DSS Talk : Plan

CNS paper : focus on Lyapunov Spectrum Rigidity for local diffeos Respecting a Polarization.
("polarized / horizontal local diffeos").

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Thm. 1.1. (Tangent Cone Structure Thm) [p. 3]

Prop. 3.4. [3.1, p. 13]

Prop. 3.8 [3.2, p. 14]

Thm. 1.4 (Arithmeticity at a "CC" Fixed Point)

[3.2, p. 18]

Thm. 1.2. (Subadditivity at a "CC" Fixed Point)

[3.2, p. 20]

Thm. 1.3: (Additivity at a Heisenberg Fixed Point)

[3.3, p. 21]

(II)

Thm. 4.2 (Foliated T.C Structure Thm) [4, p.22]

Thm. 1.6 (Foliated Arithmeticity) [4, p.24].

Notation:

$M \in \underline{\mathrm{Man}}^{\infty}$, $\mathrm{Polar}^r(M) = \left\{ H = \cancel{\mathrm{im}(\sigma)} \;\middle|\; \begin{array}{l} E \in \underline{VB}(M), \\ [\sigma : E \to TM] \in \underline{VB}^r(M) \\ H = \mathrm{im}(\sigma). \end{array} \right\}$

$\mathrm{Polar}(M) = \mathrm{Polar}^{\infty}(M)$

$CC^r(M) = \{ H \in \mathrm{Polar}^r(M) \mid H$ satisfies Hörmander $\}$

$CC^{\infty}(M) = CC(M)$.

$\forall H \in \mathrm{Polar}^r(M)$:

$$\begin{aligned}
&H^0 = 0 \in \mathrm{Polar}^{\infty}(M) \\
&\wedge\!| \\
&H^1 = H \in \mathrm{Polar}^r(M) \\
&\wedge\!| \\
&H^2 = H^1 + [H, H^1] \in \mathrm{Polar}^{r-1}(M) \\
&\wedge\!| \\
&H^3 = H^2 + [H, H^2] \in \mathrm{Polar}^{r-2}(M) \\
&\wedge\!| \\
&\vdots \\
&H^i = H^{i-1} + [H, H^{i-1}] \in \mathrm{Polar}^{r-i+1}(M), \\
&\wedge\!| \\
&\vdots
\end{aligned}$$

$$\boxed{H \in CC^r(M) \Leftrightarrow H^i \text{ stabilizes: } H^i = TM \text{ and } r - i + 1 \geq 0.}$$

<u>growth vector of H</u>: $x \mapsto (\dim(H^1_x), \dim(H^2_x), \ldots, \dim(H^i_x))$

Think of each $H \in CC^r(M)$ as endowed with a C^r positive definite sym bilin form

$g : H \otimes H \to \mathbb{R}$, or quad form $|\cdot| : H \to \mathbb{R}_{\geq 0}$

~~d~~ ~~γ :~~

* $\gamma : [0,1] \to M$ be C^0 and a.c. (diff. a.e.)

γ is <u>H-horizontal</u> if $\forall t \in I$: $\gamma'(t) \in H_{\gamma(t)}$.

~~length~~$_H$ * $\gamma : [0,1] \to M$ be H-horizontal ~~curve~~ C^0 & a.c. curve

$$\mathrm{length}_H(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)} \, dt$$

* $p, q \in M$. d_H = CC-distance induced by H.

$$d_H(p,q) = \inf \left(\left\{ \mathrm{length}_H(\gamma) \;\middle|\; \begin{array}{l} \gamma : ([0,1], 0, 1) \to (M, p, q) \\ C^0 \text{. a.c., } H\text{-horizontal.} \end{array} \right\} \right).$$

* $H \in CC(M)$. $\mathrm{Reg}(H) = \{ x \in M \mid \exists \underset{\text{open}}{U} \ni x$; growth vector of H is constant on U.$\}$

$\mathrm{Sing}(H) = M \setminus \mathrm{Reg}(H)$.

<u>Obs</u>: $\mathrm{Reg}(M) \subseteq M$ is open & dense. (yes if analytic)

(?)

GH:

$\underline{\mathrm{cMet}}$: category of compact metric spaces & $\overline{\mathrm{Lip}}$

$\underline{\mathrm{cMet}}_{pt}$: category of pointed ~~compact~~ metric spaces & $\overline{\mathrm{Lip}}$.

* $X_\bullet : \mathbb{Z}_{\geq 1} \to \underline{\mathrm{cMet}}$, $X_\infty \in \underline{\mathrm{cMet}}$.

$$\operatorname*{GH-lim}_{n\to\infty} X_n = X_\infty \text{ if}$$

$$\lim_{n\to\infty} \inf_{\substack{Z\in \underline{\mathrm{cMet}} \\ i:X_n \hookrightarrow Z \\ j:X \hookrightarrow Z \\ \text{isom. embeddings}}} \mathrm{Hausdist}\left(\vec{i}(X_n), \vec{j}(X_\infty)\right) = 0.$$

* $X_\bullet : \mathbb{Z}_{\geq 1} \to \underline{\mathrm{cMet}}_{pt}$, $X_\infty \to \underline{\mathrm{cMet}}_{pt}$.

$$\operatorname*{GH-lim}_{n\to\infty} X_n = X_\infty \text{ if}$$

$$\forall R \in \mathbb{R}_{>0} : \operatorname*{GH-lim}_{n\to\infty} X_n(*_n, \leq R) = X_\infty(x_\infty, \leq R)$$

in $\underline{\mathrm{cMet}}$.

