# DSS - Talk 2 :

**Thm 1.6.** (Foliated Lyapunov Spectrum Approximate Arithmeticity)

- $M$ $C^\infty$ compact Riemannian
- $\mathcal{F} \in \mathrm{Fol}^{\text{Hölder}, \infty}(M)$.
- $H \in \mathrm{Polar}^0(M)$, $H \leq F = T\mathcal{F} \leq TM$.

  $H$ $C^k$ along $\mathcal{F}$.

$$\mathrm{Reg}(H, \mathcal{F}) = \left\{ x \in M \;\middle|\; \begin{array}{l} \exists\, U_x \subseteq \mathcal{F}_x \text{ open, } x \in U_x, \\ H|_{U_x} \text{ is } C^\infty, \text{ satisfies Hörmander} \\ \quad \text{in } \mathcal{F}_x, \\ \text{growth vector is constant.} \end{array} \right\}.$$

- $\forall x \in \mathrm{Reg}(H, \mathcal{F})$: $TC_x \mathcal{F}_x$ is asymmetric.

  (automorphisms = homotheties)

  and $r$-step, $r \geq k$

- $f \in \mathrm{Diff}^1(M)$, $f : (H, \mathcal{F}) \circlearrowleft$.
- $\mathrm{Per}(f) \subseteq \mathrm{Reg}(H, \mathcal{F})$ $f$ $C^\infty$ along $\mathcal{F}$,
- $T f : TM \circlearrowleft$ Hölder
- $f$ is topologically transitive
- $f$ satisfies the (weak) Kalinin closing property

- $\mu$ be an $f$-invariant, $f$-ergodic Borel prob. measure on $M$. (Weg [illegible])

- $\mathrm{supp}(\mu) \cap \mathrm{Reg}(H, \mathcal{F}) \neq \emptyset$.

Let us enumerate the Lyapunov exponents of $(\mu, f, T\mathcal{F}_f)$:

$$\underbrace{X^{11}, X^{12}, X^{13}, \ldots, X^{1\delta^1}}_{=x^1}, \underbrace{X^{21}, \ldots, X^{2\delta^2}}_{=x^2}, \underbrace{X^{31}, \ldots, X^{3\delta^3}}_{=x^3}, \ldots,$$

$$\underbrace{X^{\ell 1}, X^{\ell 2}, \ldots, X^{\ell \delta^\ell}}_{=x^\ell}$$

$$\boxed{x^1 > x^2 > \ldots > x^\ell}$$

$$\sum_{i=1}^{\ell} \delta^i = \dim(\mathcal{F}).$$

Then $\forall\, \varepsilon \in \mathbb{R}_{>0}, \exists\, x(\varepsilon) \in \mathbb{R}$ ;

$\forall\, j \in \{1, 2, \ldots, \ell\}, \forall\, k \in \{1, 2, \ldots, s^{\ell}\}$ :

$$|X^{jk} - j \cdot x(\varepsilon)| < \varepsilon$$

In particular $\forall\, j \in \{1, 2, \ldots, \ell\}$:

$$|x^{j} - j\, x(\varepsilon)| < \varepsilon$$

and (unless $x(\varepsilon) = 0$), $\ell = r$ and

$\cancel{s^{1}, \ldots, s^{\ell}}$

~~Oseledets numbers~~ $\Rightarrow$

Oseledets dimensions = Mitchell dimensions.

~~$\mathrm{Spec}_x\left(f, T^{b}f|_H\right) = \{\chi^1_x, \dots, \chi^{l_x}_x\}$.~~

~~Osctb~~

<u>Oseledets</u>: $f : M \circlearrowleft$ $C^1$ diffeo.

$H \in \mathrm{Polar}(M)$, $Tf : H \circlearrowleft$.

$\mu$ $f$-inv., $f$-erg. Borel prob. measure on $M$.

