SL(2, ℝ) - Cocycles over Circle Maps, Part 2 :

(DSS - 04/08/21)

Outline :

1) Formalities ✓

2) Uniform Hyperbolicity ✓

3) Obstructions to Uniform Hyperbolicity ✓

4) Oseledets Theorem ✗

5) Examples of Lai-Sang Young. ~

1) Formalities.

& lab-pres

$f: \mathbb{Z}_{\geq 0} \curvearrowright \mathbb{T}^1$ or $f: \mathbb{Z} \curvearrowright \mathbb{T}^1$ C^1.

$\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. $G = \mathbb{Z}_{\geq 0}, \mathbb{Z}$

- A <u>cocycle</u> over f with values in $SL(2,\mathbb{R})$ is a function $u: G \times \mathbb{T}^1 \longrightarrow SL(2,\mathbb{R})$ such that

(1) $u(n,\cdot) \in C^1(\mathbb{T}^1, SL(2,\mathbb{R}))$

(2) $u(n+m,x) = u(n, f^m(x))\, u(n,x)$.

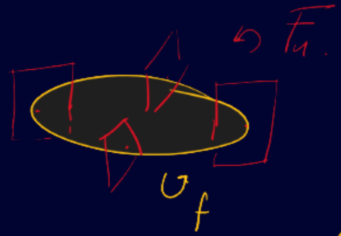
Denote by $\mathrm{Cocy}^1(f; SL(2,\mathbb{R}))$ the set of all cocycles.

Geometrically, a cocycle u allows one to extend $G \curvearrowright \mathbb{T}^1$ to some $G \curvearrowright (\mathbb{T}^1 \times \mathbb{R}^2)$.

$$\begin{tikzcd} \mathbb{T}^1 \times \mathbb{R}^2 \arrow[r, "F_u"] \arrow[d] & \mathbb{T}^1 \times \mathbb{R}^2 \arrow[d] \\ \mathbb{T}^1 \arrow[r, "f"] & \mathbb{T}^1 \end{tikzcd}$$

$$F_u^n : (x,v) \longmapsto (f^n(x), u(n,x)\, v).$$

$F_u : G \curvearrowright \mathbb{T}^1 \times \mathbb{R}^2$ is the <u>skew-product (or extension)</u> over f defined by u.



$$\cdot\ \mathrm{Coc}^1(f; SL(2,\mathbb{R})) \xrightarrow{\text{bij}} C^1(\mathbb{T}^1, SL(2,\mathbb{R}))$$

$$u \longmapsto u(1,\cdot)$$

$$u(n,x) = A(f^{n-1}(x)) \cdots A(f(x))A(x) \longleftarrow A$$

$$= A^n(x)$$

generator of the cocycle u.

$\cdot$ $C^1(\mathbb{T}^1; SL(2,\mathbb{R}))$ is a group. Pulling back this structure via the above bijection makes $\mathrm{Coc}^1(f; SL(2,\mathbb{R}))$ also a group.

Ex: $u^2(n,x) = \left(A(f^{n-1}(x))\right)^2 \cdots \left(A(f(x))\right)^2 (A(x))^2$.

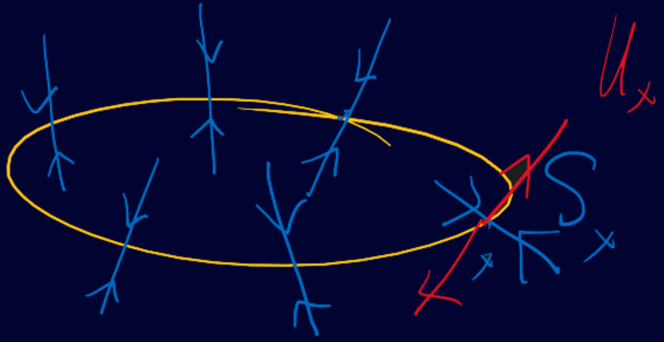
In particular we have an <u>Iwasawa decomposition</u> for $\mathrm{Coc}^1(f; SL(2,\mathbb{R}))$:

$$u(n,x) = (\widehat{nak})(n,x)$$

$$n = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \lambda > 0 \quad k = R(\theta) = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

(shear) (par.) — (expansion/contraction) (hyp.) — (rotation) (ell.)

