathcal{L}\{-t\,\varphi(t)\}(s), \checkmark.$$

$$\Rightarrow \boxed{-\partial_s\circ\mathcal{L} = \mathcal{L}\circ\mathcal{M}}$$

SW: • $\partial_s^2 \mathcal{L} = \partial_s \partial_s \mathcal{L} = -\partial_s \mathcal{L} \mathcal{M} = \mathcal{L} \mathcal{M}\mathcal{M} = \mathcal{L}\mathcal{M}^2$

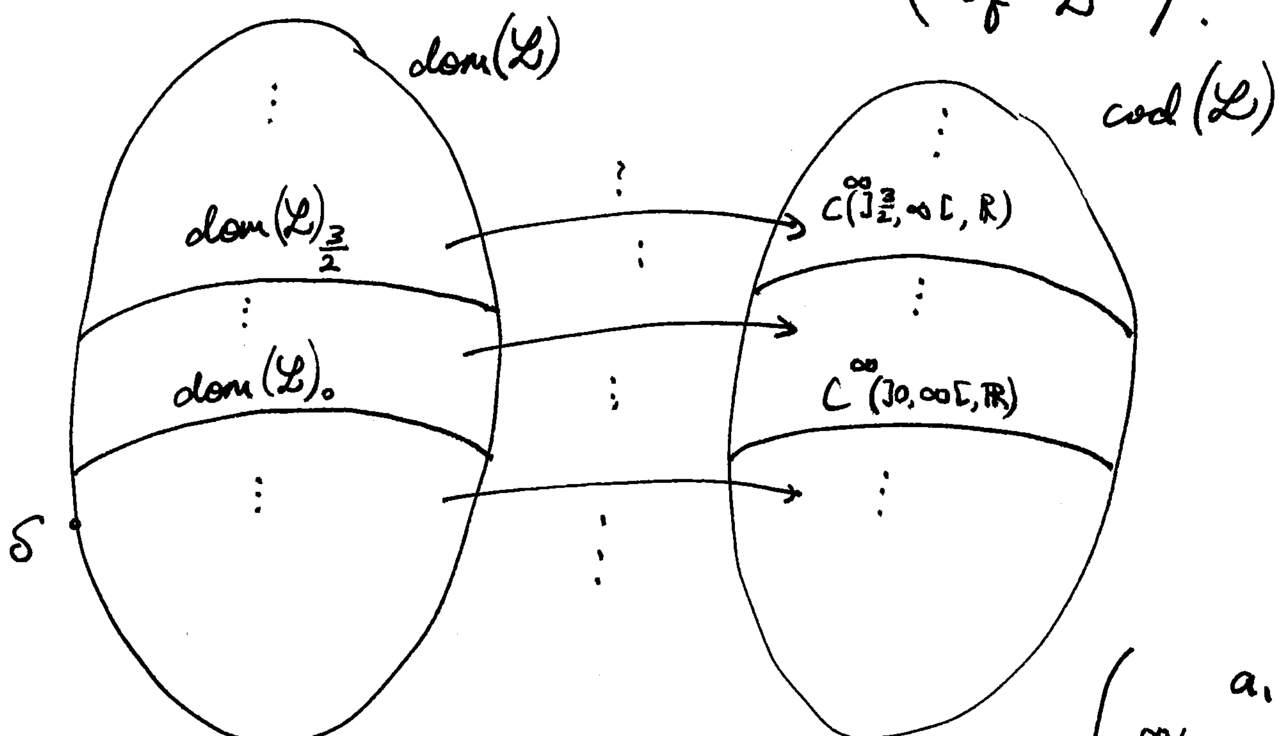
$$\Rightarrow \boxed{\partial_s^2 \circ \mathcal{L} = \mathcal{L} \circ \mathcal{M}^2}$$

ie., for all φ & s, $\mathcal{L}(\varphi)''(s) = \mathcal{L}\{t^2 \varphi(t)\}(s)$, whenever this makes sense. • Generalize to ∂_s^n.

• This means that if $\varphi \in \mathrm{dom}(\mathcal{L})_a$, then $\mathcal{L}(\varphi) \in C^\infty(]a,\infty[,\mathbb{R})$. Also $\mathcal{L}(\delta_c) \in C^\infty(]0,\infty[,\mathbb{R})$.

• Put $\mathrm{cod}(\mathcal{L}) := \bigcup_{a\in\mathbb{R}} C^\infty(]a,\infty[,\mathbb{R})$. Then we have

$$\mathcal{L} : \underset{\left(\frac{\text{domain}}{\text{of } \mathcal{L}}\right)}{\mathrm{dom}(\mathcal{L})} \longrightarrow \underset{\left(\frac{\text{codomain}}{\text{of } \mathcal{L}}\right)}{\mathrm{cod}(\mathcal{L})}.$$



$$\left(\begin{array}{c} a_1 < a_2 \\ C^\infty(]a_1,\infty[,\mathbb{R}) \rightarrowtail C^\infty(]a_2,\infty[,\mathbb{R}) \\ \varphi \longmapsto \varphi\big|_{]a_2,\infty[} \end{array} \right)$$

Ex: Let $n \geq 0$ be an integer. Then $\mathcal{M}^n(1)(t) = t^n \cdot 1 = t^n$.

