WIKIPEDIA

Almost Mathieu operator

In mathematical physics, the **almost Mathieu operator** arises in the study of the quantum Hall effect. It is given by

$$[H_\omega^{\lambda,\alpha}u](n) = u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n),$$

acting as a self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z})$. Here $\alpha, \omega \in \mathbb{T}, \lambda > 0$ are parameters. In pure mathematics, its importance comes from the fact of being one of the best-understood examples of an ergodic Schrödinger operator. For example, three problems (now all solved) of Barry Simon's fifteen problems about Schrödinger operators "for the twenty-first century" featured the almost Mathieu operator.[1]

For $\lambda = 1$, the almost Mathieu operator is sometimes called **Harper's equation**.

The spectral type

If α is a rational number, then $H_\omega^{\lambda,\alpha}$ is a periodic operator and by Floquet theory its spectrum is purely absolutely continuous.

Now to the case when α is irrational. Since the transformation $\omega \mapsto \omega + \alpha$ is minimal, it follows that the spectrum of $H_\omega^{\lambda,\alpha}$ does not depend on ω. On the other hand, by ergodicity, the supports of absolutely continuous, singular continuous, and pure point parts of the spectrum are almost surely independent of ω. It is now known, that

- For $0 < \lambda < 1$, $H_\omega^{\lambda,\alpha}$ has surely purely absolutely continuous spectrum. [2] (This was one of Simon's problems.)
- For $\lambda = 1$, $H_\omega^{\lambda,\alpha}$ has almost surely purely singular continuous spectrum.[3] (It is not known whether eigenvalues can exist for exceptional parameters.)
- For $\lambda > 1$, $H_\omega^{\lambda,\alpha}$ has almost surely pure point spectrum and exhibits Anderson localization.[4] (It is known that almost surely can not be replaced by surely.)[5][6]

That the spectral measures are singular when $\lambda \geq 1$ follows (through the work of Last and Simon) [7] from the lower bound on the Lyapunov exponent $\gamma(E)$ given by

$$\gamma(E) \geq \max\{0, \log(\lambda)\}.$$

This lower bound was proved independently by Avron, Simon and Michael Herman, after an earlier almost rigorous argument of Aubry and André. In fact, when E belongs to the spectrum, the inequality becomes an equality (the Aubry–André formula), proved by Jean Bourgain and Svetlana Jitomirskaya.[8]

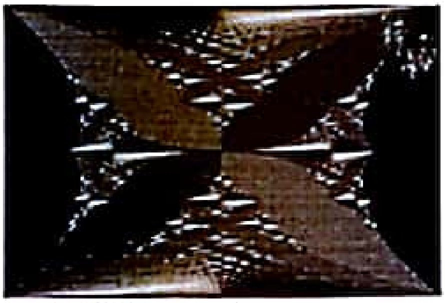
The structure of the spectrum

Another striking characteristic of the almost Mathieu operator is that its spectrum is a Cantor set for all irrational α and $\lambda > 0$. This was shown by Avila and Jitomirskaya solving the by-then famous "ten martini problem"[9] (also one of Simon's problems) after several earlier results (including generically[10] and almost surely[11] with respect to the parameters).

Furthermore, the Lebesgue measure of the spectrum of the almost Mathieu operator is known to be

$$\mathrm{Leb}(\sigma(H_\omega^{\lambda,\alpha})) = |4-4\lambda|$$

for all $\lambda > 0$. For $\lambda = 1$ this means that the spectrum has zero measure (this was first proposed by Douglas Hofstadter and later became one of Simon's problems[12]). For $\lambda \neq 1$, the formula was discovered numerically by Aubry and André and proved by Jitomirskaya and Krasovsky.



Hofstadter's butterfly

The study of the spectrum for $\lambda = 1$ leads to the Hofstadter's butterfly, where the spectrum is shown as a set.