$H \in CC(M)$. flag sheaf / sheaf of flags.

$$0 = H^0 \lneq H^1 \lneq H^2 \lneq \dots \lneq H^{r-1} \lneq H^r = TM.$$

locally

$\to \quad \forall p \in Reg(H)$: $\cancel{TC_pM} = \bigoplus_{i=1}^{r} H_p^i / H_p^{i-1}$

infinitesimal tangent cone; graded nilpotent Lie ~~group~~ algebra.
bracket structure coming from vector fields.

$$TC_pM = Lie\left(\bigoplus_{i=1}^{r} H_p^i / H_p^{i-1} \right).$$

$$M_p = lie\,(TC_pM) = \bigoplus_{i=1}^{r} \underbrace{H_p^i / H_p^{i-1}}_{= M_p^i}.$$

(vector fields homogenized to degree 1 under dilations.)

$$T_pM \geq \mathrm{span}\left(\hat{X}_{1,p}, \dots, \hat{X}_{m,p}\right)$$

$$\begin{array}{ccc} & \xrightarrow[\cong]{\text{Metivier}} & H_p \quad (\text{some nilpotent Carnot Lie algebra.}) \\ & & \big\downarrow \exp \\ & & \mathrm{Lie}(H_p) \xrightarrow[\cong]{\text{Mitchell}} TC_pM \end{array}$$

$$TC_pM \overset{\text{def}}{=}_{GH} \lim_{t\to\infty}\left(M, t\,d_H, p\right).$$

* <u>Sanity check</u>:

$$TC_p\,\mathrm{Heis}(3,\mathbb{R}) \cong \mathrm{Heis}(3,\mathbb{R})$$

as Carnot graded nilpotent Lie groups.

$\forall \varphi \in \mathrm{Aut}_{\underline{\mathrm{Lie}}}(\mathrm{Heis}(3,\mathbb{R}))$:

$$TC_p\varphi = \varphi.$$

$f \in \mathrm{Diff}^{\infty}_{loc}(M)$ "local diffeo."

$\exists$ open $U, V \subseteq M$, $f : U \xrightarrow[\text{diffeo}]{C^\infty} V$.

$H \in \mathrm{Polar}(M)$

$f : (M, H) \circlearrowleft$ (polarized / horizontal loc diff) $\overset{def}{\Leftrightarrow}$ $f \in \mathrm{Diff}_{loc}(M)$, and $f : U \to V$

$\forall p \in U : \overrightarrow{T_p f}(H_p) \leq H_{f(p)}$

Assume $\overrightarrow{T_p f}(H_p) = H_{f(p)}$

(otherwise need to say: forward / backward polarized).

$\Rightarrow$. ~~$p \in \mathrm{Reg}(H) \cap \mathrm{Fin}(f)$~~

<u>Obs</u>: $f : (M, H) \circlearrowleft$.

Then (i) Along f-orbits the growth vector is constant.

(ii) $\mathcal{F}$. $T^i f = Tf|_{H^i} : H^i \circlearrowleft$.

Thm (Oseledets for polarized derivative cocycles):

$f : M^d \circlearrowleft$ C^1 diffeo, $H \in \mathrm{Polar}^0(M)$.;

$f : (M, H) \circlearrowleft$. & $Tf : H \circlearrowleft$

Then $\exists M_0 \subseteq M$ Borel, f-inv.

$\forall x \in M_0$,

$\exists!\, \ell_x \leq d$, $\exists!\, \chi_x^1, \dots, \chi_x^{\ell_x} \in \mathbb{R}$ distinct

$\exists!\, H_x = \bigoplus_{i=1}^{\ell} L_x^i$ f-inv. $\left(\overrightarrow{T_x f}(L_x^i) = L_{f(x)}^i \right)$

$\forall v^i \in L_x^i$, $\forall$ C^0 $|\cdot|$ on M_0:

$$\lim_{|n| \to \infty} \frac{\log |T_x f^n v^i| - \chi_x^i \cdot n}{|n|} = 0$$

$\dim(L_x^i)$: dyn. multiplicity of χ^i at x

$L\mathrm{spec}_x(f, Tf|_H) = \{\chi_x^1, \dots, \chi_x^{\ell_x}\}$

Lyap spec at x

Lyap exponents

$X_x(f, Tf|_H)$ Lyap vector

$\tilde{X}_x(f, Tf|_H)$ reduced Lyap vector

* M_0 has full measure w/r/t any f-inv. measure

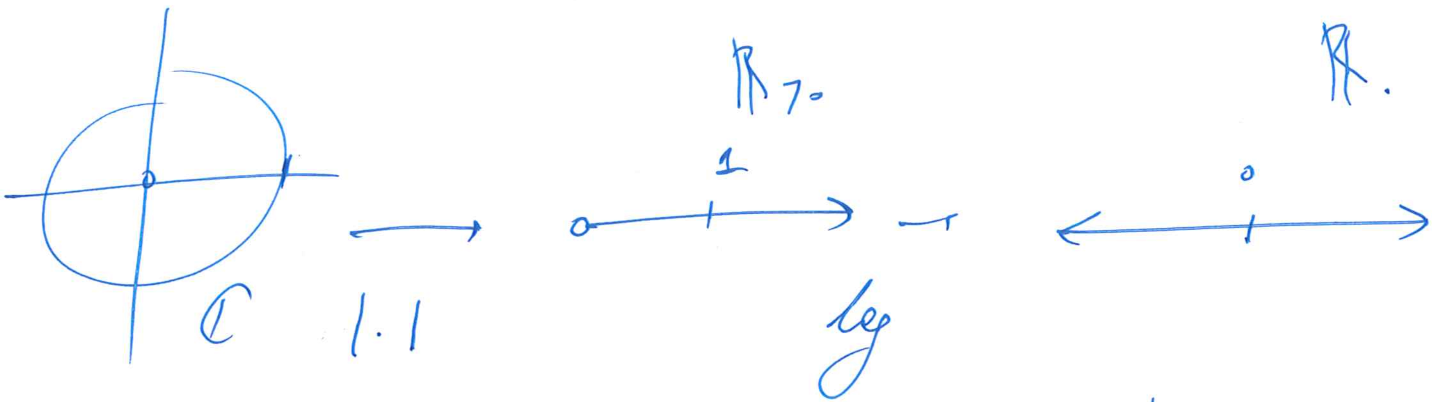
↳ All objects measurable,

* $x \mapsto \|T_x f|_{H_x}\|$, $x \mapsto \|T_x f^{-1}|_{H_x}\| \in \log^+(L^1)$ is suff. instead of compact

