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ARITHMETICITY FOR SMOOTH MAXIMAL RANK POSITIVE ENTROPY ACTIONS OF $\mathbb{R}^k$

A Dissertation in
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Abstract

This dissertation presents a contibution to the nonuniform measure rigidity program initiated in [KK01]. More specifically, we establish arithmeticity in the sense of [KRH16] of higher rank ($k \in \mathbb{Z}_{\geq 2}$) locally free actions $\alpha_\bullet : \mathbb{R}^k \curvearrowright M$ by C^r ($r \in \mathbb{R}_{>1}$) diffeomorphisms on an anonymous compact C^∞ manifold M of dimension $\dim(M) = 2k+1$ provided that there is an ergodic invariant Borel probability measure μ on M w/r/t which each time-t ($t \neq 0$) map α_t of the action has positive entropy. Arithmeticity in this context means that the action α is measure theoretically isomorphic to a constant time change of the suspension of an affine Cartan action of $\mathbb{Z}^k$ on a torus or $\pm$-infratorus of dimension $k+1$. This in particular solves, up to measure theoretical isomorphism, Problem 4 from [KKRH11, p.394].

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Acknowledgments

One realizes, upon thinking of who to acknowledge at an occasion like this, that it is nearly impossible to do justice to the deed of acknowledging everyone that needs to be acknowledged, concisely and without getting too sentimental. Thus I will try the opposite, in that I will merely focus on those who must not be left unacknowledged, and of course fail, still.

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[*] https://tikzcd.yichuanshen.de/

Chapter 1

Introduction

Statement of the Main Results We prove the following arithmeticity result. Let M be a compact C^∞ manifold, $k \in \mathbb{Z}_{\geq 1}$, $\theta \in]0,1]$, $r = (1,\theta)$, $\mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be the **ergodic theory with no potential** of $\mathbb{R}^k$ systems on M of class C^r; by definition this is the collection of all pairs (μ, α) such that μ is a Borel probability measure on M and $\alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M)$ is a group homomorphism such that $\alpha : \mathbb{R}^k \times M \to M$ is C^r and μ is α-invariant; we call any such pair (μ, α) an $\mathbb{R}^k$ **system**. We show that in the case of simplest Lyapunov geometry compatible with hyperbolicity, any system is the suspension of a hyperbolic algebraic action on some space crystal:

Theorem 1: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$. If

- $k \in \mathbb{Z}_{\geq 2}$ and $\dim(M) = 2k+1$,
- (μ, α) is locally free and ergodic,
- The system (μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes, and
- $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) = \mathrm{ent}_\mu(\alpha_t) > 0$,

then there is

- an affine Cartan action $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{Aff}(T^{k+1})$, where T^{k+1} is the torus or the $\pm$-infratorus of dimension $k+1$, and
- a $\kappa \in \mathrm{GL}(k, \mathbb{R})^\circ$

such that

$$\exists \Phi_{(\mu,\alpha)} : (\mu, \alpha) \xrightarrow{\cong_{\overline{\text{Meas}}}} (\mathrm{haar}_{\mathbb{T}^k} \otimes^\gamma \mathrm{haar}_{T^{k+1}}, \hbar^\gamma_\kappa),$$

where $\hbar^\gamma_\kappa$ is the suspension of γ with a constant time change κ. Furthermore,

- The restriction of the measure theoretical isomorphism $\Phi_{(\mu,\alpha)}$ to any global stable manifold of any Weyl chamber of (μ, α) is C^r, and

- For any $\theta' \in]0,\theta[$ there is a open subset $U_{\theta'} \subseteq M$ with $\mu(M \setminus U_{\theta'}) < \theta'$ and the measure theoretical isomorphism $\Phi_{(\mu,\alpha)}$ extends to a $C^{r-\theta'}$ injective immersion on $U_{\theta'}$.

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Here the infratorus is defined in terms of the standard action of $\mathrm{Aff}(\mathbb{R}^{k+1})$ of affine automorphisms of $\mathbb{R}^{k+1}$ on $\mathbb{R}^{k+1}$: we have that $\mathrm{Aff}(\mathbb{R}^{k+1}) \cong \mathbb{R}^{k+1} \rtimes \mathrm{GL}(k+1,\mathbb{R})$. Then the $(k+1)$-dimensional torus $\mathbb{T}^{k+1}$ is the orbit space of the subgroup $\mathbb{Z}^{k+1} \rtimes \{I_{k+1}\} \leq \mathrm{Aff}(\mathbb{R}^{k+1})$ and the $(k+1)$-dimensional **$\pm$-infratorus** is the orbit space of the subgroup $\mathbb{Z}^{k+1} \rtimes \{\pm I_{k+1}\} \leq \mathrm{Aff}(\mathbb{R}^{k+1})$. In general an **infratorus** is the orbit space of a subgroup $L \rtimes F \leq \mathrm{Aff}(\mathbb{R}^{k+1})$ for L a lattice and F a finite group. An **affine Cartan action** γ is an action by affine automorphisms of T^{k+1} such that for any $t \in \mathbb{Z}^k \setminus 0$, the linear part of the time-t map γ_t is ergodic w/r/t the Haar measure[1].

Remark 1: Some hypotheses in Thm. 1 can be weakened. More specifically, if M is not necessarily compact, $r = (q, \omega)$ for some $q \in \mathbb{Z}_{\geq 1}$, and for some modulus of continuity ω satisfying the Dini condition, so that the qth differential objects have ω as a local modulus of continuity, or $r = \infty$, $\alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M)$ is an arbitrary family of C^r diffeomorphisms with k weakly generating vector fields whose norms w/r/t some C^0 fiberwise norm are $\log^+$-$(k,1)$-Lorentz w/r/t μ and that commute on a set of full μ measure, and if α is not necessarily μ-essentially locally free as a group action in the category of standard probability spaces, all other hypotheses ditto (in particular ergodicity is certainly necessary), then the theorem is still true. For the sake of readability we don't prove this generalization (although see Rem. 18 for an indication as to the kinds of arguments that can be used to handle higher regularity).

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Remark 2: There is also some redundancy in the way the hypotheses of Thm. 1 are stated and the above statement corresponds to the most inefficient choice, see chapter 4 for a related discussion.

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Definition 1: Let us call a system (μ, α) satisfying the four hypotheses of Thm. 1 an $\mathbb{R}^k$ **maximal rank positive entropy system** (**MRPES** for short). Similarly, if N is a compact C^∞ manifold, $(\nu, \beta) \in \mathfrak{E}^r(\mathbb{Z}^k \curvearrowright N)$ is an ergodic system whose suspension system $(\mathrm{haar}_{\mathbb{T}^k} \otimes^\beta \nu, \hbar^\beta)$ is an MRPES, let us call (ν, β) a $\mathbb{Z}^k$ **maximal rank positive entropy system**.

Moreover, we call $\mathfrak{e}_{(\mu,\alpha)} : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ the **entropy gauge** of the system (μ, α).

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Many corollaries follow immediately from Thm. 1; below is a sampler:

[1] One could equivalently define γ to be an affine Cartan action iff the linear part of γ_t is a hyperbolic (infra-)toral automorphism by [KKS02, p.731, Prop.4.1]. Note that for $k = 1$, ergodicity is strictly weaker than hyperbolicity. Also note that in the literature there are nonequivalent definitions of Cartan actions; see e.g. [SV19, pp.4-5]. Our definition is compatible with [KKRH10] and is an extension of the definition in [KKS02, p.731].

Corollary 1: Any $\mathbb{R}^k$ MRPES is the suspension of a $\mathbb{Z}^k$ MRPES up to a measure theoretical isomorphism and a constant linear time change. In particular, the answer to Problem 4[2] of [KKRH11] is negative up to $\underline{\overline{\text{Meas}}}$-isomorphism.

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Corollary 2: The Jacobian cocycle of any $\mathbb{R}^k$ MRPES along any Lyapunov æ-foliation is cohomologous to a cocycle constant in space with measurable transfer that is C^r along any Lyapunov æ-foliation. In particular, this solves Conjecture 1[3] of [KKRH11] for the $\mathbb{R}^k$ case.

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Corollary 3: Any Lyapunov exponent of any time-t map of any $\mathbb{R}^k$ MRPES is the logarithm of an algebraic number.

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Corollary 4: If $k \in \mathbb{Z}_{\geq 2}$, $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ is an essentially locally free and ergodic system with exactly $k+1$ distinct Lyapunov hyperplanes, $\dim(M) = 2k+1$ and for some $t^* \in \mathbb{R}^k : (\mu, \alpha_{t^*}) \in \mathfrak{E}^r(\mathbb{Z} \curvearrowright M)$ has the K-property, then the Lyapunov hyperplanes of (μ, α) can't be in general position, and there is some $t^\dagger \in \mathbb{R}^k \setminus 0$ such that $\mathfrak{e}_{(\mu,\alpha)}(t^\dagger) = 0$.

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Corollary 5: Let $k \in \mathbb{Z}_{\geq 2}$ and $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be an essentially locally free ergodic system with exactly $k+1$ distinct Lyapunov hyperplanes. If $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) > 0$ and for some $t^* \in \mathbb{R}^k$, $(\mu, \alpha_{t^*}) \in \mathfrak{E}^r(\mathbb{Z} \curvearrowright M)$ has the K-property, then $\dim(M) \geq 2k+1$.

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Corollary 6: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be an MRPES. Then the set

$$\{t \in \mathbb{R}^k \,|\, (\mu, \alpha_t) \in \mathfrak{E}^r(\mathbb{Z} \curvearrowright M) \text{ is not ergodic}\}$$

of non-ergodic time-t systems of (μ, α) is the union of exactly countably many hyperplanes that constitute a mesh determined by an injective group homomorphism $\mathbb{Z}^k \hookrightarrow \mathbb{R}^k$.

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1.1 History and Context

The strategy we follow to prove Thm.1 is to adapt the machinery developed in [KRH16] from $\mathbb{Z}^k$ systems to $\mathbb{R}^k$ systems (along the way we will also provide significantly more details and address various gaps). Said machinery is built on previous advances on the geometric approach to measure rigidity of actions of $\mathbb{R}^{k_1} \oplus \mathbb{Z}^{k_2}$ ($k_1 + k_2 \in \mathbb{Z}_{\geq 2}$) by automorphisms of homogeneous spaces and more generally diffeomorphisms of manifolds, with

[2] See Sec.1.1 below for a discussion of this problem.

[3] This conjecture ([KKRH11, p.393]) claims that for any $\mathbb{Z}^k$ or $\mathbb{R}^k$ MRPES the derivative cocycle along any Lyapunov æ-foliation is measurably cohomologous to one constant in space. The conjecture for $\mathbb{Z}^k$ MRPES follows from [KKRH11, p.394].

some hyperbolicity in either case. In particular this dissertation is a continuation of the chain of work [KS96, KS98, KK01, KK07, KRH07, KKRH10, KKRH11, KRH10, KRH16][4].

Measure Rigidity of Higher Rank Abelian Actions Let us momentarily switch to a less precise and a more generalist language and review the broader context of measure rigidity of higher rank abelian actions. The fundamental premise of measure rigidity[5] of higher rank abelian actions is that such actions ought to preserve some objects of geometric (or even algebraic) origin[6], under some dynamical conditions that guarantee enough recurrence and infinitesimal hyperbolicity, and further that such preserved objects of geometric origin ought to be very few. The standard example of this phenomenon is that there are many Borel probability measures on $\mathbb{T}$ that are ergodic and invariant under $x \mapsto 2x$; and likewise there are many ergodic invariant Borel probability measures of $x \mapsto 3x$, but there is a unique Borel probability measure on $\mathbb{T}$ that is ergodic and invariant under the $\mathbb{Z}^2_{\geq 0}$ action generated by $x \mapsto 2x$ and $x \mapsto 3x$, provided that the measure is required to give positive entropy to at least one of these generators; further this unique measure is the Haar probability measure on $\mathbb{T}$; see [Fur67, Lyo88, Rud90, Joh92].

The geometric approach to measure rigidity, commonly accepted to have been introduced in [KS96][7], focuses on using conditional measures along various invariant æ -foliations that an action induces. One of the characteristics of the geometric approach to global rigidity is to make use of the geometry of Lyapunov exponents, Lyapunov hyperplanes and the Weyl chambers in a crucial manner; let's call the collection of all such tools the **Lyapunov geometry** toolkit. Recall that for a higher rank abelian action Lyapunov exponents of the whole action and the Lyapunov exponents of any time-t map of the action are intimately related, so that the Lyapunov exponents of the whole action can be thought of as (continuous) group homomorphisms from the acting group to a finite dimensional real vector space. Lyapunov hyperplanes are codimension-one subgroups of the acting group which are exactly the kernels of nonzero Lyapunov exponents; any connected component of the complement of the union of the Lyapunov hyperplanes is a Weyl chamber. Since the Lyapunov hyperplanes will miss all nonzero integer points under reasonable hyperbolicity assumptions[8], even in the case of discrete group actions it's beneficial to take suspensions and consider the Lyapunov hyperplanes of the suspension action to be able to use isometry properties of time-t maps where t comes from a Lyapunov hyper-

[4]Note that, as of the writing of this dissertation, [KRH17] is the latest addition to this chain of papers. See [KKRH08] for a survey of the papers preceding [KRH16]. The survey [Lin05] contextualizes the geometric method in the broader study of the ergodic theory of $\mathbb{R}^{k_1} \oplus \mathbb{Z}^{k_2}$ actions, whereas the survey [RH21] contextualizes the works in question as part of late Prof. Katok's life's work. Finally [Gor07, Sec.3] surveys the connections between measure rigidity and other areas.

[5]See Rem.11 below for a more concrete instantiation of the phenomenon called measure rigidity.

[6]We consider Borel probability measures on a manifold that are absolutely continuous w/r/t the Lebesgue class objects of geometric origin. In particular measures of Lebesgue class are objects of geometric origin, no matter how nondifferentiable the associated Radon-Nikodym derivative is. Recall that on any C^∞ manifold one can patch the Lebesgue measures on charts up to get a measure class; this is the **Lebesgue class**. Choosing a density or a Riemannian metric on the manifold fixes a unique measure in the Lebesgue class. See e.g. [Lan99, p.463, XVI.4].

[7]Although note that in [EKL06, p.520] it is reported that the same ideas were implicit in [Rud90].

[8]This is referred to as "total irrationality" in [KK01, p.625].

plane. It is well known that the suspension construction is categorical, and further many dynamical properties survive (with natural modifications at times). Thus from the Lyapunov geometry point of view in measure rigidity, it is a natural question whether or not Lyapunov geometry is adapted well to actions with discrete time in a natural way. Put differently, are there continuous time actions that are not produced by suspensions?[9]

Let us now be more precise. Let N be a compact C^∞ manifold, $r \in \mathbb{R}_{>1}$, $k \in \mathbb{Z}_{\geq 2}$, and suppose that we have a C^r group action $\beta_\bullet : \mathbb{Z}^k \to \mathrm{Diff}^r(N)$. In [KKRH11] it is shown that if ν is a Borel probability measure on N that is β-invariant and β-ergodic, then ν is absolutely continuous w/r/t the Lebesgue measure class of N, provided that the action is of maximal rank, that is, $\dim(N) = k+1$, and that for any $t \in \mathbb{Z}^k \setminus 0$ the time-t map β_t of the action β has positive metric entropy $\mathrm{ent}_\nu(\beta_t) > 0$ w/r/t ν. As mentioned above their method uses heavily the geometric properties of the Lyapunov æ-foliations, Lyapunov hyperplanes, and Weyl chambers, and the latter two gadgets in particular are better adapted to actions of connected groups as opposed to discrete ones, whence first an analogous statement for $\mathbb{R}^k$ actions is proved, and then it is noted that the suspension of a $\mathbb{Z}^k$ action satisfying the hypotheses stated satisfies the analogous hypotheses. More precisely, it is shown that if $\alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M)$ is a locally free C^r action on a compact C^∞ manifold, and if μ is a Borel probability measure that is α-invariant and α-ergodic, then $\mu \ll \mathrm{leb}_M$, provided that $\dim(M) = 2k+1 = k+(k+1)$ (maximal rank assumption for $\mathbb{R}^k$ actions) and for any $t \in \mathbb{R}^k \setminus 0$, $\mathrm{ent}_\mu(\alpha_t) > 0$. Since the suspension of a β as above produces an α as above (see Lem. 9 below), the statement for $\mathbb{Z}^k$ actions follows, as ν is the æ-conditional measure of μ along the bundle map from the suspension space onto the time factor $\mathbb{T}^k$, and the conditionals of an absolutely continuous measure must be absolutely continuous (w/r/t fiber volume measure). In the same paper they also formulate the following problem:

> **Problem[10]:** Are there any α's as above that is not a constant time change of a suspension of a β as above?

It's clear that one of the corollaries (Cor. 1 above) of our main theorem addresses precisely this question.

Discrete v. Connected Time For comparison purposes it is useful to recall the $\mathbb{Z}^k$ arithmeticity theorem from [KRH16]:

Theorem 2 (Katok - Rodríguez Hertz[11]): Let N be a compact C^∞ manifold. If $(\nu, \beta) \in \mathfrak{E}^r(\mathbb{Z}^k \curvearrowright N)$ is an MRPES, then there is

- an affine Cartan action $\delta_\bullet : \mathbb{Z}^k \to \mathrm{Aff}(T^{k+1})$, and
- an injective group homomorphism $j : \mathbb{Z}^k \hookrightarrow \mathbb{Z}^k$ with $F = \mathbb{Z}^k / \overrightarrow{j}(\mathbb{Z}^k)$ finite

[9] Note that "suspension rigidity" is a theme in dynamics in general; this question is akin to e.g. the Verjovsky conjecture for codimension-one Anosov flows [Ver74].

[10] [KKRH11, p.394]; the author's paraphrase. This problem is repeated in [KKRH08, p.80, footnote 2] also.

[11] [KRH16, pp.137-138, Thm.1]

such that

$$\exists \Phi_{(\nu,\beta)} : (\nu, \beta) \xrightarrow{\cong_{\overline{\text{Meas}}}} (\text{haar}_F \otimes^\delta \text{haar}_{T^{k+1}}, \hbar^\delta).$$

Furthermore,

- The restriction of the measure theoretical isomorphism $\Phi_{(\nu,\beta)}$ to any global stable manifold of any Weyl chamber of (ν, β) is a C^r diffeomorphism, and
- For any $\theta' \in]0, \theta[$, there is a open subset $U_{\theta'} \subseteq N$ with $\nu(N \setminus U_{\theta'}) < \theta'$ and the measure theoretical isomorphism extends to a $C^{r-\theta'}$ injective immersion on $U_{\theta'}$.

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Definition 2: We call $\Phi_{(\mu,\alpha)}$ of Thm.1 the **arithmeticity isomorphism** of (μ, α). Similarly $\Phi_{(\nu,\beta)}$ of Thm.2 is the **arithmeticity isomorphism** of (ν, β).

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Note that the positive entropy hypothesis of Thm.2 is formulated in terms of the elements of the suspension system, since the positive entropy along "irrational directions" is crucial for the proof. See Prop.2 for a characterization of the positive entropy hypothesis in terms of Lyapunov geometry. There are essentially two new complications in the proof of Thm.1 compared to the proof of Thm.2; both due to the presence of the orbit directions. The first is that in the proof of $\mathbb{Z}^k$ arithmeticity by a classical theorem of Pesin[12] the finite time crystal can be separated at the first instance (this is their "weak mixing reduction"); this classical theorem of Pesin does not work for $\mathbb{R}^k$ actions (it has indeed an analog for $\mathbb{R}^k$ actions, which provides only an alternative in terms of the eigenfunctions of the Koopman action[13]). The second complication is that in the $\mathbb{R}^k$ case the Katok - Rodríguez Hertz machinery may a priori spill over in the orbit directions. In more concrete terms, these make the biggest difference toward the end in chapter 7 and chapter 8, where we establish the suspension structure. In the $\mathbb{Z}^k$ case the homoclinic group is recognized to be a lattice in an abelian group; in our case it is recognized to be a lattice in a solvable group.

Uniform v. Non-Uniform Normal Hyperbolicity to the Orbit Foliation Our main theorem Thm.1 can also be seen as a non-uniformly normally hyperbolic (to the orbit foliation) analog of a result of Matsumoto on uniformly normally hyperbolic (to the orbit foliation):

Theorem 3 (Matsumoto[14])**:** Let M be a closed oriented C^∞ manifold, $k \in \mathbb{Z}_{\geq 2}$, $\alpha_\bullet : \mathbb{R}^k \to \text{Diff}^\infty_+(M)$ be a C^∞ action. If

- α is locally free,
- $\dim(M) = 2k + 1$,

12[Pes77a, p.94, Thm.7.9]

13[Pes77b, p.1227, Thm.9.7]; compare [Ano69, p.29, Thm.14].

14[Mat03, p.42,Thm.3.1]

- There is an Ad^α-invariant C^0 splitting $TM = O \oplus \bigoplus_{i\in\overline{k+1}} E^i$, where O is the subbundle tangent to the orbit foliation of α and each E^i is a topological line bundle with the property that there is a $\xi_\bullet : \overline{k+1} \to \mathbb{R}^k$ such that $\forall i \in \overline{k+1} : S(\alpha_{\xi_i}) = E^i$ and $U(\alpha_{\xi_i}) = \bigoplus_{j\neq i} E^j$[15],

then there is

- an affine Cartan action $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{Aff}_+(\mathbb{T}^{k+1})$, and
- $\kappa \in \mathrm{GL}(k, \mathbb{R})^\circ$

such that

$$\exists \Phi_\alpha : \alpha \xrightarrow{\cong_{C^\infty}} \hbar^\gamma_\kappa,$$

where again $\hbar^\gamma_\kappa$ is the suspension of γ with a constant time change κ. ⌟

Matsumoto calls an action of $\mathbb{R}^k$ satisfying the hypotheses of Thm.3 a "split Anosov action". The C^∞ setting is amenable to differential topological methods; as such despite the fact that Matsumoto's theorem looks formally very similar to our main theorem (partially also due to the author's paraphrasing) the methods involved are completely different.

Time Crystal Interpretation In [KK01, p.596] and [KK02, p.510][16] the k-torus over which the suspension of a $\mathbb{Z}^k$ action fibers is called a "'time' torus". We take this nomenclature more seriously and borrowing a name from contemporary physics[17] we call the base torus of the suspension constructed in Thm. 1 the **time crystal** and the fiber torus or $\pm$-infratorus of the suspension the **space crystal** of the system (μ, α). We also call the total space of the suspension the **spacetime crystal presentation** of (μ, α). This evocative language is meant to simplify referring to the similar algebraic structures along different directions. Further we may interpret our result as saying that up to a measure-preserving coordinate change the time crystal of (μ, α) does keep track of the broken continuous time symmetry, all the while being part of the state space M: a part of the state space functionally works as an internal clock which keeps track of the discrete fiber dynamics (see Figure 1.1). Analogously the time crystal of Thm.2 is the finite factor group F. The reader ought to be warned that this use of the phrase "time crystal" does not match with the use of the term in the physics literature[18].

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[15] This means that vectors along E^i are contracted exponentially fast under the flow $t \mapsto \alpha_{t\xi_i}$, and vectors along $\bigoplus_{j\neq i} E^j$ are contracted exponentially fast under the flow $t \mapsto \alpha_{-t\xi_i}$ up to a constant that is independent of the basepoint.

[16] Also see [Kat07, p.583].

[17] See [Wil12, YN18, YCLZ20].

[18] The relation between the two uses is roughly the same relation between Pugh's and Anosov's closing lemmas (see [Pug11]). Note that in classical ergodic theory typically the discrete system that is being suspended is thought of as the "horizontal" space, especially if there is a cocycle (i.e. roof function). In the study of actions of groups more complicated than $\mathbb{Z}$ or $\mathbb{R}$ the perspective is typically flipped, so that the suspended system is "vertical". In any suspension with a cocycle, existence of a time crystal signifies the

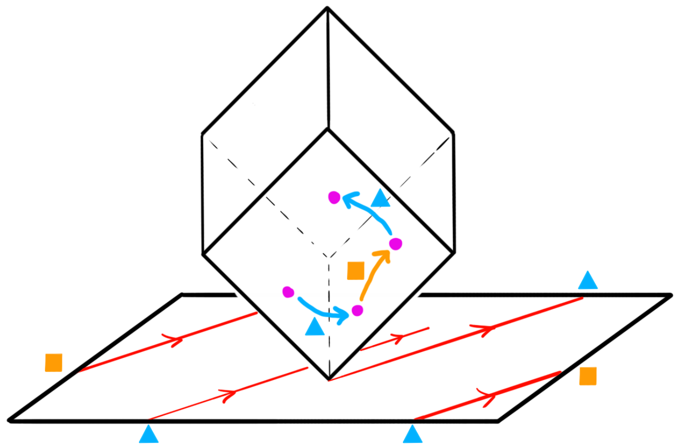

Figure 1.1: In the case of $k = 2$, we have that the time crystal of $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^2 \curvearrowright M^5)$ is $\mathbb{T}^2$ and the space crystal of (μ, α) is T^3. The space crystal carries an affine Cartan action $\gamma : \mathbb{Z}^2 \curvearrowright T^3$, with generators ▲ and ■. We think of the space crystal sliding along the time crystal (this is the translation action $\mathbb{T}^2 \curvearrowright \mathbb{T}^2$). The discrete time progresses on T^3 according to which sides of the time crystal are hit as the space crystal slides.

1.2 Miscellaneous Notation

Here we collect some miscellaneous notation; other notation we'll use will be introduced when appropriate in context, if its meaning is not clear from context. We repeat and reintroduce some notation in-text.

Overall $\bullet$ signifies and emphasizes the first priority place for an argument; e.g. we'll use $x_\bullet : \mathbb{Z}_{\geq 1} \to X$ (or simply $x_\bullet$) to denote a sequence of points in X. We put $\underline{0} = \emptyset = \overline{0}$, and for $p \in \mathbb{Z}_{\geq 1}$ we put $\underline{p} = \{0, 1, ..., p-1\} = \overline{p} - 1$, $\underline{p} + 1 = \{1, 2, ..., p\} = \overline{p}$, $\overline{-p} = \{-1, -2, ..., -p\} = -\overline{p}$ and $\underline{-p} = \{0, -1, ..., -p+1\} = -\underline{p}$. $\mathcal{P}(X), \mathcal{B}(X), \mathcal{T}(X)$ stand for the collection of all subsets, all (Borel) measurable subsets, and all open subsets of a set, measurable space, and topological space X, respectively.

For (M, d) a metric space, $x \in M$ and $r \in \mathbb{R}_{>0}$: $M[x| \leq r]$ denotes the closed ball in M centered at x with radius r; $M[x| < r]$ similarly denotes the open ball. We also write $(M, d)[x| \leq r]$ etc. if the distance needs to be emphasized.

For (X, μ) a measure space, $\forall x \in_\mu X$ means that the point x is chosen from an appropriate full μ-measure subset of X[19]. For two measurable subsets $A, B \subseteq X$, $A \subseteq_\mu B$ means that $\mu(A \setminus B) = 0$ and $A =_\mu B$ means $\mu(A \triangle B) = 0$. Similarly if f, g are two measurable functions with æ-domain X, $f =_\mu g$ means that $\forall x \in_\mu X : f(x) = g(x)$. The ligature æ("ash") stands for a measure whose name is suppressed; it also stands for the phrase "almost everywhere"; when in the form μ-æthe former usage is dropped. L^0 means the space of measurable functions æ-defined and æ-identified if a measure is fixed.

Let $P = \prod_{\alpha \in A} P_\alpha$ be a set of tuples of numbers or more general objects, where A is an anonymous indexing set, $L : P \to \mathbb{R}$ and $R : P \to \mathbb{R}$ be two functions. For two

triviality (interpreted appropriately) of the cocycle, so that time crystals are obstructions to the suspension system having the K property, and existence of a space crystal signifies algebraicity of the suspended system; the Totoki-Gurevich Theorem (see [Tot70, Gur69]) is important for this discussion, and it also makes the case that even for classical ergodic theory, and even when there is a cocycle it is better to think of the suspended system as vertical.

[19] Interpreting measures as quantifier qualifiers allows a great level of compression, however as this also brushes measure theoretical subtleties under the rug some vigilance is often needed.

subsets $Q_1, Q_2 \subseteq P$ and $p \in P$ we write $L(p) \lesssim_{Q_1}^{Q_2} R(p)$ if for some function $C : P \to \mathbb{R}_{>0}$ that is possibly nonconstant on Q_1 and certainly constant on Q_2, for any $p \in P$, we have $L(p) \leq C(p)R(p)$. If $Q_1 = P_\alpha$, $Q_2 = P_\beta$ we also write $L(p) \lesssim_{p_\alpha}^{p_\beta} R(p)$. Here neither of Q_1, Q_2 needs to be optimal.

$\overline{\underline{\text{Lie}}}$, $\overline{\underline{\text{lie}}}$, $\overline{\underline{\text{Man}}}^s$, $\overline{\underline{\text{Man}}}^s_\rho$, $\overline{\underline{\text{sMble}}}$, $\overline{\underline{\text{sMeas}}}$, $\overline{\underline{\text{sProb}}}$ stand for the categories of Lie groups, Lie algebras, C^s manifolds, C^s ρ-manifolds, standard measurable spaces, standard measure spaces, and standard probability spaces with the standard choices for arrows. The latter three "Borelesque" categories are often the associated æ-factor categories, signified by æ in diagrams. We'll conflate a category and the category of G-objects (as well as G-systems) of that category for G a group object in the category; accordingly instead of "conjugacy" we'll say "isomorphism". For f a morphism in a category $\overrightarrow{f}$ stands for the induced map under a covariant functor and $\overleftarrow{f}$ stands for the induced map under a contravariant functor. A priori non-categorical covariant and contravariant induced maps are denoted by $f_!$ and $f^!$, respectively. For any morphism object f, the notation f^{-1} will only be used when it can be interpreted as an object of the same type as f.

Imm^s and Emb^s stand for C^s injective immersions and embeddings, respectively; all such function spaces are topologized via the uniform C^s topology on compact subsets; we'll denote the associated chosen distance by d_{C^s}.

For semidirect products we always denote the group G that fits into a split short exact sequence $L \to G \to R$ (in any group object category) by $G \cong L \rtimes R$ or $G \cong R \ltimes L$. Thus L is a normal subgroup in G and R acts on L.

Chapter 2

Outline of the Argument

In this chapter we give an outline of the proof of Thm. 1. Our starting point is that by [KKRH11, p.363, Main Thm.(2)] μ is absolutely continuous w/r/t the Lebesgue measure class of M. Throughout we roughly follow the outline of [KRH16].

Nonstationary Linearizations for Lyapunov æ -Foliations For any nonorbital Lyapunov exponent χ of (μ, α), denote by $\mathcal{L}^\chi$ the associated one dimensional æ-foliation and by $L^\chi \leq TM$ the corresponding tangent æ-subbundle. We first introduce an æ-unique family of nonstationary linearizations

$$\forall x \in_\mu M : \Lambda^\chi_x : L^\chi_x \to \mathcal{L}^\chi_x$$

that depend measurably on the basepoint and is C^r if the basepoint is fixed. Within any Lusin-Pesin set the linearizations depend uniformly continuously on the basepoint in the C^r topology.

Affine Structures for Stable and Unstable æ -Foliations For any Weyl chamber $\mathcal{C}$ of (μ, α) denote by $\mathcal{S}^\mathcal{C}, \mathcal{U}^\mathcal{C}, \mathcal{O}, \mathcal{OS}^\mathcal{C}, \mathcal{OU}^\mathcal{C}$ the associated stable, unstable, orbit, orbit-stable and orbit-unstable æ-foliations and by $S^\mathcal{C}, U^\mathcal{C}, O, OS^\mathcal{C}, OU^\mathcal{C} \leq TM$ the corresponding tangent æ-subbundles. Then the nonstationary linearizations $\Lambda^\chi_\bullet$ in the previous item can be uniquely assembled into C^r affine manifold structures on global manifolds

$$\begin{aligned}
\forall x \in_\mu M : \Sigma^\mathcal{C}_x &: S^\mathcal{C}_x \to \mathcal{S}^\mathcal{C}_x, \\
\Gamma\Sigma^\mathcal{C}_x &: \mathbb{R}^k \times S^\mathcal{C}_x \to \mathcal{OS}^\mathcal{C}_x, \\
\Upsilon^\mathcal{C}_x &: U^\mathcal{C}_x \to \mathcal{U}^\mathcal{C}_x, \\
\Gamma\Upsilon^\mathcal{C}_x &: \mathbb{R}^k \times U^\mathcal{C}_x \to \mathcal{OU}^\mathcal{C}_x,
\end{aligned}$$

where the fibers of stable and unstable æ-subbundles are endowed with the natural affine manifold structure, and $\mathbb{R}^k$ is endowed with the Euclidean affine manifold structure.

Furthermore we have a diagonal property: for $x \in_\mu M$ if $(\chi, \mathcal{C}) = -$ the 1-dimensional global Lyapunov manifold $\mathcal{L}^\chi_x$ inside $\mathcal{S}^\mathcal{C}_x \subseteq \mathcal{OS}^\mathcal{C}_x$ corresponds to a coordinate axis in $\mathbb{R}^s \leq$

$\mathbb{R}^k \times \mathbb{R}^s$ and similarly for global unstable and orbit-unstable manifolds. Consequently the global manifolds admit unique C^r $\mathcal{A}$-affine manifold structures, where $\mathcal{A}$ is the subgroup of the affine group of appropriate dimension with the property that each element of $\mathcal{A}$ has diagonal linear part.

Affine Holonomies Next we show that for any Weyl chamber $\mathfrak{C}$ of (μ, α), for any $x \in_\mu M$, and for any $y \in_{æ} \mathcal{U}_x^{\mathfrak{C}}$, there is a well-defined holonomy map

$$\mathcal{U}_{y \leftarrow x}^{\mathfrak{C}} : \mathcal{OS}_x^{\mathfrak{C}} \to \mathcal{OS}_y^{\mathfrak{C}}$$

along the unstable æ-foliation associated to $\mathfrak{C}$ with the property that w/r/t the affine structures $\Gamma\Sigma_x^{\mathfrak{C}}$ and $\Gamma\Sigma_y^{\mathfrak{C}}$, the holonomy $\mathcal{U}_{y \leftarrow x}^{\mathfrak{C}}$ is affine with linear part of the form $I_k \times D_s$, where I_k is the $k \times k$ identity matrix and $D_s = D_s(\mathfrak{C}, y \leftarrow x)$ is a diagonal $s \times s$ matrix with $s = s(\mathfrak{C})$ the rank of the æ-subbundle $S^{\mathfrak{C}}$. Similarly for any $z \in_{æ} \mathcal{S}_x^{\mathfrak{C}}$ there is a well-defined holonomy map

$$\mathcal{S}_{z \leftarrow x}^{\mathfrak{C}} : \mathcal{OU}_x^{\mathfrak{C}} \to \mathcal{OU}_z^{\mathfrak{C}}$$

with the property that w/r/t the affine structures $\Gamma\Upsilon_x^{\mathfrak{C}}$ and $\Gamma\Upsilon_z^{\mathfrak{C}}$, $\mathcal{S}_{z \leftarrow x}^{\mathfrak{C}}$ is affine with linear part of the form $I_k \times D_u$, where $D_u = D_u(\mathfrak{C}, z \leftarrow x)$ is a diagonal $u \times u$ matrix with $u = u(\mathfrak{C})$ the rank of the æ-subbundle $U^{\mathfrak{C}}$. The two families of holonomy cohere, in the sense that for any $x \in_\mu M$ and for any $(y, z) \in_{æ} \mathcal{S}_x^{\mathfrak{C}} \times \mathcal{U}_x^{\mathfrak{C}} \subseteq \mathcal{OS}_x^{\mathfrak{C}} \times \mathcal{OU}_x^{\mathfrak{C}}$:

$$\mathcal{U}_{z \leftarrow x}^{\mathfrak{C}}(y) = \mathcal{S}_{y \leftarrow x}^{\mathfrak{C}}(z),$$

in particular there is no spill-over along the orbit foliations.

Measurable Covering Map and Diagonal Affine Extension For any Weyl chamber $\mathfrak{C}$ and for $x \in_\mu M$, the affine structures and the holonomy maps assemble into a measurable map $\Phi_x = \Phi_x^{\mathfrak{C}} : (T_xM, 0) \to (M, x)$ that we call the measurable covering map of (μ, α). Φ_x has a certain diagonal property in that it is compatible with the orbits and the one dimensional Lyapunov subspaces. Further, the image of Φ_x is a full μ-measure subset of M, $(\Phi_x)^{-1}$ carries the measure μ to a Haar measure on T_xM, and locally Φ_x is a measure theoretical isomorphism.