References

1. Simon, Barry (2000). "Schrödinger operators in the twenty-first century". *Mathematical Physics 2000*. London: Imp. Coll. Press. pp. 283–288. ISBN 186094230X.
2. Avila, A. (2008). "The absolutely continuous spectrum of the almost Mathieu operator". *Preprint*. arXiv:0810.2965 (https://arxiv.org/abs/0810.2965).
3. Gordon, A. Y.; Jitomirskaya, S.; Last, Y.; Simon, B. (1997). "Duality and singular continuous spectrum in the almost Mathieu equation". *Acta Math.* 178 (2): 169–183. doi:10.1007/BF02392693 (https://doi.org/10.1007%2FBF02392693).
4. Jitomirskaya, Svetlana Ya. (1999). "Metal-Insulator transition for the almost Mathieu operator". *Ann. of Math.* 150 (3): 1159–1175. JSTOR 121066 (https://www.jstor.org/stable/121066).
5. Avron, J.; Simon, B. (1982). "Singular continuous spectrum for a class of almost periodic Jacobi matrices". *Bull. Amer. Math. Soc.* 6 (1): 81–85. doi:10.1090/s0273-0979-1982-14971-0 (https://doi.org/10.1090%2Fs0273-0979-1982-14971-0). Zbl 0491.47014 (https://zbmath.org/?format=complete&q=an:0491.47014).
6. Jitomirskaya, S.; Simon, B. (1994). "Operators with singular continuous spectrum, III. Almost periodic Schrödinger operators". *Comm. Math. Phys.* 165 (1): 201–205. doi:10.1007/bf02099743 (https://doi.org/10.1007%2Fbf02099743). Zbl 0830.34074 (https://zbmath.org/?format=complete&q=an:0830.34074).
7. Last, Y.; Simon, B. (1999). "Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators". *Invent. Math.* 135 (2): 329–367. doi:10.1007/s002220050288 (https://doi.org/10.1007%2Fs002220050288).
8. Bourgain, J.; Jitomirskaya, S. (2002). "Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential". *Journal of Statistical Physics*. 108 (5–6): 1203–1218. doi:10.1023/A:1019751801035 (https://doi.org/10.1023%2FA%3A1019751801035).
9. Avila, A.; Jitomirskaya, S. (2005). "The Ten Martini problem". *Preprint*. arXiv:math/0503363 (https://arxiv.org/abs/math/0503363).
10. Bellissard, J.; Simon, B. (1982). "Cantor spectrum for the almost Mathieu equation". *J. Funct. Anal.* 48 (3): 408–419. doi:10.1016/0022-1236(82)90094-5 (https://doi.org/10.1016%2F0022-1236%2882%2990094-5).
11. Puig, Joaquim (2004). "Cantor spectrum for the almost Mathieu operator". *Comm. Math. Phys.* 244 (2): 297–309. doi:10.1007/s00220-003-0977-3 (https://doi.org/10.1007%2Fs00220-003-0977-3).
12. Avila, A.; Krikorian, R. (2006). "Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles". *Annals of Mathematics*. 164 (3): 911–940. doi:10.4007/annals.2006.164.911 (https://doi.org/10.4007%2Fannals.2006.164.911).

Almost Mathieu Operator - Wiki :

- $\forall\, \alpha, \omega \in \mathbb{T}^1,\ \forall \lambda > 0$:

$$H_\omega^{\lambda,\alpha} : \ell^2(\mathbb{Z}, \mathbb{C}) \longrightarrow \ell^2(\mathbb{Z}, \mathbb{C})$$

$$u = \{u(n)\}_n \longmapsto \left\{ \begin{array}{l} u(n+1) + u(n-1) \\ + 2\lambda \cos\big(2\pi \underbrace{(\omega + n\alpha)}_{R_\alpha^n(\omega)}\big)\, u(n) \end{array} \right\}_n$$

is an almost Mathieu operator.
(When $\lambda := 1$, $H_\omega^{1,\alpha}$ is a Harper operator).

- Almost Mathieu operators arise in the study of the quantum Hall effect in mathematical physics. They are important from a purely mathematical perspective as well, since they are [redacted] very well-understood examples of ergodic Schrödinger operators.

- $H_\omega^{\lambda,\alpha}$ is self-adjoint.

§. The Spectral Type:

- We can state what is known about H by first considering two cases:

(i) $\alpha \in \mathbb{Q}$ (mod 1)

(ii) $\alpha \notin \mathbb{Q}$ (mod 1).

(i) In this case $H_\omega^{\lambda,\alpha}$ is periodic, and (in solid-state physics: Bloch theory) Floquet theory guarantees that it will have purely absolutely continuous spectrum (for any λ and ω).

(ii) In this case
$$R_\alpha : \begin{array}{ccc} \mathbb{T}^1 & & \mathbb{T}^1 \\ \| & & \| \\ S^1 & \longrightarrow & S^1 \\ \omega & \longmapsto & \omega+\alpha \end{array}$$
is minimal (ie., $\forall \omega \in \mathbb{T}^1 : \overline{\{R_\alpha^n(\omega) \mid n \in \mathbb{Z}\}} = \mathbb{T}^1$), whence the spectrum of $H_\omega^{\lambda,\alpha}$ is independent of ω.