2) <u>UH.</u>



A cocycle $u \in \mathrm{Coc}^1(f; SL(2,\mathbb{R}))$ is

<u>uniformly hyperbolic</u> if

there there is a C^0

line bundle $S_\cdot : \mathbb{T}^1 \to Gr_1(\mathbb{R}^2)$

and $\exists C \in \mathbb{R}_{>0}$, $\exists \tau \in \mathbb{R}_{>1}$ such that

$\forall x \in \mathbb{T}^1$, $\forall n \in \mathbb{Z}_{\geq 0}$:

$$\forall s \in S_x : |u(n,x)s| \leq C \cdot \tau^{-n} |s|$$

Def: $f : \mathbb{Z} \curvearrowright \mathbb{T}^1$, $A \in \mathrm{Cocy}^1(f; SL(2,\mathbb{R}))$

A is __uniformly hyperbolic__ if

$\exists (S_\cdot, U_\cdot)$ transverse line bundles $(\Rightarrow C^0)$

over $\mathbb{T}^1$, C^0

$\exists C > 0$, $\exists \lambda \in]0,1[$, $\forall x \in \mathbb{T}^1$:

(i) $A(x)(S_x) = S_{f(x)}$, $A(x)(U_{(x)}) = U_{f(x)}$.

(ii) $\forall n \in \mathbb{Z}_{\geq 0}$, $\forall s \in S_x$: $|A^n(x)s| \leq C\lambda^n |s|$

$\forall u \in U_x$: $|A^{-n}(x)u| \leq C\lambda^n |u|$.

Prop: $f:\mathbb{Z} \circlearrowright \mathbb{T}^1$, $A \in \mathcal{C}oc_y^1(f, SL(2,\mathbb{R}))$

A is uniformly hyperbolic iff

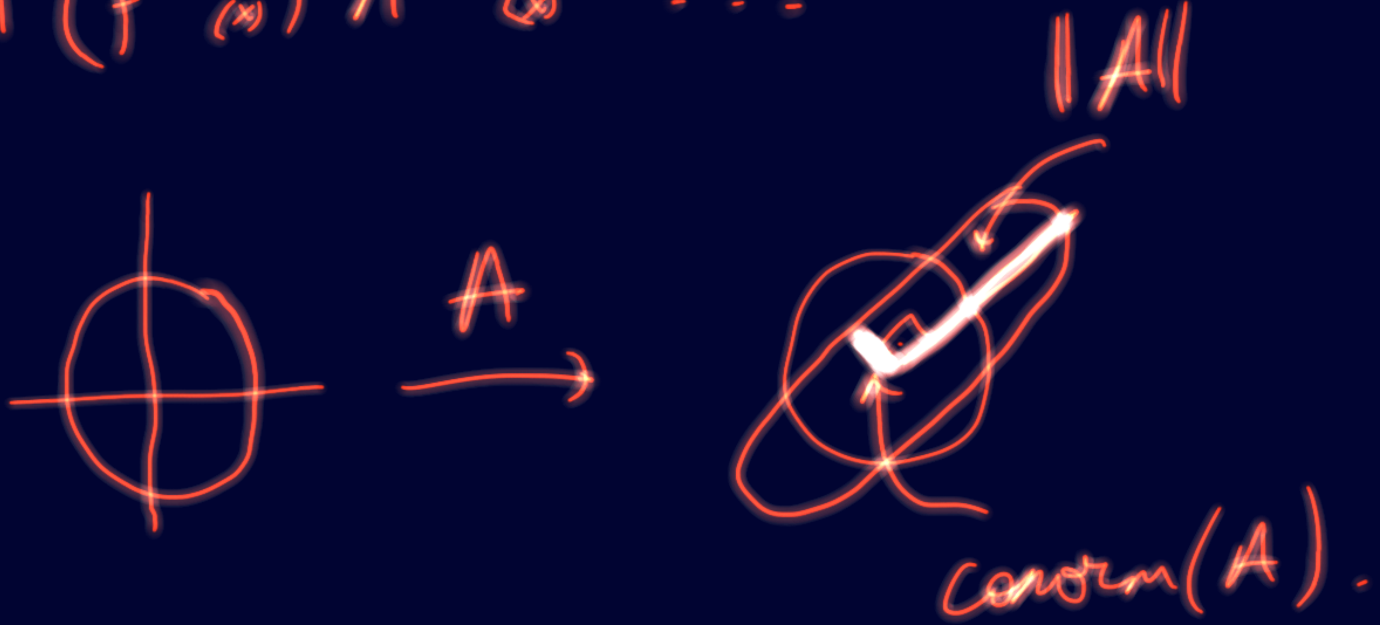
- $\exists\, c>0,\ \exists\, \sigma>1,\ \forall x \in \mathbb{T}^1,\ \forall n \in \mathbb{Z}_{\geq 0}:$

$$\|A^n_{(x)}\| \geq c\,\sigma^n \quad \text{iff}$$

$$\boxed{\begin{array}{l} \bullet\ \exists\, c>0,\ \exists\, \sigma>1,\ \forall x \in \mathbb{T}^1,\ \forall n \in \mathbb{Z}_{\geq 0}: \\ \|A^n_{(x)}\|^{-1} = \text{conorm}\left(A^n_{(x)}\right) \leq c\,\sigma^{-n} \end{array}}$$

$$A^n_{(x)} = A\left(f^{n-1}_{(x)}\right) A\left(f^{n-2}_{(x)}\right) \cdots A\left(f_{(x)}\right) A_{(x)}$$

$$A^n_{(x)} = A^1\left(f^{n-1}_{(x)}\right) A^{n-1}_{(x)} \quad \cdots$$



Def: $f : \mathbb{Z}_{\geq 0} \curvearrowright \mathbb{T}^1$, $A \in \mathrm{loc}^1(f, SL(2,\mathbb{R}))$.

