§10.5: (2)

- Up until now the differential equations that we dealt with involved derivatives with respect to not more than one variable. Accordingly our unknown functions were single variable.
- It turns out many phenomena are way too intricate to admit a mathematical model with a single variable. This leads us to consider equations whose unknowns are multivariable functions. Such equations are forced to involve derivatives with respect to more than one variable, whence they are called <u>partial differential equations</u> (<u>PDE</u>).
- We'll focus on those PDE's that can be dealt with using ODE methods, together with a method that allows us to disentangle (certain) PDE's into ODE's (called the <u>method of separation of variables</u>, <u>or eigenfunction decomposition</u>, <u>or disentanglement</u>).

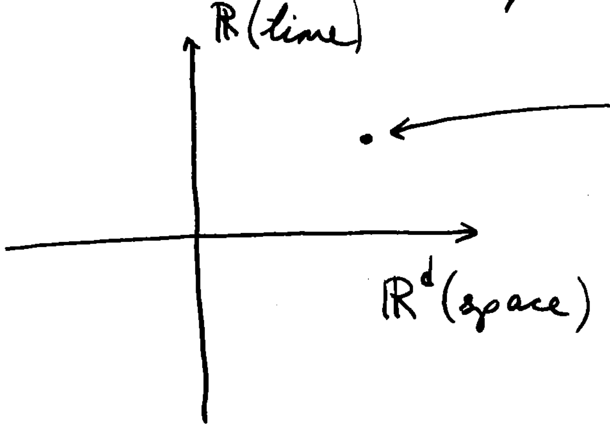
• Our new unknown functions will typically be of the form

$$u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x_1, x_2, \dots, x_d, t) \longmapsto u(x_1, x_2, \dots, x_d, t),$$

where $x_1, x_2, \dots, x_d$ are called the "space" coordinates and t is called the "time" coordinate. Mathematics does not distinguish $\mathbb{R}^d \times \mathbb{R}$ and $\mathbb{R}^{d+1}$. Indeed, both of these symbols represent the set of $(d+1)$-tuples of real numbers. In fact, forgetting the primordiality (and tyranny) of "time" will be very convenient.

Yet, since we are still trying to understand physical phenomena, the "time" coordinate should be kept separate from the "space" coordinates.

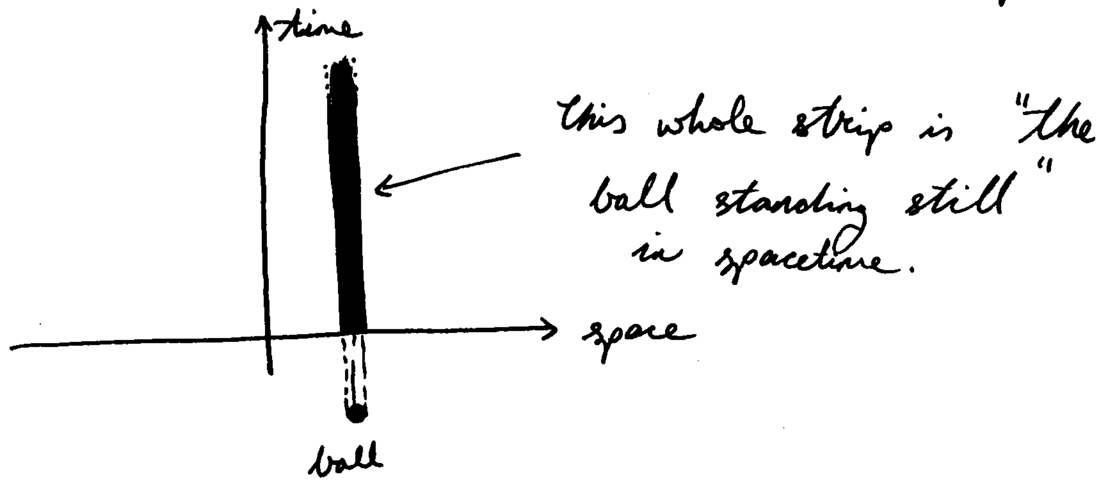
The notion of "<u>spacetime</u>" provides a compromise between these two positions:



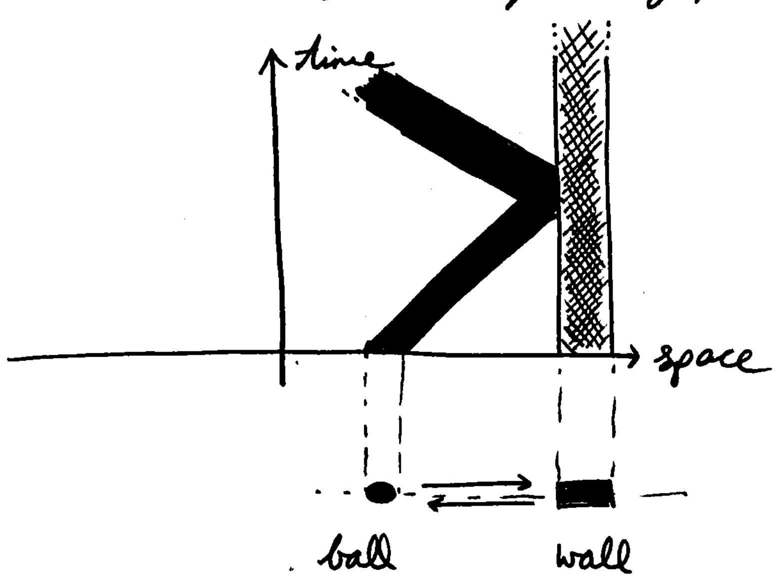
Points of the spacetime $\mathbb{R}^d \times \mathbb{R}$ are called "<u>herenow</u>"s or "<u>therethen</u>"s.

Thus we'll take the "time" coordinate to be on equal footing with the "space" coordinates (until it is time to interpret the mathematical results physically).

Here is how a ball standing still looks like in spacetime:



Here is how a ball bouncing off a wall looks like in spacetime (no bouncing angle, no gravity, no friction):

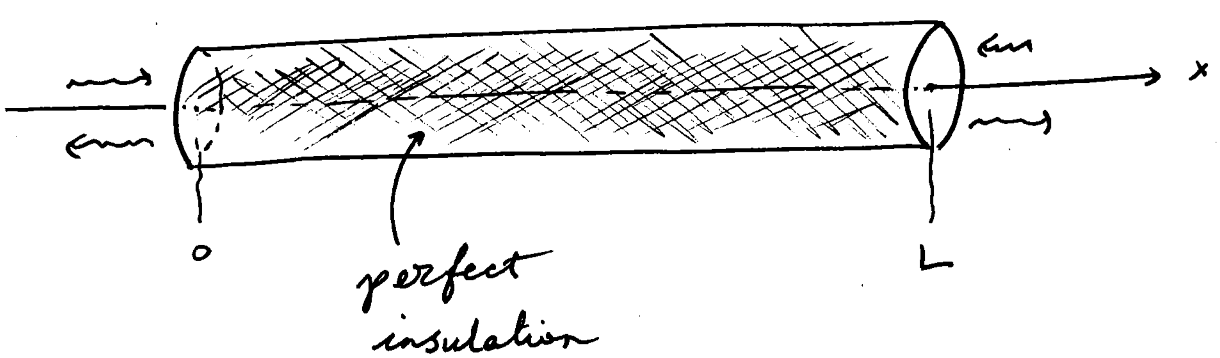


SW: (Derivation of the Heat Conduction Equation Through a Uniform Medium for $d=1$)

(This equation is based on the conception of heat as something that can flow as an incompressible fluid throughout a region of space occupied by a (uniform) substance.)

Consider a cylindrical rod made of a uniform material of length $L > 0$ with density $\rho > 0$.

Suppose the rod is perfectly insulated along its curved surface so that heat can enter or leave only at the ends. Also suppose the cross-section of the rod has such a small area that the nonnegligible heat flow is along a one-dimensional axis:



The unknown function of the heat equation is temperature $u: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$, i.e., $u(x,t)$ denotes the temperature at the herenow (x,t). Then the total thermal energy (ie., heat) E contained in the rod at time t is:

$$E(t) = \int_0^L s\rho \, u(x,t)\, dx,$$

where $s > 0$ is a physical constant called the <u>specific heat</u> of the material the rod is made of.

<u>Fourier's Law</u> of <u>Heat Conduction</u> (which is an empirical law) dictates that heat flows from hot to cold regions proportionately to the difference in temperature.

(i) Using Fourier's Law, show that

$$\partial_t E(t) = c \;\; \partial_x u(L,t) - c\, \partial_x u(0,t),$$

where $c > 0$ is another physical constant called the <u>heat conductivity</u> of the material the rod is made of.

