amen. / (T)	✓	✗
✓	$\mathbb{R}^n/\mathbb{Z}^n$, finite groups	$SL_n(\mathbb{R})$, $n \geq 3$
✗	$\mathbb{Z}^n$, $\mathbb{R}^n$	$SL_2(\mathbb{R})$, F_n.

WM—Ch. 13:

§13.1: Definition and Basic Properties:

Def. (13.1.1): A Lie group H has Kazhdan property (T) if any unitary representation of it that has almost invariant vectors has actually invariant nonzero vectors.

- Let $(\mathcal{H}, \langle \cdot,\cdot \rangle)$ be a Hilbert space. A linear $\varphi: \mathcal{H} \to \mathcal{H}$ is unitary if $\varphi^* \langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle$. $U(\mathcal{H})$ is the group of unitary operators $\varphi: \mathcal{H} \to \mathcal{H}$.

  A unitary representation of H is a Hilbert space $\mathcal{H}$ and a strongly continuous homomorphism $\rho: H \to U(\mathcal{H})$.

$$\left[\begin{array}{l} \rho: H \times \mathcal{H} \to \mathcal{H} \\ \forall v \in \mathcal{H}: \rho(\cdot, v) \text{ is cts.} \end{array}\right]$$

- A group with Kazhdan's property (T) is called a Kazhdan group.

War. (13.1.3):

(i) Kazhdan's property (T) says nothing about actions on topological vector spaces that are not Hilbert spaces. There are actions of Kazhdan groups by norm-preserving linear transformations on some Banach spaces that have almost-invariant vectors but no invariant (nonzero) ones. $\boxed{\text{EX. 13.1.\#1}}$

(ii) But one can replace "2" with any other $p \in [1, \infty[$, and consider representations into groups of L^p-norm preserving operators safely.

Prop. (13.1.4): A Lie group is compact iff it is amenable and Kazhdan.

Cor. (13.1.5): A discrete group is finite iff it is amenable and Kazhdan

Ex. (13.1.6): $\mathbb{Z}^n$ is not Kazhdan.

Prop. (13.1.7): Let Λ be a discrete Kazhdan groups. Then

(i) $\forall N \in \mathcal{P}_{\trianglelefteq}(\Lambda) \cap \tau^{c}(\Lambda)$: Λ/N has Kazhdan's property (τ).

(ii) $\Lambda/[\Lambda,\Lambda]$ is finite.

(iii) Λ is finitely generated.

Cor. (13.1.8): (Nonabelian) free groups are not Kazhdan.

Rem. (13.1.9) (*Generalization of Prop. (13.1.7)*): Let H be an arbitrary Kazhdan groups. Then

(i) $\forall N \in \mathcal{P}_{\trianglelefteq}(\mathbf{H}) \cap \tau^{c}(H)$: $\mathbf{H/N}$ is Kazhdan.

(ii) $H/[H,H]$ is compact.

(iii) H is *compactly generated*, i.e., it has a compact set of generators.

War. (13.1.11):

(i) Discrete Kazhdan groups are finitely generated by Prop. (13.1.7).(iii), but they need not be finitely presented.

(ii) (Gromov) There are uncountably many non-isomorphic discrete Kazhdan groups, countably many of which can be finitely presented.

(iii) (Y. Shalom) Every discrete Kazhdan group is a quotient of a finitely presented Kazhdan group.

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§13.2. Semisimple Kazhdan Groups:

Thm. (13.2.1) (Kazhdan): $SL(3,\mathbb{R})$ is Kazhdan.

Lem (13.2.2): If π is a unitary representation of

$$SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 = \left(\begin{array}{c|c} SL(2,\mathbb{R}) & \mathbb{R}^2 \\ \hline 0 \quad 0 & 1 \end{array}\right) \leq SL(3,\mathbb{R})$$

that has almost-invariant vectors, then it [redacted]as a nonzero $\mathbb{R}^2 \cong \begin{pmatrix} 1 & 0 & \mathbb{R}^2 \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} (\leq SL(3,\mathbb{R}))$ — invariant vector. [i.e., $(SL(2,\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative (T)]

- Let H be a topological group, $R \leq H$. Then (H,R) has the relative property (T) if any unitary representation of H that has almost-invariant vectors has nonzero R-invariant vectors.

Rem: If G is a simple Lie group with $\text{rank}_{\mathbb{R}}(G) \geq 2$, then G contains a subgroup isogenous to $SL(2,\mathbb{R}) \ltimes \mathbb{R}^n$ for some $n \geq 2$. $\boxed{\text{EX. 13.2. \#2}}$

Thus a variant of Thm (13.2.1) shows that G is Kazhdan.

- Let G be a semisimple sub-Lie-group of $SL(n,\mathbb{R})$. A closed connected subgroup $T \leq G$ is a torus if it is diagonalizable over $\mathbb{C}$, ie., $\exists\, g \in GL(n,\mathbb{C})$: gTg^{-1} consists of diagonal matrices. A torus $T \leq G$ is $\mathbb{R}$-split if it is diagonalizable over $\mathbb{R}$, ie., $\exists\, g \in GL(n,\mathbb{R})$: gTg^{-1} consists of diagonal matrices.

Maximal $\mathbb{R}$-split tori in G are conjugate, so the real rank $[\text{rank}_{\mathbb{R}}(G)]$ of G is defined to be the dimension of a maximal $\mathbb{R}$-split torus in G, while the dimension of a subgroup is by definition the $\mathbb{R}$-dimension of its Lie algebra, considered as an $\mathbb{R}$-vector space.

Ex (13.2.3): $SL(2,\mathbb{R})$ is not Kazhdan.

Thm (13.2.4): A semisimple G is Kazhdan iff no simple factor of G is isogenous to $SO(1,n)$ or $SU(1,n)$.

§ 13.4. Lattices in Kazhdan Groups:

Prop. (13.4.1): If G is Kazhdan, then so is Γ.

Cor. (13.4.2): If no simple factor of G is isogenous to $SO(1,n)$ or $SU(1,n)$, then Γ is Kazhdan.

Cor. (13.4.3): If no simple factor of G is isogenous to $SO(1,n)$ or $SU(1,n)$, then

(i) Γ is finitely [redacted] generated,

(ii) $\Gamma/[\Gamma,\Gamma]$ is finite.

Rem. (13.4.4):

(i) Any lattice in any semisimple Lie group [redacted] with finitely many connected components is finitely presented by Thm (4.7.10), whence Cor. (13.4.3). (i) is redundant.

(ii) $SO(1,n)$ and $SU(1,n)$ can have abelian quotients.

(iii) Every lattice in $SO(1,3)$ has a finite-index subgroups with an infinite abelian quotient.

(iv) Also relevant (in tandem with Margulis' Normal Subgroup theorem):

$\boxed{\text{EX. 16.1.\#3}}$, $\boxed{\text{EX 17.1.\#1}}$

• Let H be a Lie group, $\pi_1 : H \curvearrowright \mathcal{H}_1$, $\pi_2 : H \curvearrowright \mathcal{H}_2$ be unitary representations.

→ $\pi_1 \leq \pi_2$ ("π_1 is a <u>subrepresentation</u> of π_2") if

$\exists$ closed π_2-invariant linear subspace $\mathcal{H}_2' \leq \mathcal{H}_2$, and

$\exists$ a isometric linear isomorphism $T : \mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2'$ with

$$\begin{tikzcd}
H \times \mathcal{H}_1 \arrow[r, "\pi_1"] \arrow[d, hook, "id_H \times T"'] & \mathcal{H}_1 \arrow[d, hook, "T"] \\
H \times \mathcal{H}_2 \arrow[r, "\pi_2"] & \mathcal{H}_2
\end{tikzcd}$$

→ $\pi_1 \overset{*}{\leq} \pi_2$ ("π_1 is <u>weakly contained</u> in π_2") if

$\forall \varepsilon > 0, \forall K \in \mathcal{K}(\mathcal{H}_1), \forall \varphi_1, \varphi_1' \in \mathcal{H}_1, \exists \varphi_2, \varphi_2' \in \mathcal{H}_2, \forall k \in K:$

[redacted] $$\left| \langle \pi_1(k, \varphi_1), \varphi_1' \rangle - \langle \pi_2(k, \varphi_2), \varphi_2' \rangle \right| < \varepsilon.$$

<u>Rem. (13.4.6)</u>:

(i) $\pi_1 \leq \pi_2 \Rightarrow \pi_1 \overset{*}{\leq} \pi_2$. (ii) $\pi : H \curvearrowright \mathcal{H}$ has invariant vectors $\Leftrightarrow \mathbb{1} \leq \pi$.