A is *uniformly hyperbolic* if

$\exists\, c > 0,\ \exists\, \sigma > 1,\ \forall x \in \mathbb{T}^1, \forall n \in \mathbb{Z}_{\geq 0}$:

$$\mathrm{conorm}\left(A^n(x)\right) \leq c\, \sigma^{-n}$$

- In words, $A^n(x)$ contracts some sequence of vectors exponentially fast.

3) Obstructions.

Fix $\lambda > 0$ and let $\varphi \in C^1(\mathbb{T}^1, \mathbb{T}^1)$. Define

$$A : \mathbb{T}^1 \xrightarrow{C^1} SL(2,\mathbb{R})$$
$$x \longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} R(2\pi \varphi(x))$$

$\rightarrow$ $A^{\bullet} \in \mathrm{Cocy}^1(f : \mathbb{Z}_{\geq 0} \curvearrowright \mathbb{T}^1; SL(2,\mathbb{R}))$

Say $A^{\bullet}$ is uniformly hyperbolic.

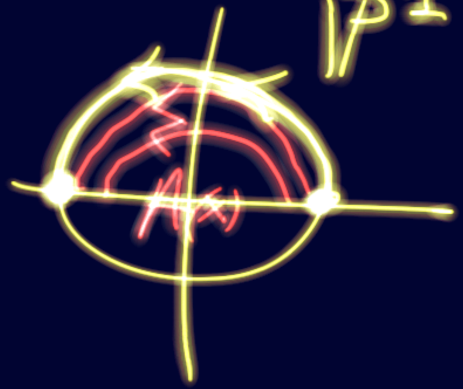
- $f : \mathbb{T}^1 \to \mathbb{T}^1 \rightsquigarrow H_1(f) : \mathbb{Z} \to \mathbb{Z}$
  $[\gamma] \mapsto [f \circ \gamma] = \# [\gamma]$

- $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$
  $$A_\varphi(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} R(2\pi \varphi(x))$$

- Fix $v_0 \in \mathbb{R}^2$.

$$\mathbb{T}^1 \longrightarrow \mathbb{R}^2 \setminus 0 \longrightarrow \mathbb{P}^1 \cong \mathbb{T}^1$$
$$x \longmapsto A(x)v \longmapsto [A(x)v]$$

$$[0, \pi[\xrightarrow{\cong} \mathbb{P}^1$$
$$t \longmapsto [\cos(t), \sin(t)]$$



$$F : \mathbb{T}^1 \times \mathbb{R}^2 \longrightarrow \mathbb{T}^1 \times \mathbb{R}^2$$
$$(x, v) \longmapsto (f(x), A_\varphi(x) v)$$

$$\mathbb{P}F : \mathbb{T}^1 \times \mathbb{P}^1 \to \mathbb{T}^1 \times \mathbb{P}^1$$
$$(x, [v]) \mapsto (f(x), [A_\varphi(x) v])$$

$$= H_1(\mathbb{T}^1 \times \mathbb{P}^1)$$
$$\rightsquigarrow H_1(\mathbb{P}F) : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$H_1(\mathbb{P}F) = \begin{pmatrix} \deg(f) & 0 \\ 2\deg(\varphi) & 1 \end{pmatrix} \qquad \deg\begin{pmatrix} E_N : \mathbb{T}^1 \to \mathbb{T}^1 \\ x \mapsto Nx \end{pmatrix} = N.$$

since A° is UH, $\qquad \mathbb{T}^1 \xrightarrow{f} \mathbb{T}^1 \xrightarrow{S_\circ} \mathbb{P}^1$

$$\boxed{S_\circ : \mathbb{T}^1 \longrightarrow \mathbb{P}^1} \text{ is } C^{\circ}$$

and $\forall x \in \mathbb{T}^1 : A(x) S_x = S_{f(x)}$

$$F(S) = S$$

$\gamma : \mathbb{T}^1 \longrightarrow \mathbb{T}^1 \times \mathbb{P}^1 \qquad C^{\circ}$

