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## Vector Functions :

$$r : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (r_1(t), r_2(t), r_3(t)) = r_1(t)\,\vec{i} + r_2(t)\,\vec{j} + r_3(t)\,\vec{k}.$$

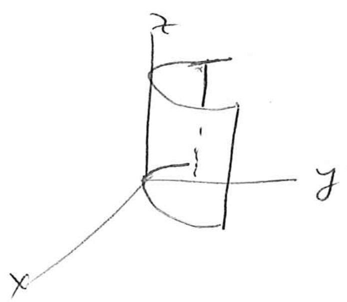
$F(\mathbb{R}, \mathbb{R}^3)$ : set of all $r : \mathbb{R} \to \mathbb{R}^3$. This is a linear space (i.e. additions and scalar mult. are defined on it).

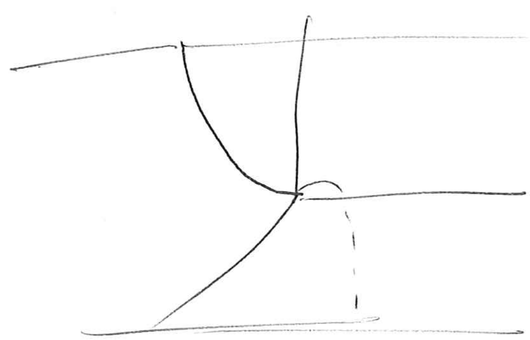
Ex. (Twisted Cubic).

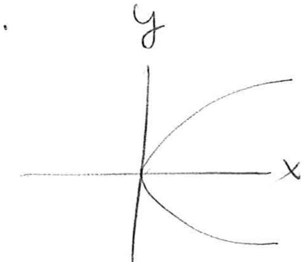
$$r : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (t, t^2, t^3)$$

parametrizes the common zero locus of

$$X^2 - Y = 0 \quad \text{and} \quad X^3 - Z = 0.$$

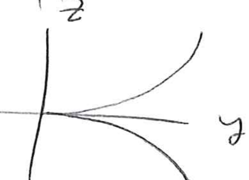






$Y^3 = Z^2$

$Z = Y^{3/2}$



Ex: (Helix).

$$r : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (\cos t, \sin t, t)$$

inside the zero locus of $X^2 + Y^2 = 1$.

SW: Take a quadric surface and an intersecting plane. Write down a parametrization for the intersection.

**Def:** $r \in F(\mathbb{R},\mathbb{R}^3)$ & $t_0 \in \mathbb{R}$. $r$ is <u>continuous</u> at $t_0$

if $\cancel{r_1, r_2, r_3 \in C^0(\mathbb{R},\mathbb{R})}$

$r_1, r_2, r_3 \in F(\mathbb{R},\mathbb{R})$ are continuous at $t_0$.

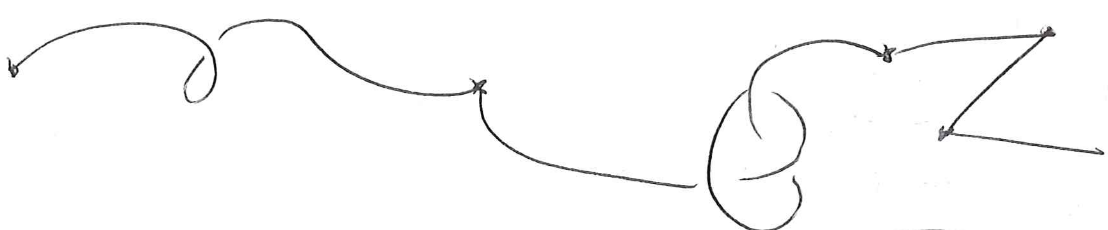
If at all $t_0 \in \mathbb{R}$: $r$ is continuous at $t_0$, then $r$ is continuous.

$C^0(\mathbb{R},\mathbb{R}^3)$ $\to$ set of continuous functions $\mathbb{R} \to \mathbb{R}^3$

<u>SW</u>: $C^0(\mathbb{R},\mathbb{R}^3)$ is a linear space.

<u>Ex</u>: Polys, trigs, — cts.

<u>SW</u>: • $C^0(\mathbb{R},\mathbb{R}^3)$ is closed under concatenation, ~~dot products~~, cross products



• $r_0, \tilde{r} \in C^0(\mathbb{R},\mathbb{R}^3)$

$\to$ $r \cdot \tilde{r} \in C^0(\mathbb{R},\mathbb{R})$.