(ii) Deduce that

$$0 = \int_0^L \left(\partial_t u(x,t) - \frac{c}{s\rho} \Delta u(x,t) \right) dx .$$

$k := \frac{c}{s\rho} > 0$ is called the thermal diffusivity of the material the rod is made of. Consequently it does not depend on $L > 0$.

$$\Rightarrow \quad 0 = \partial_L \left(\int_0^L \left(\partial_t u(x,t) - k \Delta u(x,t) \right) dx \right)$$

$$= \partial_t u(x,t) - k \Delta u(x,t)$$

$$\Rightarrow \quad \boxed{\partial_t u(x,t) - k \Delta u(x,t) = 0} . \qquad \text{(heat eq.)}$$

• A solution $u(x,t)$ of

$$\boxed{\partial_t u(x,t) - k\,\Delta u(x,t) = 0} \quad (*) \qquad \begin{pmatrix} k>0 \\ \Delta = \partial_x^2 \end{pmatrix}$$

is called <u>disentangled</u> (or <u>separated</u>) if it is not the constantly zero function and there are two functions $\pi : \mathbb{R}^1 \to \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ such that

$$u(x,t) = \pi(x)\,\tau(t).$$

Here the <u>space component</u> $\pi : \mathbb{R}^1 \to \mathbb{R}$ of u depends only on the "space" coordinate and the <u>time component</u> $\tau : \mathbb{R} \to \mathbb{R}$ of u depends only on the "time" coordinate.

If $u(x,t) = \pi(x)\,\tau(t)$ is a disentangled sol. of $\circledast$, then there are $x_0 \in \mathbb{R}^1$, $t_0 \in \mathbb{R}$:

$$\pi(x_0) \neq 0 \ , \ \tau(t_0) \neq 0.$$

$$\circledast \Rightarrow \quad \partial_t \big(\pi(x)\,\tau(t)\big) - k\,\Delta\big(\pi(x)\,\tau(t)\big) = 0$$

$$\Rightarrow \big(-\Delta\,\pi(x)\big)\big(k\,\tau(t)\big) = \big(-\partial_t\,\tau(t)\big)\ \pi(x)$$

$$\Rightarrow \quad -\Delta\,\pi(x) = \underbrace{\left.\left[\frac{-\partial_t\,\tau(t)}{k\,\tau(t)}\right]\right|_{t=t_0}}_{=:\ \lambda}\ \pi(x)$$

(this is well-defined because $\tau(t_0) \neq 0 \neq k$.)

$$\Rightarrow \quad -\Delta\,\pi(x) = \lambda\ \pi(x) \quad \Leftrightarrow \quad (\lambda, \pi(x)) \text{ is an eigenpair of } -\Delta.$$

$$\Rightarrow \big(\lambda\,\pi(x)\big)\big(k\,\tau(t)\big) = \big(-\partial_t\,\tau(t)\big)\ \pi(x)$$

$$\Rightarrow \quad \lambda\ \pi(x_0)\ \ k\ \tau(t) = -\partial_t\,\tau(t)\ \ \pi(x_0)$$

$$\underset{(\pi(x_0)\neq 0)}{\Rightarrow} \quad \partial_t\,\tau(t) = -k\,\lambda\,\tau(t) \quad \Leftrightarrow \quad (-k\lambda, \tau(t)) \text{ is an eigenpair of } \partial_t.$$

Thus $\circledast$ disentangles into

$$\boxed{\begin{array}{l} u(x,t) = \pi(x)\,\tau(t) \\ -\Delta \pi(x) = \lambda\, \pi(x) \\ \partial_t \tau(t) = -k\lambda \tau t \end{array}} \quad \circledast\circledast .$$

(Abbreviated Version:

$$u = \pi\tau \Rightarrow \circledast \Leftrightarrow \pi\dot{\tau} - k\ddot{\pi}\tau = 0$$

$$\Leftrightarrow (-\ddot{\pi})(k\tau) = (\pi)(-\dot{\tau})$$

$$\Leftrightarrow \frac{-\ddot{\pi}}{\pi} = \frac{-\dot{\tau}}{k\tau} = \lambda \text{ is constant}$$

$$\Rightarrow \boxed{\begin{array}{l} u = \pi\tau \\ -\ddot{\pi} = \lambda\pi \\ \dot{\tau} = -k\lambda\tau \end{array}}$$
)

and $\circledast\circledast$ is called a disentanglement of $\circledast$. Observe that λ in $\circledast\circledast$ is a new parameter that is not fixed.

$$\circledast\circledast \Rightarrow \pi(x) = \left\{ \begin{array}{ll} c_1 e^{-\sqrt{-\lambda}\,x} + c_2 e^{\sqrt{-\lambda}\,x}, & \text{if } \lambda < 0 \\ c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x), & \text{if } \lambda > 0 \\ c_1 + c_2 x, & \text{if } \lambda = 0 \end{array} \right\}, \quad \tau(t) = d\, e^{-k\lambda t}$$

$$\Rightarrow u(x,t) = \left\{ \begin{array}{ll} \left(d_1 e^{-\sqrt{-\lambda}\,x} + d_2 e^{\sqrt{-\lambda}\,x}\right) e^{-k\lambda t}, & \text{if } \lambda < 0 \\ \left(d_1 \cos(\sqrt{\lambda}\,x) + d_2 \sin(\sqrt{\lambda}\,x)\right) e^{-k\lambda t}, & \text{if } \lambda > 0 \\ (d_1 + d_2 x), & \text{if } \lambda = 0 \end{array} \right\}$$

is the general disentangled solution of $\circledast$.

A PDE that has a disentanglement is called disentangleable (or separable).

Ex: Disentangle $\boxed{x^2 \Delta u(x,t) - t^2 \partial_t^2 u(x,t) = 0}$ (*).

$$u = \pi\tau \Rightarrow (*) \Leftrightarrow (x^2 \ddot{\pi})(\tau) - (\pi)(t^2 \ddot{\tau}) = 0$$

$$\Leftrightarrow \frac{x^2 \ddot{\pi}}{\pi} = \frac{t^2 \ddot{\tau}}{\tau} =: \lambda \in \mathbb{R}$$

$$\Leftrightarrow \boxed{\begin{aligned} & u(x,t) = \pi(x)\,\tau(t) \\ & x^2 \Delta \pi(x) = \lambda \pi(x) \\ & t^2 \partial_t^2 \tau(t) = \lambda \tau(t) \end{aligned}}$$

λ is called the disentanglement constant (or the separation constant)

SW: Let n be a nonnegative integer, $\alpha > 0$, and consider

$$\boxed{\partial_t^n u(x,y,z,t) - \alpha \Delta u(x,y,z,t) = 0} \quad (*) \quad (\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2)$$

(i) Using $u(x,y,z,t) = \pi(x,y,z)\,\tau(t)$, disentangle (*) into

$$\boxed{\begin{aligned} & u(x,y,z,t) = \pi(x,y,z)\,\tau(t) \\ & -\Delta \pi(x,y,z) = \lambda_0 \pi(x,y,z) \\ & -\partial_t^n \tau(t) = \alpha \lambda_0 \tau(t) \end{aligned}} \quad (**)$$

(ii) Using $\pi(x,y,z) = p(x)\,q(y)\,r(z)$, disentangle (**) into

$$\boxed{\begin{aligned} & u(x,y,z,t) = p(x)\,q(y)\,r(z)\,\tau(t) \\ & -\partial_x^2 p(x) = \lambda_1 p(x) \\ & -\partial_y^2 q(y) = \lambda_2 q(y) \\ & -\partial_z^2 r(z) = (\lambda_0 - \lambda_1 - \lambda_2)\, r(z) \\ & -\partial_t^n \tau(t) = \alpha \lambda_0 \tau(t) \end{aligned}}$$

(iii) Generalize to $\Delta = \sum_{k=1}^{d} \partial_{x_k}^2$.

Ex: Allegedly $\boxed{\Delta u(x,y) + \partial_x \partial_y u(x,y) = 0}$ $\circledast$ $\left(\Delta = \partial_x^2 + \partial_y^2\right)$

does not admit a disentanglement. But consider:

(i) $u(x,y) = p(x)\, q(y)\,, \quad p(x_0) \neq 0 \neq q(y_0)$

$$\Rightarrow \circledast \Leftrightarrow \ddot{p}\, q + p\, \ddot{q} + \dot{p}\, \dot{q} = 0 \Rightarrow \underbrace{\frac{\ddot{p}(x_0)}{p(x_0)}}_{=:\lambda_4} + \underbrace{\frac{\dot{p}(x_0)}{p(x_0)}}_{=:\lambda_3} \underbrace{\frac{\dot{q}(y_0)}{q(y_0)}}_{=:\lambda_1} + \underbrace{\frac{\ddot{q}(y_0)}{q(y_0)}}_{=:\lambda_2} = 0$$

$$\Leftrightarrow \quad \lambda_1 \lambda_3 + \lambda_2 + \lambda_4 = 0.$$

$$\Rightarrow \boxed{\begin{array}{c} u(x,y) = p(x)\, q(y) \\ \ddot{p}(x) + \lambda_1 \dot{p}(x) + \lambda_2 p(x) = 0 \\ \ddot{q}(y) + \lambda_3 \dot{q}(y) + \lambda_4 q(y) = 0 \\ \lambda_1 \lambda_3 + \lambda_2 + \lambda_4 = 0 \end{array}}$$

is a disintegration of $\circledast$ with three parameters.

