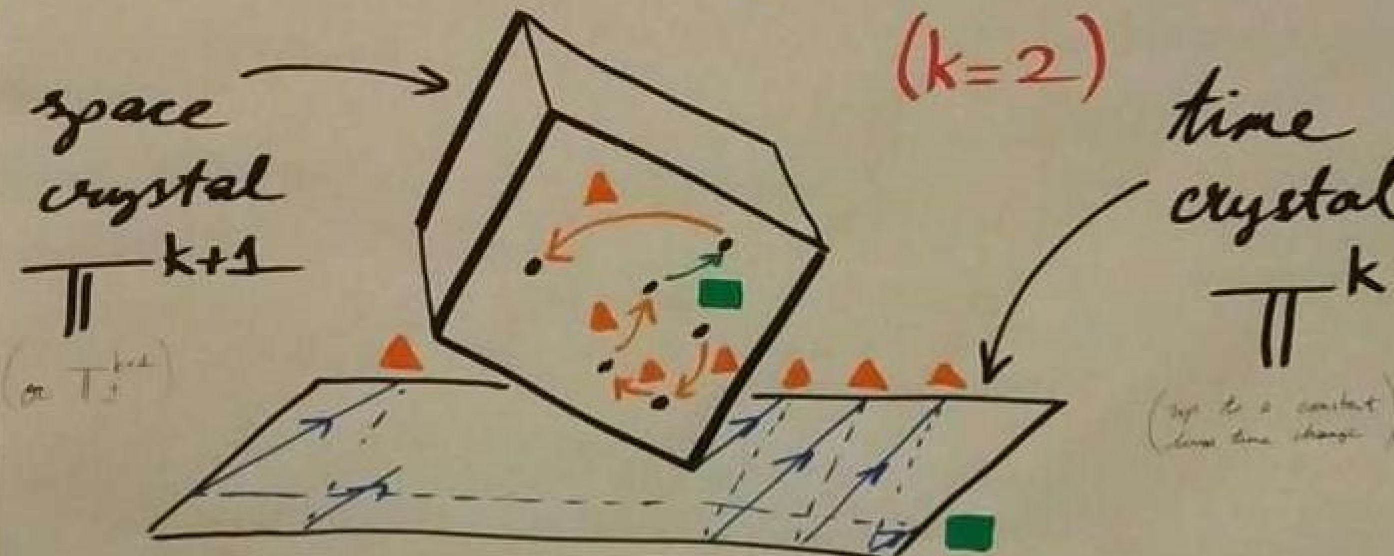
Katok – Rodriguez Hertz Arithmeticity for $\mathbb{R}^k$ - actions

by Alp Uzman, Penn State (azu4@psu.edu)

SCGP 2022 : Flexibility & Rigidity



Thm (Work in Progress): A maximal rank positive entropy action of $\mathbb{R}^k$ $(k \geq 2)$ by C^r $(r>1)$ diffeomorphisms is measure theoretically isomorphic to a $(k+1)$-dimensional space crystal carrying an affine Cartan action of $\mathbb{Z}^k$ and sliding along a k-dimensional time crystal.

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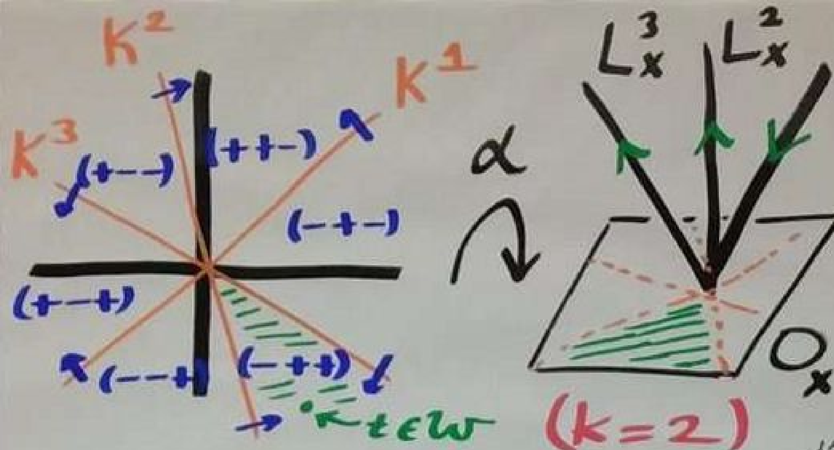


$$\begin{tikzcd}[column sep=small] T_xM \arrow[d] & = & O_x \arrow[d] & \oplus & \underset{S_x(\alpha_t)}{\overset{\|}{L^1_x}} \arrow[d] & \oplus & \underbrace{L^2_x \oplus L^3_x}_{U_x(\alpha_t)} \arrow[d] \\ M & = & \mathcal{O}_x & \uplus & \mathcal{S}_x(\alpha_t) & \uplus & \mathcal{U}_x(\alpha_t) \end{tikzcd}$$

Precise statement

We make a contribution to the study of global measure rigidity. The precise statement is as follows:

Theorem: Let $r \in \mathbb{R}_{>1}$ (regularity), $k \in \mathbb{Z}_{\geq 2}$ (rank), $d = 2k+1$ (dimension), M be a compact C^∞ manifold of dimension d, $\alpha : \mathbb{R}^k \to \mathrm{Diff}^r(M, \mu)$ be a locally free ergodic action preserving a Borel probability measure μ on M. If $\forall t \in \mathbb{R}^k \setminus 0 : \mathrm{ent}_\mu(\alpha_t) > 0$, then $\exists \kappa \in GL(k, \mathbb{R})^0$ and an affine Cartan action $\gamma : \mathbb{Z}^k \to \mathrm{Aff}(\mathbb{T}^{k+1})$ ($\mathbb{T}^{k+1} = \mathbb{T}^{k+1}$ or $\mathbb{T}^{k+1}_{\mathcal{I}} = \mathbb{T}^{k+1}/\mathcal{I}$) such that measure theoretically $(\mu, \alpha) \cong (\mathrm{Leb}_{\mathbb{T}^k} \otimes_\gamma \mathrm{Leb}_{\mathbb{T}^{k+1}}, \tilde{\alpha}^{\gamma,\kappa})$

* Here $\tilde{\alpha}^{\gamma,\kappa} : \mathbb{R}^k \to \mathrm{Diff}^\infty(\mathbb{T}^k \otimes_\gamma \mathbb{T}^{k+1})$ is the κ-time change of the suspension of γ; the "time crystal" is the factor torus $\mathbb{T}^k$ and the "space crystal" is the algebraic fiber $\mathbb{T}^{k+1}$. "Cartan" means: $\forall t \in \mathbb{Z}^k \setminus 0$: the linear part of γ_t is hyperbolic and diagonalizable.

Context & Previous Work

* One can analogously consider $\mathbb{Z}^k$ actions on $k+1$ dimensional manifolds with ent > 0, constant time changes of suspensions of such actions would satisfy the hypotheses of the above theorem. Kalinin-K-RH asked if there were other $\mathbb{R}^k$-actions satisfying the hypotheses (Problem #4 in their 2011 Annals paper); our conclusion is: no.

* The above theorem can be considered as a nonuniformly hyperbolic version of a theorem proved by Matsumoto in 2001; said theorem has the analogous statement for C^∞ actions with sufficiently many codimension 1 uniformly normally hyperbolic elements to the orbit foliation. In particular no measure is involved.

* The most important hypothesis in the above theorem is the positive entropy hypothesis ($\forall t \in \mathbb{R}^k \setminus 0 : \mathrm{ent}_\mu(\alpha_t) > 0$); dropping or weakening it would be a major advancement in the area. (Furstenberg, Rudolph, Katok, Spatzier, Kalinin, RH, ...). In our setting it implies that the Lyapunov hyperplanes provided by the higher rank Oseledets Theorem produce the "✳ picture".

* Our main strategy is to extend the machinery K-RH used in their 2016 arithmeticity paper. We extend the machinery for $\mathbb{Z}^k$-actions to $\mathbb{R}^k$-actions. Their arithmeticity theorem is the $\mathbb{Z}^k$ analog of the above theorem; their time crystal is a finite cyclic group.

Steps:

⓪ The acting group is abelian; the basic machinery for nonuniform hyperbolicity (Oseledets, Pesin, Ledrappier, Young) that works for one element works for the whole action. By the positive entropy assumption we have the ✳ picture. In particular any (rank one) Lyapunov subbundle is integrable; as any such subbundle is the stable bundle of some element of the action.

① By K-K the (dimension one) Lyapunov foliations admit measurable nonstationary linearizations, which endow them with C^r $(\mathrm{Aff}(\mathbb{R}), \mathbb{R})$-structures. The orbit foliation too has a natural affine structure. Assembling the Lyapunov foliations produces "split" C^r affine structures for stable and unstable foliations (for any Weyl chamber $\mathcal{W}$ of (μ, α)). The affine structure of the orbit foliation is compatible with these affine structures.

② By K-K-RH, $\mu \sim \mathrm{Leb}_M$; (orbit-) stable and (orbit-) unstable holonomies are well defined, they are diagonal linear in affine charts.