$\boxed{\text{all algebraic operations are continuous.}}$

Def: $r \in F(\mathbb{R}, \mathbb{R}^3)$, $t_0 \in \mathbb{R}$.

$r$ is diff. at $t_0$ if $r_1, r_2, r_3 \in F(\mathbb{R}, \mathbb{R})$ diff. at $t_0$,

$$\dot{r}(t_0) := \frac{dr}{dt}(t_0) = r'(t_0) = \left(\dot{r}_1(t_0), \dot{r}_2(t_0), \dot{r}_3(t_0)\right)$$

$r$ is diff. if it is diff. everywhere.

$D(\mathbb{R}, \mathbb{R}^3)$ is the set of all diff functions.

SW: . $D(\mathbb{R}, \mathbb{R}^3)$ is a linear space.

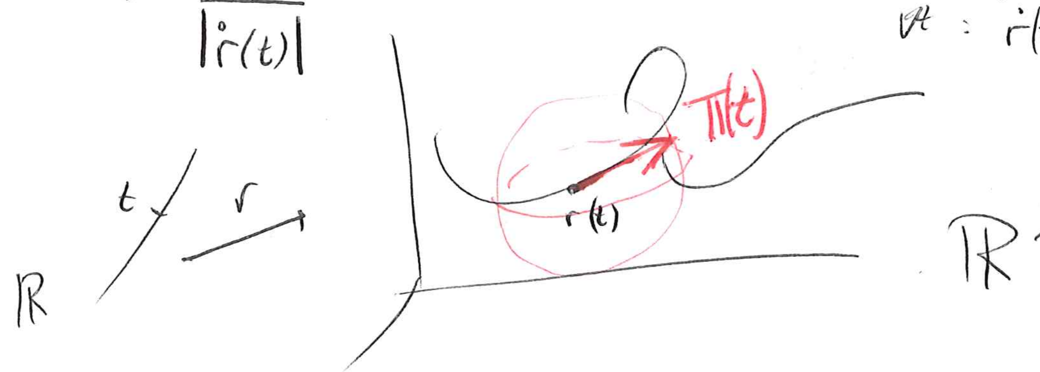
. $D(\mathbb{R}, \mathbb{R}^3) \subseteq C^0(\mathbb{R}, \mathbb{R}^3)$

diff. at a point $\rightarrow$ continuous at the same point.

. $r \in D(\mathbb{R}, \mathbb{R}^3)$. Its unit tangent vector is:

$$\begin{aligned} \mathbb{T} : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto \frac{\dot{r}(t)}{|\dot{r}(t)|} \end{aligned}$$ , defined whenever $\dot{r}(t) \neq 0$.

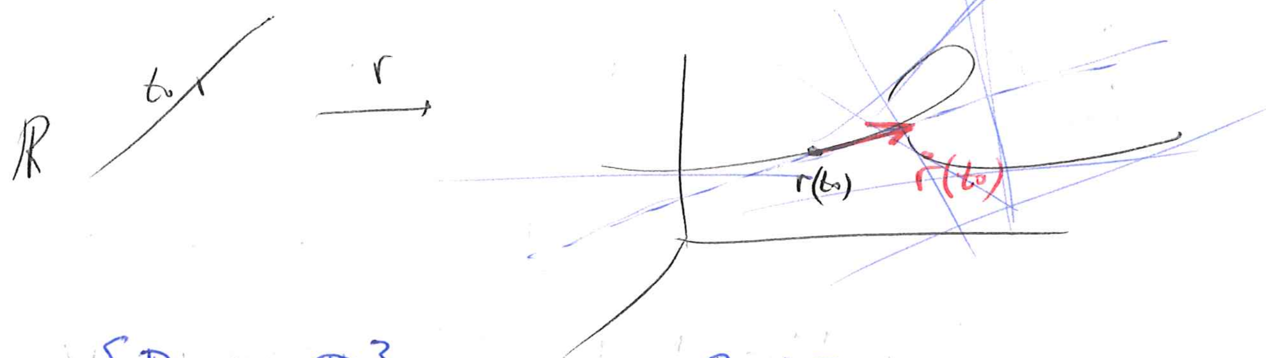
$r$ is regular if $\forall t : \dot{r}(t) \neq 0$.



Ex:

Ex. $r \in D(\mathbb{R}, \mathbb{R}^3)$, $t_0 \in \mathbb{R}$.

