Katok–Rodríguez Hertz Arithmeticity for Maximal Rank Positive Entropy Actions of $\mathbb{R}^k$.

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Penn State 2021 Workshop in Dynamical Systems and Related Topics

Alp Uzman (Penn State University) *Katok-Rodríguez Hertz Arithmeticity for Maximal Rank Positive Entropy Actions of* $\mathbb{R}^k$. We present a work in progress on an extension of an arithmeticity theorem of A. Katok and F. Rodríguez Hertz. They showed that [J. Mod. Dyn. 10 (2016), 135–172; MR3503686] given a C^{1+} maximal rank positive entropy action of $\mathbb{Z}^k$ ($k \in \mathbb{Z}_{\geq 2}$), the manifold acted upon can be decomposed into finitely many components in such a way that the action restricted to any one of these components and the finite index subgroup fixing said component is measure theoretically isomorphic to an algebraic action, and the isomorphism has certain smoothness properties compatible with the dynamical foliations. We extend their machinery to locally free C^{1+} maximal rank positive entropy actions of $\mathbb{R}^k$ ($k \in \mathbb{Z}_{\geq 2}$), and show that any such action comes from suspending an algebraic maximal rank positive entropy action of $\mathbb{Z}^k$ up to measurable time change. This in particular solves a problem in a prequel paper of Katok and Rodríguez Hertz, joint with B. Kalinin [Ann. of Math. (2) 174 (2011), no. 1, 361–400; MR2811602].

MR3503686 Reviewed
Katok, Anatole (1-PAS); Rodriguez Hertz, Federico (1-PAS)
Arithmeticity and topology of smooth actions of higher rank abelian groups.
J. Mod. Dyn. 10 (2016), 135–172.

1.1. **The arithmeticity theorem.**

THEOREM 1. *For $r = 1+\theta$, $0<\theta<1$, or $r \geq 2$ an integer, let α be a C^r maximal rank positive entropy action on a smooth manifold M of dimension $m \geq 3$. Then there exist*

- *disjoint measurable sets of equal measure $R_1, \dots, R_n \subset M$ such that $R = \bigcup_{i=1}^n R_i$ has full measure and the action α cyclically interchanges those sets. Let $\Gamma \subset \mathbb{Z}^{m-1}$ be the stationary subgroup of any of the sets R_i (Γ is of course isomorphic to $\mathbb{Z}^{m-1}$);*
- *a Cartan action α_0 of Γ by affine transformations of either the torus $\mathbb{T}^m$ or the infratorus $\mathbb{T}^m_\pm$ that we will call* the algebraic model*;*
- *measurable maps $h_i : R_i \to \mathbb{T}^m$ or $h_i : R_i \to \mathbb{T}^m_\pm$, $i = 1, \dots, n$;*

such that

(1) *h_i is bijective almost everywhere and $(h_i)_*\mu = \lambda$, the Lebesgue (Haar) measure on $\mathbb{T}^m$ (correspondingly $\mathbb{T}^m_\pm$);*
(2) *$\alpha_0 \circ h_i = h_i \circ \alpha\restriction_\Gamma$;*
(3) *for almost every $x \in M$ and every $\mathbf{n} \in \mathbb{Z}^{\mathbf{m-1}}$ the restriction of h_i to the stable manifold W^s_x of x with respect to $\alpha(\mathbf{n})$ is a C^r diffeomorphism;*
(4) *h_i is $C^{r-\epsilon}$ in the sense of Whitney on a set whose complement to R_i has arbitrarily small measure; those sets will be described in the course of proof; in particular, they are saturated by local stable manifolds.*

