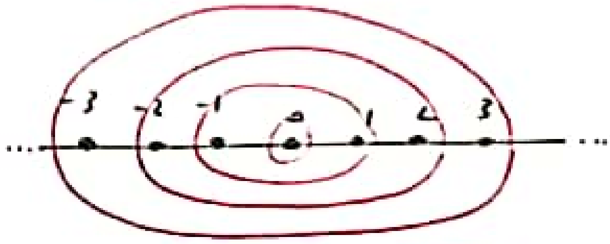
- $(0, \{0\})$. $\ell(g) = 0$

  $\gamma(n) = 1$.

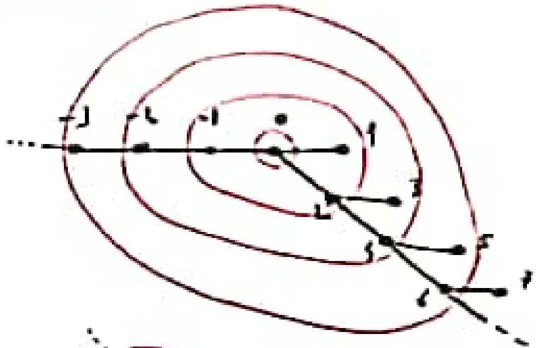
- $(\mathbb{Z}, \{\pm 1\})$ $\ell(g) = |g|$

  $\gamma(n) = 2n+1$.



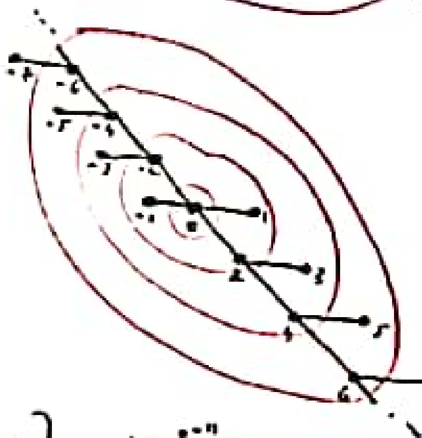
- $(\mathbb{Z}, \{\pm 1, 2\})$ $\ell(g) = \begin{cases} \frac{g}{2}, \text{ if } g \in 2\mathbb{Z}_{\geq 0} \\ |g|, \text{ if } g \in \mathbb{Z}_{<0} \\ \frac{g+1}{2}, \text{ if } g \in 2\mathbb{Z}_{\geq 0}+1 \end{cases}$

  $\gamma(n) = 3n+1$.



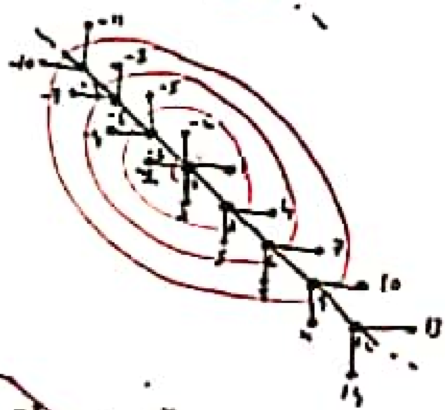
- $(\mathbb{Z}, \{\pm 1, \pm 2\})$ $\ell(g) = \begin{cases} \frac{|g|}{2}, \text{ if } g \in 2\mathbb{Z} \\ \frac{|g|+1}{2}, \text{ if } g \in 2\mathbb{Z}+1 \end{cases}$

  $\gamma(n) = 4n+1$



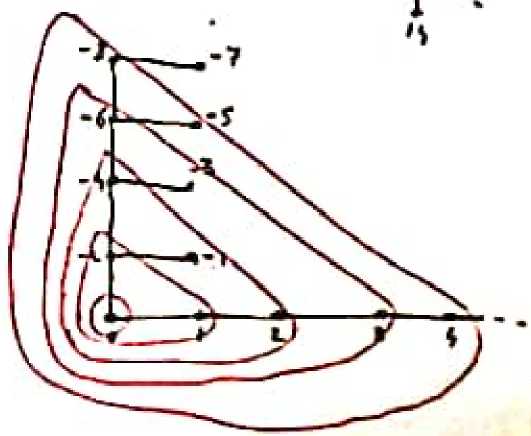
- $(\mathbb{Z}, \{\pm 1, \pm 2, \pm 3\})$ $\ell(g) = \begin{cases} \frac{|g|}{3}, \text{ if } g \in 3\mathbb{Z} \\ \frac{|g|+2}{3}, \text{ if } g \in 3\mathbb{Z}+1 \\ \frac{|g|+1}{3}, \text{ if } g \in 3\mathbb{Z}+2 \end{cases}$

  $\gamma(n) = 6n+1$.

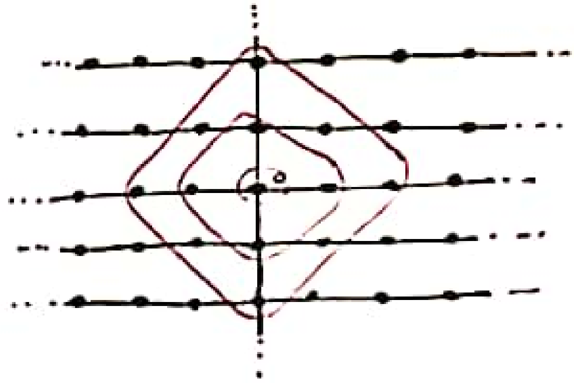


- $(\mathbb{Z}, \{1, -2\})$ $\ell(g) = \begin{cases} g, \text{ if } g \in \mathbb{Z}_{\geq 0} \\ \frac{|g|}{2}, \text{ if } g \in 2\mathbb{Z}_{<0} \\ \frac{|g|+3}{2}, \text{ if } g \in 2\mathbb{Z}_{<0}+1 \end{cases}$

  $\gamma(n) = 2n+1$.



- $\left(\mathbb{Z}^2, \{(\pm 1,0), (0,\pm 1)\}\right)$ $\quad \ell(g_1,g_2) = |g_1|+|g_2|$



$$\gamma_2(n) = 2n^2+2n+1.$$

$$\gamma_2(n) = 2\cdot\left(\sum_{k=0}^{n-1} \gamma_1(k)\right) + \gamma_1(n)$$

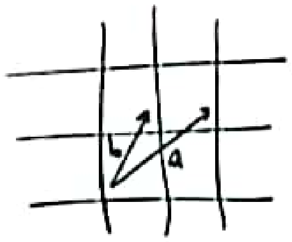
- $\left(\mathbb{Z}^3, \{(\pm 1,0,0), (0,\pm 1,0), (0,0,\pm 1)\}\right)$ $\quad \ell(g_1,g_2,g_3) = |g_1|+|g_2|+|g_3|.$

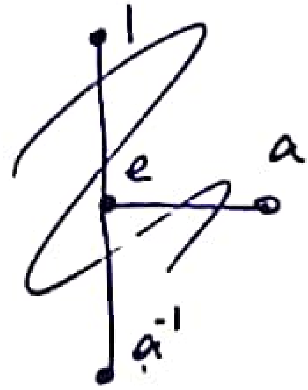
$$\gamma_3(n) = \frac{4}{3}n^3 + 2n^2 + \frac{8}{3}n + 1.$$

