Ex: $\ddot{y} - 3\dot{y} - 4y = 2e^{-t} + 3e^{2t} + 2\sin t - 8e^{t}\cos(2t)$.

$\mathcal{A} := \partial_t^2 - 3\partial_t - 4$.
$g_1(t) := 2e^{-t}$
$g_2(t) := 3e^{2t}$
$g_3(t) := 2\sin(t)$
$g_4(t) := -8e^{t}\cos(2t)$.

$\mathcal{A}(y)(t) = g_1(t) + g_2(t) + g_3(t) + g_4(t)$

splice:

ⓒ $\mathcal{A}(y)(t) = 0 \leftarrow y_c(t)$ (complementary sol.)

① $\mathcal{A}(y)(t) = g_1(t) \leftarrow y_1(t)$ (particular sol.)

② $\mathcal{A}(y)(t) = g_2(t) \leftarrow y_2(t)$ (— " —)

③ $\mathcal{A}(y)(t) = g_3(t) \leftarrow y_3(t)$ (— " —)

④ $\mathcal{A}(y)(t) = g_4(t) \leftarrow y_4(t)$ (— " —).

ⓒ $\mathrm{char}(\lambda) = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$

$$\Rightarrow \boxed{y_c(t) = c_1 e^{-t} + c_2 e^{4t}, \quad c_1, c_2 \in \mathbb{R}}$$

① Try $y_1(t) = \text{poly} \cdot e^{-t} = (At^2 + Bt + C)e^{-t}$.

$$\dot{y}_1(t) = (2At + B)e^{-t} - (At^2 + Bt + C)e^{-t}$$
$$= [(-A)t^2 + (2A - B)t + (B - C)]e^{-t}$$

$$\ddot{y}_1(t) = (-2At + 2A - B)e^{-t} - ((-A)t^2 + (2A-B)t + (B-C))e^{-t}$$
$$= [At^2 + (-4A + B)t + (2A - 2B + C)]e^{-t}.$$

$$2e^{-t} = g_1(t) = \ddot{y}_1(t) - 3\dot{y}_1(t) - 4y_1(t) = t^2e^{-t}(A + 3A - 4A)$$
$$+ te^{-t}(-4A + B - 6A + 3B - 4B)$$
$$+ e^{-t}(2A - 2B + C - 3B + 3C - 4C)$$

$$= -10A\,te^{-t} + (2A - 5B)e^{-t} \Rightarrow \left.\begin{array}{l} -10A = 0 \\ 2A - 5B = 2 \end{array}\right\} \quad \left.\begin{array}{l} A = 0 \\ B = -\frac{2}{5} \\ C \text{ free.} \end{array}\right\} \Rightarrow \boxed{y_1(t) = -\frac{2}{5}te^{-t}}$$

② Try $y_2(t) = A e^{2t}$.

$\dot{y}_2(t) = 2A e^{2t}, \quad \ddot{y}_2(t) = 4A e^{2t}$

$$3e^{2t} = g_2(t) = \ddot{y}_2(t) - 3\dot{y}_2(t) - 4y_2(t) = (4A - 6A - 4A)\, e^{2t}$$
$$= -6A\, e^{2t}$$

$$\Rightarrow A = -\frac{1}{2} \quad \Rightarrow \boxed{y_2(t) = -\frac{1}{2} e^{2t}}$$

③ Try $y_3(t) = A \sin t + B \cos t$

$\dot{y}_3(t) = -B \sin t + A \cos t$,

$\ddot{y}_3(t) = -A \sin t - B \cos t$.

$$2 \sin t = g_3(t) = \ddot{y}_3(t) - 3\dot{y}_3(t) - 4y_3(t) = \sin t\,(-A + 3B - 4A)$$
$$+ \cos t\,(-B - 3A - 4B)$$

$$\Rightarrow \left.\begin{aligned} 2 &= -5A + 3B \\ 0 &= -3A - 5B \end{aligned}\right\} \Rightarrow \begin{aligned} -6 &= 15A - 9B \\ 0 &= -15A - 25B \end{aligned} \quad \Rightarrow \begin{aligned} -6 &= -34B \\ B &= +\frac{3}{17} \quad A = \frac{-5}{17} \end{aligned}$$

$$\Rightarrow \boxed{y_3(t) = -\frac{5}{17} \sin(t) + \frac{3}{17} \cos(t)}$$

④ Try $y_4(t) = A e^t \cos(2t) + B e^t \sin(2t)$.

$$\dot{y}_4(t) = e^t \cos(2t)\,(A + 2B)$$
$$+ e^t \sin(2t)\,(-2A + B)$$

$$\ddot{y}_4(t) = e^t \cos(2t)\,(A + 2B - 4A + 2B) = e^t \cos(2t)\,(-3A + 4B)$$
$$+ e^t \sin(2t)\,(-2A - 4B - 2A + B) \quad + e^t \sin(2t)\,(-4A - 3B).$$

$$-8e^t \cos(2t) = g_4(t) = \ddot{y}_4(t) - 3\dot{y}_4(t) - 4y_4(t) = e^t \cos(2t)\,(-3A + 4B - 3A - 6B - 4A)$$
$$+ e^t \sin(2t)\,(-4A - 3B + 6A - 3B - 4B)$$

$$\left.\begin{aligned} -8 &= -10A - 2B \\ 0 &= 2A - 10B \end{aligned}\right\} \quad \begin{aligned} -8 &= -52B \\ A &\overset{\Uparrow}{=} 5B \end{aligned} \quad \Rightarrow \quad B = \frac{8}{52} = \frac{2}{13}, \quad A = \frac{10}{13}$$

$$\Rightarrow \boxed{y_4(t) = \frac{10}{13} e^{t} \cos(2t) + \frac{2}{13} e^{t} \sin(2t)}$$

$$y(t) = y_c(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t).$$

$$\Rightarrow \boxed{\begin{gathered} y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{2}{5} t e^{-t} - \frac{1}{2} e^{2t} - \frac{5}{17} \sin(t) + \frac{3}{17} \cos(t) + \frac{10}{13} e^{t} \cos(2t) + \frac{2}{13} e^{t} \sin(2t), \\ c_1, c_2 \in \mathbb{R} \end{gathered}}$$

is the gen. sol..

SW: Also add $g_5(t) = e^{-t} \cos(2t)$ to the RHS