In addition, (unique) ergodicity of R_α provides that the supports of absolutely continuous spectrum, singular continuous spectrum, and pure point spectrum of $H_\omega^{\lambda,\alpha}$ are ae independent of ω.

Facts:

(I) $\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{leb} S^1, \forall \lambda \in]0,1[$: $H_\omega^{\lambda,\alpha}$ has surely purely absolutely continuous spectrum.

(II) $\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{leb} S^1$, [redacted] $(\lambda := 1)$: $H_\omega^{1,\alpha}$ has almost surely purely singular continuous spectrum.

(though existence of eigenvalues for exceptional parameters [redacted] is open)

(III) $\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{leb} S^1, \forall \lambda \in]1,\infty[$: $H_\omega^{\lambda,\alpha}$ has almost surely, but not surely, pure point spectrum and it exhibits Anderson localization.

(in condensed matter physics: the absence of diffusion of waves in a disordered medium)

In cases (II) and (III), (i.e., when $\lambda \in [1, \infty[$), we have that the spectral measures are singular.
This comes from the following lower bound on the Lyapunov exponent $\gamma(E)$:

$$\boxed{\gamma(E) \geq \max\{0, \log(\lambda)\} \stackrel{\text{def}}{=} \log^+(\lambda)}$$

obtained by Avron, Simon, Herman and Last, Aubry, André.

When $E \in \sigma(H_\omega^{\lambda,\alpha})$, the inequality becomes an equality:

$$\boxed{\begin{array}{l} \forall \alpha \notin \mathbb{Q}, \forall \omega \in_\infty S^1, \forall \lambda \in [1, \infty[, \\ \quad \forall E \in \sigma(H_\omega^{\lambda,\alpha}) : \gamma(E) = \log^+(\lambda) \quad \begin{pmatrix}\text{Aubry–} \\ \text{André} \\ \text{formula}\end{pmatrix} \end{array}},$$

proved by Bourgain and Jitomirskaya.

§. The Structure of the Spectrum of $H_\omega^{\lambda,\alpha}$:

- Avila and Jitomirskaya's conclusive solution to the famous [redacted] "Ten Martini Problem" shows that $\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{ae} S^1, \forall \lambda \in]0,\infty[$:

  $\sigma(H_\omega^{\lambda,\alpha})$ is a Cantor set (i.e., perfect & nowhere dense) (closed & no isolated points).

?

- $$\boxed{\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{ae} S^1, \forall \lambda \in]0,\infty[: \operatorname{leb}\left(\sigma(H_\omega^{\lambda,\alpha})\right) = |4-4\lambda|} \qquad (\text{IV})$$

In particular,

?

$$\forall \alpha \notin \mathbb{Q}, \forall \omega \in_{ae} S^1 \ (\lambda := 1) : \operatorname{leb}\left(\sigma(H_\omega^{1,\alpha})\right) = 0,$$

which was conjectured earlier by Hofstadter.

(IV) was first computed [redacted] numerically by Aubry and André, and then it was proved by Jitomirskaya and Krasovsky.

- Fixing $\lambda := 1$ and varying α leads to the Hofstadter's butterfly.

———— .

Hofstadter - GEB, Ch. 5:

§. Two Striking Recursive Graphs:

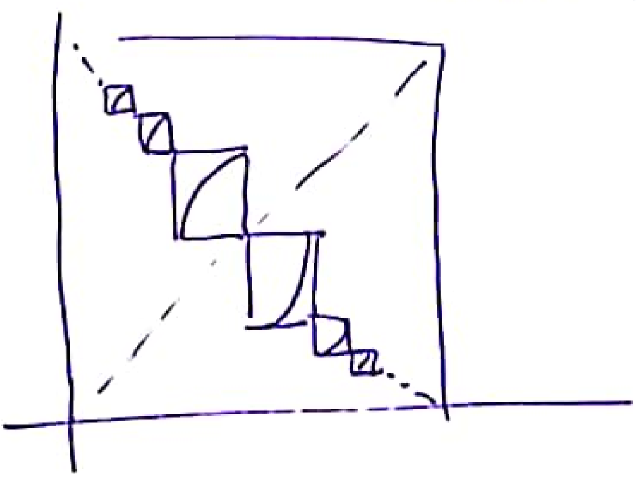
- We are interested in two shapes in this section:

(i) The graph of a particular number-theoretical function INT: $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$

(ii) Gplot, which came up in the theoretical Ph.D work of Hofstadter in solid state physics.