(ii) $$0 = \frac{\ddot{p}}{p} + \frac{\dot{p}}{p}\frac{\dot{q}}{q} + \frac{\ddot{q}}{q} \Rightarrow 0 = \partial_{xy}\left(\frac{\ddot{p}}{p} + \frac{\dot{p}}{p}\frac{\dot{q}}{q} + \frac{\ddot{q}}{q}\right)$$

$$= \partial_x\left(\frac{\dot{p}}{p}\, \partial_y\left(\frac{\dot{q}}{q}\right) + \partial_y\left(\frac{\ddot{q}}{q}\right)\right) = \partial_x\left(\frac{\dot{p}}{p}\right) \partial_y\left(\frac{\dot{q}}{q}\right)$$

$$\Rightarrow \partial_x\left(\frac{\dot{p}}{p}\right) = 0 \quad \text{or} \quad \partial_y\left(\frac{\dot{q}}{q}\right) = 0$$

$\Rightarrow$ $\dfrac{\dot{p}}{p} =: \mu_3$ is a constant or $\dfrac{\dot{q}}{q} =: \mu_1$ is a constant

If $\frac{\dot{p}}{p} = \mu_3$, then $\frac{\ddot{p}}{p} = -\frac{\mu_3 \dot{q} + \ddot{q}}{q} =: \mu_4$ is a constant

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} \dot{p} = \mu_3 p \\ \ddot{p} = \mu_4 p \end{array}\right\} \mu_4 = \mu_3^2 \\ \ddot{q} + \mu_3 \dot{q} = -\mu_4 q \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} u(x,y) = p(x)\, q(y) \\ \dot{p}(x) = \mu_3\, p(x) \\ \ddot{q}(y) + \mu_3\, \dot{q}(y) = -\mu_3^2\, q(y) \end{array}}$$

If $\frac{\dot{q}}{q} = \mu_1$, then $\frac{\ddot{q}}{q} = -\frac{\mu_1 \dot{p} + \ddot{p}}{p} =: \mu_2$ is a constant

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} \dot{q} = \mu_1 q \\ \ddot{q} = \mu_2 q \end{array}\right\} \mu_2 = \mu_1^2 \\ \ddot{p} + \mu_1 \dot{p} = -\mu_2 p \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} u(x,y) = p(x)\, q(y) \\ \dot{q}(y) = \mu_1\, q(y) \\ \ddot{p}(y) + \mu_1\, \dot{p}(y) = -\mu_1^2\, p(y) \end{array}}$$

$$\Rightarrow \boxed{\begin{array}{lcl} u(x,y) = p(x)\, q(y) & & \\ \dot{p}(x) - \mu_3 p(x) = 0 & \text{or} & \dot{q}(y) - \mu_1\, q(y) = 0 \\ \ddot{q}(y) + \mu_3\, \dot{q}(y) + \mu_3^2\, q(y) = 0 & & \ddot{p}(y) + \mu_1\, \dot{p}(y) + \mu_1^2\, p(y) = 0 \end{array}}$$

is another disintegration of ⊛ with two parameters (and two cases).

———— .

- We will typically encounter a PDE as part of an <u>initial / boundary value problem</u> (IBVP), which is a triple of the form

$$\left(\text{PDE}, \quad \begin{matrix} \text{boundary conditions} \\ \text{in terms of the} \\ \text{"space" coordinates} \end{matrix}, \quad \begin{matrix} \text{initial data} \\ \text{in terms of the} \\ \text{"time" coordinate} \end{matrix} \right)$$

- The <u>method of disentanglement</u> for solving <u>homogeneous</u> IBVP's (ie., boundary conditions $= 0$.) goes like this:

  (i) Disentangle the PDE.

  (ii) Use the boundary conditions to detect the relevant disentangled solutions.

  (iii) Any entangled solution satisfying the boundary condition is the limit of a linear combination of disentangled solutions (ie., by taking infinite sums of disentangled solutions we can obtain any solution) (This we'll take for granted.).

  (iv) Determine the coefficients for (iii) by looking at the Fourier coefficients of the initial data.

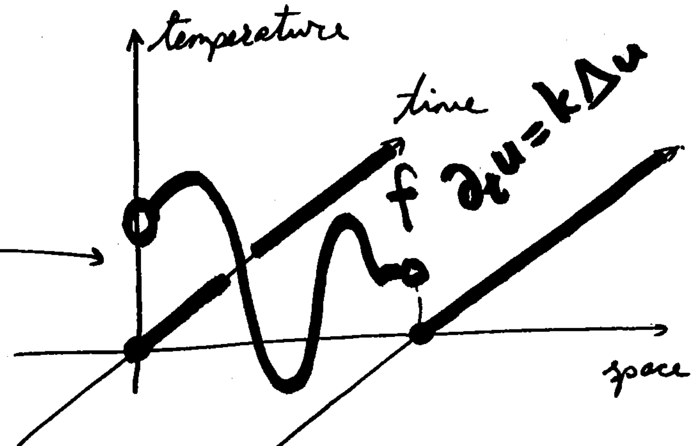
Ex: (Homogeneous Heat Conduction Problem for $d=1$)

Let $k>0$, $L>0$, $f \in R(]0,L[, \mathbb{R})$ be piecewise smooth.

Consider

$$
(*) \quad \boxed{\begin{array}{lll}
\partial_t u(x,t) - k\,\Delta u(x,t) = 0 & , \text{ for } (x,t) \in]0,L[\times]0,\infty[& \text{(PDE)} \\
u(0,t) = 0 = u(L,t) & , \text{ for } t \in [0,\infty[& \text{(BC)} \\
u(x,0) = f(x) & , \text{ for } x \in]0,L[& \text{(ID)}
\end{array}}
$$

Geometrically, we are trying to find that surface which fits into this frame whose concavity in the "space" direction is proportional to its slope in the "time" direction.



Physically, the boundary conditions mean that the ends of the rod are kept at constant zero temperature (but quite possibly there is still heat flow in and out of the rod at the ends). The initial data $f(x)$ represents the initial temperature distribution on the rod (except the ends).

(*) disentangles into :
$$\boxed{\begin{aligned} u(x,t) &= \pi(x)\, \tau(t) \\ -\Delta \pi(x) &= \lambda \pi(x) \\ \partial_t \tau(t) &= -k\lambda \tau(t) \end{aligned}}$$

(BC) disentangles into: $\pi(0)\, \tau(t) = 0 = \pi(L)\, \tau(t)$.

There is a $t_0 \geq 0 : \tau(t_0) \neq 0$

$$\Rightarrow \boxed{\pi(0) = 0 = \pi(L)}$$

(see the SW.(i) at the end of § 10.1)

<u>Relevant eigenpairs of $-\Delta$</u> :

(I) $\lambda < 0 \Rightarrow \pi(x) = c_1 e^{-\sqrt{-\lambda}\, x} + c_2 e^{\sqrt{-\lambda}\, x}$

$$\Rightarrow 0 = c_1 + c_2, \quad 0 = c_1 e^{-\sqrt{\lambda} L} + c_2 e^{\sqrt{\lambda} L}$$

$$\Rightarrow c_2 = -c_1, \quad 0 = c_1 \Big(\underbrace{e^{-\sqrt{-\lambda} L} - e^{\sqrt{-\lambda} L}}_{\neq 0} \Big).$$

$$\Rightarrow c_1 = 0 = c_2$$

$\Rightarrow$ no relevant eigenpairs.

(II) $\lambda > 0 \Rightarrow \pi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$$\Rightarrow 0 = c_1, \quad 0 = c_1 \cos(\sqrt{\lambda} L) + c_2 \sin(\sqrt{\lambda} L)$$

$$\Rightarrow 0 = c_2 \sin(\sqrt{\lambda} L).$$

$$c_2 \neq 0 \Leftrightarrow \sin(\sqrt{\lambda} L) = 0 \Leftrightarrow \sqrt{\lambda} L = \begin{matrix} \pi, 3\pi, \ldots, (2n+1)\pi, \ldots \\ 2\pi, 4\pi, \ldots, (2n)\pi, \ldots \end{matrix}$$

$\Rightarrow$ For any $n \geq 1$: $\left(\left(\frac{n\pi}{L} \right)^2, \sin\left(\frac{n\pi}{L} x \right) \right)$ is a relevant eigenpair.

(III) $\lambda = 0 \Rightarrow \quad X(x) = c_1 + c_2 x$

$$\Rightarrow \quad 0 = c_1 , \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_2 = 0$$

$\Rightarrow$ no relevant eigenpairs.

$\Rightarrow$ The relevant disentangled solutions are:

$$\boxed{\text{For any } n \geq 1: \quad \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$\Rightarrow$ Any solution is of the form

$$u(x,t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} ,$$

where the coefficients [redacted] $b_1, b_2, \dots, b_n, \dots$ are yet to be determined.