(iii) $\pi : H \curvearrowright \mathcal{H}$ has almost-invariant vectors $\Leftrightarrow \mathbb{1} \overset{*}{\leq} \pi$,

where $\mathbb{1} : H \times \mathcal{H} \to \mathcal{H}$, $(h, \varphi) \mapsto \varphi$ is the <u>trivial representation</u>.

(iv) <u>Kazhdan's property (T)</u> is the converse of Rem. (13.4.6) $\mathbb{1} \leq \pi \Rightarrow \mathbb{1} \overset{*}{\leq} \pi$:

$$\boxed{(T) \Leftrightarrow \mathbb{1} \leq \pi \Leftarrow \mathbb{1} \overset{*}{\leq} \pi}$$

• Let G be a semisimple Lie group with finitely many connected components, $\Gamma \leq G$ be a lattice, $\pi : \Gamma \curvearrowright \mathcal{H}$ be unitary.

→ A measurable $\varphi : G \to \mathcal{H}$ is <u>essentially right Γ-equivariant</u> if $\forall \gamma \in \Gamma, \forall g \in_{x} G : \varphi(g\gamma^{-1}) = \pi(\gamma, \varphi(g))$

→ $L^0_\Gamma(G, \mathcal{H}) := \{\varphi : G \to \mathcal{H} \mid \varphi \text{ is measurable \& ess. right } \Gamma\text{-equiv}\}/_{x}$

→ $\forall \varphi \in L^0_\Gamma(G, \mathcal{H}), \forall \gamma \in \Gamma, \forall g \in_{x} G : \|\varphi(g\gamma)\|_{\mathcal{H}} = \|\varphi(g)\|_{\mathcal{H}}$.

$\boxed{\text{EX. 11.3.\#2}}$

Thus we can define

$$\|\cdot\|_2 : L^0_\Gamma(G, \mathcal{H}) \to [0, \infty]$$
$$\varphi \longmapsto \left(\int_{G/\Gamma} \|\varphi(g)\|^2_{\mathcal{H}} \, dg \right)^{1/2}$$

$L^2_\Gamma(G, \mathcal{H}) := \{\varphi \in L^0_\Gamma(G, \mathcal{H}) \mid \|\varphi\|_2 < \infty\}$.

$L^2_\Gamma(G, \mathcal{H})$ is a Hilbert space. $\boxed{\text{EX. 11.3.\#3}}$

→ G acts on $L^2_\Gamma(G, \mathcal{H})$ by unitary operators:

$$\mathrm{Ind}^G_\Gamma(\pi) : G \times L^2_\Gamma(G, \mathcal{H}) \longrightarrow L^2_\Gamma(G, \mathcal{H})$$
$$(g, \varphi) \longmapsto \boxed{x \mapsto \varphi(g^{-1}x)}$$

$\mathrm{Ind}^G_\Gamma(\pi)$ is the <u>induced representation</u> of G from π.

<u>Lem. (13.4.7)</u>: $\pi_1 \overset{*}{\leq} \pi_2 \Rightarrow \mathrm{Ind}_\Gamma^G(\pi_1) \overset{*}{\leq} \mathrm{Ind}_\Gamma^G(\pi_2)$

(here $\Gamma \leq G \overset{\pi_1}{\curvearrowright} \mathcal{H}_1$, $\Gamma \leq G \overset{\pi_2}{\curvearrowright} \mathcal{H}_2$).

<u>Rem. (13.4.8)</u>: If Γ is Kazhdan and $S \subseteq \Gamma$ is a generating set (which is finite, since Γ is finitely presented), then $\exists\, \varepsilon > 0$, $\forall$ unitary $\pi : \Gamma \curvearrowright \mathcal{H}$: if π has (ε, S)-invariant unit vectors then it has invariant vectors $\boxed{\text{EX. 13.1.\#11}}$.

In many cases, including when $\Gamma = SL(n, \mathbb{Z})$, an explicit value of ε can be obtained from the algebraic structure of Γ.

<u>Rem. (13.4.9)</u>:

(i) If the relations (specifying a group) are selected randomly then the resulting group has Kazhdan's property (T).

(ii) $SL(n, \mathbb{Z}[X_1, X_2, \dots, X_k])$ is Kazhdan for $n \geq k+3$.

<u>Rem. (13.4.10)</u>:

By Prop. (12.7.22), amenability is preserved under <u>quasi-isometries</u>. However Kazhdan's property (T) is <u>not</u> preserved under quasi-isometries.

- Let $(X_1, d_1), (X_2, d_2)$ be metric spaces. A function $f: X_1 \to X_2$ is a <u>quasi-isometry</u> if $\exists\, C > 0$:

  (i) $\forall x_1, y_1 \in X_1 : d_1(x_1, y_1) > C \Rightarrow \frac{1}{C} < \frac{d_2(f(x_1), f(y_1))}{d_1(x_1, y_1)} < C$

  (ii) $\forall x_2 \in X_2,\ \exists x_1 \in X_1 : \ d_2(f(x_1), x_2) < C$.

- A property preserved under quasi-isometries is <u>geometric</u>.

<u>Ex (to Rem.(13.4.10))</u>: Let G be Kazhdan, $p: \tilde{G} \to G$ be its universal covering, $\Gamma \le G$ be a cocompact lattice, $\tilde{\Gamma} := p^{-1}(\Gamma) \le \tilde{G}$ (which is again a lattice). Then $\tilde{\Gamma}$ is Kazhdan: However if $\pi_1(G) \cong \mathbb{Z}$ (e.g. if $G = Sp(4, \mathbb{R})$) then $\tilde{\Gamma} \overset{QI}{\sim} \Gamma \times \mathbb{Z}$, and $\Gamma \times \mathbb{Z}$ is not Kazhdan.

<u>§13.5. Fixed Points in Hilbert Spaces:</u>

<u>Def. (13.5.1)</u>: Let $\mathcal{H}$ be a Hilbert space. A bijection $T: \mathcal{H} \to \mathcal{H}$ is an <u>affine isometry</u> of $\mathcal{H}$ if $\exists\, U \in \mathcal{U}(\mathcal{H}),\ \exists\, v \in \mathcal{H}$: $T = U + v$.

Group of affine isometries: $\begin{pmatrix} \mathcal{U}(\mathcal{H}) & \mathcal{H} \\ 0 & 1 \end{pmatrix} = \mathcal{U}(\mathcal{H}) \ltimes \mathcal{H}$
$\begin{pmatrix} U & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U^{-1} & U^{-1}v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} id_{\mathcal{H}} & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} U & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} id_{\mathcal{H}} & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U^{-1} & U^{-1}v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} id_{\mathcal{H}} & Ux+2v \\ 0 & 1 \end{pmatrix}$

<u>Exc. (13.5.2)</u>: Let $v \in \mathcal{H} \setminus 0$, $\pi_v: \mathbb{R} \times \mathcal{H} \to \mathcal{H}$, $(t, w) \mapsto w + tv$. Then π_v is an action [redacted] by affine isometries with no fixed points.

Def. (13.5.3): Let H be a Lie group, $\pi: H \curvearrowright \mathcal{H}$ be a unitary representation. Define

$$Z^1(H,\pi) := Z^1(H,\mathcal{H}) := \left\{ f \in C^0(H,\mathcal{H}) \,\middle|\, \begin{array}{l} \forall h_1,h_2 \in H: \\ f(h_1 h_2) = f(h_1) + \pi(h_1, f(h_2)) \end{array} \right\}$$

$$B^1(H,\pi) := B^1(H,\mathcal{H}) := \left\{ f \in C^0(H,\mathcal{H}) \,\middle|\, \begin{array}{l} \exists v \in \mathcal{H}, \forall h \in H: \\ f(h) = v - \pi(h,v) \end{array} \right\}$$

$$H^1(H,\pi) := H^1(H,\mathcal{H}) := Z^1(H,\pi) \big/ B^1(H,\pi).$$

$\boxed{\text{EX. 13.5. \# 3}}$

Thm. (13.5.4): Let H be a Lie group. Then TFAE:

(i) H is Kazhdan.