What is the eq. for the line tangent to $r$ at $r(t_0)$?




$$\left\{\begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R}^3 \\ s \longmapsto r(t_0) + s\,\dot r(t_0) \end{array}\right\}$$

~~this~~ ~~th~~ $T_{r(t_0)} C$, where $C$ is the curve in 3D drawn by $r$.

Ex. $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto (t, t^2, t^3)$ position

$\dot\gamma(t) = (1, 2t, 3t^2)$ velocity $\quad |\dot\gamma(t)| = \sqrt{1+4t^2+9t^3}$ speed.

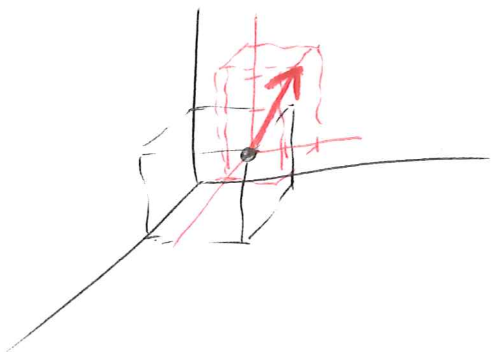
$\ddot\gamma(t) = (0, 2, 6t)$ acceleration

$\gamma(0) = (0,0,0)$

$$\mathbb{T}(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|} = \frac{1}{\sqrt{1+4t^2+9t^3}}(1, 2t, 3t^2). \quad \mathbb{T}(0) = (1,0,0).$$

$$\gamma(1) = (1,1,1).$$

$$\mathbb{T}(1) = \frac{1}{\sqrt{14}}(1,2,3).$$



$$\mathbb{T}(t) \in T_{r(t)}\mathbb{R}^3 = \{r(t)\} \times \mathbb{R}^3$$

$$= \{(r(t), v) \mid v \in \mathbb{R}^3\}$$

tangent bundle of $\mathbb{R}^3$ = phase space

$$T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3 = \{(p,v) \mid p \in \mathbb{R}^3, v \in \mathbb{R}^3\}$$

$$= \bigcup_{p \in \mathbb{R}^3} \{(p,v) \mid v \in \mathbb{R}^3\}.$$

$$= \bigcup_{p \in \mathbb{R}^3} \underbrace{T_p \mathbb{R}^3}$$

tangent space at $p$.
= set of all possible vectors emanating from $p$.

at time $t=1$:

position $= (1,1,1) = \gamma(1)$

velocity $= \dot\gamma(1) = (1,2,3)$

acceleration $= \ddot\gamma(1) = (0,2,1)$

speed $= |\dot\gamma(1)| = \sqrt{14}$.

SW:

$$\left.\begin{aligned} &\measuredangle(\gamma(t), \dot\gamma(t)) = ? \\ &\measuredangle(\dot\gamma(t), \ddot\gamma(t)) = ? \\ &\measuredangle(\gamma(t), \ddot\gamma(t)) = ? \end{aligned}\right\} \text{(as functions of } t\text{)}$$

SW : ~~$\partial_t\, c(t)\,\gamma(t)$~~

$c \in D(\mathbb{R},\mathbb{R})$ , $u, v \in D(\mathbb{R},\mathbb{R}^3)$.

$$\left.\begin{aligned}
\partial_t (cu) &= \dot{c}u + c\dot{u} \\
\partial_t (u\cdot v) &= \dot{u}\cdot v + u\cdot\dot{v} \\
\partial_t (u\times v) &= \dot{u}\times v + u\times\dot{v} \\
\partial_t (u\circ c) &= \dot{u}(c)\,\dot{c}.
\end{aligned}\right\}$$

Chain Rule.

$\frac{d}{dt}\left(u(t)\cdot v(t)\right)$

$= \frac{d}{dt}\left(u_1(t)v_1(t)+u_2(t)v_2(t)+u_3(t)v_3(t)\right)$

$= \dot{u}_1v_1+u_1\dot{v}_1+\dot{u}_2v_2+u_2\dot{v}_2+\dot{u}_3v_3+u_3\dot{v}_3$

$= \dot{u}\cdot v + u\cdot\dot{v}$

<u>Prop</u>: $\gamma\in D(\mathbb{R},\mathbb{R}^3)$. If $|\gamma(t)|$ is constant, then $\gamma\perp\dot{\gamma}$.

($\Leftrightarrow$ $\gamma$ moves in a sphere centered at $0$.)