$$\Rightarrow \mathcal{L}\{t^n\}(s) = \mathcal{L} \circ \mathcal{M}^n(1)(s) = (-1)^n \partial_s^n \circ \mathcal{L}(1)(s) = (-1)^n \partial_s^n\left(\frac{1}{s}\right).$$

$$= (-1)^n \partial_s^n (s^{-1}) = (-1)^{n-1} \partial_s^{n-1}(s^{-2}) = (-1)^{n-2}\, 2\, \partial_s^{n-2}(s^{-3}) = (-1)^{n-3}\, 3!\, \partial_s^{n-3}(s^{-4})$$

$$= \dots = \frac{n!}{s^{n+1}} \quad \boxed{\#3}$$

SW: Laplace transform of $t^n e^{ct}$? $\boxed{\#11}$

Ex: $\boxed{\begin{array}{l} \ddot{y} - y = t, \quad t > 0 \\ y(0) = 1 \\ \dot{y}(0) = 1 \end{array}}$ $\qquad \mathcal{L}(\ddot{y} - y) = \mathcal{L}(t).$

$$LHS = s^2 \mathcal{L}(y) - s\,y(0) - \dot{y}(0) - \mathcal{L}(y)$$
$$= (s^2 - 1)\,\mathcal{L}(y) - s - 1$$

$$RHS = -\partial_s \mathcal{L}(1) = \frac{1}{s^2}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s^2-1}\left(\frac{1}{s^2} + s + 1\right) = \frac{1}{s^2(s^2-1)} + \frac{1}{(s-1)} = -\frac{1}{s^2} + \frac{1}{s^2-1} + \frac{1}{s-1}$$

$$= -\mathcal{L}\{t\} + \mathcal{L}\{\sinh(t)\} + \mathcal{L}\{e^t\} = \mathcal{L}\{e^t + \sinh(t) - t\}$$

$$\Rightarrow \boxed{y(t) = e^t + \sinh(t) - t}$$

$$\left(\text{or}: \frac{1}{s^2(s^2-1)} = \frac{-1}{s^2} + \frac{1/2}{s-1} + \frac{-1/2}{s+1} \quad \dots \quad \underline{\text{SW}}\right).$$

Ex: $\boxed{\begin{array}{c}\ddot{y}-\dot{y}-2y=0\ ,\ t\geq 0\\ y(0)=1,\ \dot{y}(0)=0.\end{array}}$ $\quad \mathcal{L}(\ddot{y}-\dot{y}-2y)=0.$

$$\Rightarrow 0=\left(s^2\,\mathcal{L}(y)-s\,y(0)-\dot{y}(0)\right)-\left(s\,\mathcal{L}(y)-y(0)\right)-2\,\mathcal{L}(y)$$

$$=(s^2-s-2)\,\mathcal{L}(y)+(-s+1)\underbrace{y(0)}_{=1}+(-1)\underbrace{\dot{y}(0)}_{=0}$$

$$=(s^2-s-2)\,\mathcal{L}(y)+(-s+1)$$

$$\Rightarrow \mathcal{L}(y)=\frac{s-1}{s^2-s-2}=\frac{s-1}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$$

$$=\frac{1}{3}\cdot\frac{1}{s-2}+\frac{2}{3}\cdot\frac{1}{s+1}$$

$$=\frac{1}{3}\,\mathcal{L}\{e^{2t}\}+\frac{2}{3}\,\mathcal{L}\{e^{-t}\}$$

$$=\mathcal{L}\left\{\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}\right\}\Rightarrow \boxed{y(t)=\frac{1}{3}e^{2t}+\frac{2}{3}e^{-t}}$$

$$\left(\begin{array}{lr} s(A+B) = & s\\ +1(A-2B) & -1\\ \Rightarrow A+B=1 & \\ \quad A-2B=-1 & \\ \Rightarrow A=\frac{1}{3}\quad B=\frac{2}{3} & \end{array}\right)$$

Ex: $\boxed{\begin{array}{c}\ddot{y}+y=\sin(2t)\ ,\ t\geq 0\\ y(0)=2,\ \dot{y}(0)=1\end{array}}$ $\quad \mathcal{L}(\ddot{y}+y)=\mathcal{L}\{\sin(2t)\}$

$$\text{LHS}=s^2\,\mathcal{L}(y)-s\,y(0)-\dot{y}(0)+\mathcal{L}(y)=(s^2+1)\,\mathcal{L}(y)-s\,y(0)-\dot{y}(0)$$

$$=(s^2+1)\,\mathcal{L}(y)-2s-1$$

$$\text{RHS}=\frac{2}{s^2+4}\quad\Rightarrow\quad \mathcal{L}(y)=\frac{1}{s^2+1}\left(\frac{2}{s^2+4}+2s+1\right)$$

$$\left(\begin{array}{lr} s^2(A+B)= & 0\\ +1\ (4A+B) & +2\\ \Rightarrow A+B=0 & \\ \quad 4A+B=2 & \\ \quad A=\frac{2}{3}\quad B=-\frac{2}{3} & \end{array}\right)$$

$$=\frac{A}{s^2+1}+\frac{B}{s^2+4}+2\,\frac{s}{s^2+1}+\frac{1}{s^2+1}=\frac{5}{3}\cdot\frac{1}{s^2+1}-\frac{2}{3}\,\frac{1}{s^2+4}+2\,\frac{s}{s^2+1}$$