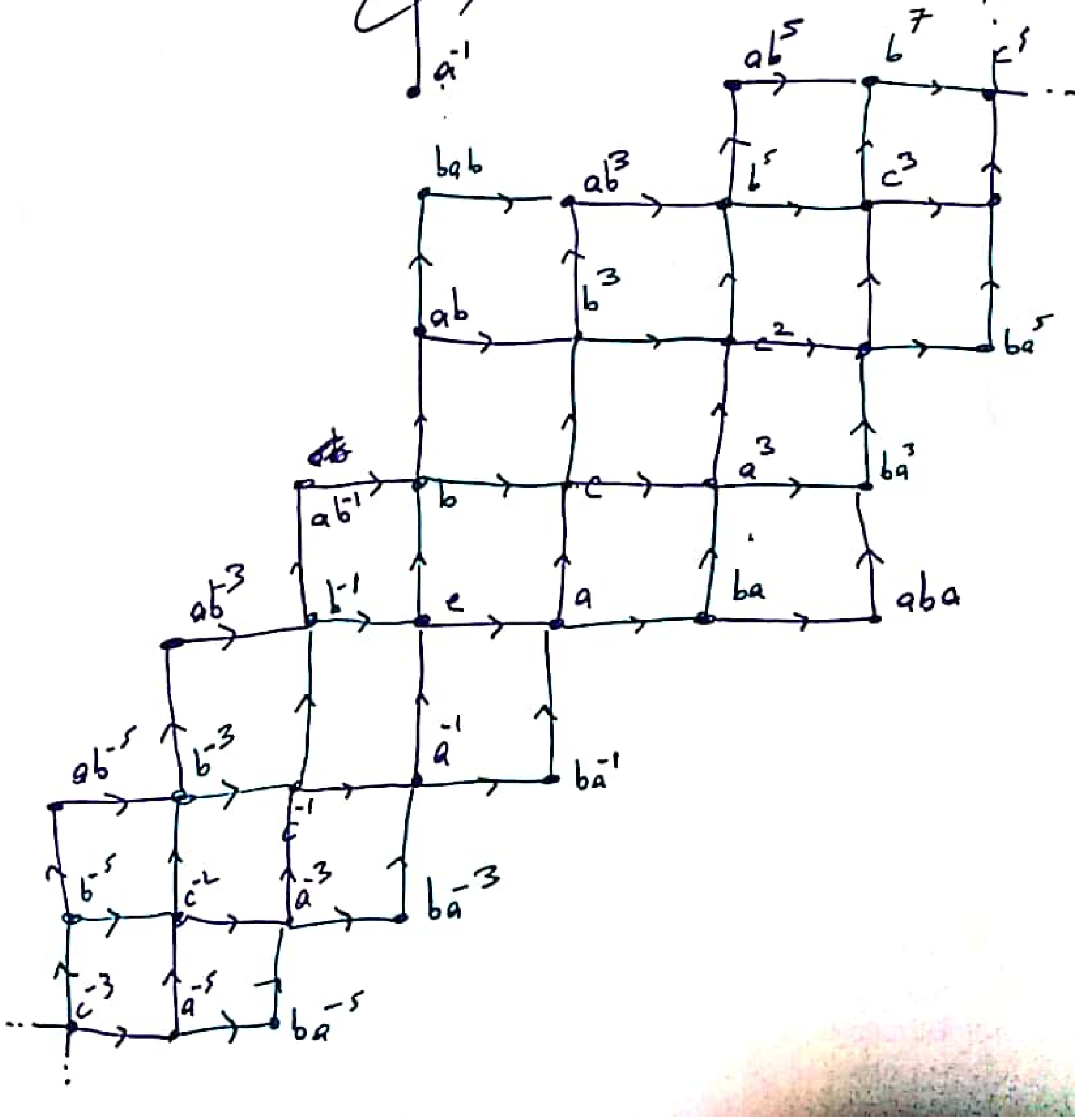
$$\gamma_3(n) = 2\cdot\left(\sum_{k=0}^{n-1} \gamma_2(k)\right) + \gamma_2(n).$$

$$\boxed{\begin{array}{l} S_p(n) := \sum_{k=1}^{n} k^p. \\ S_0(n) = n,\quad S_1(n) = \frac{n(n+1)}{2},\quad S_2(n) = \frac{n(n+1)(2n+1)}{6},\quad S_3(n) = \frac{n^2(n+1)^2}{4}, \\ S_p(n) = \frac{1}{p+1}\left[(n+1)^{p+1} - 1\right] - \frac{1}{p+1}\sum_{q=0}^{p-1}\binom{p+1}{q} S_q(n) \end{array}}$$

$\pi_1 K = \langle a, b \mid a^2 = b^2 = c \rangle.$







Grigorchuk Group:

Schwarz 50's M compact Riemannian manifold

$V(r)$ = volume of ball of radius r in $\tilde{M}$

$V(n) \sim \gamma_{\pi_1 M}(n)$.

Milnor '68: Thm: M: complete Riem. manifold. If mean curvature tensor K_{ij} is everywhere pos. def., then $\forall$ f.g. $H \leq \pi_1 M$:

$$\gamma_H(n) \leq c \cdot n^{\dim M}.$$

Thm: M: compact Riem. with all sectional curvatures < 0

$\Rightarrow \exists a > 1: \; a^n \leq \gamma_{\pi_1(M)}(n)$.

Definitions: $G \in \mathcal{G}$ be f.g.,

$S \subseteq G$ be a fin. gen. set, i.e.

$$G = \bigcup_{n \in \mathbb{Z}_{\geq 0}} (S \cup S^{-1})^n.$$

$$\ell_{G,S} : G \to \mathbb{Z}_{\geq 0} \quad \underline{\text{length function}}$$
$$g \mapsto \min\{n \in \mathbb{Z}_{\geq 0} \mid g \in (S \cup S^{-1})^n\}$$

$$\gamma_{G,S} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$$
$$n \mapsto \operatorname{card}\{g \in G \mid \ell_{G,S}(g) \leq n\}.$$

- $\gamma_{G,S}$ is increasing.
- If G is infinite, then it is strictly increasing.
- $\gamma_{G,S}(m+n) \leq \gamma_{G,S}(m)\,\gamma_{G,S}(n)$
- $\lim\limits_{n\to\infty} \dfrac{\log(\gamma_{G,S}(n))}{n} \in \mathbb{R}$ exists. ($\Rightarrow \gamma_{G,S}(n)$ is at most exponential).

Pf: $k := \left\lfloor \frac{n}{m} \right\rfloor + 1 = \begin{cases} 1, & \text{if } 0 \leq n < m \\ 2, & \text{if } m \leq n < 2m \\ \vdots \\ k, & \text{if } (k-1)m \leq n < km \end{cases}$

$mk = \begin{cases} m, & \text{if } 0 \leq n < m \\ 2m, & \text{if } m \leq n < 2n \\ \vdots & \vdots \\ km, & \text{if } (k-1)m \leq n < km \end{cases} > n.$

$\Rightarrow \gamma_{G,S}(n) \leq \gamma_{G,S}(mk) \leq \gamma_{G,S}(m)^k \leq \gamma_{G,S}(m)^{\frac{n}{m}+1}$

$\Rightarrow \gamma_{G,S}(n)^{1/n} \leq \gamma_{G,S}(m)^{\frac{n+m}{nm}}$

$\Rightarrow \limsup_n \gamma_{G,S}(n)^{1/n} \leq \gamma_{G,S}(m)^{1/m}$

$\Rightarrow \limsup_n \gamma_{G,S}(n)^{1/n} \leq \liminf_m \gamma_{G,S}(m)^{1/m}$, ✓.