<u>Pf</u>: $0 = \partial_t\left(|\gamma(t)|^2\right)$ ~~$= \gamma(t)\cdot\dot{\gamma}$~~

$= \partial_t\left(\gamma(t)\cdot\gamma(t)\right) = 2\,\dot{\gamma}(t)\cdot\gamma(t)$

$\to \dot{\gamma}(t)\cdot\gamma(t) = 0.$ ✓.

<u>Def</u>: <u>Integral</u> of a $\gamma\in C^0(\mathbb{R},\mathbb{R}^3)$ is defined entrywise

$\left(\int_0^1\gamma(t)\,dt\in\mathbb{R}^3\right.$ is the unique vector such that $\forall v\in\mathbb{R}^3: \left(\int_0^1\gamma(t)\,dt\right)\cdot v = \left.\int_0^1(\gamma(t)\cdot v)\,dt.\right)$.

Ex: particle w/ mass $m > 0$,

pos. vec. : $r(t)$.

<u>angular momentum</u> = $AM(t) := m\, r(t) \times v(t)$.

<u>torque</u> = $Tor(t) := m\, r(t) \times a(t)$.

$$\frac{d}{dt} AM(t) = m\left(\dot{r}(t) \times v(t) + r(t) \times \dot{v}(t)\right)$$

$$= m(\underbrace{v \times v}_{=0} + r \times a) = Tor(t).$$

$\Rightarrow$ $\boxed{\text{If } Tor(t) = 0, \text{ } AM(t) \text{ is constant}}$

Conservation of angular momentum

Ex: $\gamma : \mathbb{R} \to \mathbb{R}^2$
$t \mapsto (t^3, t^2)$.

pos. vector.

vel vec.? unit tangent vec?
acc. vec. ? tangent line?
speed ? at $t=1$?

$v(t) = \dot\gamma(t) = (3t^2, 2t)$ = vel. vec.

not regular!

$a(t) = \ddot\gamma(t) = (6t, 2)$ = acc. vec.

$|\dot\gamma(t)| = \sqrt{9t^4 + 4t^2} = \sqrt{t^2(9t^2+4)} = |t|\sqrt{9t^2+4}$ = speed

~~unit tangent~~

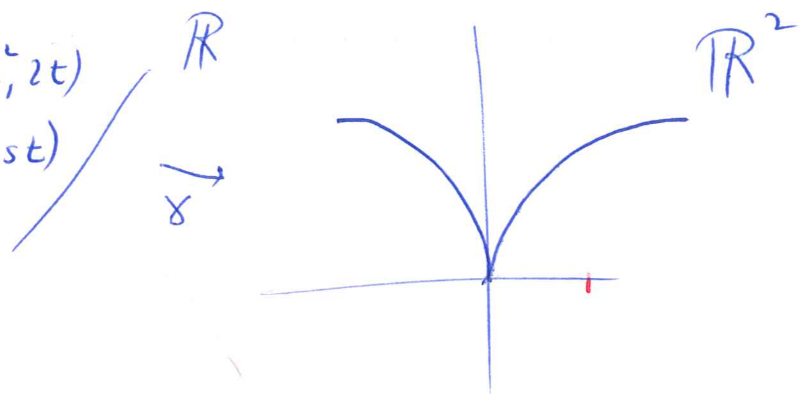
$T(t) = \dfrac{\dot\gamma(t)}{|\dot\gamma(t)|} = \dfrac{1}{|t|\sqrt{9t^2+4}}(3t^2, 2t) = \dfrac{1}{\sqrt{9t^2+4}}\left(3|t|, 2\dfrac{t}{|t|}\right)$

(for $t \neq 0$)

$\ell_t(s) = \gamma(t) + s\,\dot\gamma(t) = (t^3, t^2) + s(3t^2, 2t)$
$= (t^3 + 3st^2, t^2 + 2st)$

$\gamma(1) = (1,1)$.

