An Exposition of Cawley's "The Teichmüller space of the standard action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ is trivial"

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1 Introduction

Here we give an exposition of the main argument in [Caw92] that proves the following statement:

Theorem 1 ([1])**:** Let $T \leq \mathrm{SL}(2,\mathbb{Z})$ be a subgroup and $\sigma_\bullet : T \to \mathrm{Aut}_{\underline{\mathrm{Lie}}}(\mathbb{T}^2)$ be the standard action. Suppose there are $t^1, t^2 \in T$ such that $\sigma_{t^1} = A^1, \sigma_{t^2} = A^2$ are hyperbolic automorphisms such that $S(A^1), U(A^1), S(A^2), U(A^2)$ pairwise transverse, then for any $r \in \mathbb{Z}_{\geq 1} \times]0,1]$ and for any action $\alpha_\bullet : T \to \mathrm{Diff}^r(\mathbb{T}^2)$, if there is a homeomorphism $\Phi \in \mathrm{Homeo}(\mathbb{T}^2)$ with $\alpha_\bullet = \Phi \circ \sigma_\bullet \circ \Phi^{-1}$ and $f^1 = \alpha_{t^1}$ and $f^2 = \alpha_{t^2}$ Anosov, then $\Phi \in \mathrm{Diff}^r(\mathbb{T}^2)$.

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Remark 1: Any non-virtually-cyclic subgroup of $\mathrm{SL}(2,\mathbb{Z})$ can be taken as T in the above theorem.

As an example, one can take

$$A^1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

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Corollary 1 ([2])**:** Let $\sigma_\bullet : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Aut}_{\underline{\mathrm{Lie}}}(\mathbb{T}^2)$ be the standard action. Then for any action $\alpha_\bullet : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{Diff}^r(\mathbb{T}^2)$ for some $r \in \mathbb{Z}_{\geq 1} \times]0,1]$, if there is a homeomorphism $\Phi \in \mathrm{Homeo}(\mathbb{T}^2)$ with $\alpha_\bullet = \Phi \circ \sigma_\bullet \circ \Phi^{-1}$ with the property that $f^1 = \alpha_{t^1}$ and $f^2 = \alpha_{t^2}$ are Anosov and $A^1 = \sigma_{t^1}$ and $A^2 = \sigma_{t^2}$ are hyperbolic automorphisms, then $\Phi \in \mathrm{Diff}^r(\mathbb{T}^2)$.

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Remark 2: There is a Teichmüller space interpretation of Thm. 1, inspired by [MS98]. For this we consider an Anosov action up to topological conjugacy to be a structure, and smoothly conjugate Anosov actions are considered to be the

[1] [Caw92, p.135,Thm.1]
[2] [Caw92, p.136,Thm.2]

same; note that in this case a smooth conjugacy class of an Anosov action completely determines the smooth structure of the underlying manifold; see also Rem. 3 below.

More specifically let T be a discrete group, M be a closed C^∞ manifold, and $\alpha_\bullet : T \curvearrowright M$ be a group action by C^1 diffeomorphisms. For $r \in \mathbb{Z}_{\geq 1} \times]0,1]$, the $C^0 \to C^r$ **Anosov Teichmüller space of** α is by definition a certain set of triples (Φ, N, β) modulo a certain equivalence relation.

Here one considers all triples (Φ, N, β) where

- N is a closed C^∞ manifold,
- $\beta_\bullet : T \curvearrowright N$ is a group action by C^r diffeomorphisms,
- $\Phi : M \to N$ is a homeomorphism such that $\Phi \circ \alpha_\bullet = \beta_\bullet \circ \Phi$, and for any $t \in T$, α_t is Anosov iff β_t is Anosov,

and the equivalence relation is defined by

$$(\Phi^1, N^1, \beta^1) \sim (\Phi^2, N^2, \beta^2) \Leftrightarrow \Phi^2 \circ (\Phi^1)^{-1} \in \mathrm{Diff}^r(N^1; N^2).$$

The conclusion of Thm.1 is that the $C^0 \to C^r$ Anosov Teichmüller space of the standard action of Γ is a point, as once the conjugacy is smooth, it conjugates any Anosov element to an Anosov element.

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2 Preliminaries

Let M be a compact C^∞ manifold. $f \in \mathrm{Diff}^1(M)$ is called **Anosov** if there is a topological Ad_f-invariant splitting $TM = S(f) \oplus U(f)$, each summand of at least rank one, and there are numbers $C \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$ such that with respect to some C^0 fiberwise norm on M, for any $x \in M$ and for any $n \in \mathbb{Z}_{\geq 0}$ we have:

$$\forall v^S \in S_x(f) : |T_x f^n v^S|_{f^n(x)} \leq C e^{-\lambda n} |v^S|_x,$$

$$\forall v^U \in U_x(f) : |T_x f^{-n} v^U|_{f^{-n}(x)} \leq C e^{-\lambda n} |v^U|_x.$$

The main properties we'll use of Anosov diffeomorphisms and certain objects that can be attached to them are among those listed below:

Proposition 1 ([3])**:** Let M be a compact C^∞ manifold, $r \in \mathbb{Z}_{\geq 1} \times]0,1]$, $f \in \mathrm{Diff}^r(M)$ be Anosov. Then

(i) $S_\bullet(f), U_\bullet(f) : M \to \mathrm{Gr}(TM)$ are Hölder continuous.

[3] [Ano67], [Ano69]

(ii) $S(f)$ and $U(f)$ are uniquely integrable. More precisely, for any $x \in M$, there is a unique $\dim(S_x(f))$ dimensional C^r embedded closed ball $\mathcal{S}_{x,\mathrm{loc}}(f)$ such that $x \in \mathcal{S}_{x,\mathrm{loc}}(f)$ and $T_x\mathcal{S}_{x,\mathrm{loc}}(f) = S_x(f)$. Similarly there is a unique $\dim(U_x(f))$ dimensional C^r embedded closed ball $\mathcal{U}_{x,\mathrm{loc}}(f)$ such that $x \in \mathcal{U}_{x,\mathrm{loc}}(f)$ and $T_x\mathcal{U}_{x,\mathrm{loc}}(f) = U_x(f)$. $\mathcal{S}_{x,\mathrm{loc}}(f)$ is called the **local stable manifold** of f at x and $\mathcal{U}_{x,\mathrm{loc}}(f)$ is called the **local unstable manifold** of f at x, respectively. For some maximal $\varepsilon_0 \in \mathbb{R}_{>0}$, we have

$$\mathcal{S}_{x,\mathrm{loc}}(f) = \bigcap_{n\in\mathbb{Z}_{\geq 0}} \{y \in M \mid d_M(f^n(y), f^n(x)) \leq \varepsilon_0\},$$

$$\mathcal{U}_{x,\mathrm{loc}}(f) = \mathcal{S}_{x,\mathrm{loc}}(f^{-1}) = \bigcap_{n\in\mathbb{Z}_{\geq 0}} \left\{y \in M \,\middle|\, d_M(f^{-n}(y), f^{-n}(x)) \leq \varepsilon_0\right\}.$$

(iii) For any $x \in M$, $\mathcal{S}_x(f) = \bigcup_{n\in\mathbb{Z}_{\geq 0}} \overleftarrow{f^n}\left(\mathcal{S}_{f^n(x),\mathrm{loc}}(f)\right)$ is the **global stable manifold** and $\mathcal{U}_x(f) = \bigcup_{n\in\mathbb{Z}_{\geq 0}} \overleftarrow{f^{-n}}\left(\mathcal{U}_{f^{-n}(x),\mathrm{loc}}(f)\right)$ is the **global unstable manifold** of f at x. They are C^r injectively immersed discs of appropriate dimensions. Using the intrinsic distance function d_M on M induced by the chosen fiberwise norm, they are also characterized as follows:

$$\mathcal{S}_x(f) = \left\{y \in M \,\middle|\, \lim_{n\to\infty} d_M(f^n(y), f^n(x)) = 0\right\},$$

$$\mathcal{U}_x(f) = \mathcal{S}_x(f^{-1}) = \left\{y \in M \,\middle|\, \lim_{n\to\infty} d_M(f^{-n}(y), f^{-n}(x)) = 0\right\}.$$

(iv) $\mathcal{S}(f)$ and $\mathcal{U}(f)$ are C^0 foliations with C^r leaves.

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In particular, there is an $r_0 \in \mathbb{R}_{>0}$ such that for any $x \in M$, there is a neighborhood $N \in \mathrm{Nbhd}(x)$ and a homeomorphism

$$\phi : \left(\mathbb{R}^{\dim(S_x(f))}[0| < r_0] \times \mathbb{R}^{\dim(U_x(f))}[0| < r_0], (0,0)\right) \to (N, x)$$ [4]

such that

$$\forall a \in \mathbb{R}^{\dim(S_x(f))}[0| < r_0] : \overrightarrow{\phi}\left(\{a\} \times \mathbb{R}^{\dim(U_x(f))}[0| < r_0]\right) = N \cap \mathcal{U}_{x,\mathrm{loc}}(f),$$

$$\forall b \in \mathbb{R}^{\dim(U_x(f))}[0| < r_0] : \overrightarrow{\phi}\left(\mathbb{R}^{\dim(S_x(f))}[0| < r_0] \times \{b\}\right) = N \cap \mathcal{S}_{x,\mathrm{loc}}(f).$$

[4]For X a metric space, $X[x| < r]$ denotes the open ball centered at x of radius r.

Such a ϕ is called a **local product structure chart** associated to (the stable and unstable foliations of) f at x, and the collection of all such (U, ϕ) is called a **local product structure** associated to (the stable and unstable foliations of) f.

For A, B two arbitrary subsets, denote by $\mathrm{Hit}^f(B \leftarrow A)$ the set of all those integers n such that $\overrightarrow{f^n}(A) \cap B \neq \emptyset$; any such n is an f**-hitting time** from A to B. $x \in M$ is an f**-nonwandering point** if $\forall U \in \mathrm{Nbhd}(x) : \mathrm{Hit}^f(U \leftarrow U) \cap \mathbb{Z}_{\geq 1} \neq \emptyset$. Denote by $\mathrm{NW}(f)$ the set of f-nonwandering points.

Proposition 2 ([5])**:** Let M be a compact C^∞ manifold, $f \in \mathrm{Diff}^1(M)$ be Anosov. Then the following are equivalent:

(i) $\mathrm{NW}(f) = M$.

(ii) $\overline{\mathrm{Per}(f)} = M$.

(iii) f is topologically transitive[6].

(iv) f is topologically strong mixing[7].

(v) $\forall x \in M : \overline{\mathcal{S}_x(f)} = M = \overline{\mathcal{U}_x(f)}$.

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Theorem 2 ([8])**:** Let M be a compact C^∞ manifold, $f \in \mathrm{Diff}^1(M)$ be Anosov. Then

(Franks) If $S(f)$ or $U(f)$ is a line bundle and $\mathrm{NW}(f) = M$, then f is $\underline{\overline{\mathrm{Top}}}$-isomorphic to a hyperbolic Lie group automorphism of $\mathbb{T}^{\dim(M)}$.

(Newhouse) If $S(f)$ or $U(f)$ is a line bundle, then $\mathrm{NW}(f) = M$.

- If $S(f)$ or $U(f)$ is a line bundle, then f is $\underline{\overline{\mathrm{Top}}}$-isomorphic to a hyperbolic Lie group automorphism of $\mathbb{T}^{\dim(M)}$.

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Proposition 3 ([9])**:** Let $d \in \mathbb{Z}_{\geq 2}$ and $A \in \mathrm{Aut}_{\underline{\overline{\mathrm{Lie}}}}(\mathbb{T}^d)$ be hyperbolic. Then for any $x \in \mathbb{T}^d$, $\overline{\mathcal{S}_x(A) \cap \mathcal{U}_x(A)} = \mathbb{T}^d$.

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[5] [Kat72, p.68,Thm.4.3; p.69,Rem.4.1,Exr.4.1], also see [KH95, Ch.18]

[6] Recall that f is **topologically transitive** if it has a dense orbit, or alternatively for any two nonempty open subsets U, V, $\mathrm{Hit}^f(V \leftarrow U) \cap \mathbb{Z}_{\geq 1} \neq \emptyset$.

[7] Recall that f is **topologically strong mixing** if for any two nonempty open subsets U, V, $\mathrm{card}(\mathbb{Z}_{\geq 1} \setminus \mathrm{Hit}^f(V \leftarrow U)) < \infty$.

[8] [Fra70, p.64,Thm.6.3], [New70, p.762,Thm.1.2]

[9] [LS99, pp.597-598,Ex.3.3]

Corollary 2: Let M be a compact C^∞ manifold. If $1 < \dim(M) \leq 3$ and M carries an Anosov diffeomorphism, then M is homeomorphic to a torus via a homeomorphism which conjugates f to a hyperbolic Lie group automorphism. Further, $\overline{\mathrm{Per}(f)} = \mathrm{NW}(f) = M$, f is topologically strong mixing, and for any $x \in M$, $\overline{\mathcal{S}_x(f)} = \overline{\mathcal{U}_x(f)} = \overline{\mathcal{S}_x(f) \cap \mathcal{U}_x(f)} = M$.

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Let M be a compact C^∞ manifold, $f \in \mathrm{Diff}^1(M)$ be Anosov. Let L, R be two embedded manifolds transverse to $\mathcal{S}(f)$. A **holonomy** (or a **Poincaré transformation**[10], or **projection**[11]) $\mathcal{S}^f_{R\leftarrow L} = \mathcal{S}_{R\leftarrow L}(f) : L \rightsquigarrow R$[12] from L to R along $\mathcal{S}(f)$ (**stable holonomy** for short) is a local homeomorphism such that $\forall x \in \mathrm{dom}\left(\mathcal{S}^f_{R\leftarrow L}\right) : \mathcal{S}^f_{R\leftarrow L}(x) \in R \cap \mathcal{S}_x(f)$. Similarly, if L and R are embedded manifolds transverse to $\mathcal{U}(f)$, a **holonomy** $\mathcal{U}^f_{R\leftarrow L} = \mathcal{U}_{R\leftarrow L}(f) : L \rightsquigarrow R$ from L to R along $\mathcal{S}(f)$ (**unstable holonomy** for short) is a local homeomorphism such that $\forall x \in \mathrm{dom}\left(\mathcal{U}^f_{R\leftarrow L}\right) : \mathcal{U}^f_{R\leftarrow L}(x) \in R \cap \mathcal{U}_x(f)$. It's clear that stable and unstable holonomies exist and if the transverse manifolds are close enough they are unique.

Let μ be a probability measure induced by a C^∞ Riemannian metric on M, L, R be two embedded manifolds transverse to $\mathcal{S}(f)$, $\mathcal{S}^f_{R\leftarrow L} : L \to R$ be an everywhere defined stable holonomy. Denote by μ^L and μ^R be the Radon measures on L and R induced by the induced Riemannian metrics, respectively. We say that $\mathcal{S}^f_{R\leftarrow L}$ is **absolutely continuous** if $\left(\mathcal{S}^f_{R\leftarrow L}\right)^!(\mu_R) \ll \mu_L$, or alternatively $\mu_R \ll \overrightarrow{\mathcal{S}^f_{R\leftarrow L}}(\mu_L)$. In words, absolute continuity means that zero measure sets are sent to zero measure sets. The Radon-Nikodym derivative coming from the first absolute continuity relation is called the **generalized Jacobian** of the stable holonomy from L to R:

$$J^{\mathcal{S}}_\bullet(f; L \leftarrow R) = \frac{d\left(\mathcal{S}^f_{R\leftarrow L}\right)^!(\mu_R)}{\mu_L} : L \to \mathbb{R}_{>0}.$$

Proposition 4 ([13])**:** Let M be a compact C^∞ manifold, $r \in \mathbb{Z}_{\geq 1} \times]0,1]$, $f \in \mathrm{Diff}^r(M)$ be an Anosov diffeomorphism, L, R be two C^r embedded discs transverse to $\mathcal{S}(f)$ close enough that there is a unique stable holonomy $\mathcal{S}^f_{R\leftarrow L} : L \to R$. Then for any probability measure μ on M induced by a C^∞ Riemannian metric on M,

(i) $\mathcal{S}^f_{R\leftarrow L} : L \to R$ is Hölder and absolutely continuous, and

(ii) $J^{\mathcal{S}}_\bullet(f; L \leftarrow R) : L \to \mathbb{R}_{>0}$ is also Hölder continuous.

[10] [Mn87, p.190]
[11] [Hir01, 802]
[12] For two sets A, B, $f : A \rightsquigarrow B$ denotes a partially defined function from A to B.
[13] [Ano69, p.27,Thm.10]; note that here the holonomy is required, and proved, to be continuous with respect to small C^0 perturbations in Emb^r. One may call this "stable continuity of holonomies". Also see [Mn87, p.191,Thm.3.1].

Similarly if L, R be two C^r embedded discs transverse to $\mathcal{U}(f)$ close enough that there is a unique unstable holonomy $\mathcal{U}^f_{R\leftarrow L} : L \to R$, then for any probability measure μ on M induced by a C^∞ Riemannian metric on M,

(i) $\mathcal{U}^f_{R\leftarrow L} : L \to R$ is Hölder and absolutely continuous, and

(ii) $J^{\mathcal{U}}_\bullet(f; L \leftarrow R) : L \to \mathbb{R}_{>0}$ is also Hölder continuous.

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Proposition 5 ([14])**:** Let M be a compact C^∞ manifold, $r = (q,\theta) \in \mathbb{Z}_{\geq 1}\times]0,1]$, $f \in \mathrm{Diff}^r(M)$ be Anosov. If $U(f)$ is a corank one subbundle of TM, then for some $s = (q,\theta') \in \mathbb{Z}_{\geq 1}\times]0,\theta]$ one has:

(i) Any unstable holonomy between C^s embedded discs transverse to $\mathcal{U}(f)$ is C^s.

(ii) $\mathcal{S}(f)$ is a C^s foliation with C^r leaves[15].

Similarly if $S(f)$ is a corank one subbundle of TM, then for some $s = (q,\theta') \in \mathbb{Z}_{\geq 1}\times]0,\theta]$ one has:

(i) Any stable holonomy between C^s embedded discs transverse to $\mathcal{S}(f)$ is C^s.

(ii) $\mathcal{U}(f)$ is a C^s foliation with C^r leaves.

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Remark 3: If f is a C^r ($r = (q,\theta) \in \mathbb{Z}_{\geq 1}\times]0,1]$) Anosov diffeomorphism of a compact C^∞ surface M, as both $S(f)$ and $U(f)$ are of rank one, the local product structure associated to f is of regularity C^s for some $s = (q,\theta') \in \mathbb{Z}_{\geq 1}\times]0,\theta]$. Therefore, the C^s manifold structure of M is determined (up to C^s diffeomorphisms) by the pair of transverse foliations $\mathcal{S}(f), \mathcal{U}(f)$.

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3 The Proof

Using arguments similar to those that can be found in [dlL87], once the C^1 differentiability of Ψ is guaranteed the regularity can be upgraded to C^r. Thus in what follows we focus on showing that $\Phi \in \mathrm{Diff}^1(\mathbb{T}^2)$.

First note that by hypothesis $\mathcal{S}(A^1), \mathcal{U}(A^1), \mathcal{S}(A^2), \mathcal{U}(A^2)$ are pairwise transverse foliations of $\mathbb{T}^2$. We have that $\overrightarrow{\Phi}(\mathcal{S}(A^i)) = \mathcal{S}(f^i)$ and $\overrightarrow{\Phi}(\mathcal{U}(A^i)) = \mathcal{U}(f^i)$. We'll need that the foliations $\mathcal{S}(f^1), \mathcal{U}(f^1), \mathcal{S}(f^2), \mathcal{U}(f^2)$ are also pairwise transverse. Since both f^1 and f^2 are Anosov, $\mathcal{S}(f^1)$ and $\mathcal{U}(f^1)$; as well as $\mathcal{S}(f^2)$ and

[14] [Mn87, p.202,Exr.3.1], [PR02, p.343,Thm.2.1], [PRF09, p.11,Thm.1.6], [PSW97, p.543,Thm.6.1]

[15] More explicitly this means that $\mathcal{S}(f)$ admits foliation charts whose transitions are C^s diffeomorphisms that are C^r along (images of) leaves of $\mathcal{S}(f)$.

$\mathcal{U}(f^2)$ are pairwise transverse. By replacing a diffeomorphism by its inverse it suffices to show that $\mathcal{U}(f^1)$ and $\mathcal{S}(f^2)$ are transverse. Note also that since these foliations are one dimensional foliations of a two dimensional manifold, if their tangent fields don't span the whole tangent space, they have to coincide.

Lemma 1: Let $x \in \mathbb{T}^2$ be such that $U_x(f^1) = S_x(f^2)$. Then for any $y \in \mathcal{U}_x(f^1) \cap \mathcal{S}_x(f^1) : U_y(f^1) = S_y(f^2)$.

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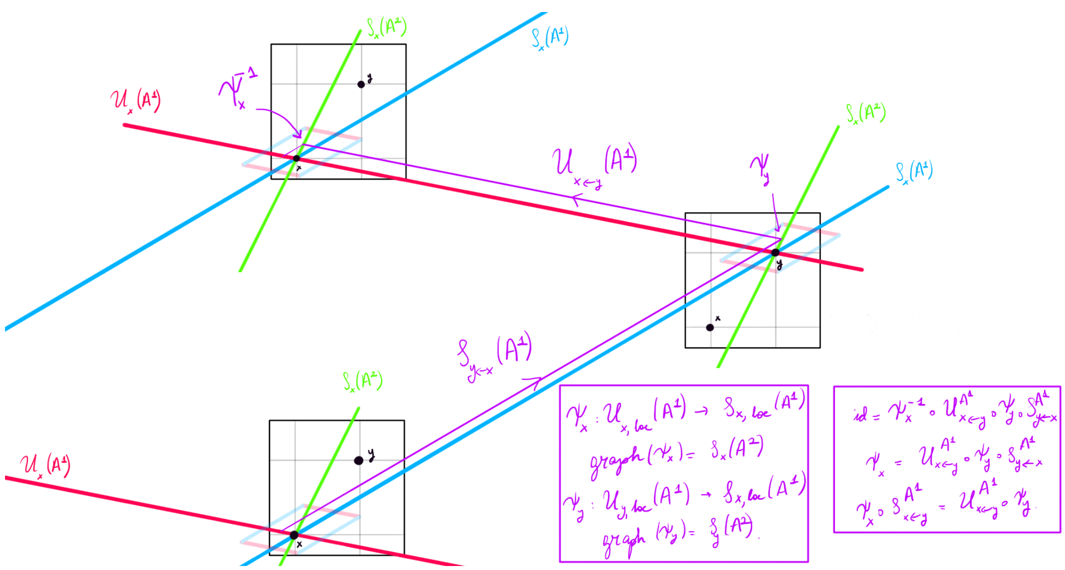
Proof: First note that for any $x \in \mathbb{T}^2$, there is an affine map $\psi_x : (\mathcal{U}_{x,\text{loc}}(A^1), x) \to (\mathcal{S}_{x,\text{loc}}(A^1), x)$ such that graph$(\psi_x) = \mathcal{S}_{x,\text{loc}}(A^2)$. Further, if $y \in \mathcal{U}_x(A^1) \cap \mathcal{S}_x(A^1)$, so that y is A^1-biasymptotic to x, then we have

$$\begin{tikzcd}
\mathcal{U}_{x,\text{loc}}(A^1) \arrow[r, "\psi_x"] \arrow[d, "\mathcal{S}^{A^1}_{y\leftarrow x}"'] & \mathcal{S}_{x,\text{loc}}(A^1) \arrow[d, "\mathcal{U}^{A^1}_{y\leftarrow x}"] \\
\mathcal{U}_{y,\text{loc}}(A^1) \arrow[r, "\psi_y"] & \mathcal{S}_{y,\text{loc}}(A^1)
\end{tikzcd}$$



Since $U_x(f^1) = S_x(f^2)$, $\mathcal{U}_{x,\text{loc}}(f^1)$ is tangent to $\mathcal{S}_{x,\text{loc}}(f^2)$ at x, as such there is a C^r map $\phi_x : (\mathcal{U}_{x,\text{loc}}(f^1), x) \to (\mathcal{S}_{x,\text{loc}}(f^1), x)$ such that graph$(\phi_x) = \mathcal{S}_{x,\text{loc}}(f^2)$ and $\phi_x'(x) = 0$. Similarly we can find a C^r map ϕ_y such that graph$(\phi_y) = \mathcal{U}_{y,\text{loc}}(f^2)$. Indeed, either $S_y(f^1) \neq S_y(f^2)$ xor $S_y(f^1) = S_y(f^2)$. In the first case there is a diffeomorphism $\phi_y : (\mathcal{U}_{y,\text{loc}}(f^1), y) \to (\mathcal{S}_{y,\text{loc}}(f^1), y)$ such that graph$(\phi_y) = \mathcal{S}_{y,\text{loc}}(f^2)$ and in the second case there is a diffeomorphism $\phi_y : (\mathcal{S}_{y,\text{loc}}(f^1), y) \to (\mathcal{U}_{y,\text{loc}}(f^1), y)$ such that graph$(\phi_y) = \mathcal{S}_{y,\text{loc}}(f^2)$. Since Φ conjugates σ and α, we thus have[16]:

[16] In the diagram, the two-sided arrow at the bottom summarizes the two cases.