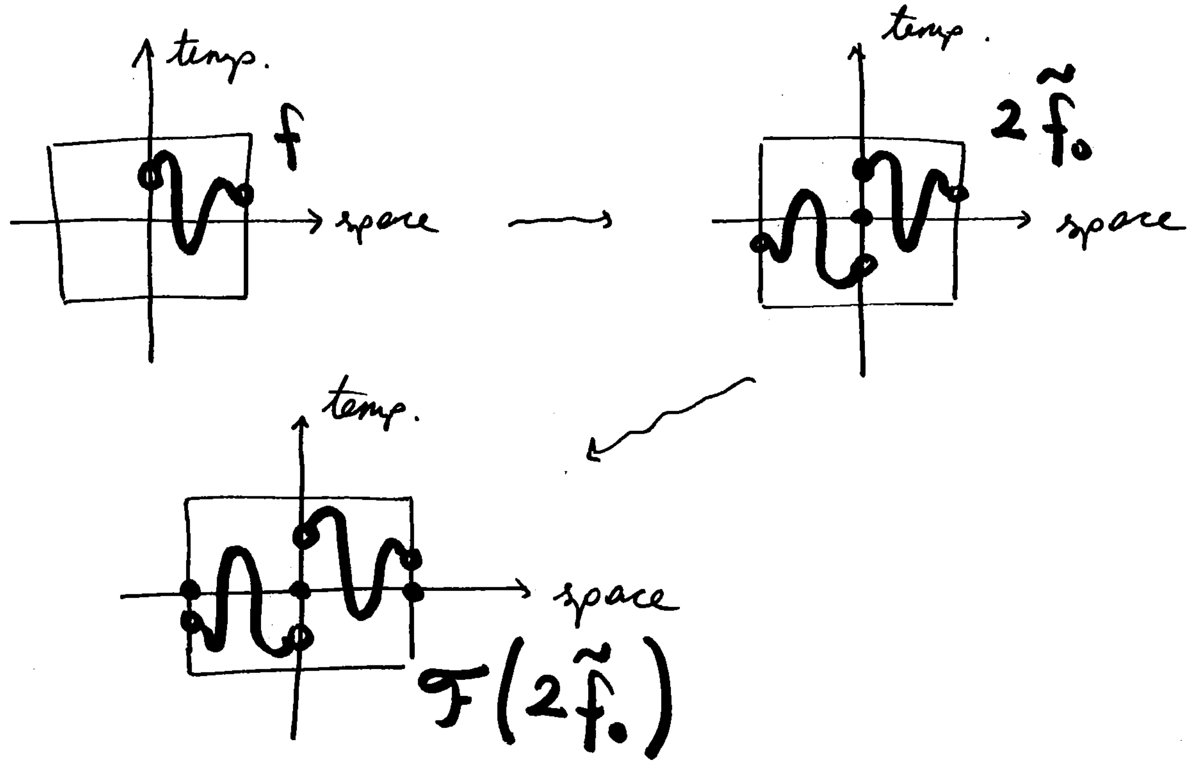
If $u(x,t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ solves (*),

then $f(x) = u(x,0) = \sum_{n \geq 1} b_n \sigma_n(x)$.

Observe that since $f:]0,L[\to \mathbb{R}$ is piecewise smooth, by the Fourier Convergence Theorem

$$\text{for all } x \in [0,L] : \mathcal{F}(2\tilde{f}_o)(x) = f(x) \chi_{]0,L[}(x)$$

where $2\tilde{f}_0$ is the odd periodic extension of f onto $]-L, L[$



$$\mathcal{F}(2\tilde{f}_0)(x) = \frac{c_0}{2} + \sum_{n\geqslant 1} c_n \gamma_n(x) + \sum_{n \geqslant 1} s_n \sigma_n(x).$$

$(n \geqslant 0)$ $\quad c_n = 0$ because $2\tilde{f}_0$ is odd.

$(n \geqslant 1)$ $$s_n = \frac{1}{L} \int_{-L}^{L} \underbrace{2\tilde{f}_0(x)\, \sigma_n(x)}_{\text{even}}\, dx = \frac{2}{L} \int_0^L 2\tilde{f}_0(x)\, \sigma_n(x)\, dx$$

$$= \frac{2}{L} \int_0^L f(x)\, \sigma_n(x)\, dx$$

$\left(\begin{array}{l} \text{for } 0 < x < L, \\ 2\tilde{f}_0(x) = f(x) \end{array}\right)$

$\Rightarrow$ for all $x \in [0, L]$: $f(x)\chi_{]0,L[}(x) = \mathcal{F}\left(2\tilde{f}_0\right)(x) = \sum_{n \geq 1} s_n \, \sigma_n(x)$,

where $\quad s_n = \frac{2}{L}\int_0^L f(x)\,\sigma_n(x)\,dx$.

$\Rightarrow \quad \sum_{n \geq 1} b_n \, \sigma_n(x) = u(x, 0) = f(x) = \sum_{n \geq 1} s_n \, \sigma_n(x)$

$\Rightarrow$ Picking $b_n := s_n$ produces the solution of $\circledast$:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n \geq 1} s_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \\ \text{where} \quad s_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}}$$

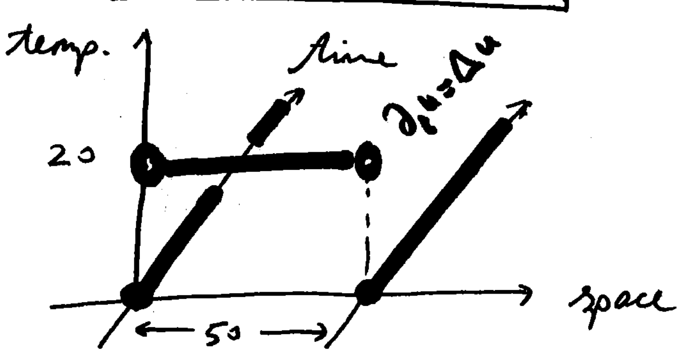
Observe that $\lim_{t \to \infty} u(x,t) = 0$.

Ex: $k := 1\left[\frac{cm^2}{s}\right]$, $L := 50$ [cm], $f :]0, 50[\to \mathbb{R}$ $[^\circ C]$,
$x \longmapsto 20$

Solve [redacted]

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &u(0,t) = 0 = u(50,t) \\ &u(x,0) = 20. \end{aligned}}$$

$$u(x,t) = \sum_{n \geq 1} s_n \sin\left(\frac{n\pi}{50}x\right) e^{-\left(\frac{n\pi}{50}\right)^2 t}$$



$$s_n = \frac{2}{L}\int_0^L f(x)\,\sigma_n(x)\,dx = \frac{4}{5}\int_0^{50}\sin\left(\frac{n\pi}{50}x\right)dx$$

$$= \frac{4}{5}\left(\frac{-50}{n\pi}\right)\left[\cos\left(\frac{n\pi}{50}x\right)\right]\Big|_0^{50} = \left(\frac{-40}{n\pi}\right)\left(\cos(n\pi) - 1\right)$$

$$= \frac{80}{n\pi}\chi_{2\mathbb{Z}+1}(n). \quad \checkmark$$

SW : (i) Replace the boundary condition of ⊛ with "$\partial_x u(0,t) = 0 = \partial_x u(L,t)$ for $t \in [0,\infty[$", then find the solution $u(x,t)$. Physically this new boundary condition means that the ends of the rod are isolated as well. Also show that

$$\lim_{t\to\infty} u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx = \text{average of the initial data}.$$

(ii) Replace the boundary condition of ⊛ with "$\partial_x u(0,t) = 0 = u(0,t)$ for $t \in [0,\infty[$", then find the solution $u(x,t)$. Interpret this new boundary condition physically. Find $\lim_{t\to\infty} u(x,t)$.

(iv) Solve ⊛ with $k := \frac{i}{2} = \frac{\sqrt{-1}}{2}$. Interpret physically. $\partial_t u(x,t) - \frac{i}{2}\Delta u(x,t) = 0$ is the free Schrödinger eq. for $d=1$.

(iii) Solve

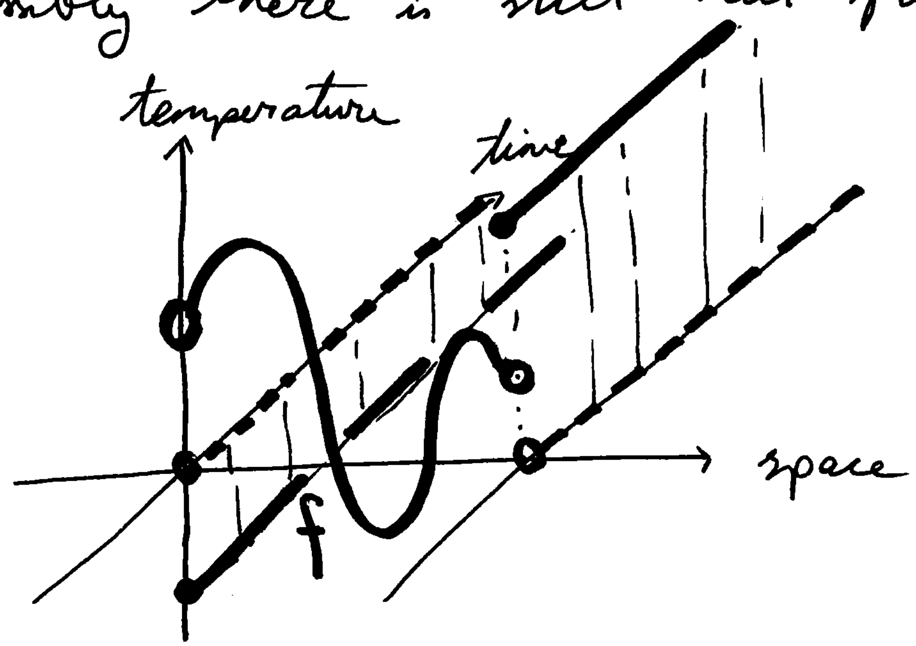
$$\boxed{\begin{aligned} &\partial_t u(x,t) - k\,\Delta u(x,t), \text{ for } (x,t) \in]-L,L[\times]0,\infty[\\ &u(-L,t) - u(L,t) = 0 = \partial_x u(-L,t) - \partial_x u(L,t), \text{ for } t \in [0,\infty[\\ &u(x,0) = f(x), \text{ for } x \in]-L,L[\end{aligned}}\,,$$

where $L>0$, $k>0$, $f \in R(]-L,L[,\mathbb{R})$ is pw. smooth. Interpret physically.