(ii) $\forall$ Hilbert $\mathcal{H}$, $\forall$ continuous $\alpha: H \to \mathcal{U}(\mathcal{H}) \ltimes \mathcal{H}$:
α has a fixed point.

(continuous action by affine isometries)

(iii) $H^1(H,\pi) = 0$, $\forall$ unitary $\pi: H \curvearrowright \mathcal{H}$.
($\forall$ Hilbert $\mathcal{H}$).

property (FH)

Def. (13.5.6): Let H be a Lie group, $\pi: H \curvearrowright \mathcal{H}$ be a unitary representation.

$$\overline{H}^1(H,\pi) := \overline{H}^1(H,\mathcal{H}) := Z^1(H,\pi) \big/ \overline{B^1(H,\pi)}$$ is the reduced first cohomology.

__Thm. (13.5.7)__: Let H be a compactly generated Lie group. Then H is Kazhdan iff $\overline{H}^1(H,\pi)=0$ for every unitary $\pi: H \curvearrowright \mathcal{H}$.

__Cor. (13.5.8)__: Let H be a compactly generated Lie group. Then H is Kazhdan iff $\overline{H}^1(H,\pi)=0$ for every irreducible unitary $\pi: H \curvearrowright \mathcal{H}$.

- Let H be a Lie group. A unitary representation $\pi: H \curvearrowright \mathcal{H}$ is __irreducible__ if the only closed π-invariant linear [redacted] subspaces of $\mathcal{H}$ are 0 and $\mathcal{H}$.

__§13.6. Functions of Positive Type__:

Let our Hilbert space be over $\mathbb{R}$.

__Def. (13.6.2)__:

(i) A symmetric $A \in Mat(n,\mathbb{R})$ is of __positive type__ (or __positive-definite__) if $\forall v \in \mathbb{R}^n: \langle Av, v\rangle \geqslant 0$.

($\Leftrightarrow \sigma(A) \subseteq [0,\infty[$ $\boxed{\text{EX. 13.6. \#1}}$) (or __positive semi-definite__)

(ii) A symmetric $A \in Mat(n,\mathbb{R})$ is of __positive type conditionally__ if $\forall v \in \mathbb{R}^n: \sum_{k=1}^{n} v_k = 1 \Rightarrow \langle Av, v\rangle \geqslant 0$; and all diagonal entries of A are 0.

(iii) Let H be a topological group. $\varphi \in C^0(H, \mathbb{R})$ is of <u>positive type [conditionally]</u> if
(or <u>positive [semi-] definite</u>)

$\forall n \geq 1$, $\forall \{h_k\}_{k=1}^{n} \subseteq H$: $\varphi(h_i^{-1} h_j) \in Mat(n, \mathbb{R})$ is symmetric and of positive type [conditionally].

<u>Lem. (13.6.5)</u>: Let H be a topological group, $\alpha: H \curvearrowright \mathcal{H}$ be continuous and by affine isometries, $v \in \mathcal{H}$. Then

(i) [redacted] $\begin{aligned} \psi_v : H &\longrightarrow \mathbb{R} \\ h &\longmapsto -\|\alpha(h,v) - v\|^2 \end{aligned}$ is positive-semi-definite.

(ii) $\begin{aligned} \varphi_v : H &\longrightarrow \mathbb{R} \\ h &\longmapsto \langle \alpha(h,v), v \rangle \end{aligned}$ is positive-definite, if $\forall h \in H: \alpha(h,0) = 0$.

<u>Prop. (13.6.6) (GNS construction)</u> (Gelfand, Naimark, Segal): Let H be a topological group. If $\varphi \in C^0(H, \mathbb{R})$ is positive-definite, then

$\exists\, \alpha: H \curvearrowright \mathcal{H}$ continuous and by affine isometries, $\exists v \in \mathcal{H}$:

$\varphi = \varphi_v$.

<u>Lem (13.6.7) (Schoenberg's Lemma)</u>: If $\varphi \in C^0(H, \mathbb{R})$ is positive-semi-definite, then $\exp \circ \varphi \in C^0(H, \mathbb{R})$ is positive-definite.

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§13.1:

Def (13.1.1). A Lie group H is Kazhdan if $\forall$ unitary $\pi : H \curvearrowright \mathcal{H}$: if π has almost inv.-vectors, then it has nonzero inv. vectors.

Warn (13.1.3) (i) This says nothing about actions on Banach spaces.

$\boxed{\text{EX. 13.1.\#1}}$ $C_0^0(H,\mathbb{R}) := \{ f \in C^0(H,\mathbb{R}) \mid \lim_{\cancel{x}\to\infty} |f(x)| = 0 \}$

with $\|f\|_{\sup} = \sup_{x\in H} |f(x)|$.

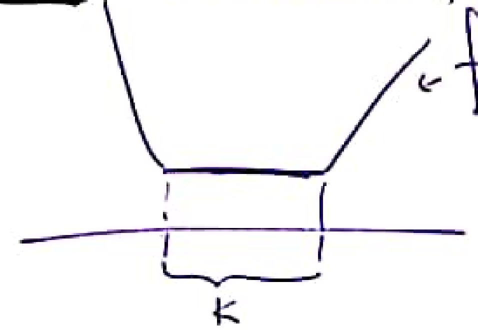
$\alpha : H \times H \to H$

$\pi : H \times C_0^0(H,\mathbb{R}) \to C_0^0(H,\mathbb{R})$

$(h, f) \longmapsto \alpha(h^{-1}, \cdot)^* f$.

Then$^{(i)}$ π has [redacted] alm. inv. vectors.

[redacted] Fix $K \subset \mathcal{K}(H)$, $f^K \in C^0(H,\mathbb{R})$.



($\lim_{\cancel{x}\to\infty} |f(x)| = 0$ means (?):

$\forall \{K_n\}_n \subseteq \mathcal{K}(H)$: $K_n \uparrow H$: and K_n^c are not precompact

$\|f \chi_{K_n^c}\|_{\sup} \longrightarrow 0$.)

(Or : Let $\hat{H} := H \cup \{\infty\}$ be the one-point compactification of H)

$\forall n:\quad f_n^K : x \longmapsto \dfrac{n}{n + f^K(x)}$ [redacted] $\rightarrow \{f_n^K\}_n \subseteq C_0^0(H,\mathbb{R})$

$\forall x \in H,\ \forall k \in K$

$$\left| f_n^K(k^{-1}x) - f_n^K(x) \right| = \frac{n\,\left|f^K(x) - f^K(k^{-1}x)\right|}{n^2 + n\left|f^K(x) + f^K(k^{-1}x)\right| + \left|f^K(x)\, f^K(k^{-1}x)\right|}$$ bdd.

$\Rightarrow \forall \varepsilon > 0,\ \forall K \in \mathcal{K}(H),\ \exists\, f_n^K \in C_0^0(H,\mathbb{R}) \setminus 0:$

$$\sup_{k\in K}\ \sup_{x \in X} \left| f_n^K(k^{-1}x) - f_n^K(x) \right| < \varepsilon.$$

(ii) If H is noncompact, $\forall h \in H:\ |f(h^{-1}x) - f(x)| = 0$

$\Rightarrow$ f is constant $\Rightarrow$ $\lim f = 0$ $\Rightarrow$ $f = 0$.

$\Rightarrow$ no nonzero alm. inv. vectors.

(iii) If H is compact, it has nonzero inv. vectors. (namely, constant functions for instance)

(ii) But if H is Kazhdan [redacted] then

$\forall p \in [1, \infty[, \ \forall \pi_p : H \longrightarrow \mathrm{End}(L^p(H, \mathbb{C}))$:
$h \longmapsto [f \mapsto [x \mapsto f(h^{-1}x)]]$

if π_p has almost inv. vectors then it has nonzero inv. vectors.

<u>Prop. (13.1.4)</u> : H is compact iff it is amenable & Kazhdan.

<u>Alternatively</u> :
<u>Cor. 7.1.9 (Zimmer p.132)</u>.
If G is amenable, then G is Kazhdan iff G is compact.

<u>Pf</u> : ($\Rightarrow$) H be compact.
Then By Prop (12.2.4)
H is amenable.

. Let $\pi : H \times \mathcal{H} \to \mathcal{H}$ be a unitary rep. with alm. inv. vectors:

$\forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists v \in \mathcal{H} : \|v\| = 1, \ \pi(H, v) \subseteq B_\varepsilon(v)$.

