Coding:

. Consider a collection $\mathcal{C}$ of things of the same kind (in some sense), e.g., all humans, all living organisms, all musical tones, all ~~pieces of~~ digital pictures all digital movies ... all points on the real line $\longleftrightarrow$, all propositions in mathematics; ...

By a coding we mean a (1:1, many:1, ∞:1, ...) correspondence that assigns to each thing in the collection $\mathcal{C}$ a unique string ~~(i.e., finite~~ or sequence of symbols.

~~Recall that a string is a list of finitely many~~

Recall that the difference between a string and a sequence is that a string is always finite, while a sequence is either infinite or biinfinite.

~~# In a Coding~~

For any coding we have a predetermined nonempty set, called an ~~alphabet~~, <u>alphabet</u> $\mathcal{A}$, from which we choose the symbols. Whatever structure $\mathcal{A}$ has will be inherited by the codes $\mathcal{A}^s$.

<u>Ex</u>: (1) $\{\text{humans}\} \longrightarrow \mathcal{A}^{10}$, $\mathcal{A} :=$ { all squiggles like

}

$H \longmapsto$ fingerprints (1 for each finger)

(2) $\{\text{living things}\} \longrightarrow \mathcal{A}^{\cancel{N}}$, $\mathcal{A} := \{$A: adenozin, T: timin, G: guanin, C: citozin$\}$

$L \longmapsto$ DNA

(3) $\{\text{PSU students, alumni, personnel}\} \longrightarrow \mathcal{A}^9$ $\quad \mathcal{A} = \{0, 1, 2, \dots, 9\}$

$H \longmapsto$ PSU ID (H) $\quad$ (SSN, passport #, bank acc #, debit card #, ...)

④ $\{\text{All spoken words in English language}\} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ $\mathcal{A} = \{\text{English alphabet}\}$

$\text{word} \longmapsto \text{its spelling}$

⑤ $\{\text{All English narratives}\} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ $\mathcal{A} = \{\text{English alphabet, together with punctuation}\}$

⑥ $\{\text{All spoken } \cancel{\text{languages}} \text{ narratives, in any language}\} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ $\mathcal{A} = \{\text{phonetic alphabet.}\}$

⑦ $\{\text{all simple melodies}\} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ $\mathcal{A} = \{A = \text{la},\ B = \text{si},\ C = \text{do},\ D = \text{re},\ E = \text{mi},\ F = \text{fa},\ G = \text{sol}\}$

(∃ upgrades to this) (𝄞, 𝄴, ♭, ♯, ...)

(8) $\left\{\begin{array}{c}\text{all digital} \\ \text{pictures}\end{array}\right\} \longrightarrow A^{\mathbb{Z}\times\mathbb{Z}}$

$A = \{\text{all colors}\}$

(similarly with movies)

(increase # of $\mathbb{Z}\times\mathbb{Z}$ boxes for higher resolution)

__Ex:__

(a) $\mathbb{R} = \left\{\begin{array}{c}\text{all real} \\ \text{numbers}\end{array}\right\} \underset{\text{def}}{=} \left\{\text{all points on } \longleftrightarrow \right\}$.

$\mathbb{R} \longrightarrow A^{\mathbb{Z}}$ $\quad A = \{0, 1, \dots, 9\}$

$x \longmapsto$ decimal expansion

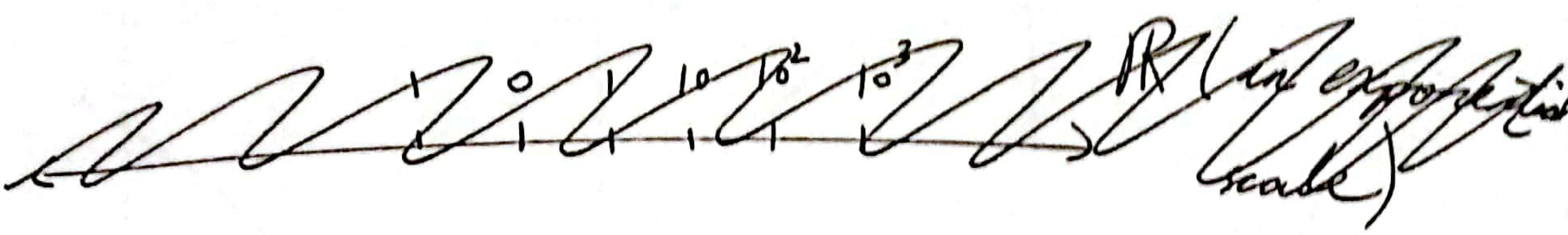
__How to do this:__ $\dots 0\, 0\, d_n \dots d_2\, d_1\, d_0 .\, d_{-1}\, d_{-2} \dots d_{-m}\, 0\, 0 \dots$

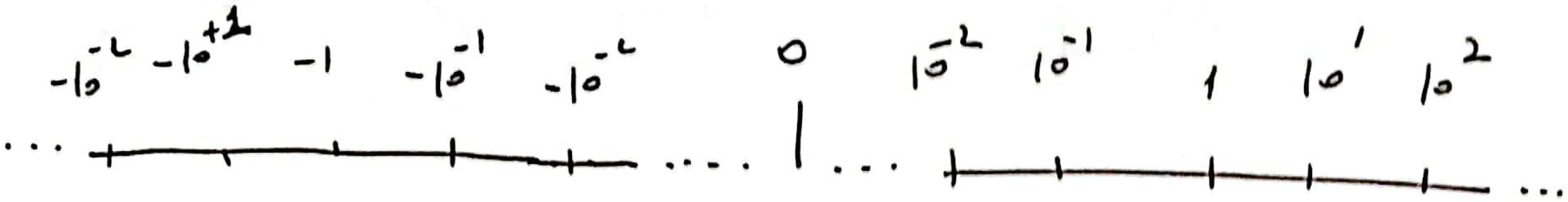
~~(i) Pick~~ (0) where x is given.

(i) Pick where 0 is.