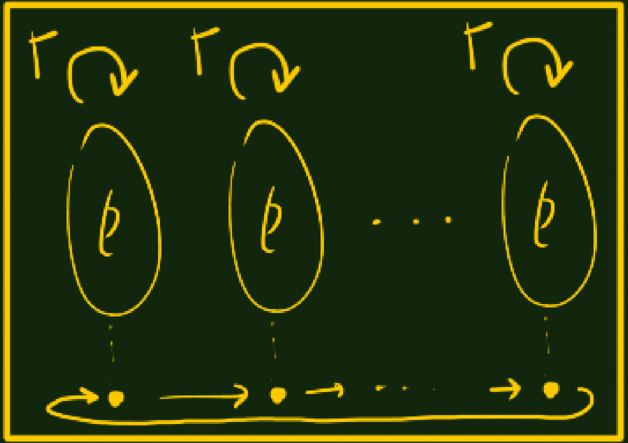
(M compact or $T\alpha \in \log(\mathbb{Z}^k, 1)$.
μ ergodic α-inv Borel prob.)

$\alpha : \mathbb{Z}^k \to \mathrm{Diff}^r(M^d)$

MRPEA $\begin{cases} d = k+1, \; k \geq 2, \; r = 1+\theta \text{ or } r \geq 2 \\ \mathrm{ent}_\mu(\alpha_t) = 0 \Rightarrow t = 0 \end{cases}$ "hyper"

$\Rightarrow$ $\boxed{\Gamma \overset{\text{aff}}{\curvearrowright} \mathbb{T}^d \text{ Cartan}} \to \boxed{\mathbb{Z}^k \overset{\alpha}{\curvearrowright} M^d \text{ MRPEA}} \to \boxed{\text{finite} \curvearrowright \text{finite cyclic Kronecker}}$

$\boxed{\alpha : \mathbb{Z}^k \curvearrowright M^d} \cong$



Note: The object under study really is the pair (μ, α).

Def: For M a C^∞ manifold, define the smooth ergodic theory $\mathcal{E}^{1+}(\mathbb{R}^k \curvearrowright M)$ of $\mathbb{R}^k$ actions on it by:

$$\mathcal{E}^{1+}(\mathbb{R}^k \curvearrowright M) = \{(\mu, \alpha)\}.$$

$$\mathcal{E}^{1+}(\mathbb{R}^k \curvearrowright M)$$

$$\swarrow \qquad \searrow$$

$$\left\{\begin{matrix}\text{Borel proba-}\\ \text{bilities on } M\end{matrix}\right\} \qquad \left\{\begin{matrix}C^{1+} \text{ actions}\\ \mathbb{R}^k \curvearrowright M\end{matrix}\right\}$$

Aim: Extend this to locally free MRPE actions

$$\alpha : \mathbb{R}^k \to \mathrm{Diff}^r(M^d).$$

Obs: α locally free $\Rightarrow$ $\exists$ orbit foliation

$\Rightarrow$ $\exists$ k orbital Lyapunov exponents that are automatically zero.

Def: An l.f. $\alpha : \mathbb{R}^k \curvearrowright M^d$ is an MRPEA if $d - k = k + 1$,

and $\mathrm{ent}_\mu(\alpha_t) = 0 \Rightarrow t = 0$.

"hyper" = l.f. & MRPEA

- Ignore orbital Lyap. exponents systematically.

Prop: A b.j. $\alpha : \mathbb{R}^k \curvearrowright M^{2k+1}$ is an MRPFA

$\Leftrightarrow$ $\mathrm{ent}_\mu(\alpha_{t^*}) > 0$ for some $t^* \in \mathbb{R}^k$

and the $k+1$ Lyap. hyperplanes $\subseteq \mathbb{R}^k$

are in gen. pos. (Lyap. hyperplane = kernel of (nonzero) Lyap. exponent)

Obs: The acting group is abelian, so the dynamical objects attached to the time-t^* diffeo α_{t^*} are respected by the time-t diffeo α_t for any $t \in \mathbb{R}^k$.

In particular $\exists$ Oseledets Thm, Lyapunov exponents $\chi \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R})$, $\exists$ Pesin theory, ...