Obs: $f : M \circlearrowleft$, $p \in \mathrm{Fix}(f)$ ~~AA~~

$\Rightarrow T_p f : T_p M \circlearrowleft$

$$\Rightarrow \quad \mathrm{LSpec}_p(f) = \{ \log(|\lambda|) \mid \lambda \in \mathrm{Spec}(T_p f \otimes \mathbb{C}) \}.$$



Spec (linear/Banach)

Spec (Mather)

LSpec Lyapunov.

Examples:

- Grushin Plane: $M = \mathbb{R}^2$, $H_{(x,y)} = \operatorname{span}\big(\underbrace{(1,0)}_{=\partial_x}, \underbrace{(0,x)}_{x\partial_y}\big) \leq T_{(x,y)} \mathbb{R}^2$.

$$[\partial_x, x\partial_y] = \partial_x(x\partial_y) - (x\partial_y)(\partial_x) = \partial_y + x\partial_{xy} - x\partial_{yx} = \partial_y, \cancel{\neq x \neq 0}$$

$$H^1_{(0,y)} = \operatorname{span}(1,0)$$

$$H^2_{(0,y)} = \operatorname{span}((1,0),(0,1)) = T_{(0,y)}\mathbb{R}^2$$

$x \neq 0$. $\quad H^1_{(x,y)} = \operatorname{span}((1,0),(0,1)) = T_{(x,y)}\mathbb{R}^2$.

$$\operatorname{Reg}(H) = \mathbb{R}^2 \setminus 0 \times \mathbb{R}.$$

Mitchell: $(x,y) \in \operatorname{Reg}(H)$ $\quad (\Rightarrow x \neq 0)$

$$\operatorname{Hausdim}(\mathbb{R}^2; H) = 1 \cdot 2 = 2.$$

$$0 = H^0_{(x,y)} < H^1_{(x,y)} = T_{(x,y)}\mathbb{R}^2$$

$$\boxed{\alpha = dz - x\,dy}$$

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- $\underline{\text{3D Heisenberg}}$: $M = \mathbb{R}^3$

$$H_{(x,y,z)} = \operatorname{span}\Big(\underbrace{(1,0,0)}_{\partial_x}, \underbrace{(0,1,x)}_{\partial_y + x\partial_z}\Big) \leq T_{(x,y,z)}\mathbb{R}^3$$

~~$\alpha = dz$~~

"standard contact structure"

$\alpha = dz - \frac{1}{2}(x\,dy - y\,dx)$

gives:

$$H = \operatorname{span}\Big(\big(1, 0, -\tfrac{y}{2}\big), \big(0, 1, +\tfrac{x}{2}\big)\Big)$$

$$[\partial_x, \partial_y + x\partial_z] = \partial_x(\partial_y + x\partial_z) - (\partial_y + x\partial_z)\partial_x$$

$$= \partial_{xy} + \partial_z + x\partial_{xz} - \partial_{yx} - x\partial_{zx} = \partial_z .$$

$$\operatorname{Reg}(H) = \mathbb{R}^3$$

$$0 = H^0_{(x,y,z)} \lneq H^1_{(x,y,z)} \lneq H^2_{(x,y,z)} = T_{(x,y,z)}\mathbb{R}^3 .$$

$\underline{\text{Mitchell}}$: $\operatorname{Hausdim}(\mathbb{R}^3, H) = 1\cdot 2 + 2 \cdot 1 = 4.$

$$\left[\partial_x - \frac{y}{2}\partial_z ,\ \partial_y + \frac{x}{2}\partial_z\right]$$

$$= \left(\partial_x - \frac{y}{2}\partial_z\right)\left(\partial_y + \frac{x}{2}\partial_z\right) - \left(\partial_y + \frac{x}{2}\partial_z\right)\left(\partial_x - \frac{y}{2}\partial_z\right)$$

$$= \left(\partial_{xy} + \frac{1}{2}\partial_z + \frac{x}{2}\partial_{xz} - \frac{y}{2}\partial_{zy} - \frac{xy}{4}\partial_{zz}\right)$$

$$- \left(\partial_{yx} - \frac{1}{2}\partial_z - \frac{y}{2}\partial_{yz} + \frac{x}{2}\partial_{zx} - \frac{xy}{4}\partial_{zz}\right)$$

$$= \partial_z .$$

• $(2n+1)$–D Heisenberg:

$$\cancel{\begin{pmatrix} x_1 & x_2 & \text{—} & x_n & z \\ & & & & y_1 \\ & & & & y_2 \\ & & & & \vdots \end{pmatrix}} \qquad \left(\begin{array}{c|ccc|c} 1 & x_1 & \text{—} & x_n & z \\ \hline 0 & & I_{n\times n} & & \begin{matrix} y_1 \\ | \\ y_n \end{matrix} \\ \hline 0 & & 0 & & 1 \end{array}\right) = \mathrm{Heis}(2n+1;\mathbb{R}) = M$$