Ex: (Nonhomogeneous Heat Conduction Problem for $d=1$)
Let $k>0$, $L>0$, $T_0, T_L \in \mathbb{R}$, $f \in R(]0,L[,\mathbb{R})$ be pw. smooth. Consider

$$(*)\quad \boxed{\begin{array}{ll} \partial_t u(x,t) - k\Delta u(x,t) = 0, & \text{for } (x,t) \in]0,L[\times]0,\infty[\\ u(0,t) - T_0 = 0 = u(L,t) - T_L, & \text{for } t \in [0,\infty[\\ u(x,0) = f(x), & \text{for } x \in]0,L[\end{array}} \quad \begin{array}{l} (PDE) \\ (BC) \\ (ID) \end{array}$$

The boundary conditions now mean that the left end of the rod is kept at the temperature T_0 and the right end is kept at T_L (again quite possibly there is still heat flow at the ends).



Recall the general disentangled solution of the heat equation:

$$u(x,t) = \begin{cases} \left(d_1 e^{-\sqrt{-\lambda}\,x} + d_2 e^{\sqrt{-\lambda}\,x}\right) e^{-k\lambda t} & , \text{ if } \lambda < 0 \\ \left(d_1 \cos(\sqrt{\lambda}x) + d_2 \sin(\sqrt{\lambda}\,x)\right) e^{-k\lambda t} & , \text{ if } \lambda > 0 \\ d_1 + d_2 x & , \text{ if } \lambda = 0. \end{cases}$$

We do not have an external source of heat (ie. forcing), whence we would expect that no disentangled solution with $\lambda<0$ will be relevant to the IBVP. When the boundary conditions were homogeneous we had also eliminated the disentangled solutions with $\lambda=0$ (except possibly constant ones, eg. when $\partial_x u(0,t) = 0 = \partial_x u(L,t)$). Since now the boundary conditions are (quite possibly) not homogeneous, we have:

$$\lambda = 0 \;\Rightarrow\; \Pi(x) = d_1 + d_2 x$$

$$\Rightarrow \left.\begin{aligned} T_0 &= \Pi(0) = d_1 \\ T_L &= \Pi(L) = d_1 + d_2 L \end{aligned}\right\} \quad T_L - T_0 = d_2 L \;\Rightarrow\; d_2 = \frac{T_L - T_0}{L}$$

$$\Rightarrow\; T_0 + \frac{T_L - T_0}{L}x \text{ is a relevant disentangled solution.}$$

Thus the general solution of ⊛ should be of the form

$$u(x,t) = \underbrace{\left(T_0 + \frac{T_L - T_0}{L}\, x\right)}_{=:\, u_E(x)} + \underbrace{\int_{\substack{]0,\infty[\\ \lambda:\ \text{relevant}}} \left(a(\lambda)\cos(\sqrt{\lambda}\, x) + b(\lambda)\sin(\sqrt{\lambda}\, x)\right) e^{-k\lambda t}\, d\lambda}_{=:\, v(x,t)} ,$$

where $a, b : \{\lambda \in]0,\infty[\mid \lambda \text{ is relevant}\} \to \mathbb{R}$ are coefficients yet to be determined.

$$\Rightarrow \lim_{t\to\infty} v(x,t) = 0$$

$$\Rightarrow \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \left(u_E(x) + v(x,t)\right) = u_E(x)$$

u_E is called the <u>equilibrium solution</u> of the IBVP (or <u>steady-state</u>)

(it is constant in time) and $v(x,t)$ is called the <u>transient solution</u> of the IBVP.

(Recall the periodically forced harmonic oscillator.)

To determine u_E we did not use f, consequently it is unreasonable to expect that u_E solves the whole IBVP. Likewise since $v(x,t)$ decays exponentially fast in time, unless $T_0 = 0 = T_L$ it won't solve the whole IBVP. But we have:

$$\partial_t u_E(x) - k\Delta u_E(x) = -k\,\partial_x^2\left(T_0 + \frac{T_L - T_0}{L}x\right) = 0$$

$$\partial_t v(x,t) - k\Delta v(x,t) = \partial_t\left(u(x,t) - u_E(x)\right) - k\Delta\left(u(x,t) - u_E(x)\right)$$

$$= \left(\partial_t u(x,t) - k\Delta u(x,t)\right) - \left(\partial_t u_E(x) - k\Delta u_E(x)\right) = 0.$$

$$u_E(0) - T_0 = 0 = u_E(L) - T_L$$

$$v(0,t) = u(0,t) - u_E(0) = T_0 - T_0 = 0$$
$$v(L,t) = u(L,t) - u_E(L) = T_L - T_L = 0.$$
$$v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x)$$

$\Rightarrow$ If $u(x,t) = u_E(x) + v(x,t)$ solves $*$, then

$u_E(x)$ solves: and $v(x,t)$ solves:

$$\boxed{\begin{aligned} &\underbrace{\partial_t u_E(x)}_{=0} - k\Delta u_E(x) = 0 \\ &u_E(0) = T_0 \\ &u_E(L) = T_L \end{aligned}} \quad (**)$$

$$\boxed{\begin{aligned} &\partial_t v(x,t) - k\Delta v(x,t) = 0 \\ &v(0,t) = 0 = v(L,t) \\ &v(x,0) = f(x) - u_E(x) \end{aligned}} \quad (***)$$

SW: Show that the converse holds as well, i.e., if $p(x)$ solves (**) and $q(x,t)$ solves (***) (with "u_E" replaced with "p"), then $r(x,t) := p(x) + q(x,t)$ solves (*) and $r_E = p$.

Observe that (***) is a homogeneous IBVP whose solution we already discovered:

$$\boxed{\begin{aligned} v(x,t) &= \sum_{n\geq 1} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \\ b_n &= \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sin\left(\frac{n\pi}{L}x\right) dx \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} u(x,t) = u_E(x) + v(x,t) &= \left(T_0 + \frac{T_L - T_0}{L}x\right) + \sum_{n\geq 1} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \\ b_n &= \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sigma_n(x)\, dx. \end{aligned}}$$

is the solution of (*).

SW: (i) Make this more explicit by using the fact that $u_E(x) = T_0 + \frac{T_L - T_0}{L}x$ in b_n's.

(ii) Find the eq. sol.s of all SW's at the end of §10.5.

Ex: Find the eq. sol. $u_E(x)$ of:

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &\partial_x u(0,t) - 8 = 0 = u(10,t) - 100 \\ &u(x,0) = 5x + 27 \end{aligned}} \longrightarrow \boxed{\begin{aligned} -\Delta u_E(x) &= 0 \\ \partial_x u_E(0) &= 8 \\ u_E(10) &= 100 \end{aligned}}$$

(This is irrelevant to u_E.)

$$\Rightarrow \left.\begin{aligned} u_E(x) &= a + bx \\ \partial_x u_E(x) &= b \end{aligned}\right\} \Rightarrow$$

[redacted]

$8 = b, \quad 100 = a + 10b$

$\Rightarrow a = 20 \Rightarrow \boxed{u_E(x) = 20 + 8x.}$

Ex: If $u(x,t)$ solves:

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &\partial_x u(0,t) - 30 = 0 = \partial_x u(10,t) - 10 \\ &u(x,0) = x^2 \end{aligned}}$$

, find $\lim_{t\to\infty} u(x,t)$.