• Chambers of α : connected components of $\mathbb{R}^k \setminus \{$all Lyap. hyperplanes$\}$

Obs: In a fixed chamber no Lyapunov exponent changes sign.

$\Rightarrow$ $\exists$ a pairing: exponents $\times$ chambers $\longrightarrow \{\pm\}$

$\Rightarrow$ lots of stable/unstable combinatorial flexibility (eg. "freezing" / synchronization)

• PE $\Rightarrow$ nonatomic $\Rightarrow$ the pairing is not all + (nor all −) on any given chamber.

Ex: $k=2$. $\alpha: \mathbb{R}^2 \curvearrowright M^5$ hyper

$$T_x M = T_x \mathrm{Orb}_x \oplus L_x(\chi^1) \oplus L_x(\chi^2) \oplus L_x(\chi^3)$$

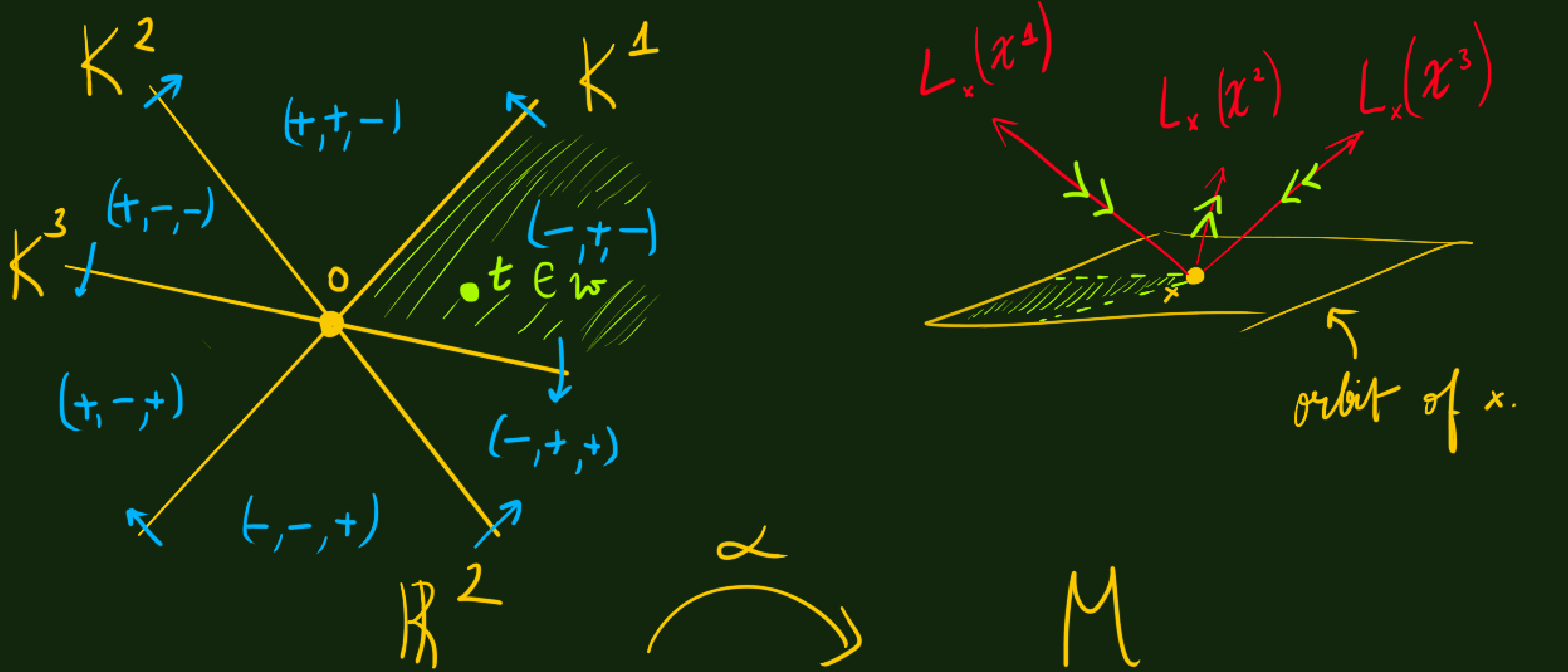
$\chi^1, \chi^2, \chi^3 : \mathbb{R}^2 \to \mathbb{R}$ linear