The change of the basepoint x for the measurable covering map Φ_x contributes a pull-back by a diagonal affine automorphism. We may change the basepoint either vertically along the tangent space T_xM or horizontally along the orbits of α; these two types of changes are intimately related. By thinking of TM as an æ-bundle over M we obtain a diagonal affine extension $\mathbb{R}^k \curvearrowright TM$ with $\Phi : TM \to M$, $(x, v) \mapsto \Phi_x(v)$ the factor map.

Homoclinic Group and the Construction of the Algebraic Model Thinking of diagonal affine isomorphisms between different tangent spaces as the symmetries of the measurable covering maps $\Phi_\bullet$ gives the homoclinic groupoid $\mathfrak{H}_{\bullet \leftarrow \bullet}$ of (μ, α), and specifying a base point $x \in_\mu M$ gives the homoclinic group $\mathfrak{H}_x \leq \mathrm{Aff}(T_xM)$. More specifically

$$
\begin{aligned}
\mathfrak{H}_{y\leftarrow x} &= \{A \in \mathrm{Aff}(T_xM; T_yM) \mid \Phi_y \circ A =_{\text{æ}} \Phi_x, A'(0) \text{ diagonal } \}, \\
\mathfrak{H}_x = \mathfrak{H}_{x\leftarrow x} &= \{A \in \mathrm{Aff}(T_xM) \mid \Phi_x \circ A =_{\text{æ}} \Phi_x, A'(0) \text{ diagonal } \}.
\end{aligned}
$$

We show that $\mathfrak{H}_x \cong (\mathbb{Z}^{k+1} \rtimes F) \rtimes \mathbb{Z}^k$, where F is a group of involutions of order at most two, and consequently factoring it out induces the isomorphism $\Phi_{(\mu,\alpha)}$ that transforms α to the suspension of an affine Cartan action and μ to the suspension measure induced by Haar measure on T^{k+1}. Finally smoothness of $\Phi_{(\mu,\alpha)}$ along the stable and unstable æ -foliations of any Weyl chamber $\mathcal{C}$ of (μ, α) as well as smoothness in the sense of Whitney is by a straightforward application of a Journé lemma.

Chapter 3

Preliminaries

In this chapter we go over the preliminaries. This chapter contains no proofs as all the material is fairly straightforward except possibly the notation or formalism. The reader is advised to skip this chapter and return to it when a clarification of definitions or notations is needed.

3.1 Manifolds with Geometric Structures

Here we include some basic definitions from the theory of geometric structures[1]; as far as the author is aware the detailed specific definitions we need in this dissertation are not available in the literature.

Definition 3: Let X be a C^∞ manifold, G be a Lie group, $s \in \mathbb{Z}_{\geq 1} \times [0,1]$, $\rho_\bullet : G \to \mathrm{Diff}^s(X)$ be a C^s action. A **local** C^s ρ**-diffeomorphism** of X is a C^s-diffeomorphism $\psi : A \to B$, where A and B are open subsets of X, such that there is some $g \in G$ with $\psi = \rho_g\big|_A$.

Let Y be a C^s manifold. A C^s ρ**-structure** (or (G,X)**-structure**, or $(G \overset{\rho}{\curvearrowright} X)$**-structure**) on Y is an open cover $\mathcal{U}$ of Y and for each $U \in \mathcal{U}$ a C^s diffeomorphism $\phi \in \mathrm{Diff}^s(U, \overrightarrow{\phi}(U))$ with $\overrightarrow{\phi}(U) \subseteq X$ open such that for any $U_1, U_2 \in \mathcal{U}$,

$$\mathfrak{R}_{U_2 \leftarrow U_1} = \phi_2 \circ \phi_1^{-1}\Big|_{\overrightarrow{\phi_1}(U_1 \cap U_2)} : \overrightarrow{\phi_1}(U_1 \cap U_2) \to \overrightarrow{\phi_2}(U_1 \cap U_2)$$

is a local C^s ρ-diffeomorphism of X. Y is a C^s ρ**-manifold** if it's endowed with a maximal C^s ρ-structure. A pair (U, ϕ) as above is a C^s ρ**-chart**. We call a C^s ρ-chart (U, ϕ) **global** if $U = Y$. A C^s **globally-**ρ**-manifold** is a C^s ρ-manifold with all C^s ρ-charts global.

If Y is a C^s ρ-manifold, then a C^s ρ**-diffeomorphism** of Y is a C^s diffeomorphism of Y that is a local C^s ρ-diffeomorphism of X in any C^s ρ-chart. Let us denote by $\mathrm{Diff}^s_\rho(Y)$ the subgroup of $\mathrm{Diff}^s(Y)$ consisting of C^s ρ-diffeomorphisms of Y.

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[1]See [Gol21, Thu97, Rat06] for versions more appropriate for geometry.

Definition 4: A C^s **affine manifold** is a C^s ρ-manifold with $\rho_\bullet : \mathrm{Aff}(\mathbb{R}^d) \hookrightarrow \mathrm{Diff}^s(\mathbb{R}^d)$ the standard affine action.

If $\mathcal{A} \leq \mathrm{Aff}(\mathbb{R}^d)$, a C^s **$\mathcal{A}$-affine manifold** is a C^s ρ-manifold, where ρ is the restriction of the standard affine action to $\mathcal{A}$; we'll denote the associated ρ-diffeomorphism group by $\mathrm{Diff}^s_{\mathcal{A}}(M)$. A C^s **globally-$\mathcal{A}$-affine manifold** is likewise a C^s globally-ρ-manifold where ρ is the restriction of the standard affine action to $\mathcal{A}$. Let us denote by $\mathrm{DGL}(\mathbb{R}^d)$ the group of diagonal invertible linear automorphisms[2] of $\mathbb{R}^d$ and $\mathrm{DAff}(\mathbb{R}^d) \leq \mathrm{Aff}(\mathbb{R}^d)$ the subgroup of $\mathrm{Aff}(\mathbb{R}^d)$ consisting of all affine automorphisms of $\mathbb{R}^d$ with diagonal linear part; so $\mathrm{DAff}(\mathbb{R}^d) = \mathbb{R}^d \rtimes \mathrm{DGL}(\mathbb{R}^d)$.

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Remark 3: In general $\mathrm{Diff}^s_\rho(Y)$ is not a closed subgroup of $\mathrm{Diff}^s(Y)$, nor are the groups $\mathrm{Diff}^s_\rho(Y)$ and $\mathrm{im}(\rho) \leq \mathrm{Diff}^s_\rho(X)$ isomorphic. If Y is a globally-ρ-manifold then any chart provides an isomorphism $\mathrm{Diff}^s_\rho(Y) \cong \mathrm{im}(\rho)$. If $\mathcal{A} \leq \mathrm{Aff}(\mathbb{R}^d)$, then $\mathrm{Diff}^s_{\mathcal{A}}(Y)$ is a closed subgroup of $\mathrm{Diff}^s(Y)$, the latter being endowed with the uniform C^s topology. Thus for any globally-$\mathcal{A}$-affine manifold Y, $\mathrm{Diff}^s_{\mathcal{A}}(Y) \cong \mathcal{A}$.

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3.2 Suspensions and Time Changes

In this section we briefly discuss the two categorical constructions that we will recover in the context of Thm.1, namely, suspensions and time changes. Let G be a unimodular Lie group, $\Gamma \leq G$ be a lattice, $s \in \mathbb{Z}_{\geq 1} \times [0,1]$, N be a C^s manifold and $(\nu, \beta) \in \mathfrak{E}^s(\Gamma \curvearrowright N)$. We do not disregard the case of discrete G; in this case Γ is a finite index subgroup. The aim of the suspension construction is to produce a system, in an æ-unique manner in the first coordinate, in $\mathfrak{E}^s(G \curvearrowright M)$ for some C^s manifold M. First define the diagonal C^s action

$$ð^\beta_\bullet : \Gamma \to \mathrm{Diff}^s(G \times N),\ t \mapsto [(g,x) \mapsto (gt^{-1}, \beta_t(x))].$$

$ð^\beta$ is a free and proper action, so that the orbit space ${}^{G \times N}\!/_{ð^\beta} = G \otimes^\beta N$ has a unique C^s manifold structure w/r/t which the canonical map $G \times N \to G \otimes^\beta N$ is a C^s submersion. There is a natural C^s submersion map $\pi^\beta : G \otimes^\beta N \to {}^{G}\!/_{\Gamma}$ which gives a C^s fiber bundle structure with each fiber C^s diffeomorphic to N. Moreover the left translation action $G \to \mathrm{Diff}^s(G \times N), t \mapsto [(g,x) \mapsto (tg, x)]$ commutes with $ð^\beta$, hence induces an action $\hbar^\beta_\bullet : G \to \mathrm{Diff}^s(G \otimes^\beta N)$ called the **suspension** of β.

Let us denote by haar_G the Haar measure on G that π^β pushes forward to the unique G invariant Borel probability measure $\mathrm{haar}_{G/\Gamma}$ on ${}^{G}\!/_{\Gamma}$. Then pushing the product measure $\mathrm{haar}_G \otimes \nu$ on $G \times N$ forward to $G \otimes^\beta N$ gives a Borel probability measure $\mathrm{haar}_{G/\Gamma} \otimes^\beta \nu$. It's

[2]Note that this notation implies that there is a chosen but suppressed splitting of $\mathbb{R}^d$ into 1 dimensional subspaces.

straightforward that $\hbar^\beta$ preserves $\mathrm{haar}_{G/\Gamma} \otimes^\beta \nu$, so that we have produced the **suspension system**

$$\left(\mathrm{haar}_{G/\Gamma} \otimes^\beta \nu, \hbar^\beta\right) \in \mathfrak{E}^s(G \curvearrowright G \otimes^\beta N).$$

Note that G/Γ also carries a left translation action l^G of G under which $\mathrm{haar}_{G/\Gamma}$ is invariant, so that $\left(\mathrm{haar}_{G/\Gamma}, l^G\right) \in \mathfrak{E}^{\mathrm{aff}}\left(G \curvearrowright G/\Gamma\right)$ which is a factor system of $\left(\mathrm{haar}_{G/\Gamma} \otimes^\beta \nu, \hbar^\beta\right)$ via π^β. For each $g \in G$ such that $g\Gamma = \Gamma g$, the fiber $\{g\Gamma\} \otimes^\beta N = \overleftarrow{\pi^\beta}(g\Gamma)$ is preserved by the suspension $\hbar^\beta$, and we have a C^s isomorphism $\iota^\beta_{g\Gamma}$: $(\nu, \beta) \xrightarrow{\cong_{C^s}} \left(\mathrm{haar}_{G/\Gamma} \otimes^\beta \nu\big|_{\{g\Gamma\}\otimes^\beta N}, \hbar^\beta\big|_{\{g\Gamma\}\otimes^\beta N}\right)$ that is a C^s embedding $N \hookrightarrow G \otimes^\beta N$.

Observation 1: In the special case of $G = \mathbb{R}^k$ and $\Gamma = \mathbb{Z}^k$, since the normalizer of $\mathbb{Z}^k$ is the whole $\mathbb{R}^k$, the suspension system restricted to any fiber of the suspension manifold is a copy of the original $\mathbb{Z}^k$ system. Moreover the factor system $\left(\mathrm{haar}_{\mathbb{T}^k}, l^{\mathbb{R}^k}\right) \in \mathfrak{E}^{\mathrm{aff}}\left(\mathbb{R}^k \curvearrowright \mathbb{T}^k\right)$ can be replaced by the translation system $\left(\mathrm{haar}_{\mathbb{T}^k}, l^{\mathbb{T}^k}\right) \in \mathfrak{E}^{\mathrm{aff}}\left(\mathbb{T}^k \curvearrowright \mathbb{T}^k\right)$. Thus we may think of the the totality of the discussion above in this section as a bundle of systems:

$$(\nu, \beta) \xhookrightarrow{\iota^\beta} \left(\mathrm{haar}_{\mathbb{T}^k} \otimes^\beta \nu, \hbar^\beta\right) \xrightarrow{\pi^\beta} \left(\mathrm{haar}_{\mathbb{T}^k}, l^{\mathbb{T}^k}\right)$$

In Thm.1 we recover such a bundle up to measure theoretical isomorphism, with the fiber action affine Cartan.

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Observation 2: Let us consider the case of $G = \mathbb{Z}^k$ and $\Gamma = \overrightarrow{j}(\mathbb{Z}^k)$ the image of an embedding. $\overrightarrow{j}(\mathbb{Z}^k) \leq \mathbb{Z}^k$ being a lattice means that the factor group $F = \mathbb{Z}^k / \overrightarrow{j}(\mathbb{Z}^k)$ is finite. Similar to Obs.1 we have a bundle of systems:

$$(\nu, \beta) \xhookrightarrow{\iota^\beta} \left(\mathrm{haar}_F \otimes^\beta \nu, \hbar^\beta\right) \xrightarrow{\pi^\beta} \left(\mathrm{haar}_F, l^F\right)$$

where haar_F is the counting measure on F normalized by $\frac{1}{\#(F)}$.

⌟

Let us now discuss time changes. We generalize the C^s version of the theory of time changes in [Tot66, Par86] and [Par81, p.74][3] to systems with acting group a unimodular Lie group. The aim of the time change construction is to reparameterize a system in an orbit-preserving manner. Let us endow G with a suppressed left translation invariant

[3] Also see [AS67] for a slightly different, infinitesimal approach.

Riemannian metric and let $(\mu, \alpha) \in \mathfrak{E}^s(G \curvearrowright M)$ be a system. Let λ be a C^s **(G-valued 1-)cocycle** over α, that is, $\lambda \in C^s(G \times M; G)$ is such that

$$\forall t_1, t_2 \in G, \forall x \in M : \lambda(t_2, x)\lambda(t_1 t_2, x)^{-1}\lambda(t_1, \alpha_{t_2}(x)) = e_G.$$

Suppose further that for each fixed $x \in M$, the map $t \mapsto \lambda(t, x)$ is an orientation preserving diffeomorphism; that is, for any $x \in M$, $\lambda(\bullet, x) \in \mathrm{Diff}^s_+(G, e_G)$. Then we have that $(\lambda, \mathrm{id}_M) : G \times M \to G \times M, (t, x) \mapsto (\lambda(t, x), x)$ is a C^s diffeomorphism with inverse of the form (κ, id_M) for some $\kappa \in C^s(G \times M; G)$ (explicitly, $\kappa : (t, x) \mapsto \lambda(\bullet, x)^{-1}(t)$). Define a family of C^s self-maps of M by $\beta_\bullet = \alpha_{\kappa\bullet} : G \to C^s(M, M), t \mapsto [x \mapsto \alpha_{\kappa(t,x)}(x)]$. As $\alpha = \beta_\lambda$ also, it's straightforward to verify that κ is a C^s cocycle over β, and $\beta = \alpha_\kappa$ is a G action by C^s diffeomorphisms; it's called the κ **time change** of the action α. Note that we have a commutative diagram (in the category $\underline{\overline{\mathrm{Man}}}^s$ of C^s manifolds):

$$\begin{array}{ccc} G \times M & \xrightarrow{\alpha=\beta_\lambda} & M \\ {\scriptstyle (\lambda,\mathrm{id}_M)=(\kappa,\mathrm{id}_M)^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_M} \\ G \times M & \xrightarrow[\beta=\alpha_\kappa]{} & M \end{array}$$

We also need to modify the α-invariant measure μ to get a β-invariant measure. Define a new Borel probability measure ν on M by

$$\nu = \mu_\kappa : \mathcal{B}(M) \to [0, 1],\ B \mapsto \frac{\int_B \det\left(T^1\lambda(e_G, x)\right)\, d\mu(x)}{\int_M \det\left(T^1\lambda(e_G, x)\right)\, d\mu(x)},$$

where $T^1\lambda(e_G, x)$ is, for fixed $x \in M$, the derivative of $G \to G, t \mapsto \lambda(t, x)$ evaluated at e_G, and the determinant is w/r/t the Riemannian metric fixed on G[4]. It's straightforward to verify that ν is β-invariant, e.g. by way of taking the Laplace transform or derivative of $G \to \mathbb{R}, t \mapsto \mathbb{E}_\nu(f \circ \beta_t)$ for some anonymous $f \in C^0(M; \mathbb{R})$. It's also straightforward that (μ, α) is ergodic iff (ν, β) is ergodic, provided that G is a group for which a pointwise ergodic theorem holds that allows one to replace α-time averages of the time change λ with its μ-space averages[5]. Finally the entropy gauges of (μ, α) and (ν, β) are proportional to each other by an Abramov formula:

Lemma 1: Let G be a unimodular Lie group. Let $(\mu, \alpha) \in \mathfrak{E}^s(G \curvearrowright M)$ and κ be a C^s time change of (μ, α). Then we have

$$\forall t \in G : \mathfrak{e}_{(\mu,\alpha)}(t) = \left(\int_M \det\left(T^1\lambda(e_G, x)\right)\, d\mu(x)\right) \mathfrak{e}_{(\mu_\kappa,\alpha_\kappa)}(t).$$

⌟

[4] Instead of the Riemannian Jacobian one could alternatively use Radon-Nikodym derivative of the pushforward of a Haar measure haar_G on G by $\lambda(\bullet, x)$ w/r/t haar_G.

[5] E.g. for G compactly generated abelian (with possibly some further reasonable conditions on G and λ); see [Lin99, Lin01, GE74] and [Pat88, p.195, Ch.5].

Observe that we may consider any orientation preserving Lie group automorphism $\lambda \in \mathrm{Aut}_{\underline{\mathrm{Lie}},+}(G)$ as a cocyle over α. In this case $\kappa = \lambda^{-1} \in \mathrm{Aut}_{\underline{\mathrm{Lie}},+}(G)$, $\nu = \mu_\kappa = \mu = \nu_\lambda$ and

$$\forall t \in G : \mathfrak{e}_{(\mu,\alpha)}(t) = \det(\mathrm{lie}(\lambda))\, \mathfrak{e}_{(\mu_\kappa,\alpha_\kappa)}(t),$$

where $\mathrm{lie}(\lambda) = T_{e_G}\lambda : \mathrm{lie}(G) \to \mathrm{lie}(G)$ is the induced Lie algebra automorphism. Since such group automorphisms λ are in bijection with cocycles over α that are constant in x, we'll call such cocycles and in particular time changes **constant in space**.

Remark 4: Let us note in closing that instead of assuming $\lambda \in C^s(G \times M; G)$ we could have assumed that $\lambda \in L^0(G \times M; G)$ with $\forall x \in_\mu M : \lambda(\bullet, x) \in \mathrm{Diff}^s_+(G)$ and

$$\forall t_1, t_2 \in G, \forall x \in_\mu M : \lambda(t_2, x)\lambda(t_1 t_2, x)^{-1}\lambda(t_1, \alpha_{t_2}(x)) = e_G.$$

Then the theory of time changes as discussed above still holds (with the commutative diagram now being in $\underline{\mathrm{sMeas}}$), and in particular we still have Lem. 1. Let us call such cocycles C^s **æ-time changes**.

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3.3 Hölder Conditions and Grassmannian-type Bundles

In this section we recall the standard Grassmannian constructions. Note that local fractional regularity C^s for $s \in \mathbb{R}_{\geq 1}$ is well-defined on C^∞ manifolds without a choice of a metric or fiberwise norm, see [BF66] and [Rue89, pp.138-151].

Definition 5: Let M be a compact C^∞ manifold and let $p : E \to M$ be a C^∞ vector bundle of rank $r \in \mathbb{Z}_{\geq 0}$. The **Grassmannian bundle** $\mathrm{Gr}(p) : \mathrm{Gr}(E) \to M$ of $p : E \to M$ is defined by $\mathrm{Gr}(E)_x = \mathrm{Gr}(E_x)$, that is, the fiber of $\mathrm{Gr}(E)$ at x is the collection of all linear subspaces of the fiber E_x. Note that $\mathrm{Gr}(E) \to M$ comes with a natural grading according to the dimension of the subspaces: $\mathrm{Gr}(E) = \bigsqcup_{k \in \underline{r+1}} \mathrm{Gr}_k(E)$, and naturally $M \xrightarrow{\cong} \mathrm{Gr}_0(E) \hookrightarrow \mathrm{Gr}(E)$ is a C^∞ embedding.

Similarly define the **total Grassmannian bundle** $\mathrm{Gr}^\infty(p) : \mathrm{Gr}^\infty(E) \to M$ of $p : E \to M$ by $\mathrm{Gr}^\infty(E)_x = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathrm{Gr}(E_x)^n$, that is, the fiber of $\mathrm{Gr}^\infty(E)$ at x is the collection of all tuples of subspaces of E_x, possibly with repetitions or intersections. Denote by $\mathrm{Spl}(p) : \mathrm{Spl}(E) \to M$ that subbundle of $\mathrm{Gr}^\infty(E) \to M$ whose fiber at x consists of tuples of subspaces of E_x which direct sum to E_x; we call $\mathrm{Spl}(E) \to M$ the **bundle of splittings** of $E \to M$.

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Observation 3: In the context of Def. 5, there are unique C^∞ manifold structures on $\mathrm{Gr}(E)$, $\mathrm{Gr}^\infty(E)$ and $\mathrm{Spl}(E)$ such that $\mathrm{Gr}(p) : \mathrm{Gr}(E) \to M$, $\mathrm{Gr}^\infty(p) : \mathrm{Gr}^\infty(E) \to M$ and $\mathrm{Spl}(p) : \mathrm{Spl}(E) \to M$ are C^∞ fiber bundles; further $\mathrm{Gr}(E) \to M$ is with compact fibers and consequently $\mathrm{Gr}(E)$ too is a compact manifold. It is clear that these constructions can be done locally, and that one can consider measurable or C^s ($s \in \mathbb{R}_{\geq 0}$) local or global sections of $\mathrm{Gr}(E) \to M$ or $\mathrm{Spl}(E) \to M$. When talking about the regularity of a polarization on M (or subbundle of TM) we interpret it as a section of $\mathrm{Gr}(TM) \to M$.

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Observation 4: In the context of Obs.3, for Hölder estimates of self-maps and polarizations on M it is often useful to have more explicit expressions in terms of distances. For this one can fix a C^∞ embedding $M \hookrightarrow \mathbb{R}^{d(M)}$ for some $d(M) \in \mathbb{Z}_{\gg 1}$[6]. Alternatively, and more intrinsically, one can fix a C^2 Riemannian metric $\mathfrak{g}$ on M whose exponentials $\exp^{\mathfrak{g}}_{\bullet}$ are C^1 local diffeomorphisms and whose parallel transports are C^1 w/r/t the basepoint. Denote by $d^{\mathfrak{g}}$ the intrinsic distance function on M induced by $\mathfrak{g}$ and continuous piecewise C^∞ paths. Note that since M is compact any other (C^0) fiberwise norm would produce an intrinsic distance on M that is Lipschitz equivalent to $d^{\mathfrak{g}}$. For any $x \in M$ denote the injectivity radius of $\exp^{\mathfrak{g}}_x : T_xM \to M$ by $r^{\mathfrak{g}}_x \in \mathbb{R}_{>0}$, the normal neighborhood at x by $N^{\mathfrak{g}}_x = \overrightarrow{\exp^{\mathfrak{g}}_x}(T_xM[0| < r^{\mathfrak{g}}_x])$, and for any $y \in N^{\mathfrak{g}}_x$ denote by $\Pi^{\mathfrak{g}}_{x\leftarrow y} : T_yM \to T_xM$ the parallel transport of $\mathfrak{g}$ from y to x along the unique (C^1) geodesic connecting them. Identify, via a $\mathfrak{g}_x$, an element E_x of $\mathrm{Gr}(T_xM)$ with the orthogonal projection operator $\mathrm{proj}^{\mathfrak{g}}(E_x) : T_xM \to T_xM$. Let $\vartheta \in]0,1]$.

- A function $f : M \to M$ is $C^{(0,\vartheta)}$ iff for some (hence for any) C^2 Riemannian metric $\mathfrak{g}$:

$$\exists C \in \mathbb{R}_{>0}, \forall x \in M, \exists U_x \in \mathrm{Nbhd}(x), \forall y \in U_x : d^{\mathfrak{g}}(f(x), f(y)) \leq C d^{\mathfrak{g}}(x,y)^{\vartheta}.$$

- A function $f : M \to M$ is $C^{(1,\vartheta)}$ iff it's C^1 and for some (hence for any) C^2 Riemannian metric $\mathfrak{g}$:

$$\begin{aligned}&\exists C \in \mathbb{R}_{>0}, \forall x \in M, \exists U_x \in \mathrm{Nbhd}(x) \cap \mathcal{P}(N^{\mathfrak{g}}_x), \forall y \in U_x : \\ &\qquad \left\| T_xf - \Pi^{\mathfrak{g}}_{f(x)\leftarrow f(y)} \circ T_yf \circ \Pi^{\mathfrak{g}}_{y\leftarrow x} \right\|_{T_{f(x)}M \leftarrow T_xM} \leq C d^{\mathfrak{g}}(x,y)^{\vartheta}.\end{aligned}$$

- A polarization $E_\bullet : M \to \mathrm{Gr}(TM)$ is $C^{(0,\vartheta)}$ iff[7] for some (hence for any) C^2 Riemannian metric $\mathfrak{g}$ on M

$$\begin{aligned}&\exists C \in \mathbb{R}_{>0}, \forall x \in M, \exists U_x \in \mathrm{Nbhd}(x) \cap \mathcal{P}(N^{\mathfrak{g}}_x), \forall y \in U_x : \\ &\qquad \left\| \mathrm{proj}^{\mathfrak{g}}(E_x) - \mathrm{proj}^{\mathfrak{g}} \circ \overrightarrow{\Pi^{\mathfrak{g}}_{x\leftarrow y}}(E_y) \right\|_{T_xM \leftarrow T_xM} \leq C d^{\mathfrak{g}}(x,y)^{\vartheta}.\end{aligned}$$

Observe that all these constructions allow one to consider global Hölder estimates as well[8].

[6] See e.g. [BS02, p.144], [BP01, pp.91-93,App.A] or [BRO21] for this approach.

[7] See [Ano67], [BK87, p.361].

[8] There are other characterizations of various Hölder estimates that become available once one makes use of natural metrics on (iterated) tangent bundles, e.g. the Sasaki metrics; a C^q ($q \in \mathbb{Z}_{\geq 0}$) Riemannian metric on M induces for $i \in \overline{q}$ a C^{q-i} Riemannian Sasaki metric on the iterated tangent bundle $T^{(i)}M$ ($T^{(0)}M = M, T^{(i)}M = T(T^{(i-1)}M)$). Note that such metrics are not natural in the categorical sense.

Definition 6: Let M be a compact C^∞ manifold, for $s, p \in \mathbb{R}_{\geq 1}$ define the **adjoint operator** by

$$\begin{aligned}\mathrm{Ad}: \mathrm{Diff}^s(M) \times C^p(\mathrm{Gr}(TM) \to M) &\to C^{\min\{s-1,p\}}(\mathrm{Gr}(TM) \to M),\\ (f, E) &\mapsto \left[x \mapsto \overrightarrow{T_{f^{-1}(x)}f}\,(E_{f^{-1}(x)})\right].\end{aligned}$$

Similarly we have adjoint operators on measurable sections, and sections of $\mathrm{Spl}(TM) \to M$. Note that $\mathrm{Ad}: \mathrm{Diff}^s(M) \curvearrowright L^0(\mathrm{Gr}(TM) \to M)$ and for $s \geq p+1$, $\mathrm{Ad}: \mathrm{Diff}^s(M) \curvearrowright C^p(\mathrm{Gr}(TM) \to M)$ is a group action. If $\alpha: \mathbb{R}^k \to \mathrm{Diff}^s(M)$ is a group action we put $\mathrm{Ad}^\alpha_\bullet = \mathrm{Ad}_{\alpha_\bullet}$.

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3.4 Oseledets Multiplicative Ergodic Theorem and Lyapunov Geometry

Lyapunov exponents we'll be using are provided by an Oseledets Theorem for the derivative cocycle of a C^1 action of $\mathbb{R}^k$ on a compact C^∞ manifold. Below we state such an Oseledets Theorem.

Theorem 4 (Oseledets[9]): Let $(\mu, \alpha) \in \mathfrak{E}^1(\mathbb{R}^k \curvearrowright M)$. If (μ, α) is locally free and ergodic, then $k \leq \dim(M)$. If $k = \dim(M)$, then the only Lyapunov exponent of (μ, α) is 0; otherwise there is a subset $\mathrm{Osel} \in \mathcal{B}(M, \alpha)$ with $\mathrm{Osel} =_\mu M$ such that

EXI1 there is a unique number $l \in \overline{\dim(M) - k}$[10]; and $\forall i \in \bar{l}, \exists \delta^i \in \overline{\dim(M)}$: $\sum_{i \in \bar{l}} \delta^i = \dim(M) - k$,

EXI2 there is a unique up to reordering linear operator

$$\begin{aligned}X &= (X^1, X^2, ..., X^{\dim(M)-k}, \underbrace{0, 0, ..., 0}_{k \text{ many}})\\ &= (\underbrace{\chi^1, ..., \chi^1}_{\delta^1 \text{ many}}, \underbrace{\chi^2, ..., \chi^2}_{\delta^2 \text{ many}}, ..., \underbrace{\chi^l, ..., \chi^l}_{\delta^l \text{ many}}, \underbrace{0, 0, ..., 0}_{k \text{ many}}) \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^{\dim(M)})\end{aligned}$$

with $i \neq j \Rightarrow \chi^i \neq \chi^j$ (but not necessarily $\chi^i \neq 0$),

EXI3 for any $x \in \mathrm{Osel}$ there is a unique splitting $T_xM = O_x \oplus \left(\bigoplus_{i \in \bar{l}} L^i_x\right)$ with $\forall i \in \bar{l}$: $\dim L^i_x = \#\{j \in \overline{\dim(M)} \mid X^j = \chi^i\} = \delta^i$

[9] [Ose68]. Also see [KKRH11, p.305, Prop.2.1]; [KN11, p.26]; [BD91, pp.19-20, Thm.1-2] [BHW16, p.5, Thm.2.4]; [Hu93a, p.74, Thm.A] (= [Hu93b, p.13, Thm.A]).

[10] Recall that we put $\bar{k} = \{1, 2, ..., k\}$ and $\underline{k} = \{0, 1, ..., k-1\}$.

such that

ASYM1 For any C^0 fiberwise norm on M,

$$\forall x \in \mathrm{Osel}, \forall i \in \bar{l} : \lim_{|t|\to\infty} \sup_{v \in L^i_x \setminus 0} \frac{\log |T_x \alpha_t v| - \chi^i(t)}{|t|} = 0,$$

ASYM2 For any C^0 density on M,

$$\forall x \in \mathrm{Osel} : \lim_{|t|\to\infty} \frac{\log \mathrm{Jac}_x(\alpha_t) - \sum_{i\in\bar{l}} \delta^i \chi^i(t)}{|t|} = 0.$$

ASYM3 For any C^0 Riemannian metric on M,

$$\forall x \in \mathrm{Osel}, \forall I \subseteq \bar{l} : \lim_{|t|\to\infty} \frac{\log \left\| \mathrm{proj}\left(\overrightarrow{T_x\alpha_t} \left(\bigoplus_{i\in I} L^j_x \right)\right) - \mathrm{proj}\left(O_{\alpha_t(x)} \right) \right\|}{|t|} = 0 \text{ and}$$
$$\lim_{|t|\to\infty} \frac{\log \left\| \mathrm{proj}\left(\overrightarrow{T_x\alpha_t} \left(\bigoplus_{i\in I} L^j_x \right)\right) - \mathrm{proj}\left(\overrightarrow{T_x\alpha_t} \left(\bigoplus_{j\in\bar{l}\setminus I} L^j_x \right)\right) \right\|}{|t|} = 0.$$

Further,

INV4 $O_\bullet \oplus \left(\bigoplus_{i\in\bar{l}} L^i_\bullet \right) \in L^0(\mathrm{Spl}(TM) \to M)$ is Ad^α-invariant.

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Here $\mathrm{Spl}(TM) \to M$ is the C^∞ bundle of splittings of TM, and Ad^α is the adjoint action on the measurable sections of the bundle of splittings: $\mathrm{Ad}^\alpha_t : E \mapsto \left[x \mapsto \overrightarrow{T_{\alpha_t^{-1}(x)}\alpha_t} \left(E_{\alpha_t^{-1}(x)} \right) \right]$. Let us now list some of the standard definitions of Lyapunov geometry.

Definition 7: In the context of Thm.4,

- Osel is the **set of Lyapunov-Perron regular points**,
- X is the **Lyapunov operator**,
- $\widetilde{X} = (\chi^1, \chi^2, ..., \chi^l) \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ is the **reduced Lyapunov operator**,
- $\dim(\ker(X)) \in \underline{k+1}$ is the **defect**,
- $\dim(\mathrm{im}(X)) \in \underline{\dim(M) - k + 1}$ is the **codefect**,
- The set $\mathrm{LSpec} = \{\chi^1, \chi^2, ..., \chi^l\}$ of linear functionals is the **(nonorbital) Lyapunov spectrum**,

- Any element $\chi \in \mathrm{LSpec}$ is a **(nonorbital) Lyapunov exponent**,
- $\mathrm{LSpec}^* = \mathrm{LSpec} \setminus 0$ is the **reduced Lyapunov spectrum**,
- For any $\chi \in \mathrm{LSpec}$ and any $x \in \mathrm{Osel}$, L_x^χ is the **Lyapunov subspace** associated to χ at x and L^χ is the **Lyapunov æ-subbundle** associated to χ,
- $TM =_{æ} \bigoplus_{i \in \bar{l}} L^i$ is the **(fine) Lyapunov æ-splitting**,
- δ^i is the **dynamical multiplicity** of χ^i.
- For $t \in \mathbb{R}^k$, the function $\mathrm{LSpec} \to \{-, 0, +\}, \chi \mapsto \mathrm{sign}(\chi(t))$ is called the **Lyapunov signature** of the time t,
- For any $t \in \mathbb{R}^k$ and $x \in \mathrm{Osel}$: the **stable, unstable, center, stable-center** and **center-unstable subspaces** of α_t at x are defined as follows, respectively:

$$S_x(\alpha_t) = \bigoplus_{\substack{\chi \in \mathrm{LSpec} \\ \chi(t) < 0}} L_x^\chi, \quad U_x(\alpha_t) = \bigoplus_{\substack{\chi \in \mathrm{LSpec} \\ \chi(t) > 0}} L_x^\chi, \quad C_x(\alpha_t) = \bigoplus_{\substack{\chi \in \mathrm{LSpec} \\ \chi(t) = 0}} L_x^\chi$$
$$SC_x(\alpha_t) = S_x(\alpha_t) \oplus C_x(\alpha_t), \quad CU_x(\alpha_t) = C_x(\alpha_t) \oplus U_x(\alpha_t);$$

  they assemble to **stable, unstable, center, stable-center** and **center-unstable æ-subbundles** of α_t, respectively.

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Remark 5: In the context of Thm. 4, we'll use the Lyapunov exponents as superscripts if we consider a Lyapunov exponent without referring to an enumeration, e.g. we may write $\bigoplus_{\chi \in \mathrm{LSpec}} L^\chi$ for the (fine) Lyapunov æ-splitting. We'll refer to this notation as the **intrinsic notation** for Lyapunov geometry. ⌟

Remark 6: Not fixing an ergodic measure at first gives a basepoint-dependent version of Thm.4, which provides a measurable subset Osel and $l_\bullet, \delta_\bullet^i, X_\bullet$ become functions on Osel such that

INV1 $\mathrm{Osel} \in \mathcal{B}(M; \alpha)$ and $\forall \mu \in \mathrm{Prob}(M; \alpha) : \mathrm{Osel} =_\mu M$,

INV2 $l_\bullet \in L^0(\mathrm{Osel}; \bar{d})$ and is α-invariant,

INV3 $X_\bullet = (X_\bullet^1, X_\bullet^2, ..., X_\bullet^d) \in L^0\left(\mathrm{Osel}; \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^d)\right)$ and is α-invariant.

In this case we consequently have basepoint-dependent Lyapunov spectrum also. The basepoint-dependent version of Oseledets Theorem is useful if no measure is fixed a priori, or if the fixed measure is not ergodic w/r/t the action, or if one is considering the action of a subgroup (e.g. the iterates of a time-t map). Also note that the set Osel in Thm. 4 can be considered to be attached to the action α instead of the system (μ, α).

⌟

Corollary 7: In the context of Thm.4, for any C^0 fiberwise norm on X and for any $\varepsilon \in \mathbb{R}_{>0}$, there is a measurable function $C_\varepsilon : \mathrm{Osel} \to \mathbb{R}_{>0}$ such that for any $x \in \mathrm{Osel}$, $i \in \bar{l}$, $v \in L^i_x$ and $t \in \mathbb{R}^k$

- $\dfrac{1}{C_\varepsilon(x)}\, e^{\chi^i(t) - \frac{1}{2}\varepsilon|t|}|v| \leq |T_x\alpha_t v| \leq C_\varepsilon(x)\, e^{\chi^i(t) + \frac{1}{2}\varepsilon|t|}|v|$, and
- $C_\varepsilon(\alpha_t(x)) \leq C_\varepsilon(x)\, e^{\varepsilon|t|}$.