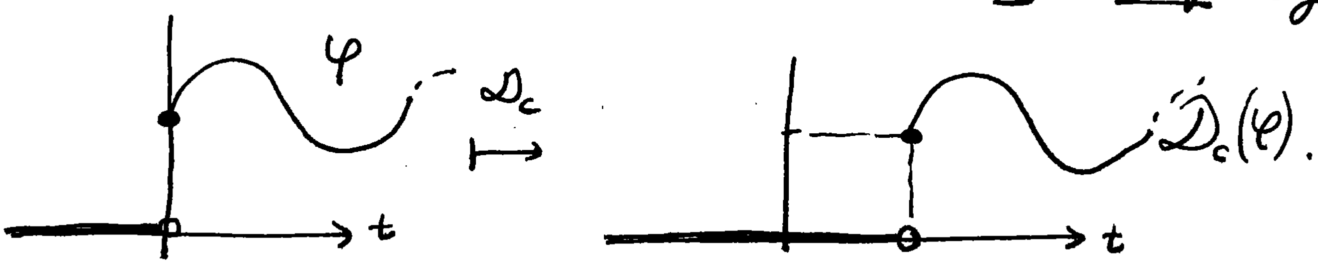
$$= \frac{5}{3} \cdot \frac{1}{s^2+1} - \frac{1}{3} \frac{2}{s^2+2^2} + 2 \frac{s}{s^2+1}$$

$$= \frac{5}{3} \mathcal{L}\{\sin(t)\} - \frac{1}{3} \mathcal{L}\{\sin(2t)\} + 2 \mathcal{L}\{\cos(t)\}$$

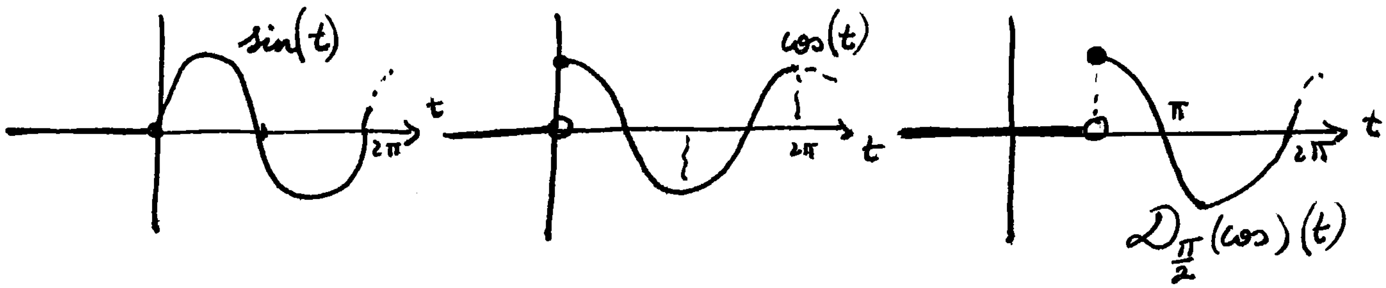
$$= \mathcal{L}\left\{\frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t) + 2\cos(t)\right\}$$

$$\Rightarrow \boxed{y(t) = \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t) + 2\cos(t)}$$

- $\forall c \in \mathbb{R} : \mathcal{D}_c : C^0_{pw}([0,\infty[, \mathbb{R}) \longrightarrow C^0_{pw}([0,\infty[, \mathbb{R})$
  $\varphi \longmapsto [\mathcal{D}_c(\varphi) : (t) \mapsto \varphi(t-c)\, u_c(t)]$ (time delay / shift by c)



Ex :



$\sin \neq \mathcal{D}_{\frac{\pi}{2}}(\cos)$ as functions in $C^0_{pw}([0,\infty[, \mathbb{R})$.

SW : For any $c \in \mathbb{R}$: $\mathcal{D}_c : \mathrm{dom}(\mathcal{L}) \longrightarrow \mathrm{dom}(\mathcal{L})$ is a linear operator.

- $\forall c \in \mathbb{R}: \mathcal{E}_c : F(\mathbb{R},\mathbb{R}) \longrightarrow F(\mathbb{R},\mathbb{R})$ "multiply by the exponential e^{ct}"

$$\varphi \longmapsto \left[\mathcal{E}_c(\varphi) : t \mapsto e^{ct}\varphi(t)\right]$$

<u>SW</u>: $\mathcal{E}_c$ is a linear operator.

<u>Thm</u>: Let $\varphi \in \mathrm{dom}(\mathcal{L})_a$, $c \in \mathbb{R}$. Then

$$\mathcal{L}\{\varphi(t-c)u_c(t)\} = e^{-cs}\mathcal{L}(\varphi) \quad \boxed{\#13}$$

<u>Pf</u>: Let $s > a$.

$$\mathcal{L}\{\varphi(t-c)u_c(t)\}(s) = \int_0^\infty e^{-st}\varphi(t-c)\,u_c(t)dt = \int_c^\infty e^{-st}\varphi(t-c)dt$$

$$= \int_0^\infty e^{-s(\tau+c)}\varphi(\tau)d\tau = e^{-cs}\int_0^\infty e^{-s\tau}\varphi(\tau)\,d\tau \qquad \begin{pmatrix} \tau = t-c \\ d\tau = dt \\ t = \tau + c \\ t \geq c \\ \tau \geq 0 \end{pmatrix}$$

$$= e^{-cs}\mathcal{L}(\varphi)(s). \quad \checkmark$$

$$\Rightarrow \boxed{\mathcal{L}\circ\mathcal{D}_c = \mathcal{E}_{-c}\circ\mathcal{L}.}$$