$\chi^1(t), \chi^2(t), \chi^3(t) \in \mathbb{R}$ are the Lyapunov exponents of the time-t diffeo α_t (in the rank-1 sense)

$$K^i = \ker(\chi^i).$$

$$S_x(w) = S_x(\alpha_t) = L_x(\chi^1) \oplus L_x(\chi^3)$$

$$U_x(w) = U_x(\alpha_t) = L_x(\chi^2)$$



Obs: If $\alpha_\cdot : \mathbb{Z}^k \curvearrowright M^d$ is hyper, then so is its suspension $\tilde{\alpha}_\cdot : \mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\alpha M^d$.

$\boxed{\text{Q: Are there any other hyper actions?}}$

K-RH asked this in a prequel paper joint with Kalinin:

MR2811602 Reviewed
Kalinin, Boris (1-SAL); Katok, Anatole (1-PAS); Rodriguez Hertz, Federico (UR-UREP)

Nonuniform measure rigidity.

Ann. of Math. (2) 174 (2011), no. 1, 361–400.

Main Theorem. (1) *Let μ be an ergodic invariant measure for a $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{Z}^k$, $k \geq 2$, on a $(k+1)$-dimensional manifold M. Suppose that the Lyapunov exponents of μ are in general position and that at least one element in $\mathbb{Z}^k$ has positive entropy with respect to μ. Then μ is absolutely continuous.*

(2) *Let μ be an ergodic invariant measure for a locally free $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{R}^k$, $k \geq 2$, on a $2k+1$-dimensional manifold M. Suppose that Lyapunov exponents of μ are in general position and that at least one element in $\mathbb{R}^k$ has positive entropy with respect to μ. Then μ is absolutely continuous.*

As already mentioned, the statement (1) is a direct corollary of (2) applied to the suspension of the $\mathbb{Z}^k$ action α. We are not aware of any examples of $\mathbb{R}^k$ actions satisfying assumptions of (2) other than time changes of suspensions of $\mathbb{Z}^k$ actions satisfying (1).

⋮

Problem 4. Are there $\mathbb{R}^k$ actions satisfying assumptions of the Main Theorem (2) which do not appear from time changes of suspensions of $\mathbb{Z}^k$ actions satisfying assumptions of the Main Theorem (1)?

Work in Progress: No.

Obs: categorical obstruction for suspensions:

$$(M^{k+1},\mu) \to \left(\mathbb{R}^k \otimes_\alpha M^{k+1}, \mathrm{Leb}_{\mathbb{T}^k} \otimes_\alpha \mu\right) \xrightarrow{\pi^\alpha} \left(\mathbb{T}^k, \mathrm{Leb}_{\mathbb{T}^k}\right)$$

in a fiber bundle, and

$$\pi^\alpha : \boxed{h^\alpha : \mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\alpha M} \twoheadrightarrow \boxed{\mathbb{R}^k \curvearrowright \mathbb{T}^k \ \text{translation}}$$