Similarly for any C^0 Riemannian metric on M and for any $\varepsilon \in \mathbb{R}_{>0}$, there are measurable functions $C_\varepsilon, K_\varepsilon : \mathrm{Osel} \to \mathbb{R}_{>0}$ such that C_ε is as above and for any $x \in \mathrm{Osel}$, $I \subseteq \bar{l}$, $v \in L^\chi_x$ and $t \in \mathbb{R}^k$

- $\left\|\mathrm{proj}\left(\bigoplus_{i\in I} L^i_x\right) - \mathrm{proj}\left(O_x\right)\right\| \geq K_\varepsilon(x)$,
- $\left\|\mathrm{proj}\left(\bigoplus_{i\in I} L^i_x\right) - \mathrm{proj}\left(\bigoplus_{j\in\bar{l}\setminus I} L^j_x\right)\right\| \geq K_\varepsilon(x)$, and
- $K_\varepsilon(\alpha_t(x)) \geq K_\varepsilon(x)\, e^{-\varepsilon|t|}$.

We will refer to $C_\bullet, K_\bullet : \mathbb{R}_{>0} \to L^0(\mathrm{Osel}; \mathbb{R}_{>0})$ as **comparison families** of (μ, α) w/r/t the chosen fiberwise norm or Riemannian metric.

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Remark 7: **ASYM3** of Thm.4 is traditionally formulated in terms of the angle between the subspaces $\overrightarrow{T_x\alpha_t}\left(\bigoplus_{i\in I} L^i_x\right)$ and $\overrightarrow{T_x\alpha_t}\left(\bigoplus_{j\in\bar{l}\setminus I} L^j_x\right)$ of $T_{\alpha_t(x)}M$. Recall that the **angle** $\sphericalangle(W_1, W_2)$ between two nonzero subspaces $W_1, W_2 \leq V$ of an inner product space V is defined as

$$\sphericalangle(W_1, W_2) = \inf\left\{ \sphericalangle(w_1, w_2) = \arccos\left(\frac{\langle w_1, w_2\rangle}{|w_1||w_2|}\right) \;\middle|\; w_1 \in W_1 \setminus 0, w_2 \in W_2 \setminus 0 \right\} \in [0, \pi/2].$$

Then **ASYM3** is equivalent to

$$\forall I \subseteq \overline{l_x} : \lim_{|t|\to\infty} \frac{\log\sin\sphericalangle\left(\overrightarrow{T_x\alpha_t}\left(\bigoplus_{i\in I} L^i_x\right), \overrightarrow{T_x\alpha_t}\left(\bigoplus_{j\in\bar{l}\setminus I} L^j_x\right)\right)}{|t|} = 0.$$

Accordingly, the first property of K_ε in Cor.7 is equivalent to

$$\sin\sphericalangle\left(\bigoplus_{i\in I} L^i_x, \bigoplus_{j\in\bar{l}\setminus I} L^j_x\right) \geq K_\varepsilon(x).$$

┘

Definition 8: In the context of Thm.4, for $\chi, \rho \in \mathrm{LSpec}$, put $\chi \propto \rho$ if for some $c \in \mathbb{R}_{>0}$, $\chi = c\rho$. $\propto$ is an equivalence relation on LSpec, and any two Lyapunov exponents χ, ρ with $\chi \propto \rho$ are called **coarsely equivalent**; let us denote by $[\chi]$ the coarse equivalence class of χ. Then

- $\mathrm{LSpec}^{\propto} = {}^{\mathrm{LSpec}}\!\diagup_{\propto}$ is the **coarse Lyapunov spectrum**,
- $\mathrm{LSpec}^{\propto,*} = \mathrm{LSpec}^{\propto} \setminus \{[0]\}$ is the **reduced coarse Lyapunov spectrum**,
- Any element $[\chi] \in \mathrm{LSpec}^{\propto}$ is called a **coarse Lyapunov exponent**,
- For any $[\chi] \in \mathrm{LSpec}^{\propto}$ and any $x \in \mathrm{Osel}$, $L_x^{[\chi]} = \bigoplus_{\rho \in [\chi]} L_x^{\rho}$ is the **coarse Lyapunov subspace** associated to $[\chi]$ at x and $L^{[\chi]}$ is the **Lyapunov æ-subbundle** associated to $[\chi]$,
- $TM =_{æ} \bigoplus_{[\chi] \in \mathrm{LSpec}^{\propto}} L^{[\chi]}$ is the **coarse Lyapunov æ-splitting** of α.

⌟

Definition 9: In the context of Thm.4,

- For $\chi \in \mathrm{LSpec}^*$, $K^{\chi} = \ker(\chi) \leq \mathbb{R}^k$ is called the **Lyapunov hyperplane** of χ.
- Any connected component of $\mathbb{R}^k \setminus \bigcup_{\chi \in \mathrm{LSpec}^*} K^{\chi}$ is called a **Weyl chamber**; let's denote the set of all Weyl chambers by $\mathrm{Cham} = \pi_0\left(\mathbb{R}^k \setminus \bigcup_{\chi \in \mathrm{LSpec}^*} K^{\chi}\right)$. Let us denote by $-$ the natural involution on Cham.
- Any element of $\mathrm{Wall}^0 = \mathbb{R}^k \setminus \bigcup_{\chi \in \mathrm{LSpec}^*} K^{\chi}$ is a **chamber time**,
- Any element of $\bigcup_{\chi \in \mathrm{LSpec}^*} K^{\chi} = \mathbb{R}^k \setminus \mathrm{Wall}^0$ is a **wall time**[11].

If $I \subseteq \mathrm{LSpec}^*$ is a subset, put

$$\mathrm{Wall}^I = \left(\bigcap_{\chi \in I} K^{\chi}\right) \setminus \left(\bigcup_{\rho \in \mathrm{LSpec}^* \setminus I} K^{\rho}\right).$$

For brevity we also put $\mathrm{Wall}^{\chi} = \mathrm{Wall}^{\{\chi\}}$. Finally put for $i \in \bar{l}$:

$$\mathrm{Wall}^i = \bigcup_{\substack{I \subseteq \mathrm{LSpec}^* \\ \#(I) = i}} \mathrm{Wall}^I.$$

We call an element of Wall^I an I**-wall time**, an element of Wall^{χ} a χ**-wall time**, and an element of Wall^i a **wall time of order** i.

Let $\chi \in \mathrm{LSpec}^*$ be a nonzero Lyapunov exponent and $\mathcal{C} \in \mathrm{Cham}$ be a Weyl chamber. $\chi|_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ is not constant, but it's either always positive or always negative. This way we have a **Lyapunov pairing** $(\bullet, \bullet) : \mathrm{LSpec}^* \times \mathrm{Cham} \to \{\pm\}$. The function $(\bullet, \mathcal{C}) :$

[11] Compared to standard terminology we prefer more descriptive nomenclature; in [KK01] a chamber time is called a generic element, a wall time is called a singular element, and a wall time in Wall^1 is called a generic singular element.

$\mathrm{LSpec}^* \to \{\pm\}$ is called the **Lyapunov signature** of the chamber $\mathcal{C}$. For any $\mathcal{C} \in \mathrm{Cham}$ and $x \in \mathrm{Osel}$ the **stable** and **unstable** subspaces of $\mathcal{C}$ at x are defined as follows, respectively:

$$S_x^{\mathcal{C}} = \bigoplus_{\substack{\chi \in \mathrm{LSpec} \\ (\chi, \mathcal{C}) = -}} L_x^{\chi}, \quad U_x^{\mathcal{C}} = \bigoplus_{\substack{\chi \in \mathrm{LSpec} \\ (\chi, \mathcal{C}) = +}} L_x^{\chi}.$$

┘

Remark 8: Note that $\chi \propto \rho$ iff $\forall \mathcal{C} \in \mathrm{Cham}$: $(\chi, \mathcal{C}) = (\rho, \mathcal{C})$, whence the Lyapunov pairing $(\bullet, \bullet)$ factors through $\mathrm{LSpec}^{\propto,*}$. Further we have $(\chi, -\mathcal{C}) = -(\chi, \mathcal{C})$. Also note that the stable and unstable subspaces of a Weyl chamber coincides with the stable and unstable subspaces of any time-t map of the action if t is chosen from the Weyl chamber in question:

$$\forall \mathcal{C} \in \mathrm{Cham}, \forall t \in \mathcal{C} : S^{\mathcal{C}} = S(\alpha_t), U^{\mathcal{C}} = U(\alpha_t).$$

Another remark is that we have a partition

$$\mathbb{R}^k = \biguplus_{i \in \underline{\dim(M)-k+1}} \mathrm{Wall}^i = \mathrm{Wall}^0 \uplus \mathrm{Wall}^1 \uplus \cdots \uplus \mathrm{Wall}^{\dim(M)-k}.$$

Finally note that in general Wall^i is not the same as Wall^{χ^i}.

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Observation 5: Let us also note the relationship between the Lyapunov exponents of α and the Lyapunov exponents of any of its time-t map α_t. **ASYM1** of Thm.4 implies that

ASYM1$_*$ For any C^0 fiberwise norm on M and for any $t \in \mathbb{R}^k$:

$$\forall x \in \mathrm{Osel}, \forall i \in \bar{l} : \lim_{|n| \to \infty} \sup_{v \in L_x^i \setminus 0} \frac{\log |T_x \alpha_t^n v|}{n} = \chi^i(t),$$

In particular, $X(t) = (X^1(t), X^2(t), X^{d-k}(t), 0, 0, ..., 0) \in \mathbb{R}^d$ has as its components the Lyapunov exponents with multiplicity of the diffeomorphism α_t, and even though the system (μ, α_t) may fail to be ergodic its Lyapunov exponents are æ-constant. Note that the components need not be ordered, as is common in rank-1 dynamics; thus we have a function $\sigma_\bullet : \mathbb{R}^k \to \mathrm{Bij}(\overline{d-k})$ which is unique up to permutations of same entries such that

$$\forall t \in \mathbb{R}^k : X^{\sigma_t(1)}(t) \geq X^{\sigma_t(2)}(t) \geq \cdots \geq X^{\sigma_t(d-k)}(t).$$

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3.5 Lyapunov Norms and Metrics; Measurable Trivializations

Let us now discuss the Lyapunov norm and metric constructions. Let $(\mu, \alpha) \in \mathfrak{E}^s(\mathbb{R}^k \curvearrowright M)$ be a locally free ergodic system for some $s \in \mathbb{Z}_{\geq 1} \times [0,1]$. Let $\mathfrak{g}$ be a C^0 Riemannian metric and $\mathfrak{q}$ be a C^0 fiberwise norm on M. The goal of the Lyapunov metric and norm is to adapt $\mathfrak{g}$ and $\mathfrak{q}$ to the action α so that however much hyperbolicity α has becomes immediately apparent. Let $\mathrm{Osel}(\alpha)$ be the set of Lyapunov-Perron regular points of α as in Thm.4. We first adapt $\mathfrak{g}$ and $\mathfrak{q}$ along a (fine) Lyapunov æ-subbundle and then patch adaptations along different Lyapunov æ-subbundles. Put for any $\chi \in \mathrm{LSpec}_x(\alpha)$, any $x \in \mathrm{Osel}(\alpha)$ and any $\varepsilon \in \mathbb{R}_{>0}$

$$\begin{aligned} \mathfrak{g}^{\chi}_{x,\varepsilon} &: L^{\chi}_x \otimes L^{\chi}_x \to \mathbb{R}, & (v,w) &\mapsto \int_{\mathbb{R}^k} \mathfrak{g}_{\alpha_t(x)}(T_x\alpha_t v, T_x\alpha_t w)\, e^{-2\varepsilon|t| - 2\chi_x(t)}\, d\,\mathrm{haar}_{\mathbb{R}^k}(t) \\ \mathfrak{q}^{\chi}_{x,\varepsilon} &: L^{\chi}_x \to \mathbb{R}_{\geq 0}, & v &\mapsto \int_{\mathbb{R}^k} \mathfrak{q}_{\alpha_t(x)}(T_x\alpha_t v)\, e^{-\varepsilon|t| - \chi_x(t)}\, d\,\mathrm{haar}_{\mathbb{R}^k}(t), \end{aligned}$$

where $\mathrm{haar}_{\mathbb{R}^k}$ is the Haar measure on $\mathbb{R}^k$ normalized so that $\mathrm{haar}_{\mathbb{R}^k}([0,1]^k) = 1$. Note that by Thm.4 and Cor.7 $\mathfrak{g}^{\chi}_{\varepsilon}$ and $\mathfrak{q}^{\chi}_{\varepsilon}$ are well defined on $\mathrm{Osel}(\alpha)$ and are measurable. Next define for $x \in \mathrm{Osel}(\alpha)$

$$\begin{aligned} \mathfrak{g}_{x,\varepsilon} &: T_xM \otimes T_xM \to \mathbb{R}, & (v,w) &\mapsto \sum_{\chi \in \mathrm{LSpec}_x(\alpha)} \mathfrak{g}^{\chi}_{x,\varepsilon}(v^{\chi}, w^{\chi}) \\ \mathfrak{q}_{x,\varepsilon} &: T_xM \to \mathbb{R}_{\geq 0}, & v &\mapsto \sum_{\chi \in \mathrm{LSpec}_x(\alpha)} \mathfrak{q}^{\chi}_{x,\varepsilon}(v^{\chi}), \end{aligned}$$

where v^{χ} is the component of $v \in T_xM$ along L^{χ}_x. Thus we obtain an æ-defined measurable Riemannian metric $\mathfrak{g}_{\varepsilon}$ and an æ-defined measurable fiberwise norm $\mathfrak{q}_{\varepsilon}$ on M; these are called the **ε-Lyapunov metric** and **ε-Lyapunov fiberwise norm** adapted to α.

Next we recall that any fiber bundle with base a compact C^∞ manifold can be measurably trivialized:

Lemma 2: Let $p : E \to M$ be a C^∞ fiber bundle with fiber model F. Then for any Borel probability measure μ on M, there is an $M_0 =_{\mu} M$ and a (not necessarily even æ-unique) $\underline{\overline{\mathrm{Mble}}}$-isomorphism $\tau : E|_{M_0} \xrightarrow{\cong} M_0 \times F$ of bundles over M_0 such that $\forall x \in M_0 : \tau_x \in \mathrm{Diff}^{\infty}(E_x; \{x\} \times F)$ with

$$\begin{tikzcd} E \arrow[rr, "\tau", "\cong"'] \arrow[dr, "p"'] & \arrow[d, phantom, "æ"] & M \times F \arrow[dl, "\mathrm{proj}_M"] \\ & M & \end{tikzcd}$$

Further, for any Borel probability measure μ on M, for any splitting $F = \prod_{i \in \bar{l}} F^i$ of the fiber into embedded C^∞ submanifolds and any measurable æ-splitting $E =_\mu \prod_{i \in \bar{l}} E^i$ with $\forall x \in_\mu M$: $\dim(F^i) = \dim(E^i_x)$ and $\forall i \in \bar{l}$: E^i_x an embedded C^∞ submanifold of E_x, there is an $M_0 =_\mu M$ and a (not necessarily æ-unique) $\underline{\overline{\mathrm{Mble}}}$-isomorphism $\tau : E|_{M_0} \xrightarrow{\cong} M_0 \times F$ of bundles over M_0 such that $\forall i \in \bar{l} : \overrightarrow{\tau}(E^i) = M \times F^i$ and $\forall x \in M_0 : \tau_x \in \mathrm{Diff}^\infty(E_x; \{x\} \times F)$ and $\tau^i_x = \tau_x|_{E^i_x} \in \mathrm{Diff}^\infty(E^i_x; \{x\} \times F^i)$ with

$$\begin{tikzcd}
E \arrow[rrrr, "\tau", "\cong"'] \arrow[dr] \arrow[ddrr, bend right, "p"'] & & & & M \times F \arrow[dl] \arrow[ddll, bend left, "\mathrm{proj}_M"] \\
& E^i \arrow[rr, "\tau^i", "\cong"'] \arrow[dr] & \arrow[d, phantom, "\text{æ}"] & M \times F^i \arrow[dl] & \\
& & M & &
\end{tikzcd}$$

⌟

Corollary 8: Let μ be a Borel probability measure on M, $TM =_\mu \bigoplus_{i \in \bar{l}} E^i$ be a measurable æ-splitting that is orthogonal w/r/t a measurable Riemannian metric on M. Then there is a $\underline{\overline{\mathrm{Mble}}}$-isomorphism $\tau : TM \xrightarrow{\cong} M \times \mathbb{R}^{\dim(M)}$ of vector bundles such that

$$\forall x \in_\mu M, \forall i \in \bar{l} : \overrightarrow{\tau_x}(E^i_x) = \{x\} \times \left(\left(\bigoplus_{j<i} 0^{\dim(E^j_x)} \right) \oplus \mathbb{R}^{\dim(E^i_x)} \oplus \left(\bigoplus_{i<j} 0^{\dim(E^j_x)} \right) \right),$$

where the target is split w/r/t the standard basis of $\mathbb{R}^{\dim(M)}$, and for any $x \in_\mu M$, $\tau_x : T_x M \to \{x\} \times \mathbb{R}^{\dim(M)}$ is an isometry, where the target is endowed with the standard inner product space structure.

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3.6 æ-Foliations, Partitions, and Conditional Measures

In this section we briefly review conditional measures of a probability measure on a manifold along the leaves of an æ-foliation. The classical reference for this section is [Roh52]; for the adaptation of the abstract theory to the geometric setting we loosely follow [RS75] and [EL10]. Let X be a C^∞ manifold of dimension $d \in \mathbb{Z}_{\geq 1}$, and let $s \in \mathbb{Z}_{\geq 1} \times [0, 1]$.

Definition 10: A **partial L^0 foliation with C^s leaves** of X is a pair $(\mathrm{supp}(\mathcal{F}), \mathcal{F})$, where $\mathrm{supp}(\mathcal{F}) \subseteq X$ is a measurable subset, called the **support of** $\mathcal{F}$, into connected injectively immersed C^s submanifolds $\{\mathcal{F}_x \mid x \in \mathrm{supp}(\mathcal{F})\}$ possibly of varying dimensions such that for any $x \in \mathrm{supp}(\mathcal{F})$, there is a $\delta_x \in \mathbb{R}_{>0}$, a measurable subset $V \subseteq \mathbb{R}^{d - \dim(\mathcal{F}_x)}[0| \leq \delta_x]$ containing 0, a measurable subset $W \subseteq \mathrm{supp}(\mathcal{F})$ containing x, and an $\underline{\overline{\mathrm{Mble}}}$-isomorphism

$$\phi : \left(\mathbb{R}^{\dim(\mathcal{F}_x)}[0| \leq \delta_x] \times V, (0,0)\right) \to (W, x)$$

such that for any $b \in V$ [12]:

$$\phi(\bullet, b) \in \mathrm{Emb}^s\left(\mathbb{R}^{\dim(\mathcal{F}_x)}[0| \leq \delta_x]; X\right) \text{ and}$$
$$\overrightarrow{\phi}\left(\mathbb{R}^{\dim(\mathcal{F}_x)}[0| \leq \delta_x] \times \{b\}\right) = W \cap \mathcal{F}_{\phi(0,b)}.$$

If μ is a Borel probability measure on X and $(\mathrm{supp}(\mathcal{F}), \mathcal{F})$ is a partial L^0 foliation with C^s leaves such that $\mathrm{supp}(\mathcal{F}) =_\mu X$, then $(\mathrm{supp}(\mathcal{F}), \mathcal{F})$ is called an **æ-foliation with C^s leaves**. We'll systematically suppress the support of a partial L^0 foliation. Denote by æ$\mathrm{Fol}^s(X, \mu)$ the collection of all æ-foliations of X with C^s leaves.

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Remark 9: In the context of Def. 10, note that the "measurable partial flowboxes" $\phi : \left(\mathbb{R}^{\dim(\mathcal{F}_x)}[0| \leq \delta] \times V, (0,0)\right) \to (W, x)$ are superfluous, and as such an æ-foliation may fail to be a foliation of its support, even when this support inherits a manifold structure from X. Still we prefer to write them as æ-foliations are indeed adaptations of foliations to the theory of nonuniform hyperbolicity.

We also remark that the æ-foliations coming from dynamics, i.e. the Lyapunov, stable, and unstable æ-foliations (see Def. 17 and Rem. 16 below) have the extra property of admitting Hölder partial flowboxes on Lusin-Pesin sets (see Lem.7 below). This extra structure will not be used in this section.

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Definition 11: Let μ be a Borel probability measure on X and let $\mathcal{F}$ be an æ-foliation of (X, μ) with C^s leaves. Let L, R be two injectively immersed C^s submanifolds of X transverse to $\mathcal{F}$. Then a **local holonomy** from L to R along $\mathcal{F}$ is a measurable function $\mathcal{F}_{L \leftarrow R} : L' \to R$ defined on a measurable subset $L' \subseteq L$ with the property that

$$\forall l \in L' : \mathcal{F}_{L \leftarrow R}(l) \in \mathcal{F}_l \cap R.$$

A **holonomy** is a local holonomy with $L' = L$. If $\mathcal{G}$ is another æ-foliation with C^s leaves that is transverse to $\mathcal{F}$, then for any $x, y \in_\mu X$ with $\mathcal{G}_x \cap \mathcal{G}_y = \emptyset$ let us make the abbreviation $\mathcal{F}_{y \leftarrow x} = \mathcal{F}_{\mathcal{G}_y \leftarrow \mathcal{G}_x}$.

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Our next aim is to define, given an anonymous Borel probability measure μ on X and an æ-foliation $\mathcal{F}$ of X, conditional measures $\mu_\bullet^{\mathcal{F}}$. One can't directly define this disintegration in situations where it would be important (including the context of this dissertation), however by systematically coarsening an æ-foliation this can be achieved under certain circumstances (including the context of this dissertation). To describe this procedure we switch to a more abstract setting momentarily.

[12]For (X, d) a metric space, $x \in X$ and $r \in \mathbb{R}_{\geq 0}$, $X[x| \leq r], X[x| < r], X[x| = r]$ is the closed ball, open ball and sphere, respectively, centered at x of radius r.

Definition 12: A **standard measurable space** Y is a measurable space $\underline{\overline{\mathrm{Mble}}}$-isomorphic to a Borel measurable subset of a compact metric space. A **standard probability space** (Y,η) is a standard measurable space Y endowed with a probability measure η. An **æ -partition** $\mathcal{P}$ of a standard probability space (Y,η) is a collection of measurable subsets that are æ-disjoint and that æ-cover the whole space, that is, $\forall P_1, P_2 \in \mathcal{P} : P_1 \cap P_2 =_\eta \varnothing$, and $\bigcup_{P\in\mathcal{P}} P =_\eta Y$; syntactically we allow η-negligible subsets to be elements of partitions. Given an æ-partition $\mathcal{P}$ of (Y,η), denote by $\pi^{\mathcal{P}} : (Y,\eta) \to \left({}^{Y}\!/_{\mathcal{P}}, \overrightarrow{\pi^{\mathcal{P}}}(\eta)\right)$ the associated quotient map in $\underline{\overline{\mathrm{Meas}}}$. A **measurable æ-partition** is an æ-partition $\mathcal{P}$ such that $\left({}^{Y}\!/_{\mathcal{P}}, \overrightarrow{\pi^{\mathcal{P}}}(\eta)\right)$ is a standard probability space. Denote by æPar(Y,η) the filtered category of æ-partitions with with $\mathcal{P} \to \mathcal{Q}$ iff $\mathcal{P}$ æ-refines $\mathcal{Q}$ and by æmPar(Y,η) the full subcategory of measurable æ-partitions. We write $\mathcal{P} \sqcap \mathcal{Q}$ for the coarsest common refiner of $\mathcal{P}$ and $\mathcal{Q}$ and $\mathcal{P} \sqcup \mathcal{Q}$ for the finest common refinee of $\mathcal{P}$ and $\mathcal{Q}$[13]. $\mathbf{0}_\eta$ and $\mathbf{1}_\eta$ are the measurable æ-partition into points and the one with exactly one element, respectively. Thus we have a diagram in æmPar(Y,η) with the lower the term the coarser the partition:

$$\begin{tikzcd} & \mathbf{0}_\eta \arrow[d] & \\ & \mathcal{P}\sqcap\mathcal{Q} \arrow[dl] \arrow[dr] & \\ \mathcal{P} \arrow[dr] & & \mathcal{Q} \arrow[dl] \\ & \mathcal{P}\sqcup\mathcal{Q} \arrow[d] & \\ & \mathbf{1}_\eta & \end{tikzcd}$$

$\lrcorner$

Lemma 3: Let (Y,η) be a standard probability space. Then the following three categories are isomorphic:

(i) The slice category of standard probability spaces under (Y,η), where the arrows are factor maps.

(ii) The category æmPar(Y,η) of measurable æ-partitions of (Y,η).

(iii) The slice category $\mathrm{Sub}_{\underline{\overline{\mathrm{Boo}}}}(\mathcal{B}(Y,\eta))$ of sub-σ-algebras of the measure algebra of (Y,η), where the arrows are inclusions.

[13] In standard notation $\mathcal{P} \to \mathcal{Q}$, $\mathcal{P} \sqcap \mathcal{Q}$, $\mathcal{P} \sqcup \mathcal{Q}$ are denoted by $\mathcal{P} \succeq \mathcal{Q}$, $\mathcal{P} \vee \mathcal{Q}$, $\mathcal{P} \wedge \mathcal{Q}$, (or the other way around occasionally) respectively; we instead follow categorical conventions. Accordingly under the Galois correspondence between measurable æ-partitions and sub-σ-algebras of the measure algebra the symbols for binary operations and relations flip.

The isomorphism between the first two is by way of the fiber partition, and the isomorphism between the last two is by way of saturation. Under these isomorphisms the diagram in Def.12 transforms as follows:

$$\begin{tikzcd}[column sep=small]
 & (Y,\eta) \arrow[d, two heads] & \\
 & \left({}^{Y}\!/_{\mathcal{P}\sqcap\mathcal{Q}}, \overrightarrow{\pi^{\mathcal{P}\sqcap\mathcal{Q}}}(\eta)\right) \arrow[dl, two heads] \arrow[dr, two heads] & \\
\left({}^{Y}\!/_{\mathcal{P}}, \overrightarrow{\pi^{\mathcal{P}}}(\eta)\right) \arrow[dr, two heads] & & \left({}^{Y}\!/_{\mathcal{Q}}, \overrightarrow{\pi^{\mathcal{Q}}}(\eta)\right) \arrow[dl, two heads] \\
 & \left({}^{Y}\!/_{\mathcal{P}\sqcup\mathcal{Q}}, \overrightarrow{\pi^{\mathcal{P}\sqcup\mathcal{Q}}}(\eta)\right) \arrow[d, two heads] & \\
 & (\mathrm{pt}, \delta_{\mathrm{pt}}) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbf{0}_\eta \arrow[d] & \\
 & \mathcal{P}\sqcap\mathcal{Q} \arrow[dl] \arrow[dr] & \\
\mathcal{P} \arrow[dr] & & \mathcal{Q} \arrow[dl] \\
 & \mathcal{P}\sqcup\mathcal{Q} \arrow[d] & \\
 & \mathbf{1}_\eta &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathcal{B}(Y,\eta) & \\
 & \mathcal{B}(Y,\eta;\mathcal{P}) \sqcup \mathcal{B}(Y,\eta;\mathcal{Q}) \arrow[u, tail] & \\
\mathcal{B}(Y,\eta;\mathcal{P}) \arrow[ur, tail] & & \mathcal{B}(Y,\eta;\mathcal{Q}) \arrow[ul, tail] \\
 & \mathcal{B}(Y,\eta;\mathcal{P}) \sqcap \mathcal{B}(Y,\eta;\mathcal{Q}) \arrow[ul, tail] \arrow[ur, tail] & \\
 & {}^{\{\emptyset, X\}}\!/_{\eta} \arrow[u, tail] &
\end{tikzcd}$$

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Definition 13: Let (Y,η) and (Z,ζ) be standard probability spaces and let $\pi : (Y,\eta) \to (Z,\zeta)$ be a measurable measure preserving map. Denote by $\mathrm{vRadon}(\pi) : \mathrm{vRadon}(Y,\eta) \to (Z,\zeta)$ the **vertical Radon measure bundle** of (Y,η) associated to π; by definition $\mathrm{vRadon}(\pi)_z = \mathrm{Radon}(\overleftarrow{\pi}(z))$ is the space of Radon measures with support a subset of the fiber $\overleftarrow{\pi}(z)$ at z for $z \in_\zeta Z$. A section $\sigma_\bullet$ of $\mathrm{vRadon}(\pi)$ is **measurable** if for any bounded measurable function $\phi \in L^0_b(Y;\mathbb{R})$, the function $Z \to \mathbb{R}, z \mapsto \int_Y \phi(y)\, d\sigma_z(y)$ is measurable. A **disintegration** (or **system of conditional measures**) of η w/r/t π is a measurable section $\sigma_\bullet$ of $\mathrm{vRadon}(\pi)$ such that

$$\begin{aligned}
&\forall \phi \in L^0_b(Y;\mathbb{R}), \forall \psi \in L^0_b(Z;\mathbb{R}) : \\
&\qquad \int_Y \psi \circ \pi(y)\, \phi(y)\, d\eta(y) = \int_Z \psi(z) \int_{\overleftarrow{\pi}(z)} \phi(y)\, d\sigma_z(y)\, d\zeta(z).
\end{aligned}$$

┘

Lemma 4: Let Y be a standard measurable space. Then for any probability measure η on Y and for any measurable æ-partition $\mathcal{P}$ of (Y,η), there is an æ-unique disintegration $\eta^{\mathcal{P}}_\bullet$ of η w/r/t $\pi^{\mathcal{P}}$. Explicitly this means:

$$\begin{aligned}
\forall \phi \in L^0_b(Y;\mathbb{R}) : \int_Y \phi(y)\, d\eta(y) &= \int_{{}^{Y}\!/_{\mathcal{P}}} \int_P \phi(y)\, d\eta^{\mathcal{P}}_P(y)\, d\,\overrightarrow{\pi^{\mathcal{P}}}(\eta)(P) \\
&= \int_Y \int_Y \phi(y_2)\, d\eta^{\mathcal{P}}_{y_1}(y_2)\, d\eta(y_1),
\end{aligned}$$

where in the last expression we made the abbreviation $\eta^{\mathcal{P}}_{y_1} = \eta^{\mathcal{P}}_{\mathcal{P}_{y_1}} = \eta^{\mathcal{P}}_{\pi^{\mathcal{P}}(y_1)}$.

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Definition 14: Let μ be a Borel probability measure on X, $\mathcal{F} \in \text{æFol}^s(X,\mu)$ be an æ-foliation and let $\mathcal{P} \in \text{æmPar}(X,\mu)$ be a measurable æ-partition. $\mathcal{P}$ is **subordinate** to $\mathcal{F}$ if $\mathcal{P} \to \mathcal{F}$ and for any $x \in_\mu X$, there is a neighborhood $N_x \subseteq \mathcal{F}_x$ of x in the intrinsic topology of $\mathcal{F}_x$ such that $x \in N_x \subseteq \mathcal{P}_x \subseteq \mathcal{F}_x$.

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Lemma 5: Let μ be a Borel probability measure on X and $\mathcal{F} \in \text{æFol}^s(X,\mu)$ be an æ-foliation. Let $\mathcal{P}, \mathcal{Q} \in \text{æmPar}(X,\mu)$ be two measurable æ-partitions. If both $\mathcal{P}$ and $\mathcal{Q}$ are subordinate to $\mathcal{F}$, then $\mathcal{P} \sqcap \mathcal{Q}$ is also a measurable æ-partition subordinate to $\mathcal{F}$ and there is an æ-unique measurable $c_\bullet(\mathcal{Q} \leftarrow \mathcal{P}) \in L^0(M; \mathbb{R}_{>0})$ such that

$$\forall x \in_\mu M : \mu_x^{\mathcal{Q}}\Big|_{\mathcal{B}(X,\mu;\mathcal{P}\sqcap\mathcal{Q})} = c_x(\mathcal{Q} \leftarrow \mathcal{P})\; \mu_x^{\mathcal{P}}\Big|_{\mathcal{B}(X,\mu;\mathcal{P}\sqcap\mathcal{Q})},$$

where $\mathcal{B}(X,\mu;\mathcal{P} \sqcap \mathcal{Q})$ is the sub-σ-algebra of the measure algebra of (X,μ) that corresponds to $\mathcal{P} \sqcap \mathcal{Q}$[14]. In words, the two disintegrations differ by a multiplicative constant that only depends on the fiber *and a priori also on the basepoint of the fiber*, when restricted to the common refinement of the two æ-partitions.

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Definition 15: Let μ be a Borel probability measure on X and $\mathcal{F} \in$ æ$\mathrm{Fol}^s(X,\mu)$ be an æ -foliation. Whenever there is a measurable æ-partition $\mathcal{P}$ of (X,μ) subordinate to $\mathcal{F}$, we define for $x \in_\mu X$, the **conditional measure** $\mu_x^{\mathcal{F}}$ to be $\mu_x^{\mathcal{P}}$. In light of Lem.5, this defines $\mu_\bullet^{\mathcal{F}}$ up to a multiplicative scalar that a priori may depend on the choice of the basepoint.

⌟

Remark 10: In Def.15 we are not claiming that in this generality one can always find a measurable æ-partition subordinate to an æ-foliation, although given any æ-partition $\mathcal{P}$ there is a finest measurable æ-partition $\mathrm{Hull}_\eta(\mathcal{P})$ that is a refinee of $\mathcal{P}$, called the **measurable hull** of $\mathcal{P}$ w/r/t η. $\mathrm{Hull}_\eta :$ æ$\mathrm{Par}(Y,\eta) \to$ æm$\mathrm{Par}(Y,\eta)$ is a functor that is identity on æm$\mathrm{Par}(Y,\eta)$ and is defined by

$$\begin{array}{ccc}
\text{æPar}(Y,\eta) & \xrightarrow{\text{saturation}} & \mathrm{Sub}_{\underline{\mathrm{Boo}}}(\mathcal{B}(Y)) \\
{\scriptstyle \mathrm{Hull}_\eta}\big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow{\scriptstyle \text{canonical}} \\
\text{æmPar}(Y,\eta) & \xrightarrow[\text{saturation}]{\cong} & \mathrm{Sub}_{\underline{\mathrm{Boo}}}(\mathcal{B}(Y,\eta))
\end{array}$$

Here the rightmost arrow is the functor that sends a sub-σ-algebra $\mathcal{B}$ of the σ-algebra of Y to the sub-σ-algebra of the measure algebra of (Y,η) that is the factor of $\mathcal{B}$ by the σ-ideal of η-negligible subsets of Y. In particular if G is a locally compact second countable group and $(\eta,\epsilon) \in \mathfrak{C}_{\underline{\mathrm{Mble}}}(G \curvearrowright Y)$ is a system in the measurable category $\underline{\mathrm{Mble}}$, for $\mathcal{O}(\epsilon)$ the orbit æ-partition we call $\mathrm{Erg}(\eta,\epsilon) = \mathrm{Hull}_\eta(\mathcal{O}(\epsilon))$ the **ergodic decomposition** of the system (η,ϵ); indeed the restriction of (η,ϵ) on each cell of $\mathrm{Erg}(\eta,\epsilon)$ is ergodic. Thus approximation from without works in broad generality whereas approximation from within requires extra smooth ergodic theoretical hypotheses; see Lem.8.

⌟

Remark 11: Also note that in Def.15 at times $\mu_\bullet^{\mathcal{F}}$ is called a system of leaf-wise measures instead of conditional measures to emphasize the difference in the natures of these two objects; we don't make this terminological distinction. Formally $\mu_\bullet^{\mathcal{P}}$ is a section of the vertical Radon measure bundle $\mathrm{vRadon}(\pi^{\mathcal{P}})$, whereas $\mu_\bullet^{\mathcal{F}}$ is a section of the projectivization

[14] Under the Galois correspondence mentioned in footnote 13, $\mathcal{B}(X,\mu;\mathcal{P} \sqcap \mathcal{Q}) \cong_\mu \mathcal{B}(X,\mu;\mathcal{P}) \sqcup \mathcal{B}(X,\mu;\mathcal{Q})$ is the smallest sub-σ-algebra of the measure algebra of (X,μ) that contains both $\mathcal{B}(X,\mu;\mathcal{P})$ and $\mathcal{B}(X,\mu;\mathcal{Q})$.