(i) Graph of INT:

What corresponds to the "bottom" in the definition of INT is a picture composed of many boxes, showing where the copies go, and how they are distorted; called the "skeleton" of the graph INT:



To construct the graph of INT from its skeleton, we follow the following algorithm:

(1) For each box in the skeleton, do two operations:

(1.1) Put a small curved copy of the skeleton inside the box, using the curved line inside as a guide;

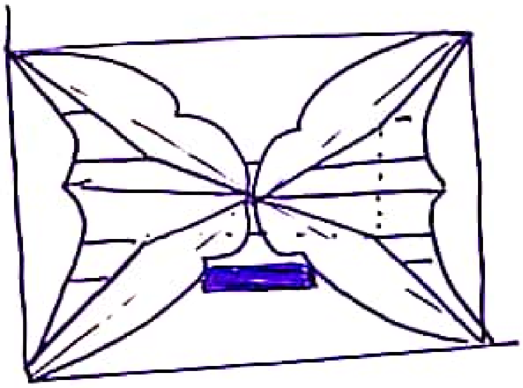
(1.2) Erase the containing box and its curved line.

(2) Repeat the process one level down, with all the baby skeletons.

IFS?

By nesting the skeleton inside itself over and over again, one gradually constructs the graph of INT "from out of nothing". But in fact "nothing" was not nothing — it was a picture.

(ii) Applying the same procedure with a different skeleton produces the Gplot:



Gplot is thus a member of the INT-family. It is a distant relative, because its skeleton is quite different from — and considerably more complicated than — that of INT. However, the recursive part of the definition is identical, and therein lies the family tie.

Gplot comes from a highly idealized version of the following question:

"What are the allowed energies of electrons in a crystal in a magnetic field?"

This problem is interesting because it is a cross between two very simple and fundamental physics questions:

→ an electron in a perfect crystal, and
→ an electron in a homogeneous magnetic field.

These two simpler problems are both well understood, and their characteristic solutions seem almost incompatible with each other. Therefore, it is of quite some interest to see how nature manages to reconcile the two.

As it happens, the crystal-without-magnetic field situation and the magnetic-field-without-crystal situation do have one feature in common: in each of them, the <u>electron behaves periodically in time</u>. When the two situations are combined, the ratio of their two time periods is the key parameter. In fact, that ratio holds all the information about the distribution of allowed electron energies — but it only gives up its secret upon being expanded into a continued fraction.

Gplot shows that distribution. The horizontal axis represents energy, and the vertical axis represents the above-mentioned ratio of time periods, denoted by α. At the bottom, $\alpha = 0$; and at the top, $\alpha = 1$.

When $\alpha = 0$, there is no magnetic field. Each of the line segments making up Gplot is an "energy band" — that is, it represents allowed values of energy. The empty swaths traversing Gplot on all different size scales are therefore regions of forbidden energy. (Can we recover the no crystal situation?)

When $\alpha = p/q \in \mathbb{Q}$ $((p,q) = 1)$, there are exactly q such energy bands. When q is even, two of the energy [redacted] bands "kiss" in the middle.

When α is irrational, the bands shrink to points, of which there are infinitely many, very sparsely distributed in a so-called "Cantor set".

"Ten Martini Problem"

"You might well wonder whether such an intricate structure would ever show up in an experiment. Frankly, I would be the most surprised person in the world if Gplot came out of an experiment. The physicality of Gplot lies in the fact that it points the way on the proper mathematical treatment of less idealized problems of this sort. In other words, Gplot is purely a contribution to theoretical physics, not a hint to experimentalists as to what to expect to see! An agnostic friend of mine once was so struck by Gplot's infinitely many infinities that he called it "a picture of God," which I don't think is blasphemous at all."

(experimented!)

Wilkinson – What are Lyapunov Exponents, and Why are they Interesting?