- $\mathcal{L}\{\varphi(t)u_c(t)\}(s) = \mathcal{L}\{\varphi(t+c-c)u_c(t)\}(s) = \mathcal{L}\{\psi(t-c)u_c(t)\}(s)$

$$(\psi(t) = \varphi(t+c))$$

$$= \mathcal{L}\circ\mathcal{D}_c(\psi)(s) = \mathcal{E}_{-c}\circ\mathcal{L}(\psi)(s) = e^{-cs}\mathcal{L}\{\varphi(t+c)\}(s).$$

Ex : $\varphi : [0,\infty[\longrightarrow \mathbb{R}$

$$t \longmapsto \begin{cases} \sin(t), \text{ if } 0 \le t < \frac{\pi}{4} \\ \sin(t) + \cos\left(t - \frac{\pi}{4}\right), \text{ if } \frac{\pi}{4} \le t \end{cases}.$$

$\varphi \in \mathrm{dom}(\mathcal{L})_0 \Rightarrow \mathcal{L}(\varphi) :]0,\infty[\longrightarrow \mathbb{R}$ is well-defined.

$$\varphi(t) = \sin(t)\left(1 - u_{\frac{\pi}{4}}(t)\right) + \left(\sin(t) + \cos\left(t - \frac{\pi}{4}\right)\right) u_{\frac{\pi}{4}}(t)$$

$$= \sin(t) + \cos\left(t - \frac{\pi}{4}\right) u_{\frac{\pi}{4}}(t) = \sin(t) + \mathcal{D}_{\frac{\pi}{4}}(\cos)(t)$$

$$\Rightarrow \mathcal{L}(\varphi) = \mathcal{L}(\sin) + \mathcal{L} \circ \mathcal{D}_{\frac{\pi}{4}}(\cos) = \frac{1}{s^2+1} + \mathcal{E}_{-\frac{\pi}{4}} \circ \mathcal{L}(\cos)$$

$$= \frac{1}{s^2+1} + \frac{s e^{-\frac{\pi}{4}s}}{s^2+1} = \frac{1 + s e^{-\frac{\pi}{4}s}}{s^2+1}. \quad s > 0.$$

Ex : $\mathcal{L}\left\{\sin(t)\, u_{\frac{\pi}{2}}(t)\right\} = \mathcal{L}\left\{\sin\left(t + \frac{\pi}{2} - \frac{\pi}{2}\right) u_{\frac{\pi}{2}}(t)\right\}$

$$= e^{-\frac{\pi}{2}s} \mathcal{L}\left\{\sin\left(t + \frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s} \mathcal{L}(\cos) = \frac{s e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\mathcal{L}\left\{\sin\left(t - \frac{\pi}{2}\right) u_{\frac{\pi}{2}}(t)\right\} = e^{-\frac{\pi}{2}s} \mathcal{L}\{\sin(t)\} = \frac{e^{-\frac{\pi}{2}}}{s^2+1}$$

Ex : Find a $\varphi \in \mathrm{dom}(\mathcal{L}) : \mathcal{L}(\varphi)(s) = \dfrac{1 - e^{-2s}}{s^2}$ (φ is the <u>inverse Laplace transform</u> of $\dfrac{1-e^{2s}}{s^2}$, $\varphi = \mathcal{L}^{-1}\left\{\dfrac{1-e^{2s}}{s^2}\right\}$)

$$\frac{1 - e^{-2s}}{s^2} = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} = \mathcal{L} \circ \mathcal{M} - \mathcal{E}_{-2} \circ \mathcal{L} \circ \mathcal{M}$$

$$= \mathcal{L}\left(\mathcal{M} - \mathcal{D}_2 \circ \mathcal{M}\right)$$

$$\Rightarrow \varphi(t) = (\mathcal{M} - \mathcal{D}_2 \circ \mathcal{M})(t) = t - (t-2)\, u_2(t) = \begin{cases} t, \text{ if } 0 \le t < 2 \\ 2, \text{ if } 2 \le t \end{cases}$$

Thm: Let $\varphi \in \mathrm{dom}(\mathcal{L})_a$, $c \geq 0$. Then $\mathcal{E}_c(\varphi) \in \mathrm{dom}(\mathcal{L})_{a+c}$
and $\mathcal{L}\{e^{ct}\varphi(t)\}(s) = \mathcal{L}\{\varphi(t)\}(s-c)$ for $s > a+c$. $\boxed{\#14}$

Pf: · $|\mathcal{E}_c(\varphi)(t)| = |e^{ct}\varphi(t)| \leq e^{ct} C e^{at} = C e^{(a+c)t}$.

$$\Rightarrow \mathcal{E}_c(\varphi) \in \mathrm{dom}(\mathcal{L})_{a+c}$$

$$\Rightarrow \mathcal{L} \circ \mathcal{E}_c(\varphi) :]a+c, \infty[\longrightarrow \mathbb{R} \text{ is well-defined.}$$

· Let $s > a+c$.