$x \longmapsto (x, S_x)$

$\operatorname{im}(\gamma) = \Gamma$
is a loop

$$\mathbb{T}^1 \xrightarrow{\gamma} \mathbb{T}^1 \times \mathbb{P}^1 \xrightarrow{F} \mathbb{T}^1 \times \mathbb{P}^1$$

$$F \circ \gamma(x) = F(x, S_x) = (f(x), S_{f(x)})$$

$$= \gamma \circ f(x) \to \boxed{F \circ \gamma = \gamma \circ f}$$

$$[\gamma] \in H_1(\mathbb{T}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$$

$$[\gamma] = (1, \#(S))$$

$$(\deg(f), \underline{2\deg(\psi) + \#(S)}) = H_1(\mathbb{P}F)\,[\gamma] = [\gamma \circ f] = (\deg(f), \underline{\deg(f)\#S})$$

$$H_1 \longrightarrow \begin{pmatrix} \deg(f) & 0 \\ 2\deg(\psi) & 1 \end{pmatrix}$$

$$\Rightarrow \quad 2\deg(\varphi) + \#(S) = \deg(f)\#(S)$$

$$\Rightarrow \quad \boxed{2\deg(\varphi) = (\deg(f)-1)\,\#(S)}$$

__Ex 1__: $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, $\quad f(x) = E_N x = Nx$

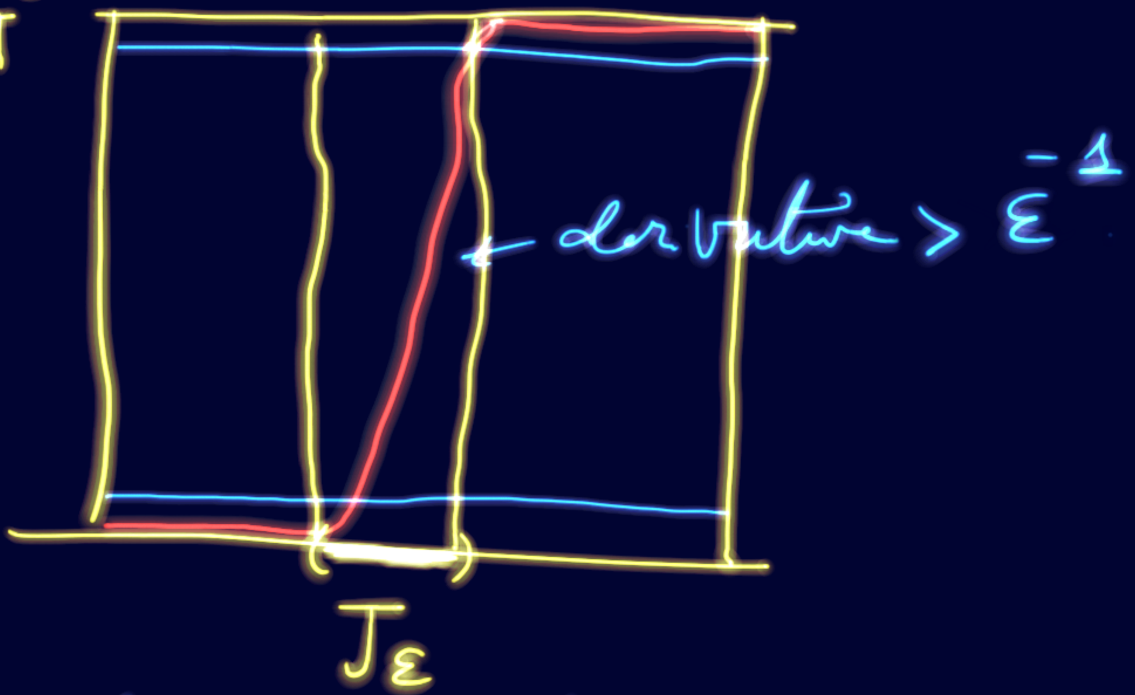
$\Rightarrow \quad 2 = (N-1)\,\#(S)$

$\Rightarrow$ For $N = 4$, no cocycle is hyperbolic.

Lai-Sang Young Examples:

$\varphi_\varepsilon : \mathbb{T}^1 \to \mathbb{T}^1$

C^1



$$A_\varepsilon(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} R\left(2\pi \varphi_\varepsilon(x)\right)$$

over $E_N = f : z \mapsto N \ \mathbb{T}^1$

Thm (Young):

For $\varepsilon > 0$ small enough,

$\exists\, N_\varepsilon \subseteq C^1(\mathbb{T}^1, SL(2,\mathbb{R}))$

open nbhd, $A_\varepsilon \in N_\varepsilon$

$\forall B \in N_\varepsilon$: $\lambda_+(B,x) = \lim_{n\to\infty} \frac{1}{n} \log |B^n(x) v|$ for some v

$\lambda_+(B,x) > 0$. $\rightarrow$ B is nonuniformly hyperbolic

$\lambda_+(B,x) + \lambda_-(B,x) = 0$.