§4. Hofstadter's Butterfly:

- Define $\forall x \in \mathbb{R}$, $\forall \alpha \in [0,1]$:

$$H_x^\alpha : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C})$$
$$u = \{u(n)\}_n \mapsto \begin{Bmatrix} u(n+1) + u(n-1) \\ + 2\cos(2\pi(x+\alpha n))u(n) \end{Bmatrix}_n .$$

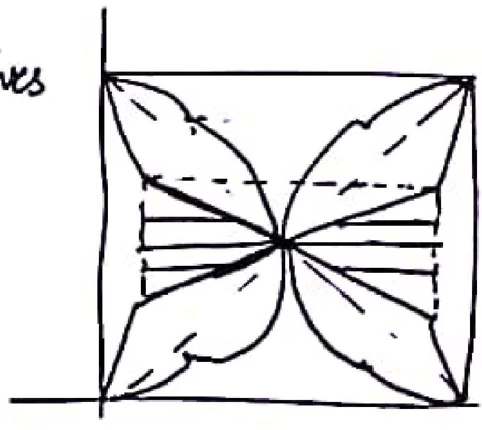
Here x is the <u>phase</u> and α is the <u>frequency</u>.

The <u>spectrum</u> of H_x^α is $\sigma(H_x^\alpha) \overset{\text{def}}{=} \{E \in \mathbb{C} \mid H_x^\alpha - E\cdot \mathrm{id} \text{ is not invertible}\}$,

any $E \in \bigsqcup_{\substack{\alpha \in [0,1] \\ x \in \mathbb{R}}} \sigma(H_x^\alpha)$ is an <u>energy</u>, any

$E \in \{E \in \sigma(H_x^\alpha) \mid H_x^\alpha - E \text{ is not injective}\}$ is an <u>eigenvalue</u> of H_x^α.

- Plotting $\mathcal{HB} := [-4,4] \times \bigsqcup_{\alpha \in [0,1]} \sigma(H_x^\alpha)$ gives the <u>Hofstadter Butterfly</u>.



. This fractal picture was discovered by Hofstadter while modelling the behavior of electrons in a crystal lattice under the force of a magnetic field.

. The operator H_x^α plays a central role in the theory of the integer quantum Hall effect, and, as predicted theoretically, the butterfly has indeed appeared in certain experiments.

<u>Theorem</u> (AK - Red. or NUH):

$$\forall \alpha \in [0,1] \setminus \mathbb{Q} : \; leb\left(\sigma\left(H_x^\alpha\right)\right) = 0$$

. Other properties of the butterfly, e.g., its Hausdorff dimension, remain unknown (2016).

<u>Cor</u>: $leb^2(\mathcal{HB}) = 0$.

. There is an interesting relation between the spectrum of this operator and cocycles. (See the Key Observation below)

• Define

$$\forall \alpha \in [0,1] : \; R_\alpha : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$x \longmapsto x + \alpha \pmod 1$$

$$\forall E \in [-4,4] : \; A_E : \mathbb{R}/\mathbb{Z} \longrightarrow SL(2,\mathbb{R})$$
$$x \longmapsto \begin{pmatrix} E - 2\cos(2\pi x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we can consider A_E as a <u>cocycle</u> over R_α:

$\forall \alpha \in [0,1], \forall E \in [-4,4]$:

$$\mathcal{S}_{\alpha,E} : \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \longrightarrow SL(2,\mathbb{R})$$
$$\vdots$$
$$(\omega, n) \longmapsto A_E(R_\alpha^{n-1}(\omega)) \dots A_E(\omega) = \prod_{k=n-1}^{0} A_E(R_\alpha^k(\omega))$$
$$\vdots$$
$$(\omega, 1) \longmapsto A_E(\omega)$$
$$(\omega, 0) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$(\omega, -1) \longmapsto \left(A_E(R_\alpha^{-1}(\omega))\right)^{-1}$$
$$\vdots$$
$$(\omega, -n) \longmapsto A_E(R_\alpha^{-n}(\omega))^{-1} \dots A_E(R_\alpha^{-1}(\omega))^{-1} = \left(A_E(R_\alpha^{-1}(\omega)) \dots A_E(R_\alpha^{-n}(\omega))\right)^{-1}$$

■ [redacted] $= \mathcal{S}_{\alpha,E}(R_\alpha^{-n}(\omega), n)^{-1}$

$$\begin{tikzcd}
\cdots & & x \arrow[dll, "2"'] \arrow[dl, "1"'] \arrow[d, "0"] \arrow[dr, "-1"] \arrow[drr, "-2"] & & \cdots \\
A_E(R_\alpha(\omega))A_E(\omega)(x) & A_E(\omega)(x) \arrow[l, "A_E(R_\alpha(\omega))"] & x \arrow[l, "A_E(\omega)"] & A_E(R_\alpha^{-1}(\omega))^{-1}(x) \arrow[l, "A_E(R_\alpha^{-1}(\omega))"] & A_E(R_\alpha^{-2}(\omega))^{-1}A_E(R_\alpha^{-1}(\omega))^{-1}(x) \arrow[l, "A_E(R_\alpha^{-2}(\omega))"]
\end{tikzcd},$$

whence we also have the (a) <u>skew product</u>

$\forall \alpha \in [0,1], \forall E \in [-4,4]$:

[redacted] $$F_{\alpha,E} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$$
$$(\omega, p) \longmapsto \left(R_\alpha(\omega), \mathcal{S}_{\alpha,E}(\omega)(p)\right).$$

Here $\mathcal{S}_{\alpha,E}$ is called the (a) <u>Schrödinger cocycle</u>.

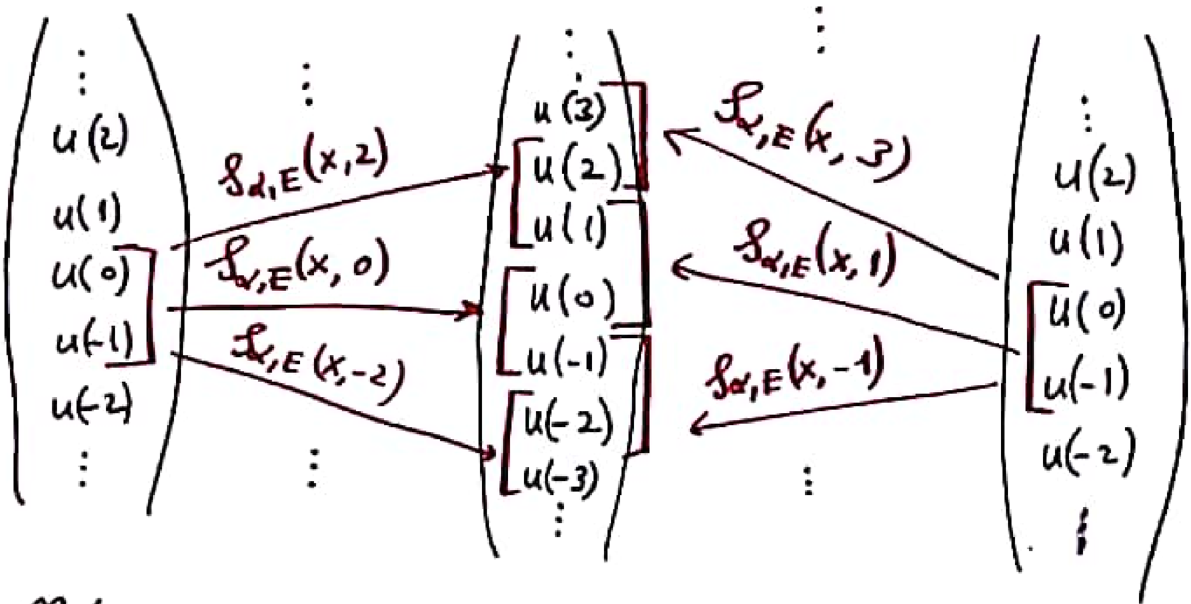
<u>Key Observation</u>: TFAE, $\forall \alpha \in [0,1], \forall x \in \mathbb{R}, \forall E \in [-4,4]$:

(i) [redacted] $u \in \mathbb{C}^{\mathbb{Z}}$: $H_x^\alpha u = E u$

(ii) $\forall n \in \mathbb{Z}$: $A_E(R_\alpha^n(x))\begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} = \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}$

(iii) $\forall n \in \mathbb{Z}$: $A_E(R_\alpha^n(x))\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$

(iv) $\forall n \in \mathbb{Z}$: $\mathcal{S}_{\alpha,E}(x,n)\begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$



Pf: $H_x^\alpha u = E u$

$\Leftrightarrow \forall n \in \mathbb{Z}$: [redacted] $H_x^\alpha u(n) = E u(n)$

$\Leftrightarrow \forall n \in \mathbb{Z} : u(n+1) + u(n-1) + 2\cos(2\pi(x+n\alpha))\, u(n) = E u(n)$

$\Leftrightarrow \forall n \in \mathbb{Z} : (E - 2\cos(2\pi R_\alpha^n(x)))\, u(n)$ [redacted] $- u(n+1)$ [redacted] $- u(n-1) = 0$

$\Leftrightarrow \forall n \in \mathbb{Z} : \begin{pmatrix} E - 2\cos(2\pi R_\alpha^n(x)) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} = \begin{pmatrix} u(n\text{[redacted]}) \\ u(n) \end{pmatrix}$ ((i)$\Leftrightarrow$(ii) ok!)