$gro(G,S) := \lim_{n\to\infty} \frac{\log(\gamma_{G,S}(n))}{n} \in \mathbb{R}$.

is the <u>growth rate</u> of (G,S)

. Let $\gamma_1, \gamma_2 : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be two functions

$\gamma_1 \preceq \gamma_2$ if $\exists k \in \mathbb{Z}_{\geq 0}, \forall n \in \mathbb{Z}_{\geq 0} : \gamma_1(n) \leq \gamma_2(kn)$

$\gamma_1 \sim \gamma_2$ if $\gamma_1 \preceq \gamma_2 \preceq \gamma_1$.

$\sim$ is an eq rel on $F(\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0})$.

$[\gamma]$ be the eq. class of γ.

γ is <u>exponential</u> if $\gamma(n) \sim e^n \sim 2^n$

<u>polynomial</u> if $\gamma(n) \sim n^d$, $d \in \mathbb{Z}_{\geq 0}$

<u>intermediate</u> if $\gamma(n) \sim e^{n^\alpha}$ $\alpha \in]0,1[$

$\gamma(n) \sim e^{n^{\log\log n}}$

$\gamma(n) \sim e^{n/\log n}$.

⋮

(many more).

Cannot happen with $\gamma_{G,S}$.

$\Big\{$ <u>superexponential</u> if $\gamma(n) \sim n^n$

$\gamma(n) \sim n^{\log\log n}$

⋮

(many more)

$\cancel{\mathrm{a}}$. $\mathrm{gro}(G,S) > 0 \Leftrightarrow$ exponential. $\gamma_{G,S}$.

$\mathrm{gro}(G,S) = 0 \Leftarrow$ polynomial $\gamma_{G,S}$

$\overset{?}{\Rightarrow}$

$\uparrow$

Grigorchuk '84 : No.

$\exists$ PB δ. | Grigorchuk '84 : $\exists$ f.g. G : (3-gen infinite non f.p.)

$$e^{\sqrt{n}} \precsim \gamma_G(n) \precsim e^{n^\beta} \qquad \beta = 0.787\ldots$$

- G : f.g., $H \leq G$: $[G:H] < \infty$
$$\Rightarrow \gamma_H \sim \gamma_G.$$
- H, G : f.g. , $H \leq G \Rightarrow \gamma_H \precsim \gamma_G$.
- $G \twoheadrightarrow H$, G : f.g. $\Rightarrow \gamma_H \precsim \gamma_G$.
- $\gamma_{G,S_1} \sim \gamma_{G,S_2}$
- $[\gamma_{G,S}]$ is an invariant under quasi-isometries (of Cayley graphs of) f.g. groups.

X, Y be metric spaces.

$\varphi: X \to Y$ is a q.i. if

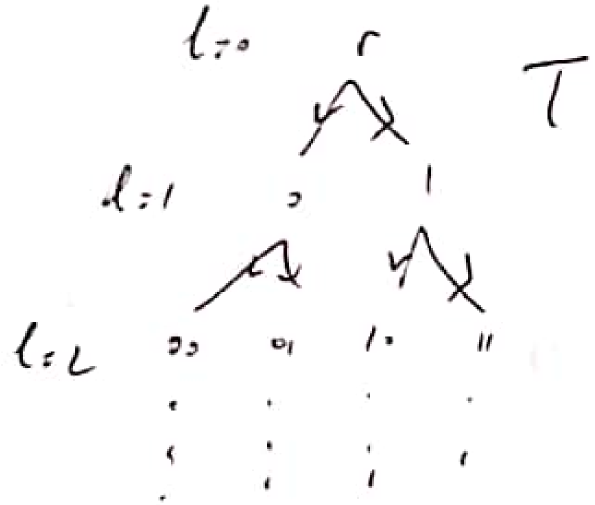
(i) $\exists C \geq 1, D \geq 0$: $\frac{d_X}{C} - D \leq \varphi^*(d_Y) \leq C d_X + D$

(ii) $\exists L > 0$, $\forall y \in Y$ $d_Y(y, \operatorname{im}\varphi) \leq L$.

<u>Thm (Gromov)</u>. G be f.g. $\gamma_{G,S}(n) \sim n^d$ for some d

$\iff$ G is virtually nilpotent.

· $G \le \mathrm{Aut}(T)$, where T is the binary rooted tree.



$$V(T) = \bigcup_{n \in \mathbb{Z}_{\ge 0}} \{0,1\}^n$$

$\cancel{\{(v,v0) \text{ or } (v,v1)\}}$

$$E(T) = \{(v, vx) \mid {v \in V \atop x \in \{0,1\}}\}.$$

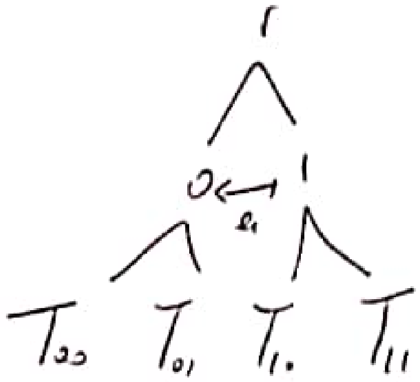
$\tau \in \mathrm{Aut}(T)$ iff it's an auto. of CW complexes.

- If $\tau \in \mathrm{Aut}(T)$, $\tau(r) = r$,

and by induction, $|\tau(v)| = v$.

· $\partial T = F(\mathbb{Z}_{\ge 1}, \{0,1\}) \cong$ Cantor Set $\subseteq I$.

$a_1 = a_r \quad 0x_1x_2\cdots x_n \longleftrightarrow 1x_1x_2\cdots x_n$



or

$$a^2 = I.$$

$\forall v \in V(T).\ a_v : v0x_2x_3\cdots x_n \leftrightarrow v1x_2x_3\cdots x_n.$

$$\mathrm{Aut}(T_v) = \left\{ \tau \in \mathrm{Aut}(T) \;\middle|\; \begin{array}{l} \tau(T_v) = T_v \\ \tau|_{T\setminus T_v} = I \end{array} \right\}$$

$$i_v : T \xrightarrow{\cong} T_v$$

$$\begin{tikzcd} T \arrow[r, "\sigma"] \arrow[d, "i_v"'] & T \arrow[d, "i_v"] \\ T_v \arrow[r, "\sigma_v"] & T_v. \end{tikzcd}$$