<u>Obs</u>: $\forall v \in \mathcal{H} : \|v\| = 1 \Rightarrow \bar{v} := \beta(f_{v*}\mu) \in \mathcal{H}$,

where μ is Haar on H, $f_v : H \longrightarrow \mathcal{H}$
$\qquad h \longmapsto \pi(h,v)$,

$$\beta(f_{v*}\mu) = \int_{\mathcal{H}} w \, d f_{v*}\mu(w) = \int_H \pi(h,v)\, d\mu(h)$$

is the barycenter of $f_{v*}\mu$. $\bar{v}$ is H-inv.

$\varepsilon := 1/2$, $K := H \in \mathcal{K}(H)$. $v \in \mathcal{H} : \|v\| = 1, \pi(H,v) \subseteq B_{1/2}(v) \not\ni 0$

$\Rightarrow \bar{v} \in B_{1/2}(v)$, it is H-inv. & $\bar{v} \neq 0$. ✓

($\Leftarrow$) If H is amenable, then by $\boxed{\text{AMES}}$,

$\pi_{reg} : H \times L^2(H, \mathbb{R}) \longrightarrow L^2(H, \mathbb{R})$ unitary rep.

has alm. inv. vec. Then by Kaz., π_{reg} has nonzero inv. vectors.

Let $f \in L^2(H,\mathbb{R})\setminus 0$ be π_{reg}-inv.:

$\Rightarrow \forall h \in H, \forall x \in_{ae} H: \quad f(h^{-1}x) = f(x)$

$\Rightarrow f$ is const. ae. $\Rightarrow \exists c \in \mathbb{R}\setminus 0: f =_{ae} c \neq 0$

$\Rightarrow \infty > \int_H |f|^2 = \mu(H)\, c^2 \Rightarrow \mu(H) < \infty$.

$\overset{(*)}{\Leftrightarrow}$ H is compact., ✓.

(*) Folland, Prop. 11.4.d pp. 341-342:

H is compact $\Leftrightarrow \mu(H) < \infty$

Pf: ($\Rightarrow$) Since Haars are Radon, so finite for compacta.

($\Leftarrow$) Suppose H is not compact. Let $K \in \mathcal{N}_{cl}(e)$.

Then $\forall \{h_k\}_{k=1}^n \subseteq H: H \setminus \left(\bigcup_{k=1}^n h_k K\right) \neq \emptyset$.

$\Rightarrow \exists \{h_n\}_n \subseteq H, \forall n: h_n \in H \setminus \left(\bigcup_{k=1}^{n-1} h_k K\right)$.

Let $L \in \mathcal{N}_{symm}(e): L^2 \subseteq K$.

If $\exists m > n: h_m L \cap h_n L \neq \emptyset$

$h_m^{-1}x, h_n^{-1}x \in L \Rightarrow$ [redacted] $h_n^{-1}h_m = h_n^{-1}x x^{-1} h_m$

$= (h_n^{-1}x)(h_m^{-1}x)^{-1} \in LL^{-1} = L^2 \subseteq K \Rightarrow h_m \in h_n K$, ↯.

$\Rightarrow h_m L \cap h_n L = \emptyset, \forall m,n$.

$\Rightarrow 0 < \underbrace{\mu\left(\biguplus_n h_n L\right)}_{= \sum_n \mu(h_n L) = \sum_n \mu(L)} \leq \mu(H) \Rightarrow \mu(H) = \infty$, ✓.

Cor. (13.1.5): Λ be discrete. Λ is finite $\Leftrightarrow$ amenable & Kazhdan.

Ex. (13.1.6): $\mathbb{Z}^n$ is not Kazhdan.

Prop. (13.1.7): Λ be discrete. If Λ is Kazhdan, then

(i) $\forall\, N \in \mathcal{P}_{\triangleleft}(\Lambda) \cap \tau^c(\Lambda)$: Λ/N is Kazhdan.

(ii) $\Lambda/[\Lambda,\Lambda]$ is finite.

(iii) Λ is finitely generated.

Pf: (i) Prop. (7.1.6) (Zimmer, p. 131): If $\varphi: G \to H$ is a continuous homomorphism with dense image, then if G is Kazhdan, then so is H.

Pf.: Let $\pi : H \times \mathcal{H} \to \mathcal{H}$ be unitary with alm. inv. vects. $\Rightarrow$ $\varphi^{*}\pi : G \times \mathcal{H} \longrightarrow \mathcal{H}$ is unitary with alm. inv. vects.

$$(g, v) \mapsto \pi(\varphi(g), v)$$

G Kaz. $\Rightarrow \exists\, v \in \mathcal{H} : \underset{\displaystyle \pi(\operatorname{im}\varphi, v)}{\underset{\|}{\varphi^{*}\pi(G, v)}} = \{v\}$

$\overline{\operatorname{im}\varphi} = H \Rightarrow \pi(H, v) = \{v\} \Rightarrow H$ Kaz. ✓

con: $H \to H/N$ is surjective, ✓

For this one we did not need that Λ is discrete.

(ii) By Cor. (12.2.3) ($\underline{\text{Kakutani–Markov FPT}}$)

$\Lambda/[\Lambda, \Lambda]$ is amenable (since it is abelian).

By (i) $\Lambda/[\Lambda, \Lambda]$ is Kazhdan.

By Prop. (13.1.4), (or Cor. (13.1.5)) $\Lambda/[\Lambda, \Lambda]$ finite (compact). ✓

For this one again we did not need that Λ is discrete.

(iii) $\{\Lambda_n\}_n \subseteq \mathcal{P}_{\leq}(\Lambda)$ be all finitely generated subgroups of Λ

$$\alpha_n : \Lambda \times \Lambda/\Lambda_n \longrightarrow \Lambda/\Lambda_n \quad (\lambda, x\Lambda_n) \mapsto \lambda x \Lambda_n$$

$$\forall n : \pi_n : \Lambda \times L^2(\Lambda/\Lambda_n, \mathbb{R}) \longrightarrow L^2(\Lambda/\Lambda_n, \mathbb{R})$$

$$(\lambda, \text{[redacted]} f) \mapsto \alpha_n(\lambda^{-1}, \cdot)^* f$$

$\forall n$: π_n is unitary.

$$\langle \pi_n(\lambda, f), \pi_n(\lambda, g) \rangle = \int_{\Lambda/\Lambda_n} f(\lambda^{-1} x \Lambda_n)\, g(\lambda^{-1} x \Lambda_n)\, d(x\Lambda_n)$$

$$= \langle f, g \rangle$$ because α is transitive.

$$\mathcal{H} := \bigoplus_{n \geq 1} L^2(\Lambda/\Lambda_n, \mathbb{R})$$

$$\langle \{f_n\}_n, \{g_n\}_n \rangle = \sum_{n \geq 1} \langle f_n, g_n \rangle_{L^2(\Lambda/\Lambda_n, \mathbb{R})}$$

is a Hilbert space.

$$\Pi := \bigoplus_{n \geq 1} \Pi_n : \Lambda \times \mathcal{H} \longrightarrow \mathcal{H} \quad \text{is unitary.}$$

$$(\lambda, \{f_n\}_n) \longmapsto \{\Pi_n(\lambda\Lambda_n, f_n)\}_n.$$

<u>Claim</u>: Π has alm. inv. vects.

$\varepsilon > 0$, $K \in \mathcal{K}(\Lambda) \Rightarrow K$ is finite $\Rightarrow \exists \underset{\underset{m(K)}{\parallel}}{m}: K \subseteq \Lambda_m$.

$$\Rightarrow \Pi_m\left(K/\Lambda_m, \chi_{\{e\Lambda_m\}}\right) = \chi_{\{e\Lambda_m\}}$$

$\Rightarrow \forall K \in \mathcal{K}(\Lambda), \forall \varepsilon > 0$: $\quad \|\chi_{\{e\Lambda_m\}}\| = \nu_m(\{e\Lambda_m\}) = \mu(\Lambda_m) > 0.$

$\{\chi_{\{e\Lambda_m\}}\, \delta_{nm}\}_n \in \mathcal{H}$ is an almost-invariant vector of Π.

Λ is Kazhdan $\Rightarrow \Pi : \Lambda \times \mathcal{H} \to \mathcal{H}$ has a nonzero inv. vec., $\{f_n\}_n \in \mathcal{H} \setminus 0$.