Then $\exists\, M_0 \subseteq M$, $f$-inv, $\mu(M_0) = 1$

$\exists!\, l \leq \cancel{\mathrm{di}}\ \mathrm{rk}(H)$, $\exists!\, \chi^1 > \dots > \chi^l \in \mathbb{R}$

$\cancel{\exists}$ $\forall x \in M_0$, $\exists!$ $H_x = \bigoplus_{j=1}^{l} L^i_x$

$$\overrightarrow{T^H_x f}(L^i_x) = L^i_{f(x)}$$

and $\forall v^i \in L^i_x|_0$, $\forall$ $c^0$ $|\cdot|$ on $H/_{M_0}$:

$$\lim_{|n| \to \infty} \frac{\log|T_x f^n v^i| - \chi^i \cdot n}{|n|} = 0.$$

$\cancel{H \leq \mathcal{F}}$

$H \leq F = T\mathcal{F}$.

$$\mathrm{Reg}(H, \mathcal{F}) = \left\{ x \in M \;\middle|\; \begin{array}{l} \exists\, U_x \subseteq \mathcal{F}_x \text{ open}, x \in U_x \\ H|_{U_x} \text{ is } C^\infty, \text{ satisfies} \\ \text{Hörmander in } F_x \\ \text{and the growth} \\ \text{vector is constant} \end{array} \right\}$$

Say ~~$\forall x \in \mathrm{Reg}(H,\mathcal{F})$, $T\mathcal{F}_x$~~

Say $\forall x \in \mathrm{Reg}(H,\mathcal{F})$ $\forall y \in \mathrm{Reg}(H,\mathcal{F}) \cap \mathcal{F}_x$: $T_y\mathcal{F}_x$ is asymmetric.

$f : (M, \mathcal{F}, H) \circlearrowleft$.

$\mu$ $f$-inv. erg. wlog nonatomic.

$\mathrm{supp}(\mu) \cap \mathrm{Reg}(H, \mathcal{F}) \neq \emptyset$

$f$ top. trans. with weak Kalinin property.

$\mathrm{Lyap}(\mu, f, T^{\mathcal{F}} f)$:

$$\chi^1 > \chi^2 > \dots > \chi^\ell$$

$\delta^j = \dim(L^j)$, $\sum_{j=1}^{\ell} \delta^j = \dim(\mathcal{F})$.

$$0 = H^0 \lneq H^1 \lneq \dots \lneq H^{r-1} \lneq H^r = F = T\mathcal{F}$$

locally on $\mathrm{Reg}(H,\mathcal{F})$ along $\mathcal{F}$.

* ~~Thm [illegible]~~:

Then $\exists!$ Carnot group $C$, $\forall x \in \mathrm{Reg}(H, \mathcal{F})$

$$\cancel{C \hookrightarrow RC}$$

$$\cancel{FC_x \mathcal{F}_x} \quad TC\mathcal{F} \underset{\substack{\text{loc} \\ \text{along } \mathcal{F} \\ \text{at } x}}{\cong} \mathcal{F}_x \times C .$$

$\mathrm{Reg}(H, \mathcal{F})$ is $f$-inv., $\mathrm{supp}(\mu) \cap \mathrm{Reg}(H, \mathcal{F}) \neq \emptyset$.

$$\Rightarrow \mu\big(\mathrm{Reg}(H, \mathcal{F})\big) = 1 .$$

Why we may ~~assume~~ replace

$M_0$ with $M_1 = M_0 \cap \mathrm{Reg}(H, \mathcal{F})$.

$x \in M_1$ — both CC-regular along $\mathcal{F}$

— and LP-regular ~~along~~ ~~$\mathcal{F}$~~

(measurable existence, $f$-invariance **, full ** $\mu$-measure) not topologically nice

$\oplus$ $+\,?$ $+$

But on $M_1$ Lyapunov objects/estimates are measurable (not necessarily continuous).

$M_2^{(?)}$

$M_2 \subseteq M_1$ Pesin set. Luzin-Pesin set

(~~Hölder~~ continuous existence, possibly not invariant, arbitrarily large measure, ~~the~~ compact))

• ~~Th~~ Thm. 1.4 (Arithmeticity at a periodic point

$p \cancel{\in Per(f)}$ $p \in Reg(H, \mathcal{F}) \cap Per(f)$.