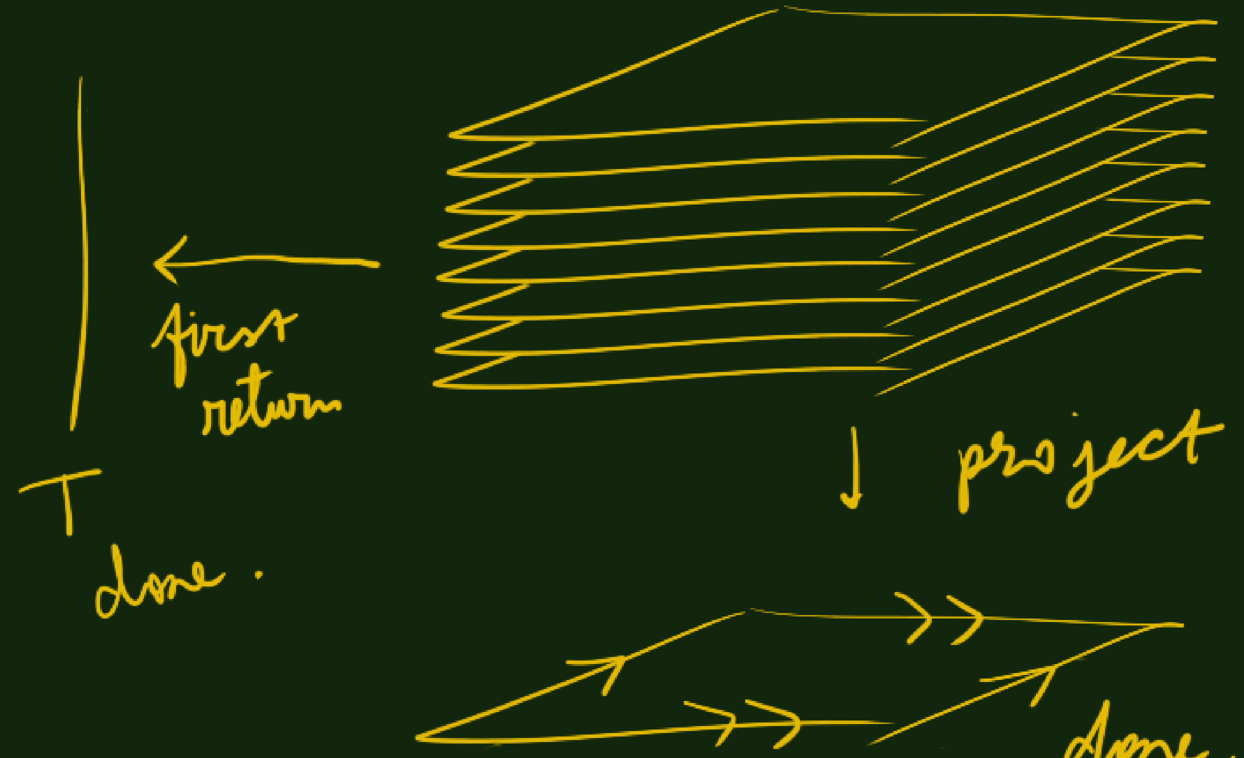
is a factor.

Thm (AU [WiP]) : Let $(\mu, \alpha.) \in \mathcal{E}^{1+\theta}(\mathbb{R}^k \curvearrowright M^d)$ be hyper. Then $\exists$ affine Cartan $(\mathrm{leb}_{\mathbb{T}^{k+1}}, \gamma.) \in \mathcal{E}^{\mathrm{aff}}(\mathbb{Z}^k \curvearrowright \mathbb{T}^{k+1})$ ($\mathbb{T}^{k+1}$ = torus or infratorus), $\exists$ measure theoretical isomorphism

$$\Phi : \boxed{(\mu, \alpha.) : \mathbb{R}^k \curvearrowright M^d} \xrightarrow{\cong} \boxed{\left(\mathrm{leb}_{\mathbb{T}^k} \otimes_\gamma \mathrm{leb}_{\mathbb{T}^{k+1}}, \hbar^\gamma_{\cdot} : \mathbb{R}^k \curvearrowright \mathbb{R}^k \otimes_\gamma \mathbb{T}^{k+1}\right)}$$

where $\hbar^\gamma_{\cdot}$ is a measurable time change of the suspension $\hbar^\gamma_{\cdot}$ of $\gamma.$