(ii) Pick where 1 is. These determine where

$\dots, -10, \boxed{-1}, \frac{-1}{10}, \frac{-1}{100}, \boxed{0}, \frac{1}{100}, \frac{1}{10}, \boxed{1}, 10, 100, 1000, \text{—}$

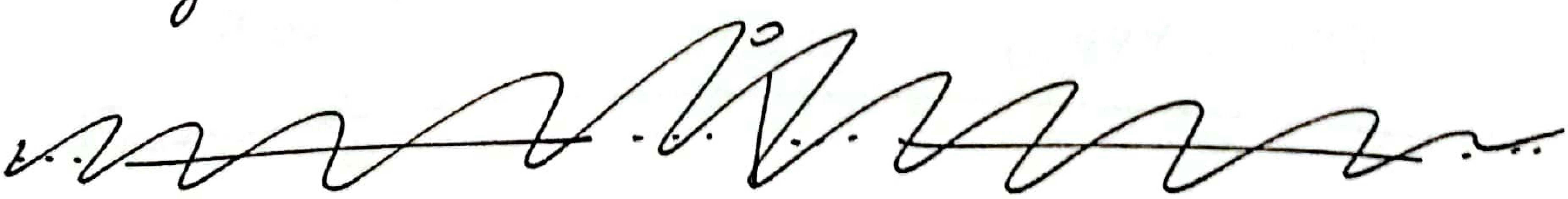


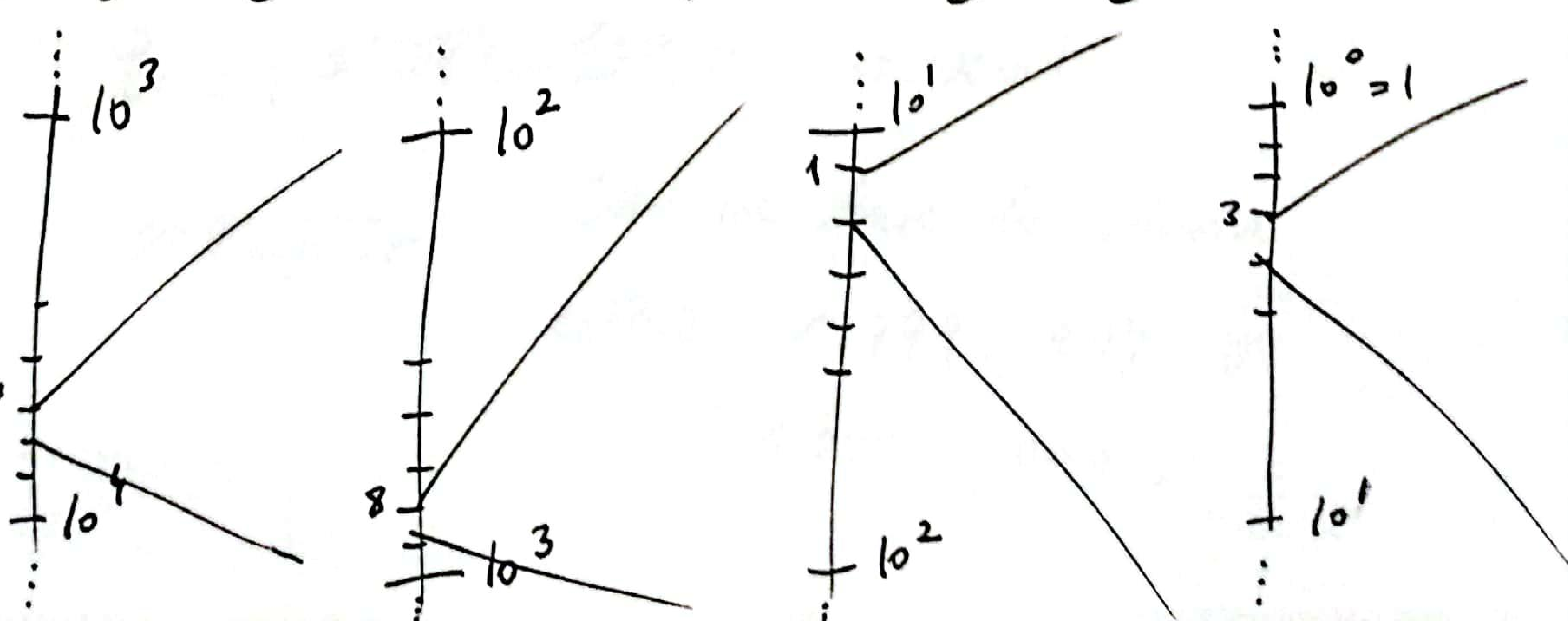


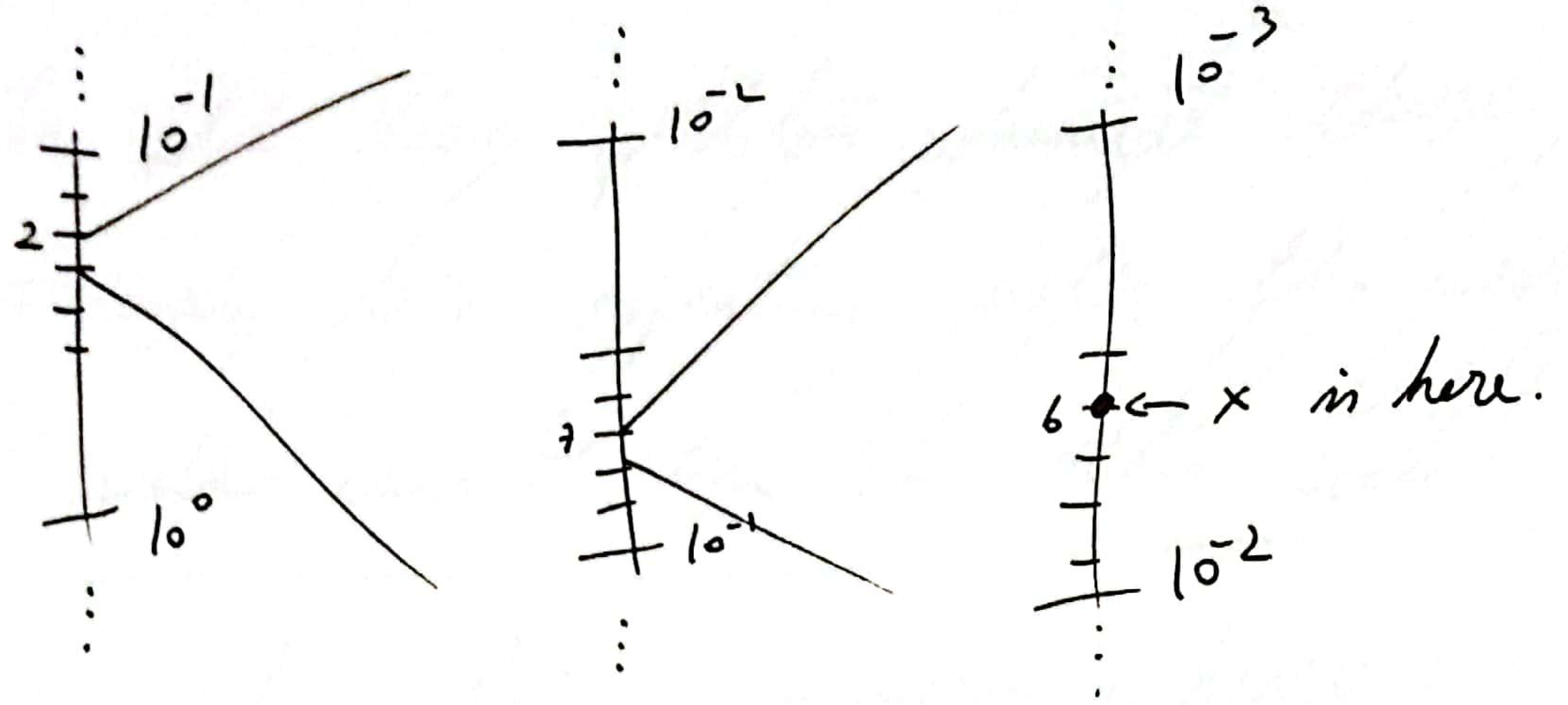
$\mathbb{R}$ ↗ (in exponential scale)