Standard contact structure:

$$\alpha = dz + 2\sum_{k=1}^{n}\left(x_k\, dy_k - y_k\, dx_k\right).$$

$$H^1 = H = \ker(\alpha)$$

$$\hookrightarrow \boxed{\text{or } \alpha = dz - \sum_{k=1}^{n} y_k\, dx_k}$$

$$H^2 = TM.$$

<u>Mitchell</u>: $\mathrm{Hausdim}\left(\mathrm{Heis}(2n+1, \mathbb{R}), \ker(\alpha)\right)$

$$= 1\cdot 2n + 2\cdot 1 = 2n+2.$$

· *Martinet*: $M = \mathbb{R}^3$

$$H_{(x,y,z)} = \operatorname{span}\Big(\underbrace{\left(1, 0, \tfrac{y^2}{2}\right)}_{\partial_x + \frac{y^2}{2}\partial_z}, \underbrace{(0, 1, 0)}_{\partial_y .} \Big) \leq T_{(x,y,z)}\mathbb{R}^3$$

$$\left[\partial_x + \frac{y^2}{2}\partial_z, \partial_y\right] = \left(\partial_x + \frac{y^2}{2}\partial_z\right)\partial_y - \partial_y\left(\partial_x + \frac{y^2}{2}\partial_z\right)$$

$$= \partial_{xy} + \frac{y^2}{2}\partial_{zy} - \partial_{yx} - y\partial_z - \frac{y^2}{2}\partial_{yz} = y\partial_z$$

$$[\partial_y, y\partial_z] = \partial_y(y\partial_z) - (y\partial_z)\partial_y = \partial_z + y\partial_{yz} - y\partial_{zy} = \partial_z .$$

$$H^1_{(x,y,z)} = \operatorname{span}\left(\left(1, 0, \tfrac{y^2}{2}\right), (0, 1, 0)\right).$$

$$H^2_{(x,y,z)} = \operatorname{span}\left(\left(1, 0, \tfrac{y^2}{2}\right), (0, 1, 0), (0, 0, y)\right)$$

$$H^3_{(x,y,z)} = \operatorname{span}\left(\left(1, 0, \tfrac{y^2}{2}\right), (0, 1, 0), (0, 0, y), (0, 0, 1)\right) = T_{(x,y,z)}\mathbb{R}^3 .$$

$$\operatorname{Reg}(H) = \mathbb{R}^3 \setminus \mathbb{R} \times 0 \times \mathbb{R} .$$

Mitchell: $y \neq 0$.

$$\operatorname{Hausdim}(\mathbb{R}^3, H) = 1 \cdot 2 + 2 \cdot 1 = 4 .$$

(Sinic)

- PH: $f : M \circlearrowleft$ M compact, f partially hyperbolic, su-accessible. (low regularity)

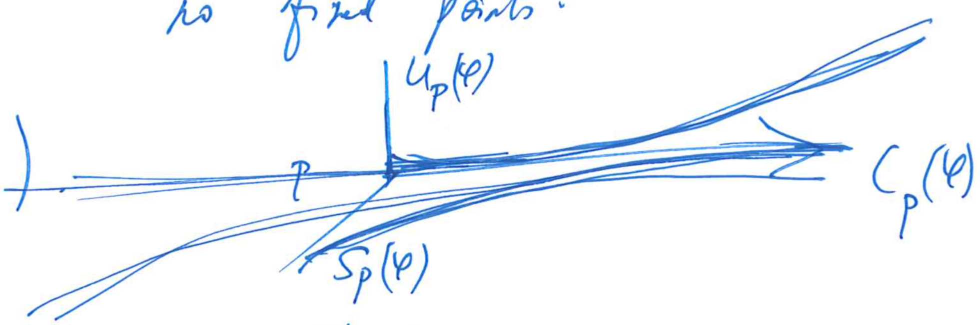
(particular case)

- Contact Anosov flows.

$\varphi_\cdot : \mathbb{R} \curvearrowright M$ Anosov. $TM = S(\varphi) \oplus C(\varphi) \oplus U(\varphi)$.

$X(p) = \frac{\partial}{\partial t}\Big|_{t=0} \varphi_t(p)$. no fixed points.

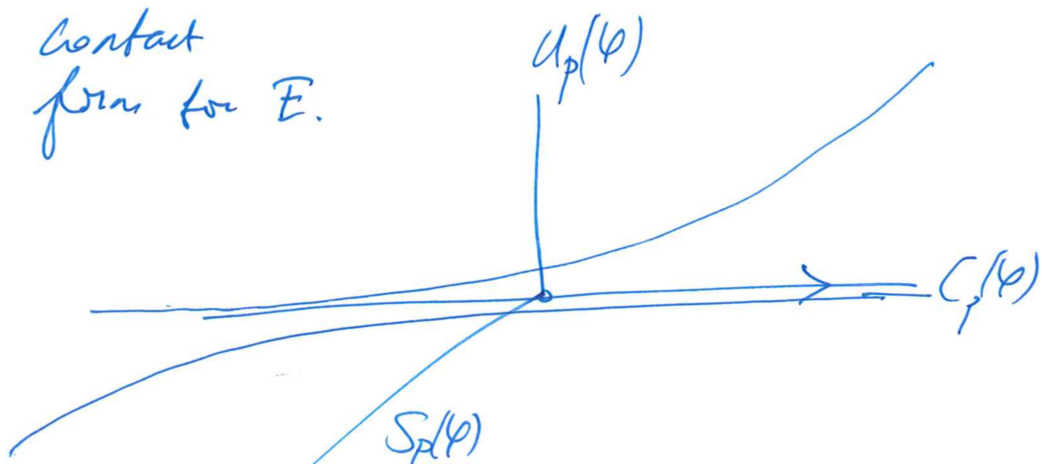
$C_p(\varphi) = \mathrm{Span}(X_p)$



contact structure on M^{2k+1}, C^1 $(2k)$-plane field E that is completely nonintegrable:

$\exists$ C^1 1-form α : $\alpha \wedge (d\alpha)^{\wedge k}$ volume form, & $E = \ker(\alpha)$.

contact form for E.