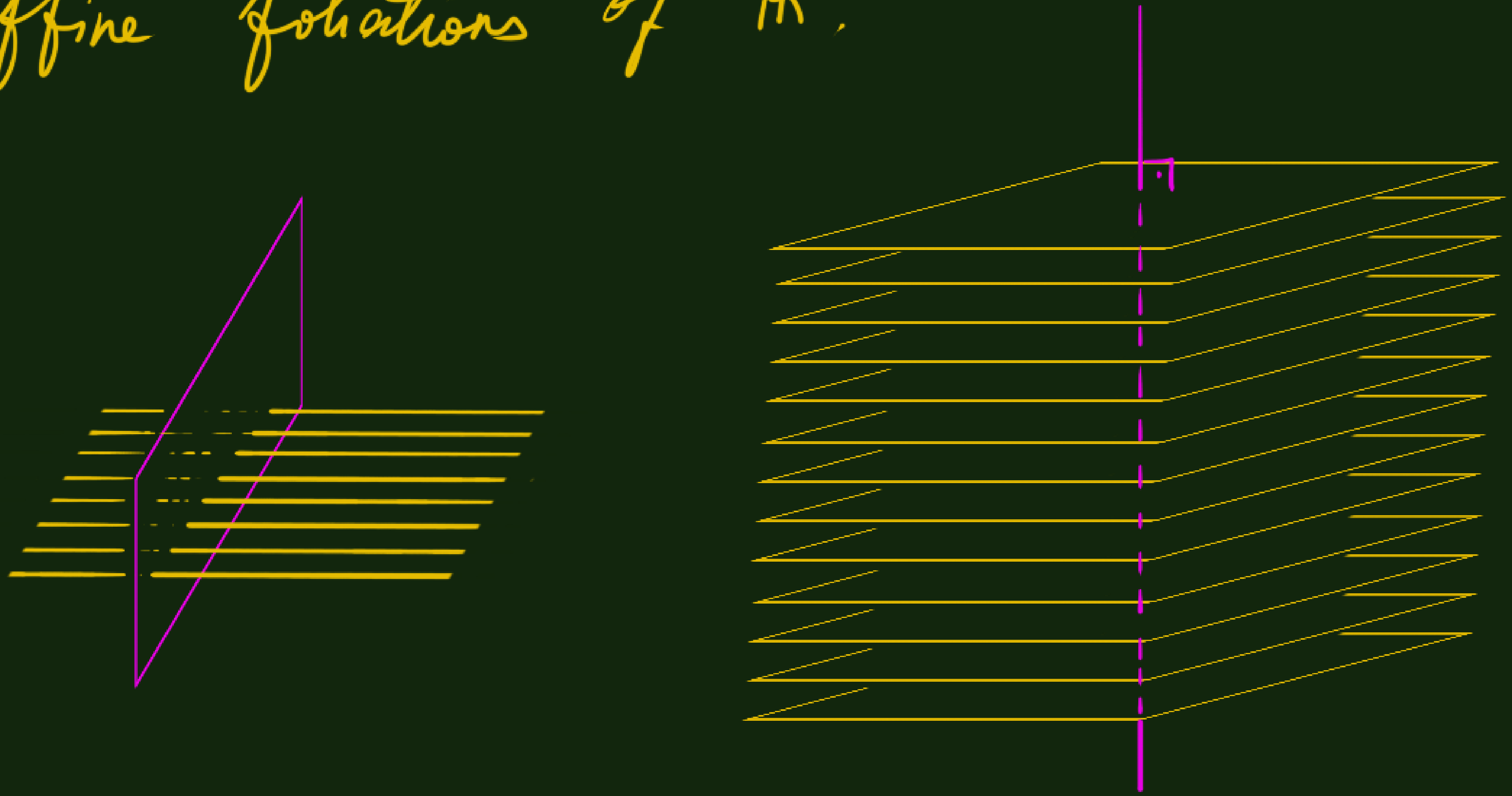
Naive attempt: Start with a hyper $\alpha : \mathbb{R}^k \curvearrowright M^d$, take the orbit foliation. Take a nice transversal T, collapse it to handle the obstruction, first return to it to recover a hyper $\mathbb{Z}^k \curvearrowright T$.