$\mathbb{P}\,\mathrm{vRadon}(\pi^{\mathcal{P}})$ of the vertical Radon measure bundle associated to some subordinated measurable æ-partition.

The fact that we define $\mu^{\mathcal{F}}_{\bullet}$ with a multiplicative scalar ambiguity and a subordinating measurable æ-partition ambiguity also means that the support of $\mu^{\mathcal{F}}_{x}$ is not a cell of any measurable æ-partition. If a sequence $\mathcal{P}_{\bullet}$ of subordinated measurable æ-partitions have cells that æ-cover the leaves of $\mathcal{F}$, then $\mu^{\mathcal{F}}_{\bullet}$ can be defined as the associated sheafy object. This procedure reduces the two ambiguities mentioned above to the former one, which ambiguity is there to stay (the cone of Haar measures on the additive group $\mathbb{R}$ is one dimensional). Functionally the phenomenon of **measure rigidity** is said to be observed when a local choice of a multiplicative scalar can be upgraded to a global choice of a scalar function that is æ-constant.

⌟

3.7 Pesin Theory

In this section we define Lusin-Pesin sets and state Pesin's Invariant Manifold Theorem, which ought to be considered as a nonlinear analog of the Oseledets Theorem. Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be a locally free and ergodic system with $\mathrm{LSpec}^*(\mu, \alpha) \neq \emptyset$. Let us fix a C^∞ Riemannian metric on M with intrinsic distance function d_M on M and comparison families $C_\bullet, K_\bullet : \mathbb{R}_{>0} \to L^0(\mathrm{Osel}(\alpha); \mathbb{R}_{>0})$ as in Rem.7.

Definition 16: For any α-invariant measurable subset $\Pi \subseteq \mathrm{Osel}(\alpha)$, and any $(\varepsilon, \Lambda) \in \mathbb{R}^2_{>0}$ let us define the **Pesin set** $\underline{\Pi}(\varepsilon, \Lambda)$ by

$$\underline{\Pi}(\varepsilon, \Lambda) = \Pi \cap \overleftarrow{C_\varepsilon}\,(]0, \Lambda]) \cap \overleftarrow{K_\varepsilon}\,([1/\Lambda, \infty[).$$

Next, let $\eta \in \,]0, \mu\,(\underline{\Pi}(\varepsilon, \Lambda))[$, and for each $n \in \mathbb{Z}_{\geq 0}$ let $\Phi_n : M \to Y_n$ be a measurable function with target a second countable topological space. We define as a **Lusin-Pesin set** $\Pi(\varepsilon, \Lambda, \eta, \Phi)$ any compact subset of $\underline{\Pi}(\varepsilon, \Lambda)$ with the properties that

- $\mu\,(\underline{\Pi}(\varepsilon, \Lambda)) - \eta < \mu(\Pi(\varepsilon, \Lambda, \eta, \Phi)) \leq \mu(\underline{\Pi}(\varepsilon, \Lambda))$,
- $\forall n \in \mathbb{Z}_{\geq 0} : \Phi_n|_{\Pi(\varepsilon,\Lambda,\eta,\Phi)} : \Pi(\varepsilon, \Lambda, \eta, \Phi) \to Y_n$ is continuous.

We further assume that for any $\eta_1, \eta_2 \in \,]0, \mu\,(\underline{\Pi}(\varepsilon, \Lambda))[$, if $\eta_1 < \eta_2$ then $\Pi(\varepsilon, \Lambda, \eta_1, \Phi) \supseteq \Pi(\varepsilon, \Lambda, \eta_2, \Phi)$ by replacing the Lusin-Pesin set with parameter η_1 with the union of the two sets if necessary. Let us also call any α-invariant measurable subset $\Pi \subseteq \mathrm{Osel}(\alpha)$ a **pre-Lusin-Pesin set**.

⌟

Observation 6: In the context of Def.16, for any pre-Lusin-Pesin set Π and for any $\varepsilon \in \mathbb{R}_{>0}$ we have that strict Lusin-Pesin sets $\underline{\Pi}(\varepsilon, \Lambda)$ increase Λ increases. Further we have:

$$\Pi = \bigcup_{\Lambda \in \mathbb{R}_{>0}} \underline{\Pi}(\varepsilon, \Lambda) =_\mu \bigcup_{\Lambda \in \mathbb{R}_{>0}} \; \bigcup_{\eta \in]0, \mu(\underline{\Pi}(\varepsilon,\Lambda))[} \Pi(\varepsilon, \Lambda, \eta, \Phi),$$

so that Lusin-Pesin sets æ-cover the underlying pre-Lusin-Pesin set and in particular μ-ae Lyapunov-Perron regular point is in some Lusin-Pesin set (with parameters depending on the point).

⌟

Remark 12: Note that a Pesin set as defined in Def. 16 is not necessarily compact nor α-invariant and a Lusin-Pesin set is compact but not necessarily α-invariant. As a general rule we suppress η and Φ and write $\Pi(\varepsilon, \Lambda) = \Pi(\varepsilon, \Lambda, \eta, \Phi)$; we will also not declare if and where Pesin sets would suffice. Contextually η is meant to be small enough and Φ is meant to contain all functions that are needed to be continuous; there are at most countably many such functions for the purposes of this dissertation.

An alternative approach to compact Lusin-Pesin sets is to take the closures of Pesin sets[15]; we prefer not to do this to stay inside the set $\mathrm{Osel}(\alpha)$ of Lyapunov-Perron regular points.

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Lemma 6 ([16])**:** Let $\Pi \subseteq \mathrm{Osel}(\alpha)$ be an α-invariant measurable subset. Then for any $(\varepsilon, \Lambda) \in \mathbb{R}^2_{>0}$ and for any $t \in \mathbb{R}^k[0| = 1]$[17] with $\mathrm{LSpec}(\mu, \alpha_t) \cap \mathbb{R}_{<0} \neq \emptyset$, we have that $S(\alpha_t) \in C^{(0,+)}\left(\mathrm{Gr}\left(T_{\Pi(\varepsilon,\Lambda)}M\right) \to \Pi(\varepsilon, \Lambda)\right)$, where the latter space is the space of continuous sections of the associated Grassmannian bundle that satisfy a local Hölder estimate with anonymous exponent.

⌟

Theorem 5 (Pesin[18])**:** Let $\Pi \subseteq \mathrm{Osel}(\alpha)$ be an α-invariant measurable subset. Then for any $t \in \mathbb{R}^k[0| = 1]$ with $\chi \in \mathrm{LSpec}(\mu, \alpha_t) \cap \mathbb{R}_{<0}$, $\varepsilon \in \mathbb{R}_{>0}$, and $x \in \Pi$ we have:

- there is a number $r_{\varepsilon,t}(x) \in \mathbb{R}_{>0}$,
- there is a measurable function $D_{\varepsilon,x,t} : \Pi \to \mathbb{R}_{>0}$, and
- an embedded C^r submanifold with unique C^r germ at x

$$\left(\mathbb{R}^{\dim S_x(\alpha_{\bullet t})}[0| < r_{\varepsilon,t}(x)], 0\right) \xrightarrow{\cong_{C^r}} (\mathcal{S}_{x,\mathrm{loc}}(\alpha_{\bullet t}), x) \hookrightarrow (M, x)$$

such that

- $T_x\mathcal{S}_{x,\mathrm{loc}}(\alpha_{\bullet t}) = S_x(\alpha_{\bullet t}) \geq L^{\chi}_x(\alpha)$,
- $\forall \tau \in \mathbb{R} : D_{\varepsilon,x,t}(\alpha_{\tau t}(x)) \leq D_{\varepsilon,x,t}(x)\, e^{10\varepsilon|\tau|}$,
- $\forall y \in \mathcal{S}_{x,\mathrm{loc}}(\alpha_t), \forall \tau \in \mathbb{R}_{\geq 0} : d_M(\alpha_{\tau t}(x), \alpha_{\tau t}(y)) \leq D_{\varepsilon,x,t}(x) e^{\chi(\tau t)+\varepsilon\tau} d_M(x, y)$.

---

[15] See e.g. [BP02, pp.47,81,101] for this approach.

[16] [BK87, pp.372-373, Cor.5.3], [BP01, pp.91-93, App.A].

[17] In accordance with footnote 12, $\mathbb{R}^k[0| = 1]$ denotes the unit sphere in $\mathbb{R}^k$ centered at 0.

[18] [Pes77a, pp.100-101, 9.3], [Pes76, p.1287, Thm.2.2.1], [LY85a, pp.516, Prop.2.2.1], [FHY83, p.195, Thm.16].

For such a $t \in \mathbb{R}^k$ we also put $\mathcal{U}_{x,\mathrm{loc}}(\alpha_{\bullet t}) = \mathcal{S}_{x,\mathrm{loc}}\left(\alpha_{\bullet(-t)}\right)$. Then $T_x\mathcal{U}_{x,\mathrm{loc}}(\alpha_{\bullet t}) = U_x(\alpha_{\bullet t}) \leq \bigoplus_{\substack{\rho\in\mathrm{LSpec}(\alpha)\\ \rho\neq\chi}} L_x^{\rho}(\alpha)$.

Furthermore,

- $r_{\varepsilon,t} : \Pi \to \mathbb{R}_{>0}$ is measurable,
- There is a constant $r^*_{\varepsilon,t} \in \mathbb{R}_{>0}$ such that for any $x \in \Pi$ and $\tau \in \mathbb{R}$ we have $r_{\varepsilon,t}(\alpha_{\tau t}(x)) \geq r^*_{\varepsilon,t}\, e^{-\varepsilon|\tau|}\, r_{\varepsilon,t}(x)$,
- $\mathcal{S}_{\bullet,\mathrm{loc}}(\alpha_{\bullet t}) : \Pi \to \bigsqcup_{n\in\mathbb{Z}_{\geq 0}} \mathrm{Emb}^r(\mathbb{R}^n; M)$ is measurable, where the target is endowed with the coproduct Borel σ-algebra.

⌟

Remark 13: In the context of Thm. 5, it is clear that if $t \in \mathbb{R}^k$ is not in K^χ, then one can apply the theorem to $\frac{t}{|t|}$ to construct the local stable and unstable manifolds. Alternatively one can observe that in this case t and $\frac{t}{|t|}$ define the same flow up to a constant time change (see Sec. 3.2), thus the local stable and unstable manifolds constructed for them are the same (possibly after one takes r and D smaller). With this understanding we will drop $\bullet$ from the notations. We will also conflate a local stable manifold and its Euclidean parameterization.

Finally one may use a C^2 Riemannian metric to construct the local stable and unstable manifolds, though this will alter the relationship between the regularity of the manifolds and the regularity of the action. In fact it is reasonable to expect that with minor modifications one ought to be able to construct local stable and unstable manifolds for a $C^{(1,1)}$ Riemannian metric; for a Riemannian metric (or a more general spray) less regular than $C^{(1,1)}$ a more substantial adaptation might be required; see [KSS14, SS18].

⌟

Definition 17: In the context of Thm. 5, $\mathcal{S}_{x,\mathrm{loc}}(\alpha_t)$ and $\mathcal{U}_{x,\mathrm{loc}}(\alpha_t)$ are called the **local stable** and **unstable manifolds** of $\alpha_{\bullet t} : \mathbb{R} \to \mathrm{Diff}^r(M)$ at x, respectively. In either case $r_{\varepsilon,t}(x)$ is referred to as the **size** of the local manifold of α_t at x.

Further, for any $x \in \Pi$ the **global stable** and **unstable manifolds** $\mathcal{S}_x(\alpha_t), \mathcal{U}_x(\alpha_t)$ of α_t at x are defined respectively by

$$\mathcal{S}_x(\alpha_t) = \bigcup_{\tau\in\mathbb{R}_{\geq 0}} \overleftarrow{\alpha_{\tau t}}\left(\mathcal{S}_{\alpha_{\tau t}(x),\mathrm{loc}}(\alpha_{\bullet t})\right),\ \mathcal{U}_x(\alpha_t) = \mathcal{S}_x\left(\alpha_{-t}\right).$$

⌟

Remark 14: The global stable and unstable manifolds of α_t at x defined in Def. 17 can be characterized as global stable and unstable sets of α_t at x of exponential rate:

$$\mathcal{S}_x(\alpha_t) = \bigcup_{\lambda\in\mathbb{R}_{>0}} \left\{ y \in M \,\middle|\, d_M(\alpha_{nt}(y), \alpha_{nt}(x)) = O_{n\to\infty}(e^{-\lambda n}) \right\}.$$

The global stable and unstable manifolds are injectively immersed C^r submanifolds parameterized by vector spaces of appropriate dimensions, and as submanifolds are independent of the auxilliary Riemannian metric, ε and $r_{\varepsilon,t}(x)$ and are uniquely defined.

Also note that since $\mathbb{R}^k$ is abelian, the time-t map α_t of the action α for any $t \in \mathbb{R}^k$ permutes the global stable manifolds of α_{t^*} for any $t^* \in \mathbb{R}^k$ with $\chi(t^*) < 0$ for some χ. Similarly the global unstable manifolds are permuted by the action.

⌟

Observation 7: In the context of Thm.5, for any $x, y \in \Pi$ we have $\mathcal{S}_x(\alpha_t) \cap \mathcal{S}_y(\alpha_t) \neq \emptyset \Rightarrow \mathcal{S}_x(\alpha_t) = \mathcal{S}_y(\alpha_t)$, so that the global stable manifolds partition Π.

⌟

Definition 18: In the context of Thm.5, and in light of Obs.7, for $\Pi = \mathrm{Osel}(\alpha)$ if $t \in \mathbb{R}^k$ and $\mathrm{LSpec}(\mu, \alpha_t) \cap \mathbb{R}_{<0} \neq \emptyset$ we call $\mathcal{S}(\alpha_t)$ the **stable æ-foliation** of M associated to the time-t map α_t of the action α. Similarly if $\mathrm{LSpec}(\mu, \alpha_t) \cap \mathbb{R}_{>0} \neq \emptyset$ we call $\mathcal{U}(\alpha_t) = \mathcal{S}(\alpha_{-t})$ the **unstable æ-foliation** of M associated to the time-t map α_t of the action α.

⌟

Lemma 7: In the context of Thm.5, let also $\Lambda \in \mathbb{R}_{>0}$. Then

- $\inf_{x \in \Pi(\varepsilon,\Lambda)} r_{\varepsilon,t}(x) > 0$,
- $\mathcal{S}_{\bullet,\mathrm{loc}}(\alpha_t) : \Pi(\varepsilon, \Lambda) \to \mathrm{Emb}^r\left(\mathbb{R}^{s(\varepsilon,\Lambda)}; M\right)$ is in $C^{(0,+)}$, where $s(\varepsilon, \Lambda) \in \mathbb{Z}_{\geq 1}$ is the common dimension of local stable manifolds on the Lusin-Pesin set $\Pi(\varepsilon, \Lambda)$.

⌟

Lemma 8 ([19])**:** In the context of Thm.5, for $\Pi = \mathrm{Osel}(\alpha)$ there is a measurable æ-partition $\mathcal{P} \in \text{æmPar}(M, \mu)$ subordinate to $\mathcal{S}(\alpha_t) \in \text{æFol}^r(M, \mu)$ such that

- $\mathcal{P} \to \overleftarrow{\alpha_t}(\mathcal{P})$,
- $\bigsqcap_{n \in \mathbb{Z}_{\geq 0}} \overleftarrow{\alpha_{-nt}}(\mathcal{P}) = \mathbf{0}_\mu$,
- $\bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \overleftarrow{\alpha_{nt}}(\mathcal{P}) = \mathrm{Hull}_\mu(\mathcal{S}(\alpha_t))$,

⌟

Thus if we have an essentially locally free ergodic system $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ on a compact C^∞ manifold, and $t \in \mathbb{R}^k$ is such that at each Lyapunov-Perron regular point $x \in \mathrm{Osel}(\alpha)$, α_t has at least one negative Lyapunov exponent, then $\mathcal{S}(\alpha_t)$ is an æ-partition of M into injectively immersed connected C^r submanifolds, possibly of variable dimension, that depend measurably on the basepoint, with the property that, w/r/t any C^∞ Riemannian metric, on each Lusin-Pesin set the dependency of the C^r germs on the basepoint is Hölder and the dimension of the submanifolds is constant. Further, μ can be disintegrated along leaves of $\mathcal{S}(\alpha_t)$ with the caveats discussed in Rem.11.

[19] [LS82, pp.210-211, Prop.3.1]

Chapter 4

On the Maximal Rank Positive Entropy Hypotheses

In this chapter we establish some consequences of the maximal rank positive entropy hypotheses of Thm. 1 in terms of Lyapunov geometry. Recall that for $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$, $\mathfrak{e}_{(\mu,\alpha)} : \mathbb{R}^k \to \mathbb{R}_{\geq 0}, t \mapsto \mathrm{ent}_\mu(\alpha_t)$ is the entropy gauge of the system (μ, α). Note that by an Abramov formula[1] the entropy gauge is absolutely homogeneous, i.e., $\forall \tau \in \mathbb{R}, \forall t \in \mathbb{R}^k : \mathfrak{e}_{(\mu,\alpha)}(\tau t) = |\tau|\, \mathfrak{e}_{(\mu,\alpha)}(t)$, and by the thesis work of Hu[2] it is subadditive, i.e. $\forall t_1, t_2 \in \mathbb{R}^k : \mathfrak{e}_{(\mu,\alpha)}(t_1 + t_2) \leq \mathfrak{e}_{(\mu,\alpha)}(t_1) + \mathfrak{e}_{(\mu,\alpha)}(t_2)$, so that $\mathfrak{e}_{(\mu,\alpha)}$ is a seminorm of $\mathbb{R}^k$. Thus we may think of the ergodic theory $\mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ as parameterizing a family of seminorms on $\mathbb{R}^k$.

Definition 19: The **Fried entropy map** on $\mathbb{R}^k$ is defined as:

$$\begin{aligned}\mathrm{Fried} : \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M) \to \mathbb{R}_{\geq 0},\ (\mu, \alpha) \mapsto & \frac{\mathrm{haar}_{\mathbb{R}^k}(\{t \in \mathbb{R}^k \mid |t|_{\ell^1} \leq 1\})}{\mathrm{haar}_{\mathbb{R}^k}(\{t \in \mathbb{R}^k \mid \mathfrak{e}_{(\mu,\alpha)}(t) \leq 1\})} \\ = & \frac{2^k/k!}{\mathrm{haar}_{\mathbb{R}^k}(\{t \in \mathbb{R}^k \mid \mathfrak{e}_{(\mu,\alpha)}(t) \leq 1\})},\end{aligned}$$

where $\mathrm{haar}_{\mathbb{R}^k}$ is the Haar measure on $\mathbb{R}^k$ normalized so that $\mathrm{haar}_{\mathbb{R}^k}([0,1]^k) = 1$. If the denominator is infinite we take Fried entropy to be zero.

⌟

We now prove several straightforward propositions regarding the maximal rank positive entropy hypotheses. In the literature one can find $\mathbb{Z}^k$ analogs of versions of these propositions[3]. The proofs are mostly linear algebraic.

Observation 8: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be locally free and ergodic. If $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) > 0$, then $\mathrm{Fried}(\mu, \alpha) > 0$ by the homogeneity of entropy.

[1] [Abr59, p.169, Thm.2]

[2] [Hu93a, p.75, Thm.B] (= [Hu93b, p.15, Thm.B]); subadditivity of the entropy gauge in the case of $\mu \ll \mathrm{leb}_M$ was observed earlier in [Fri83, p.113, Prop.3].

[3] See e.g. [KKRH11, p.368], [KKRH14, p.1209, Prop.3.1], [KRH16, p.136, Prop.A.].

┘

Proposition 1: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be locally free and ergodic. If $\mathrm{Fried}(\mu, \alpha) > 0$, then (μ, α) has at least $k+1$ distinct Lyapunov hyperplanes and they are **in general position**, i.e.

$$\forall I \subseteq \overline{k+1} : \dim\left(\bigcap_{i \in I} K^i\right) = \max\{k - \#(I), 0\}.$$

Consequently we also have $2k + 1 \leq \dim(M)$.

┘

Proof: First note that the vanishing of Fried entropy is equivalent to an infinite volume of time parameters with low entropy. Geometrically one can consider the boundary of the unit ball w/r/t $\mathfrak{e}_{(\mu,\alpha)}$; zero Fried entropy would correspond to affine codimension-1 cones parallel to Lyapunov hyperplanes that extend without bounds as parts of the boundary in the case of positive $\mathfrak{e}_{(\mu,\alpha)}$, or to codimension-0 cones in the case of zero $\mathfrak{e}_{(\mu,\alpha)}$.

Let us first see the case $k = 2$ for illustration purposes. If there are less than three hyperplanes, then either there is a quadrant or a halfplane of elements with low entropy, whence Fried entropy vanishes. Thus there are exactly three hyperplanes. If a pair of hyperplanes coincide and have the same orientation, or else the opposite orientation, then there is a half-infinite strip in a quadrant with low entropy; in either case Fried entropy vanishes. Thus the hyperplanes must be distinct, which is equivalent to them being in general position in dimension two. Further they must be oriented so as to see all signatures but $(+,+,+)$ and $(-,-,-)$ (see Cor.9 below for more on this).

Now consider the general case. If there are less than $k+1$ Lyapunov hyperplanes, then there is a 2^k-ant of elements with zero entropy, whence Fried entropy vanishes. Thus there are at least $k+1$ Lyapunov hyperplanes, say there are $\ell \in \mathbb{Z}_{\geq k+1}$ many of them. Say they are not in general position, and let $I \subseteq \overline{\ell}$ be a maximal collection of indices such that the corresponding hyperplanes $\{K^i \mid i \in I\}$ are not in general position. Putting $K^I = \bigcap_{i \in I} K^i$, this means that K^I is at least one dimensional and $\dim(K^I) \geq k + 1 - \#(I)$. Let $i \in \overline{\ell} \setminus I$. Then by the maximality assumption $K^j + K^I = \mathbb{R}^k$, so that by the dimension formula (sum of the dimensions of the intersection and sum is the sum of the dimensions), we have that $K^j \cap K^I \leq K^I$ is a hyperplane. Thus $\{K^j \cap K^I \mid j \in \overline{\ell} \setminus I\}$ is a collection of $\ell - \#(I) \geq k + 1 - \#(I)$ many hyperplanes of K^I which has at least $k + 1 - \#(I)$ dimensions. Then there is at the very least a 2^k-ant of elements with zero entropy, whence again Fried entropy vanishes. Thus there has to be at least $k+1$ Lyapunov hyperplanes and they have to be in general position.

┘

Proposition 2: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be locally free and ergodic. If (μ, α) has exactly $k+1$ distinct Lyapunov hyperplanes that are in general position, and there exists a $t^* \in \mathbb{R}^k$ such that $\mathfrak{e}_{(\mu,\alpha)}(t^*) > 0$, then $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) > 0$.

┘

Proof: Suppose there are exactly $k+1$ distinct Lyapunov hyperplanes and they are in general position, and let $t^* \in \mathbb{R}^k\setminus$ be such that $\mathfrak{e}_{(\mu,\alpha)}(t^*) > 0$. First we claim that for any chamber time $t \in \mathrm{Wall}^0(\mu,\alpha)$: $\mathfrak{e}_{(\mu,\alpha)}(t) > 0$. By the Ledrappier-Young entropy formula[4] (and Abramov formula), we have that for any $t \in \mathbb{R}^k$:

$$\mathfrak{e}_{(\mu,\alpha)}(t) > 0 \Leftrightarrow \forall x \in_\mu M : \mu_x^{\mathcal{S}(\alpha_t)} \text{ is nonatomic } \Leftrightarrow \forall x \in_\mu M : \mu_x^{\mathcal{U}(\alpha_t)} \text{ is nonatomic.}$$

Since $\mu_\bullet^{\mathcal{S}(\alpha_{t^*})}$ is nonatomic μ-æ, there is a $\chi \in \mathrm{LSpec}^*(\mu,\alpha)$ such that $L^\chi = S^{\mathcal{W}(\chi)} \not\leq U(\alpha_{t^*})$. Thus $U(\alpha_{t^*}) \leq U^{\mathcal{W}(\chi)} = \bigoplus_{\rho\neq\chi} L^\rho$. Since $\mu_\bullet^{\mathcal{U}(\alpha_{t^*})}$ is nonatomic μ-æ, so is $\mu_\bullet^{\mathcal{U}^{\mathcal{W}(\chi)}}$, whence $\forall t \in \mathcal{W}(\chi) : \mathfrak{e}_{(\mu,\alpha)}(t) > 0$, and $\mu_\bullet^{S^{\mathcal{W}(\chi)}}$ is nonatomic μ-æ. Next let $\rho \in \mathrm{LSpec}^*(\mu,\alpha)$ be such that $\rho \neq \chi$. Then $U^{\mathcal{W}(\rho)} = \bigoplus_{\zeta\neq\rho} L^\zeta \geq L^\chi = S^{\mathcal{W}(\chi)}$, so that $\mu_\bullet^{U^{\mathcal{W}(\rho)}}$ is nonatomic μ-æ, hence we have that for any chamber time t, $\mathfrak{e}_{(\mu,\alpha)}(t) > 0$.

To extend the statement to wall times, first note that by the assumption on Lyapunov hyperplanes and wall time of order i is an element of a $k-i$ dimensional intersection that is not an element of a $k-i+1$ dimensional intersection. In particular, $\mathrm{Wall}^k = \emptyset$ and $\mathrm{Wall}^{k+1} = \{0\}$.

Let $I \subseteq \overline{k+1}$ and put $K^I = \bigcap_{i\in I} K^i$. Let $t \in \mathrm{Wall}^I$ and suppose $\forall j \in \overline{k+1} \setminus I : \chi^j(t) < 0$, so that the signature at t is $(\underbrace{0,\cdots,0}_{\#(I)\text{ many}}, \underbrace{-,\cdots,-}_{k+1-\#(I)\text{ many}})$. We claim that there is a perturbation of t such that the signature becomes $(-,\cdots,-)$, hence giving a contradiction to all chamber times having positive entropy. First note that by the general position assumption for $\varepsilon \in \mathbb{R}_{>0}$ small enough any element in $\mathbb{R}^k[|t| < \varepsilon]$ would have a signature of the form $(\epsilon_1,\cdots,\epsilon_{\#(I)}, \underbrace{-,\cdots,-}_{k+1-\#(I)\text{ many}})$. Then by induction one can turn each 0 into a $-$ one by one, as by the general position assumption $\{K^j \cap K^I | j \in \overline{k+1} \setminus I\}$ is a collection of hyperplanes in K^I that are in general position.

⌟

Corollary 9: Let $(\mu,\alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be essentially locally free and ergodic. If (μ,α) is an MRPES, then there are exactly $2^{k+1}-2$ Weyl chambers parameterized by signatures $(\epsilon_1,\epsilon_2,...,\epsilon_{k+1})$ for $\epsilon_i \in \{\pm\}$ except for $(+,+,\cdots,+)$ and $(-,-,\cdots,-)$. In particular there is a unique injective map

$$\mathcal{W} : \mathrm{LSpec}^*(\mu,\alpha) \to \mathrm{Cham}(\mu,\alpha)$$

such that

$$\forall\chi \in \mathrm{LSpec}^*(\mu,\alpha) : L^\chi = S^{\mathcal{W}(\chi)} = U^{-\mathcal{W}(\chi)}, \bigoplus_{\substack{\rho\in\mathrm{LSpec}^*(\mu,\alpha)\\ \rho\neq\chi}} L^\rho = U^{\mathcal{W}(\chi)} = S^{-\mathcal{W}(\chi)}.$$

⌟

[4] [LY85b, p.547, Thm.C]

Proof: We use induction. For $k = 2$ the statement is straightforward; the fact that we don't see $(+,+,+)$ or $(-,-,-)$ is due to the positive entropy assumption. Say the statement is true for $k = m$. If $K^1, K^2, ..., K^m, K^{m+1} \leq \mathbb{R}^m$ were in general position, then so would $K^1, K^2, ..., K^m$. Then these first m hyperplanes would separate $\mathbb{R}^m$ into 2^m chambers and all signatures $(\epsilon_1, \cdots, \epsilon_m)$ would be available (†). Again by the entropy assumption K^{m+1} must be positioned so as to not separate the chambers with signatures $(+, \cdots, +)$ and $(-, \cdots, -)$ and bisect all other chambers. This gives $(2^m - 2)2 + 2 = 2^{m+1} - 2$ chambers with signatures $(\epsilon_1, \cdots, \epsilon_m, +), (\epsilon_1, \cdots, \epsilon_m, -), (+, \cdots, +, -), (-, \cdots, -, +)$.

(†) also easily follows from induction. Indeed, the statement is clearly true for $k = 2$, and if the statement is true for $k = m$, then for $k = m + 1$, by the general position assumption $K^1 \cap \cdots \cap K^m$ is one dimensional and $K^1 \cap \cdots \cap K^m \cap K^{m+1}$ is zero dimensional. Thus $K^1, \cdots, K^m$ splits $\mathbb{R}^m$ into 2^m chambers and K^{m+1} bisects each one of these 2^m chambers.

⌟

Observation 9: As mentioned in Rem.2, we see that by Obs.8, Prop.1 and Prop.2 the third hypothesis of Thm.1 is redundant; in particular among locally free ergodic $\mathbb{R}^k$ systems on $2k+1$ dimensional manifolds for $k \in \mathbb{Z}_{\geq 2}$, being a maximal rank positive entropy system is a $\underline{\overline{\text{Meas}}}$-isomorphism invariant. Alternatively, one can reformulate the hypotheses of Thm. 1 to be more in line with the standard hypothesis scheme in measure rigidity: a locally free ergodic $\mathbb{R}^k$ system is an MRPES iff there are exactly $k+1$ distinct Lyapunov hyperplanes that are in general position and at least one time-t map of the system has positive entropy.

⌟

The next lemma addresses suspension hereditarity and time change hereditarity for the MRPES hypotheses.

Lemma 9: Let N be a compact C^∞ manifold. If $(\nu, \beta) \in \mathfrak{E}^r(\mathbb{Z}^k \curvearrowright N)$ is a $\mathbb{Z}^k$ MRPES and $\kappa : \mathbb{R}^k \times (\mathbb{R}^k \otimes^\beta N) \to \mathbb{R}^k$ is a C^r time change of $\hbar^\beta$, then $\left((\mathrm{haar}_{\mathbb{T}^k} \otimes^\beta \nu)_\kappa, \hbar^\beta_\kappa\right)$ is an $\mathbb{R}^k$ MRPES.

⌟

Proof: By definition (ν, β) is a $\mathbb{Z}^k$ MRPES iff its suspension system (μ, α) is an $\mathbb{R}^k$ MRPES. Thus it suffices to show that being an MRPES is invariant under time changes. It's clear that a time change κ does not change the dimension. After footnote 5 it's also clear that ergodicity is preserved, and if $x \in M$ and $t \in \mathrm{Stab}_x(\alpha_\kappa)$, then $\kappa(t, x) \in \mathrm{Stab}_x(\alpha)$, and since $\kappa(\bullet, x)$ is a diffeomorphism and $\mathrm{Stab}_x(\alpha)$ is discrete, $\mathrm{Stab}_x(\alpha_\kappa)$ is discrete so that local freeness is also preserved. Finally by Lem.1,

$$\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) > 0 \Leftrightarrow \forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu_\kappa,\alpha_\kappa)}(t) > 0,$$

and after Obs.9 this is sufficient for the time change of the suspension $(\mu_\kappa, \alpha_\kappa)$ to be an $\mathbb{R}^k$ MRPES.

⌟

Remark 15: After Rem. 4, if in Lem. 9 we take κ to be a C^s æ-time change, then $\left((\mathrm{haar}_{\mathbb{T}^k} \otimes^\beta \nu)_\kappa, \hbar^\beta_\kappa\right)$ is still an $\mathbb{R}^k$ MRPES, although now possibly it's not a system of diffeomorphisms.

⌟

Remark 16: In the context of Cor.9, and in light of the comments at the end of Sec.3.7, we have that for any $\chi \in \mathrm{LSpec}^*(\mu, \alpha), \forall t \in \mathcal{W}(\chi) \neq \emptyset : L^\chi(\alpha) = S(\alpha_t)$, whence we may define the **(fine) Lyapunov æ-foliation** of α associated to χ by

$$\mathcal{L}^\chi(\alpha) = \mathcal{S}(\alpha_t).$$

We will also write $\mathcal{L}^\chi_{x,\mathrm{loc}}(\alpha) = \mathcal{S}_{x,\mathrm{loc}}(\alpha_t)$. Note that this notation is doubly ambiguous, as the size of any local stable manifold is suppressed, and the size of the local Lyapunov manifold is dependent on the time parameter t.

Thus in the context of Thm.1, we have, for each nonzero Lyapunov exponent χ of the system (μ, α), an α-invariant measurable æ-foliation $\mathcal{L}^\chi$ of M each of whose leaves is a 1-dimensional injectively immersed and connected C^r submanifold, and on any Lusin-Pesin set w/r/t any C^∞ Riemannian metric the C^r germs of leaves of $\mathcal{L}^\chi$ depend Hölder continuously. Further, for $x \in_\mu M$, there is a Radon measure μ^χ_x on $\mathcal{L}^\chi_x(\alpha)$ that is well defined up to a multiplicative scalar that a priori depends on the basepoint, even along the same leaf, such that

$$\forall \phi \in L^0_b(M; \mathbb{R}) : \int_M \phi(x)\, d\mu(x) = \int_M \int_{\mathcal{L}^\chi_x} \phi(y)\, d\mu^\chi_x(y)\, d\mu(x).$$

Here the righthand side is ambiguous; see Rem.11.

⌟

We make one final observation regarding the orbit type of an MRPES.

Lemma 10: Let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be ergodic. If $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) > 0$, then (μ, α) is essentially free, that is,

$$\forall x \in_\mu M : \alpha_\bullet(x) : \mathbb{R}^k \xrightarrow{\cong_{C^r}} \mathcal{O}_x.$$

⌟

Proof: Let us consider the collection $\mathfrak{C}(\mathbb{R}^k)$ of closed subgroups of $\mathbb{R}^k$ endowed with the Chabauty topology[5]. Then $\mathfrak{C}(\mathbb{R}^k)$ is a compact Polish space and we have

$$\mathrm{Stab}_\bullet(\alpha) : M \to \mathfrak{C}(\mathbb{R}^k).$$

Using the basis elements of the Chabauty topology it's straightforward to verify that $\mathrm{Stab}_\bullet(\alpha)$ is Borel measurable, hence by ergodicity it's constant μ-æ. If (μ, α) weren't essentially free then $\exists t^* \in \mathbb{R}^k \setminus 0, \forall x \in_\mu M : t^* \in \mathrm{Stab}_x(\alpha)$, i.e. $\alpha_{t^*} = \mathrm{id}_M$, whence $\mathfrak{e}_{(\mu,\alpha)}(t^*) = 0$, a contradiction.

⌟

[5] See [dlH08].

Chapter 5

Affine Structures for Leaves of Invariant æ-Foliations

From now on until the end of the dissertation let $(\mu, \alpha) \in \mathfrak{E}^r(\mathbb{R}^k \curvearrowright M)$ be an MRPES as in Thm. 1. Recall that therefore by [KKRH11, p.363, Main Thm.(2)] we have that $\mu \ll \mathrm{leb}_M$; by Cor. 9 and Rem. 16, all nonorbital Lyapunov exponents are simple and non-zero, no two distinct nonorbital Lyapunov exponents are neither positively nor negatively proportional, and there is a unique map

$$\mathcal{W} : \mathrm{LSpec}^*(\mu, \alpha) \to \mathrm{Cham}(\mu, \alpha),\ \chi^i \mapsto \mathcal{W}(\chi^i) = \mathcal{W}^i$$

with the property that $\forall i \in \bar{l}, \forall t \in \mathcal{W}^i : L^i(\alpha) = S^{\mathcal{W}^i}(\alpha) = S(\alpha_t)$. We also fix a C^∞ Riemannian metric $\mathfrak{g}$ on M with comparison families $C_\bullet, K_\bullet : \mathbb{R}_{>0} \to L^0(\mathrm{Osel}(\alpha); \mathbb{R}_{>0})$ (see Cor. 7) and intrinsic distance function d_M on M. For an æ-foliation $\mathcal{L}$ of M denote by $d_{\mathcal{L}_x}$ the intrinsic distance function on the leaf $\mathcal{L}_x$ induced by the induced Riemannian metric $\mathfrak{g}_x^{\mathcal{L}}$ on $\mathcal{L}_x$.