Anosov flow $\varphi_\cdot : \mathbb{R} \curvearrowright M^{2k+1}$ is <u>contact</u> if

(i) $S(\varphi) \oplus U(\varphi)$ is a contact structure,

and

(ii) For some contact 1-form α of $S(\varphi) \oplus U(\varphi)$,

X is the Reeb <u>vector field</u> of α:

$$\alpha(X) = 1$$
$$d\alpha(X, \cdot) = 0.$$

* $f : M \circlearrowleft$ accessible partially hyperbolic C^1.
M compact.

(say with $S(f) \oplus U(f)$ C^∞ if needed)

(if not restrict to some embedded accessibility class).

$g \in \text{Cen}^1(f)$ $\quad f \circ g = g \circ f$.

(This is a rare situation, if $g \neq f^n$). BCWV.

$\Rightarrow g : (M, S(f) \oplus U(f)) \circlearrowleft$

$S(f) \oplus U(f)$ cc.

* Can replace $f : M \circlearrowleft$ with a contact Anosov flow. $\varphi_t : \mathbb{R} \curvearrowright M$.

* $M = \mathbb{R}^2 \times \mathbb{R}$ (for CNS)

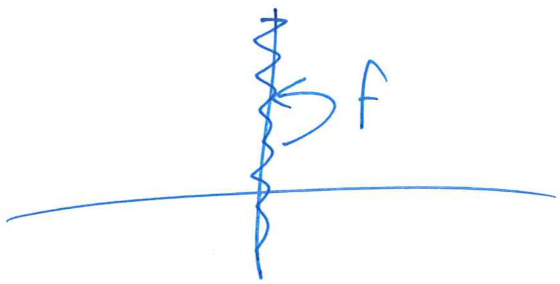
$\uparrow$

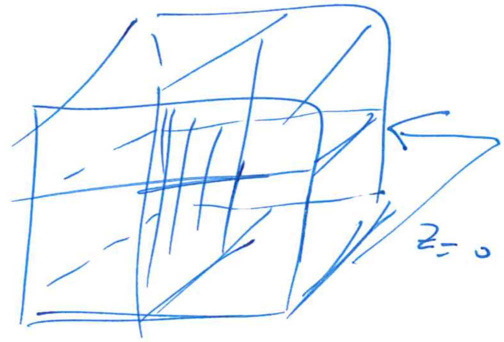
Grushin.

$$f : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2 \times \mathbb{R}$$

$$(x, y, z) \longmapsto \big(x \cdot \varphi(x,y),\ \psi(x,y),\ u(z)\big)$$

$$f(0, y, z) = \big(0,\ \text{—},\ u_{(0,y)}(z)\big)$$





$$Df = \begin{pmatrix} \varphi + x\partial_x\varphi & \partial_y\varphi & 0 \\ \partial_x\psi & \partial_y\psi & 0 \\ 0 & 0 & \partial_z u \end{pmatrix}.$$

Borel-Smale Ex:

Ex. $G = Heis(3;\mathbb{R}) \times Heis(3;\mathbb{R})$

$$X_1 = \begin{pmatrix} \cancel{1} & 1 & 0 \\ 0 & \cancel{1} & 0 \\ 0 & 0 & \cancel{1} \end{pmatrix} \qquad [X_2, Y_2] = Z_2.$$

$$Y_1 = \begin{pmatrix} \cancel{1} & 0 & 0 \\ 0 & \cancel{1} & 1 \\ 0 & 0 & \cancel{1} \end{pmatrix}$$

$$Z_1 = [X_1, Y_1] = \begin{pmatrix} \cancel{1} & 0 & 1 \\ 0 & \cancel{1} & 0 \\ 0 & 0 & \cancel{1} \end{pmatrix}$$

$$\mathrm{lie}(G) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\Gamma = \exp\left(\left\{ (A, \sigma(A)) \in \mathrm{lie}(G) \;\middle|\; \sigma(x+y\sqrt{5}) = x - y\sqrt{5} \right\}\right)$$

$\leq G$ lattice.

$$M = G/\Gamma.$$

$$\Phi : \quad \begin{aligned} X_1 &\longmapsto \lambda_1 X_1 \\ Y_1 &\longmapsto \lambda_1^2 Y_1 \\ Z_1 &\longmapsto \lambda_1^3 Z_1 \\ X_2 &\longmapsto \lambda_1^{-1} X_2 \\ Y_2 &\longmapsto \lambda_1^{-2} Y_2 \\ Z_2 &\longmapsto \lambda_1^{-3} Z_2 \end{aligned}$$

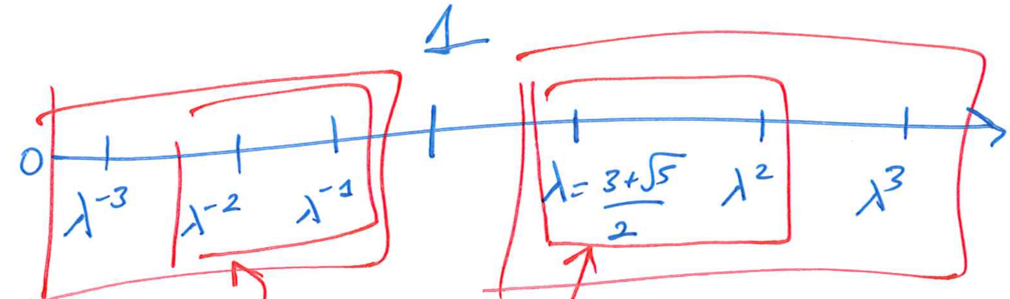
$$\lambda_1 = \frac{3+\sqrt{5}}{2} > 1.$$

$$\lambda_1^{-1} = \frac{3-\sqrt{5}}{2} < 1$$

$\Phi \in \mathrm{Aut}(\mathrm{Lie}(G))$,

preserves $\log(\Gamma)$.

$\rightsquigarrow$ defines $\varphi \in \mathrm{Aut}_\Gamma(G)$. Anosov diffeo.