⚠ ⚠ ⚠ ⚠ :

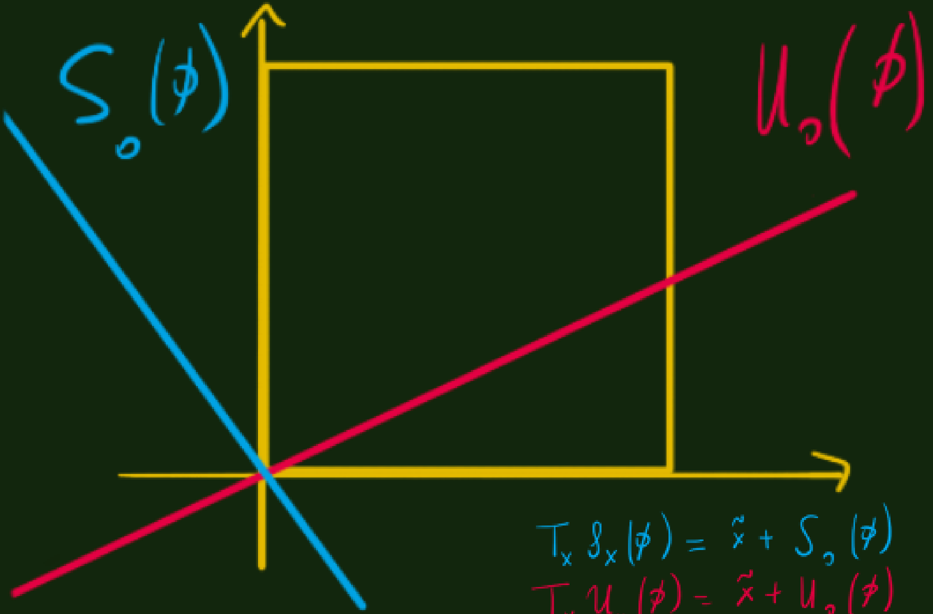
- How to find the nice transversal?
- Fundamental domain for the base?
- "First" return?
- How to ensure survival of hyper? ...

Obs: It is very easy to find transversals to affine foliations of $\mathbb{R}^d$.



Obs:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\tilde{\phi} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\ \mathbb{T}^2 \arrow[r, "\phi"] & \mathbb{T}^2 \end{tikzcd}$$



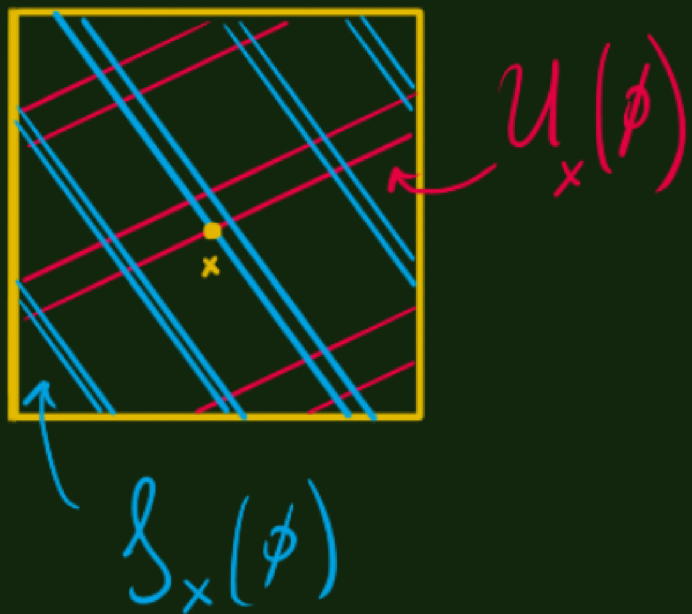
$T_x \mathcal{S}_x(\phi) = \tilde{x} + S_0(\phi)$

$T_x \mathcal{U}_x(\phi) = \tilde{x} + U_0(\phi)$

homoclinic points of x:

$$\Lambda_x(\phi) = \mathcal{S}_x(\phi) \cap \mathcal{U}_x(\phi)$$

is a dense subgroup of $\mathbb{T}^2$.