$\forall v \in V(T)$: <u>sign at v</u> . $\varepsilon_v : Aut(T) \to \mathbb{Z}/2\mathbb{Z}$

$$\tau \longmapsto \begin{cases} 0, \text{ if } & \tau(v0) = \tau(v)0 \\ & \tau(v1) = \tau(v)1 \\ 1, \text{ if } & \tau(v0) = \tau(v)1 \\ & \tau(v1) = \tau(v)0 \end{cases}.$$

is a group hom

$$\cdot \; F\left(V(T), \mathbb{Z}/2\mathbb{Z}\right) \xleftarrow{\text{bij}} Aut(T)$$
$$[v \mapsto \varepsilon_v(\tau)] \longleftarrow\!\!\shortmid \tau$$

$\Rightarrow$ $Aut(T)$ is uncountable

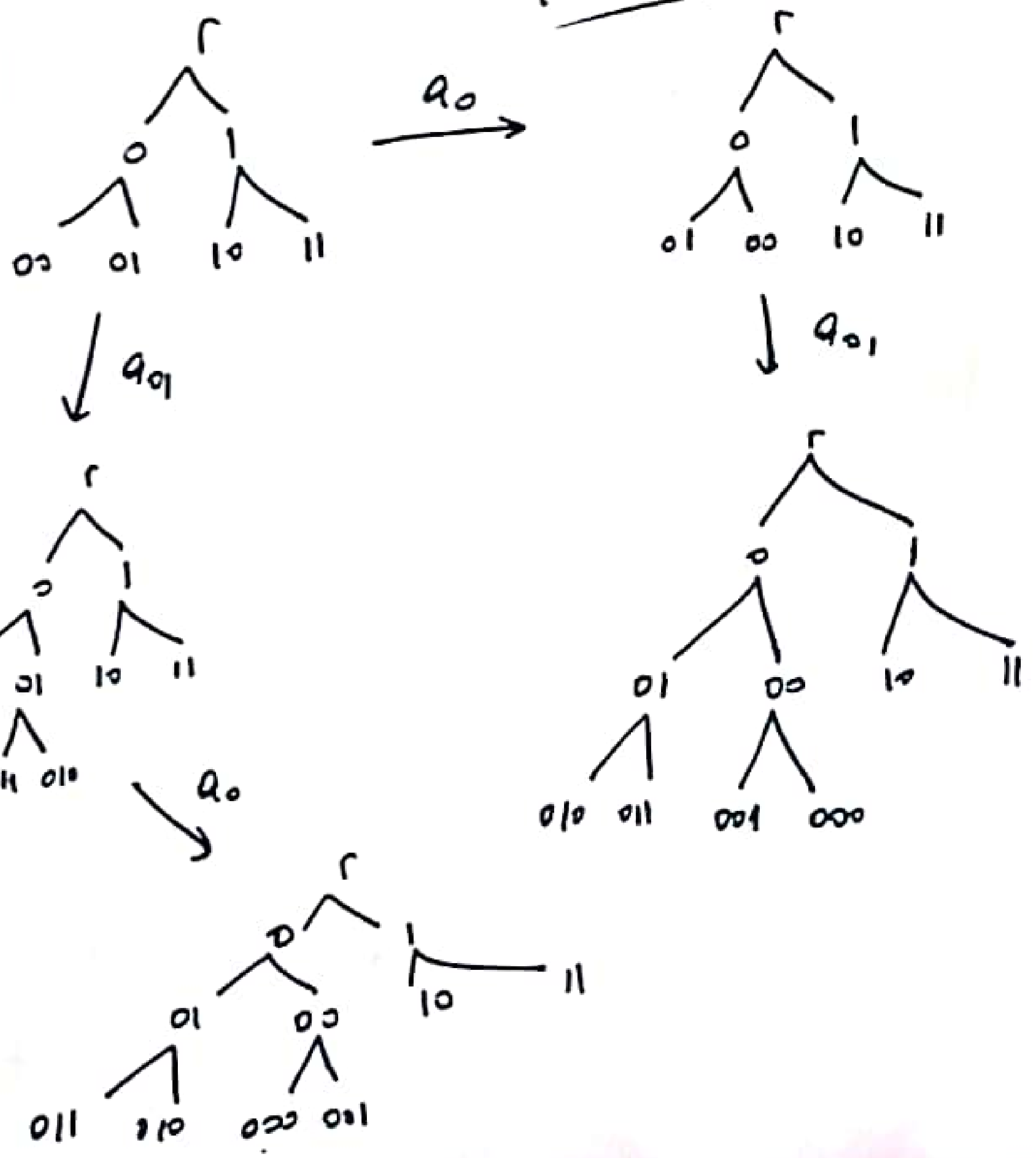
(→ can not be f.g.)

$$\Phi : Aut(T) \times Aut(T) \to Aut(T)$$

$$(\tau, \sigma) \mapsto \varphi(\tau_0) \cdot \varphi(\sigma_1).$$

$$\boxed{a_0 \, a_{01} \stackrel{?}{=} a_{01} \, a_0}$$

$$\boxed{a_0 \, a_{01} \neq a_{01} \, a_0}$$



G : Grp. <u>wreath product</u>:

$G \wr \mathbb{Z}_2$ is defined as ~~the~~ a semidirect product

$$(G \times G) \rtimes \mathbb{Z}_2 \text{ with}$$

$\mathbb{Z}_2$ acting by exchanging two copies of G.

$G \rtimes H$ $\qquad$ $\mathrm{Aff} = GL \ltimes \mathbb{R}^2$

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right).$$

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} B & w \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & Aw+v \\ \hline 0 & 1 \end{array}\right).$$

$$(v, A)\ (w, B) = (Aw+v, AB).$$

Φ $H \rtimes G$. $\qquad H \trianglelefteq H \rtimes G$

$$1 \longrightarrow H \longrightarrow H \rtimes G \longrightarrow G \longrightarrow 1$$

$$h \longmapsto (h, e_G)$$

$$(h, g) \longmapsto g$$

$$(h_1, g_1)(h_2, g_2) = h_1 h_2 h_2^{-1} g_1 h_2 g_2$$
$$= (h_1 h_2, h_2^{-1} g_1 h_2 \cdot g_2).$$

$(v, A)\,(w, B) =$

<u>Semidirect Product</u> : $B \times_\theta H$.

$\theta : H \longrightarrow Aut(B)$ be a morphism of groups.

$\theta(h)(b) = h * b$.