$\Rightarrow \exists n$: $f_n \in L^2(\Lambda/\Lambda_n, \mathbb{R})^0$ is Π_n-inv.

$\Rightarrow \exists\, c_n \in \mathbb{R}\setminus 0 : \; f_n =_{\nu_n} c_n ,$

$\nu_n = \mathrm{can}_{n*} \mu$

Haar on μ.

$$\infty > \int_{\Lambda/\Lambda_n} |f_n|^2 \, d\nu_n = \nu_n\left(\Lambda/\Lambda_n\right) c_n^2$$

$$\Rightarrow \nu_n\left(\Lambda/\Lambda_n\right) < \infty .$$

$\Rightarrow$ By [Folland Prop 11.4.d, pp. 341–342]

Λ/Λ_n is compact $\Rightarrow$ finite

Λ_n is finitely generated $\Rightarrow \Lambda$ is finitely generated, ✓.

---.

From Kazhdan's paper (last sentence):

If $\Gamma \leq G$ is a lattice, $U \leq G$ is the maximal compact subgroup, and G is a simple real group with $\mathrm{rank}_{\mathbb{R}}(G) \underset{(\geq)}{>} 2$, then

- $\pi_1(U \backslash G / \Gamma)$ is finitely generated and
- $H_1(U \backslash G/\Gamma, \mathbb{Z})$ is finite.

Cor. (13.1.8): (Nonabelian) free groups are not Kazhdan.

Pf: $F_n / [F_n, F_n] \underset{(*)}{\cong} \mathbb{Z}^n$ is infinite, ✓

(*) Thm 27.2 (Armstrong, p. 168):

$F_n = *\langle g_1, \dots, g_n \rangle \Rightarrow F_n/[F_n,F_n]$ is generated by

$\{g_1[F_n,F_n], \dots, g_n[F_n,F_n]\}$:

$\Rightarrow \forall\, x[F_n,F_n],\ \exists (r_1,\dots,r_n)_{(x)} \in \mathbb{Z}^n : x[F_n,F_n] = \prod_{k=1}^{n} g_k^{r_k}[F_n,F_n]$

$=$ [redacted] $\left(\prod_{k=1}^{n} g_k^{r_k}\right)[F_n,F_n]. \rightsquigarrow x \mapsto (r_1,\dots,r_n) \quad F_n/[F_n,F_n] \cong \mathbb{Z}^n$.

Rem. (13.1.9): Prop. (13.1.7) holds [redacted] for any group, but "finite" should be [redacted] replaced by "compact" (Prop (13.1.7) (i), (ii) have been dealt with already.)

For (iii) "If H is any Lie group, then $H/[H,H]$ is compactly generated.", $\boxed{\text{EX. 13.1.\#15}}$

$\boxed{\text{EX. 13.1.\#14}}$ A Lie group H is compactly generated $\Leftrightarrow$ H/H° is finitely generated.

Pf: $(\Rightarrow)$ If $K \in \mathcal{K}(H)$ is a generating set, $\{kH^\circ \mid k \in K\} = \mathrm{can}(K)$ is a compact generating set for H/H°, which is discrete, so $\mathrm{can}(K)$ is finite, ✓.

$(\Leftarrow)$ H° is connected $\Leftrightarrow$ [redacted] $\forall K \in \mathcal{N}_{\mathcal{K}(H)}(e): H^\circ = \bigcup_{h \in H^\circ} hK$

If $S = \{s_1H^\circ, \dots, s_nH^\circ\} \subseteq H/H^\circ$ is a generating set for H/H°, $s_1K \cup \dots \cup s_nK \in \mathcal{K}(H)$ is a generating set for H, ✓.

$\boxed{\text{EX. 13.1 \#15}}$ Let H be Kazhdan. Then by Prop (7.1.6) (of Zimmer), H/H° is Kazhdan, and is discrete. Then By Prop. (13.1.7.) iii, H/H° is finitely generated ⇒ By $\boxed{\text{Ex 13.1 \#14}}$ H is compactly generated. ✓.

$\underline{\S\, 13.2}$:

$\boxed{\text{Def.}(11.3.1)}$

$$\boxed{\begin{array}{l} G \text{ be semisimple with } |G/G^\circ| < \infty \\ \Gamma \leq G \text{ be a lattice.} \end{array}}$$

Let $\pi : \Gamma \curvearrowright \mathcal{H}$ be unitary.

. $\varphi \in L^\circ(G, \mathcal{H})$ is <u>right ess. Γ-equivariant</u> if for $R : \Gamma \times G \to G$
$(\gamma, g) \mapsto g\gamma^{-1}$

$$\begin{tikzcd} \Gamma \times G \arrow[r, "R"] \arrow[d, "id_\Gamma \times \varphi"'] & G \arrow[d, "\varphi"] \\ \Gamma \times \mathcal{H} \arrow[r, "\pi"] & \mathcal{H} \end{tikzcd}$$

$\left(\text{, i.e. } \forall \gamma \in \Gamma, \forall g \in_x G : \varphi(g\gamma^{-1}) = \pi(\gamma, \varphi(g))\right)$.

$L^\circ_\Gamma(G, \mathcal{H})$ denotes the set of all such maps, (modulo x).

$\boxed{\text{EX. 11.3 \#2}}$ $\forall\, \varphi, \psi \in \mathcal{L}^0_\Gamma(G, \mathcal{H}),\ \forall \gamma \in \Gamma, \forall g \in_{\mathfrak{x}} G,$

$$\langle \varphi(g\gamma), \psi(g\gamma) \rangle_{\mathcal{H}} = \langle \varphi(g), \psi(g) \rangle_{\mathcal{H}}.$$

Pf: $\langle \varphi(g\gamma), \psi(g\gamma) \rangle_{\mathcal{H}} = \langle \pi(\gamma^{-1}\varphi(g)), \pi(\gamma^{-1}\psi(g)) \rangle_{\mathcal{H}}$

$= \langle \varphi(g), \psi(g) \rangle_{\mathcal{H}}$, ✓.

$$\Rightarrow F: \mathcal{L}^0_\Gamma(G, \mathcal{H})^2 \longrightarrow \mathcal{L}^0(G/\Gamma \text{ [redacted] } [0,\infty])$$
$$(\varphi, \psi) \longmapsto [g\Gamma \mapsto \langle \varphi(g), \psi(g) \rangle_{\mathcal{H}}]$$

is well defined.

$$\Rightarrow \langle \cdot, \cdot \rangle_\Gamma : \mathcal{L}^0_\Gamma(G, \mathcal{H}) \times \mathcal{L}^0_\Gamma(G, \mathcal{H}) \longrightarrow \text{[redacted] } [0,\infty]$$
$$(\varphi, \psi) \longmapsto \int_{G/\Gamma} F(\varphi,\psi)(g\Gamma)\, d(g\Gamma)$$
$$= \int_{G/\Gamma} \langle \varphi(g), \psi(g) \rangle_{\mathcal{H}}\, d(g\Gamma)$$

is well-def.,

[redacted]

$\|\varphi\|_\Gamma := (\langle \varphi, \varphi \rangle_\Gamma)^{1/2}$

$$\Rightarrow \mathcal{L}^2_\Gamma(G,\mathcal{H}) := \{\varphi \in \mathcal{L}^0_\Gamma(G,\mathcal{H}) \mid \|\varphi\|_\Gamma < \infty\}$$

is a Hilbert space with $\langle\cdot,\cdot\rangle_\Gamma$. $\boxed{\text{EX 11.3.\#3}}$

$$\mathrm{Ind}_\Gamma^G(\pi) : G \times \mathcal{L}^2_\Gamma(G,\mathcal{H}) \longrightarrow \mathcal{L}^2_\Gamma(G,\mathcal{H})$$
$$(g\ ,\ \varphi) \longmapsto [x \mapsto \varphi(g^{-1}x)]$$

is a unitary rep., called the <u>representation of G induced from π</u>.