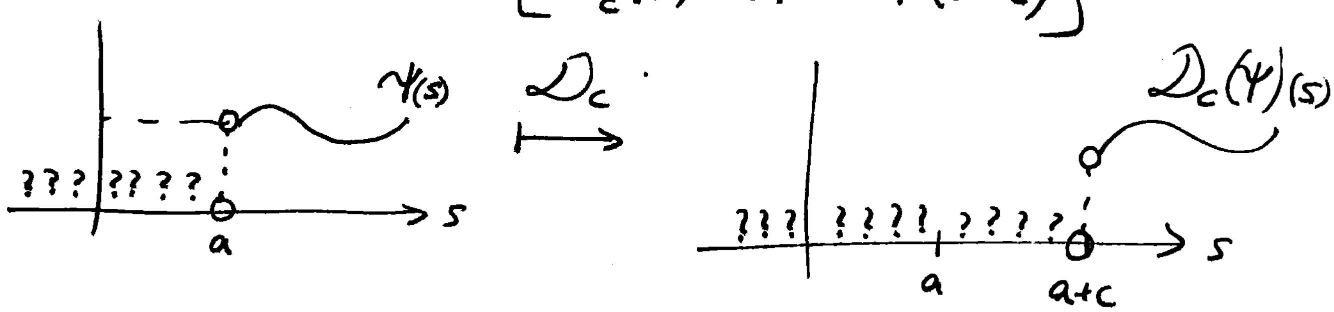
$$\mathcal{L} \circ \mathcal{E}_c(\varphi)(s) = \mathcal{L}\{e^{ct}\varphi(t)\}(s) = \int_0^\infty e^{-st} e^{ct}\varphi(t)\,dt$$

$$= \int_0^\infty e^{-(s-c)t}\varphi(t)\,dt = \mathcal{L}(\varphi)(s-c), \checkmark.$$

$$\Rightarrow \boxed{\mathcal{L} \circ \mathcal{E}_c = \mathcal{D}_c \circ \mathcal{L}}$$

This $\mathcal{D}_c$ is the <u>frequency shift/delay</u> by c:

$$\mathcal{D}_c : F(]a, \infty[, \mathbb{R}) \longrightarrow F(]a+c, \infty[, \mathbb{R})$$
$$\psi \longmapsto [\mathcal{D}_c(\psi) : s \longmapsto \psi(s-c)]$$



<u>SW</u>: · Laplace transform of $t^n e^{ct}$ (again)? $\boxed{\#11}$

$$t^n e^{ct} = \mathcal{M}^n \circ \mathcal{E}_c(1)(t) = \mathcal{E}_c \circ \mathcal{M}^n(1)(t).$$

· Laplace transform of $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$? $\boxed{\#9, \#10}$
of $e^{(\alpha+i\beta)t}$? $\boxed{\#2, \#5, \#6}$

__Ex__ : Find a $\varphi \in \mathrm{dom}(\mathcal{L})$: $\mathcal{L}(\varphi)(s) = \dfrac{1}{s^2-4s+5}$.

$$\frac{1}{s^2-4s+5} = \frac{1}{(s-2)^2+1^2} = \mathcal{D}_2\left\{\frac{1}{s^2+1^2}\right\} = \mathcal{D}_2 \circ \mathcal{L}(\sin)$$

$$= \mathcal{L} \circ \mathcal{E}_2(\sin) \quad \Rightarrow \quad \varphi(t) = \mathcal{E}_2(\sin)(t) = e^{2t}\sin(t).$$

__Ex__ : $$\boxed{\begin{array}{l} 2\ddot{y} + \dot{y} + 2y = \chi_{[5,20[}\,, \quad t \geq 0 \\ y(0) = 0 = \dot{y}(0) \end{array}}$$

$$LHS = \mathcal{L}(2\ddot{y}+\dot{y}+2y) = 2\left(s^2\mathcal{L}(y) - s\,y(0) - \dot{y}(0)\right) + \left(s\mathcal{L}(y) - y(0)\right)$$
$$+ 2\mathcal{L}(y) = (2s^2+s+2)\mathcal{L}(y) + (-2s-1)\underbrace{y(0)}_{=0} + (-2)\underbrace{\dot{y}(0)}_{=0}$$
$$= (2s^2+s+2)\mathcal{L}(y).$$

$$RHS = \mathcal{L}(\chi_{[5,20[}) = \frac{e^{-5s} - e^{-20s}}{s}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s(2s^2+s+2)} \cdot \left(e^{-5s} - e^{-20s}\right)$$

$$= \left(\frac{A}{s} + \frac{Bs+C}{2s^2+s+2}\right)\left(e^{-5s} - e^{-20s}\right)$$

$$\left(\begin{array}{ll} s^2(2A+B) & = 0 \\ +s\,(A+C) & +0 \\ +1\,(2A) & +1 \\ \Rightarrow A = \frac{1}{2},\ B = -1,\ C = -\frac{1}{2} & \end{array}\right)$$

$$= \left(\frac{1}{2}\cdot\frac{1}{s} + \frac{-s-\frac{1}{2}}{2s^2+s+2}\right)\left(e^{-5s} - e^{-20s}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{-s-\frac{1}{2}}{2\left(\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2\right)}\left(e^{-5s} - e^{-20s}\right)$$

$$\left(\begin{array}{l} 2s^2+s+2 = 2\left(s^2+\frac{1}{2}s+1\right) \\ = 2\left(\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2\right) \end{array}\right)$$

$$= \left(\frac{1}{2} \cdot \frac{1}{s} + \frac{-\left(s+\frac{1}{4}\right) - \frac{1}{4}}{2\left(\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2\right)} \right) \left(e^{-5s} - e^{-20s} \right)$$

$$= \left(\frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{2\sqrt{15}} \cdot \frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} \right) \left(e^{-5s} - e^{-20s} \right)$$