$\lim_{t\to\infty} u(x,t) = u_E(x)$ and

$u_E(x)$ solves:

$$\boxed{\begin{aligned} \Delta u_E(x) &= 0 \\ \partial_x u_E(0) &= 30 \\ \partial_x u(10) &= 10 \end{aligned}}$$

$\Rightarrow u_E(x) = a + bx$

$\partial_x u_E(x) = b$

$$\left.\begin{aligned} \Rightarrow 30 &= \partial_x u_E(0) = b \\ 10 &= \partial_x u_E(10) = b \end{aligned}\right\} \Rightarrow b = 10 = 30$$

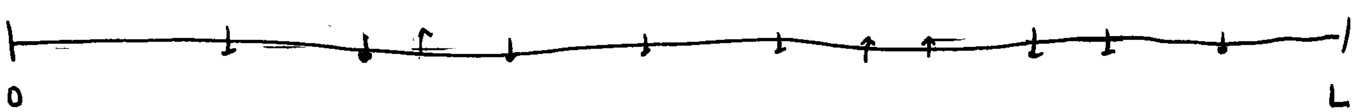
↯.

$\Rightarrow$ The limit does not exist.

SW: Can $u(x,t)$ exist?

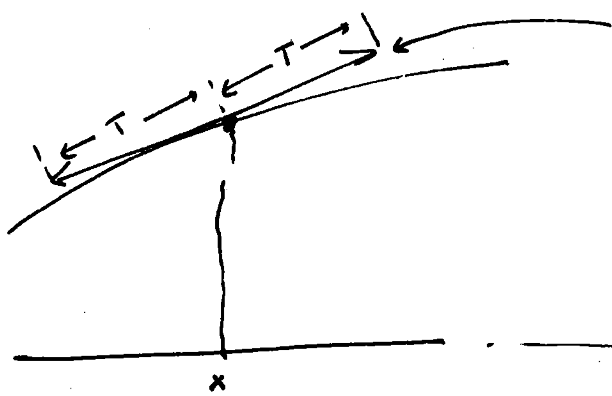
SW: (Derivation of the Wave Equation for a Uniform Medium with Small Vibrations for $d=1$)

Consider a perfectly elastic and flexible string of length $L>0$ and density $\rho>0$ stretched out. Suppose the string undergoes small transverse vibrations and remains in a plane:



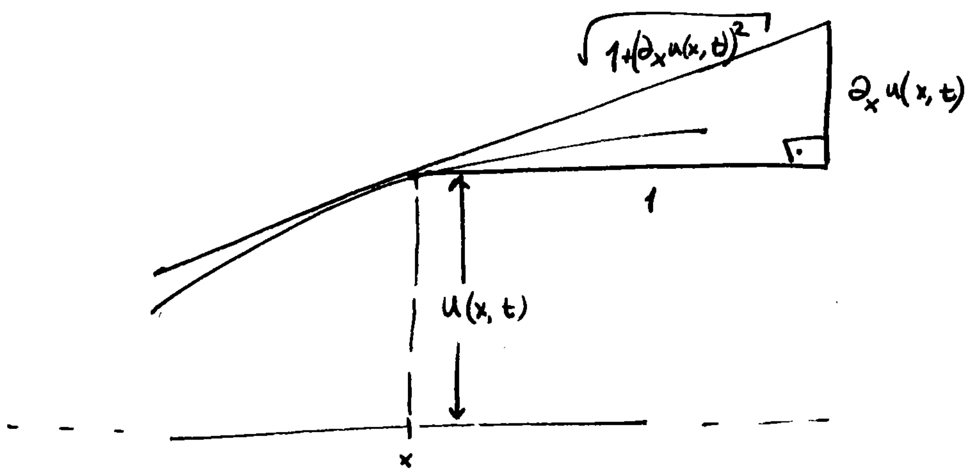
The only force acting on the string is the tension force. We assume that its magnitude is a constant $T>0$. Since the string is flexible the tension force is always [redacted] tangent to the string:

If $x \in]0,L[$,

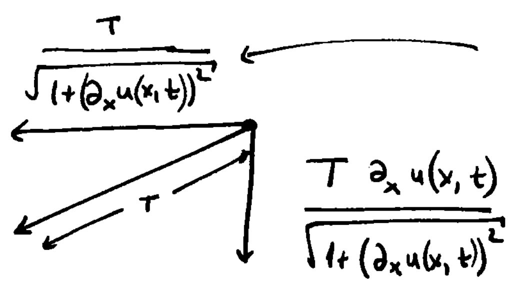


the tension force at (x,t) acting on the point marked as x is zero by the action-reaction principle.

The unknown function of the wave equation is the displacement $u: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$, i.e., $u(x,t)$ denotes the vertical displacement of the string from its equilibrium position at the herenow (x,t):



$\Rightarrow$



$\leftarrow$ horizontal tension force

$\leftarrow$ vertical tension force.

(i) Using Newton's 2nd Law and the fact that there is no lateral motion, show that if $x_0, x_1 \in [0, L] : x_0 < x_1$, then

$$\left[\frac{T}{\sqrt{1+(\partial_x u(x,t))^2}}\right]\Bigg|_{x_0}^{x_1} = 0 \quad \text{and} \quad \left[\frac{T\,\partial_x u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}}\right]\Bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho\, \partial_t^2 u(x,t)\, dx$$

(ii) Differentiating the second equality with respect to x_1, deduce that

$$\rho\, \partial_t^2 u(x,t) = T \frac{\partial_x^2 u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}^{\,3}}.$$

Define $c := \sqrt{\frac{T}{\rho}} > 0$. c is called the velocity of propagation of waves along the string.

(or: the standing waves' speed of propagation along the string)

$$\Rightarrow \partial_t^2 u(x,t) - c^2 \frac{\partial_x^2 u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}^{\,3}} = 0.$$

Since we assumed that the vibrations of the string are small, $\partial_x u(x,t) \approx 0 \Rightarrow \sqrt{1+(\partial_x u(x,t))^2}^{\,3} \approx 1$.

$$\Rightarrow \boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0} \quad \text{(wave eq.)}$$

<u>D'Alembert's Theorem</u>: Let $c \neq 0$. The general solution of

$$\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$$

is

$$\boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)},$$

where $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$ are arbitrary.

<u>Pf</u>: Define $A := \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \in Mat(2\times 2, \mathbb{R})$.

$\det(A) = -2c \neq 0$, so A is invertible with

inverse $A^{-1} = \frac{1}{-2c} \begin{pmatrix} -c & -c \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} \end{pmatrix}$

Thus we have a coordinate change for spacetime:

$$T_A : \mathbb{R}^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ t \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x+ct \\ x-ct \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$T_{A^{-1}} = (T_A)^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \times \mathbb{R}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto A^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{\xi+\eta}{2} \\ \frac{\xi-\eta}{2c} \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}.$$

If $u: \mathbb{R}^1 \times \mathbb{R} \longrightarrow \mathbb{R}$, then $\tilde{u}: \mathbb{R}^2 \to \mathbb{R}$

$$(\xi, \eta) \longmapsto u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

is the unique function that fits into

$$\begin{tikzcd}
\mathbb{R}^1 \times \mathbb{R} \arrow[rr, "u"] \arrow[dr, "T_A"'] & & \mathbb{R} \\
& \mathbb{R}^2 \arrow[ur, "\tilde{u}"'] &
\end{tikzcd}$$

.

<u>SW</u>: Conversely, if $v: \mathbb{R}^2 \to \mathbb{R}$, then $\underset{\sim}{v}: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$

$$(x,t) \mapsto v(x+ct, x-ct)$$

is the unique function that fits into

$$\begin{tikzcd}
\mathbb{R}^1 \times \mathbb{R} \arrow[rr, "\underset{\sim}{v}"] & & \mathbb{R} \\
& \mathbb{R}^2 \arrow[ul, "T_{A^{-1}}"] \arrow[ur, "v"'] &
\end{tikzcd}$$

and $\widetilde{(\underset{\sim}{v})} = v$.

The Chain Rule applied to the second triangle gives:

for any $(\xi,\eta) \in \mathbb{R}^2$: $D_{(\xi,\eta)}(v) = D_{T_{A^{-1}}(\xi,\eta)}(\underset{\sim}{v}) \cdot \underbrace{D_{(\xi,\eta)}\left(T_{A^{-1}}\right)}_{= A^{-1}}$

$\Rightarrow$ for any $(\xi,\eta) \in \mathbb{R}^2$:

$$\left(\partial_\xi v(\xi,\eta) \quad \partial_\eta v(\xi,\eta)\right)_{1\times 2} = \left(\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \quad \partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)\right)_{1\times 2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} \end{pmatrix}_{2\times 2}$$

$$= \left(\frac{\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2} + \frac{\partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2c} \;\middle|\; \frac{\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2} - \frac{\partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2c} \right)_{1\times 2}$$

$$= \left(\left(\frac{\partial_t + c\partial_x}{2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \;\middle|\; \left(\frac{\partial_t - c\partial_x}{-2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \right)_{1\times 2}$$

$$\Rightarrow \quad \partial_\xi v(\xi,\eta) = \left(\frac{\partial_t + c\partial_x}{2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \quad \text{and}$$

$$\partial_\eta v(\xi,\eta) = \left(\frac{\partial_t - c\partial_x}{-2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)$$

$$\Rightarrow \quad \partial_\xi \tilde{u}(x+ct,\, x-ct) = \left(\frac{\partial_t + c\partial_x}{2c}\right) u(x,t)$$

$$\partial_\eta \tilde{u}(x+ct,\, x-ct) = \left(\frac{\partial_t - c\partial_x}{-2c}\right) u(x,t).$$

$$\Rightarrow \quad (\partial_t + c\partial_x)\, u(x,t) = 2c\, \partial_\xi \tilde{u}(x+ct,\, x-ct)$$

$$(\partial_t - c\partial_x)\, u(x,t) = -2c\, \partial_\eta \tilde{u}(x+ct,\, x-ct).$$