(Def. (11.3.1) ends)

$\boxed{\text{EX. 11. 3. \# 5}}$ Consider the <u>trivial rep.</u> $\mathbb{1}_\Gamma : \Gamma \times \mathbb{C} \to \mathbb{C}$, $(\gamma, z) \mapsto z$. Then $\mathrm{Ind}_\Gamma^G(\mathbb{1}_\Gamma) \cong \tilde{L}$, where [redacted]

$$\tilde{L} : G \times L^2(G/\Gamma, \mathbb{C}) \longrightarrow L^2(G/\Gamma, \mathbb{C})$$
$$(g,\ \varphi) \longmapsto [x\Gamma \mapsto \varphi(g^{-1}x\Gamma)].$$

Pf:
$$\Phi : \mathcal{L}^2_\Gamma(G,\mathbb{C}) \to L^2(G/\Gamma, \mathbb{C})$$
$$f \longmapsto [x\Gamma \mapsto f(x)]$$

gives an isomorphism:

$$\begin{array}{ccc} G \times \mathcal{L}^2_\Gamma(G,\mathbb{C}) & \xrightarrow{\mathrm{Ind}_\Gamma^G(\mathbb{1}_\Gamma)} & \mathcal{L}^2_\Gamma(G,\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{id}_G \times \Phi} & & \downarrow{\scriptstyle \Phi} \\ G \times L^2(G/\Gamma,\mathbb{C}) & \xrightarrow{\tilde{L}} & L^2(G/\Gamma,\mathbb{C}) \end{array}, \checkmark.$$

$\mathbb{1}_\Gamma : \Gamma \times \mathbb{C} \to \mathbb{C}$

$\mathrm{Ind}_\Gamma^G(\mathbb{1}_\Gamma) . G \times \mathcal{L}^2_\Gamma(G,\mathbb{C}) \to \mathcal{L}^2_\Gamma(G,\mathbb{C})$

$(g, f) \longmapsto [x \mapsto f(g^{-1}x)]$

$\forall \gamma \in \Gamma, \forall g \in_{a.e.} G : f(g\gamma^{-1}) = f(g)$

Not.(13.4.5): Let H be a Lie group, $\pi_1 : H \curvearrowright \mathcal{H}_1$, $\pi_2 : H \curvearrowright \mathcal{H}_2$ be unitary:

- π_1 is a subrepresentation of π_2 $(\pi_1 \leq \pi_2)$ if $\exists$ closed π_2-inv. linear subspace $\mathcal{H}_2' \leq \mathcal{H}_2$, $\exists$ an isometric linear isom. $T : \mathcal{H}_1 \xrightarrow{\cong} \mathcal{H}_2'$:

$$\begin{tikzcd} H \times \mathcal{H}_1 \arrow[r, "\pi_1"] \arrow[d, hook, "id_H \times T"'] & \mathcal{H}_1 \arrow[d, hook, "T"] \\ H \times \mathcal{H}_2 \arrow[r, "\pi_2"] & \mathcal{H}_2 \end{tikzcd}.$$

- π_1 is contained weakly in π_2 $(\pi_1 \underset{Z}{\prec} \pi_2)$
(in the sense of Zimmer)
if $\forall \varepsilon > 0, \forall K \in \mathcal{K}(G)$ $\forall$ finite orthonormal $S_1 \subseteq \mathcal{H}_1$, $\exists$ finite orthonormal $S_2 \subseteq \mathcal{H}_2$: $\quad card(S_1) = card(S_2)$ and

$$\sup_{k \in K} \left\| \Phi_{S_1}(k) - \Phi_{S_2}(k) \right\| < \varepsilon, \text{ where}$$

$\forall$ finite orthonormal $S := \{s_1, \dots, s_n\}$:

$$\Phi_S : H \to Mat(n, \mathbb{C})$$
$$h \longmapsto \{\langle \pi(h, s_i), s_j \rangle_{\mathcal{H}}\}_{ij}$$

is the <u>submatrix</u> associated to S.

<u>Rem. (13.5.6)</u>:

(i) $\pi_1 \leq \pi_2 \implies \pi_1 \underset{Z}{\lesssim} \pi_2$.

Pf:

$$\begin{tikzcd} H \times \mathcal{H}_1 \arrow[r, "\pi_1"] \arrow[d, "id_H \times T"'] & \mathcal{H}_1 \arrow[d, "T"] \\ H \times \mathcal{H}_2 \arrow[r, "\pi_2"] & \mathcal{H}_2 \end{tikzcd}$$

$T : \mathcal{H}_1 \xrightarrow{\cong} \mathcal{H}_2' \leq \mathcal{H}_2$

$S_1 := \{s_1, \dots, s_n\}$

$S_2 := T(S_1) = \{T(s_1), \dots, T(s_n)\}$.

$$|\langle \pi_1(k, s_i), s_j \rangle_{\mathcal{H}_1} - \langle \pi_2(k, T(s_i)), T(s_j) \rangle_{\mathcal{H}_2}|$$
$$= |\langle \pi_1(k, s_i), s_j \rangle_{\mathcal{H}_1} - \langle T(\pi_1(k, s_i)), T(s_j) \rangle_{\mathcal{H}_2}|$$
$$= 0. \checkmark.$$

(ii) $\pi : H \curvearrowright \mathcal{H}$ has inv. vectors

$\Leftrightarrow \mathbb{1}_H \leq \pi$.

Pf: Both are equiv. to

$$\begin{tikzcd} H \times \mathcal{H} \arrow[r, "\mathbb{1}_H"] \arrow[d, "id_H \times id_{\mathcal{H}}"'] & \mathcal{H} \arrow[d, "id_{\mathcal{H}}"] \\ H \times \mathcal{H} \arrow[r, "\pi"] & \mathcal{H} \end{tikzcd}$$

, ✓.

(iii) $\pi : H \curvearrowright \mathcal{H}$ has alm. inv. vects. iff

$\mathbb{1}_H \underset{Z}{\leq} \pi$.

Pf of $(\Leftarrow)$: $\mathbb{1}_H \leq_Z \pi$ implies by taking $S = \{s\}$, $\|s\| = 1$

$\forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists t \in \mathcal{H}, \forall k \in K$:

$$\underbrace{|\langle \pi(k,s), s\rangle - \langle t,t\rangle|} = |\langle \pi(k,s), s\rangle - \langle \mathbb{1}(k,t), t\rangle| < \varepsilon$$

$= |1 - \langle \pi(k,s), s\rangle|$

Recall:

$$\boxed{\begin{array}{l} \forall h \in H, \forall v \in \mathcal{H} : \|v\|_{\mathcal{H}} = 1 : \\ (i) \|\pi(h,v) - v\|^2 = 2\left(1 - \mathrm{Re}\langle \pi(h,v), v\rangle\right) \leq 2\left|1 - \langle\pi(h,v),v\rangle\right| \\ (ii) \left|1 - \langle \pi(h,v), v\rangle\right|^2 \leq 2\left(1 - \mathrm{Re}\langle\pi(h,v),v\rangle\right) = \|\pi(h,v) - v\|^2 \end{array}}$$

$\overset{(i)}{\Rightarrow}$ $\forall k \in K$:

$$\|\pi(k,s) - s\|^2 < 2\varepsilon \Rightarrow \|\pi(k,s) - s\| < \sqrt{2\varepsilon}$$

$\Rightarrow$ s is a $(\sqrt{2\varepsilon}, K)$-inv. vector. ✓.

[redacted]

$(\Rightarrow)$: $\Big($ Bekka, Harpe, Valette p. 399
Prop. (F.1.4) Cor. (F.1.5) $\Big)$.

(iv) H is (T) $\iff$ $\forall$ unitary $\pi : H \curvearrowright \mathcal{H}$:

$\mathbb{1}_H \leq \pi \Leftarrow \mathbb{1}_H \underset{Z}{\prec} \pi$.

<u>Lem (13.4.7)</u>: If π_1 [redacted]: $\Gamma \curvearrowright \mathcal{H}_1$, $\pi_2 : \Gamma \curvearrowright \mathcal{H}_2$:

$\pi_1 \underset{Z}{\prec} \pi_2$, then $\mathrm{Ind}_\Gamma^G(\pi_1) \underset{Z}{\prec} \mathrm{Ind}_\Gamma^G(\pi_2)$.

<u>Thm (F.3.5) (Continuity of induction) (Bekka, Harpe, Valette, p.403)</u>

Let G be locally compact, $H \leq G$ be closed,

$\sigma, \tau \in \hat{H}$: $\sigma \prec \tau \Rightarrow \mathrm{Ind}_H^G(\sigma) \prec \mathrm{Ind}_H^G(\tau)$.

<u>Pf</u>: Skipped!

$\boxed{\text{EX. 13.4.\#1}}$

<u>Ex. (12.2.3)</u>: $SL(2,\mathbb{R})$ is not Kazhdan.