5.1 Nonstationary Linearizations for Lyapunov æ-Foliations

In this section we'll define the nonstationary linearizations for Lyapunov æ-foliations and discuss its main properties. The main strategy is to use the fact that each leaf of each Lyapunov æ-foliation is one dimensional; whence an affinely chosen real number completely characterizes the position of any point on any leaf once a basepoint on that leaf is chosen. From a structural point of view this is also the first instance the $\pm$-ambiguity in the space crystal in Thm. 1 can be observed; for this reason the account below is written in invariant language and identifications are kept track of pedantically. We start with an adaptation of a well-known lemma to our case[1]:

[1] Compare e.g. [AS67, p.151], [Sin72, p.34, Thm.4], [PS82, pp.418-419, Prop.2], [dlLMM86, p.573], [dlL92, p.298, Cor.4.4], [dlL97, p.658, Lem.3.1], [KK07, pp.131-134], [dlLMM86, p.572, Lem.2.2], [dlL92, p.298, p.303], [dlL97, p.659], [Dol01, p.5].

Proposition 3: Let $\chi \in \mathrm{LSpec}^*(\mu, \alpha)$, $t \in \mathcal{W}(\chi)$, $\varepsilon \in \mathbb{R}_{>0}$ be such that $\varepsilon < -\chi(t)$. Define for any $x \in_\mu M$

$$\Delta_x^\chi = \Delta_{x,t}^\chi : (\mathcal{L}^\chi_{x,\mathrm{loc}}, x) \to (\mathbb{R}_{>0}, 1),\ z \mapsto \lim_{n\to\infty} \frac{\left\| T_z^\chi \alpha_{nt} \right\|}{\left\| T_x^\chi \alpha_{nt} \right\|} = \prod_{n \in \mathbb{Z}_{\geq 0}} \frac{\left\| T^\chi_{\alpha_{nt}(z)} \alpha_t \right\|}{\left\| T^\chi_{\alpha_{nt}(x)} \alpha_t \right\|},$$

where $T^\chi \alpha_s = T\alpha_s|_{L^\chi}$. Then for any $x \in_\mu M$, Δ_x^χ is well-defined. Moreover,

(i) $\forall x \in_\mu M : \Delta_x^\chi \in C^{(0,\theta)}(\mathcal{L}^\chi_{x,\mathrm{loc}}, x; \mathbb{R}_{>0}, 1)$.

(ii) $\forall x \in_\mu M, \forall z_1, z_2 \in \mathcal{L}^\chi_{x,\mathrm{loc}} : \dfrac{\Delta_x^\chi(z_1)}{\Delta_x^\chi(z_2)} = \Delta^\chi_{z_2}(z_1)$. In particular also $\Delta^\chi_{z_2}(z_1) = \dfrac{1}{\Delta^\chi_{z_1}(z_2)}$ and $\Delta^\chi_{z_2}(x)\Delta^\chi_x(z_1) = \Delta^\chi_{z_2}(z_1)$.

(iii) $\forall x \in_\mu M, \forall z \in \mathcal{L}^\chi_{x,\mathrm{loc}}, \forall s \in \mathbb{R}^k : \Delta^\chi_{\alpha_s(x)}(\alpha_s(z)) \left\| T_z^\chi \alpha_s \right\| = \left\| T_x^\chi \alpha_s \right\| \Delta_x^\chi(z)$.

(iv) $\Delta^\chi_\bullet(\bullet) : \{(x,z) \in M \times M \,|\, x \in \mathrm{Osel}(\alpha), z \in \mathcal{L}^\chi_{x,\mathrm{loc}}\} \to \mathbb{R}_{>0}$ is measurable and everywhere defined.

(v) On any Lusin-Pesin set the limit defining $\Delta^\chi_\bullet(\bullet)$ is uniform. More precisely, for any pre-Lusin-Pesin set Π, any $\Lambda \in \mathbb{R}_{>0}$, any $r_0 \in \mathbb{R}_{>0}$ that bounds from below the sizes of local stable manifolds of α_t on $\Pi(\varepsilon, \Lambda)$, and $\eta \in \mathbb{R}_{>0}$, there is an $N \in \mathbb{Z}_{\geq 1}$ such that for any $n \in \mathbb{Z}_{\geq N}$:

$$\sup_{x \in \Pi(\varepsilon,\Lambda)} \sup_{z \in \mathcal{L}^\chi_{x,\mathrm{loc}}[x| \leq r_0]} \left| \prod_{i \in \underline{n}} \frac{\left\| T^\chi_{\alpha_{it}(z)} \alpha_t \right\|}{\left\| T^\chi_{\alpha_{it}(x)} \alpha_t \right\|} - \Delta_x^\chi(z) \right| < \eta.$$

(vi) On any Lusin-Pesin set $\Delta^\chi_\bullet(\bullet)$ is uniformly continuous. More precisely, for any pre-Lusin-Pesin set Π, any $\Lambda \in \mathbb{R}_{>0}$, any $r_0 \in \mathbb{R}_{>0}$ that bounds from below the sizes of local stable manifolds of α_t, $\Delta^\chi_\bullet(\bullet) : \{(x,z) \in M \times M \,|\, x \in \Pi(\varepsilon, \Lambda), z \in \mathcal{L}^\chi_{x,\mathrm{loc}}[x| \leq r_0]\} \to \mathbb{R}_{>0}$ is uniformly continuous.

(vii) On any Lusin-Pesin set $\Delta^\chi_\bullet$ is uniformly continuous. More precisely, for any pre-Lusin-Pesin set Π, any $\Lambda \in \mathbb{R}_{>0}$, any $r_0 \in \mathbb{R}_{>0}$ that bounds from below the sizes of local stable manifolds of α_t, w/r/t the Euclidean coordinates on local manifolds we have that $\Delta^\chi_\bullet : \Pi(\varepsilon, \Lambda) \to C^{(0,\theta)}(\mathbb{R}[0| \leq r_0], 0; \mathbb{R}_{>0}, 1)$ is uniformly continuous.

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Proof: Since $t \in \mathcal{W}(\chi)$ we have $L^\chi(\alpha) = S(\alpha_t)$. Let us first see that Δ_x^χ is well-defined. If $z \in \mathcal{L}^\chi_{x,\mathrm{loc}}$ (recall Rem.16), then

$$
\begin{aligned}
&\left|\,\left\|T^{\chi}_{\alpha_{nt}(z)}\alpha_t\right\| - \left\|T^{\chi}_{\alpha_{nt}(x)}\alpha_t\right\|\,\right| \\
&\qquad\qquad = \left|\,\left\|T_{\alpha_{nt}(z)}\alpha_t : L^{\chi}_{\alpha_{nt}(z)} \to T_{\alpha_{(n+1)t}(z)}M\right\|\right. \\
&\qquad\qquad\qquad\qquad \left. - \left\|T_{\alpha_{nt}(x)}\alpha_t : L^{\chi}_{\alpha_{nt}(x)} \to T_{\alpha_{(n+1)t}(x)}M\right\|\,\right| \\
&\qquad\qquad \leq \left|\,\left\|T_{\alpha_{nt}(z)}\alpha_t : L^{\chi}_{\alpha_{nt}(z)} \to T_{\alpha_{(n+1)t}(z)}M\right\|\right. \\
&\qquad\qquad\qquad\qquad \left. - \left\|T_{\alpha_{nt}(x)}\alpha_t : \overrightarrow{\Pi_{\alpha_{nt}(x)\leftarrow\alpha_{nt}(z)}}(L^{\chi}_{\alpha_{nt}(z)}) \to T_{\alpha_{(n+1)t}(x)}M\right\|\,\right| \\
&\qquad\qquad + \left|\,\left\|T_{\alpha_{nt}(x)}\alpha_t : \overrightarrow{\Pi_{\alpha_{nt}(x)\leftarrow\alpha_{nt}(z)}}(L^{\chi}_{\alpha_{nt}(z)}) \to T_{\alpha_{(n+1)t}(x)}M\right\|\right. \\
&\qquad\qquad\qquad\qquad \left. - \left\|T_{\alpha_{nt}(x)}\alpha_t : L^{\chi}_{\alpha_{nt}(x)} \to T_{\alpha_{(n+1)t}(x)}M\right\|\,\right|,
\end{aligned}
$$

where $\Pi_{q\leftarrow p}$ is the parallel transport from T_pM to T_qM. In the final expression the first difference has a θ-Hölder modulus of continuity since α_t is C^r for $r = (1, \theta)$ and parallel transports are C^∞. Similarly the second difference has a θ-Hölder modulus of continuity since $L^{\chi}_{\alpha_{nt}(z)} = T_{\alpha_{nt}(z)}\mathcal{L}^{\chi}_{\alpha_{nt}(x),\text{loc}}$, $\mathcal{L}^{\chi}_{\alpha_{nt}(x),\text{loc}}$ is a C^r manifold by Thm. 5, α_{nt} is C^r and parallel transports are C^∞. Again by Thm. 5 we thus have

$$
\left|\,\left\|T^{\chi}_{\alpha_{nt}(z)}\alpha_t\right\| - \left\|T^{\chi}_{\alpha_{nt}(x)}\alpha_t\right\|\,\right| \lesssim_{t,x,\varepsilon} e^{(\chi(nt)+\varepsilon n)\theta} d_M(z,x)^{\theta}.
$$

Thus

$$
\left|\frac{\left\|T^{\chi}_{\alpha_{nt}(z)}\alpha_t\right\|}{\left\|T^{\chi}_{\alpha_{nt}(x)}\alpha_t\right\|} - 1\right| \lesssim_{t,x,\varepsilon} d_{C^1}(\text{id}_M, \alpha_t) e^{(\chi(nt)+\varepsilon n)\theta} d_M(z,x)^{\theta},
$$

and so

$$
\prod_{n\in\mathbb{Z}_{\geq 0}} \frac{\left\|T^{\chi}_{\alpha_{nt}(z)}\alpha_t\right\|}{\left\|T^{\chi}_{\alpha_{nt}(x)}\alpha_t\right\|} \leq \exp\left(\sum_{n\in\mathbb{Z}_{\geq 0}} \left|\frac{\left\|T^{\chi}_{\alpha_{nt}(z)}\alpha_t\right\|}{\left\|T^{\chi}_{\alpha_{nt}(x)}\alpha_t\right\|} - 1\right|\right) < \infty,
$$

so that Δ^{χ}_x is well-defined and as stated in the first item has a θ-Hölder modulus of continuity. The second and third items are syntactic (for item (ii) note that the above argument works for x replaced with any other point in $\mathcal{L}^{\chi}_x$ and item (iii) is valid germinally in that one may need to reduce the size of the local manifolds, e.g. if s and t is in the same chamber). The fourth item follows from the fact that local Lyapunov manifolds depend measurably on the basepoint by Thm. 5, as in our case we may choose $\Pi =_\mu M$, so that Δ^{χ} is defined as the limit of the restrictions, to a measurable subset, of functions $M \times M \to \mathbb{R}_{>0}$ that are $C^{(0,\theta)}$ in each component. Fifth and sixth items follow from the fact that on a Lusin-Pesin set with $r_0 \in \mathbb{R}_{>0}$ a lower bound on the local Lyapunov manifolds we have, for $z \in \mathcal{L}^{\chi}_{x,\text{loc}}[x| \leq r_0]$:

$$\left| \frac{\left\| T^{\chi}_{\alpha_{nt}(z)} \alpha_t \right\|}{\left\| T^{\chi}_{\alpha_{nt}(x)} \alpha_t \right\|} - 1 \right| \lesssim^{x,z}_{t,\varepsilon,\Lambda} d_{C^1}(\mathrm{id}_M, \alpha_t) e^{(\chi(nt)+\varepsilon n)\theta} r_0^{\theta}.$$

Here $\lesssim^B_A$ means that the inequality holds up to a multiplicative constant that may depend on A but does not depend on B. The seventh item follows from the previous two items (here we conflate an embedded submanifold with its parameterization; see Rem. 13).

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Remark 17: Note that in Prop.3 to define Δ^{χ}_x one could alternatively use the limit

$$z \mapsto \lim_{\tau \to \infty} \frac{\left\| T^{\chi}_z \alpha_{\tau t} \right\|}{\left\| T^{\chi}_x \alpha_{\tau t} \right\|} = \int_0^{\infty} \frac{\left\| T^{\chi}_{\alpha_{\tau t}(z)} \alpha_t \right\|}{\left\| T^{\chi}_{\alpha_{\tau t}(x)} \alpha_t \right\|} d\tau.$$

Also note that even though we suppressed the dependency of $\Delta^{\chi}_x = \Delta^{\chi}_{x,t}$ on t, it may very well depend on t beyond the ambiguities mentioned in Rem.16.

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Remark 18: In Prop.3 if instead of $r = (1, \theta)$, we had $r = (q, \theta)$ for $q \in \mathbb{Z}_{\geq 2}$, one could argue as follows to get the more specific $C^{(q-1,\theta)}$ regularity instead of $C^{(0,\theta)}$ in the first item: For the sake of readability we switch to Euclidean coordinates, suppress different spaces of sections involved and treat all objects simply as functions, and apply a soft Faà di Bruno argument. For $\mathcal{E}$ an anonymous collection of functions, which we'll call a collection of elementary functions, and $S, P, C \in \mathbb{Z}_{\geq 1}$, denote by $\sum_S \prod_P \bigcirc_C(\mathcal{E})$ the collection of all functions that can be written as the sum of not more than S summands, each of which can be written as the product[2] of not more than P factors, each of which can be written as the composition of not more than C elements in $\mathcal{E}$. For a collection $\mathcal{F}$ of functions and $s \in \mathbb{Z}_{\geq 0}$ let's also denote by $\mathcal{F}^{(s)}$ the collection of functions that are the s-th derivatives of functions in $\mathcal{F}$, and put $\mathcal{F}^{(\leq s)} = \bigcup_{k \in \underline{s+1}} \mathcal{F}^{(k)}$. Then a straightforward induction argument shows that

$$\left[\sum_S \prod_P \bigcirc_C (\mathcal{E}) \right]^{(s)} \subseteq \sum_{S \prod_{k \in \underline{s}} (P+kC)} \prod_{P+sC} \bigcirc_C \left(\mathcal{E}^{(\leq s)} \right) \subseteq \sum_{S(P+C)^s (s-1)!} \prod_{P+sC} \bigcirc_C \left(\mathcal{E}^{(\leq s)} \right).$$

In our context the collection $\mathcal{E}$ will correspond to $\left\{ T^{\chi}_{\alpha_{nt}(z)} \alpha_t \,\middle|\, z \in \mathcal{L}^{\chi}_{x,\mathrm{loc}}, n \in \mathbb{Z}_{\geq 0} \right\}$, so that e.g. $T^{\chi}_z \alpha_{nt} \in \sum_1 \prod_1 \bigcirc_n(\mathcal{E}) \cap \sum_1 \prod_n \bigcirc_1(\mathcal{E})$. As such, for the q-th derivatives it suffices to control $[\sum_1 \prod_n \bigcirc_1(\mathcal{E})]^{(q)}$. The strategy is to use the exponential decay of elements in $\mathcal{E}$ to offset the growth due to the sums of products of functions in $\mathcal{E}^{(\leq q)}$. Indeed, each summand of $[\sum_1 \prod_n \bigcirc_1(\mathcal{E})]^{(q)}$ has at most q factors with leading term a derivative of order more than 1, and there are $n + q$ factors of each summand, whence for $\theta' \in]0, \theta]$ we have

[2] Note that intrinsically "products" really are contractions of tensors.

the following uniform upper bound (which may depend on t, x, and ε) for differential objects of same type at x and z:

$$d_{C^q}(\mathrm{id}_M, \alpha_t)(n+1)^q(q-1)!\, e^{(\chi(nt)+\varepsilon n)\theta'} d_M(z,x)^{\theta'}.$$

$\lrcorner$

Proposition 4: In the context of Prop. 3, for any $x \in_\mu M$ there is a unique $\Lambda_x^\chi \in \mathrm{Diff}^r\left(L_x^\chi, 0; \mathcal{L}_x^\chi, x\right)$ such that

(i) $\forall x \in_\mu M : T_0^\chi \Lambda_x^\chi \cong \mathrm{id}_{L_x^\chi} : L_x^\chi \cong T_0 L_x^\chi \to T_x \mathcal{L}_x^\chi = L_x^\chi$,

(ii) $\forall x \in_\mu M, \forall v \in L_x^\chi, \forall s \in \mathbb{R}^k : \left|\Lambda_{\alpha_s(x)}^\chi \circ T_x^\chi \alpha_s(v)\right| = \left|\alpha_s \circ \Lambda_x^\chi(v)\right|$,

(iii) W/r/t Euclidean coordinates on global manifolds $\Lambda_\bullet^\chi : M \to \mathrm{Imm}^r(\mathbb{R}; M)$ is measurable,

(iv) On any Lusin-Pesin set $\Lambda_\bullet^\chi$ is uniformly continuous. More precisely, for any pre-Lusin-Pesin set Π and for any $\Lambda \in \mathbb{R}_{>0}$, w/r/t Euclidean coordinates on global manifolds we have that $\Lambda_\bullet^\chi : \Pi(\varepsilon, \Lambda) \to \mathrm{Imm}^r(\mathbb{R}; M)$ is uniformly continuous, where the target is endowed with the C^r compact-open topology,

(v) $\forall x \in_\mu M, \forall y \in \mathcal{L}_x^\chi : \left(\Lambda_y^\chi\right)^{-1} \circ \Lambda_x^\chi : L_x^\chi \to L_y^\chi$ is affine.

$\lrcorner$

Proof: Fix a $t \in \mathcal{W}(\chi)$ so that $L^\chi(\alpha) = S(\alpha_t)$, let $\varepsilon \in \mathbb{R}_{>0}$ be such that $\varepsilon < -\chi(t)$. Let us also pick a section $\xi \in L^0(L^\chi[0| = 1] \to M)$[3] to act as an orientation in such a way that the identification $L^\chi \cong_\mu M \times \mathbb{R}$, $c\xi_x \mapsto c$ preserves the orientation of each fiber. Note that ξ also naturally orients each leaf of $\mathcal{L}^\chi$. We'll first define a map $\Phi_{x,\mathrm{loc}} : \mathcal{L}_{x,\mathrm{loc}}^\chi \to L_x^\chi$ and then extend it to a map $\Phi_x : \mathcal{L}_x^\chi \to L_x^\chi$; Λ_x^χ will then be defined as the inverse of Φ_x. Define $\Phi_{x,\mathrm{loc}} = \Phi_{x,\mathrm{loc},\xi} : (\mathcal{L}_{x,\mathrm{loc}}^\chi, x) \to (L_x^\chi, 0)$ by

$$y \mapsto \left\{\begin{array}{ll} \left(\int_x^y \Delta_x^\chi(z)\,dz\right)\xi_x & \text{, if the segment of } \mathcal{L}_x^\chi \text{ from } x \text{ to } y \text{ is positively oriented} \\ -\left(\int_y^x \Delta_x^\chi(z)\,dz\right)\xi_x & \text{, if the segment of } \mathcal{L}_x^\chi \text{ from } x \text{ to } y \text{ is negatively oriented} \end{array}\right\},$$

where the integrant is the function established in Prop.3 and the integral is w/r/t the Riemannian metric on $\mathcal{L}_x^\chi$ induced by $\mathfrak{g}$, so that under the identification $L_x^\chi \cong \mathbb{R}$, $\Delta_x^\chi(y)$ is the derivative of $\Phi_{x,\mathrm{loc}}$ along $\mathcal{L}_{x,\mathrm{loc}}^\chi$ evaluated at y; in particular $\Phi_{x,\mathrm{loc}}(x) = 0 \in T_x M$, $T_x^\chi \Phi_{x,\mathrm{loc}} \cong \mathrm{id}_{L_x^\chi}$ ($\dagger_1$) and $\Phi_{x,\mathrm{loc}}$ is a C^r diffeomorphism onto its image:

$$\begin{tikzcd} (\mathcal{L}_{x,\mathrm{loc}}, x) \arrow[r, "\Phi_{x,\mathrm{loc}}"] \arrow[dr, "\int_x^\bullet \Delta_x(z)\,dz"'] & (L_x, 0) \arrow[d, "\cong"] \\ & (\mathbb{R}, 0) \end{tikzcd}$$

[3]In accordance with footnote 12, this means that ξ is an æ-vector field along L^χ with unit norm.

By item (iii) of Prop.3 using the Euclidean parameterizations for local Lyapunov manifolds we have for any $s \in \mathbb{R}^k$

$$\frac{d}{dy}\left[\int_{\alpha_s(x)}^{\alpha_s(y)} \Delta_{\alpha_s(x)}(z)\,dz\right] = \Delta_{\alpha_s(x)}(\alpha_s(y))\,\left\|T_y^\chi \alpha_s\right\|$$
$$= \left\|T_x^\chi \alpha_s\right\| \Delta_x^\chi(y) = \frac{d}{dy}\left[\left\|T_x^\chi \alpha_s\right\| \int_x^y \Delta_x(z)\,dz\right]$$

Taking the antiderivative of this equation w/r/t y and evaluating at x we obtain

$$|\Phi_{\alpha_s(x),\mathrm{loc}} \circ (\alpha_s(y))| = |T_x^\chi \alpha_s \circ (\Phi_{x,\mathrm{loc}}(y))| \qquad (\dagger_2),$$

that is, we have a diagram that commutes up to $\pm$:

$$\begin{tikzcd}
\mathcal{L}^\chi_{x,\mathrm{loc}} \arrow[r, "\alpha_s"] \arrow[d, "\Phi_{x,\mathrm{loc}}"'] \arrow[dr, phantom, "\pm"] & \mathcal{L}^\chi_{\alpha_s(x),\mathrm{loc}} \arrow[d, "\Phi_{\alpha_s(x),\mathrm{loc}}"] \\
L^\chi_x \arrow[r, "T^\chi_x \alpha_s"'] & L^\chi_{\alpha_s(x)}
\end{tikzcd}$$

Next we extend $\Phi_{x,\mathrm{loc}}$ to a map $\Phi_x : (\mathcal{L}_x^\chi, x) \to (L_x^\chi, 0)$. If $y \in \mathcal{L}_x^\chi$, then put $n_y = \inf\left\{n \in \mathbb{Z}_{\geq 0} \,\middle|\, \alpha_{nt}(y) \in \mathcal{L}^\chi_{\alpha_{nt}(x),\mathrm{loc}}\right\} \in \mathbb{Z}_{\geq 0}$. Note that as $\mathcal{L}^\chi = \mathcal{S}(\alpha_t)$, by Thm.5 even if the sizes of local manifolds may decrease $n_\bullet : \mathcal{L}_x^\chi \to \mathbb{Z}_{\geq 0}$ is a well-defined locally constant function. Thus we may define $\Phi_x(y) = \left(T_x^\chi \alpha_{n_y t}\right)^{-1} \circ \Phi_{\alpha_{n_y t(x),\mathrm{loc}}} \circ \alpha_{n_y t}(y)$. Further, by $(\dagger_2)$ we have:

$$\forall y \in \mathcal{L}_x^\chi, \forall n \in \mathbb{Z}_{\geq n_y} : \left(T_x^\chi \alpha_{nt}\right)^{-1} \circ \Phi_{\alpha_{nt(x),\mathrm{loc}}} \circ \alpha_{nt}(y) = \left(T_x^\chi \alpha_{n_y t}\right)^{-1} \circ \Phi_{\alpha_{n_y t(x),\mathrm{loc}}} \circ \alpha_{n_y t}(y),$$

so that the definition of Φ_x on an open ball in $\mathcal{L}_x$ restricted to a smaller ball in $\mathcal{L}_x$ coincides with the definition of Φ_x on said smaller ball; consequently $\Phi_x : (\mathcal{L}_x^\chi, x) \to (L_x^\chi, 0)$ is well-defined. In Euclidean coordinates Φ_x is strictly increasing, whence Φ_x is a diffeomorphism onto its image which is by definition L_x^χ (see Def.17). Thus we may put $\Lambda_x^\chi = (\Phi_x)^{-1} : (L_x^\chi, 0) \to (\mathcal{L}_x^\chi, x)$.

Let us now verify the declared properties of $\Lambda_\bullet^\chi$. Let $x \in_\mu M$. By item (i) of Prop.3 since $n_\bullet$ is locally constant $\Phi_x = \left(\Lambda_x^\chi\right)^{-1}$ is a C^r diffeomorphism, whence together with $(\dagger_1)$ above we have item (i) and $\Lambda_x^\chi \in \mathrm{Diff}^r(L_x^\chi, 0; \mathcal{L}_x^\chi, x)$. Item (ii) follows from $(\dagger_2)$ above. Item (iii) follows from Currying item (iv) of Prop.3 and using Euclidean coordinates for Lyapunov manifolds. Similarly item (iv) follows from item (vii) of Prop.3.

Let us next verify item (v). First let $y \in \mathcal{L}_{x,\mathrm{loc}}^\chi$ and $v \in L_x^\chi$ be such that $\Lambda_x^\chi(v) \in \mathcal{L}_{x,\mathrm{loc}}$. Then in Euclidean coordinates we have

$$\frac{d}{dv}\left(\Lambda_y^\chi\right)^{-1}\circ\Lambda_x^\chi(v)=\frac{\Delta_y^\chi(\Lambda_x^\chi(v))}{\Delta_x^\chi(\Lambda_x^\chi(v))}=\Delta_y^\chi(\Lambda_x^\chi(v))\Delta^\chi_{\Lambda_x^\chi(v)}(x)=\Delta_y^\chi(x).$$

Here in the first equality we used the fact that $y=\int_x^{\Lambda_x^\chi(y)}\Delta_x^\chi(z)\,dz$ and in the last two equalities we used item (ii) of Prop. 3. Consequently $\left(\Lambda_y^\chi\right)^{-1}\circ\Lambda_x^\chi$ has locally constant derivative. Next let $y\in\mathcal{L}_x^\chi$ and $v\in L_x^\chi$ be arbitrary. For any appropriately chosen $s\in\mathbb{R}^k$ we have that

$$\begin{aligned}\left|\left(\Lambda^\chi_{\alpha_s(y)}\right)^{-1}\circ\Lambda^\chi_{\alpha_s(x)}\circ T_x^\chi\alpha_s(v)\right|&=\left|\left(\Lambda^\chi_{\alpha_s(y)}\right)^{-1}\circ\alpha_s\circ\Lambda_x^\chi(v)\right|\\&=\left|T_y^\chi\alpha_s\circ\left(\Lambda_y^\chi\right)^{-1}\circ\Lambda^\chi_{\alpha_s(x)}(v)\right|.\end{aligned}$$

Here in the first equality we use item (ii) and in the second equality we use $(\dagger_2)$. As for the appropriate choice of s we need both y and $\Lambda_x^\chi(v)$ to be near x simultaneously, so one can take s to be any time of the form nt where $n\in\mathbb{Z}$ is such that

$$n\geq\max\left\{n_y,n_{\Lambda_x^\chi(v)}\right\}.$$

Finally let us verify uniqueness. Say for $x\in_\mu M$, we have a diffeomorphism $K_x\in\mathrm{Diff}^r(L_x^\chi,0;\mathcal{L}_x^\chi,x)$ satisfying the properties (i) to (v). We claim that $\forall x\in_\mu M:K_x=\Lambda_x^\chi$. Put $R_x=(K_x)^{-1}\circ\Lambda_x^\chi:L_x^\chi\to L_x^\chi$. As in the proof of item (v) once $R_x(v)=v$ is verified locally the equation can be extended to the whole Lyapunov subspace. Let $v\in L_x^\chi$ be such that $\Lambda_x^\chi(v)\in\mathcal{L}^\chi_{x,\mathrm{loc}}$. Then by item (ii) we have for any $s\in\mathbb{R}^k$:

$$\left|R_{\alpha_s(x)}\circ T_x^\chi\alpha_s(v)\right|=\left|T_x^\chi\alpha_s\circ R_x(v)\right|.$$

Then by the one-dimensionality of L_x^χ we have:

$$|R_x(v)|=\frac{\left|R_{\alpha_s(x)}\left(T_x^\chi\alpha_s(v)\right)\right|}{\left\|T_x^\chi\alpha_s\right\|}=\frac{\left|R_{\alpha_s(x)}\left(T_x^\chi\alpha_s(v)\right)\right|}{\left|R_x\left(T_x^\chi\alpha_s(v)\right)\right|}\,\frac{\left|R_x\left(T_x^\chi\alpha_s(v)\right)\right|}{\left|T_x^\chi\alpha_s(v)\right|}\,|v|.\qquad(\dagger_3)$$

We will chose s appropriately and take a limit. By Obs. 6, we have that for $x\in_\mu M$, there is a $\Lambda=\Lambda_x\in\mathbb{R}_{>0}$ large enough such that $x\in\Pi(\varepsilon,\Lambda)$ and that this Lusin-Pesin set has positive μ-measure, where ε is the same ε we fixed in the beginning of the proof. Further we may assume that x is a density point of $\Pi(\varepsilon,\Lambda)$ without hurting the æ -quantifiers. Then by Poincaré Recurrence Theorem there is a sequence $n_k\subseteq\mathbb{Z}_{\geq1}$ such that $\alpha_{n_kt}(x)\in\Pi(\varepsilon,\Lambda)$ and $\lim_{k\to\infty}\alpha_{n_kt}(x)=x$. Plugging in $s=n_kt$ in $(\dagger_3)$ and taking the limit $k\to\infty$ we obtain:

$$|R_x(v)|=\frac{\left|R_{\alpha_{n_kt}(x)}\left(T_x^\chi\alpha_{n_kt}(v)\right)\right|}{\left|R_x\left(T_x^\chi\alpha_{n_kt}(v)\right)\right|}\,\frac{\left|R_x\left(T_x^\chi\alpha_{n_kt}(v)\right)\right|}{\left|T_x^\chi\alpha_{n_kt}(v)\right|}\,|v|\xrightarrow{k\to\infty}|v|.$$

Here the first factor goes to 1 since $L^{\chi} = S(\alpha_t)$ (so that the arguments converge to 0 in Euclidean coordinates) and by item (iv) we have that $R_\bullet$ is uniformly continuous on the Lusin-Pesin set $\Pi(\varepsilon, \Lambda)$, and similarly the second factor goes to $|T_0 R_x|$ which is equal to 1 by item (i). Thus we have $|R_x(v)| = |v|$. Again item (i) implies that R_x is orientation preserving, whence $R_x(v) = v$, that is, $K_x(v) = \Lambda_x^{\chi}(v)$ for $|v|$ small. The extension to an arbitrary $v \in L_x^{\chi}$ is similar to the extension argument used in the proof of item (v); we omit writing it.

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Remark 19: In the context of Prop. 4 for the uniqueness part one does not need all the properties already proved in the proof. More specifically, for uniqueness we only need that the map whose uniqueness is under scrutiny is a C^r diffeomorphism $(L_x^{\chi}, 0) \to (\mathcal{L}_x^{\chi}, x)$ that satisfies item (i), item (ii) only with $s = t$, item (iii) and item (iv).

Another important point is that after Rem. 17, compared to the derivatives defined in Prop. 3, $\Lambda_\bullet^{\chi}$ does not depend on t, nor does it depend on the section ξ. Thus the family $\Lambda_\bullet^{\chi}$ is truly a global object attached to the system (μ, α).

Let us also note that the affine maps established in item (v) are orientation preserving only for y near x; for an arbitrary $y \in \mathcal{L}_x^{\chi}$ the compositions may reverse orientation.

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Definition 20: In the context of Prop. 4, $\Lambda_\bullet^{\chi}$ is called the **nonstationary linearization** of the Lyapunov æ-foliation $\mathcal{L}^{\chi}(\alpha)$ associated to the Lyapunov exponent $\chi \in \mathrm{LSpec}^*(\mu, \alpha)$.

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Consequently $\left\{ \left(\Lambda_y^{\chi}\right)^{-1} \,\middle|\, y \in \mathcal{L}_x^{\chi} \right\}$ is a C^r Aff($\mathbb{R}$)-manifold structure on $\mathcal{L}_x^{\chi}$ w/r/t which $\mathcal{L}_x^{\chi}$ is a globally-affine manifold. In particular each $\mathcal{L}_x^{\chi}$ is $\overline{\underline{\mathrm{Man}}}^{r}_{\mathrm{Aff}(\mathbb{R})}$-isomorphic to $\mathbb{R}$ with its standard affine manifold structure. By Rem. 3 we thus have $\forall x \in_\mu M$: $\mathrm{Diff}^r_{\mathrm{Aff}(\mathbb{R})}(\mathcal{L}_x^{\chi}) \cong \mathrm{Aff}(\mathbb{R})$. Let us finally note that as for $x \in_\mu M$ the conditional measure μ_x^{χ} is absolutely continuous w/r/t the Lebesgue class $\mathrm{leb}_{\mathcal{L}_x^{\chi}}$, we have that the affine manifold structures push $\mu_\bullet^{\chi}$ to a family of Haar measures:

Lemma 11 ([4])**:** For any $x \in_\mu M$: $\overrightarrow{\left(\Lambda_x^{\chi}\right)^{-1}} (\mu_x^{\chi})$ is a Haar measure on L_x^{χ}.

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5.2 Affine Structures for Stable, Unstable, Orbit-Stable and Orbit-Unstable æ-Foliations

In this section we assemble the nonstationary linearizations for Lyapunov æ-foliations to obtain affine structures for the stable, unstable, orbit-stable and orbit-unstable æ-foliations of any Weyl chamber and we'll discuss the main properties of such affine structures. Note that as a consequence of Cor. 9, we may write the stable æ-foliation of any

[4] [KK07, p.138, Lem.3.10], [KKRH11, p.392, Lem.7.4]

Weyl chamber as the intersection of codimension-$(k+1)$ stable æ-foliations. Indeed, let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then

$$\begin{aligned}
\bigcap_{(\chi,\mathcal{C})=+} S^{-\mathcal{W}(\chi)} &= \bigcap_{(\chi,\mathcal{C})=+} \bigoplus_{\rho \neq \chi} L^{\rho} \\
&= \bigcap_{(\chi,\mathcal{C})=+} \left[\left(\bigoplus_{\substack{\rho \neq \chi \\ (\rho,\mathcal{C})=+}} L^{\rho} \right) \oplus \left(\bigoplus_{\substack{\rho \neq \chi \\ (\rho,\mathcal{C})=-}} L^{\rho} \right) \right] \\
&= \bigoplus_{(\chi,\mathcal{C})=-} L^{\chi} = S^{\mathcal{C}},
\end{aligned}$$

and in particular we have

$$\forall \chi \in \mathrm{LSpec}^*(\mu, \alpha) : L^{\chi} = S^{\mathcal{W}(\chi)} = \bigcap_{(\rho,\mathcal{W}(\chi))=+} S^{-\mathcal{W}(\rho)} = \bigcap_{\rho \neq \chi} S^{-\mathcal{W}(\rho)}.$$

Now we establish affine structures for stable æ-foliations. We only give an outline of the proof as the normal form theory is already well-developed.

Corollary 10: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and $s(\mathcal{C}) = \mathrm{rank}(S^{\mathcal{C}}) \in \overline{k}$. Then for any $x \in_{\mu} M$ there is a unique a diffeomorphism $\Sigma_x^{\mathcal{C}} \in \mathrm{Diff}^r\left(S_x^{\mathcal{C}}, 0\,;\, \mathcal{S}_x^{\mathcal{C}}, x\right)$ such that

(i) $\forall x \in_{\mu} M, \forall \chi \in \mathrm{LSpec}^*(\mu, \alpha) : (\chi, \mathcal{C}) = - \implies \Sigma_x^{\mathcal{C}}\big|_{L_x^{\chi}} \in \mathrm{Diff}^r(L_x^{\chi}, 0; \mathcal{L}_x^{\chi}, x)$,

(ii) $\phi^{\Sigma^{\mathcal{C}}} : \mathbb{R}^k \times M \to \mathrm{DGL}\left(\mathbb{R}^{s(\mathcal{C})}\right), (t, x) \mapsto \left(\Sigma_{\alpha_t(x)}^{\mathcal{C}}\right)^{-1} \circ \alpha_t \circ \Sigma_x^{\mathcal{C}}$ defines a measurable cocycle over α taking values in the group of diagonal linear automorphisms of $\mathbb{R}^{s(\mathcal{C})}$ w/r/t the Euclidean coordinates on global stable manifolds,

(iii) W/r/t Euclidean coordinates on global manifolds $\Sigma_{\bullet}^{\mathcal{C}} : M \to \mathrm{Imm}^r(\mathbb{R}^{s(\mathcal{C})}; M)$ is measurable,

(iv) On any Lusin-Pesin set $\Sigma_{\bullet}^{\mathcal{C}}$ is uniformly continuous. More precisely, for any pre-Lusin-Pesin set Π and for any $\Lambda \in \mathbb{R}_{>0}$, w/r/t Euclidean coordinates on global manifolds we have that $\Sigma_{\bullet}^{\mathcal{C}} : \Pi(\varepsilon, \Lambda) \to \mathrm{Imm}^r(\mathbb{R}^{s(\mathcal{C})}; M)$ is uniformly continuous, where the target is endowed with the C^r compact-open topology,

(v) $\forall x \in_{\mu} M, \forall y \in \mathcal{S}_x^{\mathcal{C}} : \left(\Sigma_y^{\mathcal{C}}\right)^{-1} \circ \Sigma_x^{\mathcal{C}} : S_x^{\mathcal{C}} \to S_y^{\mathcal{C}}$ is affine with diagonal linear part.