$\Rightarrow$ If $u(x,t)$ solves $\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$, then

$$0 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u(x,t) = (\partial_t - c\partial_x)\; 2c\, \underbrace{\partial_\xi \tilde{u}(x+ct, x-ct)}_{=:\, w(x,t)}$$

$$= 2c\, (\partial_t - c\partial_x)\, w(x,t) = 2c\,(-2c)\, \partial_\eta \tilde{w}(x+ct, x-ct)$$

$$= (-4c^2)\, \partial_\eta \partial_\xi \tilde{u}(x+ct, x-ct) \underset{\begin{pmatrix} \xi := x+ct \\ \eta := x-ct \end{pmatrix}}{\overset{?}{=}} (-4c^2)\, \partial_\eta \partial_\xi \tilde{u}(\xi, \eta)$$

$$\Rightarrow \quad 0 = \partial_\eta \partial_\xi \tilde{u}(\xi, \eta)$$

$$\Rightarrow \quad \partial_\xi \tilde{u}(\xi, \eta) = \varphi_1(\xi)$$

$$\Rightarrow \quad \tilde{u}(\xi, \eta) = \varphi(\xi) + \psi(\eta)$$

$$\Rightarrow \quad u(x,t) = \varphi(x+ct) + \psi(x-ct), \checkmark.$$

SW: Conversely, any function $u(x,t)$ of the form

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

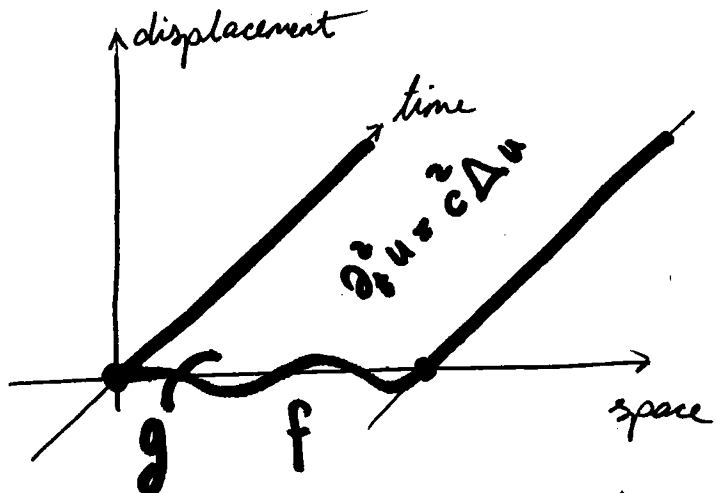
solves $\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$. $\checkmark$.

Ex: (Homogeneous Vibration Problem for $d=1$)

Let $c \neq 0$, $L>0$, $f, g \in C^0([0,L], \mathbb{R})$ be piecewise smooth and $f(0) = 0 = f(L)$, $g(0) = 0 = g(L)$. Consider

$$
(*)\quad \boxed{\begin{array}{ll}
\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0, & \text{for } (x,t) \in]0,L[\times]0,\infty[\\
u(0,t) = 0 = u(L,t), & \text{for } t \in [0,\infty[\\
u(x,0) - f(x) = 0 = \partial_t u(x,0) - g(x), & \text{for } x \in [0,L]
\end{array}}
\quad
\begin{array}{l}
\text{(PDE)} \\
\text{(BC)} \\
\text{(ID)}
\end{array}
$$

SW: Give the geometric and physical interpretation of the initial data and the boundary conditions.



By D'Alembert's Theorem we know that the solution of the PDE is of the form

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

for two yet to be determined functions $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$.

Let $f_1, g_1 \in R(\mathbb{R}, \mathbb{R})$ be piecewise smooth and extend f, g, respectively (i.e., for any $x \in [0, L]$: $f_1(x) = f(x)$ and $g_1(x) = g(x)$, but as opposed to f and g, f_1 and g_1 are defined everywhere).

$\Rightarrow$ (1D) gives: $\varphi(x) + \psi(x) = f_1(x)$ & $c(\dot{\varphi}(x) - \dot{\psi}(x)) = g_1(x)$

$$\Rightarrow \frac{d}{dx}(\varphi - \psi)(x) = \frac{1}{c} g_1(x) \Rightarrow (\varphi - \psi)(x) = \frac{1}{c} \int_{-\infty}^{x} g_1(y)\,dy + D$$

$(D \in \mathbb{R})$

$$\Rightarrow \varphi(x) = \frac{1}{2}\left(f_1(x) + \frac{1}{c}\int_{-\infty}^{x} g_1(y)\,dy + D\right)$$

$$\psi(x) = \frac{1}{2}\left(f_1(x) - \frac{1}{c}\int_{-\infty}^{x} g_1(y)\,dy - D\right)$$

$$\Rightarrow u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\left(\int_{-\infty}^{x+ct} g_1(y)\,dy - \int_{-\infty}^{x-ct} g_1(y)\,dy\right)$$

$$\underset{(x-ct \le x+ct)}{=} \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(y)\,dy.$$

(BC) gives: $\varphi(ct) + \psi(-ct) = 0 = \varphi(L+ct) + \psi(L-ct)$

$$\Rightarrow \quad 0 = \frac{1}{2}\left(f_1(ct) + f_1(-ct) \right) + \frac{1}{2c} \int_{-ct}^{ct} g_1(y)\,dy$$

$$0 = \frac{1}{2}\left(f_1(L+ct) + f_1(L-ct) \right) + \frac{1}{2c} \int_{L-ct}^{L+ct} g_1(y)\,dy ,$$

but these are not sufficient to specify f_1 & g_1 (and consequently $u(x,t)$).

Thus we turn to the method of disentanglement:

$$u(x,t) := \pi(x)\,\tau(t) \Rightarrow (*) \Rightarrow \pi\,\ddot{\tau} - c^2\,\ddot{\pi}\,\tau = 0 \Leftrightarrow \frac{-\ddot{\tau}}{c^2\tau} = \frac{-\ddot{\pi}}{\pi} = \lambda$$

$$\Rightarrow \boxed{\begin{array}{l} u(x,t) = \pi(x)\,\tau(t) \\ -\Delta\pi(x) = \lambda\,\pi(x) \\ \pi(0) = 0 = \pi(L) \\ -\partial_t^2\tau(t) = c^2\lambda\,\tau(t) \end{array}}$$

is a disentanglement of $(*)$.

$\pi(0) = 0 = \pi(L) \Rightarrow$ For any $n \geq 1$: $\left(\left(\frac{n\pi}{L}\right)^2, \sin\left(\frac{n\pi}{L}x\right) \right)$ is a relevant eigenpair of $-\Delta$.

$$\Rightarrow \; -\partial_t^2\,\tau(t) = \underbrace{\left(\frac{cn\pi}{L}\right)^2}_{>0} \tau(t) \Rightarrow \tau(t) = c_1 \cos\left(\frac{cn\pi}{L}t\right) + c_2 \sin\left(\frac{cn\pi}{L}t\right)$$

$\Rightarrow$ The relevant disentangled solutions of ⊛ are:

For any $n \geq 1$:

$$\boxed{\begin{aligned} \sin\left(\tfrac{n\pi}{L}x\right)\cos\left(c\tfrac{n\pi}{L}t\right) &= \sigma_n(x)\,\gamma_n(ct) = \tfrac{1}{2}\left(\sigma_n(x+ct) + \sigma_n(x-ct)\right) \\ \sin\left(\tfrac{n\pi}{L}x\right)\sin\left(c\tfrac{n\pi}{L}t\right) &= \sigma_n(x)\,\sigma_n(ct) = -\tfrac{1}{2}\left(\gamma_n(x+ct) - \gamma_n(x-ct)\right) \end{aligned}}$$

<u>SW</u>: Verify the (nontrivial) equalities above.

If $u(x,t) = \sum_{n\geq 1} a_n\,\sigma_n(x)\,\gamma_n(ct) + \sum_{n\geq 1} b_n\,\sigma_n(x)\,\sigma_n(ct)$ solves ⊛,

then $f(x) = u(x,0) = \sum_{n\geq 1} a_n\,\sigma_n(x)$ and

$$g(x) = \partial_t u(x,0) = \left[\sum_{n\geq 1} b_n\,\frac{cn\pi}{L}\,\sigma_n(x)\,\gamma_n(ct)\right]\Bigg|_{t=0} = \sum_{n\geq 1} b_n\,\frac{cn\pi}{L}\,\sigma_n(x).$$

Both f and g are piecewise smooth, so by the Fourier Convergence Theorem

for all $x \in [0, L]$: $\mathcal{F}(2\tilde{f}_o)(x) = f(x)$

and $\mathcal{F}(2\tilde{g}_o)(x) = g(x)$

$$\mathcal{F}(2\tilde{f}_o)(x) = \frac{c_o^f}{2} + \sum_{n\geq 1} c_n^f \gamma_n(x) + \sum_{n\geq 1} s_n^f \sigma_n(x).$$

$(n\geq 0)$ $c_n^f = 0$ because $2\tilde{f}_o$ is odd.