$$= \frac{e^{-5s} - e^{-20s}}{2} \cdot \left(\frac{1}{s} - \frac{s+1/4}{\left(s+1/4\right)^2 + \left(\sqrt{15}/4\right)^2} - \frac{1}{\sqrt{15}} \cdot \frac{\sqrt{15}/4}{\left(s+1/4\right)^2 + \left(\sqrt{15}/4\right)^2} \right)$$

$$= \frac{1}{2} \left(\mathcal{E}_{-5} - \mathcal{E}_{-20} \right) \left(\mathcal{L}(1) - \mathcal{D}_{-\frac{1}{4}} \mathcal{L}\left\{ \cos\left(\frac{\sqrt{15}}{4} t \right) \right\} - \frac{1}{\sqrt{15}} \mathcal{D}_{-\frac{1}{4}} \mathcal{L}\left\{ \sin\left(\frac{\sqrt{15}}{4} t \right) \right\} \right)$$

$$= \mathcal{L}\left(\frac{1}{2} \cdot \left(\mathcal{D}_5 - \mathcal{D}_{20} \right) \left(1 - \mathcal{E}_{-\frac{1}{4}} \left\{ \cos\left(\frac{\sqrt{15}}{4} t \right) \right\} - \frac{1}{\sqrt{15}} \mathcal{E}_{-\frac{1}{4}} \left\{ \sin\left(\frac{\sqrt{15}}{4} t \right) \right\} \right) \right)$$

$$\Rightarrow y(t) = \frac{1}{2} \left(\mathcal{D}_5 - \mathcal{D}_{20} \right) \left(1 - \mathcal{E}_{-\frac{1}{4}} \left\{ \cos\left(\frac{\sqrt{15}}{4} t \right) \right\} - \frac{1}{\sqrt{15}} \mathcal{E}_{-\frac{1}{4}} \left\{ \sin\left(\frac{\sqrt{15}}{4} t \right) \right\} \right)$$

$$= \frac{1}{2} \left(\mathcal{D}_5 - \mathcal{D}_{20} \right) \left(1 - e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}}{4} t \right) - \frac{1}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t \right) \right)$$

<u>SW</u>: Completely expand.

$\underline{Ex}$: $\boxed{\begin{array}{l} \ddot{y} + 9y = \cos(2t) - u_{4\pi}(t)\cos(2t) \\ y(0) = 0 = \dot{y}(0) \qquad t \geq 0 \end{array}}$

$$\mathcal{L}(\ddot{y} + 9y) = \mathcal{L}\{\cos(2t) - u_{4\pi}(t)\cos(2t)\}$$

$$LHS = \left(s^2\mathcal{L}(y) - s\,y(0) - \dot{y}(0)\right) + 9\,\mathcal{L}(y) = (s^2+9)\,\mathcal{L}(y).$$

$$RHS = \frac{s}{s^2+4} - \mathcal{L}\{\cos(2t+4\pi-4\pi)u_{4\pi}(t)\}$$

$$= \frac{s}{s^2+4} - \mathcal{L}\circ\mathcal{D}_{4\pi}\{\cos(2t+4\pi)\}$$

$$= \frac{s}{s^2+4} - \mathcal{E}_{-4\pi}\circ\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}\left(1 - e^{-4\pi s}\right)$$

$$\Rightarrow \mathcal{L}(y) = \frac{s}{(s^2+9)(s^2+4)}\left(1 - e^{-4\pi s}\right)$$

$$= \left(\frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}\right)\left(1 - e^{-4\pi s}\right)$$

$$\left(\begin{array}{lr} s^3(A+C) & = 0 \\ +s^2(B+D) & +0 \\ +s(4A+9C) & +1 \\ +1(4B+9D) & +0 \\ \Rightarrow B = 0 = D & \\ A = -\frac{1}{5} \quad C = \frac{1}{5} & \end{array}\right)$$

$$= \left(-\frac{1}{5}\cdot\frac{s}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4}\right)\left(1 - e^{-4\pi s}\right)$$

$$= \left(\mathcal{E}_0 - \mathcal{E}_{-4\pi}\right)\left(-\frac{1}{5}\mathcal{L}\{\cos(3t)\} + \frac{1}{5}\mathcal{L}\{\cos(2t)\}\right)$$

$$= \mathcal{L}\left\{\frac{1}{5}\left(\mathcal{D}_0 - \mathcal{D}_{4\pi}\right)\left(-\cos(3t) + \cos(2t)\right)\right\}$$

$$\Rightarrow y(t) = \frac{1}{5}\left(-\cos(3t) + \cos(2t)\right) - \frac{1}{5}\left(-\cos(3t-12\pi)u_{4\pi}(t) + \cos(2t-8\pi)u_{4\pi}(t)\right)$$

$$= \frac{1}{5}\left(-\cos(3t) + \cos(2t)\right) - \frac{1}{5}\left(-\cos(3t)\,u_{4\pi}(t) + \cos(2t)\,u_{4\pi}(t)\right)$$

$$= \frac{1}{5}\left(-\cos(3t) + \cos(2t)\right)\chi_{[0,4\pi[}(t).$$

• Let $J : C^0([0,\infty[,\mathbb{R}) \to \mathbb{R}$ be a linear <u>functional</u>, ie., a function of functions with real outputs.

<u>Ex</u>: $\forall c \geq 0$: $\delta_c : C^0([0,\infty[,\mathbb{R}) \longrightarrow \mathbb{R}$

$$\varphi \longmapsto \varphi(c)$$

<u>Ex</u>: Any $\psi \in C^0_{pw}([0,\infty[,\mathbb{R})$ can be interpreted as a linear functional:

$$\langle \cdot, \psi \rangle : C^0([0,\infty[,\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\varphi \longmapsto \int_0^\infty \varphi(t)\psi(t)\,dt$$

"integrate against ψ".