$h * (b_1 b_2) = (h * b_1)(h * b_2)$

$(h_1 h_2) * b = h_1 * (h_2 * b)$

$$(H \ltimes B) \times (H \ltimes B) \longrightarrow H \ltimes B$$

$$\big((h_1, b_1), (h_2, b_2)\big) \mapsto \big(h_1 h_2, b_1 (h_1 * b_2)\big)$$

$$\underbrace{h_1 b_1 \; h_2}\, b_2 = h_1 \underbrace{b_1 h_2 b_1^{-1} b_1} b_2$$

$$H \xrightarrow{\theta} Aut(B)$$

$$(h_1, b_1)(h_2, b_2) = \big(h_1 h_2, b_1 \; \theta(h_1, b_2)\big).$$

$$\begin{tikzcd}
H \times_\theta B \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & B \arrow[d] \\
H \arrow[r, "\theta"'] & Aut(B)
\end{tikzcd}$$

$$\gamma : B \to Aut(B)$$
$$b \mapsto \gamma_b : c \mapsto b c b^{-1}$$

$$\begin{tikzcd}
(h, b) \arrow[r, mapsto] \arrow[d, mapsto] & b \arrow[d, mapsto] \\
h \arrow[r, mapsto] & \theta_h(c) = \gamma_b(c)
\end{tikzcd}$$

$$\mathbb{1} : H \to \mathrm{Aut}(B)$$
$$h \mapsto id_B .$$

$$(h_1, b_1)(h_2, b_2) = (h_1 h_2,\ b_1 \cdot \mathbb{1}(h, b_2))$$
$$= (h_1 h_2,\ b_1 b_2)$$

$$\Rightarrow \boxed{H \rtimes_{\mathbb{1}} B \cong H \times B}$$

//

$H := \mathrm{Aut}(B)$.

$$\theta = id_{\mathrm{Aut}(B)} : \mathrm{Aut}(B) \to \mathrm{Aut}(B)$$

$$(\varphi_1, b_1)(\varphi_2, b_2) = (\varphi_1 \circ \varphi_2,\ b_1 \cdot \theta(\varphi_1, b_2))$$
$$= (\varphi_1 \circ \varphi_2,\ b_1 \cdot \varphi_1(b_2)).$$

$$\boxed{\mathcal{H}ol(B) := \mathrm{Aut}(B) \rtimes_{id_{\mathrm{Aut}(B)}} \mathrm{Aut}(B)}$$

holomorph group of B.

Euclidean Motion Group.

~~Eucl~~ $\mathrm{Euc}(\mathbb{R}^2). = \mathbb{R}^2 \rtimes SO(2).$

$\mathbb{R}^2 \longrightarrow \mathrm{Aut}(SO(2)).$

$v \longmapsto$

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} B & w \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & Aw+v \\ \hline 0 & 1 \end{array}\right).$$

$$(v, A)(w, B) = (Aw+v, AB)$$

$$\theta : \overbrace{SO(2)}^{H} \rightarrow \mathrm{Aut}(\overbrace{\mathbb{R}^2}^{?})$$

$$A \longmapsto [v \longmapsto Av]$$

~~$H \times_\theta B$~~

$H \xrightarrow{\theta} \mathrm{Aut}(B)$

~~$1 \rightarrow B \trianglelefteq H \times_\theta B \rightarrow H \rightarrow 1$~~

$B \times_\theta H.$

$H \xrightarrow{\theta} \mathrm{Aut}(B)$

$1 \rightarrow B \rightarrow B \times_\theta H \rightarrow H \rightarrow 1.$

$(b_1, h_1)(b_2, h_2)$

$= (b_1 \cdot \theta(h_1)(b_2),\ h_1 h_2)$

~~$1 \to \theta$~~

$$1 \rightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times_\theta SO(2) \rightarrow SO(2) \rightarrow 1 \quad B \trianglelefteq B \times_\theta H.$$

$$v \longmapsto (v, 1)$$

$$(v, A) \longmapsto A.$$

$\boxed{G \wr \mathbb{Z}_2.}$ $\qquad \theta : \mathbb{Z}_2 \longrightarrow Aut(G \times G)$

$$1 \longmapsto \left[\tilde{\omega} : (g_1, g_2) \longmapsto (g_2, g_1) \right].$$

$$1 \to \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times_\theta (G \times G) \to G \times G \to 1.$$

$$G \wr \mathbb{Z}_2 = \mathbb{Z}_2 \times_\theta (G \times G) = \mathbb{Z}_2 \ltimes (G \times G).$$

Prop. 3.1: $\cancel{G}\ Aut(T) \cong Aut(T) \wr \mathbb{Z}_2.$

$$\Phi : Aut(T) \times Aut(T) \to Aut(T)$$
$$(\tau, \sigma) \longmapsto \tau_0 \sigma_1$$

$$\hat{\Phi} : [Aut(T) \times Aut(T)] \ltimes \mathbb{Z}_2 \to Aut(T)$$
$$((\tau, \sigma), 1) \longmapsto \tau_0 \sigma_1 \cdot a_r.$$
$$= \Phi(\tau, \sigma) \cdot a_r.$$

$\Phi(\tau, \sigma)$

$$\boxed{Aut(T) \wr \mathbb{Z}_2 \cong Aut(T)}$$

$$\hat{\Phi} : (Aut(T) \times Aut(T)) \rtimes \mathbb{Z}_2 \longrightarrow Aut(T)$$

$$((\tau, \sigma), 1) \longmapsto \left[\begin{array}{l} \tau_0 \sigma_1 \cdot a_r \\ = \Phi(\tau, \sigma) \cdot a_r \end{array}\right]$$

$$\rho : \mathbb{Z}_2 \to Aut(Aut(T) \times Aut(T))$$
$$1 \longmapsto [\ : (\tau, \sigma) \mapsto (\sigma, \tau)]$$

$$(\tau_1, \sigma_1, z_1)(\tau_2, \sigma_2, z_2)$$

$$=$$

$\cancel{Aut}$ $\cancel{\mathbb{Z}_2 \times (Aut(T) \times Aut(T))}$

$$\theta : \mathbb{Z}_2 \to Aut(Aut(T)^2).$$

$$G \wr \mathbb{Z}_2$$

$\cancel{(g_1, g_2, z_1, \ldots)}$

$$((g_1, h_1) z_1)((g_2, h_2), z_2)$$

$$= ((g_1, h_1) \cdot \theta(z_1, (g_2, h_2)), z_1 + z_2)$$

$$= \begin{cases} ((g_1, h_1)(g_2, h_2), z_2), & \text{if } z_1 = 0 \\ ((g_1, h_1)(h_2, g_2), 1 + z_2), & \text{if } z_1 = 1 \end{cases}$$

$$= \begin{cases} ((g_1 g_2, h_1 h_2), z_2), & \text{if } z_1 = 0 \\ ((g_1 h_2, h_1 g_2), 1 + z_2), & \text{if } z_1 = 1 \end{cases}.$$

$\cancel{1 \to Aut(T)^2 \to Aut(T) \wr \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1}$

$$1 \to \mathbb{Z}_2$$

$$H \xrightarrow{\theta} Aut(B)$$

$$1 \to B \to B \times_\theta H \to H \to 1$$

$$(b_1, h_1)(b_2, h_2) = (b_1 \cdot \theta(h_1, b_2), h_1 h_2)$$

$$\mathbb{Z}_2 \to Aut(G \times G)$$

$$1 \to G \times G \to G \wr \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1$$

$$\Rightarrow \hat{\Phi} : Aut(T) \wr \mathbb{Z}_2 \longrightarrow Aut(T)$$

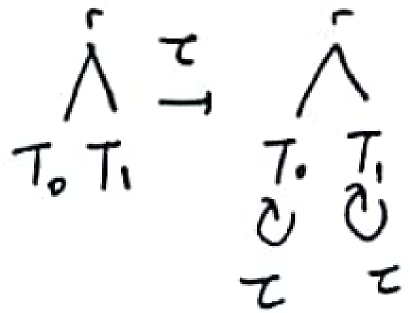
is a group homomorphism.