(13) Using these marks, read off the place of x on $\mathbb{R}$.

eg. the code 7813.276 is assigned only to:







But, there is a problem! More than one code is assigned to some points! but a coding must be sth : 1 always!

eg., $0.999\ldots9\ldots = x = 1$:

$$10x = 10.(0.999\ldots9\ldots) = 9.999\ldots999\ldots$$
$$x = 0.999\ldots999\ldots$$

$$9x = 9\cancel{.999\ldots999} \Rightarrow x = 1.$$

$\cancel{\Rightarrow x = 0.1x}$

Thus we have to choose either $0.999\ldots999$ or $1.000\ldots000$.

~~To solve this problem, whenever there are ∞ many 0's repeating after the dot we will choose the ~~co~~ other code!~~

To solve this problem, whenever there are two codes are assigned to one point, we will choose the one obtained by zooming in from the left.

(in the example we will thus choose 0.999...999...)

Observe that this problem will only ocur when we try to code one of the marker ~~boundarie~~s we choose in (i) ~~or~~ and (ii).
In particular there are exactly $\aleph$ many points at which this problem ocurs.

Also observe that we chose ~~to~~ powers of 10 arbitrarily, i.e. there is nothing special about ~~the~~ 10.

- ~~We~~ If we choose to use powers of 2, we have $A = \{0,1\}$ and $A^{\mathbb{Z}}$ is the set of all <u>binary sequences</u>.

- If we choose to use powers of 3, we have $A = \{0,1,2\}$ and $A^{\mathbb{Z}}$ is the set of all <u>ternary sequences</u>.

After we deal with the aforementioned problem, these codings are 1:1 :

e.g., $x \longmapsto$ its decimal exp.

$$\sum_{n\in\mathbb{Z}} d_n 10^n \longleftrightarrow \ldots 00 d_n \ldots d_0 . d_{-1} \ldots d_{-m} \cancel{00} \text{———}$$

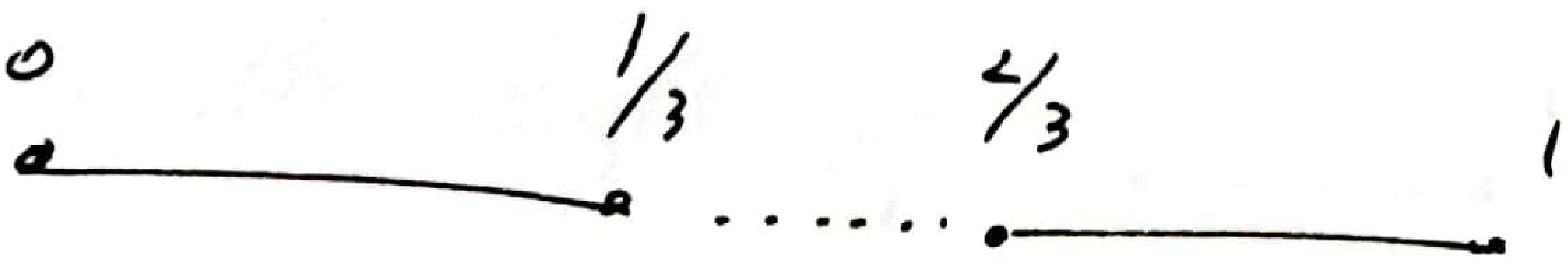
Cantor Set :

• One can build the Cantor set like this:

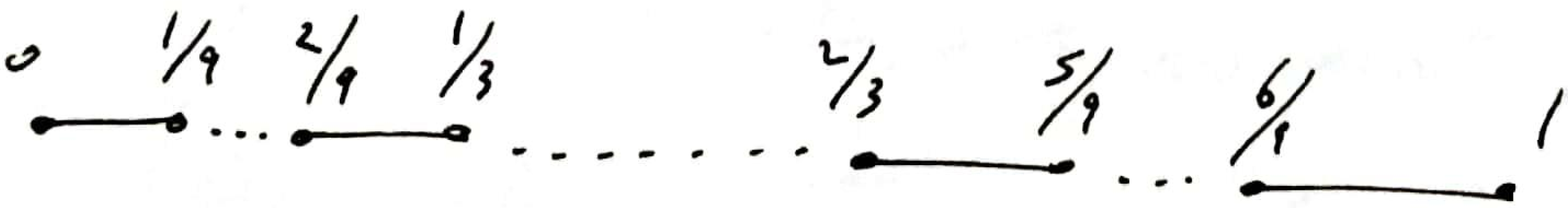
① 

is taken (0, 1 for convenience).

② Erase middle third:



③ Erase middle thirds :



④ Rinse and repeat until the end of time

⑤ What remains is the Cantor set!

(More precisely it's middle-thirds Cantor set $C_{1/3}$.)

Things to know about the Cantor set:

~~(I) It has no meat to it~~

~~(i.e. contains to no b inte~~

(I) It has no "length" as in

"the length of $\overset{0}{\bullet}\!\!\longrightarrow\!\!\overset{1}{\bullet}$ is 1"

- In particular ~~it has~~ there no meat to it (i.e. it contains no line segment)

(II) There are ∞ many points in $C_{1/3}$.

- In fact there are more the points in $C_{1/3}$ than there are natural numbers.

(III) For any error margin ε and for any point x in $C_{1/3}$, there is at least another point y in $C_{1/3}$ that is not further ~~away~~ away than ε from x.