• If $J : C^0([0,\infty[,\mathbb{R}) \to \mathbb{R}$ is a linear functional and $\{\psi_n\}_n \subseteq C^0_{pw}([0,\infty[,\mathbb{R})$ is a sequence such that

$$\lim_{n\to\infty} \langle \cdot, \psi_n \rangle = J$$

$$\left(\text{ie., } \forall \varphi \in C^0([0,\infty[,\mathbb{R}) : \lim_{n\to\infty} \int_0^\infty \varphi(t)\psi_n(t)\,dt = J(\varphi)\right),$$

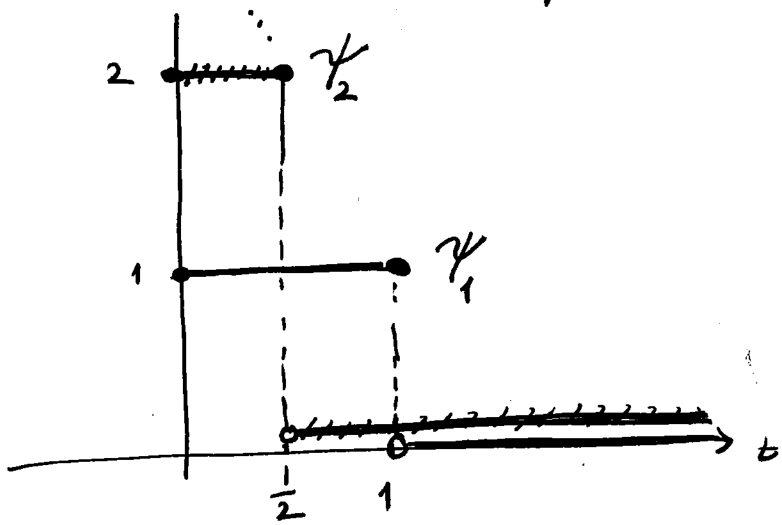
then J is a <u>weak limit</u> of $\{\psi_n\}_n$. Any linear functional that can be obtained as a weak limit of a sequence of C^0_{pw} functions is a <u>distribution</u> / <u>generalized function</u>.

Thm: $\delta_0 = \int_0^\infty \cdot\ \delta(t)\,dt : C^0([0,\infty[,\mathbb{R}) \longrightarrow \mathbb{R}$

$$\varphi \longmapsto \left[\varphi(0) = \int_0^\infty \varphi(t)\,\delta(t)\,dt\right]$$

is a distribution.

Pf: It suffices to verify that δ_0 can be obtained as the weak limit of some sequence of C^0_{pw} functions.



$\psi_1 := \chi_{[0,1]}$

$\psi_2 := 2\,\chi_{[0,1/2]}$

$\psi_3 := 4\,\chi_{[0,1/4]}$

$\vdots$

$\psi_n := 2^{n+1}\,\chi_{[0,2^{-(n+1)}]}$.

Let $\varphi \in C^0([0,\infty[,\mathbb{R})$ be arbitrary.

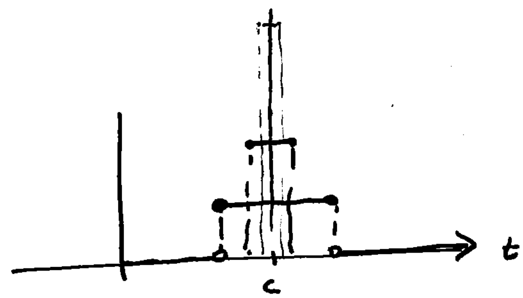
$$\lim_{n\to\infty} \langle \varphi, \psi_n \rangle = \lim_{n\to\infty} \int_0^\infty \varphi(t)\,\psi_n(t)\,dt = \lim_{n\to\infty} 2^{n+1}\int_0^{2^{-(n+1)}} \varphi(t)\,dt$$

$$= \lim_{n\to\infty} 2^{n+1}\left(2^{-(n+1)}\varphi(t_n)\right) = \lim_{n\to\infty} \varphi(t_n) = \varphi(0) = \delta_0(\varphi).$$

(Mean Value Theorem
$\forall n: 0 < t_n < 2^{-(n+1)}$)

(φ is continuous)

SW: $\forall c \geq 0$: δ_c is a distribution.



$\Rightarrow \quad \forall c \geqslant 0,\ \forall \varphi \in C^{0}_{pw}([0,\infty[, \mathbb{R})$: if φ is continuous at c, then

$$\boxed{\delta_c(\varphi) = \varphi(c) = \int_0^\infty \varphi(t)\ \delta(t-c)\,dt} .$$

- $\forall c \geqslant 0$: $\mathcal{L}\{\delta(t-c)\}(s) = \int_0^\infty e^{-st}\ \delta(t-c)\,dt = e^{-cs}$. $\boxed{\#17}$