$\Leftrightarrow \forall n \in \mathbb{Z} : \begin{pmatrix} E - 2\cos(2\pi R_\alpha^n(x)) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$ ((i)$\Leftrightarrow$(iii) ok!)

$\Leftrightarrow$ [redacted] $\underbrace{A_E(x)}_{= S_{\alpha,E}(x,1)} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$, $\underset{(n>0)}{S_{\alpha,E}(x,n)} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = A_E(R_\alpha^{n-1}(x)) \cdots A_E(x) \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$

$\underbrace{A_E(R_\alpha^{-1}(x))}_{= S_{\alpha,E}(x,-1)} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = \begin{pmatrix} u(-1) \\ u(-2) \end{pmatrix}$, $S_{\alpha,E}(x,-n) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} = A_E(R_\alpha^{-n}(x))^{-1} \cdots A_E(R_\alpha^{-1}(x))^{-1} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} u(-n) \\ u(-n-1) \end{pmatrix}$

(and by induction, (iii)$\Leftrightarrow$(iv) ok!)

• A cocycle $\mathcal{S}: \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \to SL(2,\mathbb{R})$ over $R: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is <u>uniformly hyperbolic</u> if

$\exists\, E^s, E^u \in C^0\left(\mathbb{R}/\mathbb{Z},\ \mathbb{P}(\mathbb{R}^2)\right),\ \exists\, C \in\,]0,\infty[,\ \exists\, \lambda \in\,]0,1[,$ $\forall x \in \mathbb{R}/\mathbb{Z},\ \forall n \geq 1$:

(i) $E^s(x) \oplus E^u(x) = \mathbb{R}^2$, (This splitting will be automatically unique and hence $\mathcal{S}$-invariant, i.e., $\mathcal{S}(x,n)\, E^{s,u}(x) = E^{s,u}(R^{n}$ [redacted] $(x))$)

(ii) $\left\| \mathcal{S}(x,n) \big|_{E^s(x)} \right\| \leq C \lambda^n$,

(iii) $\left\| \mathcal{S}(x,-n) \big|_{E^u(x)} \right\| \leq C \lambda^n$.

• Suppose $\exists\, \alpha \in [0,1], \exists\, E \in [-4,4]$: $S_{\alpha,E}$ is uniformly hyperbolic. Let $x \in \mathbb{R}$ be an arbitrary phase. Then if $u \in \mathbb{C}^{\mathbb{Z}}$ is a solution to $H_x^\alpha u = Eu$, it mustn't be polynomially bounded in both stable and unstable directions simultaneously, whence $u \notin \ell^2(\mathbb{Z}, \mathbb{C})$, i.e., $E \notin \sigma(H_x^\alpha)$. It turns out the converse is also true; in fact we have:

<u>Thm</u> (<u>Johnson</u>): $\forall \alpha \in [0,1] \setminus \mathbb{Q}$, $\forall x \in [0,1]$:

$$\sigma(H_x^\alpha) = \{E \in \mathbb{C} \mid S_{\alpha,E} \text{ is not uniformly hyperbolic}\}.$$

Via Johnson's Theorem, for irrational α, $\Sigma_\alpha := \sigma(H_x^\alpha)$ is independent of the phase parameter x.

Thus for irrational α, Σ_α corresponds to the colored regions of the α-horizontal slice of $\mathcal{HB}$.

<u>Cor</u>: Uniform hyperbolicity is an open condition (for both
(Obs)
α and E), whence $\mathcal{HB}$ is closed.

?

• $\mathcal{HB}$ is therefore both a dynamical and a spectral picture. On the one hand it depicts the spectrum of a family of operators $\{H_x^\alpha\}_{\alpha\in[0,1]}$, and on the other hand it depicts the set of parameters $(E,\alpha) \in [-4,4]\times[0,1]$ (energy × frequency) for which $S_{\alpha,E}$ is not uniformly hyperbolic.