Pf of (I):

Step	Shape	Total length
0	$0 \overset{1}{—} 1$	1
1	1 2	$\frac{1}{3}+\frac{1}{3} = \frac{2}{3} = 1-\frac{1}{3}$
2	1 2 3 4	$\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9} = \underbrace{\frac{4}{9}}_{}$ $\left(=\left(\frac{2}{3}\right)^2\right)$ $= 1-\frac{5}{9} = 1-\left(\frac{1}{3}+\frac{2}{9}\right)$
3	1 2 3 4 5 6 7 8	$8\cdot\frac{1}{27} = \frac{8}{27}$ $\cancel{= 1-\left(\frac{1}{3}+\frac{2}{9}+\frac{4}{27}\right)}$ $= 1-\left(\frac{1}{3}+\frac{2}{9}+\frac{4}{27}\right)$ $=\left(\frac{2}{3}\right)^3$
4	painful (16 pieces)	$\frac{16}{81} = \left(\frac{2}{3}\right)^4 = 1-\left(\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}\right)$

There is a pattern!

at step n the remaining length is

$$\left(\frac{2}{3}\right)^n = 1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots + \frac{2^{n-1}}{3^n}\right).$$

Thus at the end we will have

$$1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \text{———} \cdots\right)$$

length remaining. But how to compute the infinite sum inside parantheses?

Like so:

$$\left(\frac{1}{3}+\frac{2}{9}+\dots+\frac{2^{n-1}}{3^n}\right)=\frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots+\left(\frac{2}{3}\right)^{n-1}\right)$$

$$=\frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots+\left(\frac{2}{3}\right)^{n-1}\right)\frac{\left(1-\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)}$$

$$=\frac{1}{3}\,\frac{\left(1+\cancel{2/3}+\cancel{(2/3)^2}+\dots+\cancel{(2/3)^{n-1}} - \left(\cancel{2/3}+\cancel{(2/3)^2}+\dots+\cancel{(2/3)^{n-1}}+(2/3)^n\right)\right)}{1-2/3}$$

$$=\frac{1}{3}\,\frac{1-\left(\frac{2}{3}\right)^n}{1/3}=1-\left(\frac{2}{3}\right)^n$$

$$\Rightarrow\left(\frac{1}{3}+\frac{2}{9}+\dots\right)=1-\left(\frac{2}{3}\right)^\infty=1$$

$$\Rightarrow 1-\left(\frac{1}{3}+\frac{2}{3}+\dots\right)=0.$$

· This also prohibits $C_{1/3}$ to ~~to~~ contain a line segment, for otherwise it would have positive length.

Pf of (II): Use ternary expansions to see that by definition there is a 1:1 correspondence between the points in the Cantor set and the ternary sequences in $\{0,1,2\}^{\mathbb{Z}}$, ~~which is infinite~~ which has no 1's in ~~it~~ them.

Cantor's Diagonal argument:

To see that $C_{1/3}$ has more points than $\mathbb{N}$ has, suppose otherwise, i.e. there is a list of all points in $C_{1/3}$.

Then look at the diagonal sequence.

1) 0.□□□□ . . —

2) 0.□□□□ . . .

3) 0.□□□□ . . .

4) 0.□□□□ . . .

⋮

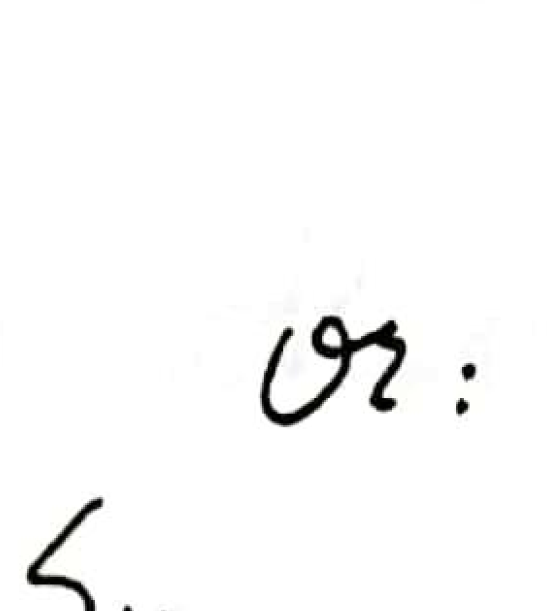
Or:

$\{x_1, x_2, x_3, x_4 \dots\} = C_{1/3}$

Look at the first digit of x_1. It is either 0 or 2.

using codes

If it is 0, go to



Otherwise (if it is 2) go to

. Rinse & repeat. You'll stay in $C_{1/3}$ and end up at some unlisted place.

~~In fact~~, Extending the list by this new column does not cut it!
(Just do the same thing by putting the new point ~~at~~ the first place)

<u>Pf of III</u>: Look at the ternary expansion of x. Observe that two points are close iff the beginnings of their codes coincide. Depending on the desired error ~~~~ margin, choose the beginning of ~~the~~ a new code to be ~~exactly~~ the same as x's, (say, the first $10^{10^{10^{10}}}$ digits), then ~~change~~ use 0 if x has 2, use 2 if x has 0.
this ~~go~~ ~~code~~ ~~list~~

This code is for another unique point, and that point is in $C_{1/3}$.

Symbolic Dynamics.

• Observe that reading a code (from left to right) corresponds to the Bernoulli shift

$$\sigma : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$$
$$\{f_n\}_n \longmapsto \{f_{n+1}\}_n .$$

• How can we use σ to ~~to~~ ~~come up with~~ talk about things we have coded?

In general this is a big and open problem (eg. ~~to~~ if we could answer this in the context of DNA we could possibly cure all diseases ~~by~~ ~~...~~, create new organisms, ...)

Let's focus on mathematics.

<u>Ex</u>: Let's use binary codes for points on the real line $\mathbb{R}$. Then

$$\cdots 0 d_n \cdots d_2 d_1 d_0 . d_{-1} d_{-2} \cdots \overset{\sigma}{\longmapsto} \cdots 0 d_n d_{n-1} \cdots d_1 d_0 d_{-1} . d_{-2} \cdots$$

corresponds to "multiplying by 2.

$$\begin{tikzcd}
x \arrow[r, mapsto, "\text{``mult. by 2''}"] \arrow[d, "\text{coding}"'] & 2x \arrow[d, "\text{coding}"] \\
\begin{array}{c}\text{binary}\\ \text{code}\\ \text{of } x\end{array} \arrow[r, mapsto, "\sigma"] & \begin{array}{c}\text{binary}\\ \text{code}\\ \text{of } 2x.\end{array}
\end{tikzcd}$$

Observe that this correspondence is very natural, since we did the coding first by ~~the~~ taking where

$$\vdots$$

$$-2^{-1} = \ldots 0 \ldots 0.10 \ldots$$

$$\vdots$$

$$0 = \ldots 0 \ldots 0 \ldots 0.0 \ldots$$

$$\vdots$$

$$2^{-2} = \ldots 0 \ldots 0.01 \ldots$$

$$2^{-1} = \ldots 0 \ldots 0.10 \ldots$$

$$2^{0} = \ldots 0 \ldots 1.00 \ldots$$

$$2^{1} = \ldots 0 \ldots 10.00 \ldots$$

$$\vdots$$

This naturality thus comes from a hidden group action.