  $\mathcal{L}\{\delta(t-c)\} :]0,\infty[\longrightarrow \mathbb{R}$

SW : $\mathcal{L}$ is continuous for weak limits :

$$\lim_{n\to\infty} \mathcal{L}(\gamma_n) = \mathcal{L}\{\delta(t)\} = 1 .$$

$$\left(\gamma_n = 2^{n+1} \chi_{[0, 2^{-(n+1)}]}\right)$$

Ex :

$$\boxed{\begin{array}{ll} \ddot{y} + 6\dot{y} + 5y = \delta(t) + \delta(t-2) & \\ y(0) = 1 & t \geqslant 0. \\ \dot{y}(0) = 0 & \end{array}}$$

$$\mathcal{L}(\ddot{y} + 6\dot{y} + 5y) = \mathcal{L}\{\delta(t) + \delta(t-2)\}$$

$$LHS = \left(s^2 \mathcal{L}(y) - s\,y(0) - \dot{y}(0)\right) + 6\left(s\,\mathcal{L}(y) - y(0)\right) + 5\,\mathcal{L}(y)$$

$$= (s^2 + 6s + 5)\cdot \mathcal{L}(y) + (-s-6)\underbrace{y(0)}_{=1} - \underbrace{\dot{y}(0)}_{=0} = (s^2 + 6s + 5)\mathcal{L}(y) + (-s-6).$$

$$RHS = 1 + e^{-2s}$$

$$\Rightarrow \mathcal{L}(y) = \frac{1}{(s^2+6s+5)}\left(1 + e^{-2s} + s + 6\right) = \frac{1}{(s+5)(s+1)}\left((s+7) + e^{-2s}\right)$$

$$= \left(\frac{A}{s+5} + \frac{B}{s+1}\right) + \left(\frac{C}{s+5} + \frac{D}{s+1}\right) e^{-2s}$$

$$\left(\begin{array}{l|l} s(A+B) = s & s(C+D) = 0 \\ +1(A+5B) \quad +7 & +1(C+5D) \quad +1 \\ \Rightarrow A+B = 1 & \Rightarrow C = -1/4 \\ \quad A+5B = 7 & \quad D = 1/4 \\ A = -\frac{1}{2} \quad B = \frac{3}{2} & \end{array}\right)$$

$$= \left(-\frac{1}{2}\cdot\frac{1}{s+5} + \frac{3}{2}\cdot\frac{1}{s+1}\right) + \left(-\frac{1}{4}\cdot\frac{1}{s+5} + \frac{1}{4}\cdot\frac{1}{s+1}\right) e^{-2s}$$

$$= -\frac{1}{2}\ \mathcal{D}_{-5}\circ\mathcal{L}(1) + \frac{3}{2}\ \mathcal{D}_{-1}\circ\mathcal{L}(1) - \frac{1}{4}\ \mathcal{E}_{-2}\circ\mathcal{D}_{-5}\circ\mathcal{L}(1) + \frac{1}{4}\ \mathcal{E}_{-2}\circ\mathcal{D}_{-1}\circ\mathcal{L}(1)$$

$$= \mathcal{L}\left(-\frac{1}{2}\ \mathcal{E}_{-5}(1) + \frac{3}{2}\ \mathcal{E}_{-1}(1) - \frac{1}{4}\ \mathcal{D}_{2}\circ\mathcal{E}_{-5}(1) + \frac{1}{4}\ \mathcal{D}_{2}\circ\mathcal{E}_{-1}(1)\right)$$

$$\Rightarrow y(t) = \left(-\frac{1}{2}\ \mathcal{E}_{-5} + \frac{3}{2}\ \mathcal{E}_{-1} - \frac{1}{4}\ \mathcal{D}_{2}\circ\mathcal{E}_{-5} + \frac{1}{4}\ \mathcal{D}_{2}\circ\mathcal{E}_{-1}\right)(1)(t)$$

$$= -\frac{1}{2}\ e^{-5t} + \frac{3}{2}\ e^{-t} - \frac{1}{4}\ e^{-5(t-2)} u_2(t) + \frac{1}{4}\ e^{-(t-2)} u_2(t).$$

$\longrightarrow$.

Table of Laplace Transforms:

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}$
~~4.~~	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\frac{a}{s^2+a^2}$
6.	$\cos at$	$\frac{s}{s^2+a^2}$
7.	$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$	$\frac{a}{s^2-a^2}$
8.	$\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$	$\frac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
~~16.~~	$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17.	$\delta(t-c) = \delta_c$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

SW: (i) Deduce the rules that are not crossed out from the ones that are boxed.

(ii) $p(t) = a_0 + a_1 t + \cdots + a_n t^n$

$\Rightarrow \mathcal{L}(p) = ?$

(iii) $c(t) = (\cos(t))^2$, $s(t) = (\sin(t))^2$

$\Rightarrow \mathcal{L}(c) = ?$ $\mathcal{L}(s) = ?$

(iv) Fix $T > 0$, $f : [0,\infty[\longrightarrow \mathbb{R}$ be Laplaceable. Suppose f is T-periodic:

for any $t \geq 0$: $f(t+T) = f(t)$.

Then

$$\mathcal{L}(f)(s) = \frac{\int_0^T e^{-s\tau} f(\tau)\,d\tau}{1 - e^{-sT}}.$$

$$\mathcal{L} \circ \mathcal{D}_c = \mathcal{E}_{-c} \circ \mathcal{L}$$

$$\mathcal{L} \circ \mathcal{E}_c = \mathcal{D}_c \circ \mathcal{L}.$$

$$\mathcal{L} \circ \partial_t = \mathcal{M}^1 \circ \mathcal{L} - \delta_0$$

$$\mathcal{L} \circ \mathcal{M}^1 = -\partial_s \circ \mathcal{L}$$