Moreover analogous statements establish a family $\Upsilon_{\bullet}^{\mathcal{C}}$ for the unstable æ-foliation of $\mathcal{C}$.

⌟

Proof: Let $\chi^1, ..., \chi^s$ be the only Lyapunov exponents that are negative on $\mathcal{C}$. Note that by the rank-nullity theorem $\dim(\ker(\chi^1 - \chi^2)) = k - 1$ and by induction $\dim(\{t \in \mathbb{R}^k \,|\, \chi^1(t) = \chi^2(t) = \cdots = \chi^s(t)\}) = k - s + 1 > 0$ is a positive dimensional linear space. Consequently there is a $t \in \mathcal{C}$ such that $\chi^1(t) = \chi^2(t) = \cdots = \chi^s(t)$. Then the normal form theory presented in [KRH16, App. A] applies to the diffeomorphism α_t and guarantees all statements but uniqueness. Alternatively one can apply [KS17, pp. 344-345, Thm.2.3] to α_t which guarantees uniqueness as well. Let us also note that initially the arguments in [KRH16] gives item (v) for $y \in_{æ} \mathcal{S}^{\mathcal{C}}_x$; here we let the first $x \in_\mu M$ absorb the cases item (v) does not hold for every $y \in \mathcal{S}^{\mathcal{C}}_x$; this is valid since the conditionals $\mu^\chi_\bullet$ are absolutely continuous w/r/t the Lebesgue classes $\mathrm{leb}_{\mathcal{L}^\chi_\bullet}$.

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Remark 20: Le us also note that by comparing Cor.10 with Prop.4, for any $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and for any $\chi \in \mathrm{LSpec}^*(\mu, \alpha)$ with $(\chi, \mathcal{C}) = -$, we have $S^{\mathcal{C}}_x|_{L^\chi_x} = \Lambda^\chi_x$. Further, we have:

$$\forall x \in_\mu M, \forall t \in \mathbb{R}^k, \forall v \in S^{\mathcal{C}}_x : \phi^{\Sigma^{\mathcal{C}}}(t, x)(v) = \pm T_x\alpha_t|_{S^{\mathcal{C}}_x}(v),$$
$$\forall x \in_\mu M, \forall t \in \mathbb{R}^k, \forall v \in U^{\mathcal{C}}_x : \phi^{\Upsilon^{\mathcal{C}}}(t, x)(v) = \pm T_x\alpha_t|_{U^{\mathcal{C}}_x}(v).$$

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From Cor. 10 we also have the following assembly properties; they will come in very handy in establishing the independence of many maps of the choice of the chamber that is used to define them:

Corollary 11: (i) $\forall \mathcal{C}, \mathcal{D} \in \mathrm{Cham}(\mu, \alpha) : \mathcal{S}^{\mathcal{C}} \subseteq_\mu \mathcal{S}^{\mathcal{D}} \implies \Sigma^{\mathcal{D}}|_{S^{\mathcal{C}}} =_\mu \Sigma^{\mathcal{C}}$.

(ii) $\forall \mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cham}(\mu, \alpha) : S^{\mathcal{C}} =_\mu S^{\mathcal{D}} \oplus S^{\mathcal{E}} \implies \Sigma^{\mathcal{C}} =_\mu \Sigma^{\mathcal{D}} \times \Sigma^{\mathcal{E}}$.

(iii) $\forall \mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cham}(\mu, \alpha) : S^{\mathcal{C}} =_\mu S^{\mathcal{D}} \cap S^{\mathcal{E}} \implies \forall x, y \in_\mu M : z \in \mathcal{S}^{\mathcal{D}}_x \cap \mathcal{S}^{\mathcal{E}}_y \implies \mathcal{S}^{\mathcal{C}}_z \subseteq \mathcal{S}^{\mathcal{D}}_x \cap \mathcal{S}^{\mathcal{E}}_y$ and $\Sigma^{\mathcal{C}}_z = \Sigma^{\mathcal{D}}_x|_{S^{\mathcal{C}}_z} = \Sigma^{\mathcal{E}}_y\Big|_{S^{\mathcal{C}}_z}$.

Moreover there are analogous assembly properties of unstable æ-foliations $\mathcal{U}^{\mathcal{C}}$ w/r/t the family $\Upsilon^{\mathcal{C}}_\bullet$.

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As in the case of Lyapunov manifolds, we have that for any chamber $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$, and for $x \in_\mu M$, $\left\{ \left(\Sigma^{\mathcal{C}}_y\right)^{-1} \,\middle|\, y \in \mathcal{S}^{\mathcal{C}}_x \right\}$ is a C^r $\mathrm{DAff}(\mathbb{R}^{s(\mathcal{C})})$-manifold structure on $\mathcal{S}^{\mathcal{C}}_x$ w/r/t which $\mathcal{S}^{\mathcal{C}}_x$ is a globally-$\mathrm{DAff}(\mathbb{R}^{s(\mathcal{C})})$-affine manifold. The assembly properties Cor.11 guarantee that these affine manifold structures on different stable manifolds are compatible. There are similar statements for unstable manifolds of chambers.

Finally we extend the affine manifold structures defined on the stable and unstable manifolds to orbit-stable and orbit-unstable manifolds. Recall that by Lem. 10 (μ, α) is essentially free and hence each orbit carries a canonical affine manifold structure isomorphic to the Euclidean structure of $\mathbb{R}^k$.

Proposition 5: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and $s(\mathcal{C}) = \mathrm{rank}(S^{\mathcal{C}}) \in \overline{k}$. Then for any $x \in_\mu M$ there is a unique a diffeomorphism $\Gamma\Sigma^{\mathcal{C}}_x \in \mathrm{Diff}^r\left(\mathbb{R}^k \times S^{\mathcal{C}}_x, 0\,;\, \mathcal{O}\mathcal{S}^{\mathcal{C}}_x, x\right)$ such that

(i) $\forall x \in_\mu M : \Gamma\Sigma^{\mathcal{C}}_x|_{\mathbb{R}^k} = \alpha_\bullet(x) \in \mathrm{Diff}^r\left(\mathbb{R}^k; \mathcal{O}_x\right)$,

(ii) $\forall x \in_\mu M : \Gamma\Sigma^{\mathcal{C}}_x|_{S^{\mathcal{C}}_x} = \Sigma^{\mathcal{C}}_x \in \mathrm{Diff}^r(S^{\mathcal{C}}_x; \mathcal{S}^{\mathcal{C}}_x)$,

(iii) $\phi^{\Gamma\Sigma^{\mathcal{C}}} \;:\; \mathbb{R}^k \times M \;\to\; (\mathbb{R}^k \times \mathbb{R}^{s(\mathcal{C})}) \rtimes \left(\{I_k\} \times \mathrm{DGL}\left(\mathbb{R}^{s(\mathcal{C})}\right)\right) \;\leq\; \mathrm{DAff}(\mathbb{R}^k \times \mathbb{R}^{s(\mathcal{C})})$, $(t, x) \mapsto \left(\Gamma\Sigma^{\mathcal{C}}_{\alpha_t(x)}\right)^{-1} \circ \alpha_t \circ \Gamma\Sigma^{\mathcal{C}}_x$ defines a measurable cocycle over α,

(iv) W/r/t Euclidean coordinates on orbits and on global invariant manifolds $\Gamma\Sigma^{\mathcal{C}}_\bullet : M \to \mathrm{Imm}^r(\mathbb{R}^k \times \mathbb{R}^{s(\mathcal{C})}; M)$ is measurable,

(v) On any Lusin-Pesin set $\Gamma\Sigma^{\mathcal{C}}_\bullet$ is uniformly continuous. More precisely, for any pre-Lusin-Pesin set Π and for any $\Lambda \in \mathbb{R}_{>0}$, w/r/t Euclidean coordinates on orbits and global manifolds we have that $\Gamma\Sigma^{\mathcal{C}}_\bullet : \Pi(\varepsilon, \Lambda) \to \mathrm{Imm}^r(\mathbb{R}^k \times \mathbb{R}^{s(\mathcal{C})}; M)$ is uniformly continuous, where the target is endowed with the C^r compact-open topology,

(vi) $\forall x \in_\mu M, \forall y \in \mathcal{S}^{\mathcal{C}}_x : \left(\Gamma\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \Gamma\Sigma^{\mathcal{C}}_x : \mathbb{R}^k \times S^{\mathcal{C}}_x \to \mathbb{R}^k \times S^{\mathcal{C}}_y$ is affine with diagonal linear part and the linear part of the first component I_k.

Moreover analogous statements establish a family $\Gamma\Upsilon^{\mathcal{C}}_\bullet$ for the orbit-unstable æ-foliation of $\mathcal{C}$.

⌟

Proof: By definition we have $\mathcal{O}\mathcal{S}^{\mathcal{C}}_x = \{\alpha_t(y) \,|\, t \in \mathbb{R}^k, y \in \mathcal{S}^{\mathcal{C}}_x\}$, and since we have already established that $\mathcal{S}^{\mathcal{C}}_x$ is affinely isomorphic to $\mathbb{R}^{s(\mathcal{C})}$ this is a bijective description. Thus

$$\Gamma\Sigma^{\mathcal{C}}_x : (t, v) \mapsto \alpha_t \circ \Sigma^{\mathcal{C}}_x(v)$$

is a diffeomorphism from $\mathbb{R}^k \times \mathcal{S}^{\mathcal{C}}_x$ onto $\mathcal{O}\mathcal{S}^{\mathcal{C}}_x$. The first two items are syntactic and the remaining items as well as uniqueness follow from Prop.4 and Rem.19.

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Lem.11 together with the assembly properties Cor.11 gives:

Lemma 12: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then $\forall x \in_\mu M$: $\overrightarrow{\left(\Sigma^{\mathcal{C}}_x\right)^{-1}}\left(\mu^{S^{\mathcal{C}}}_x\right)$, $\overrightarrow{\left(\Upsilon^{\mathcal{C}}_x\right)^{-1}}\left(\mu^{U^{\mathcal{C}}}_x\right)$ are Haar measures on $S^{\mathcal{C}}_x$ and $U^{\mathcal{C}}_x$.

⌟

Let us observe in closing that by the second item in Prop.5 assembly properties analogous to those in Cor.11 are available after orbit saturation.

Chapter 6

Affine Holonomies along Invariant æ -Foliations

We had defined in Def.11 holonomies along an æ-foliation. In this chapter we establish the existence of global holonomies along the stable and unstable æ-foliations of any chamber using the affine structures established in chapter 5. Here the higher and maximal rank assumptions play a key role; note that in the rank-one case holonomies are available locally and on positive measure subsets of transversals. We start with an observation.

Observation 10: Let Y be a C^s complete globally affine manifold, $\mathcal{F}$ be a C^s foliation of Y with each leaf a C^s injectively immersed affine submanifold of Y, and let L, R be C^s injectively immersed complete affine submanifolds of Y transverse to $\mathcal{F}$. Then there are C^s global affine charts on Y such that w/r/t these charts both L and R are horizontal and each leaf of $\mathcal{F}$ is vertical. Furthermore substituting the first (that is, horizontal) component w/r/t these charts defines the holonomy $\mathcal{F}_{R\leftarrow L} : L \to R$ from L to R along the leaves of $\mathcal{F}$ that is a C^s affine diffeomorphism.

⌟

We construct our global holonomies piece by piece using the above observation with appropriately chosen invariant manifolds. We start with the simpler case of a chamber of the form $\mathcal{C} = \mathcal{W}(\chi)$ (see Cor.9).

Proposition 6: Let $\chi \in \mathrm{LSpec}^*(\mu, \alpha)$ and put $\mathcal{C} = \mathcal{W}(\chi)$. Then $\forall x \in_\mu M, \forall y \in_{æ} \mathcal{U}^{\mathcal{C}}_x$, there is a holonomy $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x} : \mathcal{OS}^{\mathcal{C}}_x \to \mathcal{OS}^{\mathcal{C}}_y$ along the unstable æ-foliation $\mathcal{U}^{\mathcal{C}}$ of $\mathcal{C}$ that is a C^r affine diffeomorphism with the property that

$$\forall t \in \mathbb{R}^k, \forall x \in_\mu M, \forall y \in_{æ} \mathcal{U}^{\mathcal{C}}_x : \mathcal{U}^{\mathcal{C}}_{\alpha_t(y)\leftarrow\alpha_t(x)}\Big|_{\mathcal{S}^{\mathcal{C}}_{\alpha_t(x)}} \circ \alpha_t = \alpha_t \circ \mathcal{U}^{\mathcal{C}}_{y\leftarrow x}\Big|_{\mathcal{S}^{\mathcal{C}}_x}.$$

Consequently, w/r/t affine coordinates the linear part of the first component of $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}$ is I_k.

⌟

Proof: First we define holonomies along the unstable æ-foliation between global stable manifolds and then we extend the definition to be between global orbit-stable manifolds. Note that as $\mathcal{C} = \mathcal{W}(\chi)$ and there are exactly $k + 1$ distinct Lyapunov exponents, there are at least two Lyapunov exponents of (μ, α) distinct from χ, so that there are two chambers $\mathcal{D}, \mathcal{E} \in \mathrm{Cham}(\mu, \alpha)$ such that $U^{\mathcal{C}} =_\mu U^{\mathcal{D}} \oplus U^{\mathcal{E}} =_\mu S^{-\mathcal{C}}$; thus we also have

$$\begin{aligned} S^{\mathcal{C}} &=_\mu S^{\mathcal{D}} \cap S^{\mathcal{E}} \\ S^{\mathcal{E}} &=_\mu S^{\mathcal{C}} \oplus U^{\mathcal{D}} \qquad (\dagger) \\ S^{\mathcal{D}} &=_\mu S^{\mathcal{C}} \oplus U^{\mathcal{E}} \end{aligned}$$

By Pesin theory, there is an $M_0 =_\mu M$ such that $\forall x \in M_0$, $\forall y \in M_0 \cap \mathcal{U}^{\mathcal{C}}_x$, there are $a, b \in \mathrm{Osel}(\alpha) \cap \mathcal{U}^{\mathcal{C}}_x$ such that

$$a \in \mathcal{U}^{\mathcal{D}}_x, b \in \mathcal{U}^{\mathcal{E}}_a, y \in \mathcal{U}^{\mathcal{D}}_b.$$

By the absolute continuity of μ we may rearrange the quantifiers for x and y to match the statement of the proposition. Note that indeed only one pivot point from x to y in $\mathcal{U}^{\mathcal{C}}_x$ may fail to be sufficient, but two pivot points are sufficient. We shall define the desired holonomy as the composition of three holonomies with the required properties. Since by the second equation in $(\dagger)$ we have $S^{\mathcal{C}}_x \oplus U^{\mathcal{D}}_x = U^{-\mathcal{E}}_x$, by Cor. 10 and Cor. 11, following Obs.10 we may define a C^r affine diffeomorphism $\mathcal{U}^{\mathcal{D}}_{a \leftarrow x} : \mathcal{S}^{\mathcal{C}}_x \to \mathcal{S}^{\mathcal{C}}_a$ by

$$\mathcal{U}^{\mathcal{D}}_{a \leftarrow x} = \Upsilon^{-\mathcal{E}}_x \circ \begin{pmatrix} \text{component} \\ \text{substitution} \end{pmatrix} \circ \left(\Upsilon^{-\mathcal{E}}_x \Big|_{S^{\mathcal{C}}_x} \right)^{-1}.$$

In particular for any $z \in \mathcal{S}^{\mathcal{C}}_x$ we have $\mathcal{U}^{\mathcal{D}}_{a \leftarrow x}(z) \in \mathcal{U}^{\mathcal{D}}_z \cap \mathcal{S}^{\mathcal{C}}_a \subseteq \mathcal{U}^{\mathcal{D}}_z \cap \mathcal{S}^{\mathcal{D}}_a$. Note that Pesin theory provides this only locally and for a positive $\mathrm{leb}_{\mathcal{S}^{\mathcal{C}}_x}$-measure set; the holonomy above is everywhere defined on the manifold $\mathcal{S}^{\mathcal{C}}_x$ by virtue of it being defined in terms of affine coordinates; further whenever a holonomy is provided by Pesin theory its value coincides with the value of the holonomy above. Similarly, by the third and second equations in $(\dagger)$, respectively, we may define C^r affine diffeomorphisms $\mathcal{U}^{\mathcal{E}}_{b \leftarrow a} : \mathcal{S}^{\mathcal{C}}_a \to \mathcal{S}^{\mathcal{C}}_b$ and $\mathcal{U}^{\mathcal{D}}_{y \leftarrow b} : \mathcal{S}^{\mathcal{C}}_b \to \mathcal{S}^{\mathcal{C}}_y$ by

$$\mathcal{U}^{\mathcal{E}}_{b \leftarrow a} = \Upsilon^{-\mathcal{D}}_a \circ \begin{pmatrix} \text{component} \\ \text{substitution} \end{pmatrix} \circ \left(\Upsilon^{-\mathcal{D}}_a \Big|_{S^{\mathcal{C}}_a} \right)^{-1}, \text{ and}$$
$$\mathcal{U}^{\mathcal{D}}_{y \leftarrow b} = \Upsilon^{-\mathcal{E}}_b \circ \begin{pmatrix} \text{component} \\ \text{substitution} \end{pmatrix} \circ \left(\Upsilon^{-\mathcal{E}}_b \Big|_{S^{\mathcal{C}}_b} \right)^{-1},$$

respectively. Thus defining

$$\mathcal{U}^{\mathcal{C}}_{y \leftarrow x} = \mathcal{U}^{\mathcal{D}}_{y \leftarrow b} \circ \mathcal{U}^{\mathcal{E}}_{b \leftarrow a} \circ \mathcal{U}^{\mathcal{D}}_{a \leftarrow x}$$

gives us a C^r affine diffeomorphism from $\mathcal{S}^{\mathcal{C}}_x$ onto $\mathcal{S}^{\mathcal{C}}_y$. Note that by construction for any $z \in \mathcal{S}^{\mathcal{C}}_x$ we have $\mathcal{U}^{\mathcal{C}}_{y \leftarrow x}(z) \in \mathcal{U}^{\mathcal{C}}_z \cap \mathcal{S}^{\mathcal{C}}_y$. Next let us verify the α-invariance. Our earlier choices of pivot points now translate into

$$\alpha_t(a) \in \mathcal{U}^{\mathcal{D}}_{\alpha_t(x)}, \alpha_t(b) \in \mathcal{U}^{\mathcal{E}}_{\alpha_t(a)}, \alpha_t(y) \in \mathcal{U}^{\mathcal{D}}_{\alpha_t(b)}.$$

The main property that facilitates α-invariance is the fact that the cocycles $\phi^{\bullet}$ attached to the stable and unstable affine parameters in item (ii) of Cor. 10 have no translational parts. Indeed we have

$$\mathrm{Y}^{-\mathcal{E}}_{\alpha_t(x)} = \alpha_t \circ \mathrm{Y}^{-\mathcal{E}}_x \circ \left(\phi^{\mathrm{Y}^{-\mathcal{E}}}(t,x)\right)^{-1},$$

and consequently

$$\begin{aligned}
\mathcal{U}^{\mathcal{D}}_{\alpha_t(a)\leftarrow\alpha_t(x)} &= \mathrm{Y}^{-\mathcal{E}}_{\alpha_t(x)} \circ \begin{pmatrix}\text{component}\\ \text{substitution}\end{pmatrix} \circ \left(\mathrm{Y}^{-\mathcal{E}}_{\alpha_t(x)}\Big|_{S^{\mathcal{C}}_{\alpha_t(x)}}\right)^{-1} \\
&= \alpha_t \circ \mathrm{Y}^{-\mathcal{E}}_x \\
&\qquad \circ \left[\left(\phi^{\mathrm{Y}^{-\mathcal{E}}}(t,x)\right)^{-1} \circ \begin{pmatrix}\text{component}\\ \text{substitution}\end{pmatrix} \circ \phi^{\mathrm{Y}^{-\mathcal{E}}}(t,x)\right] \\
&\qquad\qquad \circ \left(\mathrm{Y}^{-\mathcal{E}}_x\Big|_{S^{\mathcal{C}}_x}\right)^{-1} \circ \alpha_{-t}|_{\mathcal{S}^{\mathcal{C}}_{\alpha_t(x)}} \\
&= \alpha_t \circ \mathcal{U}^{\mathcal{D}}_{a\leftarrow x} \circ \alpha_{-t}|_{\mathcal{S}^{\mathcal{C}}_{\alpha_t(x)}}.
\end{aligned}$$

Applying this to the other components that make up $\mathcal{U}^{\mathcal{D}}_{\alpha_t(y)\leftarrow\alpha_t(x)}$ we obtain the invariance property. Finally we may extend

$$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x} = \Gamma\Sigma^{\mathcal{C}}_y \circ \left(I_k \times \left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{U}^{\mathcal{C}}_{y\leftarrow x} \circ \Sigma^{\mathcal{C}}_x\right)\right) \circ \left(\Gamma\Sigma^{\mathcal{C}}_x\right)^{-1} : \mathcal{OS}^{\mathcal{C}}_x \to \mathcal{OS}^{\mathcal{C}}_y.$$

By construction this is a C^r affine diffeomorphism such that w/r/t affine coordinates the linear part of the first component is I_k. Let us also verify that this is a holonomy between the orbit-stable manifolds of $\mathcal{C}$ along the unstable æ-foliation of $\mathcal{C}$. After the proof of Prop.5, let $z \in \mathcal{OS}^{\mathcal{C}}_x$ and let $(t,v) \in \mathbb{R}^k \times S^{\mathcal{C}}_x$ be the unique affine coordinates so that $\Gamma\Sigma^{\mathcal{C}}_x(t,v) = z$. Then

$$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}(z) = \Gamma\Sigma^{\mathcal{C}}_y\left(t, \left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{U}^{\mathcal{C}}_{y\leftarrow x} \circ \alpha_{-t}(z)\right) = \alpha_t \circ \underbrace{\mathcal{U}^{\mathcal{C}}_{y\leftarrow x} \circ \alpha_{-t}(z)}_{\in \mathcal{U}^{\mathcal{C}}_{\alpha_{-t}(z)} \cap \mathcal{S}^{\mathcal{C}}_y} \in \mathcal{U}^{\mathcal{C}}_z \cap \mathcal{OS}^{\mathcal{C}}_y.$$

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Remark 21: The affine holonomies we construct in Prop.6 are independent of the splitting chambers $\mathcal{D}$ and $\mathcal{E}$ by the assembly properties Cor. 11. They are also independent of the pivot points a, b, however we will establish this independence in Cor. 12 only after establishing the coherence of affine holonomies along stable and unstable æ-foliations in Prop. 10 below. Until then we will need to be careful about the choices of these pivot points.

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Next we establish global holonomies that are C^r affine diffeomorphisms for anonymous chambers. For this we will use induction. Since at this point the independence of the affine holonomy on the pivot points is not established we will update these pivot points in such a way that all affine holonomies that need to be predetermined before a step are determined appropriately.

Proposition 7: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then $\forall x \in_\mu M, \forall y \in_{æ} \mathcal{U}^{\mathcal{C}}_x$, there is a holonomy $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x} : \mathcal{OS}^{\mathcal{C}}_x \to \mathcal{OS}^{\mathcal{C}}_y$ along the unstable æ-foliation $\mathcal{U}^{\mathcal{C}}$ of $\mathcal{C}$ that is a C^r $\mathrm{DAff}(\mathbb{R}^k \times \mathbb{R}^{s(\mathcal{C})})$-affine diffeomorphism with the property that

$$\forall t \in \mathbb{R}^k, \forall x \in_\mu M, \forall y \in_{æ} \mathcal{U}^{\mathcal{C}}_x : \mathcal{U}^{\mathcal{C}}_{\alpha_t(y)\leftarrow\alpha_t(x)}\Big|_{\mathcal{S}^{\mathcal{C}}_{\alpha_t(x)}} \circ \alpha_t = \alpha_t \circ \mathcal{U}^{\mathcal{C}}_{y\leftarrow x}\Big|_{\mathcal{S}^{\mathcal{C}}_x}.$$

Consequently, w/r/t affine coordinates the linear part of the first component of $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}$ is I_k. Similarly there is a family $\mathcal{S}^{\mathcal{C}}_{\bullet\leftarrow\bullet}$ of holonomies along the stable æ-foliation $\mathcal{S}^{\mathcal{C}}$ of $\mathcal{C}$ that are C^r $\mathrm{DAff}(\mathbb{R}^k \times \mathbb{R}^{u(\mathcal{C})})$-affine diffeomorphisms.

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Proof: It suffices to establish only the affine holonomies along the unstable æ-foliations. We shall use induction on the rank $s(\mathcal{C})$ of the stable æ-folation of the chamber $\mathcal{C}$. Prop.6 covers the case $s(\mathcal{C}) = 1$. Suppose $s(\mathcal{C}) \geq 2$, and that for any lower rank the statement is true. Then there are two chambers $\mathcal{D}, \mathcal{E} \in \mathrm{Cham}(\mu, \alpha)$ such that $S^{\mathcal{C}} =_\mu S^{\mathcal{D}} \oplus S^{\mathcal{E}} = U^{-\mathcal{C}}$; consequently we also have

$$\begin{aligned} U^{\mathcal{C}} &=_\mu U^{\mathcal{D}} \cap U^{\mathcal{E}} \\ U^{\mathcal{E}} &=_\mu U^{\mathcal{C}} \oplus S^{\mathcal{D}} \qquad (\dagger) \\ U^{\mathcal{D}} &=_\mu U^{\mathcal{C}} \oplus S^{\mathcal{E}} \end{aligned}$$

Let $x \in_\mu M$ and $y \in_{æ} \mathcal{U}^{\mathcal{C}}_x$. On the one hand, by the second equation in $(\dagger)$, $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}\Big|_{\mathcal{S}^{\mathcal{D}}_x}$ is an affine holonomy given by component substitution w/r/t the $\mathrm{Y}^{\mathcal{E}}_x$ coordinates with image $\mathcal{S}^{\mathcal{D}}_y$. On the other hand, by induction hypothesis, since $s(\mathcal{D}) < s(\mathcal{C})$, there is an affine holonomy $\mathcal{U}^{\mathcal{D}}_{y\leftarrow x}$ that a priori depends on the choice of pivot points. By the first equation in $(\dagger)$, we may choose the pivot points involved in $\mathcal{U}^{\mathcal{D}}_{y\leftarrow x}$ so that

$$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}\Big|_{\mathcal{S}^{\mathcal{D}}_x} = \mathcal{U}^{\mathcal{D}}_{y\leftarrow x} : \mathcal{S}^{\mathcal{D}}_x \to \mathcal{S}^{\mathcal{D}}_y.$$

Let $z \in_{æ} \mathcal{S}^{\mathcal{C}}_x$. Then since $\mathcal{S}^{\mathcal{D}}$ and $\mathcal{S}^{\mathcal{E}}$ transversely sub-æ-foliate $\mathcal{S}^{\mathcal{C}}_x$, there is a unique $a = a_z \in \mathcal{S}^{\mathcal{D}}_x \cap \mathcal{S}^{\mathcal{E}}_z$. Put $b = b_z = \mathcal{U}^{\mathcal{C}}_{y\leftarrow x}\Big|_{\mathcal{S}^{\mathcal{D}}_x}(a)$. On the one hand $\mathcal{U}^{\mathcal{C}}_{b\leftarrow a}|_{\mathcal{S}^{\mathcal{E}}_a}$ is an affine holonomy given by component substitution w/r/t the $\mathrm{Y}^{\mathcal{D}}_a$ coordinates with image $\mathcal{S}^{\mathcal{E}}_b$ by the third equation in $(\dagger)$. On the other hand, by induction hypothesis there is an affine holonomy $\mathcal{U}^{\mathcal{E}}_{b\leftarrow a}$. Then by the first equation in $(\dagger)$, we may choose the pivot points involved in the affine holonomy provided by the induction hypothesis so that

$$\mathcal{U}^{\mathcal{C}}_{b\leftarrow a}\Big|_{\mathcal{S}^{\mathcal{E}}_a} = \mathcal{U}^{\mathcal{E}}_{b\leftarrow a} : \mathcal{S}^{\mathcal{E}}_a \to \mathcal{S}^{\mathcal{E}}_b.$$

Applying this holonomy to $z \in \mathcal{S}^{\mathcal{E}}_a$, we have that $\mathcal{U}^{\mathcal{C}}_{b\leftarrow a}|_{\mathcal{S}^{\mathcal{E}}_a}(z) = \mathcal{U}^{\mathcal{E}}_{b\leftarrow a}(z) \in \mathcal{U}^{\mathcal{C}}_z \cap \mathcal{S}^{\mathcal{E}}_b \subseteq \mathcal{U}^{\mathcal{E}}_z \cap \mathcal{S}^{\mathcal{E}}_b$; we may thus define $\mathcal{U}^{\mathcal{E}}_{y\leftarrow x}(z)$ to be this point. Finally we may extend $\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}$ to an affine holonomy between orbit-stables using the formula we used for the same purpose in the proof of Prop.7:

$$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x} = \Gamma\Sigma^{\mathcal{C}}_y \circ \left(I_k \times \left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{U}^{\mathcal{C}}_{y\leftarrow x} \circ \Sigma^{\mathcal{C}}_x\right)\right) \circ \left(\Gamma\Sigma^{\mathcal{C}}_x\right)^{-1} : \mathcal{OS}^{\mathcal{C}}_x \to \mathcal{OS}^{\mathcal{C}}_y.$$

The verification that this is indeed a holonomy along $\mathcal{U}^{\mathcal{C}}$ as well as the α-invariance property is straightforward and is as in the previous proof.

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Remark 22: Note that we stated the α-invariance properties of the affine holonomies in Prop. 6 and Prop. 7 for the restrictions of these holonomies to stable or unstable manifolds. That there is invariance in exactly this sense allows us to define affine holonomies between orbit-stable manifolds and orbit-unstable manifolds. Moreover when affine holonomies are not restricted to stable or unstable manifolds, the points x and $\alpha_t(x)$ determine the same orbit-stable and orbit-unstable manifolds, so we may indeed write $\mathcal{U}^{\mathcal{C}}_{\alpha_t(y)\leftarrow\alpha_t(x)} = \mathcal{U}^{\mathcal{C}}_{y\leftarrow x}$.

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Proposition 8: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then the affine holonomies along the unstable manifolds of $\mathcal{C}$ preserve the conditional measures of μ along the orbit-stable æ-foliation of $\mathcal{C}$. More precisely, there is a measurable $c_{\bullet\to\bullet} : \{(x,y) \in M \times M | x \in_\mu M, y \in_{æ} \mathcal{U}^{\mathcal{C}}_x\} \to \mathbb{R}_{>0}$ such that

$$\forall x \in_\mu M, \forall y \in_{æ} \mathcal{U}^{\mathcal{C}}_x, \overrightarrow{\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}}\,(\mu^{\mathcal{OS}^{\mathcal{C}}}_x) = c_{x\to y}\, \mu^{\mathcal{OS}^{\mathcal{C}}}_y.$$

There is a similar statement for the affine holonomies $\mathcal{S}^{\mathcal{C}}_{\bullet\leftarrow\bullet}$.

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Proof: By Lem. 12 the affine structures on stable and unstable manifolds transform the appropriate conditionals of μ to Haar measures, and affine holonomies are defined compositions of affine structures and component substitution. The proportionality function $c_{\bullet\to\bullet}$ is due to the fact that there is no natural way to normalize the stable and unstable manifolds for a general chamber (as opposed to a chamber of the form $\mathcal{C} = \mathcal{W}(\chi)$; see Prop.4); see also Lem.5 and Def.15.

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Let us next qualify the dependence of the families $\mathcal{S}^{\mathcal{C}}_{\bullet\leftarrow\bullet}$ and $\mathcal{U}^{\mathcal{C}}_{\bullet\leftarrow\bullet}$ on the base points:

Proposition 9: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then

(i) $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$ depends measurably on the basepoints. More precisely, w/r/t the Euclidean coordinates on orbit-stable manifolds of $\mathfrak{C}$, $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet} : \{(x,y) \in M \times M \,|\, x \in_\mu M, y \in_{æ} \mathcal{U}^{\mathfrak{C}}_x\} \to \mathrm{DAff}(\mathbb{R}^k \times \mathbb{R}^{s(\mathfrak{C})})$ is measurable.

(ii) On any Lusin-Pesin set $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$ is uniformly continuous. More precisely, w/r/t the Euclidean coordinates on orbit-stable manifolds of $\mathfrak{C}$, for any pre-Lusin-Pesin set Π, any $\Lambda \in \mathbb{R}_{>0}$, any $r_0 \in \mathbb{R}_{>0}$ that bounds from below the sizes of local unstable manifolds of $\mathfrak{C}$, $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet} : \{(x,y) \in M \times M \,|\, x \in \Pi(\varepsilon,\Lambda), y \in \mathcal{U}^{\mathfrak{C}}_{x,\mathrm{loc}}[x| \leq r_0]\} \to \mathrm{DAff}(\mathbb{R}^k \times \mathbb{R}^{s(\mathfrak{C})})$ is uniformly continuous.

There are similar statements for the family $\mathcal{S}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$.

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Proof: Note that any function involved in the definition of $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$ is either a component substitution, which is C^r since the affine manifold in question is C^r, or an affine parameter. Thus the statement follows from the third and fourth items in Cor.10.

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Finally we establish the coherence of the families $\mathcal{S}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$ and $\mathcal{U}^{\mathfrak{C}}_{\bullet\leftarrow\bullet}$ of affine holonomies along the stable and unstable æ-foliations for an anonymous chamber $\mathfrak{C}$. If the orbit saturations are not involved there are no obstructions to coherence, whereas in the presence of orbit saturations the only obstruction to coherence is matching of the temporal components:

Proposition 10: Let $\mathfrak{C} \in \mathrm{Cham}(\mu,\alpha)$. Then

(i) $\forall x \in_\mu M, \forall (y,z) \in_{æ} \mathcal{S}^{\mathfrak{C}}_x \times \mathcal{U}^{\mathfrak{C}}_x : \mathcal{U}^{\mathfrak{C}}_{z\leftarrow x}(y) = \mathcal{S}^{\mathfrak{C}}_{y\leftarrow x}(z)$.

(ii) More generally, $\forall x \in_\mu M$, $\forall (y,z) \in_{æ} \mathcal{OS}^{\mathfrak{C}}_x \times \mathcal{OU}^{\mathfrak{C}}_x$: $\mathrm{proj}_{\mathbb{R}^k} \circ \left(\Gamma\Sigma^{\mathfrak{C}}_x\right)^{-1}(y) = \mathrm{proj}_{\mathbb{R}^k} \circ \left(\Gamma\Upsilon^{\mathfrak{C}}_x\right)^{-1}(z) \Leftrightarrow \mathcal{U}^{\mathfrak{C}}_{z\leftarrow x}(y) = \mathcal{S}^{\mathfrak{C}}_{y\leftarrow x}(z)$.