<u>Obs</u>: $SL(2,\mathbb{R})$ contains a copy of $\mathbb{F}_2$ as a lattice.

$$\mathbb{F}_2 \xrightarrow{\sim} \Gamma \leq SL(2,\mathbb{R})$$

$$a \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$b \longmapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Consider $\mathrm{Ind}_{[\Gamma,\Gamma]}^{G}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right) : G \times L^2_{[\Gamma,\Gamma]}(G,\mathbb{C}) \to L^2_{[\Gamma,\Gamma]}(G,\mathbb{C})$

$$(g,\varphi) \longmapsto \varphi(g^{-1}\cdot).$$

<u>Claim</u>: $\mathrm{Ind}_{[\Gamma,\Gamma]}^{G}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)$ has almost inv. vectors.

$\Gamma/[\Gamma,\Gamma]$ is abelian $\overset{(\text{Cor. }12.2.3)}{\Rightarrow}$ amenable

$\boxed{\text{IAMES}}$ $\Rightarrow$ aux : $\Gamma/[\Gamma,\Gamma] \times L^2(\Gamma/[\Gamma,\Gamma], \mathbb{C}) \to L^2(\Gamma/[\Gamma,\Gamma], \mathbb{C})$

$$(\gamma[\Gamma,\Gamma], \varphi) \longmapsto \varphi(\gamma^{-1}\cdot)$$

has almost inv. vectors.

$$\Rightarrow \mathrm{Ind}_{[\Gamma,\Gamma]}^{\Gamma}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right) : \Gamma \times L^2\left(\Gamma/[\Gamma,\Gamma], \mathbb{C}\right) \to L^2\left(\Gamma/[\Gamma,\Gamma], \mathbb{C}\right)$$

$(\gamma, \varphi) \longmapsto$ [redacted]

aux $(\gamma[\Gamma,\Gamma], \varphi)$

has alm. inv. vectors

$$\Rightarrow \mathbb{1}_{\Gamma} \prec \mathrm{Ind}_{[\Gamma,\Gamma]}^{\Gamma}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)$$

$\boxed{\text{Rem}(13.4.6)(ii)}$

$\Rightarrow$

$\boxed{\text{Lem.}(13.4.7)}$

$$\mathrm{Ind}_{\Gamma}^{G}\left(\mathbb{1}_{\Gamma}\right) \prec \mathrm{Ind}_{\Gamma}^{G}\left(\mathrm{Ind}_{[\Gamma,\Gamma]}^{\Gamma}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)\right)$$
$$= \mathrm{Ind}_{[\Gamma,\Gamma]}^{G}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)$$

$\Gamma \leq G$ is a lattice $\Rightarrow$ G/Γ has finite measure $\Rightarrow$ $\Phi : \mathbb{C} \hookrightarrow L^2$[redacted]$(G/\Gamma, \mathbb{C}) \cong L^2_{\Gamma}(G, \mathbb{C})$

$$z \longmapsto \left[c_z : g\Gamma \longmapsto z\right]$$

is an embedding of $\mathbb{1}_G$ into $\mathrm{Ind}_{\Gamma}^{G}(\mathbb{1}_{\Gamma})$:

$$\begin{array}{ccc} G \times \mathbb{C} & \xrightarrow{\mathbb{1}_G} & \mathbb{C} \\ \downarrow{\scriptstyle id_G \times \Phi} & & \downarrow{\scriptstyle \Phi} \\ G \times L^2_{\Gamma}(G, \mathbb{C}) & \longrightarrow & L^2_{\Gamma}(G, \mathbb{C}) \end{array} \quad \Rightarrow \quad \mathbb{1}_G \leq \mathrm{Ind}_{\Gamma}^{G}(\mathbb{1}_{\Gamma})$$

$\Rightarrow$ $\boxed{\text{Rem}(13.4.6)(i)}$

$$\boxed{\mathbb{1}_G \prec \mathrm{Ind}_{\Gamma}^{G}(\mathbb{1}_{\Gamma}) \prec \mathrm{Ind}_{[\Gamma,\Gamma]}^{G}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)}$$

<u>Claim</u>: $\mathrm{Ind}_{[\Gamma,\Gamma]}^{G}\left(\mathbb{1}_{[\Gamma,\Gamma]}\right)$ has no nontrivial invariant vectors.

Suppose otherwise, i.e.

$\exists\, \varphi \in L^2_{[\Gamma,\Gamma]}(G, \mathbb{C})$:

$$\forall g \in G,\ \forall x \in_{ae} G:\ \varphi(g^{-1}x) = \varphi(x),$$

$$\left(\begin{array}{l} \forall \gamma_1, \gamma_2 \in \Gamma,\ \forall x \in_{ae} G:\ \varphi\left(x\,[\gamma_1,\gamma_2]^{-1}\right) = \varphi(x), \\ \displaystyle\int_{G/[\Gamma,\Gamma]} \| \varphi(x) \|^2 \, d\left(x[\Gamma,\Gamma]\right) < \infty \end{array} \right)$$

$\boxed{\text{BLACKBOX}} \Rightarrow \exists$ G-inv. prob. meas. on $G/[\Gamma,\Gamma]$?

Γ is a lattice $\Rightarrow$ $\Gamma/[\Gamma,\Gamma] \leq G/[\Gamma,\Gamma]$ is finite, but $\Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}^2$, ↯.

<u>Thm (13.2.1)(Kazhdan)</u>: $SL(n,\mathbb{R})$ is Kazhdan for $n \geq 3$. (<u>Pf</u>: Using Lem (13.2.2) below)

<u>Def</u>: Let H be a [redacted] topological group, $R \leq H$. (H,R) is <u>relatively Kazhdan</u> if $\forall$ unitary $\pi : H \curvearrowright \mathcal{H}$: if π has alm. inv. vectors, then $\pi|_R$ has inv. nonzero vectors.

<u>Lem (13.2.2)</u>: $(SL(2,\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has the relative property (T)

$\boxed{\text{Pf: in } \S 13.3}$

Pf of Thm (13.2.1):

$$SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 \cong \left(\begin{array}{cc|c|c} \multicolumn{2}{c|}{SL(2,\mathbb{R})} & \mathbb{R}^2 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{n-3} \end{array}\right) \leq SL(n,\mathbb{R})$$

$$\mathbb{R}^2 \cong \left(\begin{array}{cc|c|c} \multicolumn{2}{c|}{I_2} & \mathbb{R}^2 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 & I_{n-3} \end{array}\right) \leq SL(n,\mathbb{R})$$

By Lem (13.2.2), $(SL(2,\mathbb{R}) \ltimes \mathbb{R}^2; \mathbb{R}^2)$ has relative property (T).

Let $\pi : SL(n,\mathbb{R}) \curvearrowright \mathcal{H}$ be unitary with almost inv. vectors

$\Rightarrow$ $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ has alm. inv. vects.

$\Rightarrow$ $\mathbb{R}^2$ has nonzero inv. vects, say v.

(*)

$\Rightarrow$ v is inv. for

$$SL(2,\mathbb{R}) \cong \left\{ \left(\begin{array}{ccc|c} a & 0 & b & \\ 0 & 1 & 0 & 0 \\ c & 0 & d & \\ \hline & 0 & & I_{n-3} \end{array} \right) \;\middle|\; ad-bc=1 \right\} \leq SL(n,\mathbb{R})$$

(**)

$\Rightarrow$ v is inv. for $SL(n,\mathbb{R})$, v.

(*) Lem. (1.4.9) (BHV, p. 45): If v is invariant for $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cong \left(\begin{array}{ccc|c} 1 & 0 & 4 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & I_{n-3} \end{array} \right)$,

then it is inv. for

$$SL(2,\mathbb{R}) \cong \left(\begin{array}{ccc|c} * & 0 & * & \\ 0 & 1 & 0 & 0 \\ * & 0 & * & \\ \hline & 0 & & I_{n-3} \end{array} \right).$$

(**) Prop. (1.4.11)(BHP, pp. 46-47): If v is invariant under any standard embedding $SL(2,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R})$, then v is invariant for all.

• (Standard Embeddings of SL into SL)

$n \geq 2$, $n > m$, $\{e_{i_1}, \dots, e_{i_m}\} \subseteq \{e_1, \dots, e_n\} \subset \mathbb{K}^n$

$$\varphi_{\{e_{i_1},\dots,e_{i_m}\}} : SL_{n-m}(\mathbb{K}) \hookrightarrow SL_n(\mathbb{K})$$

$$g \longmapsto$$ fix $\{e_{i_1}, \dots, e_{i_m}\}$

fix the span of the remainings.