$\Rightarrow \exists n \in \mathbb{Z}_{>0} : p \in Fix(f^n)$

$T_p f^n : (T_p M, F_p, H_p) \circlearrowleft$

$\sim \; TC_p^{\mathcal{F}} f^n : TC_p \mathcal{F}_p \circlearrowleft$ homothety.

$$\Rightarrow \exists!\, \chi(p) \in \mathbb{R} :$$

$$\mathrm{LSpec}_p\left(f^n, T^{\mathcal{F}} f^n\right) = \{\chi(p), 2\chi(p), \ldots, r\chi(p)\}.$$

with multiplicities = Mitchell numbers.

$$\Rightarrow \mathrm{LSpec}_p\left(f, T^{\mathcal{F}} f\right) = \{n \cdot \chi(p), n \cdot 2\chi(p), \ldots, n \cdot r \cdot \chi(p)\}$$

with multiplicities = Mitchell numbers.

$$\mathcal{P} = \left\{(n,p) \in \mathbb{Z}_{>0} \times M \;\middle|\; \begin{array}{l} p \in \mathrm{Reg}(H,\mathcal{F}) \cap \cancel{\mathrm{Per}(f)} \\ \cancel{f^n(p)}, \quad \cap \mathrm{Fix}(f^n) \end{array} \right\}$$

$$\mathcal{P} \to \mathrm{Reg}(H,\mathcal{F}) \cap \mathrm{Per}(f) \quad \text{discrete}.$$

$$\Rightarrow \boxed{\exists!\ \chi : \mathcal{P} \to \mathbb{R}}$$

Let $x \in M_2$ be a point of density.

$\Rightarrow$ For $\eta \approx 0$, $\frac{1}{\eta}\mu\left(M_2 \cap M[x, \eta]\right) \gg 0$.

$\uparrow$ w.r.t. some other metric

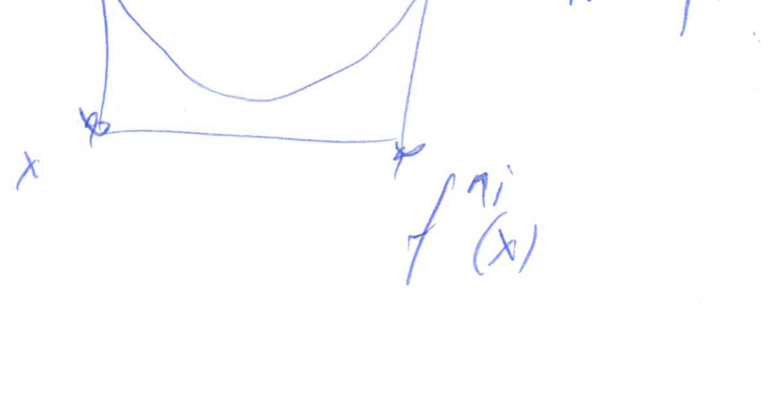
Poincaré Recurrence $\exists\, n_i \uparrow \infty$:

$\forall i$ : $f^{n_i}(x) \in M_2 \cap M[x, \eta]$

For $\eta$ small enough, by the weak Kalinin property,

$\exists i$

$\exists\, p_i = p_i(\eta) \in \mathrm{Per}(f)$, $f^{n_i}(p_i) = p_i$,



Say $\boxed{Per(f) \subseteq Reg(H, \mathcal{F})}$

Maybe $Per(f) \not\subseteq M_0$,

so maybe $Per(f) \not\subseteq M_1$.

$\angle Spec_{p_i}(f, T\mathcal{F}_f) = \{x(p_i), 2x(p_i), \dots, r\,x(p_i)\}$

with Mitchell numbers.

$\Rightarrow$ $|x^1 - r\,x(p_i)| < \varepsilon$.

Kalinin

$|x^2 - (r-1)\,x(p_i)| < \varepsilon$

with matching mitchell numbers.

$|x^3 - (r-2)\,x(p_i)| < \varepsilon$

$\vdots$

$|x^e - x(p_i)| < \varepsilon$.