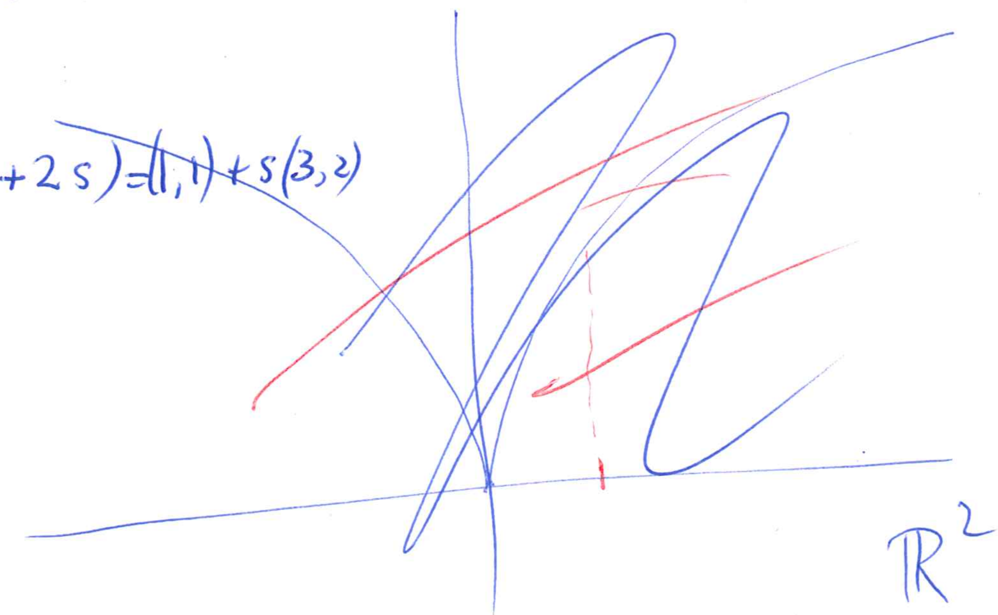
$\dot\gamma(1) = (3,2)$

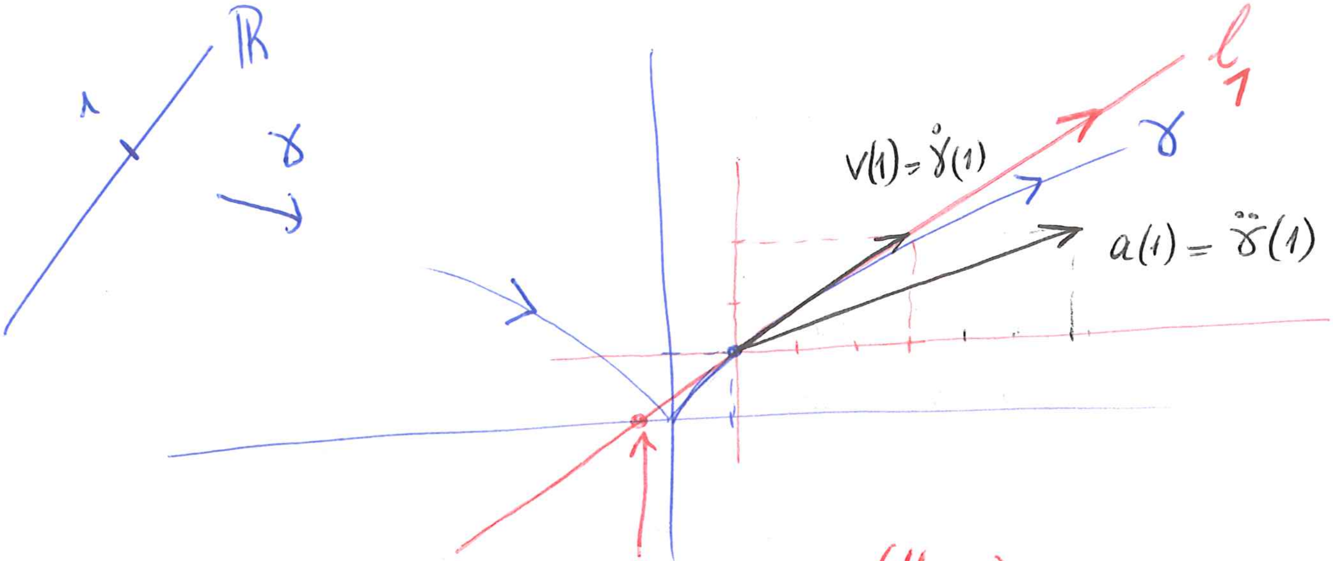



$T(1) = \dfrac{1}{\sqrt{13}}(3,2)$

$\ddot\gamma(1) = (6,2)$

$\ell_1(s) = (1+3s, 1+2s) = (1,1) + s(3,2)$




$$s : \ell_1(s) = (\#, 0)$$

$$(1+3s,\ 1+2s) = (\#, 0)$$

$$s = -\frac{1}{2} \to \# = -\frac{1}{2}.$$

<u>Ex</u>: $\gamma(t) = (t, t^2, t^3)$.