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Proof: Let us first prove the first item. Let us write $y' = \mathcal{U}^{\mathfrak{C}}_{z\leftarrow x}(y)$ for the sake of brevity; our aim is to realize y' as the value of the affine holonomy $\mathcal{S}^{\mathfrak{C}}_{y\leftarrow x}$ (with appropriately chosen pivot points) at the point $z \in \mathcal{U}^{\mathfrak{C}}_x$. Since $s(\mathfrak{C}) + u(\mathfrak{C}) = k + 1 \geq 3$, by considering $-\mathfrak{C}$ if necessary, we may assume that $s(\mathfrak{C}) \geq 2$. Thus there are two chambers $\mathfrak{D}, \mathfrak{E} \in \mathrm{Cham}(\mu,\alpha)$ such that $S^{\mathfrak{C}} =_\mu S^{\mathfrak{D}} \oplus S^{\mathfrak{E}}$, and consequently also

$$\begin{aligned} U^{\mathfrak{C}} &=_\mu U^{\mathfrak{D}} \cap U^{\mathfrak{E}} \\ U^{\mathfrak{E}} &=_\mu U^{\mathfrak{C}} \oplus S^{\mathfrak{D}} \qquad (\dagger) \\ U^{\mathfrak{D}} &=_\mu U^{\mathfrak{C}} \oplus S^{\mathfrak{E}} \end{aligned}$$

By Pesin theory, there are $a, b \in \mathrm{Osel}(\alpha) \cap \mathcal{S}^{\mathfrak{C}}_x$ such that

$$a \in \mathcal{S}_x^{\mathfrak{D}}, b \in \mathcal{S}_a^{\mathfrak{E}}, y \in \mathcal{S}_b^{\mathfrak{D}},$$
$$a' = \mathcal{U}_{z\leftarrow x}^{\mathfrak{C}}(a) \in \mathrm{Osel}(\alpha) \cap \mathcal{S}_z^{\mathfrak{C}},$$
$$b' = \mathcal{U}_{a'\leftarrow a}^{\mathfrak{C}}(b) \in \mathrm{Osel}(\alpha) \cap \mathcal{S}_z^{\mathfrak{C}}.$$

By the second equation in (†), w/r/t $\mathrm{Y}_x^{\mathfrak{E}}$ coordinates, after Obs. 10, $\mathcal{U}_{z\leftarrow x}^{\mathfrak{C}}|_{\mathcal{S}_x^{\mathfrak{D}}}$ and $\mathcal{S}_{a\leftarrow x}^{\mathfrak{D}}|_{\mathcal{U}_x^{\mathfrak{C}}}$ are given by substitution of complementary components (that is, if the former is given by substitution of the horizontal component then the latter is given by substitution of the vertical component). Consequently the pivot points for $\mathcal{U}_{z\leftarrow x}^{\mathfrak{D}}$ and $\mathcal{S}_{a\leftarrow x}^{\mathfrak{C}}$ can be chosen so that

$$\begin{aligned} a' &= \mathcal{U}_{z\leftarrow x}^{\mathfrak{C}}\Big|_{\mathcal{S}_x^{\mathfrak{D}}}(a) = \mathcal{U}_{z\leftarrow x}^{\mathfrak{D}}(a) \\ &= \mathcal{S}_{a\leftarrow x}^{\mathfrak{D}}\Big|_{\mathcal{U}_x^{\mathfrak{C}}}(z) = \mathcal{S}_{a\leftarrow x}^{\mathfrak{C}}(z) \in \mathcal{S}_z^{\mathfrak{D}}. \end{aligned}$$

Next, by the third equation in (†), w/r/t $\mathrm{Y}_a^{\mathfrak{D}}$ coordinates, $\mathcal{U}_{a'\leftarrow a}^{\mathfrak{C}}|_{\mathcal{U}_a^{\mathfrak{C}}}$ and $\mathcal{S}_{b\leftarrow a}^{\mathfrak{E}}|_{\mathcal{U}_a^{\mathfrak{C}}}$ are given by substitution of complementary components. Consequently the pivot points for $\mathcal{U}_{a'\leftarrow a}^{\mathfrak{E}}$ and $\mathcal{S}_{b\leftarrow a}^{\mathfrak{C}}$ can be chosen so that

$$\begin{aligned} b' &= \mathcal{U}_{a'\leftarrow a}^{\mathfrak{C}}\Big|_{\mathcal{S}_a^{\mathfrak{E}}}(b) = \mathcal{U}_{a'\leftarrow a}^{\mathfrak{E}}(b) \\ &= \mathcal{S}_{b\leftarrow a}^{\mathfrak{E}}\Big|_{\mathcal{U}_a^{\mathfrak{C}}}(a') = \mathcal{S}_{b\leftarrow a}^{\mathfrak{C}}(a') \in \mathcal{S}_{a'}^{\mathfrak{E}}. \end{aligned}$$

Finally again by the second equation in (†), w/r/t $\mathrm{Y}_b^{\mathfrak{E}}$ coordinates, $\mathcal{U}_{b'\leftarrow b}^{\mathfrak{C}}|_{\mathcal{S}_b^{\mathfrak{D}}}$ and $\mathcal{S}_{y\leftarrow b}^{\mathfrak{D}}\Big|_{\mathcal{U}_b^{\mathfrak{C}}}$ are given by substitution of complementary components and consequently the pivot points for $\mathcal{U}_{b'\leftarrow b}^{\mathfrak{D}}$ and $\mathcal{S}_{y\leftarrow b}^{\mathfrak{C}}$ can be so chosen that

$$\begin{aligned} &\mathcal{U}_{b'\leftarrow b}^{\mathfrak{C}}\Big|_{\mathcal{S}_b^{\mathfrak{D}}}(y) = \mathcal{U}_{b'\leftarrow b}^{\mathfrak{D}}(y) \\ &= \mathcal{S}_{y\leftarrow b}^{\mathfrak{D}}\Big|_{\mathcal{U}_b^{\mathfrak{C}}}(b') = \mathcal{S}_{y\leftarrow b}^{\mathfrak{C}}(b') \in \mathcal{S}_{b'}^{\mathfrak{D}}. \end{aligned}$$

Since $y \in \mathcal{S}_b^{\mathfrak{D}}$, this last point also coincides with $y' = \mathcal{U}_{z\leftarrow x}^{\mathfrak{C}}(y)$ due to the first description of the point. Therefore

$$\begin{aligned} y' &= \mathcal{U}_{z\leftarrow x}^{\mathfrak{C}}(y) = \mathcal{S}_{y\leftarrow b}^{\mathfrak{C}}(b') = \mathcal{S}_{y\leftarrow b}^{\mathfrak{C}} \circ \mathcal{S}_{b\leftarrow a}^{\mathfrak{C}}(a') \\ &= \mathcal{S}_{y\leftarrow b}^{\mathfrak{C}} \circ \mathcal{S}_{b\leftarrow a}^{\mathfrak{C}} \circ \mathcal{S}_{a\leftarrow x}^{\mathfrak{C}}(z) = \mathcal{S}_{y\leftarrow x}^{\mathfrak{C}}(z), \end{aligned}$$

where in the last equality we have used the fact that all the pivot points for the affine holonomies that are being composed are compatible.

For the second item, for sufficiency we may simply apply the argument for the first item with $\alpha_t(x)$ instead of x, where $t = \mathrm{proj}_{\mathbb{R}^k} \circ \left(\Gamma\Sigma_x^{\mathcal{C}}\right)^{-1}(y) = \mathrm{proj}_{\mathbb{R}^k} \circ \left(\Gamma\Upsilon_x^{\mathcal{C}}\right)^{-1}(z)$. Necessity of this condition is straightforward.

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Corollary 12: In Prop.6, and consequently in Prop.7 and Prop.10, the C^r affine holonomies are independent of the pivot points.

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Proof: Let A, B be points alternative to a, b in the proof of Prop.6, respectively. Independence of affine holonomies on the pivot points means that

$$\mathcal{U}^{\mathcal{D}}_{y\leftarrow b} \circ \mathcal{U}^{\mathcal{E}}_{b\leftarrow a} \circ \mathcal{U}^{\mathcal{D}}_{a\leftarrow x} = \mathcal{U}^{\mathcal{D}}_{y\leftarrow B} \circ \mathcal{U}^{\mathcal{E}}_{B\leftarrow A} \circ \mathcal{U}^{\mathcal{D}}_{A\leftarrow x}, \quad (\star)$$

where each one of the composed holonomies is given by component substitution w/r/t some affine coordinate. By factoring holonomies

$$\mathcal{U}^{\mathcal{D}}_{A\leftarrow x} = \mathcal{U}^{\mathcal{D}}_{A\leftarrow a} \circ \mathcal{U}^{\mathcal{D}}_{a\leftarrow x}, \qquad \mathcal{U}^{\mathcal{D}}_{y\leftarrow b} = \mathcal{U}^{\mathcal{D}}_{y\leftarrow B} \circ \mathcal{U}^{\mathcal{D}}_{B\leftarrow b},$$

one sees that $(\star)$ is in turn equivalent to

$$\mathcal{U}^{\mathcal{E}}_{B\leftarrow A} \circ \mathcal{U}^{\mathcal{D}}_{A\leftarrow a} = \mathcal{U}^{\mathcal{D}}_{B\leftarrow b} \circ \mathcal{U}^{\mathcal{E}}_{b\leftarrow a}.$$

Since $A \in \mathcal{U}^{\mathcal{D}}_a = \mathcal{S}^{-\mathcal{D}}_a$ and $b \in \mathcal{U}^{\mathcal{E}}_a \subseteq \mathcal{U}^{-\mathcal{D}}_a$, applying Prop.10 with $\mathcal{C}$ replaced with $-\mathcal{D}$ guarantees this equality.

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Chapter 7

Measurable Covering Map and Diagonal Affine Extension

In this section we put together the affine structures as well as the affine holonomies we have established in the previous sections to obtain a diagonal affine extension of (μ, α). We shall use invariant language to describe this affine extension; so that its phase space is TM considered as an æ-bundle over M. Using measurable trivializations as discussed in Sec.3.5 we will also extract a version of this extension with the phase space $\mathbb{R}^{2k+1}$.

Remark 23: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Note that for $x \in_\mu$ we have $T_xM \cong \mathbb{R}^k \times \left(S_x^{\mathcal{C}} \oplus U_x^{\mathcal{C}}\right)$ as vector spaces. Let us denote for $v \in T_xM$ the corresponding components by $v = (v^o, v^s, v^u)$. Here and in what follows we use a direct product between the orbit and normal to the orbit directions instead direct sums (as opposed to Thm. 4); we will indeed recover a certain solvable group structure on T_xM where the twist involved in the semidirect product occurs exactly between the orbit directions $O_x \cong \mathbb{R}^k$ and the directions $S_x^{\mathcal{C}} \oplus U_x^{\mathcal{C}}$ normal to the orbit (see Prop. 16 below). More precisely we want a *split* short exact sequence (in $\underline{\overline{\mathrm{Lie}}}$)

$$\begin{tikzcd} S_x^{\mathcal{C}} \oplus U_x^{\mathcal{C}} \arrow[r, tail] & T_xM \arrow[r, two heads] & \mathbb{R}^k \\ & \mathbb{R}^k \arrow[u] \arrow[ur, "\mathrm{id}_{\mathbb{R}^k}"'] & \end{tikzcd}$$

where the action $\mathbb{R}^k \curvearrowright S_x^{\mathcal{C}} \oplus U_x^{\mathcal{C}}$ is *not* trivial.

$\lrcorner$

7.1 Two Guiding Examples

Before going any further let us try to justify Rem.23 by reverse engineering the algebraic model we want to end up with in Thm. 1. We follow Spatzier & Vinhage's description of the suspension of algebraic actions on tori as homogeneous flows on a solvable Lie groups[1].

[1]See [SV19, pp.39-40]

Example 1: Let $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{GL}(n, \mathbb{Z})$ be a linear Cartan action for some $n \in \mathbb{Z}_{\geq 1}$; for simplicity let us assume that the generators of γ have all eigenvalues positive. Let $\widetilde{\gamma}_\bullet : \mathbb{R}^k \to \mathrm{GL}(n, \mathbb{R})$ be a lift of γ so that for $t \in \mathbb{Z}^k$, $\widetilde{\gamma}_t = \gamma_{-t}$. Then $G = \mathbb{R}^k \ltimes_{\widetilde{\gamma}} \mathbb{R}^n$ is a solvable Lie group; as a manifold it is C^∞ diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^n$ and we have the following identities in G:

$$\begin{aligned}
(r,x)(s,y) &= (r+s, \widetilde{\gamma}_s(x) + y) \\
(r,x)^{-1} &= (-r, -\widetilde{\gamma}_{-r}(x)) \\
(r,0)(0,y) &= (r,y) \\
(0,x)(s,0) &= (s, \widetilde{\gamma}_s(x)) \\
(t,0)(r,x) &= (t+r, x) \\
(r,x)(t,0) &= (r+t, \widetilde{\gamma}_t(x)).
\end{aligned}$$

Let us put $\Gamma = \mathbb{Z}^k \ltimes_{\widetilde{\gamma}} \mathbb{Z}^n \leq G$. Then we have

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, tail] \arrow[d, tail] & \Gamma \arrow[r, two heads] \arrow[d, tail] & \mathbb{Z}^k \arrow[d, tail] \\
\mathbb{R}^n \arrow[r, tail] \arrow[d, two heads] & G \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R}^k \arrow[d, two heads] \\
\mathbb{T}^n \arrow[r, tail] & {}^{G}\!/_{\Gamma} \arrow[r, two heads] & \mathbb{T}^k
\end{tikzcd}$$

Thus ${}^{G}\!/_{\Gamma}$ is a $\mathbb{T}^n$-bundle over $\mathbb{T}^k$; consequently $\Gamma \leq G$ is a cocompact lattice. Note that ${}^{G}\!/_{\Gamma}$ has a canonical C^∞ affine manifold structure. Let $l_\bullet : \mathbb{R}^k \curvearrowright G$, $l_t(r,x) = (t,0)(r,x)$ be the left translation action restricted to the $\mathbb{R}^k$ subgroup of G. Then $l_\bullet$ descends to an affine Cartan action $\hbar^\gamma_\bullet : \mathbb{R}^k \to \mathrm{Aff}\left({}^{G}\!/_{\Gamma}\right)$. Note that $\hbar^\gamma_t((r,x)\Gamma) = (r,y)\Gamma$ iff $t \in \mathbb{Z}^k$ and $x - \widetilde{\gamma}_t(y) = x - \gamma_{-t}(y) \in \mathbb{Z}^n$. Thus for $t \in \mathbb{Z}^k$ and $r \in \mathbb{T}^k$,

$$\hbar^\gamma_t((r,x)\Gamma) = (r, \gamma_t(x))\Gamma,$$

that is for any $r \in \mathbb{T}^k$, the fiber in ${}^{G}\!/_{\Gamma}$ at r carries a copy of the action γ and $\hbar^\gamma$ is the suspension of γ. Note that in terms of crystals (see Sec.1.1) $\mathbb{T}^k$ is the time crystal and $\mathbb{T}^n$ is the space crystal.

⌟

Instead of trying to reverse engineer this solvable group structure on T_xM directly, we'll instead first identify a discrete subgroup of it; more specifically this discrete subgroup will be the translation part of some group of affine isomorphisms of T_xM; which group we'll call the homoclinic group. As another guiding example let us discuss homoclinic groups of hyperbolic toral automorphisms:

Example 2 ([2])**:** Let X be a Polish group, d be a left-invariant metric on X and let $f : X \to X$ be a topological group isomorphism. Put for any $x \in X$, $H_x(f) = \{y \in X \mid \lim_{|n|\to\infty} d(f^n(y), f^n(x)) = 0\}$. Then H_{e_X} is a (not necessarily closed nor not closed) subgroup of X, for any $x, y \in X$ we have $y \in H_x \Leftrightarrow x^{-1}y \in H_{e_X}$ and consequently $y \sim_f x$ iff $y \in H_x(f)$ is an equivalence relation.

If $X = \mathbb{T}^d$ and f is hyperbolic, then $H_x(f) \cong \pi_1(\mathbb{T}^d, x) \cong \mathbb{Z}^d$ and since $H_x(f)$ is by definition the intersection of the global stable and unstable manifolds of f at x, $H_0(f) \leq \mathbb{T}^d$ is a countable dense subgroup. Further, $\{x \mapsto x + y | y \in H_0(f)\} \cong H_{\mathrm{id}_{\mathbb{T}^d}}(\Gamma_f)$, where $\Gamma_f : \mathrm{Homeo}(\mathbb{T}^d) \to \mathrm{Homeo}(\mathbb{T}^d), g \mapsto f \circ g \circ f^{-1}$ is the conjugation action.

⌟

7.2 Measurable Covering Map

Proposition 11: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then for $x \in_\mu M$, the following formula defines a measurable map that is æ-defined w/r/t the Haar measure class on $T_x M$:

$$\begin{aligned} \Phi_x = \Phi_x^{\mathcal{C}} :&(T_x M, 0) \cong (\mathbb{R}^k \times (\mathcal{S}_x^{\mathcal{C}} \oplus \mathcal{U}_x^{\mathcal{C}}), (0,0,0)) \to (M, x), \\ &v = (v^o, v^s, v^u) \mapsto \mathcal{U}^{\mathcal{C}}_{\Upsilon_x^{\mathcal{C}}(v^u) \leftarrow x} \circ \Gamma\Sigma_x^{\mathcal{C}}(v^o, v^s). \end{aligned}$$

Moreover,

(i) $\Phi : TM \to M$, $(x, v) \mapsto \Phi_x(v)$ is measurable.

(ii) The same map is also æ-defined by the following formulas:

$$\begin{aligned} &\forall x \in_\mu M, \forall (v^o, v^s, v^u) \in_{æ} T_x M : \\ &\quad \Phi_x^{\mathcal{C}}(v^o, v^s, v^u) = \alpha_{v^o} \circ \left[\mathcal{U}^{\mathcal{C}}_{\Gamma\Upsilon_x^{\mathcal{C}}(-v^o, v^u) \leftarrow \alpha_{-v^o}(x)}\right]\Big|_{\mathcal{S}^{\mathcal{C}}_{\alpha_{-v^o}(x)}} \circ \Sigma_x^{\mathcal{C}}(v^s) \\ &\qquad = \alpha_{v^o} \circ \mathcal{U}^{\mathcal{C}}_{\Upsilon_x^{\mathcal{C}}(v^u) \leftarrow x} \circ \Sigma_x^{\mathcal{C}}(v^s) \\ &\qquad = \mathcal{S}^{\mathcal{C}}_{\Sigma_x^{\mathcal{C}}(v^s) \leftarrow x} \circ \Gamma\Upsilon_x^{\mathcal{C}}(v^o, v^u) \\ &\qquad = \alpha_{v^o} \circ \left[\mathcal{S}^{\mathcal{C}}_{\Gamma\Sigma_x^{\mathcal{C}}(-v^o, v^s) \leftarrow \alpha_{-v^o}(x)}\right]\Big|_{\mathcal{U}^{\mathcal{C}}_{\alpha_{-v^o}(x)}} \circ \Upsilon_x^{\mathcal{C}}(v^u) \\ &\qquad = \alpha_{v^o} \circ \mathcal{S}^{\mathcal{C}}_{\Sigma_x^{\mathcal{C}}(v^s) \leftarrow x} \circ \Upsilon_x^{\mathcal{C}}(v^u). \end{aligned}$$

(iii) For any $x \in_\mu M$, Φ_x has a diagonal property: each coordinate line according to the splitting $T_x M = O_x \times \left(\bigoplus_{\chi \in \mathrm{LSpec}^*(\mu,\alpha)} L_x^\chi\right)$ is either mapped onto the leaf of a 1-dimensional subfoliation of $\mathcal{O}_x$ given by one of the infinitesimal generators of the action α xor to a 1-dimensional Lyapunov manifold $\mathcal{L}_x^\chi$.

[2]See [Rue78a, pp. 129-130, Ex.], [Rue78b, p. 131], [Gor95], [LS99, p. 957, Ex.3.3].

(iv) For any $x \in_\mu M$, Φ_x is independent of the chamber $\mathcal{C}$. That is, if $\mathcal{D} \in \mathrm{Cham}(\mu, \alpha)$ is an anonymous chamber, then

$$\forall x \in_\mu M, \forall v \in_{æ} T_x M : \Phi_x^{\mathcal{C}}(v) = \Phi_x^{\mathcal{D}}(v).$$

(v) We have for any $x \in_\mu M$, for any $v \in_{æ} T_x M$ and for any chamber $\mathcal{D} \in \mathrm{Cham}(\mu, \alpha)$:

$$\begin{aligned} v \in O_x &\implies \Phi_x(v) = \alpha_v(x), \\ v \in S_x^{\mathcal{D}} &\implies \Phi_x(v) = \Sigma_x^{\mathcal{D}}(v) \\ v \in U_x^{\mathcal{D}} &\implies \Phi_x(v) = \Upsilon_x^{\mathcal{D}}(v). \end{aligned}$$

⌟

Proof: We have all the ingredients established already. That $\Phi_x = \Phi_x^{\mathcal{C}}$ is well-defined follows from Prop. 5, Prop. 7 and Prop. 10. That both the basepointed version Φ_x and the global version Φ are measurable follows from item (iii) of Cor. 10, item (iv) of Prop. 5 and item (i) of Prop. 9. Item (ii) is syntactic (see also Rem. 22). Items (iii) and (iv) follow from the assembly properties Cor. 11 together with the observation that all diagonal affine holonomies can be ultimately decomposed as affine holonomies along Lyapunov æ -foliations. Finally item (v) follows from the previous three items.

⌟

Next we establish further properties of Φ_x that will justify considering it as a measurable covering map. Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and define $\forall x \in_\mu M : O_x^\perp = S_x^{\mathcal{C}} \oplus U_x^{\mathcal{C}}$; $O_x^\perp$ is independent of the choice of the chamber $\mathcal{C}$ (indeed it is canonically isomorphic to ${T_x M}/{O_x}$ as a vector space). We also write $v = (v^o, v^\perp) \in O_x \times O_x^\perp$.

Proposition 12: For $x \in_\mu M$, let $\Phi_x : T_x M \to M$ be as in Prop. 11. Denote for any $\delta \in \mathbb{R}_{>0}$ by $\eta_{x,\delta}$ the Haar measure on $T_x M$ with $\eta_{x,\delta}(O_x[0| \leq \delta] \times O_x^\perp[0| \leq \delta]) = 1$. Then for any $x \in_\mu M$ and for any $\varepsilon \in]0,1[$, there is a $\delta \in \mathbb{R}_{>0}$, $c \in \mathbb{R}_{>0}$ and a measurable $B_{x,\varepsilon} \subseteq O_x[0| \leq \delta] \times O_x^\perp[0| \leq \delta]$ such that

(i) $0 \in B_{x,\varepsilon}$,

(ii) $1 - \varepsilon < \eta_{x,\delta}(B_{x,\varepsilon})$ and $1 - \varepsilon < (\eta_{x,\delta})_0^{O_x}(B_{x,\varepsilon} \cap \{0\} \times O_x^\perp)$, where we disintegrate $\eta_{x,\delta}$ along $T_x M \to O_x$[3].

(iii) $\Phi_x|_{B_{x,\varepsilon}} : (B_{x,\varepsilon}, 0) \to (M, x)$ is injective,

(iv) $\overrightarrow{\Phi_x}(\eta_{x,\delta}|_{B_{x,\varepsilon}}) = c\, \mu|_{\overrightarrow{\Phi_x}(B_{x,\varepsilon})}$.

(v) Furthermore we have $\mu\left(\overrightarrow{\Phi_x}(T_x M)\right) = 1$.

[3]See Lem.4.

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Proof: Let $\varepsilon \in]0,1[$. By Obs.6 and the Lebesgue Density Theorem, for $x \in_\mu M$, $\exists \Lambda \in \mathbb{R}_{>0}$ such that there is a $\delta' \in \mathbb{R}_{>0}$ with

$$1 - \varepsilon < \mu(M[x| \leq \delta'] \mid \Pi(\varepsilon, \Lambda));$$

here we condition μ on the Lusin-Pesin set $\Pi(\varepsilon, \Lambda)$. By items (iv) and (ii) in Cor.10 and Prop.9, respectively, and the absolute continuity of holonomies on Pesin sets, fixing a chamber $\mathcal{C}$ and defining

$$B_{x,\varepsilon} = \overleftarrow{\Phi_x} \left(\mathcal{O}_{x,\text{loc}} \times \mathcal{S}^{\mathcal{C}}_{x,\text{loc}} \times \mathcal{U}^{\mathcal{C}}_{x,\text{loc}} \cap \Pi(\varepsilon, \Lambda)\right)$$

with the sizes of local leaves not larger than some $\delta \in \mathbb{R}_{>0}$ gives the first three items. Note that $\overrightarrow{\Phi_x} (B_{x,\varepsilon})$ is measurable by the continuity of Φ on Pesin sets, item (iii) and the Lusin-Souslin Theorem[4]; hence item (iv) follows from Lem.12. For item (v) note first that $\overrightarrow{\Phi_x} (T_x M)$ is α-invariant; thus it suffices by ergodicity to verify that it's measurable. That it's indeed measurable follows from items (iii) and (iv), together with Prop.8.

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Definition 21: In light of Prop.12, we call the measurable map $\Phi_x : (T_x M, 0) \to (M, x)$ defined in Prop.11 the **measurable covering** of the system (μ, α) at $x \in_\mu M$.

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7.3 Diagonal Affine Extension

Next we discuss the effect of changing the basepoints on Φ_x. One can change the basepoint either horizontally or vertically (or both simultaneously). The horizontal basepoint change is the change of x along the manifold M. The vertical basepoint change is the change of the marked origin $0 \in T_x M$ to some other vector; note that we consider $T_x M$ as an affine space. Of course under the measurable covering map Φ_x vertical basepoint changes transform into horizontal basepoint changes. We start with special cases.

Proposition 13: Let $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and $x \in_\mu M$. Then

(i) For $v^s \in_{æ} S^{\mathcal{C}}_x$ put $y = \Phi_x(v^s) \in \mathcal{S}^{\mathcal{C}}_x$. Then there is a unique affine isomorphism $\Phi_{(y,0)\leftarrow(x,v^s)} \in \mathrm{DAff}(T_x M; T_y M)$ such that $\Phi_{(y,0)\leftarrow(x,v^s)}(v^s) = 0$ and $\Phi_y \circ \Phi_{(y,0)\leftarrow(x,v^s)} =_{æ} \Phi_x$ on $T_x M$.

(ii) Similarly for $v^u \in_{æ} U^{\mathcal{C}}_x$, putting $z = \Phi_x(v^u) \in \mathcal{U}^{\mathcal{C}}_x$, there is a unique affine isomorphism $\Phi_{(z,0)\leftarrow(x,v^u)} \in \mathrm{DAff}(T_x M; T_z M)$ such that $\Phi_{(z,0)\leftarrow(x,v^u)}(v^u) = 0$ and $\Phi_z \circ \Phi_{(z,0)\leftarrow(x,v^u)} =_{æ} \Phi_x$ on $T_x M$.

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[4] See [Kec95, p.89, Thm.15.1].

Proof: The proof is a matter of unfolding and transforming the definitions. We only verify the first item and merely give the analogous formula for the second item for future use. Let $w = (w^o, w^s, w^u) \in_{æ} T_x M$. Then

$$\begin{aligned}
\Phi_x(w) &= \alpha_{w^o} \circ \mathcal{S}^{\mathcal{C}}_{\Sigma^{\mathcal{C}}_x(w^s)\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x(w^u) \\
&= \alpha_{w^o} \circ \mathcal{S}^{\mathcal{C}}_{\Sigma^{\mathcal{C}}_y\left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1}\circ\Sigma^{\mathcal{C}}_x(w^s)\right)\leftarrow y} \circ \mathcal{S}^{\mathcal{C}}_{y\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x(w^u) \\
&= \alpha_{w^o} \circ \mathcal{S}^{\mathcal{C}}_{\Sigma^{\mathcal{C}}_y\left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1}\circ\Sigma^{\mathcal{C}}_x(w^s)\right)\leftarrow y} \circ \Upsilon^{\mathcal{C}}_y\left(\left(\Upsilon^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{S}^{\mathcal{C}}_{y\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x(w^u)\right) \\
&= \Phi_y\left(w^o, \left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \Sigma^{\mathcal{C}}_x(w^s), \left(\Upsilon^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{S}^{\mathcal{C}}_{y\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x(w^u)\right) \\
&= \Phi_y \circ \left[I_k \times \left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \Sigma^{\mathcal{C}}_x\right) \times \left(\left(\Upsilon^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{S}^{\mathcal{C}}_{y\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x\right)\right](w),
\end{aligned}$$

so that the following formula defines the affine isomorphism $\Phi_{(y,0)\leftarrow(x,v^s)}$ with the required properties æ-uniquely:

$$\Phi_{(y,0)\leftarrow(x,v^s)} = I_k \times \left(\left(\Sigma^{\mathcal{C}}_y\right)^{-1} \circ \Sigma^{\mathcal{C}}_x\right) \times \left(\left(\Upsilon^{\mathcal{C}}_y\right)^{-1} \circ \mathcal{S}^{\mathcal{C}}_{y\leftarrow x} \circ \Upsilon^{\mathcal{C}}_x\right) \qquad (\star^s)$$

$$\text{for } y = \Phi_x(v^s), v^s \in_{æ} S^{\mathcal{C}}_x.$$

Similarly we have

$$\Phi_{(z,0)\leftarrow(x,v^u)} = I_k \times \left(\left(\Sigma^{\mathcal{C}}_z\right)^{-1} \circ \mathcal{U}^{\mathcal{C}}_{z\leftarrow x} \circ \Sigma^{\mathcal{C}}_x\right) \times \left(\left(\Upsilon^{\mathcal{C}}_z\right)^{-1} \circ \Upsilon^{\mathcal{C}}_x\right) \qquad (\star^u)$$

$$\text{for } z = \Phi_x(v^u), v^u \in_{æ} U^{\mathcal{C}}_x.$$

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In Prop.13 we have discussed the effect on Φ of changing the basepoint vertically along the stable and unstable directions of a chamber; next we discuss the effect of changing the basepoint along the orbit directions. Note that since $\forall x \in_\mu M$, $\mathbb{R}^k \cong O_x \cong \mathcal{O}_x$ canonically, we may interpret this change of basepoint either vertically or horizontally, without reference to the measurable covering map Φ_x.

Proposition 14: Let $x \in_\mu M$, $v^o \in O_x$ and put $y = \Phi_x(v^o) \in \mathcal{O}_x$. Then there is a unique affine isomorphism $\Phi_{(y,0)\leftarrow(x,v^o)} \in \mathrm{DAff}(T_x M; T_y M)$ such that $\Phi_{(y,0)\leftarrow(x,v^o)}(v^o) = 0$ and $\Phi_y \circ \Phi_{(y,0)\leftarrow(x,v^o)} =_{æ} \Phi_x$ on $T_x M$.

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Proof: Let us fix a chamber $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$ and consider the cocycles $\phi^{\Sigma^{\mathcal{C}}}$ and $\phi^{\Upsilon^{\mathcal{C}}}$ attached to the stable and unstable affine parameters of $\mathcal{C}$ in item (ii) of Cor.10. Then arguing as in Prop.13 we have for $w = (w^o, w^s, w^u) \in_{æ} T_x M$:

$$\Phi_x(w) = \Phi_y\left(w^o - v^o, \phi^{\Sigma^{\mathcal{C}}}(v^o,x)(w^s), \phi^{\Upsilon^{\mathcal{C}}}(v^o,x)(w^u)\right).$$

Thus the following formula defines the affine isomorphism $\Phi_{(y,0)\leftarrow(x,v^o)}$ with the desired properties:

$$\Phi_{(y,0)\leftarrow(x,v^o)} = \left(I_k \times \phi^{\Sigma^{\mathcal{C}}}(v^o,x) \times \phi^{\Upsilon^{\mathcal{C}}}(v^o,x)\right) - v^o \qquad (\star^o)$$

$$\text{for } y = \Phi_x(v^o), v^o \in O_x.$$

Note that by virtue of the assembly properties Cor. 11, this formula defines an affine isomorphism independently of the choice of the chamber $\mathcal{C}$.

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Corollary 13: Let $x \in_\mu M$ and $t \in \mathbb{R}^k$. Then

$$\alpha_t \circ \Phi_x =_{æ} \pm \Phi_{\alpha_t(x)} \circ T_x\alpha_t.$$

Consequently $\Phi : TM \to M$ displays (μ, α) as the factor map of some system that acts via diagonal isomorphisms fiberwise.

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Proof: The canonical identification $\forall x \in_\mu M$, $\mathbb{R}^k \cong O_x \cong \mathcal{O}_x$ explicitly refers to the fact that $(\alpha_\bullet(\alpha_t(x)))^{-1} \circ \alpha_t \circ \alpha_\bullet(x) = \mathrm{id}_{\mathbb{R}^k}$, that is

$$\begin{tikzcd} \mathcal{O}_x \arrow[r, "\alpha_t"] & \mathcal{O}_{\alpha_t(x)} \\ \mathbb{R}^k \arrow[u, "\alpha_\bullet(x)"] \arrow[r, "\mathrm{id}_{\mathbb{R}^k}"'] & \mathbb{R}^k \arrow[u, "\alpha_\bullet(\alpha_t(x))"'] \end{tikzcd}$$

Thus $T_x\alpha_t$ fixes any vector in O_x, when the Euclidean coordinates are used. By Rem. 20 we also have that the cocycles $\phi^{\Sigma^{\mathcal{C}}}$ and $\phi^{\Upsilon^{\mathcal{C}}}$ coincide with the derivative cocycles of α restricted to the stable and unstable æ-subbundles of the chamber $\mathcal{C}$ up to a sign, respectively; thus $(\star^o)$ in the proof Prop. 14 immediately gives the statement with v^o replaced by t. The extension of (μ, α) that $\Phi : TM \to M$ is a factor map that coincides with the derivative cocycle up to a sign that depends on $x \in_\mu M$ and $t \in \mathbb{R}^k$.

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Proposition 15: Let $x, y \in_\mu M$, $v \in_{æ} T_xM$ and $w \in_{æ} T_yM$. If $p = \Phi_x(v) = \Phi_y(w)$, then there is a unique affine isomorphism $\Phi_{(y,w)\leftarrow(x,v)} \in \mathrm{DAff}(T_xM; T_yM)$ with $\Phi_{(y,w)\leftarrow(x,v)}(v) = w$ such that $\Phi_y \circ \Phi_{(y,w)\leftarrow(x,v)} =_{æ} \Phi_x$, that is,

$$\begin{tikzcd}
(T_x M, v) \arrow[rr, "\Phi_{(y,w)\leftarrow(x,v)}"] \arrow[dr, "\Phi_x"'] & \text{æ} & (T_y M, w) \arrow[dl, "\Phi_y"] \\
& (M, p) &
\end{tikzcd}$$

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Proof: The proof is based on the previous two propositions Prop.13 and Prop.14; together with good choices of points that are to be abbreviated. Let us first establish the affine isomorphism $\Phi_{(p,0)\leftarrow(x,v)} : (T_x M, v) \to (T_p M, 0)$. Let $v = (v^o, v^s, v^u)$ and let us put

$$q = \alpha_{-v^o}(p), r = \Phi_x(v^s) \in \mathcal{S}^{\mathfrak{C}}_x, z^u = (\Phi_r)^{-1}(q) \in U^{\mathfrak{C}}_r.$$

We have by Prop.14 that $\Phi_q =_{\text{æ}} \Phi_p \circ \Phi_{(p,0)\leftarrow(q,v^o)}$. Since we also have

$$q = \Phi_x(0, v^s, v^u) = \mathcal{U}^{\mathfrak{C}}_{\Upsilon^{\mathfrak{C}}_x(v^u)\leftarrow x} \circ \Sigma^{\mathfrak{C}}_x(v^s) = \mathcal{U}^{\mathfrak{C}}_{\Upsilon^{\mathfrak{C}}_x(v^u)\leftarrow x}(r) \in \mathcal{U}^{\mathfrak{C}}_r,$$

by Prop. 13 we have $\Phi_r =_{\text{æ}} \Phi_q \circ \Phi_{(q,0)\leftarrow(r,z^u)}$. Again by Prop. 13 we have $\Phi_x =_{\text{æ}} \Phi_r \circ \Phi_{(r,0)\leftarrow(x,v^s)}$; thus we may æ-uniquely define the affine isomorphism $\Phi_{(p,0)\leftarrow(x,v)} : T_x M \to T_p M$ by

$$\Phi_{(p,0)\leftarrow(x,v)} =_{\text{æ}} \Phi_{(p,0)\leftarrow(q,v^o)} \circ \Phi_{(q,0)\leftarrow(r,z^u)} \circ \Phi_{(r,0)\leftarrow(x,v^s)}.$$

Composing the formulas $(\star^s)$, $(\star^u)$ and $(\star^o)$ in the proofs of Prop.13 and Prop.14 with appropriate point substitutions we obtain:

$$\begin{aligned}
\Phi_{(p,0)\leftarrow(x,v)} =_{\text{æ}} \; & I_k \\
& \times \left(\phi^{\Sigma^{\mathfrak{C}}}(v^o, q) \circ \left(\Sigma^{\mathfrak{C}}_q\right)^{-1} \circ \mathcal{U}^{\mathfrak{C}}_{q\leftarrow r} \circ \Sigma^{\mathfrak{C}}_x \right) \\
& \times \left(\phi^{\Upsilon^{\mathfrak{C}}}(v^o, q) \circ \left(\Upsilon^{\mathfrak{C}}_q\right)^{-1} \circ \mathcal{S}^{\mathfrak{C}}_{r\leftarrow x} \circ \Upsilon^{\mathfrak{C}}_x \right) - v^o. \\
& \quad \text{for } p = \Phi_x(v), q = \alpha_{-v^o}(p), r = \Phi_x(v^s), v = (v^o, v^s, v^u) \in_{\text{æ}} T_x M.
\end{aligned}$$

With this formula it's also straightforward that indeed $\Phi_{(p,0)\leftarrow(x,v)}(v) = 0$. Finally we may define the affine isomorphism

$$\Phi_{(y,w)\leftarrow(x,v)} = \left(\Phi_{(p,0)\leftarrow(y,w)}\right)^{-1} \circ \Phi_{(p,0)\leftarrow(x,v)} : (T_x M, v) \mapsto (T_y M, w).$$

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Chapter 8

Homoclinic Group and Conclusion of Proof of the Main Theorem

In this chapter we introduce the homoclinic groupoid and the homoclinic group and conclude the proof of Thm. 1. In the literature homoclinic groups are either defined already with reference to algebraic structure (i.e. for actions on groups by automorphisms)[1], or else with reference to a Smale structure (i.e. an abstract local product structure)[2]. Our approach is closer to the latter perspective; the measurable covering map Φ_x we constructed is a measurable analog of a global product structure and the recovery of the algebraic structure is essentially based on the fact that the two approaches coincide. Let us also note that often homoclinic groups are considered for discrete group systems. In fact the author was unable to find a case study for the homoclinic group of a non-discrete group system in the literature.