Affine parameters:

$\Sigma_x(\phi) : S_0(\phi) \to \mathcal{S}_x(\phi)$ $\quad$ $Y_x(\phi) : U_0(\phi) \to \mathcal{U}_x(\phi)$

$v \longmapsto \exp_{\tilde{x}}(v + \tilde{x})$ $\quad$ $v \longmapsto \exp_{\tilde{x}}(v + \tilde{x})$

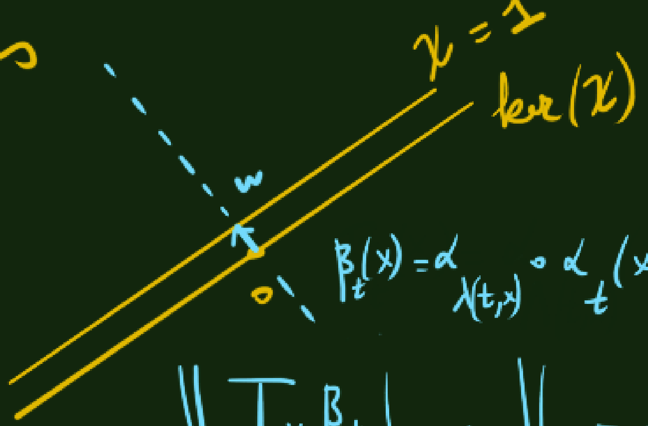
$\Rightarrow$ development map: $\Sigma_x(\phi) \times Y_x(\phi) : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$.

$\boxed{\text{(\$) The symmetry group of the development map is isomorphic to } \Lambda_x(\phi).}$

$\Rightarrow$ Can forget/remember that $\mathbb{T}^2$ is a group.

Geometric Method for Nonuniform Measure Rigidity

0) Emphasis on conditional measures along invariant foliations



$\beta_t(x) = \alpha_{\lambda(t,x)} \circ \alpha_t(x)$

$\| T_x \beta_t |_{L_x(\chi)} \|_{x,\varepsilon} = e^{\chi(t)}$

1) Lyap. metric / charts

1.a) Freezing / synchronization

1.b) Affine parameters (nonstationary normal forms) on invariant foliations $\Sigma_x(w)$, $\Upsilon_x(w)$

2) [AaT] Homoclinic group $\Lambda_x(w) \doteq \mathcal{S}_x(w) \cap \mathcal{U}_x(w)$

Strategy: Replace:

* $\Sigma_x(w) : (S_x(w), \circ) \to (\mathcal{S}_x(w), \times)$ with
  $\Gamma\Sigma_x(w) : (T_x \mathrm{Orb}_x \oplus S_x(w), (0, \circ)) \to (\mathrm{Orb}_x \pitchfork \mathcal{S}_x(w), \times)$

* $\mathcal{U}_{y \leftarrow x}(w) : \mathcal{S}_x(w) \to \mathcal{S}_y(w)$

with $\mathcal{U}_{y \leftarrow x}(w) : \mathcal{O}\mathcal{S}_x(w) \to \mathcal{O}\mathcal{S}_y(w)$

* $\Sigma_x(w) \times \mathcal{U}_x(w) : \mathbb{R}^d \to M$ with
  $\Gamma\Sigma_x(w) \times \mathcal{U}_x(w) : \mathbb{R}^d \to M$

Thank you for Listening!