$(n\geq 1)$ $$s_n^f = \frac{1}{L}\int_{-L}^{L} \underbrace{2\tilde{f}_o(x)\,\sigma_n(x)}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\,\sigma_n(x)\,dx$$

$$\mathcal{F}(2\tilde{g}_o)(x) = \frac{c_o^g}{2} + \sum_{n\geq 1} c_n^g \gamma_n(x) + \sum_{n\geq 1} s_n^g \sigma_n(x)$$

$(n\geq 0)$ $c_n^g = 0$ because $2\tilde{g}_o$ is odd.

$(n\geq 1)$ $$s_n^g = \frac{1}{L}\int_{-L}^{L} \underbrace{2\tilde{g}_o(x)\,\sigma_n(x)}_{\text{even}}\,dx = \frac{2}{L}\int_0^L g(x)\,\sigma_n(x)\,dx$$

→ Picking $a_n := s_n^f$ and $b_n := \frac{L}{cn\pi} s_n^g$ produces the solution of ⊛:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n\geq 1} s_n^f \sin\left(\frac{n\pi}{L}x\right)\cos\left(c\frac{n\pi}{L}t\right) + \sum_{n\geq 1}\frac{L}{cn\pi} s_n^g \sin\left(\frac{n\pi}{L}x\right)\sin\left(c\frac{n\pi}{L}t\right), \\ \text{where } s_n^f &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ s_n^g &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned}}$$

Alternatively, we could rewrite the general solution as:

$$u(x,t) = \sum_{n\geq 1} a_n \sigma_n(x)\gamma_n(ct) + \sum_{n\geq 1} b_n \sigma_n(x)\sigma_n(ct)$$

$$= \sum_{n\geq 1} a_n \left(\frac{\sigma_n(x+ct)+\sigma_n(x-ct)}{2}\right) + \sum_{n\geq 1} b_n \left(\frac{\gamma_n(x+ct)-\gamma_n(x-ct)}{-2}\right)$$

$$= \frac{1}{2}\left(\sum_{n\geq 1} a_n \sigma_n(x+ct) + \sum_{n\geq 1} a_n \sigma_n(x-ct)\right)$$

$$-\frac{1}{2}\left(\sum_{n\geq 1} b_n \gamma_n(x+ct) - \sum_{n\geq 1} b_n \gamma(x-ct)\right)$$

$$\underset{\ast}{=} \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) - \frac{1}{2}\left(h_1(x+ct) - h_1(x-ct)\right).$$

$$\left(\begin{array}{l} f_1 := \sum_{n\geq 1} a_n \sigma_n \\ h_1 := \sum_{n\geq 1} b_n \gamma_n \end{array}\right) \qquad = \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) - \frac{1}{2}\int_{x-ct}^{x+ct} \dot{h}_1(y)\,dy$$

$$\left(\begin{array}{l} \dot{h}_1 = -\sum_{n\geq 1} b_n \frac{n\pi}{L}\sigma_n \\ g_1 := -c\,\dot{h}_1 \end{array}\right) = \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(y)\,dy.$$

$$f(x) = u(x,0) = f_1(x) \overset{\text{def}}{=} \sum_{n\geq 1} a_n \sigma_n(x)$$

$$g(x) = \partial_t u(x,0) = g_1(x) \overset{\text{def}}{=} \sum_{n\geq 1} b_n \frac{cn\pi}{L}\sigma_n(x).$$

$\Rightarrow$ f_1 is the odd periodic extension of f

& g_1 is closely related to the odd periodic extension of g (as before).

SW : Verify.

. The point is that D'Alembert's Theorem provide a large class of solutions, but determining which particular solution actually satisfies the boundary conditions as well is hard. On the other hand, the method of entanglement provides solutions in terms of limits of linear combinations of disentangled solutions, but since no disentangled solution decays exponentially fast in time it is not easy to see that the limits in question actually make sense.

SW : Replace the boundary condition of ⊛ with your favorite boundary condition, then find the solution.

Ex: Let $c := 3$, $L := 5$, $f : [0,5] \to \mathbb{R}$

$$x \mapsto 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x)$$

$$g : [0,5] \to \mathbb{R}$$

$$x \mapsto 0.$$

Solve

$$\boxed{\begin{aligned} &\partial_t^2 u(x,t) - 9\,\Delta u(x,t) = 0 \\ &u(0,t) = 0 = u(5,t) \\ &u(x,0) = f(x) \\ &\partial_t u(x,0) = g(x) = 0. \end{aligned}}$$

$$f(x) = 4\sin\left(\frac{5\pi}{5}x\right) - \sin\left(\frac{10\pi}{5}x\right) - 3\sin\left(\frac{25\pi}{5}x\right)$$

is odd and equal to its Fourier series:

$$\mathcal{F}(f) = \sum_{n\geq 1} s_n \sigma_n, \text{ where } s_n = \begin{cases} 4, & \text{if } n = 5 \\ -1, & \text{if } n = 10 \\ -3, & \text{if } n = 25 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{F}(g) = g = 0$$

D'Alembert's Theorem dictates:

$$\boxed{u(x,t) = \frac{1}{2}\Big(f(x+3t) + f(x-3t)\Big)}$$

$$u(0,t) = \frac{1}{2}\Big(f(3t) + f(-3t)\Big) = 0 \quad (f \text{ is odd})$$

[redacted]

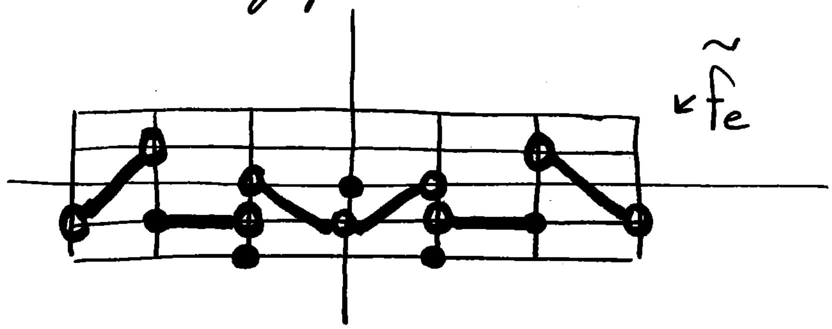
$$u(5,t) = \frac{1}{2}\Big(f(5+3t) + f(5-3t)\Big) = 0.$$

SW: (i) Verify this

(ii) Solve it via disentanglement.

Errata for the First Part:

- On p. 20, the graph on the bottom left corner should be:



- On pp. 24–25, "$(x,y),(z,t) \in \mathbb{R}^2$ are orthogonal if $\langle (x,y),(z,t) \rangle = 0$ and $(x,y) \neq (0,0) \neq (z,t)$."

- On p. 31, the third line from the bottom should end like so:

"... series of f, respectively."

- On pp. 36–37, all instances of "8" should be replaced by "4".
- On p. 39, the third integral from the top should read:

$$\text{"} \int_0^2 \frac{x}{2} \sin\left(\frac{n\pi}{2} x\right) dx \text{"}$$

- On p. 41, in the statement of the Fourier Convergence Theorem, $f \in R(I, \mathbb{R})$ itself should be piecewise continuous, $\partial_x f(x)$ should exist except possibly at finitely many points $x \in I$, and whenever $\partial_x f$ exist it should be continuous except at finitely many points. Let us abbreviate this by "piecewise smooth".

• p. 38 should read like this:

"Onto the coeff.s of $\mathcal{F}_{\mathbb{R}}(\tilde{f}_e)$:

$$c_0 = \frac{1}{2}\int_{-2}^{2} \tilde{f}_e(x)\,dx = \int_0^2 \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]\Bigg|_0^2 = 1$$

$$(n \geq 1) \quad c_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\gamma_n(x)}_{\text{even}}\,dx = \int_0^2 \frac{x}{2}\cos\left(\frac{n\pi}{2}x\right)dx = \begin{cases} \frac{-4}{\pi^2}\cdot\frac{1}{n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$(n \geq 1) \quad s_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\sigma_n(x)}_{\text{odd}}\,dx = 0$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}(\tilde{f}_e)(x) = \frac{1}{2} + \sum_{\substack{n \geq 1 \\ n:\,\text{odd}}} \left(\frac{-4}{\pi^2}\cdot\frac{1}{n^2}\right)\gamma_n(x)$$

$$= \boxed{\frac{1}{2} + \left(\frac{-4}{\pi^2}\right)\sum_{n \geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}$$

$$\Rightarrow \boxed{\mathcal{F}_{\mathbb{R}}(2\tilde{f}_e)(x) = 1 + \left(\frac{-8}{\pi^2}\right)\sum_{n \geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}$$

is the Fourier series of the even periodic extension of f."