Note that by Prop. 15, for $x \in_\mu M$ and for $v, w \in_{æ} T_xM$, if $\Phi_x(v) = \Phi_x(w)$, then there is a unique affine isomorphism $\Phi_{(x,w)\leftarrow(x,v)} : (T_xM, v) \to (T_xM, w)$ such that $\Phi_x \circ \Phi_{(x,w)\leftarrow(x,v)} =_{æ} \Phi_x$, thus in light of Prop.12 we may consider $\Phi_{(x,w)\leftarrow(x,v)}$ as a measurable deck transformation of the measurable covering map Φ_x.

Definition 22: We define the **homoclinic groupoid** $\mathfrak{H}$ of (μ, α) by

$$\mathfrak{H}_{y\leftarrow x} = \{A \in \mathrm{DAff}(T_xM; T_yM) \mid \Phi_y \circ A =_{æ} \Phi_x\}$$

for $x, y \in_\mu M$. Similarly for $x \in_\mu M$, we define the **homoclinic group** $\mathfrak{H}_x$ of (μ, α) at x by

$$\mathfrak{H}_x = \mathfrak{H}_{x\leftarrow x} = \{A \in \mathrm{DAff}(T_xM) \mid \Phi_x \circ A =_{æ} \Phi_x\}.$$

⌟

One can see the reasoning behind calling $\mathfrak{H}_x$ the homoclinic group as follows: fix a chamber $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$, and let for $x \in_\mu M$, $y \in \mathcal{S}_x^{\mathcal{C}} \cap \mathcal{U}_x^{\mathcal{C}}$, so that y is homoclinic to x w/r/t α_t for any $t \in \mathcal{C} \cup -\mathcal{C}$. But if $y \neq x$, then we have two distinct vectors $v^s, v^u \in T_xM$

[1] See e.g. [KS95, MS99, BS03, CF04, LSV13, CL15, BGRL22].
[2] See e.g. [Cap76, Rue88, Tho10a, Tho10b].

that are both sent to y via the measurable covering map Φ_x, namely, $v^s = \left(\Sigma_x^{\mathfrak{C}}\right)^{-1}(y) \in S_x^{\mathfrak{C}}$, and $v^u = \left(\Upsilon_x^{\mathfrak{C}}\right)^{-1}(y) \in U_x^{\mathfrak{C}}$. Then by Prop.15, we have a unique element $A = A_y \in \mathfrak{H}_x$ such that $A : v^s \mapsto v^u$.

Let us denote by $\mathbb{L} : \mathrm{DAff}(T_xM) \to \mathrm{DGL}(T_xM)$ the homomorphism that takes an affine automorphism to its linear part, and $\mathbb{I} : T_xM \to \mathrm{DAff}(T_xM)$ be the inclusion of the kernel of $\mathbb{L}$. Denote by $\mathfrak{T}\mathfrak{H}_x = \overleftarrow{\mathbb{I}}(\mathfrak{H}_x) \cong \ker(\mathbb{L}) \cap \mathfrak{H}_x$ the normal subgroup of $\mathfrak{H}_x$ that is the translation part. We shall write w for an element of $\mathfrak{T}\mathfrak{H}_x$ when we want to emphasize that w is a vector in T_xM and $\mathbb{I}(w)$ when we want to emphasize that w is a symmetry of Φ_x.

Observation 11: Let $x \in_\mu M$, $t \in \mathbb{R}^k$. Immediately by Cor.13 we have that $T_x\alpha_t : T_xM \to T_{\alpha_t(x)}M$ conjugates the homoclinic group at x and $\alpha_t(x)$. More precisely;

$$\forall A \in \mathfrak{H}_x : T_x\alpha_t \circ A \circ (T_x\alpha_t)^{-1} \in \mathfrak{H}_{\alpha_t(x)}.$$

It's also straightforward that $\forall w \in \mathfrak{T}\mathfrak{H}_x : T_x\alpha_t \circ \mathbb{I}(w) \circ (T_x\alpha_t)^{-1} = \mathbb{I} \circ T_x\alpha_t(w)$, thus

$$\overrightarrow{T_x\alpha_t}(\mathfrak{T}\mathfrak{H}_x) = \mathfrak{T}\mathfrak{H}_{\alpha_t(x)}.$$

⌟

By Obs. 11, the linear span $\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet) \leq TM$ of the translation part $\mathfrak{T}\mathfrak{H}_\bullet$ of the family of the homoclinic groups is Ad^α-invariant; it's also straightforward that $\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet)$ is a measurable[3] polarization. Thus by ergodicity $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet)) : M \to \underline{2k+2}$ is æ -constant. Note that by essential freeness of (μ, α) and Prop. 14, for $x \in_\mu M$, $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_x)) \cap O_x = 0$, that is, $\mathrm{Span}(\mathfrak{T}\mathfrak{H}_x) \leq O_x^\perp$; we'll show that indeed $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet)) =_\mu k+1$. We split the proof into two; first we'll show that $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet))$ is not degenerate and then we'll show that its rank is maximal, subject to the complementarity to the orbit directions $O_\bullet$.

Lemma 13: For $x \in_\mu M$, $\mathfrak{T}\mathfrak{H}_x$ is a discrete subgroup of $O_x^\perp$. Here we consider $O_x^\perp$ with its standard abelian group structure.

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Proof: Suppose otherwise. Then there is a sequence $w_\bullet \subseteq \mathfrak{T}\mathfrak{H}_x$ of vectors such that $\lim_{n\to\infty} w_n \to 0$. Fix $\varepsilon \in]0,1[$, and let $B_{x,\varepsilon}$ be as in Prop.12. If $v \in B_{x,\varepsilon} \cap (B_{x,\varepsilon} + w_n)$, then $v, v - w_n = \mathbb{I}(-w_n)(v) \in B_{x,\varepsilon}$. But then $\Phi_x(v) = \Phi_x \circ \mathbb{I}(w_n)(v)$, so that $(\eta_{x,\delta})_0^{O_x}(B_{x,\varepsilon} \cap (B_{x,\varepsilon} + w_n)) = 0$ (recall that $(\eta_{x,\delta})_0^{O_x}$ is the fiber measure at 0 of a Haar measure on T_xM along the projection $T_xM \to O_x$). We also have $\lim_{n\to\infty}(\eta_{x,\delta})_0^{O_x}(B_{x,\varepsilon} \cap (B_{x,\varepsilon} + w_n)) = (\eta_{x,\delta})_0^{O_x}(B_{x,\varepsilon}) > 1 - \varepsilon > 0$, a contradiction.

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Lemma 14: For $x \in_\mu M$: $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_x)) > 0$.

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[3] See the proof of Lem.15 for an argument for measurability.

Proof: Suppose to the contrary $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet)) =_\mu 0$, so that for $x \in_\mu M$, $\mathfrak{H}_x$ has no translation part. We'll see that this implies that $\mathfrak{H}_x$ is trivial, which in turn implies by Prop.15 that $\Phi_x : T_x M \to M$ is æ-injective. Conjugating (μ, α) via Φ_x; we have by Cor.13,

$$(\Phi_x)^{-1} \circ \alpha_t \circ \Phi_x =_{æ} \pm (\Phi_x)^{-1} \circ \Phi_{\alpha_t(x)} \circ T_x\alpha_t,$$

so that the conjugate system on $T_x M$ is by affine automorphisms. By item (iv) of Prop. 12 the transformed probability measure is of Haar class; but no affine automorphism of a vector space admits a *probability* measure of positive entropy[4], a contradiction.

Thus it suffices to show that $\mathfrak{H}_x = \{\mathrm{id}_{T_x M}\}$. Suppose not. For the sake of readability we'll fix a basis $T_x M \cong \mathbb{R}^{2k+1}$ and write e.g. (A, a) for the affine automorphism $v \mapsto Av + a$. Since $\mathfrak{T}\mathfrak{H}_x$ is trivial, $\mathbb{L} : \mathfrak{H}_x \hookrightarrow \mathrm{DGL}(T_x M)$ is an embedding; in particular $\mathfrak{H}_x$ is abelian. Then for $(A, a), (B, b) \in \mathfrak{H}_x$, the abelian nature of $\mathfrak{H}_x$ implies that $(A - I)(b) = (B - I)(a)$. If either of A or B is I this equation is vacuous; for $A \neq I \neq B$ however we have $(B - I)^{-1}(b) = (A - I)^{-1}(a)$, so that the set

$$\{-(A - I)^{-1}(a) \,|\, (A, a) \in \mathfrak{H}_x, A \neq I\}$$

is a singleton; let us denote by v_x^* its unique element. A straightforward computation shows that $v_x^* \in \mathrm{Fix}(\mathfrak{H}_x) = \{v \in T_x M \,|\, \forall A \in \mathfrak{H}_x : A(v) = v\}$, or alternatively that the affine automorphism $(I, v_x^*) \in \mathrm{DAff}(T_x M)$ conjugates the affine action $\mathfrak{H}_x \curvearrowright T_x M$ to the linear action $\overrightarrow{\mathbb{L}}(\mathfrak{H}_x) \curvearrowright T_x M$. Let us put $F_x = \mathrm{Fix}(\mathfrak{H}_x)$ and $F_x^0 = \mathrm{Fix}(\overrightarrow{\mathbb{L}}(\mathfrak{H}_x))$, so that we have

$$F_x = F_x^0 + v_x^*.$$

Since elements in $\mathrm{DGL}(T_x M)$ are diagonal w/r/t the Oseledets splitting $T_x M = O_x \oplus \bigoplus_\chi L_x^\chi$, F_x^0 is a direct sum of (possibly some) orbit directions and (possibly) some Lyapunov subspaces. In particular both $F_\bullet$ and $F_\bullet^0$ are measurable and Ad^α invariant by Lem.14. Further, by item (v) of Prop.11 and Lem.10, $O_x \leq F_x^0$. Hence there is a unique vector $v_x^\dagger \in T_x M$ that is an element of the sum of the Lyapunov subspaces that are not included in F_x^0 such that

$$F_x = F_x^0 + v_x^\dagger.$$

$v_\bullet^\dagger : M \to TM$ is an æ-defined measurable vector field that is Ad^α-invariant. If it didn't vanish μ-æ, we could find a $t^\dagger \in \mathbb{R}^k$ such that $n \mapsto |v_{\alpha_{nt^\dagger}(x)}|$ grows exponentially; which contradicts Poincaré Recurrence; whence $v_\bullet^\dagger =_\mu 0$, so $F_\bullet =_\mu F_\bullet^0$, and $0 \in F_x$ for $x \in_\mu M$. Thus if $(A, a) \in \mathfrak{H}_x$, then $a = 0$; i.e. $\mathfrak{H}_x$ is *equal* to $\mathbb{L}(\mathfrak{H}_x)$ under the standard embedding $\mathrm{DGL}(T_x M) \hookrightarrow \mathrm{DAff}(T_x M)$, $A \mapsto (A, 0)$.

Let us fix a chamber $\mathcal{C} \in \mathrm{Cham}(\mu, \alpha)$. Then any $A \in \mathfrak{H}_x$ preserves $S_x^{\mathcal{C}}$. Let $v^s \in S_x^{\mathcal{C}}$, fix $\varepsilon \in]0, 1[$ and let $B_{\bullet,\varepsilon}$ be as in Prop.12. Then there is a $t^\star \in \mathcal{C}$ such that $T_x\alpha_{t^\star}(v^s), T_x\alpha_{t^\star} \circ$

[4]Indeed, by Poincaré Recurrence the support of any probability measure invariant under an affine automorphism is in the closure of set of recurrent points; the linear part of the affine automorphism restricted to this set must have all eigenvalues of modulus 1, hence on the support of the invariant measure trajectories can separate at most polynomially.

$A(v^s) \in B_{\alpha_{t^\star}(x),\varepsilon}$; by Obs.11, the injectivity of $\Phi_{\alpha_{t^\star}(x)}$ on $B_{\alpha_{t^\star}(x),\varepsilon}$ forces $A(v^s) = v^s$; that is to say $S_x^{\mathcal{C}} \subseteq F_x$. Similarly $U_x^{\mathcal{C}} \subseteq F_x$, so that $F_x = T_x M$. Thus $\mathfrak{H}_x$ is trivial, a contradiction.

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Lemma 15: For $x \in_\mu M : \mathrm{Span}(\mathfrak{T}\mathfrak{H}_x) = O_x^\perp$ and consequently also $\mathfrak{T}\mathfrak{H}_x \cong \mathbb{Z}^{k+1}$.

⌟

Proof: Suppose otherwise, so that, after Lem.14, for $x \in_\mu M$, $0 < \dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_x)) < k+1$. Let us denote by d the μ-æconstant value of $\dim(\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet))$. For $x \in_\mu M$, let us denote by ω_x the volume of the d-dimensional torus ${}^{\mathrm{Span}(\mathfrak{T}\mathfrak{H}_x)}\!/\!_{\mathfrak{T}\mathfrak{H}_x}$; recall that at the beginning of chapter 5 we had fixed a C^∞ Riemannian metric $\mathfrak{g}$ and via this ω_x is well defined for $x \in_\mu M$. Note that $\omega_\bullet : M \to \mathbb{R}_{>0}$ is measurable; indeed we may fix a basepoint $x^* \in_\mu M$ and by item (v) of Prop.12 and by the proof of Prop.15 $\mathfrak{H}_\bullet$ is the image of $\mathfrak{H}_{x^*}$ under a measurable family of diagonal affine maps; similarly $\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet))$ and consequently the restriction of $\mathfrak{g}$ to $\mathrm{Span}(\mathfrak{T}\mathfrak{H}_\bullet))$ is also measurable. We may bunch the Lyapunov exponents of (μ,α) via an argument similar to the one we used in the proof of Cor.10 to obtain a time $t^\star$ such that $n \mapsto \omega_{\alpha_{nt^\star}(x)}$ grows exponentially by the Oseledets Theorem[5]; this contradicts Poincaré Recurrence. That $\mathfrak{T}\mathfrak{H}_x$ is isomorphic to $\mathbb{Z}^{k+1}$ follows from Lem.13.

⌟

Next observe more generally that O_x is $\mathfrak{H}_x$ invariant. Indeed, let $A \in \mathfrak{H}_x$ and $v^o \in O_x$. We may write $A(v^o,0) = (A^o(v^o) + a^o, a^\perp)$. If $a^\perp \neq 0$, then as in the final paragraph of the proof of Lem.14, choosing an appropriate $t^\star \in \mathbb{R}^k$ we get a contradiction to the local injectivity of $\Phi_{\alpha_{t^\star}(x)}$. There is a similar invariance property in the $O_x^\perp$ direction. Fix a chamber $\mathcal{C} \in \mathrm{Cham}(\mu,\alpha)$ and let $A \in \mathfrak{H}_x$. If $v^\perp = (v^s, v^u) \in_{æ} O_x^\perp$, then by Prop.14 we have

$$\begin{aligned}\Phi_x(0, v^s, v^u) &= \Phi_x(a^o, A^s(v^s) + a^s, A^u(v^u) + a^u) \\ &= \Phi_x(0, \phi^{\Sigma^{\mathcal{C}}} \circ A^s(v^s) + \phi^{\Sigma^{\mathcal{C}}}(a^o,x)(a^s), \phi^{\Upsilon^{\mathcal{C}}} \circ A^u(v^u) + \phi^{\Upsilon^{\mathcal{C}}}(a^o,x)(a^u)).\end{aligned}$$

Factoring $\mathfrak{T}\mathfrak{H}_x$ out from $T_x M$ we have a $\mathbb{T}^{k+1}$ factor coming from $O_x^\perp$ and further that $\mathbb{R}^k \cong O_x$ acts on the factor ${}^{T_x M}\!/\!_{\mathfrak{T}\mathfrak{H}_x}$. The return times of the $\mathbb{R}^k$ action on the $\mathbb{T}^{k+1}$ factor is a cocompact subgroup; note that identifying this subgroup as $\mathbb{Z}^k$ produces the constant time change κ in the statement of Thm.1. Thus we get that $\mathfrak{H}_x$ has a subgroup isomorphic to $\mathbb{Z}^k$ acts on the $\mathbb{T}^{k+1}$ factor by affine automorphisms and by the above calculation the $\mathbb{R}^k$ action is a suspension of the $\mathbb{Z}^k$ action on the $\mathbb{T}^{k+1}$ factor. Furthermore since the Lyapunov hyperplanes are in general position this $\mathbb{Z}^k$ action on $\mathbb{T}^{k+1}$ is maximal Cartan. Again using Prop.12 we have that the implied subgroup $\mathbb{Z}^{k+1} \rtimes \mathbb{Z}^k$ is of finite index in $\mathfrak{H}_x$ and indeed the factor $F = {}^{\mathfrak{H}_x}\!/\!_{\mathbb{Z}^{k+1} \rtimes \mathbb{Z}^k}$ is a finite group of automorphisms of $\mathbb{T}^{k+1}$. Since the $\mathbb{Z}^k$ action on $\mathbb{T}^{k+1}$ is maximal Cartan F is either $\{I_{k+1}\}$ xor $\{\pm I_{k+1}\}$. We have just proved:

[5]More specifically here we use the intermediate-dimensional version of **ASYM2** of Thm.4.

Proposition 16: Let $x \in_\mu M$. Then the homoclinic group $\mathfrak{H}_x$ is isomorphic to $(\mathbb{Z}^{k+1} \rtimes F) \rtimes \mathbb{Z}^k$ for $F = \{I_{k+1}\}$ xor $F = \{\pm I_{k+1}\}$. and consequently ${T_xM}/{\mathfrak{H}_x}$ is a T^{k+1} bundle over $\mathbb{T}^k$, where T^{k+1} is either $\mathbb{T}^{k+1}$ xor the $\pm$-infratorus ${\mathbb{T}^{k+1}}/{\pm I_{k+1}}$.

┘

By definition Φ_x descends to a measure theoretical isomorphism ${\Phi_x}/{\mathfrak{H}_x}$: $\left({T_xM}/{\mathfrak{H}_x}, \mathfrak{H}_x\right) \to (M, x)$ whose inverse $\Phi_{(\mu,\alpha)} = \Phi_{(\mu,\alpha),x} = \left({\Phi_x}/{\mathfrak{H}_x}\right)^{-1}$ transforms α to the suspension of an affine Cartan action and μ to the suspension measure induced by Haar measure on T^{k+1}. Applying a Journé lemma by de la Llave[6] gives the smoothness properties of $\Phi_{(\mu,\alpha)}$ and concludes the proof.

[6] [dlL92, pp.304-305, Thm.5.7; pp.312-313, Prop.5.13]

Bibliography

[Abr59] L. M. Abramov, *On the entropy of a flow*, Dokl. Akad. Nauk SSSR **128** (1959), 873–875, English translation in: American Mathematical Society translations 49 (1966), 167-170. MR 0113985 35

[Ano67] D. V. Anosov, *Tangential fields of transversal foliations in* Y*-systems*, Mat. Zametki **2** (1967), 539–548, This article has appeared in English translation [Math. Notes 2 (1967), 818–823]. MR 242190 18

[Ano69] ———, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder, American Mathematical Society, Providence, R.I., 1969. MR 0242194 6

[AS67] D. V. Anosov and Ja. G. Sinaĭ, *Certain smooth ergodic systems*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 107–172, English translation: Russian Math. Surveys 22 (1967), no. 5, 103–167. MR 0224771 15, 40

[BD91] Daniel Boivin and Yves Derriennic, *The ergodic theorem for additive cocycles of* $\mathbf{Z}^d$ *or* $\mathbf{R}^d$, Ergodic Theory Dynam. Systems **11** (1991), no. 1, 19–39. MR 1101082 19

[BF66] Robert Bonic and John Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. **15** (1966), 877–898. MR 0198492 17

[BGRL22] Sebastián Barbieri, Felipe García-Ramos, and Hanfeng Li, *Markovian properties of continuous group actions: algebraic actions, entropy and the homoclinic group*, Adv. Math. **397** (2022), Paper No. 108196, 52. MR 4366230 67

[BHW16] Aaron Brown, Federico Rodriguez Hertz, and Zhiren Wang, *Smooth ergodic theory of* $\mathbb{Z}^d$*-actions*, 2016. 19

[BK87] M. Brin and Yu. Kifer, *Dynamics of Markov chains and stable manifolds for random diffeomorphisms*, Ergodic Theory Dynam. Systems **7** (1987), no. 3, 351–374. MR 912374 18, 32

[BP01] L. Barreira and Ya. Pesin, *Lectures on Lyapunov exponents and smooth ergodic theory*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc.

Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, Appendix A by M. Brin and Appendix B by D. Dolgopyat, H. Hu and Pesin, pp. 3–106. MR 1858534 18, 32

[BP02] Luis Barreira and Yakov B. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, vol. 23, American Mathematical Society, Providence, RI, 2002. MR 1862379 32

[BRO21] AARON BROWN, *Smoothness of stable holonomies inside center-stable manifolds*, Ergodic Theory and Dynamical Systems (2021), 1–26. 18

[BS02] Michael Brin and Garrett Stuck, *Introduction to dynamical systems*, Cambridge University Press, Cambridge, 2002. MR 1963683 18

[BS03] Siddhartha Bhattacharya and Klaus Schmidt, *Homoclinic points and isomorphism rigidity of algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups*, Israel J. Math. **137** (2003), 189–209. MR 2013356 67

[Cap76] D. Capocaccia, *A definition of Gibbs state for a compact set with Z^v action*, Comm. Math. Phys. **48** (1976), no. 1, 85–88. MR 415675 67

[CF04] Alex Clark and Robbert Fokkink, *On a homoclinic group that is not isomorphic to the character group*, Qual. Theory Dyn. Syst. **5** (2004), no. 2, 361–365. MR 2275445 67

[CL15] Nhan-Phu Chung and Hanfeng Li, *Homoclinic groups, IE groups, and expansive algebraic actions*, Invent. Math. **199** (2015), no. 3, 805–858. MR 3314515 67

[dlH08] Pierre de la Harpe, *Spaces of closed subgroups of locally compact groups*, Online, 2008, `https://arxiv.org/abs/0807.2030v2`. 39

[dlL92] R. de la Llave, *Smooth conjugacy and S-R-B measures for uniformly and non-uniformly hyperbolic systems*, Comm. Math. Phys. **150** (1992), no. 2, 289–320. MR 1194019 40, 71

[dlL97] ______, *Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 649–662. MR 1452186 40

[dlLMM86] R. de la Llave, J. M. Marco, and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math. (2) **123** (1986), no. 3, 537–611. MR 840722 40

[Dol01] D Dolgopyat, *Lectures on u-gibbs states*, Online, 2001, `http://www2.math.umd.edu/~dolgop/ugibbs.pdf`. 40

[EKL06] Manfred Einsiedler, Anatole Katok, and Elon Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. (2) **164** (2006), no. 2, 513–560. MR 2247967 4

[EL10] M. Einsiedler and E. Lindenstrauss, *Diagonal actions on locally homogeneous spaces*, Homogeneous flows, moduli spaces and arithmetic, Clay Math. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 2010, pp. 155–241. MR 2648695 26

[FHY83] A. Fathi, M.-R. Herman, and J.-C. Yoccoz, *A proof of Pesin's stable manifold theorem*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 177–215. MR 730270 32

[Fri83] David Fried, *Entropy for smooth abelian actions*, Proc. Amer. Math. Soc. **87** (1983), no. 1, 111–116. MR 677244 35

[Fur67] Harry Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49. MR 213508 4

[GE74] Frederick P. Greenleaf and William R. Emerson, *Group structure and the pointwise ergodic theorem for connected amenable groups*, Advances in Math. **14** (1974), 153–172. MR 384997 16

[Gol21] W.M. Goldman, *Geometric structures on manifolds*, Online, 2021, `https://www.math.umd.edu/~wmg/gstom.pdf`. 13

[Gor95] M. I. Gordin, *Some remarks on homoclinic groups of hyperbolic automorphisms of tori*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **223** (1995), no. Teor. Predstav. Din. Sistemy, Kombin. i Algoritm. Metody. I, 140–147, 339. MR 1374317 61

[Gor07] Alexander Gorodnik, *Open problems in dynamics and related fields*, J. Mod. Dyn. **1** (2007), no. 1, 1–35. MR 2261070 4

[Gur69] B. M. Gurevič, *A certain condition for the existence of a K-decomposition for a special flow*, Uspehi Mat. Nauk **24** (1969), no. 5 (149), 233–234. MR 0265554 8

[Hu93a] Hu Yi Hu, *Some ergodic properties of commuting diffeomorphisms*, Ergodic Theory Dynam. Systems **13** (1993), no. 1, 73–100. MR 1213080 19, 35

[Hu93b] Huyi Hu, *Subadditivity of entropies of commuting diffeomorphisms and examples of non-existence of SBR measures*, ProQuest LLC, Ann Arbor, MI, 1993, Thesis (Ph.D.)–The University of Arizona. MR 2690742 19, 35

[Joh92] Aimee S. A. Johnson, *Measures on the circle invariant under multiplication by a nonlacunary subsemigroup of the integers*, Israel J. Math. **77** (1992), no. 1-2, 211–240. MR 1194793 4

[Kat07] Anatole Katok, *Fifty years of entropy in dynamics: 1958–2007*, J. Mod. Dyn. **1** (2007), no. 4, 545–596. MR 2342699 7

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 63

[KK01] Boris Kalinin and Anatole Katok, *Invariant measures for actions of higher rank abelian groups*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, pp. 593–637. MR 1858547 iii, 4, 7, 23

[KK02] ———, *Measurable rigidity and disjointness for $\mathbb{Z}^k$ actions by toral automorphisms*, Ergodic Theory Dynam. Systems **22** (2002), no. 2, 507–523. MR 1898802 7

[KK07] ———, *Measure rigidity beyond uniform hyperbolicity: invariant measures for Cartan actions on tori*, J. Mod. Dyn. **1** (2007), no. 1, 123–146. MR 2261075 4, 40, 47

[KKRH08] Boris Kalinin, Anatole Katok, and Federico Rodriguez Hertz, *New progress in nonuniform measure and cocycle rigidity*, Electron. Res. Announc. Math. Sci. **15** (2008), 79–92. MR 2457050 4, 5

[KKRH10] ———, *Errata to "Measure rigidity beyond uniform hyperbolicity: invariant measures for Cartan actions on tori" and "Uniqueness of large invariant measures for $\mathbb{Z}^k$ actions with Cartan homotopy data" [mr2261075; mr2285730]*, J. Mod. Dyn. **4** (2010), no. 1, 207–209. MR 2643892 2, 4

[KKRH11] ———, *Nonuniform measure rigidity*, Ann. of Math. (2) **174** (2011), no. 1, 361–400. MR 2811602 iii, 3, 4, 5, 10, 19, 35, 40, 47

[KKRH14] Anatole Katok, Svetlana Katok, and Federico Rodriguez Hertz, *The Fried average entropy and slow entropy for actions of higher rank abelian groups*, Geom. Funct. Anal. **24** (2014), no. 4, 1204–1228. MR 3248484 35

[KKS02] Anatole Katok, Svetlana Katok, and Klaus Schmidt, *Rigidity of measurable structure for $\mathbb{Z}^d$-actions by automorphisms of a torus*, Comment. Math. Helv. **77** (2002), no. 4, 718–745. MR 1949111 2

[KN11] Anatole Katok and Viorel Niţică, *Rigidity in higher rank abelian group actions. Volume I*, Cambridge Tracts in Mathematics, vol. 185, Cambridge University Press, Cambridge, 2011, Introduction and cocycle problem. MR 2798364 19

[KRH07] Anatole Katok and Federico Rodriguez Hertz, *Uniqueness of large invariant measures for $\mathbb{Z}^k$ actions with Cartan homotopy data*, J. Mod. Dyn. **1** (2007), no. 2, 287–300. MR 2285730 4

[KRH10] ———, *Measure and cocycle rigidity for certain nonuniformly hyperbolic actions of higher-rank abelian groups*, J. Mod. Dyn. **4** (2010), no. 3, 487–515. MR 2729332 4

[KRH16] ———, *Arithmeticity and topology of smooth actions of higher rank abelian groups*, J. Mod. Dyn. **10** (2016), 135–172. MR 3503686 iii, 3, 4, 5, 10, 35, 49

[KRH17] ———, *Non-uniform measure rigidity for $\mathbb{Z}^k$ actions of symplectic type*, Modern theory of dynamical systems, Contemp. Math., vol. 692, Amer. Math. Soc., Providence, RI, 2017, pp. 195–208. MR 3666074 4

[KS95] Anatole B. Katok and Klaus Schmidt, *The cohomology of expansive $\mathbf{Z}^d$-actions by automorphisms of compact, abelian groups*, Pacific J. Math. **170** (1995), no. 1, 105–142. MR 1359974 67

[KS96] A. Katok and R. J. Spatzier, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 751–778. MR 1406432 4

[KS98] ———, *Corrections to: "Invariant measures for higher-rank hyperbolic abelian actions" [Ergodic Theory Dynam. Systems* **16** *(1996), no. 4, 751–778; MR1406432 (97d:58116)]*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 503–507. MR 1619571 4

[KS17] Boris Kalinin and Victoria Sadovskaya, *Normal forms for non-uniform contractions*, J. Mod. Dyn. **11** (2017), 341–368. MR 3642250 49

[KSS14] Michael Kunzinger, Roland Steinbauer, and Milena Stojković, *The exponential map of a $C^{1,1}$-metric*, Differential Geom. Appl. **34** (2014), 14–24. MR 3209534 33

[Lan99] Serge Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics, vol. 191, Springer-Verlag, New York, 1999. MR 1666820 4

[Lin99] Elon Lindenstrauss, *Pointwise theorems for amenable groups*, Electron. Res. Announc. Amer. Math. Soc. **5** (1999), 82–90. MR 1696824 16

[Lin01] ———, *Pointwise theorems for amenable groups*, Invent. Math. **146** (2001), no. 2, 259–295. MR 1865397 16

[Lin05] ———, *Rigidity of multiparameter actions*, Israel J. Math. **149** (2005), 199–226, Probability in mathematics. MR 2191215 4

[LS82] François Ledrappier and Jean-Marie Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory Dynam. Systems **2** (1982), no. 2, 203–219 (1983). MR 693976 34

[LS99] Douglas Lind and Klaus Schmidt, *Homoclinic points of algebraic $\mathbf{Z}^d$-actions*, J. Amer. Math. Soc. **12** (1999), no. 4, 953–980. MR 1678035 61

[LSV13] Douglas Lind, Klaus Schmidt, and Evgeny Verbitskiy, *Homoclinic points, atoral polynomials, and periodic points of algebraic $\mathbb{Z}^d$-actions*, Ergodic Theory Dynam. Systems **33** (2013), no. 4, 1060–1081. MR 3082539 67

[LY85a] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122** (1985), no. 3, 509–539. MR 819556 32

[LY85b] ———, *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), no. 3, 540–574. MR 819557 37

[Lyo88] Russell Lyons, *On measures simultaneously* 2*- and* 3*-invariant*, Israel J. Math. **61** (1988), no. 2, 219–224. MR 941238 4

[Mat03] Shigenori Matsumoto, *On the global rigidity of split Anosov* $\mathbb{R}^n$*-actions*, J. Math. Soc. Japan **55** (2003), no. 1, 39–46. MR 1939183 6

[MS99] Anthony Manning and Klaus Schmidt, *Common homoclinic points of commuting toral automorphisms*, Israel J. Math. **114** (1999), 289–299. MR 1738686 67

[Ose68] V. I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trudy Moskov. Mat. Obšč. **19** (1968), 179–210. MR 0240280 19

[Par81] William Parry, *Topics in ergodic theory*, Cambridge Tracts in Mathematics, vol. 75, Cambridge University Press, Cambridge-New York, 1981. MR 614142 15

[Par86] ———, *Synchronisation of canonical measures for hyperbolic attractors*, Comm. Math. Phys. **106** (1986), no. 2, 267–275. MR 855312 15

[Pat88] Alan L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, vol. 29, American Mathematical Society, Providence, RI, 1988. MR 961261 16

[Pes76] Ja. B. Pesin, *Families of invariant manifolds that correspond to nonzero characteristic exponents*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 6, 1332–1379, 1440. MR 0458490 32

[Pes77a] ———, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Uspehi Mat. Nauk **32** (1977), no. 4 (196), 55–112, 287. MR 0466791 6, 32

[Pes77b] ———, *Geodesic flows in closed Riemannian manifolds without focal points*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 6, 1252–1288, 1447. MR 0488169 6

[PS82] Ya. B. Pesin and Ya. G. Sinaĭ, *Gibbs measures for partially hyperbolic attractors*, Ergodic Theory Dynam. Systems **2** (1982), no. 3-4, 417–438 (1983). MR 721733 40

[Pug11] Charles Pugh, *The closing lemma in retrospect*, Dynamics, games and science. I, Springer Proc. Math., vol. 1, Springer, Heidelberg, 2011, pp. 721–741. MR 3059636 7

[Rat06] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer, New York, 2006. MR 2249478 13

[RH21] F. Rodríguez Hertz, *Anatole Katok's work on measure rigidity*, Preprint, 2021. 4

[Roh52] V. A. Rohlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Translation **1952** (1952), no. 71, 55. MR 0047744 26

[RS75] David Ruelle and Dennis Sullivan, *Currents, flows and diffeomorphisms*, Topology **14** (1975), no. 4, 319–327. MR 415679 26

[Rud90] Daniel J. Rudolph, *$\times 2$ and $\times 3$ invariant measures and entropy*, Ergodic Theory Dynam. Systems **10** (1990), no. 2, 395–406. MR 1062766 4

[Rue78a] David Ruelle, *Integral representation of measures associated with a foliation*, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 127–132. MR 516915 61

[Rue78b] ______, *Thermodynamic formalism*, Encyclopedia of Mathematics and its Applications, vol. 5, Addison-Wesley Publishing Co., Reading, Mass., 1978, The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota. MR 511655 61

[Rue88] ______, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93** (1988), no. 1, 1–13. MR 943921 67

[Rue89] ______, *Elements of differentiable dynamics and bifurcation theory*, Academic Press, Inc., Boston, MA, 1989. MR 982930 17

[Sin72] Ja. G. Sinaĭ, *Gibbs measures in ergodic theory*, Uspehi Mat. Nauk **27** (1972), no. 4(166), 21–64, English translation: Russian Math. Surveys 27 (1972), no. 4, 21–69. MR 0399421 40

[SS18] Clemens Sämann and Roland Steinbauer, *On geodesics in low regularity*, J. Phys. Conf. Ser. **968** (2018), 012010, 14. MR 3919953 33

[SV19] Ralf Spatzier and Kurt Vinhage, *Cartan actions of higher rank abelian groups and their classification*, 2019. 2, 59

[Tho10a] Klaus Thomsen, *C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics*, Mem. Amer. Math. Soc. **206** (2010), no. 970, x+122. MR 2667385 67

[Tho10b] ______, *The homoclinic and heteroclinic C^*-algebras of a generalized one-dimensional solenoid*, Ergodic Theory Dynam. Systems **30** (2010), no. 1, 263–308. MR 2586354 67

[Thu97] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy. MR 1435975 13

[Tot66] Haruo Totoki, *Time changes of flows*, Mem. Fac. Sci. Kyushu Univ. Ser. A **20** (1966), 27–55. MR 201606 15

[Tot70] ______, *On a class of special flows*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **15** (1970), 157–167. MR 279279 8

[Ver74] Alberto Verjovsky, *Codimension one Anosov flows*, Bol. Soc. Mat. Mexicana (2) **19** (1974), no. 2, 49–77. MR 431281 5

[Wil12] Frank Wilczek, *Quantum time crystals*, Phys. Rev. Lett. **109** (2012), 160401. 7

[YCLZ20] Norman Y. Yao, Nayak Chetan, Balents Leon, and Michael P. Zaletel, *Classical discrete time crystals*, Nature Physics **16** (2020), no. 4, 438–447. 7

[YN18] Norman Y. Yao and Chetan Nayak, *Time crystals in periodically driven systems*, Physics Today **71** (2018), no. 9, 40–47. 7

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