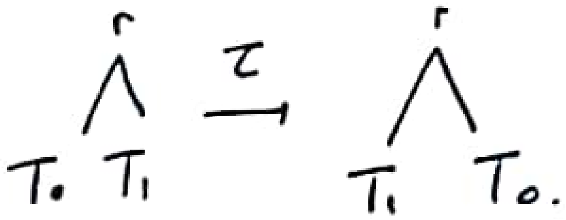
$$(\tau, \sigma, z) \longmapsto \Phi(\tau, \sigma)\, a_r^{z}$$

<u>Inverse of $\hat{\Phi}$?</u>

$\tau \in Aut(T)$.

$\varepsilon_r(\tau) = 0$ or $\varepsilon_r(\tau) = 1$





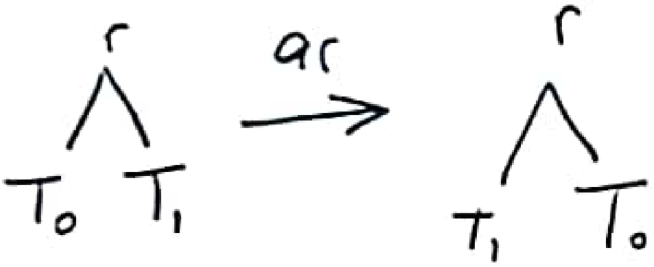
$$\tau = \Phi\left(\tau|_{T_0}, \tau|_{T_1}\right)$$

$$\tau = \Phi\left(\tau|_{T_1}, \tau|_{T_0}\right) a_r$$

$$\Phi\left(i_0^{-1}\circ\tau|_{T_1}\circ i_1,\ i_1^{-1}\circ\tau|_{T_0}\circ i_0\right)\ a_r$$

$$=\left(i_0\circ i_0^{-1}\circ\tau|_{T_1}\circ i_1\circ i_0^{-1}\cdot i_1\ i_1^{-1}\ \tau|_{T_0}\ i_0\ i_1^{-1}\right)\ a_r$$

$$=\ \tau|_{T_1}\circ i_1\circ i_0^{-1}\cdot\tau|_{T_0}\ i_0\ i_1^{-1}\cdot a_r.$$



$$\Rightarrow\ \tau=\begin{cases}\Phi\left(\tau|_{T_0},\ \tau|_{T_1}\right)a_r^0, & \text{if } \varepsilon_r(\tau)=0\\ \Phi\left(i_0^{-1}\circ\tau|_{T_1}\circ i_1,\ i_1^{-1}\circ\tau|_{T_0}\circ i_0\right)a_r, & \text{if } \varepsilon_r(\tau)=1\end{cases}$$

$$\Rightarrow\ \hat{\Phi}:\mathrm{Aut}(T)\wr\mathbb{Z}_2\longrightarrow\mathrm{Aut}(T)$$

is an onto group hom.

$$\forall n \in \mathbb{Z}_{\geq 0} : \cancel{A} \quad A_n := \bigcap_{\substack{v \in V \\ |v| \geq n}} \ker(\varepsilon_v) \; . \; \leq \mathrm{Aut}(T).$$

$$A_0 = \ker(\varepsilon_r) \cap \ker(\varepsilon_0) \cap \ker(\varepsilon_1) \cap \ker(\varepsilon_{00}) \cap \ker(\varepsilon_{01}) \cap \ker(\varepsilon_{10}) \cap \ker(\varepsilon_{11}) \cap \ldots$$

$$= \cancel{\{I\}} \quad \{I\}.$$

$$A_1 = \ker(\varepsilon_0) \cap \ker(\varepsilon_1) \cap \ldots$$

$$= \{I, a_r\} \cong \mathbb{Z}_2 .$$

<u>Exr. 3.2</u>: $$A_n \cong \overbrace{\mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \cdots \wr \mathbb{Z}_2}^{n\text{-many}}$$

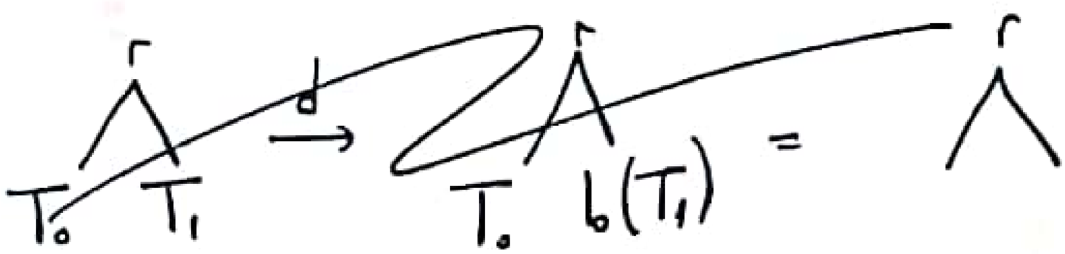
$$= \mathbb{Z}_2^{\wr n}$$

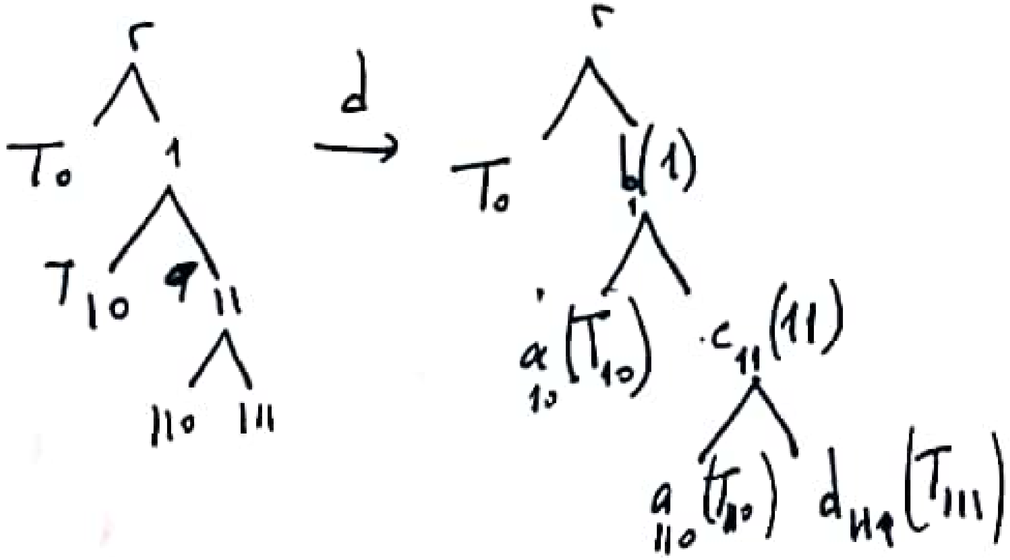
$$= \mathbb{Z}_2^{\oplus n}$$

Implicit def. of $\mathfrak{G}$:

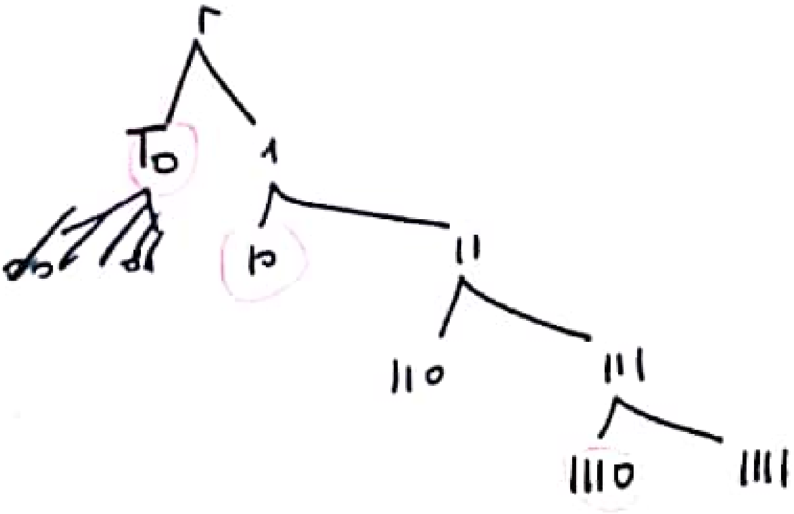
Take $b, c, d \in \mathrm{Aut}(T)$ such that

$$\left.\begin{aligned} b &= \Phi(a, c) \\ c &= \Phi(a, d) \\ d &= \Phi(I, b) \\ \text{where } a &= a_r \end{aligned}\right\} \quad \text{and put} \quad \mathfrak{G} := \langle a, b, c, d \rangle$$





$$b = \left(a_0 \cdot a_{1110} \cdot a_{1111110} \cdots\right)\left(a_{10} \cdot a_{11110} \cdots\right)$$
$$\quad\;\; 1^0 0 \qquad 1^3 0 \qquad\quad 1^6 0 \qquad\qquad\quad 1^1 0 \qquad 1^4 0$$



<u>Main Thm</u> : $G = \langle a, b, c, d \rangle \leq \mathrm{Aut}(T)$

has intermediate growth.

$$a = a_r$$

$$b = \left(a_0 \cdot a_{1^3 0} \cdot a_{1^6 0} \cdots\cdots a_{1^{3n} 0} \cdots\right) \cdot$$
$$\left(a_{10} \cdot a_{1^4 0} \cdot a_{1^7 0} \cdots\; a_{1^{3n+1} 0} \cdots\right)$$

$$c = \left(a_0 \cdot a_{1^3 0} \cdot a_{1^6 0} \cdots\cdots a_{1^{3n} 0} \cdots\right)$$
$$\left(a_{1^2 0} \cdot a_{1^5 0} \cdot a_{1^8 0} \cdots a_{1^{3n+2} 0} \cdots\right)$$

$$d = \left(a_{10} \cdot a_{1^4 0} \cdot a_{1^7 0} \cdots\; a_{1^{3n+1} 0} \cdots\right)$$
$$\left(a_{1^2 0} \cdot a_{1^5 0} \cdot a_{1^8 0} \cdots a_{1^{3n+2} 0} \cdots\right)$$

$n \geq 0$.

$$\mathrm{St}_G(n) = \left\{ \tau \in G \;\middle|\; \begin{array}{l} \forall v \in V : \\ |v| = n \\ \Rightarrow \tau(v) = v \end{array} \right\}$$

$$= \bigcap_{\substack{v \in V \\ |v| = n}} \{ \tau \in G \mid \tau(v) = v \}$$

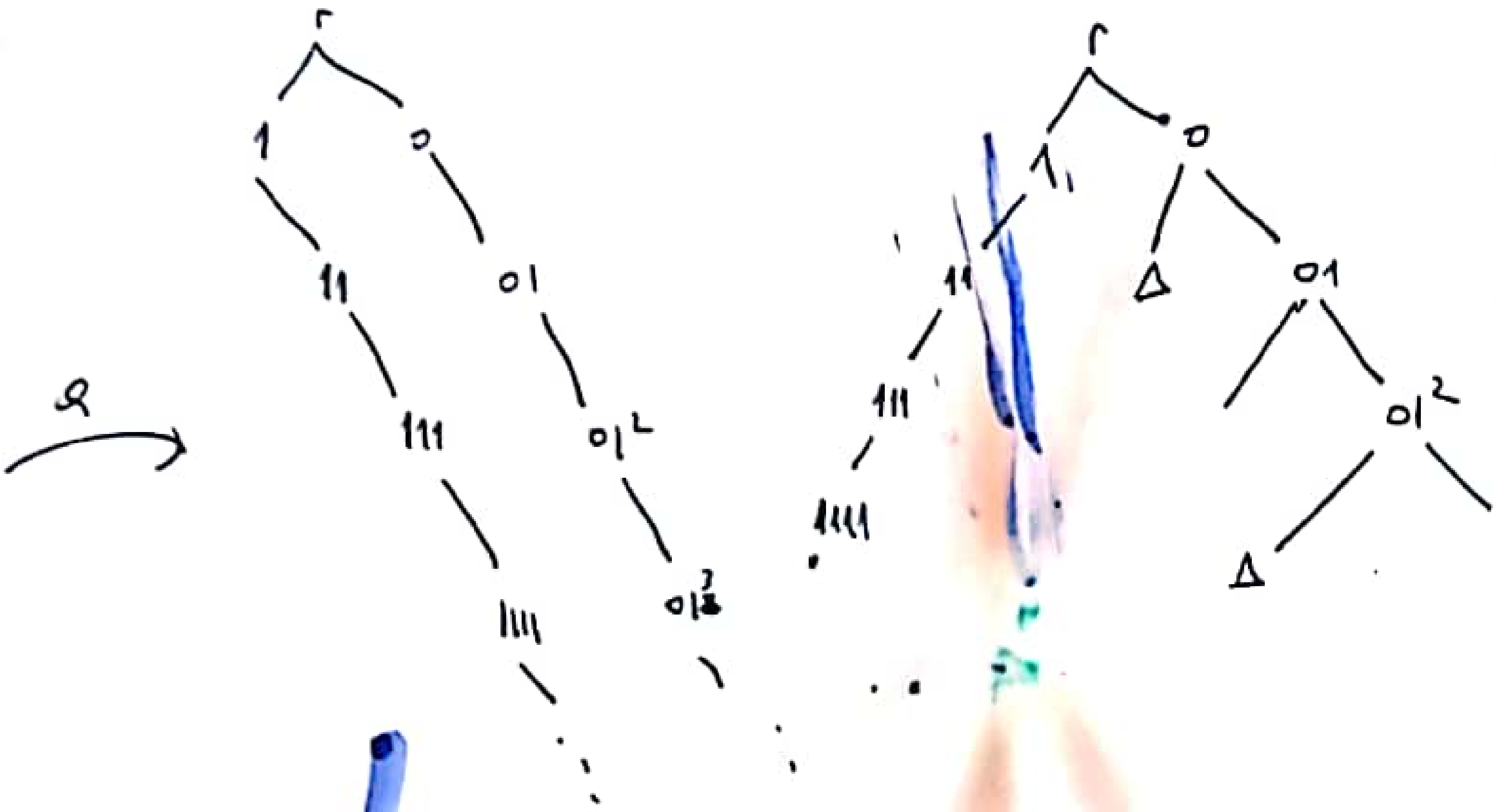
$$= \bigcap_{\substack{v \in V \\ |v| = n}} \mathrm{Stab}_G(v)$$

$H := \mathrm{St}_G(1)$ is the <u>fundamental subgroup</u> ~~of~~ of G

$$H = \mathrm{St}_G(1) = \mathrm{Stab}_G(0) \cap \mathrm{Stab}_G(1).$$

<u>Lem</u>: $H = \langle b, c, d, b^a, c^a, d^a \rangle \trianglelefteq G$,

$$[G : H] = 2.$$



<u>Exr</u>: Rewriting rules

$$\eta : \begin{array}{l} a \longmapsto aba \\ b \longmapsto d \\ c \longmapsto b \\ d \longmapsto c \end{array}$$

a seq. of elements in $\mathcal{B}$.