• If $\alpha \in [0,1]\setminus\mathbb{Q}$, then R_α is (uniquely) ergodic. Then if $S_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbf{SL}(2,\mathbb{R})$ is a cocycle over R_α, it satisfies the <u>integrability</u> <u>condition</u> ?

$$\int_{\mathbb{R}/\mathbb{Z}} \Big[\log\big(\|S_\alpha(x,1)\|\big) + \log\big(\|S_\alpha(x,-1)\|\big)\Big]\, d\,\mathrm{leb}_{\mathbb{R}/\mathbb{Z}}(x) < \infty.$$

Whence Oseledets' Multiplicative Ergodic Theorem guarantees the following:

$$\forall x \in_{leb} \mathbb{R}/2\mathbb{Z},\ \forall p \in \mathbb{R}^2 \setminus 0 : \lim_{n\to\infty} \frac{\log \| S_\alpha(x;n)(p) \|}{n} =: \chi(x,p),$$

which is called the Lyapunov exponent of S_α at x in the direction p, exists; and we have the dichotomy $\forall x \in_{leb} \mathbb{R}/2\mathbb{Z}$:

$$\chi(x,\cdot) \equiv 0 \quad \text{XOR} \quad \{\chi(x,\cdot) > 0\} \uplus \{\chi(x,\cdot) < 0\} = \mathbb{R}^2 \setminus 0.$$

Further, again by ergodicity, χ takes constant values $leb_{\mathbb{R}/2\mathbb{Z}}$-ae, whence the dichotomy turns into:

$$\underset{leb_{\mathbb{R}/2\mathbb{Z}}}{\chi = 0} \quad \text{XOR} \quad \exists\, \chi^+, \chi^- : \chi^- < 0 < \chi^+;\ \chi \in_{leb} \{\chi^+, \chi^-\}$$

(here $\chi^- = -\chi^+$)

• Applying these results to the Schrödinger cocycle $S_{\alpha,E}$, we have a dynamical splitting of Σ_α for $\alpha \in [0,1] \setminus \mathbb{Q}$:

$$\mathbb{C} = \Sigma_\alpha \uplus \{E \in \mathbb{C} \mid S_{\alpha,E} \text{ is uniformly hyperbolic}\}$$
$$\underset{\mathrm{leb}_{\mathbb{R}/\mathbb{Z}}}{=} \{\chi = 0\} \uplus \{\chi^- \overset{\cap +}{<} 0 < \chi^+\}$$

$$\Rightarrow \Sigma_\alpha \underset{\mathrm{leb}_{\mathbb{R}/\mathbb{Z}}}{=} \overbrace{\left(\Sigma_\alpha \cap \{\chi = 0\}\right)}^{=: \Sigma_\alpha^0} \uplus \overbrace{\left(\Sigma_\alpha \cap \{\chi^- < 0 < \chi^+\}\right)}^{=: \Sigma_\alpha^+}$$

($S_{\alpha,E}$ is called <u>non-uniformly</u> <u>hyperbolic</u> if $E \in \Sigma_\alpha^+$.)

• On the other hand we have a different decomposition coming from spectral theory:

$\forall x \in \mathbb{R}, \forall \alpha \in [0,1] \setminus \mathbb{Q}$:

$$\sigma(H_x^\alpha) = \underset{\substack{\uparrow \\ \text{absolutely} \\ \text{continuous spectrum}}}{\sigma_{ac}(H_x^\alpha)} \uplus \underset{\substack{\uparrow \\ \text{singular} \\ \text{spectrum}}}{\sigma_{sc}(H_x^\alpha)} \uplus \underset{\substack{\uparrow \\ \text{pure point} \\ \text{spectrum}}}{\sigma_{pp}(H_x^\alpha)},$$

which is invariant $leb_{\mathbb{R}}$-a.e. x; though $\Sigma_{\alpha,ac} := \sigma(H_x^\alpha)$ does not depend on x.

<u>Thm (Kotani)</u>: $\forall \alpha \in [0,1] \setminus \mathbb{Q}$: $\Sigma_{\alpha,ac} = \overline{\Sigma_\alpha^\circ}^{leb}$.

(Here $\overline{\Sigma_\alpha^\circ}^{leb}$ is the <u>essential closure</u> of Σ_α°, i.e., the closure of the Lebesgue density points of Σ_α°.)

<u>Cor</u>: If $\forall E \in_{ae} \Sigma_\alpha$: $S_{\alpha,E}$ is nonuniformly hyperbolic, then $\Sigma_{\alpha,ac} = \emptyset$.

<u>§5. Spaces of Dynamical Systems and Metadynamics</u>:

- Work of Bourgain and Jitomirskaya implies that the butterfly is precisely the set of parameter values (E,α) where the Lyapunov exponents of $S_{\alpha,E}$ vanish for some $x \in \mathbb{R}$.