Let X be a dummy variable,

$$\mathcal{X} := \{\dots, X^{-n}, \dots, X^{-1}, X^{0}, X^{1}, \dots\}.$$

Then $\mathcal{X}$ is an abelian group under

$$X^{n} \oplus X^{m} = X^{n+m} \qquad (X^{0} \text{ is the identity})$$

$\to$ $\mathcal{X}$ is indiscernable from $(\mathbb{Z}, +)$

or B_2 (braid group on two strings).

Let $\mathcal{A}$ be an alphabet. Then there is a group action of $\mathcal{X}$ on $\mathcal{A}^{\mathbb{Z}}$, called the Bernoulli's shift action, given by

$$\mathrm{act} : \mathcal{X} \times \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

$$(X^{m}, f) \mapsto \sigma^{m}(f)$$

Here σ^0 does nothing, σ^2 shifts to the right by 2, σ^{-3} shifts to the left by 3 etc.

- Def: A ring $(R, +, \times)$ consists of

(i) ~~A nonempty se~~ An abelian group $(R, +)$ identity: 0.

(ii) $\times$ is associative

(iii) $\times$ has an ~~...~~ ~~...~~ identity, denoted by 1

(iv) $\times$ is commutative.

(v) ~~+~~ is distributive over $\times$.

$$a(b+c) = ab + ac.$$

Obs: $\times$ need not be ~~...~~ inverses.

Ex: $(\mathbb{Z}, +, \times)$ is a ring.

Ex (no need: only group) By our ~~rule~~ inheritance remark earlier, once we choose $A = \mathbb{Z}$, $\mathbb{Z}^{\mathbb{Z}}$ too is a ring. The ~~$\mathbb{Z}$~~ Fibonacci sequence is an organism in the environment $\mathbb{Z}^{\mathbb{Z}}$.

~~a_n~~

$$\cdots \; \big| \; a_{\underline{2}} \; \big| \; a_{\underline{1}} \; \big| \; a_0 \; \big\| \; a_{+1} \; \big| \; a_{+2} \; \big| \; \cdots \quad \oplus \quad \cdots \; \big| \; b_{\underline{2}} \; \big| \; b_{\underline{1}} \; \big| \; b_0 \; \big\| \; b_{+1} \; \big| \; b_{+2} \; \big| \; \cdots$$

$$:= \quad \cdots \; \big| \; a_{\underline{1}}+b_{\underline{1}} \; \big| \; a_0+b_0 \; \big\| \; a_{+1}+b_{+1} \; \big| \; a_{+2}+b_{+2} \; \big| \; \cdots$$

and

$$\cdots \; \big| \; a_{\underline{2}} \; \big| \; a_{\underline{1}} \; \big| \; a_0 \; \big\| \; a_{+1} \; \big| \; a_{+2} \; \big| \; \cdots \quad \otimes \quad \cdots \; \big| \; b_{\underline{2}} \; \big| \; b_{\underline{1}} \; \big| \; b_0 \; \big\| \; b_{+1} \; \big| \; b_{+2} \; \big| \; \cdots$$

$$:= \quad \cdots \; \big| \; a_{\underline{2}}b_{\underline{2}} \; \big| \; a_{\underline{1}}b_{\underline{1}} \; \big| \; a_0 b_0 \; \big\| \; a_{+1}b_{+1} \; \big| \; a_{+2}b_{+2} \; \big| \; \cdots$$

(we flipped the order of the tapes for convenience!)

identity for $\oplus$: $\cdots \,|\, 0 \,|\, 0 \,\|\, 0 \,|\, 0 \,|\, \cdots =: 0$

identity for $\otimes$: $\cdots \,1 \,|\, 1 \,|\, 1 \,\|\, 1 \,|\, 1 \,|\, 1 \,|\, \cdots =: 1$

<u>Ex</u>: ~~The~~ There is a ~~smallest ring~~ smallest ~~set~~ ring $\mathbb{Z}[x]$ containing both $\mathbb{Z}$ and X is ~~a ring~~ the set of all <u>universal Laurent polynomials</u>:

$$a_{-n}X^{-n} + \cdots + a_{-1}X^{-1} + a_0 X^0 + a_1 X^1 + \cdots + a_n X^n$$

$a_{-n}, \dots, a_n \in \mathbb{Z}$.

addition: $\left(\sum_{|k|\le n} a_k X^k\right) \oplus \left(\sum_{|k|\le n} b_k X^k\right) = \sum_{|k|\le n} (a_k + b_k) X^k$

multiplication: $\left(\sum_{|k|\le n} a_k X^k\right) \otimes \left(\sum_{|\ell|\le m} b_\ell X^\ell\right) = \sum_{|k|\le n+m} \left(\sum_{i+j=k} a_i b_j\right) X^k$

~~Def: Let $(X,+)$ be~~

<u>Def</u>: Let (X, law) be an ~~com~~ abelian group, $(R, \oplus, \otimes)$ be a ring. A <u>ring action</u> of R on X is a binary op. $\mathrm{act}: R \times X \longrightarrow X$ such that

(1) It is a group action of the group $(R,+)$

(2) $\mathrm{act}(1,x) = x, \quad \forall x \in X$

(3) $\mathrm{act}(r \otimes s, x) = \mathrm{act}(r, \mathrm{act}(s,x)), \quad \forall r,s \in R \quad \forall x \in X$

(4) $\mathrm{act}(r \oplus s, x) = \cancel{\mathrm{act}}\ \mathrm{law}(\mathrm{act}(r,x), \mathrm{act}(s,x)), \quad \forall r,s \in R \quad \forall x \in X$

(5) $\mathrm{act}(r, \mathrm{law}(x,y)) = \cancel{\mathrm{act}}$ $\mathrm{law}(\mathrm{act}(r,x), \mathrm{act}(r,y)), \quad \forall r \in R \quad \forall x,y \in X.$

Ex: We have the following upgrade of the group action $act : X \times \mathbb{Z}^{\mathbb{Z}} \longrightarrow \mathbb{Z}^{\mathbb{Z}}$ to ~~the~~ a ring action $act: \mathbb{Z}[X] \times \mathbb{Z}^{\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}}$:

$$act : \mathbb{Z}[X] \times \mathbb{Z}^{\mathbb{Z}} \longrightarrow \mathbb{Z}$$
$$(p, \{f_n\}_n) \longmapsto p(\sigma)\{f_n\}_n$$

(ie., substitute σ for X).