$x_1 := a, \quad x_{i+1} = \eta(x_i)$.

$a \longmapsto a|b|a \longmapsto aba|d|aba \mapsto abadabacabadaba$

P ~~$\pi\pi$~~ $\quad P[X] \in \mathbb{R}[X]$.

$P : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$

~~P_{∞}~~ $\quad P \sim \left[n \mapsto n^{\deg(P)} \right]$.

~~$P(n)$~~

$P(n) = a_0 + a_1 n + \cdots + a_n n^d$

$f \preceq g$

$\exists k : f(n) \leq g(kn)$

$\limsup_n \frac{f(n)}{g(kn)} \leq 1$.

~~a~~ ~~$(kn)^d$~~

$\leq (a_0 + \cdots + a_n) \, n^d$

$\leq k^d \, n^d = (kn)^d \rightarrow P(n) \leq (kn)^d$.

choose k large ~~eny~~ enough.

~~a_n~~ $\quad n^d \leq (a_n k^d) \, n^d \quad \ldots$

Choose k

large enough

$0 < \alpha < \beta < 1.$

$e^{n^\alpha} \not\propto e^{n^\beta}$

$\partial_x e^{x^\alpha} = e^{x^\alpha} \alpha x^{\alpha-1}$

$\partial_x e^{x^{1/4}} = e^{x^{1/4}} \frac{1}{4x^{3/4}}$

$\partial_x e^{x^{3/4}} = e^{x^{3/4}} \frac{3}{\;}$

$\partial_x e^{x^{1/2}} = \frac{e^{x^{1/2}}}{2x^{1/2}}$ $\quad \partial_x e^{x^{3/4}} = \frac{3e^{x^{3/4}}}{4x^{1/4}}$

$\partial_x \frac{1}{1} \quad \frac{1}{1}$

$e \sim 2.7$

$e^{1^{1/2}} = e^1$

$n^{1/2}$ ✓ $n^{3/4}$

$\int_0 \quad \int_0$

$\frac{1}{1} \frac{1}{n^2} \quad \frac{2}{3} \frac{1}{n^3}$

$\frac{1}{8} = \frac{1}{3}.$

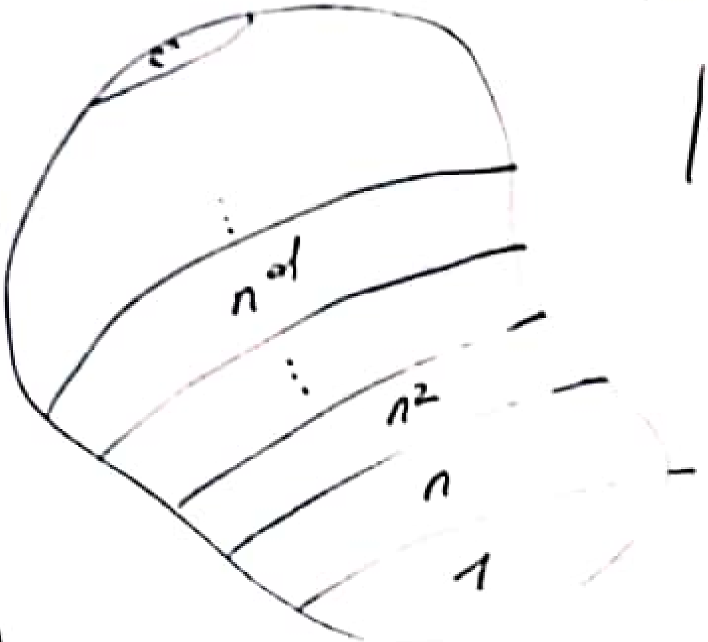
$\frac{3}{1} \quad \frac{1}{16}$

$\frac{3}{2}.$

$\mathcal{X} = \{ [\gamma_G(n)] \mid G \text{ f.g. groups} \}$

$(\mathcal{X}, \preceq)$: partially ordered set.

(not linearly ordered)

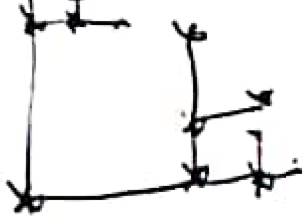


$|\mathcal{X}| = 2^{\aleph}$.

$[\gamma_G]$ is an invariant of q.i.

P1. Are there finitely presented groups of intermediate growth?

Conj: A f.p. group either contains a free subsemigroup on two generators or is virtually nilpotent



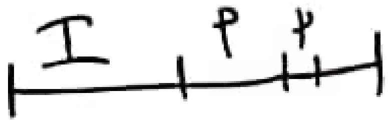
$$\gamma_{G\times H}(n) \sim \gamma_G(n)\cdot\gamma_H(n).$$

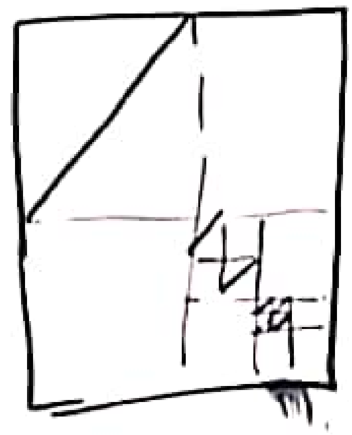
$$\cancel{(g,h)} = \quad \overset{\uparrow}{S} \qquad \overset{\uparrow}{T}$$

$$(S\times e_H)\cup(e_G\times T).$$

$$1\to N \rightarrowtail G \twoheadrightarrow K \to 1$$

$$\gamma_N\,\gamma_K \preceq \gamma_G$$

d: 



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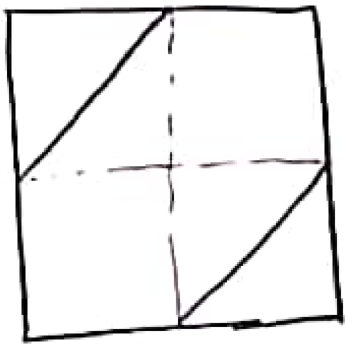

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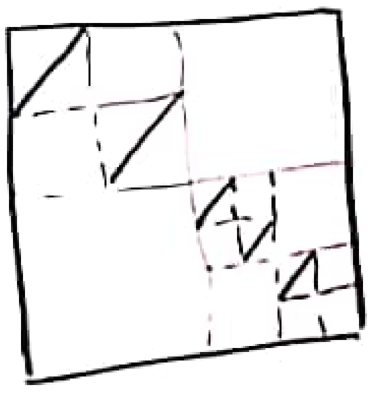

c: