Handout for MATH 022.202: The Dictionary of Obscure Symbols*

*This document is in draft stage and subject to change.

Alp Uzman

Last Revised: July 12, 2017

> Why does mathematics occupy such an isolated position in the intellectual firmament? Why is it good form, for intellectuals, to shudder and announce that they can't bear it, or, at the very least, to giggle and announce that they never could understand it? One reason, perhaps, is that mathematics is a language. Mathematics is a precise and subtle language designed to express certain kinds of ideas more briefly, more accurately, and more usefully than ordinary language. What I do mean by saying that mathematics is a language is sketchily and inadequately illustrated by the difference between the following two sentences. (1) If each of two numbers is multiplied by itself, the difference of the two results is the same as the product of the sum of the two given numbers by their difference. (2) $x^2 - y^2 = (x+y)(x-y)$.[1]
>
> — P. R. Halmos

Introduction, or "How to Use this Dictionary"

I try not to use too many symbols in class, but since MATH 022 is a mathematics course we are bound to use some symbols aside from numbers to communicate at some point. In particular I might use a particular symbol or refer to a mathematical concept in class without defining or reminding you of its meaning inadvertently. I am hoping this handout will prove to be a decent countermeasure for this possibility. Usually there is more than one way to go about doing mathematics, and in terms of definitions and concepts often I chose the fastest and most efficient (in my opinion) way of defining them. Because of this some concepts may look familiar and bizarre at the same time. Also I tried to include as many examples and expository remarks as possible, Michelin being a highlight, but **the main utility of this handout is due to the index at the end**, where you can find a list of most, if not all, phrases and symbols we might, or might not, admittedly, encounter.

Needless to say, **this is an early draft**. Accordingly typos, mistakes and inconsistencies should not be unexpected. Hence you should use this handout with utmost skepticism, in the sense that if some part doesn't make sense to you, then it is very likely that it doesn't make sense per se. I would be grateful if you could let me know of such problems.

Sets

Definition 1: A **set**[2] is a well-defined collection of well-defined objects.[3]

Example 1:

$$L := \{\text{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}\}$$

is the set of all letters used in English language.

Notation 1 (Calling Names)**:** If we have a "mathematical object", such as a set, and a "name", such as a letter, then

$$\text{"name"} := \text{"mathematical object"}$$

[1] [Hal68, p. 386] "... (Note: the longer formulation is not only awkward, it is also incomplete.)"

[2] **Boldface** denotes the word or phrase that is defined by the sentence it is in, starting with this.

[3] This is a vague definition, as it uses undefined phrases such as "collection" and "well-defined", and indeed using self-referentiality ("Can an omnipotent being create a rock he cannot lift?") one can create paradoxes using only this vague definition. But for our purposes this definition is sufficiently good.

denotes that we will call that particular object that particular name from now on. When we use the same name with $:=$ we forget whatever object that name used to denote.

Notation 2: We use $\overset{\text{def}}{=}$ to signify that said equality is immediately due to the definition of a certain object. Both $:=$ from Not. 1 and $\overset{\text{def}}{=}$ are primarily for book-keeping or better exposition, and in practice they are often replaced by their original version, that is $=$.

Example 2: There are a few famous **number sets**, whose members are numbers:

Natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$.

Integers $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Rational numbers $\mathbb{Q}$ is defined to be the set of all ratios that can be obtained by dividing an integer by a natural number.

Real numbers $\mathbb{R}$ is defined to be the set of all numbers on the line that is drawn for eternity and which contains all integers.

Complex numbers $\mathbb{C}$ is defined to be the set of all points on the plane that is everexpanding and whose x-axis is the real numbers.

Remark 1 (How to define a Set)**:** There are more than one way of defining a set, as one can see in Ex.2:

Listing The most explicit way of defining a set is to list all of its elements, since a set is completely determined by precisely which objects it has as a member. This is how we defined L in Ex.1 for instance. The disadvantage of this is that listing the members of a set is often rather cumbersome.

Incomplete Listing It may very well be the case that the set we are interested in is infinite, e.g. $\mathbb{N}$; or else the number of elements of the set might be big. In such cases we list the elements of the set until we are sick of it, and then put "...". Observe that there is no a priori reason why we don't take the symbol "..." to be one element of the set, instead this symbol is read by convention as "and so on"[4].

Declaring the General Rule Instead of listing all the elements of the set we want to work with, we can also give a general rule that we want the elements of the set to obey. For instance in Ex.1 the list of elements is in fact redundant: we could as well just say that L is the set of all letters used in English language.

Drawing We can describe a set by its geometric shape as well[5].

Notation 3: We use the curly brackets "{" and "}" to denote where a certain set begins and ends, respectively. In this regard $\emptyset := \{\}$ denotes the **empty set**. When listing the elements of a set, reordering and repetitions do not change the set (see Ex.3).

Example 3: $\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\} = \{0, 1, 1, -1, -1, -1, 2, 2, 2, 2, -2, -2, -2, -2, -2, \cdots\}$.

Notation 4: We may use the curly brackets also when we want to define a set by declaring the general rule. In this case write:

$$\{\text{generic object} \quad | \quad \text{general rule each generic object satisfies}\},$$

so that we write the form of a generic object in the set to the left, use a separator, and write the general rule that defines the set to the right.

Example 4: Continuing from Ex.2,

[4] It is said that primitive human civilizations had words for a few "small" numbers, but after them they just used some word corresponding to "many". In this regard we did not go too far.

[5] To talk about the "shape" of a set we actually need further structure on this set, which we call **topology** (τόπος ≃ topos ≃ place, λōγōς ≃ logos ≃ knowledge). For our purposes $\mathbb{R}$ will always be a line and $\mathbb{C}$ will always be a plane. Thus we also call $\mathbb{R}$ the **real line** and $\mathbb{C}$ the **complex plane**.

(i) $\mathbb{Q} = \left\{ \dfrac{p}{q} \;\middle|\; p \text{ is an integer and } q \text{ is a natural number} \right\}$.

(ii) $\mathbb{R} = \longleftrightarrow$

Definition 2: Let S and T be two sets. S is a **subset** of T if every member of S is also a member of T. We denote this by $S \subseteq T$. If both $S \subseteq T$ and $T \subseteq S$, so that

$$S \subseteq T \subseteq S,$$

then S and T are **equal**[6]. If $S \subseteq T$ but $S \neq T$, we say that S is a **proper subset** of T, and we denote this by $S \subsetneq T$.

Definition 3: Let a, b be members of $\mathbb{R}$ such that $a < b$. Then any subset of $\mathbb{R}$ that has one of the following forms is an **interval**:

Open interval $]a,b[:= \{x \mid a < x < b\}$,

Closed interval $[a,b] := \{x \mid a \leq x \leq b\}$,

Open-closed interval $]a,b] := \{x \mid a < x \leq b\}$,

Closed-open interval $[a,b[:= \{x \mid a \leq x < b\}$,

Left-closed ray $[a,\infty[:= \{x \mid a \leq x\}$,

Left-open ray $]a,\infty[:= \{x \mid a < x\}$,

Right-closed ray $]-\infty,b] := \{x \mid x \leq b\}$,

Right-open ray $]-\infty,b[:= \{x \mid x < b\}$.

Often instead of using these specific names we will call such a subset simply an interval and in addition specify its endpoints (if it has any).

Furthermore, sometimes we also use the following notations and call them **degenerate intervals**:

$$]a,a[:= \emptyset, [a,a] := \{a\},]-\infty,\infty[:= \mathbb{R}$$[7].

Remark 2: Observe that Def.2 is the first time we defined what the symbol "=" means when used for sets. One of the common practices we have in mathematics is to use some symbol or concept before actually defining what it means[8]. Also observe that this definition is not that surprising: A set is defined by its elements, hence if two sets share the exact same elements, they must be identical[9].

Notation 5: Any mathematical symbol that is asymmetric in shape, like $:=$ or $\subseteq$, can be used in whichever direction desired, provided that one is careful with the directions. In this regard, equality in Def.2 can also be stated as:

Two sets S, T are equal whenever $S \subseteq T$ and $S \supseteq T$.

Here we will always take a side when introducing new notation to cut down redundant repetition.

Example 5:

(i) If S is a set, then $\emptyset \subseteq S$.

(ii) If S is a set such that $\emptyset \supseteq S$, then $S = \emptyset$.

(iii) If S is a set, then $S \subseteq S$.

(iv) If S is a set, then $S = S$.

Notation 6: If S is a set and s is a member of S, then we denote this by $s \in S$[10].

[6] Observe the similarity of $\subseteq$ to $\leq$ we use for numbers on the real line.

[7] It is more common to use parantheses whenever a square bracket is outwards. For instance, it is more common to see $(0,1)$ instead of $]0,1[$. However the notation we introduced above is much more meaningful geometrically, and there is less chance of confusing an interval with an ordered pair (see Def.10) when consistently using square brackets.

[8] In this regard the similarity of mathematics with regular languages like English or Spanish is apparent: When learning a new language we are often exposed and used to a rule before learning what it means. This is called **circumspectivity**: The more one can understand the whole narrative the easier to guess the meanings of certain words, and vice versa.

[9] Throughout we will always care about whether two mathematical objects are the "same" (in a certain sense). Even though the notion of "sameness" is not that clear-cut in real life, it always will be when we are doing mathematics. This is one of the reasons why mathematics works beautifully.

[10] $\in$ for $\in$l$\in$m$\in$nt.

Notation 7: We put a "/" on a mathematical symbol to denote that it does not hold, e.g., if S is a set and t is not an element of S, we denote this by $t \notin S$.

Example 6:

(i) Continuing from Ex.1, $a \in L$ but $\ddot{a} \notin L$.

(ii) Continuing from Ex.2, $\mathbb{Q} = \{p/q \in \mathbb{R} \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$.

Definition 4: Let S be a set. The set $\mathcal{P}(S)$ of all subsets of S is the **power set** of S[11].

Remark 3: Sets and their elements are strongly related. Indeed, we define a set by its elements, and often we will also differentiate two objects by the sets which they are elements of. Thus Not. 6 hints at yet another common practice: we devise notation for things we use the most[12].

Also observe that a "set" is at somewhat a meta-level when compared to an "object". Even though this is the case, it turns out that considering a "set" as a "(meta-)object" is very fruitful.[13]

Example 7 (Recap of Ex.5)**:**

(i) If S is a set, then $\varnothing \in \mathcal{P}(S)$.

(ii) If S is a set such that $S \in \mathcal{P}(\varnothing)$, then $S = \varnothing$.

(iii) If S is a set, then $S \in \mathcal{P}(S)$.

Example 8: Put $S := \{1,2,3\}$. Then

$$\begin{aligned}
\mathcal{P}(S) &\overset{\text{def}}{=} \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S\}, \\
\mathcal{P}(\mathcal{P}(S)) &\overset{\text{def}}{=} \{\varnothing, \{\varnothing\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1,2\}\}, \{\{1,3\}\}, \{\{2,3\}\}, \{S\}, \{\varnothing, \{1\}\}, \dots, \mathcal{P}(S)\}, \\
\mathcal{P}(\mathcal{P}(\mathcal{P}(S))) &\overset{\text{def}}{=} \{\varnothing, \{\varnothing\}, \dots, \{\{\varnothing\}\}, \dots, \mathcal{P}(\mathcal{P}(S))\}, \\
&\vdots
\end{aligned}$$

Observe that $S \not\subseteq \mathcal{P}(S)$, but $S \in \mathcal{P}(S)$.

Definition 5: Let U[14] be a set, $S, T \subseteq U$. Then we have the following **set operations** to obtain new sets by combining S and T:

Union The **union** $S \cup T$ of S and T is defined to be the smallest subset of U that contains both S and T as subsets, i.e.,

$$S \cup T = \{x \in U \mid x \in S \text{ or } x \in T\}.$$

Intersection The **intersection** $S \cap T$ of S and T is defined to be the largest subset of U that is contained in both S and T as a subset, i.e.,

$$S \cap T = \{x \in U \mid x \in S \text{ and } x \in T\}.$$

[11] $\mathcal{P}$ for $\mathcal{P}$ower.

[12] Considering notation as a piece of "compact technology", this is in parallel to how humanity progresses: In general smaller objects which are capable of doing more things are valued more.

[13] Continuing from footnote 3, such abuse of self-referentiality can be created by considering sets S such that $S \in S$. A priori we don't have any constraints in using $\in$ in this manner, but for our purposes we might as well declare this usage to be illegal, so that if S is a set, then $S \notin S$.

[14] U for Universal. To avoid most, if not all, foundational problems as self-referentiality from footnote 3 and to make things more concrete we often fix an **ambient set** where all objects and subsets are well-defined. For our purposes the ambient set is going to be most commonly $\mathbb{R}$ and sometimes $\mathbb{C}$. When defining sets we declare the ambient set on the left-hand side of the separator, see for instance Ex.6. Ambient sets are often not declared explicitly but the audience is expected to understand what they are from the context. We often make use of a plethora of ambient sets for abstract endeavors, as they allow us to consider power sets (see Def.4) and Cartesian products (see Def.10).

Complement The **complement** $S^{\complement}$ of S is defined to be the subset of U that consists of all elements of U not contained in S, i.e.,

$$S^{\complement} \overset{\text{def}}{=} \{x \in U \mid x \notin S\}.$$

Difference The **set difference** $S \setminus T$ of S and T is defined to be the subset of U that consists of all elements of S not contained in T, i.e.,

$$S \setminus T \overset{\text{def}}{=} \{x \in U \mid x \in S \text{ and } x \notin T\}$$[15].

Remark 4: In mathematics we define a certain class of objects, and oftentimes we would like to have a means of manipulating them. We established in Rem.3 that sets themselves can be considered as mathematical objects, and Def.5 introduces the basic toolbox to manipulate sets as objects.

Example 9: Let U be a set, $S, T \subseteq U$. Then

(i) $S \cup \varnothing = S, \quad S \cup U = U.$

(ii) $S \cap \varnothing = \varnothing, \quad S \cap U = S.$

(iii) $U^{\complement} = \varnothing, \quad \varnothing^{\complement} = U, \quad \left(S^{\complement}\right)^{\complement} = S.$

(iv) $(S \cup T)^{\complement} = S^{\complement} \cap T^{\complement}, \quad (S \cap T)^{\complement} = S^{\complement} \cup T^{\complement}$[16].

(v) $S \setminus T = S \cap T^{\complement}.$

(vi) $S \setminus U = \varnothing, \quad S \setminus \varnothing = S$

(vii) $U \setminus S = S^{\complement}, \quad \varnothing \setminus S = \varnothing.$

Example 10: Let $I := [0,5], J := [-2,1[, S := \{0,5\} \subseteq \mathbb{R}$. Then

(i) $I \cup J = [-2,5].$

(ii) $I \cap S = S.$

(iii) $J \setminus S = [-2,0[\, \cup\,]0,1[.$

(iv) $S \setminus J = \{5\}.$

(v) $(I \cap J) \setminus S = [0,1[\, \setminus S =]0,1[.$

(vi) $S^{\complement} =]-\infty,0[\, \cup\,]0,5[\, \cup\,]5,\infty[$[17].

Logic

Definition 6:

(i) A **truth value** is an element of a set consisting of two elements. We denote this set usually by $\{T, F\}$, $\{\top, \bot\}$, or, if numerical values are needed, by $\{1, 0\}$. In this case $T = \top = 1$ is called **truth** and $F = \bot = 0$ is called **falsehood**.

(ii) A **statement** is a phrase that has a definite truth value. A statement with a truth value of 1 is a **true statement** and a statement with a truth value of 0 is a **false statement**.

(iii) An **open statement** is a phrase with finitely many parameters from a specified set that becomes a statement once its parameters are fixed.

(iv) An open statement that is true for any of its parameters is a **tautology**.

(v) An open statement $P(x)$ that is false for any of its parameters is a **contradiction**.

Example 11:

[15] It's not an accident that the first two equalities in the definition are not $\overset{\text{def}}{=}$. In fact, both for the union and for the intersection two different definitions are stated. In this case these definitions turn out to be the same. Again we use words without defining them properly, viz., "smallest" and "largest".

[16] These are the so-called **De Morgan's Laws**.

[17] Recall footnote 7.

(i) $''5+7=12''$ and $''5+7\neq 12''$ are both statements, where the former is true and the latter is false.

(ii) $P(x) :='' x+7=12''$ is an open statement for real numbers x. $P(5)$ is a true statement, while for any other real number x $P(x)$ is a false statement.

(iii) $P(x) :='' x+7=\frac{25}{2}''$ is an open statement for integers x. It is a contradiction.

(iv) Any statement can be considered to be an open statement with $\emptyset$ as the parameter set.

Remark 5: The reason why we define a truth value to be a member of a set with two elements in Def.6 is that we would like to abide by the so-called **Law of the Excluded Middle**, which states that a statement can only be true (and not false) or false (and not true)[18]. This property has an analog for real numbers, called the **Positivity Axiom**, which dictates that if a and b are any two real numbers, then exactly one of the following can be true: $a-b\geq 0$ or $a-b<0$.

Example for how to use the law of the excluded middle, hard vs soft mathematics a, b such that a^b is irrational.

Notation 8: When writing open statements we usually drop the reference to its supposed truth value. For instance the open statement $''x+7=12''$ stands for $''x+7=12$ is true $''$ (for a predetermined x).

Remark 6 (Extentionality)**:** There is a very intimate relationship between the language of sets and the language of logic, via **extentionality**. To be more precise, let U be a set. Then we have the following two correspondences:

(i) If $S\subseteq U$, then $P_S(x) :='' x\in S''$ is an open statement with U as its parameter set.

(ii) If $P(x)$ is an open statement with U as its parameter set, then $S_P := \{x\in U \mid P(x) \text{ is true}\}$ is a subset of U.

Definition 7: Let U be a set and let $P(x)$ and $Q(x)$ be two open statements with U as their parameter set. $P(x)$ **implies** $Q(x)$ if $S_P\subseteq S_Q$. We denote this by $P(x)\Rightarrow Q(x)$. If $S_P=S_Q$, then $P(x)$ and $Q(x)$ are **equivalent**. In this case we write $P(x)\Leftrightarrow Q(x)$ (or $P(x)\equiv Q(x)$).

Notation 9: Let U be a set and let $P(x)$ and $Q(x)$ be two open statements with U as their parameter set. Suppose $P(x)\Leftrightarrow Q(x)$. Then we also use the following to denote their equivalence:

(i) $P(x)$ **if and only if** $Q(x)$.

(ii) $P(x)$ **iff** $Q(x)$[19].

(iii) $P(x)$ is a **necessary and sufficient condition** for $Q(x)$.

Definition 8 (Logical Def.5)**:** Let U be a set and let $P(x)$ and $Q(x)$ be two open statements with U as their parameter set. Then, using the notation of Rem.6, we have the following **logical operations** to obtain new open statements by combining $P(x)$ and $Q(x)$:

Disjunction The **disjunction** $(P\vee Q)(x)$ of $P(x)$ and $Q(x)$ is defined to be the open statement "$x\in S_P\cup S_Q$", i.e., $(P\vee Q)(x)$ is the open statement such that

$$S_{P\vee Q}=S_P\cup S_Q.$$

Conjunction The **conjunction** $(P\wedge Q)(x)$ of $P(x)$ and $Q(x)$ is defined to be the open statement "$x\in S_P\cap S_Q$", i.e., $(P\wedge Q)(x)$ is the open statement such that

[18] We could have chosen a set with more (or less) than two elements as the set of truth values, though this would result in a different logical framework. For instance we could be infatuated by a particular person, to the extent that her statements defy all logical rules. In that case we could consider a set $\{\top,\bot,\heartsuit\}$ consisting of three members as our set of truth values. In this case surely all interactions still would need to be fleshed out. This is why romantic literature exists.

[19] iff for if(and only i)f.

$$S_{P\wedge Q} = S_P \cap S_Q$$[20].

Negation The **negation** $(\neg P)(x)$ is defined to be the open statement "$x \in {S_P}^{\complement}$", i.e., $(\neg P)(x)$ is the open statement such that

$$S_{\neg P} = {S_P}^{\complement}.$$

Implication The **implication** $(P \Rightarrow Q)(x)$ from $P(x)$ to $Q(x)$ is defined to be the open statement "$x \in (S_P \setminus S_Q)^{\complement}$", i.e., $(P \Rightarrow Q)(x)$ is the open statement such that

$$S_{P\Rightarrow Q} = (S_P \setminus S_Q)^{\complement}.$$

Converse The **converse implication** (or simply **converse**) of $(P \Rightarrow Q)(x)$ is defined to be $(P \Leftarrow Q)(x)$.

Remark 7: Observe that "implies" from Def.7 and "implication" from Def.8 refer to different, but closely related, notions. Namely, the former provides a means to compare two open statements, just as "being a subset of" does for sets; while the latter provides a means to construct a new set from two sets, just as "set difference" does for sets.

Remark 8: Let U be a set and let $P(x)$ and $Q(x)$ be two open statements with U as their parameter set. Then we have the following equivalent expressions:

Disjunction

(i) $(P \vee Q)(x)$.
(ii) "$(P \vee Q)(x)$ is true".
(iii) "At least one of $P(x)$ or Q(x) is true".
(iv) "Either $P(x)$ is true, or $Q(x)$ is true, or both are true".

Conjunction

(i) $(P \wedge Q)(x)$.
(ii) "$(P \wedge Q)(x)$ is true".
(iii) "Both $P(x)$ and $Q(x)$ is true".

Negation

(i) $(\neg P)(x)$.
(ii) "$(\neg P)(x)$ is true".
(iii) " $P(x)$ is not true".
(iv) " $P(x)$ is false".

Implication

(i) $(P \Rightarrow Q)(x)$.
(ii) "$(P \Rightarrow Q)(x)$ is true".
(iii) "Whenever $Q(x)$ is false, it is guaranteed that $P(x)$ is false as well".
(iv) "$Q(x)$ is a **necessary condition** for $P(x)$".
(v) "Whenever $P(x)$ is true, it is guaranteed that $Q(x)$ is true as well".
(vi) "$P(x)$ is a **sufficient condition** for $Q(x)$".

Equivalence

(i) $P(x) \Leftrightarrow Q(x)$.
(ii) "$P(x) \Leftrightarrow Q(x)$ is true".
(iii) "$P(x) \Rightarrow Q(x)$ and $P(x) \Leftarrow Q(x)$".
(iv) "$P(x) \Rightarrow Q(x) \Rightarrow P(x)$".
(v) $[P(x) \Rightarrow Q(x)] \wedge [Q(x) \Rightarrow P(x)]$.

Remark 9: Recall that in Rem.2 we noted the equality of two sets to be precisely having exactly the same elements. This property translates to logic as well. Namely, if we have a set U and two open statements $P(x)$ and $Q(x)$ with U as their parameter set, then $P(x) \Leftrightarrow Q(x)$ is equivalent to stating that for any fixed $x \in U$, the truth values of $P(x)$ and $Q(x)$ coincide. Accordingly, when the parameter set is small (e.g. $\varnothing$, so what we have statements) a common method to see logical equivalence is to make a **truth table**, where all possible combinations of truth values are listed (see Ex.12).

[20] As our main concern is not logic as an intellectual discipline, and the symbols $\vee$ and $\wedge$ are used in different mathematical contexts as well, we will usually refrain from using them, and instead use "or" and "and", respectively. Furthermore, whenever we use the words "or" and "and", we will use them with logical precision in mind. In fact we have never been doing otherwise throughout these notes already. Also observe the similarities between $\vee$ and $\cup$; and between $\wedge$ and $\cap$ (See Rem.12)

Example 12: Let P and Q be statements. Then

(i) $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$:

P	$\neg P$	Q	$P \Rightarrow Q$	$(\neg P) \vee Q$
1	0	1	$(1 \Rightarrow 1) \equiv 1$	$(0 \vee 1) \equiv 1$
1	0	0	$(1 \Rightarrow 0) \equiv 0$	$(0 \vee 0) \equiv 0$
0	1	1	$(0 \Rightarrow 1) \equiv 1$	$(1 \vee 1) \equiv 1$
0	1	0	$(0 \Rightarrow 0) \equiv 1$	$(1 \vee 0) \equiv 1$

(ii) $P \Rightarrow Q$ is equivalent to $(\neg Q) \Rightarrow (\neg P)$:

P	$\neg P$	Q	$\neg Q$	$P \Rightarrow Q$	$(\neg Q) \Rightarrow (\neg P)$
1	0	1	0	$(1 \Rightarrow 1) \equiv 1$	$(0 \Rightarrow 0) \equiv 1$
1	0	0	1	$(1 \Rightarrow 0) \equiv 0$	$(1 \Rightarrow 0) \equiv 0$
0	1	1	0	$(0 \Rightarrow 1) \equiv 1$	$(0 \Rightarrow 1) \equiv 1$
0	1	0	1	$(0 \Rightarrow 0) \equiv 1$	$(1 \Rightarrow 1) \equiv 1$

Remark 10 (Common Proof Methods)**:** There are a few proof methods induced by certain logical forms. Although there are other proof methods, these are the most universal. Let U be a set, $\text{Hyp}(x)$ and $\text{Res}(x)$ be two open statements with U as their parameter set[21]. Suppose we are trying to prove that for a fixed x, $(\text{Hyp} \Rightarrow \text{Res})(x)$ is a true statement. Then we have the following options:

Direct Proof Assume that $\text{Hyp}(x)$ is true and show (by utilizing extra properties provided by the mathematical structure we have at hand) that $\text{Res}(x)$ must be true as well, i.e., make use of the following equivalence:

$$(\text{Hyp} \Rightarrow \text{Res})(x) \Leftrightarrow [\text{Hyp}(x) \Rightarrow \text{Res}(x)].$$

Proof by Contrapositive Assume that $\text{Res}(x)$ is false and show (by utilizing extra properties provided by the mathematical structure we have at hand) that $\text{Hyp}(x)$ must be false as well, i.e., make use of the following equivalence:

$$(\text{Hyp} \Rightarrow \text{Res})(x) \Leftrightarrow [(\neg\,\text{Res})(x) \Rightarrow (\neg\,\text{Hyp})(x)].$$

Proof by Contradiction Assume that $\text{Hyp}(x)$ is true, but (that is, and) $\text{Res}(x)$ is false and reach to a falsehood (by utilizing extra properties provided by the mathematical structure we have at hand), i.e. make use of the following equivalence:

$$(\text{Hyp} \Rightarrow \text{Res})(x) \Leftrightarrow \big[[\text{Hyp}(x) \wedge (\neg\,\text{Res})(x)] \Rightarrow \perp\big].$$

Definition 9: We would also like to determine the abundance of parameters of an open statement for which we have 1 as a truth value. For this we introduce **logical quantifiers**:

Universal Quantifier $\forall$ is the **universal quantifier**; it stands for "for any" or "for all"[22].

Existential Quantifier $\exists$ is the **existential quantifier**; it stands for "there exists" or "there is"[23].

Uniquely Existential Quantifier $\exists!$ is the **uniquely existential quantifier**; it stands for "there exists a unique" or "there is one and only one". (See Ex.13 for the usage of these quantifiers.)[24]

Example 13:

(i) $P(x) :='' \forall x \in \mathbb{C} : x = x''$ is a true statement, as any complex number is equal to itself.

(ii) $P(x) :='' \forall x \in \mathbb{R} : x \leq x''$ is a true statement, as any real number is less than or equal to itself.

(iii) $P(x) :='' \exists x \in \mathbb{R} : x \leq x''$ is a true statement, as since any real number is less than or equal to itself, then surely there is some real number less than or equal to itself, for instance 0.

[21] Hyp for Hypothesis and Res for Result.
[22] $\forall$ for "for $\forall$ll".
[23] $\exists$ for "th$\exists$r$\exists$ $\exists$xists".
[24] Continuing from footnote 20, even though we will occasionally use logical quantifiers, whenever we use their verbal versions we will be using them with logical precision. Again this is what we have been doing already.

(iv) $P(x) :='' \exists! x \in \mathbb{R} : x \leq x''$ is a false statement, as there are more than one real number that is less than or equal to itself.

(v) $P(x,y) :='' \forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y''$ is a true statement, as can be seen by putting $y := x+1$.

(vi) $P(x,y) :='' \exists y \in \mathbb{R}, \forall x \in \mathbb{R} : x < y''$ is a false statement, since if we fix $y \in \mathbb{R}$, then $x := y+1$ too is a real number, but it is not the case that $x < y$.

(vii) $P(x,y) :='' \forall x \in \mathbb{R}, \forall y \in \mathbb{R} : x^2 - y^2 = (x-y)(x+y)''$ is a true statement, and it is equivalent to $''\forall y \in \mathbb{R}, \forall x \in \mathbb{R} : x^2 - y^2 = (x-y)(x+y)''$.

(viii) $P(x,y) :='' \exists x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y > 0''$ is a true statement, and it is equivalent to $''\exists y \in \mathbb{R}, \exists x \in \mathbb{R} : x + y > 0''$.

Remark 11: Continuing from Ex.13,

(i) Logical quantifiers of the same type **commute**, i.e., the order in which they are used does not matter, but logical quantifiers of different type do not commute in general. Because of this, whenever we are using parameters that are from the same set and have the same logical quantifier, we will drop redundant repetitions. For instance we will write

$$''\forall x, y \in \mathbb{R} : [\cdots]'' \text{ instead of } ''\forall x \in \mathbb{R}, \forall y \in \mathbb{R} : [\cdots]''.$$

(ii) Any statement containing logical quantifiers can be paraphrased in such a way that all the logical quantifiers are in the beginning of the statement[25].

(iii) Let P be a statement containing logical quantifiers in the beginning and an open statement at the end, i.e.,

$$P :='' L_1 x_1 \in S_1, L_2 x_2 \in S_2, \ldots, L_n x_n \in S_n : P(x_1, x_2, \ldots, x_n)'',$$

where $\forall k \in \{1, 2, \ldots, n\} : L_k \in \{\forall, \exists\}$. Then we have the following **distribution property** of negation:

$$\neg P ='' \overline{L}_1 x_1 \in S_1, \overline{L}_2 x_2 \in S_2, \ldots, \overline{L}_n x_n \in S_n : \neg P(x_1, x_2, \ldots, x_n)'',$$

where $\forall k \in \{1, 2, \ldots, n\} : \overline{L}_k \in \{\forall, \exists\} \setminus \{L_k\}$.

(iv) Let U be a set and $P(x)$ be an open statement with parameter set U. Then

$$''\forall x \in U : P(x) \equiv \top'' \Rightarrow \text{“}\exists x \in U : P(x) \equiv \top''.$$

(v) Let U be a set and $P(x)$ be an open statement with parameter set U. Then

$$''\exists! x \in U : P(x) \equiv \top'' \Leftrightarrow \text{“}\exists x \in U : \big[P(x) \wedge [\forall y \in U : P(y) \equiv \top \Rightarrow y = x]\big]''.$$

Remark 12: Recall that in Rem.6 we established the relation between the language of sets and the language of statements. Further, observe that logical quantifiers provide a means to construct new statements from an open statement by combining its parameters in a particular manner. Accordingly, logical quantifiers and set operations have a close relationship as well. To be more precise, let U be a set, $\{S_\alpha\}_{\alpha \in A} \subseteq \mathcal{P}(U)$ be a collection of subsets of U[26]. Then

(i) $\forall x \in U : [\forall \alpha \in A : x \in S_\alpha] \Leftrightarrow x \in \bigcap_{\alpha \in A} S_\alpha$.

(ii) $\forall x \in U : [\exists \alpha \in A : x \in S_\alpha] \Leftrightarrow x \in \bigcup_{\alpha \in A} S_\alpha$.

[25] [End01, pp. 288-289] This is known as the Skolem Normal Form Theorem.

[26] Here A is an arbitrary set which we use as a set of indices for our collection. For instance A can be $\{1, 2, \ldots, n\}$, for a fixed natura number n, $\mathbb{N}$, $[0,1]$, $\mathbb{R}$, or something completely abstract. For α see footnote 34.

• Hence in total we have the following associations:

$$\cap \simeq \text{"and"} \simeq \wedge \simeq \forall \qquad \cup \simeq \text{"or"} \simeq \vee \simeq \exists.$$

Example 14:

(i) $[0,1] = \bigcap_{n\geq 1} \left]-\frac{1}{n}, 1+\frac{1}{n}\right[$

(ii) $]0,1[= \bigcup_{n\geq 2} \left[\frac{1}{n}, 1-\frac{1}{n}\right]$

(iii) $\{0\} = \bigcap_{n\geq 1} \left[0, \frac{1}{n}\right] = \bigcap_{n\geq 1} \left]-\frac{1}{n}, \frac{1}{n}\right[$

(iv) $\varnothing = \bigcap_{n\geq 1} [n, \infty[= \bigcap_{n\geq 1}]n, \infty[= \bigcap_{n\geq 1} \left]0, \frac{1}{n}\right[$.

(v) $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [-n, n] = \bigcup_{n\geq\mathbb{Z}} [n, n+1[$.

Remark 13: All equalities in Ex.14 collectively are called the **Archimedean Property** of $\mathbb{R}$. Two most conventional forms of the Archimedean Principle are as follows:

(i) $\forall r > 0, \exists n \in \mathbb{N} : r < n$.

(ii) $\forall r > 0, \exists n \in \mathbb{N} : \frac{1}{n} < r$.

Relations & Functions

Definition 10: Let U be a set, $S, T \subseteq U$ both be nonempty. Then we define a new set by

$$U^2 := \{(u_1, u_2) \mid u_1, u_2 \in U\}$$

and call it the **Cartesian square** of U (or simply U**-squared**). $(u_1, u_2) \in U^2$ is an **ordered pair** of elements of U. Likewise we define the **Cartesian product** of S and T to be

$$S \times T := \{(s,t) \in U^2 \mid s \in S, t \in T\}^{27}.$$

Remark 14:

(i) Observe that in Def.10 we are actually modifying our ambient space[28] to obtain a new set.

(ii) $U^2 \overset{\text{def}}{=} U \times U$[29].

(iii) Similarly we can define

$$\begin{aligned}
U^3 &:= \{(u_1, u_2, u_3) \mid u_1, u_2, u_3 \in U\}, \\
U^4 &:= \{(u_1, u_2, u_3, u_4) \mid u_1, u_2, u_3, u_4 \in U\}, \\
&\vdots \\
U^n &:= \{(u_1, u_2, \cdots, u_n) \mid u_1, u_2, \ldots, u_n \in U\} \\
&\vdots
\end{aligned}$$

and call (u_1, u_2, u_3) an **ordered triple**, (u_1, u_2, u_3, u_4) an **ordered quadruple**,$(u_1, u_2, \cdots, u_n)$ an **ordered n-tuple** and what have you.

[27] Both of these names are in honor of René Descartes.

[28] See footnote 14.

[29] We would like to consider $\mathbb{R}^2$ as if it is obtained by crossing two copies of the real line, so that $\mathbb{R}^2$ signifies a plane geometrically, as does $\mathbb{C}$. However it is usually better not to identify them, as $\mathbb{C}$ has more structure than $\mathbb{R}^2$ has, for instance the positive square root $\imath$ of -1 is an element of $\mathbb{C}$ but not of $\mathbb{R}$. For this reason we will distinguise these two planes by calling $\mathbb{R}^2$ the **Euclidean plane**.

Definition 11: Let U be a set.

(i) Any subset of U^2 is a **relation** on U.

(ii) Let $R \in \mathcal{P}(U^2)$ be a relation, $(u_1, u_2) \in R$. Then we say that u_1 is R**-related** to u_2 (or R **relates** u_1 to u_2, or u_1 is **related** to u_2 via R). In this case we also use the following pieces of notation:

$$(u_1, u_2) \in R \quad , \quad R(u_1) = u_2 \quad , \quad u_1 R u_2 \quad , \quad u_1 \sim_R u_2, u_1 \sim u_2.$$

Example 15: Each one of the following defines a relation on $\mathbb{R}$ (i.e. is a subset of $\mathbb{R}^2$):

Circle $x \sim y \Leftrightarrow x^2 + y^2 = 1$. This relation contains the ordered pair (x, y) exactly when (x, y) lies on the unit circle centered at the origin.

x**-axis** $x \sim y \Leftrightarrow y = 0$. This relation contains the ordered pair (x, y) exactly when (x, y) lies on the x-axis. Observe that in this case we have no constraints on x, hence we include the totality of the x-axis.

y**-axis** $x \sim y \Leftrightarrow x = 0$. This relation contains the ordered pair (x, y) exactly when (x, y) lies on the y-axis. Similar to the previous case, in this case we have no constraints on y, hence we include the totality of the y-axis.

Main Diagonal and Below $x \sim y \Leftrightarrow x \leq y$. The main diagonal $y = x$ divides the Euclidean plane $\mathbb{R}^2$ into two parts. The relation at hand consists precisely of the lower half.

Cross $x \sim y \Leftrightarrow |x| = |y|$. This relation contains the ordered pair (x, y) exactly when the distances of x and y to the origin turn out to be the same. This means that when written in set notation, we can denote this set by $\{(x, x) \mid x \in \mathbb{R}\} \cup \{(x, -x) \mid x \in \mathbb{R}\}$. Thus this relation consists precisely of a cross centered at the origin.

Example 16 (Symbols as Relations)**:** Here are some of the familiar symbols we have already been using, only reinterpreted as relations. For the sake of some psychological ease exclusively to this example we will denote the same relation both by R and Rel_R.

(i) Define $\mathrm{Rel}_{\leq} \subseteq \mathbb{R}^2$ by $(x, y) \in \mathrm{Rel}_{\leq} \;\Leftrightarrow\; x \leq y$. $\mathrm{Rel}_{\leq}$ is the **"less than or equal to" relation**.

(ii) Define $\mathrm{Rel}_{>} \subseteq \mathbb{R}^2$ by $(x, y) \in \mathrm{Rel}_{>} \;\Leftrightarrow\; x > y$. $\mathrm{Rel}_{>}$ is the **"greater than" relation**.

(iii) Define $\mathrm{Rel}_{=} \subseteq \mathbb{C}^2$ by $(z, w) \in \mathrm{Rel}_{=} \;\Leftrightarrow\; z = w$. $\mathrm{Rel}_{=}$ is the **"equal to" relation**.

(iv) Let U be a set, define $\mathrm{Rel}_{\subseteq} \subseteq (\mathcal{P}(U))^2$ by $(S, T) \in \mathrm{Rel}_{\subseteq} \;\Leftrightarrow\; S \subseteq T$. $\mathrm{Rel}_{\subseteq}$ is the **"subset of" relation**[30].

Example 17: Here are some other popular relations, used for creating new objects out of old ones.

Making Donut Shells Let $x, y \in \mathbb{R}^2$ and define $x \sim y \Leftrightarrow x - y \in \mathbb{Z}^2$. This relation is used to make a donut shell (surface of a donut) out of the Euclidean plane $\mathbb{R}^2$. This **donut shell** (or **2-torus**) is denoted by $\mathbb{R}^2 / \mathbb{Z}^2$.

Making Circles Let $x, y \in \mathbb{R}$ and define $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. This relation is used to make a circle out of the real line $\mathbb{R}$. This **circle** (or **1-torus**) is denoted by $\mathbb{R} / \mathbb{Z}$.

Modular Arithmetic Fix $n \in \mathbb{N}$, let $x, y \in \mathbb{Z}$ and define $x \sim y \Leftrightarrow$ "$x - y$ is divisible by n". This is the framework of counting in repetition, which we do for instance for time[31].

Definition 12: Let U be a set and $S, T \subseteq U$.

(i) Any $R \subseteq S \times T$ is a **relation** from S to T.

[30] Disregarding footnote 13 one can give a natural meaning to $\mathrm{Rel}_{\in}$ as the **"element of" relation** as well.

[31] For someone whose memory resets every 24 hours, so that he has no perception of a new day, counting 15 hours forwards from 10 o'clock is the same as counting 9 hours backwards from 10 o'clock, as both result in 1 o'clock (Here we fixed $n := 24$).

(ii) A relation $f \subseteq S \times T$ is called a **function** (or **map**) if $\forall s \in S, \exists! t \in T : f(s) = t$.

(iii) If $f \subseteq S \times T$ is a function, $\operatorname{dom} f := S$ is the **domain** of f and $\operatorname{cod} f := T$ is the **codomain** (or **target**) of f.

(iv) If $f \subseteq S \times T$ is a function, we use the following notation more often than the notations introduced for relations:

$$\begin{aligned} f\colon S &\to T \\ s &\mapsto t = f(s), \end{aligned}$$

where $\to$ signifies the domain and codomain of f and $\mapsto$ signifies the pointwise rule of f.

Maybe include the possibility of f not being defined everywhere on S.

Remark 15:

(i) In Def.12.(ii), the definition of a function contains the following three pieces of information:

- f takes inputs from the set S and its outputs are in the set T.
- f is a relation that relates any member of S to at least one member of T.
- f is a relation that relates a member of S to at most one member of T.

(ii) The definition of a function we introduced in Def.12 defines a function by identifying it with its **graph**, i.e., we identify a function $f : S \to T$ with

$$\Gamma_f := \{(s, f(s)) \in U^2 \mid s \in S\}^{32}.$$

Example 18 (Michelin)**:** Suppose we are interested in the procedure of grading different restaurants by a picky food critic. For concreteness let's call our food critic Jacques, and suppose that when Jacques goes to a restaurant he takes into account a few factors, gives each such factor a numerical value varying from 0 to 10, then adds them all up to obtain a single numerical value, and finally translates that single numerical value into a grade ranging from ★ to ★★★★★. For instance those factors could be

- A: the location of the restaurant and the general atmosphere, whether or not the tables are well organized etc.,
- S: how good a service the restaurant provides, responsiveness of the waiter etc.,
- C: how clean the restaurant and the kitchen are, whether or not they conform to hygiene standards,
- L: how appetizing the plates look when first brought to table,
- T: how delicious the meals are, whether or not they are served at the optimum temperature etc..

Next, suppose the star cutoffs are as follows:

- ★★★★★, if the total score is more than 47.
- ★★★★, if the total score is more than 43 but not more than 47.
- ★★★, if the total score is more than 40 but not more than 43.
- ★★, if the total score is more than 35 but not more than 40.
- ★, if the total score is not more than 35.

[32] Γ ("gamma") is the Greek analog of "G", hence Γ for Γraph.

Then we can consider Jacques grading restaurants to be a function, which we will denote by j. The domain $\operatorname{dom} j$ of j is the set of all restaurants in the world, i.e.,

$$\operatorname{dom} j = ᚱ := \{R \mid R \text{ is a restaurant on Earth}\}^{33}$$

and the codomain of j is

$$\operatorname{cod} j = \{\bigstar, \bigstar\bigstar, \bigstar\bigstar\bigstar, \bigstar\bigstar\bigstar\bigstar, \bigstar\bigstar\bigstar\bigstar\bigstar\},$$

though Jacques being a picky critic, whenever he goes to a restaurant he makes sure that he deduces at least 3 points, so that he never gives five stars as a personal principle. That said, he is an experienced food critic, and accordingly he uses all other grades frequently. Thus the image of j is

$$\operatorname{im} j = \{\bigstar, \bigstar\bigstar, \bigstar\bigstar\bigstar, \bigstar\bigstar\bigstar\bigstar\}.$$

Example 19:

First Letter Take S to be the set of all alphabets and T to be the set of all letters of all alphabets. Then $f : S \to T$ that takes an alphabet to its first letter is a function. For instance we have:

$$f : \text{Greek} \mapsto \alpha, \text{Latin} \mapsto \text{a}, \text{Hebrew} \mapsto \aleph, \text{Futhark} \mapsto ᚠ^{34}.$$

Basic Arithmetic All four basic operations $+, -, \times, \div$ are functions between conveniently chosen sets. For instance we can consider addition $+$ as $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. When denoting these functions, instead of writing $+(x, y)$ we write $x + y$, so that we denote these as if these were mere relations.

Identity Let S be a set and $\mathrm{id}_S : S \to S, s \mapsto s$. Then id_S is a function, called the **identity function** of S. Observe that when $S := \mathbb{R}$, the graph $\Gamma_{\mathrm{id}_\mathbb{R}}$ of $\mathrm{id}_\mathbb{R}$ coincides with the main diagonal $y = x$ of the Euclidean plane $\mathbb{R}^2$.

Reciprocal The function $\mathbb{R} \setminus \{0\} \to \mathbb{R}, x \mapsto \dfrac{1}{x}$ is the **reciprocal function**.

Absolute Value Absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}, x \mapsto |x|$ is a function. Indeed, any real number has an absolute value, and any real number admits exactly one nonnegative real number as its absolute value.

Open Balls $B : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathcal{P}(\mathbb{R}), (x, r) \mapsto \{y \in \mathbb{R} \mid |x - y| < r\}$ is the function that takes a point x on the real line and a positive value r and sends them to the open interval $]x - r, x + r[$ centered at x with radius r. B is the **open ball function**.

Closed Balls $\overline{B} : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathcal{P}(\mathbb{R}), (x, r) \mapsto \{y \in \mathbb{R} \mid |x - y| \leq r\}$ is the function that takes a point x on the real line and a positive value r and sends them to the closed interval $[x - r, x + r]$ centered at x with radius r. B is the **closed ball function**.

Spheres $S^0 : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathcal{P}(\mathbb{R}), (x, r) \mapsto \{y \in \mathbb{R} \mid |x - y| = r\}$ is the function that takes a point x on the real line and a positive value r and sends them to the set $\{x - r, x + r\}$ which consists of all points on the real line that is r distant to x. S^0 is the **sphere function**.

other examples of functions from math 22: polynomial functions, rational functions, absolute value, reciprocal function, exponential and logarithmic functions.

Remark 16: Functions are closely related to sets, and hence to open statements. To be more precise, let U be a set, $S \subseteq U$, $u \in U$.

[33] As opposed to, say, at the end of the universe.

[34] Here α ("alpha") is the Greek analog and $\aleph$ ("aleph") is the Hebrew analog of "a". ᚠ ("fehu") is the runic analog of "f".

(i) The function defined by

$$\begin{aligned} \chi_S \colon U &\to \{0,1\} \\ t &\mapsto \begin{Bmatrix} 1, \text{ if } t \in S \\ 0, \text{ if } t \notin S \end{Bmatrix} \end{aligned}$$

is the **characteristic function** (or **indicator function**) of S[35].

(ii) Instead of focusing on the set S we could have focused on the point u and defined by

$$\begin{aligned} \delta_u \colon \mathcal{P}(U) &\to \{0,1\} \\ T &\mapsto \begin{Bmatrix} 1, \text{ if } u \in T \\ 0, \text{ if } u \notin T \end{Bmatrix} \end{aligned}$$

the **point mass** (or **Dirac delta**) at u[36].

• Observe that in this way we could also view open statements as functions into the set of truth values.

Definition 13: Let $f : S \to T$ be a function. Then we have two induced functions between the power sets of S and T:

(i) $f_* : \mathcal{P}(S) \to \mathcal{P}(T), A \mapsto \{f(s) \in T \mid s \in A\}$ is the **image function** (or **pushforward in SET**) associated to f.

(ii) $f^* : \mathcal{P}(T) \to \mathcal{P}(S), B \mapsto \{s \in S \mid f(s) \in B\}$ is the **preimage function** (or **pullback in SET**) associated to f.

(iii) $\operatorname{im} f := f_*(S)$ is the **image** (or **range**) of f.

Notation 10: Usually we abuse the notation when denoting functions introduced in Def.13 and write $f := f_*$ and $f^{-1} := f^*$. However the latter does not mean that f has an inverse (see Def.16).

Remark 17:

(i) Observe that denoting a function as $f : S \to T$, where S and T are two sets, means that $\operatorname{dom} f = S$, i.e., the expression $f(s)$ has a determined meaning for any member s of S. But this notation does not guarantee that $\operatorname{im} f = T$ (see Rem.18). Indeed, in general we only have $\operatorname{im} f \subseteq T \overset{\text{def}}{=} \operatorname{cod} f$. For instance we could have introduced the absolute value function from Ex.19 by

$$\begin{aligned} |\cdot| \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|, \end{aligned}$$

even though it is the case that $\operatorname{im} |\cdot| = \mathbb{R}_{\geq 0}$.

(ii) Continuing from the first part of this remark, the image of a function gives a lower bound on how much we can shrink the codomain of it, that is, if $f : S \to T$ is a function, then T cannot be a set that does not contain $\operatorname{im} f$. If we had to consider a smaller T as the codomain, we would have to change the domain S accordingly as well, which technically means to consider a function different from the one we started with. For instance if we wanted to consider the absolute value function having the unit interval $[0,1]$ as its codomain, this would mean that we want $|x|$ to make sense as a nonnegative number not greater than 1. Thus we would have to restrict the domain to some subset of $[-1,1]$. If we wanted $[0,1]$ to be not only the codomain but also the image of the absolute value function, we would have to take exactly $[-1,1]$ as its domain. In such cases, to remind ourselves that we are considering

[35] χ ("chi") is the Greek analog of "ch", hence χ for χaracheristic.
[36] δ ("delta") is the Greek analog of d, it is used here to honor Paul Dirac.

a function $f : S \to T$ on a subset A of its domain S, we put a bar after the name of the function and note the subset we consider, i.e. we write $f|_A : A \to T$. Observe that we can leave the codomain T unchanged as we have extensive freedom regarding the codomain of a function. For instance we could consider $|\cdot||_{[-1,1]\cap\mathbb{Q}} : [-1,1]\cap\mathbb{Q} \to [0,1]$, which has domain $[-1,1]\cap\mathbb{Q}$, codomain $[0,1]$ and image $[0,1]\cap\mathbb{Q}$.

Definition 14: Let $f : S \to T$ be a function. Then

(i) f is **injective** (or **1:1**) if $\forall s_1, s_2 \in S : f(s_1) = f(s_2) \Rightarrow s_1 = s_2$.

(ii) f is **surjective** (or **onto**) if $\forall t \in T, \exists s \in S : f(s) = t$.

(iii) f is **bijective** if it is both injective and surjective[37].

Remark 18: Recall that in Rem.15 we emphasized that the definition of a function contains three pieces of information:

- $\operatorname{dom} f = S, \operatorname{cod} f = T$.
- $\operatorname{im} f \subseteq T$.
- $\forall s_1, s_2 \in S : s_1 = s_2 \Rightarrow f(s_1) = f(s_2)$.

Among these, surjectivity and injectivity introduced in Def.14 address strengthen the claims of the second and third, respectively:

- f is surjective iff $\operatorname{im} f = T$.
- f is injective iff $\forall s_1, s_2 \in S : s_1 = s_2 \Leftrightarrow f(s_1) = f(s_2)$.

Definition 15: Let $f : S \to T$ and $g : T \to U$ be two functions. Then $g \circ f : S \to U$ is a function, called the **composition** of g and f.

Definition 16: Let $f : S \to T$ be a function. If there is a function $g : T \to S$ such that $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$, then $f^{-1} := g$ is called the **inverse** of f. In this case f is called **invertible**.

Remark 19: Let $f : S \to T$ be a function. Then

(i) f is invertible iff it is bijective.

(ii) If f is invertible, then so is f^{-1}, with $(f^{-1})^{-1} = f$.

(iii) If f is invertible, then f^{-1} is unique.

Remark 20: Let $f : S \to T$ and $g : T \to U$ be two functions. Then

(i) $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

(ii) $\operatorname{im}(g \circ f) = g(\operatorname{im} f)$[38].

Example 20: Continuing from Ex.18, observe that we can write j as a composition of three functions:

$$\begin{array}{cccccccc} j: & \mathsf{R} & \to & \{0,1,\ldots,10\}^5 & \to & \{0,1,\ldots,50\} & \to & \{\bigstar, \bigstar\bigstar, \bigstar\bigstar\bigstar, \bigstar\bigstar\bigstar\bigstar, \bigstar\bigstar\bigstar\bigstar\bigstar\} \\ & R & \mapsto & (A,S,C,L,T) & \mapsto & A+S+C+L+T & \mapsto & \text{star value based on the cutoff} \end{array}$$

[37]We also use the following nouns instead of adjectives from time to time: **injection** for an injective function, **surjection** for a surjective function, and **bijection** for a bijective function.

[38]See Not. 10.

Index

References

[CPZ08] Chartrand, G.; Polimeni, A. D.; Zhang, P.. *Mathematical Proofs: a Transition to Advanced Mathematics*, 2e. Pearson International Edition. Pearson. 2008.

[End01] Enderton, H. B.. *A Mathematical Introduction to Logic*, 2e. Harcourt Academic Press. 2001.

[Hal68] Halmos, P. R.. "Mathematics as a Creative Art". *American Scientist*, Vol. 56, No. 4. 1968. pp. 375-389.

[Mac86] Mac Lane, S.. *Mathematics Form and Function*. Springer. 1986.