$S(\varphi)$

$U(\varphi)$

$U^{slow}(\varphi)$

horizontal along $U(\varphi)$

$S^{slow}(\varphi)$

horizontal along $S(\varphi)$.

Statements of Results: I

Thm. 1.1: (TC_pM Str Thm)

$f : (M, H) \circlearrowleft$ local.

Then

(i) $Reg(H)$ is open and dense (?) & f-inv.

(ii) $\forall x \in Reg(H)$: $TC_x M$ exists and is a Carnot graded nilpotent Lie group.

$$TC_\bullet M : Reg(H) \to \underline{\mathcal{M}et}_{pt}$$

is continuous.

graded: $M = \bigoplus_{i=1}^{r} M_i$

$[M_i, M_j] \leq M_{i+j}$

Carnot: graded & $[M_1, M_j] = M_{1+j}$

(iii) $\forall x \in Reg(H)$, $\exists!\ TC_x f : TC_x M \to TC_{f(x)} M$

an isomorphism in $\underline{CgrLie}$; called

Carnot / Pansu / cc derivative.

(Pf: diffeo $\Rightarrow$ q.c.; polarized $\Rightarrow$ horizontals commute)

Thm 1.4 (Arithmeticity at a Fixed Point):

$f:(M,H) \circlearrowleft$ local.

If $\exists p \in Reg(H) \cap Fix(f)$ and

$TC_p f : TC_p M \circlearrowleft$ is a homothety;

$\hookrightarrow$

$F : X \to Y$

homothety

if $\dfrac{\overleftarrow{F}(d_x)}{d_x} = \text{constant}$

or "asymmetric"

then $\exists ! \, x \in \mathbb{R} \; \forall i :$

$$LSpec_p(f, T^i f) = \{x, 2x, \dots, (i)x\}.$$

with dynamical multiplicities

= growth vector of H.

$$X_p(f, Tf) = (x, 2x, \dots, rx).$$

$\Rightarrow LSpec_p(f, Tf) = \{x, 2x, \dots, rx\}.$

$\Rightarrow f$ is either isom at p,

nor expanding at p

nor contracting at p.

Thm. 1.2 (Subadditivity at a cc Fixed Point):

$f : (M, H) \circlearrowleft$ local.

If $\exists p \in \mathrm{Reg}(H) \cap \mathrm{Fix}(f)$,

and TC_pM is r-step. (redundant?)

$\boxed{\text{cc Lyap. exponent}}$

Write

$\boxed{\begin{array}{l} \mathfrak{n}_p = \mathrm{Lie}(TC_pM). \\ \mathfrak{n}_p \rhd_{\neq} [\mathfrak{n}_p, \mathfrak{n}_p] \rhd_{\neq} [\mathfrak{n}_p, [\mathfrak{n}_p, \mathfrak{n}_p]] \\ \rhd_{\neq} \dots \rhd_{\neq} 0 \\ \#\text{ of triangles} = \text{"step"} \end{array}}$

$\angle\mathrm{Spec}_p(f, T^1 f) \leftrightarrow x^{11} \leq x^{12} \leq \dots \leq x^{1v^1}$
(with mult)

$\cancel{\angle\mathrm{Spec}_p(f, T^2 f) \leftrightarrow x^{11} \leq x^{1}}$

$\angle\mathrm{Spec}_p(f, T^2 f / T^1 f) \leftrightarrow x^{21} \leq x^{22} \leq \dots \leq x^{2v^2}$

$\vdots$

$\angle\mathrm{Spec}_p(f, T^i f / T^{i-1} f) \leftrightarrow x^{i1} \leq \dots \leq x^{iv^i}$

$\vdots$

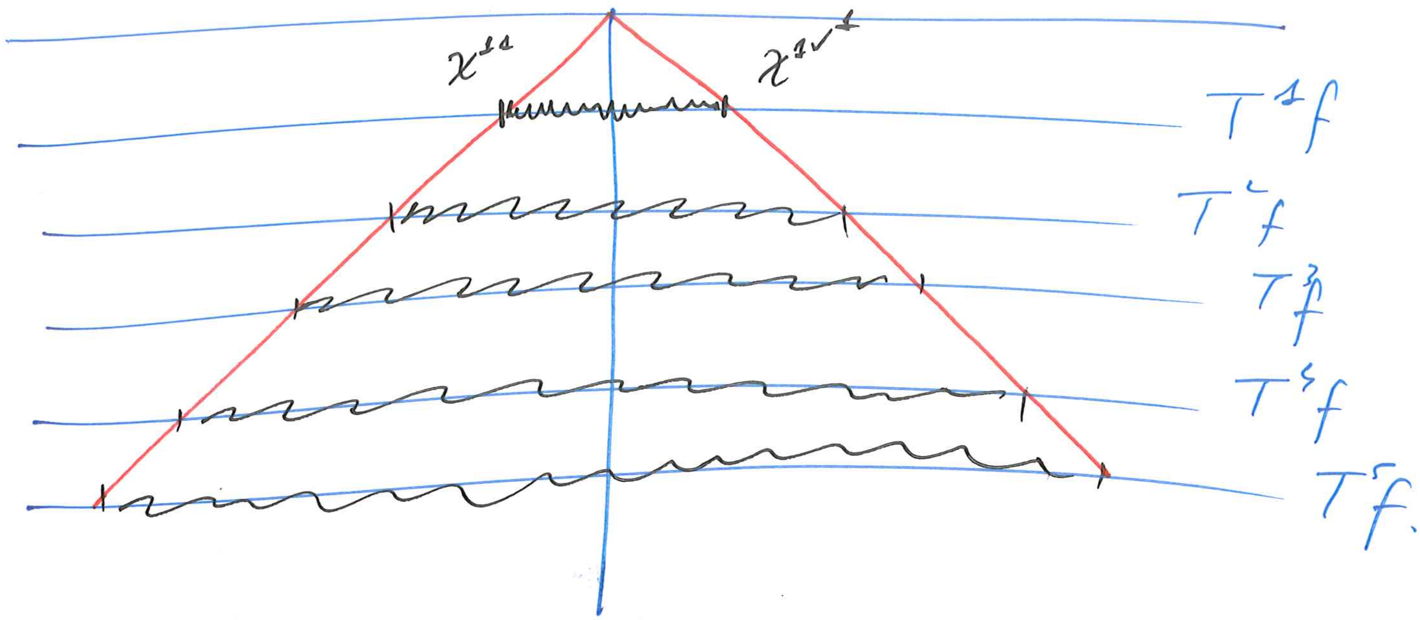
$\Rightarrow \quad x^{11}, \dots, x^{1v^1} \in [x^{11}, x^{1v^1}]$ obvious