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$$\begin{tikzcd}
\mathcal{U}_{x,\mathrm{loc}}(f^1) \arrow[rrr, "\phi_x"] \arrow[ddd, "\mathcal{S}^{f^1}_{y\leftarrow x}"'] & & & \mathcal{S}_{x,\mathrm{loc}}(f^1) \arrow[ddd, "\mathcal{U}^{f^1}_{y\leftarrow x}"] \\
& \mathcal{U}_{\Phi^{-1}(x),\mathrm{loc}}(A^1) \arrow[ul, "\Phi"] \arrow[r, "\psi_{\Phi^{-1}(x)}"] \arrow[d, "\mathcal{S}^{A^1}_{\Phi^{-1}(y)\leftarrow\Phi^{-1}(x)}"'] & \mathcal{S}_{\Phi^{-1}(x),\mathrm{loc}}(A^1) \arrow[ur, "\Phi"'] \arrow[d, "\mathcal{U}^{A^1}_{\Phi^{-1}(y)\leftarrow\Phi^{-1}(x)}"] & \\
& \mathcal{U}_{\Phi^{-1}(y),\mathrm{loc}}(A^1) \arrow[dl, "\Phi"] \arrow[r, "\psi_{\Phi^{-1}(y)}"] & \mathcal{S}_{\Phi^{-1}(y),\mathrm{loc}}(A^1) \arrow[dr, "\Phi"] & \\
\mathcal{U}_{y,\mathrm{loc}}(f^1) \arrow[rrr, leftrightarrow, "\phi_y"'] & & & \mathcal{S}_{y,\mathrm{loc}}(f^1)
\end{tikzcd}$$

Thus if $a \in \mathcal{U}_{x,\mathrm{loc}}(f)$, then in the first case above we have

$$\phi_y \circ \mathcal{S}^{f^1}_{y\leftarrow x}(a) = \mathcal{U}^{f^1}_{y\leftarrow x} \circ \phi_x(a)$$

and in the second case we have

$$\mathcal{S}^{f^1}_{y\leftarrow x}(a) = \phi_y \circ \mathcal{U}^{f^1}_{y\leftarrow x} \circ \phi_x(a).$$

Differentiating these equations with respect to a and evaluating at $a = x$, we get in the first case

$$\phi_y'(y)\frac{d\mathcal{S}^{f^1}_{y\leftarrow x}}{dx}(x) = \frac{d\mathcal{U}^{f^1}_{y\leftarrow x}}{dx}(x)\phi_x'(x) = 0,$$

and in the second case

$$\frac{d\mathcal{S}^{f^1}_{y\leftarrow x}}{dx}(x) = \phi_y'(y)\frac{d\mathcal{U}^{f^1}_{y\leftarrow x}}{dx}(x)\phi_x'(x) = 0.$$

Note that the second equation gives a contradiction since the stable holonomies are invertible (consequently it must be the case that $S_y(f^1) \neq S_y(f^2)$), and the only way the first equation could be valid is if $\phi_y'(y) = 0$, that is, $U_y(f^1) = S_y(f^2)$, as was to be shown.

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Lemma 2: $\mathcal{U}(f^1)$ and $\mathcal{S}(f^2)$ are transverse foliations.

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Proof: Suppose not. Then there is an $x \in \mathbb{T}^2$ such that $U_x(f^1) = S_x(f^2)$. By the previous lemma, $U_\bullet(f^1)$ and $S_\bullet(f^2)$ coincide on $\mathcal{U}_x(f^1) \cap \mathcal{S}_x(f^1)$. Note that

$$\begin{aligned}
\overline{\mathcal{U}_x(f^1) \cap \mathcal{S}_x(f^1)} &= \overline{\overrightarrow{\Phi}(\mathcal{U}_{\Phi^{-1}(x)}(A^1) \cap \mathcal{S}_{\Phi^{-1}(x)}(A^1))} \\
&= \overrightarrow{\Phi}\left(\overline{\mathcal{U}_{\Phi^{-1}(x)}(A^1) \cap \mathcal{S}_{\Phi^{-1}(x)}(A^1)}\right) \overset{(\dagger)}{=} \overrightarrow{\Phi}(\mathbb{T}^2) = \mathbb{T}^2.
\end{aligned}$$

Here the equality with (†) is due to Prop.3.

Since $U_\bullet(f^1)$ and $S_\bullet(f^2)$ are continuous and coincide on a dense subset, they have to be equal. This implies that $\mathcal{U}(f^1)$ and $\mathcal{S}(f^2)$ coincide everywhere, and consequently $\mathcal{U}(A^1)$ and $\mathcal{S}(A^2)$ coincide everywhere, a contradiction.

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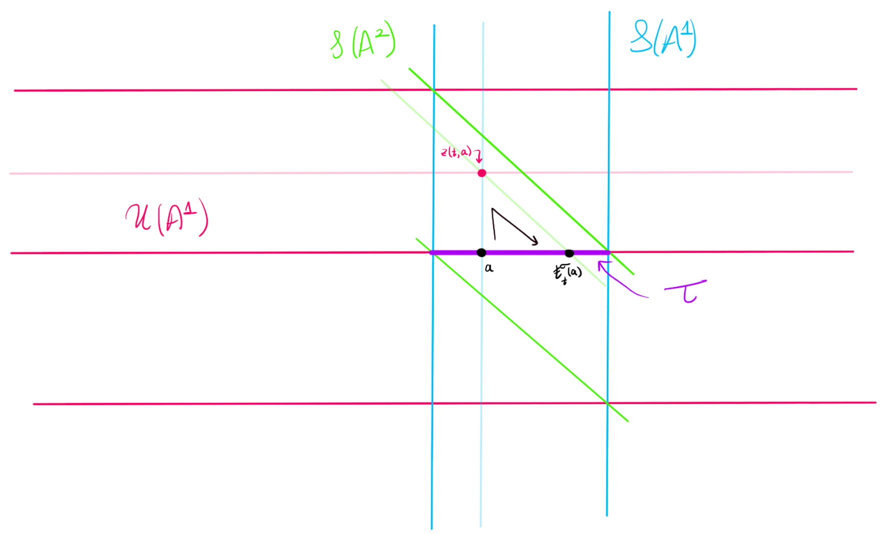
Next fix a point $x \in \mathbb{T}^2$ and let τ^σ be a compact embedded interval in $\mathcal{U}_x(A^1)$. Note that since $\mathcal{U}_x(A^1)$ is an injectively immersed line in $\mathbb{T}^2$, it carries a natural affine structure. Consequently there is a local $\mathbb{R}$ action $\hbar^\sigma : \mathbb{R} \times \tau^\sigma \rightsquigarrow \tau^\sigma$ on τ^σ by translations. This local action can be uniquely factored into stable and unstable holonomies based on the following caricature:



Thus we have that $\forall a \in \tau^\sigma, \exists \lambda_a, \rho_a \in \mathbb{R}$ with $\lambda_a < 0 < \rho_a$ such that $\forall t \in [\lambda_a, \rho_a], \exists! z^\sigma(t,a) \in \mathcal{S}_a(A^1)$:

$$t \geq 0 \Rightarrow \hbar_t^\sigma(a) = \mathcal{S}^{A^2}_{\tau^\sigma \leftarrow \mathcal{U}_{z^\sigma(t,a)}(A^1)} \circ \mathcal{S}^{A^1}_{\mathcal{U}_{z^\sigma(t,a)}(A^1) \leftarrow \tau^\sigma}(a)$$
$$t \leq 0 \Rightarrow \hbar_t^\sigma(a) = \mathcal{S}^{A^1}_{\mathcal{U}_{z^\sigma(t,a)}(A^1) \leftarrow \tau^\sigma} \circ \mathcal{S}^{A^2}_{\tau^\sigma \leftarrow \mathcal{U}_{z^\sigma(t,a)}(A^1)}(a).$$

Let us now conjugate $\hbar^\sigma$ using Φ. Put $\tau^\alpha = \overrightarrow{\Phi}(\tau^\sigma)$, $\hbar_t^\alpha(b) = \Phi \circ \hbar_t^\sigma \circ \Phi^{-1}(b)$ and $z^\alpha(t,b) = \Phi(z^\sigma(t, \Phi^{-1}(b)))$ for all $\lambda_{\Phi^{-1}(b)} \leq t \leq \rho_{\Phi^{-1}(b)}$ and for all $b \in \tau^\alpha$. Note that now $\hbar^\alpha : \mathbb{R} \times \tau^\alpha \rightsquigarrow \tau^\alpha$ is a local action by homeomorphisms.

Lemma 3: $\hbar_t^\alpha(b)$ is C^r in the t variable and C^s in the b variable.

⌟

Proof: We'll use the holonomy factorization of $\hbar^\sigma$. If $t \geq 0$, we have

$$\begin{aligned}
\hbar^\alpha_t(b) &= \Phi \circ \hbar^\sigma_t \circ \Phi^{-1}(b) \\
&= \Phi \circ \mathcal{S}^{A^2}_{\tau^\sigma \leftarrow \mathcal{U}_{z^\sigma(t,\Phi^{-1}(b))}(A^1)} \circ \Phi^{-1} \circ \Phi \circ \mathcal{S}^{A^1}_{\mathcal{U}_{z^\sigma(t,\Phi^{-1}(b))}(A^1) \leftarrow \tau^\sigma} \circ \Phi^{-1}(b) \\
&= \mathcal{S}^{f^2}_{\tau^\alpha \leftarrow \mathcal{U}_{z^\alpha(t,b)}(f^1)} \circ \mathcal{S}^{f^1}_{\mathcal{U}_{z^\alpha(t,b)}(f^1) \leftarrow \tau^\alpha}(b).
\end{aligned}$$

Similarly we have for $t \leq 0$,

$$\hbar^\alpha_t(b) = \mathcal{S}^{f^1}_{\mathcal{U}_{z^\alpha(t,b)}(f^1) \leftarrow \tau^\alpha} \circ \mathcal{S}^{f^2}_{\tau^\alpha \leftarrow \mathcal{U}_{z^\alpha(t,b)}(f^1)}(b).$$

The lemma follows from these formulas and Prop.5.

⌟

Lemma 4: Let $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (t,x) \mapsto x + t$ and $S : \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be a local group action by homeomorphisms. Suppose there is a homeomorphism $\Psi \in \mathrm{Homeo}(\mathbb{R})$ such that $T_\bullet = \Psi \circ S_\bullet \circ \Psi^{-1}$. Then $\partial_2 S : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exists and is continuous iff $\Psi \in \mathrm{Diff}^1(\mathbb{R})$.

⌟

Proof: ($\Leftarrow$) is clear. For ($\Rightarrow$), first note that since S is a local group action by invertible maps and $\partial_2 S$ exists and is continuous, $\mathrm{im}(\partial_2 S) \subseteq \mathbb{R}_{>0}$ xor $\mathrm{im}(\partial_2 S) \subseteq \mathbb{R}_{<0}$; wlog let us assume the former. Further $\partial_2 S : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{>0}$ is a cocycle over S, that is,

$$\partial_2 S(t_1 + t_2, x) = \partial_2 S(t_1, S(t_2, x))\, \partial_2 S(t_2, x).$$

Fix a $y_0 \in \mathbb{R}$ and define $\mathcal{T} = \mathcal{T}_{y_0} : (\mathbb{R}, y_0) \to (\mathbb{R}, 0), y \mapsto -\Psi(y) + \Psi(y_0)$. Note that $\mathcal{T}$ is a homeomorphism. Further, for $y \in \mathbb{R}$, $S(t,y) = y_0$ implies $\Psi(y_0) = T(t, \Psi(y)) = \Psi(y) + t$, so that $t = \mathcal{T}(y)$ is the unique solution to the equation $S(t,y) = y_0$. Note that

$$S(\mathcal{T}(y), y) = y_0 = S(\mathcal{T}(S(t,y)), S(t,y)) = S(\mathcal{T}(S(t,y)) + t, y),$$

whence by the uniqueness of $\mathcal{T}$ we have $\mathcal{T} \circ S(t,y) = \mathcal{T}(y) - t$. Define $\Theta : (\mathbb{R}, y_0) \to (\mathbb{R}, 0)$, $y \mapsto \int_{y_0}^{y} \partial_2 S(\mathcal{T}(x), x)\, dx$. Since the integrand is always positive and is continuous, Θ is a C^1 diffeomorphism. Further, $\Theta'(y) = \partial_2 S(\mathcal{T}(y), y)$. Put $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(t,z) \mapsto \Theta \circ S_t \circ \Theta^{-1}(z)$. Then we have, putting $y = \Theta^{-1}(z)$ and $u = \partial_2 S$,

$$\begin{aligned}
\partial_2 R(t,z) = R_t'(z) =& (\Theta \circ S_t \circ \Theta^{-1})'(z) \\
&= \Theta'(S(t,y))S_t'(y)(\Theta^{-1})'(z) \\
&= \frac{\Theta'(S(t,y))S_t'(y)}{\Theta'(y)} \\
&= \frac{u(\mathfrak{T} \circ S(t,y), S(t,y))\, u(t,y)}{u(\mathfrak{T}(y),y)} \\
&= \frac{u(\mathfrak{T}(y)-t, S(t,y))\, u(t,y)}{u(\mathfrak{T}(y),y)} \\
&= \frac{u(\mathfrak{T}(y),y)\, u(-t,S(t,y))\, u(t,y)}{u(\mathfrak{T}(y),y)} = 1.
\end{aligned}$$

Thus for some continuous function $\kappa : \mathbb{R} \to \mathbb{R}$, $R(t,z) = z + \kappa(t)$. By the group property $z + \kappa(t_1 + t_2) = z + \kappa(t_2) + \kappa(t_1)$, so that $\kappa : \mathbb{R} \to \mathbb{R}$ is a continuous group homomorphism; whence $\kappa(t) = kt$ for $k = \kappa'(0) \neq 0$, so that $R(t,z) = z + kt$. Putting $\Xi : \mathbb{R} \to \mathbb{R}, z \mapsto \frac{z}{k}$, we have that $\Psi \circ S_\bullet \circ \Psi^{-1} = T_\bullet = \Xi \circ R_\bullet \circ \Xi^{-1} = (\Xi \circ \Theta) \circ S_\bullet \circ (\Xi \circ \Theta)^{-1}$, thus

$$T_\bullet = \Psi \circ (\Xi \circ \Theta)^{-1} \circ T_\bullet \circ (\Xi \circ \Theta) \circ \Psi^{-1}.$$

Put $\Lambda = \Psi \circ (\Xi \circ \Theta)^{-1}$, so that $T_\bullet = \Lambda \circ T_\bullet \circ \Lambda^{-1}$. Then for any $t, x \in \mathbb{R}$, $\Lambda(x+t) = \Lambda(x) + t$, so that $\Lambda(t) = l + t$ for $l = \Lambda(0)$. Therefore

$$\Psi(y) = \Lambda \circ \Xi \circ \Theta(y) = \Lambda(0) + \Xi \circ \Theta(y) = l + \frac{1}{k}\int_{y_0}^{y} \partial_2 S(\mathfrak{T}(x),x)\,dx$$

and consequently, $\Psi \in \mathrm{Diff}^1(\mathbb{R})$. In fact, chasing the definitions we have the more explicit formula

$$\Psi(y) = \Psi(y_0) + \frac{1}{\partial_2 S(0,y_0)\,\partial_1 S(0,y_0)} \int_{y_0}^{y} \partial_2 S(-\Psi(x) + \Psi(y_0), x)\,dx.$$

⌟

Applying the above lemma with T as the translation action on $\mathcal{U}_x(A^1)$ and with S as $\hbar^\alpha : \mathbb{R} \times \tau^\alpha \rightsquigarrow \tau^\alpha$, we have that $\Phi \in \mathrm{Diff}^1(\tau^\sigma; \tau^\alpha)$, that is, the conjugacy Φ is C^1 along the global unstable manifolds of A^1. An analogous argument shows that Φ is C^1 along the global stable manifolds of A^1. Since, as mentioned in Rem.3 above, the C^1 manifold structure of $\mathbb{T}^2$ is determined up to C^1 diffeomorphism by the pair of stable and unstable foliations of A^1, we have that $\Phi \in \mathrm{Diff}^1(\mathbb{T}^2)$.

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