§ 3.8 : (1)

Ex : (Harmonic Oscillator with Periodic Forcing)

$$\boxed{m\, \partial_t^2 u(t) + \gamma\, \partial_t u(t) + k u(t) = F_0 \cos(\omega t)} \quad (*)$$

where $m, k > 0$ and $\gamma \geq 0$ are as before and

$F_0 > 0$ is the <u>amplitude</u> and
$\omega > 0$ is the <u>frequency</u> of the external force.

$$\boxed{\partial_t U(t) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix} U(t) + \begin{pmatrix} 0 \\ \frac{1}{m} F_0 \cos(\omega t) \end{pmatrix}}$$

Let's suppose that the system is either undamped or very lightly damped (i.e. $\gamma \approx 0$). In particular

$\gamma << 2\sqrt{km}$.

Then according to our earlier discussion

$$\boxed{u_c(t) = e^{-\frac{\gamma}{2m} t} \left(c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \right)}$$

is the complementary solution, where $\omega_0 := \dfrac{\sqrt{-\gamma^2 + 4km}}{2m}$.

Recall that ω_0 is the <u>natural frequency</u> of the system if $\gamma = 0$ and the <u>quasi-frequency</u> if $\gamma > 0$.

To find a solution of the nonhomogeneous equation we need to consider two cases:

① $\omega \neq \omega_0$ ② $\omega = \omega_0$.

① Try $\tilde{u}(t) = A\cos(\omega t) + B\sin(\omega t)$.

$$\partial_t \tilde{u}(t) = (\omega B)\cos(\omega t) + (-\omega A)\sin(\omega t)$$

$$\partial_t^2 \tilde{u}(t) = (-\omega^2 A)\cos(\omega t) + (-\omega^2 B)\sin(\omega t)$$

$$\Rightarrow F_0\cos(\omega t) = m\partial_t^2 \tilde{u}(t) + \gamma \partial_t \tilde{u}(t) + k\,\tilde{u}(t)$$

$$= (-m\omega^2 A + \gamma\omega B + kA)\cos(\omega t) + (-m\omega^2 B - \gamma\omega A + kB)\sin(\omega t) = ((k-m\omega^2)A + (\gamma\omega)B)\cos(\omega t) + ((-\gamma\omega)A + (k-m\omega^2)B)\sin(\omega t)$$

$$\Rightarrow \underbrace{\begin{pmatrix} k-m\omega^2 & \gamma\omega \\ -\gamma\omega & k-m\omega^2 \end{pmatrix}}_{=:M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$\det(M) = (k-m\omega^2)^2 + (\gamma\omega)^2$. If $\gamma = 0$, $\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow \omega \neq \omega_0 = \sqrt{\frac{k}{m}}$

$\Rightarrow (k-m\omega^2) > 0 \Rightarrow \det(M) > 0$.

If $\gamma > 0$, $\gamma\omega > 0 \Rightarrow \det(M) > 0$

$\Rightarrow$ No matter what γ is, $\det(M) \neq 0$, so M is invertible

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\det(M)} \begin{pmatrix} k - m\omega^2 & -\gamma\omega \\ \gamma\omega & k - m\omega^2 \end{pmatrix} \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \frac{F_0}{\det(M)} (k - m\omega^2), \quad B = \frac{F_0}{\det(M)} \gamma\omega.$$

$$\Rightarrow \boxed{\tilde{u}(t) = \frac{F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \left((k - m\omega^2)\cos(\omega t) + (\gamma\omega)\sin(\omega t) \right)}$$

is a particular solution of ⊛ when $\omega \neq \omega_0$.

$\Rightarrow$ When $\omega \neq \omega_0$, the general solution of ⊛ is:

$$\boxed{\begin{aligned} u(t) &= e^{-\frac{\gamma}{2m}t} \left(c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \right) \\ &+ \frac{F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \left((k - m\omega^2)\cos(\omega t) + (\gamma\omega)\sin(\omega t) \right) \end{aligned}}$$

② Try $\tilde{u}(t) = t\left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right)$.

$$\partial_t \tilde{u}(t) = \left(A\cos(\omega_0 t) + B\sin(\omega_0 t)\right) + t\left(B\omega_0\cos(\omega_0 t) - A\omega_0\sin(\omega_0 t)\right)$$

$$= A\cos(\omega_0 t) + B\omega_0\, t\cos(\omega_0 t)$$
$$+ B\sin(\omega_0 t) - A\omega_0\, t\sin(\omega_0 t).$$

$$\partial_t^2 \tilde{u}(t) = -A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t) - B\omega_0^2\, t\sin(\omega_0 t)$$
$$+ B\omega_0\cos(\omega_0 t) - A\omega_0\sin(\omega_0 t) - A\omega_0^2\, t\cos(\omega_0 t)$$

$$= 2B\omega_0\cos(\omega_0 t) + (-A\omega_0^2)\, t\cos(\omega_0 t)$$
$$+ (-2A\omega_0)\sin(\omega_0 t) + (-B\omega_0^2)\, t\sin(\omega_0 t).$$

$$\Rightarrow \begin{aligned} &(m\,2B\omega_0 + \gamma A)\cos(\omega_0 t) \\ &+ (-m A\omega_0^2 + \gamma B\omega_0 + kA)\, t\cos(\omega_0 t) \\ &+ (-m\,2A\omega_0 + \gamma B)\sin(\omega_0 t) \\ &+ (-m B\omega_0^2 - \gamma A\omega_0 + kB)\, t\sin(\omega_0 t) \end{aligned} = \begin{aligned} &F_0 \cos(\omega_0 t) \\ &+ 0\; t\cos(\omega_0 t) \\ &+ 0\; \sin(\omega_0 t) \\ &+ 0\; t\sin(\omega_0 t) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} \gamma A + (2m\omega_0)B &= F_0 \\ (k - m\omega_0^2)A + (\gamma\omega_0)B &= 0 \\ (-2m\omega_0)A + \gamma B &= 0 \\ (-\gamma\omega_0)A + (k - m\omega_0^2)B &= 0 \end{aligned}\right\}$$

this is an overdetermined system (4 equations with 2 unknowns A, B).

If $\gamma = 0$, $\omega_0 = \sqrt{\frac{k}{m}}$ $\rightarrow$ $2\sqrt{km}\,B = F_0$, $-2\sqrt{km}\,A = 0$ $\Rightarrow$ $\boxed{A = 0,\ B = \frac{F_0}{2\sqrt{km}}}$

If $\gamma > 0$, $\omega_0 = \frac{\sqrt{-\gamma^2 + 4km}}{2m}$, $A = \frac{F_0 - 2m\omega_0 B}{\gamma}$

$$\Rightarrow \frac{(k - m\omega_0^2)(F_0 - 2m\omega_0 B)}{\gamma} + (\gamma\omega_0)B = 0$$

$$\Rightarrow \left(\frac{k-m\omega_0^2}{\gamma}\right) F_0 + \left(\gamma\omega_0 - \frac{(k-m\omega_0^2)2m\omega_0}{\gamma}\right) B = 0$$

$$\Rightarrow (k-m\omega_0^2) F_0 + \left(\gamma^2\omega_0 - (k-m\omega_0^2) 2m\omega_0\right) B = 0.$$

$$k-m\omega_0^2 = k - m\left(\frac{-\gamma^2+4km}{4m^2}\right) = k + \frac{\gamma^2}{4m} - k = \frac{\gamma^2}{4m} > 0.$$

$$\Rightarrow 0 < F_0 = \frac{\left((k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0\right) B}{k-m\omega_0^2}$$

Since this is nonzero, none of the factors can be 0.

$$\Rightarrow \left((k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0\right) \neq 0,$$

$$\boxed{B = \frac{F_0 (k-m\omega_0^2)}{(k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0}}$$

$$A = \frac{F_0}{\gamma} - \frac{2m\omega_0}{\gamma} B = \frac{F_0}{\gamma} - \frac{2m\omega_0}{\gamma} \frac{F_0 (k-m\omega_0^2)}{(k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0}$$

$$= \frac{F_0 \left((k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0\right) - 2m\omega_0 F_0 (k-m\omega_0^2)}{\gamma\left((k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0\right)}$$

$$= \frac{-F_0\gamma^2\omega_0}{\gamma\left((k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0\right)} = \frac{F_0 (-\gamma\omega_0)}{(k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0}$$

$$\Rightarrow \boxed{A = \frac{F_0 (-\gamma\omega_0)}{(k-m\omega_0^2) 2m\omega_0 - \gamma^2\omega_0}}$$

Thus

$$\boxed{\tilde{u}(t) = \begin{cases} t\,\dfrac{F_0}{2m\omega_0}\sin(\omega_0 t) & , \text{ if } \gamma = 0 \\ t\,\dfrac{F_0}{(k-m\omega_0^2)\,2m\omega_0 - \gamma^2\omega_0}\Big((-\gamma\omega_0)\cos(\omega_0 t) + (k-m\omega_0^2)\sin(\omega_0 t)\Big), & \text{if } \gamma > 0 \end{cases}}$$

is a particular solution of ⊛ when $\omega = \omega_0$.

$\Rightarrow$ When $\omega = \omega_0$, the general solution of ⊛ is:

$$\boxed{\begin{aligned} u(t) &= e^{\frac{-\gamma}{2m}t}\big(c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)\big) \\ &+ \begin{cases} t\,\dfrac{F_0}{2m\omega_0}\sin(\omega_0 t) & , \text{ if } \gamma = 0 \\ t\,\dfrac{F_0}{(k-m\omega_0^2)\,2m\omega_0 - \gamma^2\omega_0}\Big((-\gamma\omega_0)\cos(\omega_0 t) + (k-m\omega_0^2)\sin(\omega_0 t)\Big) & \text{if } \gamma > 0 \end{cases} \end{aligned}}$$

To summarize, the general solution of ⊛ is:

$$u(t)=\begin{cases} c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+\dfrac{F_0}{k-m\omega^2}\cos(\omega t), & \text{if } \omega\neq\omega_0,\ \gamma=0 \\[2ex] e^{\frac{-\gamma}{2m}t}\left(c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)\right)+\dfrac{F_0}{(k-m\omega^2)^2+(\gamma\omega)^2}\left((k-m\omega^2)\cos(\omega t)+(\gamma\omega)\sin(\omega t)\right), & \text{if } \omega\neq\omega_0,\ \underset{2\sqrt{km}\gg\gamma>0}{} \\[2ex] c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)+t\,\dfrac{F_0}{2m\omega_0}\sin(\omega_0 t), & \text{if } \omega=\omega_0,\ \gamma=0 \\[2ex] e^{\frac{-\gamma}{2m}t}\left(c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)\right)+t\,\dfrac{F_0}{(k-m\omega_0^2)2m\omega_0-\gamma^2\omega_0}\left((-\gamma\omega_0)\cos(\omega_0 t)+(k-m\omega_0^2)\sin(\omega_0 t)\right), & \text{if } \omega=\omega_0,\ \underset{2\sqrt{km}\gg\gamma}{\gamma>0}. \end{cases}$$

- When $\gamma>0$, $e^{\frac{-\gamma}{2m}t}\left(c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)\right)$ is the <u>transient solution</u>,

  $u(t)-e^{\frac{-\gamma}{2m}t}\left(c_1\cos(\omega_0 t)+c_2\sin(\omega_0 t)\right)$ is the steady-state solution.

- When $\omega=\omega_0$ and $\gamma=0$, $u(t)$ is in <u>resonance</u>. ✓

- **Beats:**

Let $\gamma = 0$ and $\omega \neq \omega_0$. Then we know that the general solution to ⊛ is:

$$\boxed{u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{k - m\omega^2} \cos(\omega t)}$$

Let's find c_1 and c_2 for the initial data

$$\boxed{u(0) = 0 = \dot{u}(0).}$$

$$\dot{u} = -\omega_0 c_1 \sin(\omega_0 t) + \omega_0 c_2 \cos(\omega_0 t) + \frac{-\omega F_0}{k - m\omega^2} \sin(\omega t).$$

$$0 = u(0) = c_1 + \frac{F_0}{k - m\omega^2} \Rightarrow \boxed{c_1 = \frac{-F_0}{k - m\omega^2}}$$

$$0 = \dot{u}(0) = \underbrace{\omega_0}_{\neq 0} c_2 \Rightarrow c_2 = 0.$$

$$\Rightarrow u(t) = \frac{F_0}{k - m\omega^2} \Big(\cos(\underbrace{\omega t}_{=A+B}) - \cos(\underbrace{\omega_0 t}_{=A-B}) \Big)$$

$$= \frac{-2F_0}{k - m\omega^2} \underbrace{\sin\left(\frac{1}{2}(\omega + \omega_0) t\right)}_{\text{fast oscillation.}} \underbrace{\sin\left(\frac{1}{2}(\omega - \omega_0) t\right)}_{\substack{\text{slow oscillation} \\ \text{(beat when } \omega \approx \omega_0)}}.$$

$u(t)$ describes the case when the motion is purely due to the external force.

SW: Derive the trig formulas from Euler's.

$$\cos(A+B) - \cos(A-B)$$
$$= \cos A \cos B - \sin A \sin B$$
$$-(\cos A \cos B + \sin A \sin B)$$
$$= -2 \sin A \sin B.$$

SW: Do the same with $\gamma > 0$.

SW: $\lim_{\omega \to \omega_0} u(t) = ?$

(infinite beats)