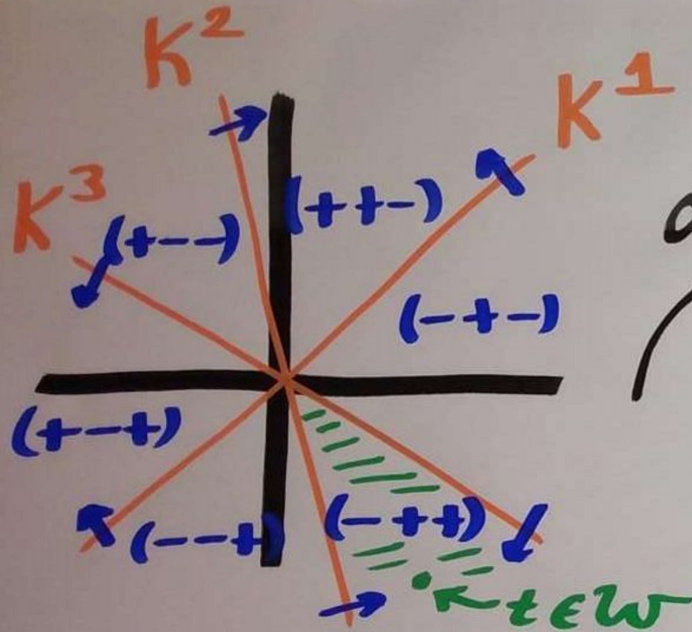
③ Using the dynamical holonomies and affine structures we can define a developing map $\mathrm{dev}^{\mathcal{W}}_x : \mathbb{R}^s \times \mathbb{R}^k \times \mathbb{R}^u \to M$. Unstable saturations of small pieces of orbit-stable foliations cover M μ-a.e., so $\mathrm{dev}^{\mathcal{W}}_x$ pushes $\mathrm{Leb}_{\mathbb{R}^d}$ forward to μ.

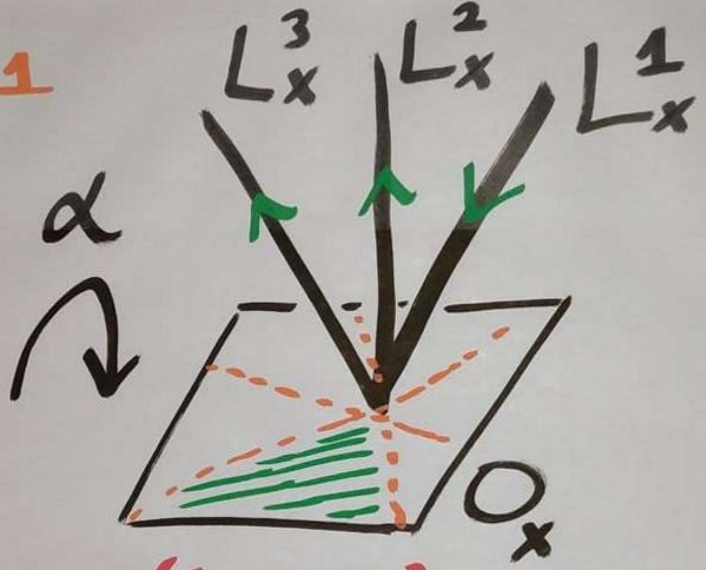
④ By construction the symmetry group $\mathcal{G}^{\mathcal{W}}_x$ of $\mathrm{dev}^{\mathcal{W}}_x$ is a subgroup of $\mathbb{R}^d \rtimes D_d \leq \mathrm{Aff}(\mathbb{R}^d)$ ($D_d = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$); $\mathcal{G}^{\mathcal{W}}_x$ is the transitive group of (μ, α). There is a natural splitting $\mathcal{G}^{\mathcal{W}}_x = \mathcal{G}^{su}_x \cdot \mathcal{G}^k_x$

⑤ We have

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "\mathrm{dev}^{\mathcal{W}}_x"] \arrow[d] & M \\ \mathbb{R}^d / \mathcal{G}^{\mathcal{W}}_x \arrow[ur, "\cong"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbb{R}^{k+1} / \mathcal{G}^{su}_x \arrow[r] & \mathbb{R}^d / \mathcal{G}^{\mathcal{W}}_x \arrow[d] \\ \text{space crystal} \arrow[ur] & \text{time crystal} \arrow[r] & \mathbb{R}^d / \mathbb{R}^{k+1} \times \mathcal{G}^k_x \end{tikzcd}$$

Finally $\mathcal{G}^k_x \cong \mathbb{Z}^k$ and $\mathcal{G}^{su}_x \cong \mathbb{Z}^{k+1}$ or $\mathbb{Z}^{k+1} \times [t]$.





$(k=2)$

$$T_xM = O_x \oplus \underbrace{L^1_x}_{\substack{\shortparallel \\ S_x(\alpha_t)}} \oplus \underbrace{L^2_x \oplus L^3_x}_{U_x(\alpha_t)}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$M = O_x \uplus S_x(\alpha_t) \uplus U_x(\alpha_t)$$

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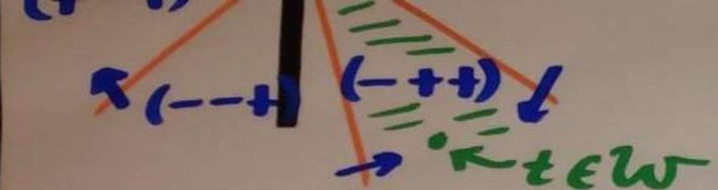

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$$\begin{tikzcd} \text{space crystal} \arrow[r] & \mathbb{R}^{k+1} / \mathcal{G}_x^{su} \arrow[r] & \mathbb{R}^d / \mathcal{G}_x^w \arrow[d] \\ & \text{time crystal} \arrow[r] & \mathbb{R}^d / \mathbb{R}^{k+1} \times \mathcal{G}_x^k \end{tikzcd}$$

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