<u>Ex</u>: $\{e_2, e_4\} \subseteq \{e_1, e_2, e_3, e_4, e_5\} \subseteq \mathbb{K}^5$

$$SL_3(\mathbb{K}) \hookrightarrow SL_5(\mathbb{K})$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & b & 0 & c \\ 0 & 1 & 0 & 0 & 0 \\ d & 0 & e & 0 & f \\ 0 & 0 & 0 & 1 & 0 \\ g & 0 & h & 0 & i \end{pmatrix}$$

<u>Thm</u> (13.2.4): G is Kazhdan iff no simple factor of G is isogenous to $SO(1,n)$ or $SU(1,n)$.

§13.5:

· $\mathcal{H}$: Hilbert space.

[redacted] $\mathrm{Aff}(\mathcal{H}) := U(\mathcal{H}) \ltimes \mathcal{H}$

Def. (13.5.1)

Ex (13.5.2)

· w [redacted] $\in \mathcal{H} \setminus 0$

$$\varphi_{\text{[redacted]}\,w} : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}.$$
$$(t, v) \longmapsto v + tw$$

$\varphi_w(t, \cdot)$ [redacted] $: v \longmapsto v + tw$

$$\varphi_w(t, \cdot) := (\mathrm{id}_{\mathcal{H}}, tw).$$

[redacted]

Let H be a Lie group. H has <u>property (FH)</u> if

$\forall$ Hilbert $\mathcal{H}$, $\forall$ continuous

$\alpha : H \longrightarrow$ [redacted] $\mathrm{Aff}(\mathcal{H})$

(U, v) $\quad \alpha(h, \cdot) := U + v.$

[redacted]

α has a fixed point; **i.e.**,

$\exists v \in \mathcal{H}, \ \forall h \in H:$

$\alpha(h, v) = v$ $\quad \boxed{v = U_h v + w_h.}$

• $\pi : H \curvearrowright \mathcal{H}$ be unitary.

$$Z^1(H,\pi) := \left\{ f \in C^0(H,\mathcal{H}) \;\middle|\; \begin{tikzcd} H \times H \arrow[r, "L"] \arrow[d, "f"'] & H \arrow[d, "f"] \\ H \times \mathcal{H} \arrow[r, "\pi"'] & \mathcal{H} \end{tikzcd} \right\}$$

$$\begin{tikzcd} (g,h) \arrow[r, mapsto] \arrow[d, mapsto] & gh \arrow[d, mapsto] \\ & f(gh) \\ (g, f(h)) \arrow[r, mapsto] & \pi(g, f(h)) \end{tikzcd}$$

$\pi : H \curvearrowright \mathcal{H}$ unitary

$\pi : H \times \mathcal{H} \to \mathcal{H}$

$$(\pi, f) : H \times \mathcal{H} \longrightarrow \mathcal{H}$$
$$(h, v) \longmapsto \pi(h,v) + f(h).$$

$\pi : H \longrightarrow \mathcal{U}(\mathcal{H})$ $\qquad$ $f : H \longrightarrow \mathcal{H}$

$(\pi \ltimes f) : H \longrightarrow \mathrm{Aff}(\mathcal{H})$

$h \longmapsto (\pi(h), f(h))$

$$Z^1(H, \pi) := \left\{ f \in C^0(H, \mathcal{H}) \;\middle|\; \begin{tikzcd} H \times H \arrow[r, "L"] \arrow[d, "\mathrm{id}_H \times f"'] & H \arrow[d, "f"] \\ H \times \mathcal{H} \arrow[r, "(\pi \ltimes f)"] & \mathcal{H} \end{tikzcd} \right\}$$

$$\begin{tikzcd} (h_1, h_2) \arrow[d, mapsto] & h_1 h_2 \arrow[l, mapsto] \arrow[d, mapsto] \\ & f(h_1 h_2) \\ (h_1, f(h_2)) \arrow[r, mapsto, "\pi \ltimes f"] & \pi(h_1, f(h_2)) + f(h_1). \end{tikzcd}$$

[redacted]

$$\pi \ltimes f : H \times \mathcal{H} \longrightarrow \mathcal{H}$$
$$(h, v) \longmapsto \pi(h,v) + f(h)$$

$$\pi \ltimes f(h,v) = v$$

[redacted]

$$B^1(H,\pi) := \left\{ f \in C^0(H, \mathcal{H}) \;\middle|\; \begin{array}{l} \exists v \in \mathcal{H} : \\ \pi \ltimes f(H,v) = \{v\} \end{array} \right\}.$$

<u>Claim</u>: $B^1(H,\pi) \leq Z^1(H,\pi)$.

$f : H \to \mathcal{H}$

$\exists v_0 \in \mathcal{H}$:

$\pi \ltimes f(H, v_0) = \{v_0\}$.

$$\begin{tikzcd} H \times H \arrow[r, "L"] \arrow[d, "id_H \times f"'] \arrow[dr, phantom, "?"] & H \arrow[d, "f"] \\ H \times \mathcal{H} \arrow[r, "\pi \ltimes f"'] & \mathcal{H} \end{tikzcd}$$

$$\begin{tikzcd}
(h_1, h_2) \arrow[r, mapsto] \arrow[d, mapsto] & h_1 h_2 \arrow[d, mapsto] \\
(h_1, f(h_2)) \arrow[r, mapsto, "\pi \ltimes f"'] & \begin{array}{c} f(h_1 h_2) \\ \shortparallel \\ \pi(h_1, f(h_2)) + f(h_1) \end{array}
\end{tikzcd}$$

$$f(h_1 h_2) = \pi(h_1, f(h_2)) + f(h_1) \quad ?$$

$$\forall h \in H : \pi(h, v_0) + f(h) = v_0.$$
$$f(h) = v_0 - \pi(h, v_0).$$

$$f(h_1 h_2) = v_0 - \pi(h_1 h_2, v_0)$$

$$\pi(h_1, f(h_2)) + f(h_1) = \pi(h_1, v_0 - \pi(h_2, v_0)) + v_0 - \pi(h_1, v_0)$$
$$= \cancel{\pi(h_1, v_0)} - \pi(h_1 h_2, v_0) + v_0 - \cancel{\pi(h_1, v_0)}$$

$$\Rightarrow \boxed{B^1(H, \pi) \leq Z^1(H, \pi)}$$

(Delorme – Guichardet)

Thm. (13.5.4): H be lcsc. Then TFAE:

(i) H is Kazhdan.

(ii) H has Property (FH)

(i.e., $\forall$ Hilbert $\mathcal{H}$, $\forall$ cts aff. $\alpha : H \curvearrowright \mathcal{H}$ $\exists\, v \in \mathcal{H}$: $\alpha(H, v) = \{v\}$.)

(iii) $H^1(H,\pi) := \dfrac{Z^1(H,\pi)}{B^1(H,\pi)} = 0.$

(iv) $\forall$ unitary $\pi : H \curvearrowright \mathcal{H}$

$$\mathbb{1}_H \leq \pi \iff \mathbb{1}_H \prec \pi.$$

($\Leftarrow$).

Problems:

① $B^1(H,\pi) \le$ [redacted] $Z^1(H,\pi)$ may not be closed.

② It is more convenient to consider only irreducible unitary reps $\pi : H \curvearrowright \mathcal{H}$.

$\pi : H \curvearrowright \mathcal{H}$ is irred. if $\forall \mathcal{H}' \le \mathcal{H}$: $\pi_{\mathcal{H}'} \le \pi_{\mathcal{H}} \Rightarrow \mathcal{H}' \in \{0, \mathcal{H}\}$.

Thm. (13.5.7)
Cor. (13.5.8), (Korevaar-Schoen-Shalom)

Let H be a Lie group. If H is compactly generated, then TFAE:

(i) H is Kazhdan

(ii) $\forall$ unitary $\pi : H \curvearrowright \mathcal{H}$: $\overline{H}^1(H,\pi) := \dfrac{Z^1(H,\pi)}{\overline{B^1(H,\pi)}} = 0$.

(iii) $\forall$ irreducible unitary $\pi : H \curvearrowright \mathcal{H}$, $\overline{H}^1(H,\pi) = 0$.