Def: Define $\forall p \in \mathbb{Z}[X]$:

$$S(p) := \{ f \in \mathbb{Z}^{\mathbb{Z}} \mid act(p,f) = \underbrace{\ldots 00 \ldots 0.0 \ldots}_{=0} \}$$

Thus $S(p)$ is the collection of all sequences f whose behavior is dictated by p. It is called the annihilator of p.

Ex: $p(X) := X^2 - X - 1 \in \mathbb{Z}[X]$.

Then $\mathcal{S}(p)$ is the collection of all sequences $f = \{f_n\}_n$ that satisfy the Fibonacci rule. $\boxed{f_{n+2} = f_{n+1} + f_n}$. Indeed,

$$f = \{f_n\}_n = \cdots \begin{array}{|c|c||c|c|} \hline f_{-1} & f_0 & f_1 & f_2 \\ \hline \end{array} \cdots \in \mathcal{S}(X^2 - X - 1)$$

$\Leftrightarrow$ act $(X^2 - X - 1, f) = 0 \Leftrightarrow (\sigma^2 - \sigma - 1)(f) = 0$

$$\Leftrightarrow \cdots \begin{array}{|c|c|c||c|c|} \hline f_0 & f_1 & f_2 & f_3 & f_4 \\ -f_{-1} & -f_0 & -f_1 & -f_2 & -f_3 \\ -f_{-2} & -f_{-1} & -f_0 & -f_1 & -f_2 \\ \hline \end{array} \cdots$$

$\Leftrightarrow$

$$\begin{array}{c} \vdots \\ f_0 - f_{-1} - f_{-2} = 0 \\ f_1 - f_0 - f_{-1} = 0 \\ f_2 - f_1 - f_0 = 0 \\ f_3 - f_2 - f_1 = 0 \\ \vdots \end{array}$$

$\Leftrightarrow \boxed{f_{n+2} = f_{n+1} + f_n}$ for any $n \in \mathbb{Z}$.

✓.

Remark. Earlier in the course we had ~~established~~ ~~proved~~ mentioned that sums of ~~integer or~~ integer multiples of shifts of sequences satisfying the Fibonacci rule also satisfy the Fibonacci rule.

Now this result is seen to be immediately due to be ~~the~~ ~~way~~ our setup. Indeed, let $q \in \mathbb{Z}[x]$ be another universal polynomial, $f \in \mathcal{S}(p)$ be a Fibonacci sequence, i.e., $\mathrm{act}(p, f) = 0$.

We claim that $\mathrm{act}(q, f) \in \mathcal{S}(p)$, i.e.,

$$\mathrm{act}\big(p, \mathrm{act}(q, f)\big) = 0.$$

Here is the proof:

$$\mathrm{act}(p, \mathrm{act}(q, f)) \underset{\substack{\uparrow \\ \text{axiom (4)} \\ \text{of a ring} \\ \text{action}}}{=} \mathrm{act}(p \otimes q, f)$$

$$\underset{\substack{\uparrow \\ \text{commutativity} \\ \text{axiom for} \\ \text{a ring (in} \\ \text{this case } \mathbb{Z}[x])}}{=} \mathrm{act}(q \otimes p, f) = \mathrm{act}(q, \mathrm{act}(p, f)) = \mathrm{act}(q, 0) = 0$$

$0 = \cdots | 0 || 0 . \,\_$

stays put, no matter ~~too~~ what. ✓

Thus changing the rule corresponds to changing the Laurent polynomial.

$S(X^2-X-1) = \{ \cdots \boxed{\cdots \mid a \mid\mid b \mid a+b} \cdots \mid a,b \in \mathbb{Z} \}$

$S(X^3-2) = \{ \cdots \boxed{\tfrac{a}{2} \mid a \mid\mid b \mid c \mid 2a} \cdots \mid a,b,c \in \mathbb{Z} \}$

$S(X^2-4X+4) = \{ \cdots \boxed{a \mid\mid b \mid 4b-4a} \cdots \mid a,b \in \mathbb{Z} \}$

$S(X-2) = \{ \cdots \boxed{a \mid\mid 2a \mid 4a} \cdots \mid a \in \mathbb{Z} \}$

$S(0) = \mathbb{Z}^{\mathbb{Z}}$ $\qquad$ $S(1) = \{0\}$

$0 = 0 \cdot X^0$ $\qquad$ $1 = 1 \cdot X^0$

$S(X) = \overset{0}{\cancel{\{ \cdots \boxed{a \mid\mid a \mid a} \cdots \mid a \in \mathbb{Z} \}}}$

$S(X^2-1) = \{ \cdots \boxed{b \mid a \mid\mid b \mid a \mid b} \cdots \mid a,b \in \mathbb{Z} \}$

$S(X-1) = \{ \cdots \boxed{\cdots a \mid a \mid\mid a \mid a} \cdots \mid a \in \mathbb{Z} \}$

$S(X+1) = \{ \cdots \boxed{-a \mid a \mid\mid -a} \cdots \mid a \in \mathbb{Z} \}$

$S(2X^2-3) = 0.$

22. If $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & -4 & 2 \\ 3 & 5 & -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, then ...

a) $A(\vec{u}+\vec{v}) = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$.

b) $A(\vec{u}+\vec{v}) = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}$.

c) $A(\vec{u}+\vec{v}) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

d) $A(\vec{u}+\vec{v}) = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}$.

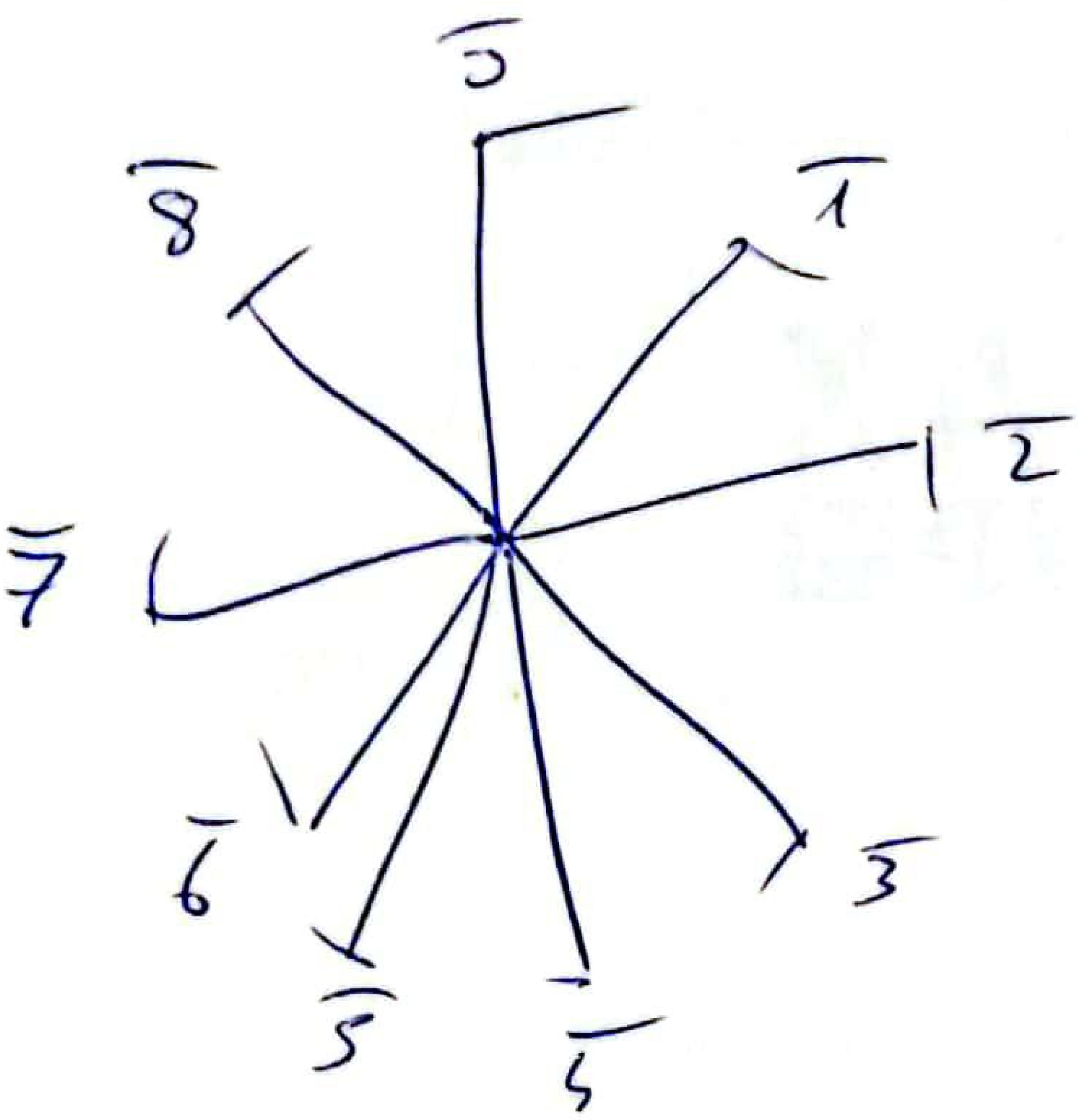
e) $A(\vec{u}+\vec{v})$ is not defined.

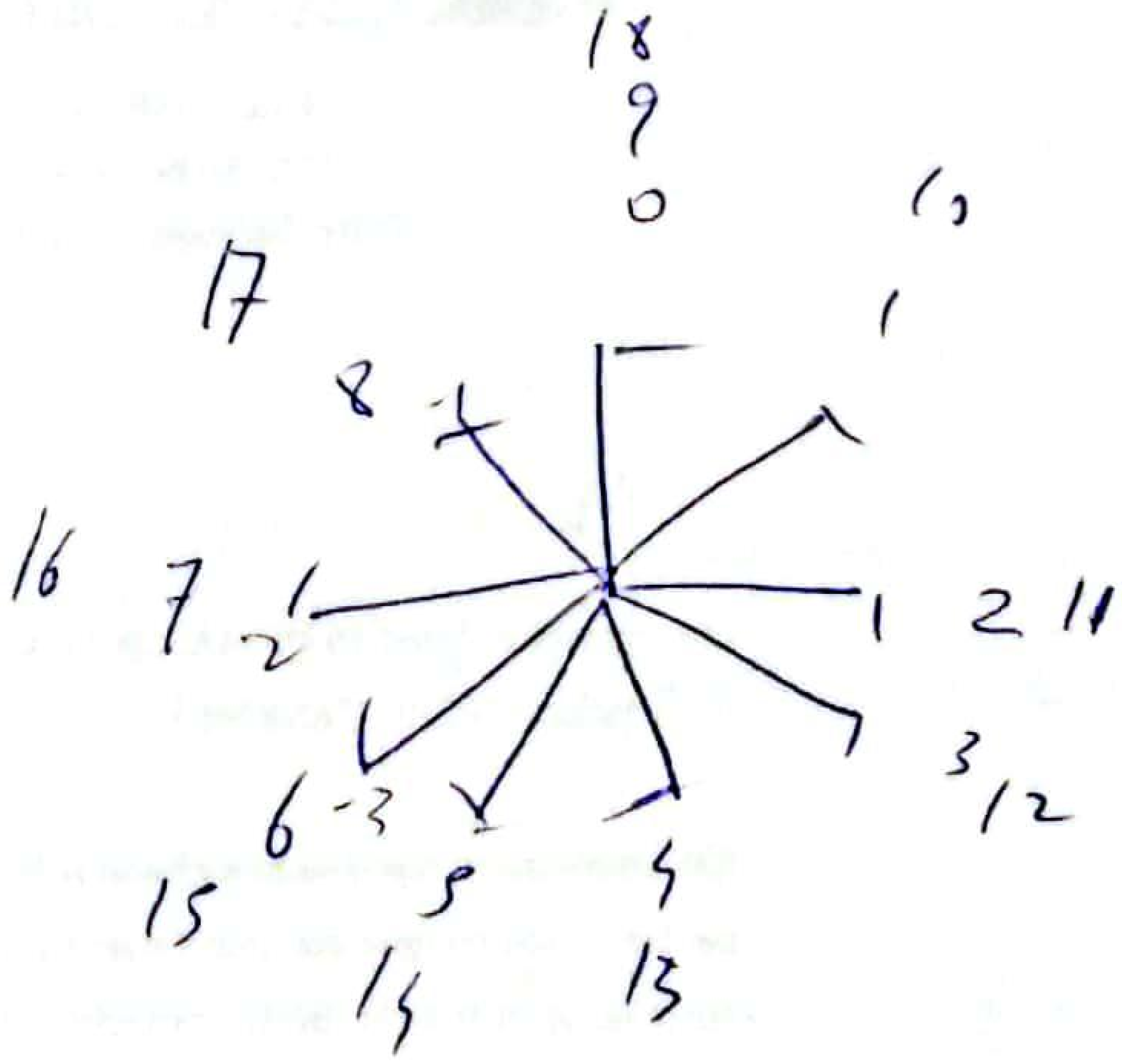
① Coding $\infty : 1$ correspondence.

$$c : \mathbb{Z} \longrightarrow A$$

$A = \{\text{all wings}\}$

$= \{\bar{0}, \bar{1}, \dots, \bar{8}\}$





③ e an integer is given the code

$\bar{0} \Leftrightarrow$ divisible by 9

$\bar{1} \Leftrightarrow$ ~~remainder~~ remainder $= 1$

$\bar{2} \Leftrightarrow$ remainder $= 2$

$\vdots$ In particular, $c(9) = c(99) = c(999) = \dots = \bar{0}$

$c(1) = c(10) = c(100) = \dots = \bar{1}$.

(3) Verify that this is indeed $\infty:1$. (i.e., each code $\bar{0}, \bar{1}, \ldots, \bar{8}$ is given to the same "amount" of integers)

$$\Leftrightarrow \left\{\begin{matrix}\#\text{ whose}\\ \text{code is } \bar{0}\end{matrix}\right\} \xleftarrow{1:1} \left\{\begin{matrix}\#\text{ whose}\\ \text{code is } \bar{1}\end{matrix}\right\}$$

$$\updownarrow 1:1$$

$$\left\{\begin{matrix}\#\text{ whose}\\ \text{code is } \bar{8}\end{matrix}\right\} \ldots \xleftrightarrow{1:1} \left\{\begin{matrix}\#\text{ whose}\\ \text{code is } \bar{2}\end{matrix}\right\}$$

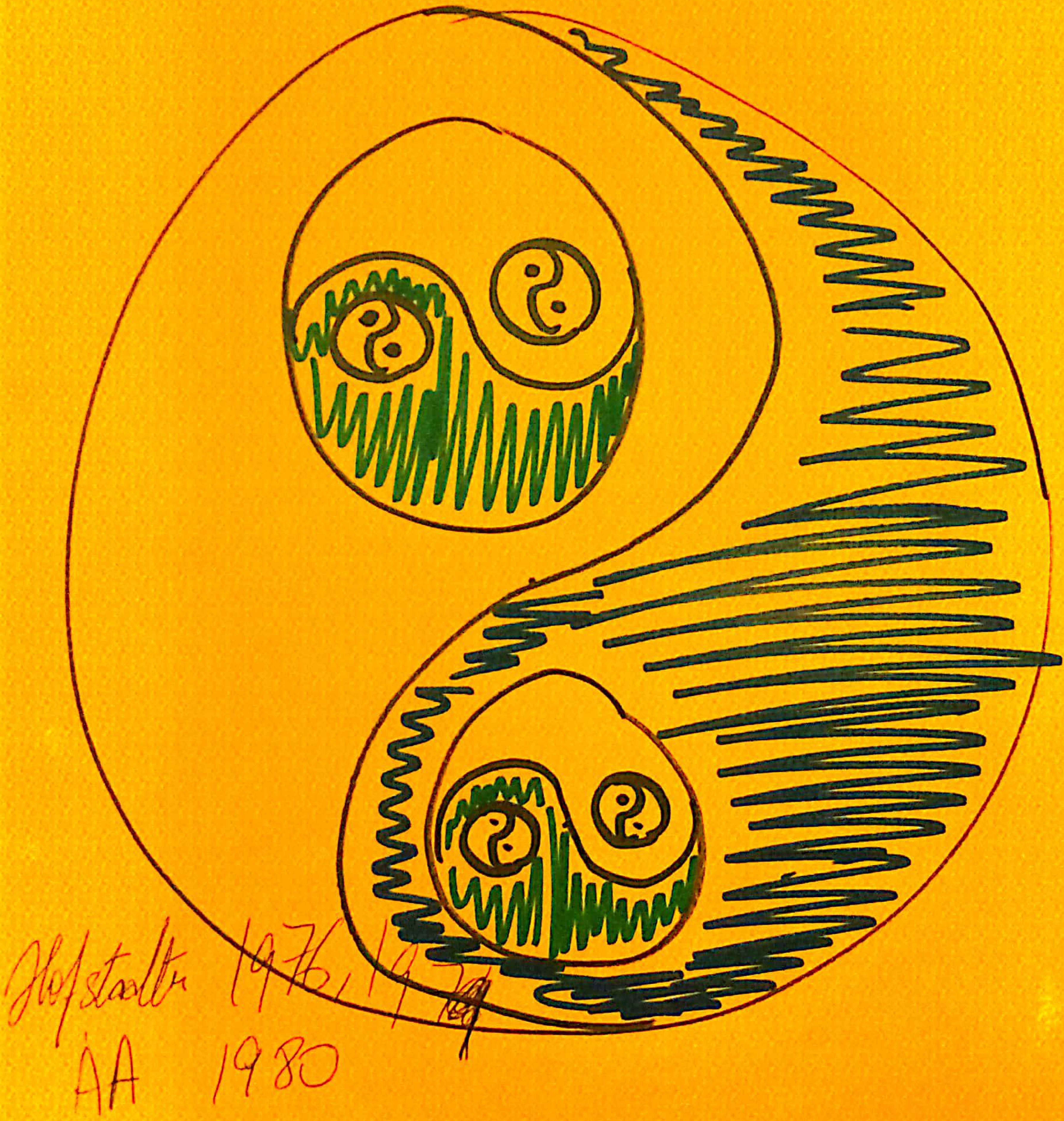
Pf: "Add 1" is a 1:1 correspondence.

(4) ~~$\mathbb{Z}$~~ C_9 acts on A.

$$\begin{tikzcd} \bar{0} \arrow[r] & \bar{1} \arrow[r] & \bar{2} \arrow[r] & \bar{3} \arrow[r] & \bar{4} \arrow[r] & \bar{5} \arrow[r] & \bar{6} \arrow[r] & \bar{7} \arrow[r] & \bar{8} \arrow[llllllll, bend left] \end{tikzcd}$$

$\mathbb{Z}$ acts on itself.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\text{add } 1"] \arrow[d, "c"'] & \mathbb{Z} \arrow[d, "c"] \\ A \arrow[r, "\bar{1}"] & A \end{tikzcd} \qquad \begin{tikzcd} n \arrow[r] \arrow[d, "c"'] & n+1 \arrow[d] \\ \bar{n} \arrow[r, "\bar{1}"] & \overline{n+1} \end{tikzcd} \checkmark.$$


Hofstadter 1976, 1979
AA 1980