$x^{21}, \dots, x^{2v^2} \in [2 \cdot x^{11}, 2 \cdot x^{1v^1}]$

$\vdots$

$x^{i1}, \dots, x^{iv^i} \in i \cdot [x^{11}, i \cdot x^{1v^1}].$

a



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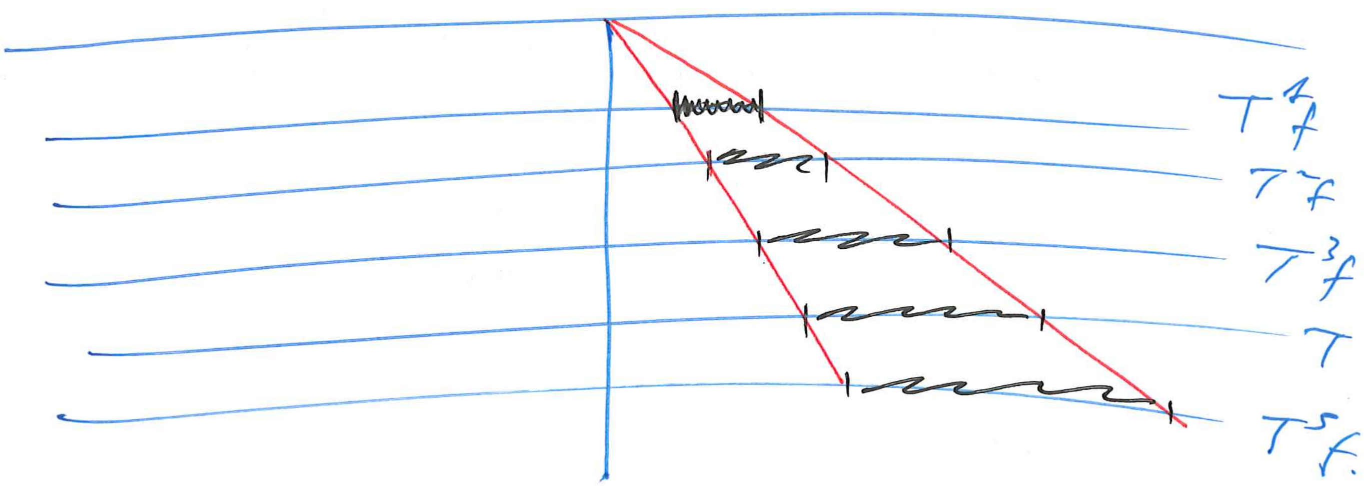


Thm. 1.3 (Additivity of a Heisenberg Time Point)

$f : (M, H) \circlearrowleft$ local

If $\exists p \in Reg(H) \cap Fin(f)$

and $TG_pM \cong Heis(2n+1, \mathbb{R})$.

$\Rightarrow$ ($H \subseteq TM$ is a hyperplane distribution)

$X(f, T^H f) = (x^1 \leq x^2 \leq \dots \leq x^{2n})$,

x^{2n+1} = remaining Lyap. exp. of (f, Tf).

Then

$$x^{2n+1} = \frac{1}{n} \sum_{i=1}^{2n} x^i.$$

More Notation – Foliated CC:

$M \in \underline{\mathrm{Man}}^\infty$ compact

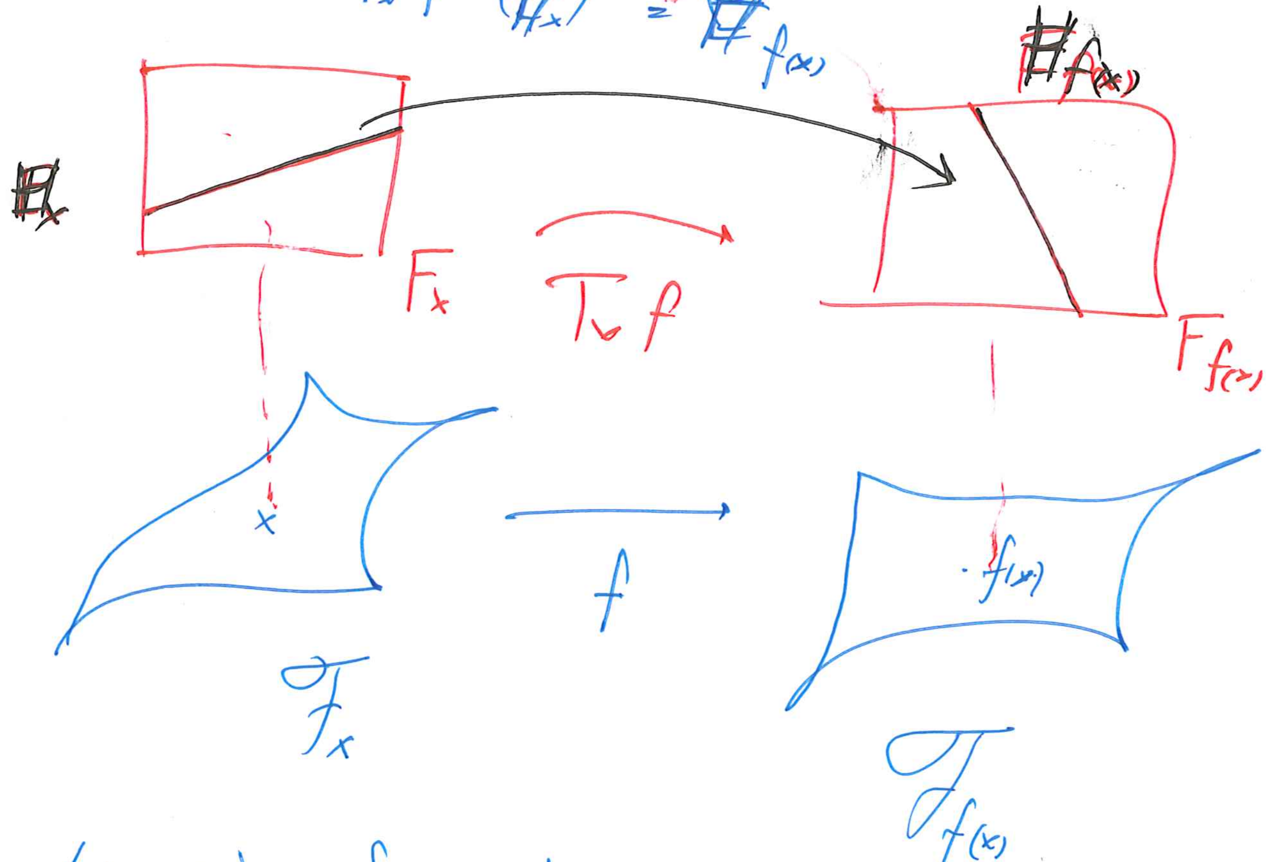
$\mathcal{F} \in \mathrm{Fol}^{0,\infty}(M)$, $F = T\mathcal{F} \in \mathrm{Polar}^0(M)$

$H \in \mathrm{Polar}^0(M)$, $H \leq F$

$f \in \mathrm{Diff}^1(M, \mathcal{F}, H)$ "doubly polarized"

i.e., $\vec{f}(\mathcal{F}_x) = \mathcal{F}_{f(x)}$,

$\overrightarrow{T_x f}(H_x) = H_{f(x)}$



$$\mathrm{Reg}(H, \mathcal{F}) = \left\{ x \in M \;\middle|\; \begin{array}{l} \exists\, U_x \subseteq \mathcal{F}_x \text{ open } x \in U_x, \\ H|_{U_x} \text{ is } C^\infty, \text{ satisfies Hörmander in } \mathcal{F}_x, \\ \text{and the growth vector is constant} \end{array} \right\}$$

Thm (Foliated $TC_p(M,\mathcal{F})$ Str Thm):

$$f : (M, \mathcal{F}, H) \circlearrowleft \quad C^1. \quad (\text{or } C^1 \text{ flow})$$

(i) $\mathrm{Reg}(H,\mathcal{F})$ is f-inv.

(ii) $\forall x \in \mathrm{Reg}(H,\mathcal{F})$: $\mathrm{Reg}(H,\mathcal{F}) \cap \mathcal{F}_x$ is open in $\mathcal{F}_x$

(iii) $\forall x \in \mathrm{Reg}(H,\mathcal{F})$, $\forall y \in \mathrm{Reg}(H,\mathcal{F}) \cap \mathcal{F}_x$:

$TC_y \mathcal{F}_x$ exists and is a Carnot graded nilpotent Lie group.

(iv) $\forall x \in \mathrm{Reg}(H,\mathcal{F})$: $TC_\circ \mathcal{F}_x : \mathrm{Reg}(H,\mathcal{F}) \cap \mathcal{F}_x \to \underline{\mathcal{M}et}_{pt}$

is continuous.

(v) $\forall x \in \mathrm{Reg}(H,\mathcal{F})$, $\exists!\, TC_x f : TC_x \mathcal{F}_x \longrightarrow TC_{f(x)} \mathcal{F}_{f(x)}$

a $\underline{\text{GrLie}}$-isomorphism, called leaf wise

cc/Carnot/Pansu derivative.

(vi) If H is uniformly C^k along $\mathcal{F}$ ($k \geq$ local geom. step),

then $\mathrm{Reg}(H,\mathcal{F}) \subseteq M$ is open.

and $TC_\circ \mathcal{F}_\circ : \mathrm{Reg}(H,\mathcal{F}) \to \underline{\mathcal{M}et}_{pt}$ is C^0.

Thm (1.6) (Foliated Arithmeticity)

$$f : (M, \mathcal{F}, H) \circlearrowleft \quad C^1.$$

If (i) H is uniformly C^k along $\mathcal{F}$ ($k \geq$ local geom step),

(ii) $\mathcal{F} \in \mathcal{F}ol^{\text{Hölder}, \infty}(M)$,

(iii) f is top. trans. (∃ dense orbit) &

satisfied by Anosov diffeos, hyperbolic autos, ...

(iv) f satisfies the $\boxed{\text{Kalinin closing property}}$ &

$\boxed{\text{needed for Livsic}}$

(v) $T^F f$ Hölder

(vi) $\forall p \in \mathrm{Reg}(H, \mathcal{F}) : T C_p \mathcal{F}_p$ is asymmetric and r-step.

meaning the only graded automorphisms are dilatations up to compact error.

(vii) μ be an f-inv. prob. ergodic with

$$\mathrm{supp}(\mu) \cap \mathrm{Reg}(H, \mathcal{F}) \neq \emptyset.$$

Then $\exists!\, x \in \mathbb{R}$:

$$X(f, T^F f) = (x, 2x, \ldots, r x)$$

with multiplicities Mitchell numbers.