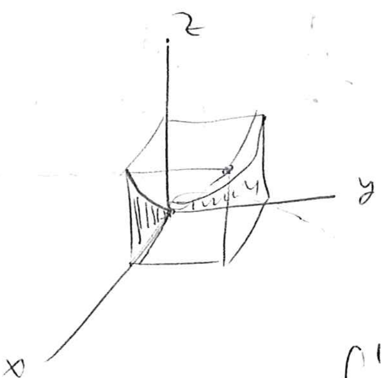
$$\int_0^1 \gamma(t)\,dt = \left( \int_0^1 t\,dt, \int_0^1 t^2 dt, \int_0^1 t^3 dt \right)$$

$$= \left( \left[\frac{t^2}{2}\right]\Big|_0^1, \left[\frac{t^3}{3}\right]\Big|_0^1, \left[\frac{t^4}{4}\right]\Big|_0^1 \right)$$

$$= \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right).$$

← SW: What is this? "Area underneath"?

$$\gamma: \mathbb{R} \to \mathbb{R}^2$$
$$t \mapsto (t, t^2).$$





$$\int_0^1 (\gamma(t) \cdot v)\,dt = \left( \int_0^1 \gamma(t)\,dt \right) \cdot v$$

$$= \frac{v_1}{2} + \frac{v_2}{3} + \frac{v_3}{4}.$$

---

$$\gamma: \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t)).$$

$$\int_0^1 \gamma(t)\,dt := \left( \int_0^1 \gamma_1(t)\,dt, \int_0^1 \gamma_2(t)\,dt, \int_0^1 \gamma_3(t)\,dt \right)$$

$$\left( \int_0^1 \gamma(t)\,dt \right) \cdot v = \int_0^1 \gamma(t) \cdot v \; dt$$

Ex: position vector of an object in the plane:

$$\gamma(t) = (t^2, e^t, te^t)$$

$$\Rightarrow \dot{\gamma}(t) = (2t, e^t, e^t + te^t) = \text{vel. vec}$$

$$\ddot{\gamma}(t) = (2, e^t, e^t + e^t + te^t) = \text{acc. vec.}$$

$$|\dot{\gamma}(t)| = \sqrt{4t^2 + e^{2t} + (t+1)^2 e^{2t}} = \text{speed.}$$

Ex: initial position $= r(0) = (1, 0, 0)$

initial velocity $= v(0) = (1, -1, 1)$

acc. $= a(t) = (4t, 6t, 1)$

$r(t) = ?$ $v(t) = ?$

$\dot{r}(t) = v(t)$

$\dot{v}(t) = a(t)$

$$v(t) - v(0) = \int_0^t \dot{v}(\tau)\, d\tau$$

$$= \int_0^t a(\tau)\, d\tau$$

$$= \left[(2\tau^2, 3\tau^2, \tau)\right]\Big|_0^t$$

$$= (2t^2, 3t^2, t)$$

$$v(t) = v(0) + (2t^2, 3t^2, t)$$

$$= (1+2t^2, -1+3t^2, 1+t).$$

$$r(t) - r(0) = \int_0^t \dot{r}(\tau)\,d\tau = \int_0^t v(\tau)\,d\tau$$

$$= \left[\left(\tau + \frac{2}{3}\tau^3, -\tau+\tau^3, \tau + \frac{\tau^2}{2}\right)\right]\Bigg|_0^t$$

$$= \left(t + \frac{2}{3}t^3, -t+t^3, t+\frac{t^2}{2}\right).$$

$$r(t) = r(0) + \left(t+\frac{2}{3}t^3, -t+t^3, t+\frac{t^2}{2}\right)$$

$$= \left(1+t+\frac{2}{3}t^3, -t+t^3, t+\frac{t^2}{2}\right)$$

---

- Newton's 2nd Law. $\boxed{F(t) = m\,a(t).}$

$m > 0$, $F : \mathbb{R} \to \mathbb{R}^3$ ← given

$a : \mathbb{R} \to \mathbb{R}^3$ ← unknown

mass of point

force acting on the point

acceleration of the point.

$F : \mathbb{R}^d \to \mathbb{R}^3$

$$\boxed{F\left(t, r(t), \dot{r}(t), \ddot{r}(t), \dddot{r}(t), \dots, \overset{(d-1)}{r}(t)\right) = m\,\ddot{r}(t).}$$

w/ $m = 10\,[kg]$.

Ex: An object has position vector

~~$r(t) = (a\cos(\omega t), \ldots$~~

$$r(t) = (\cos(t), \sin(t)).\ [m] = [m, m].$$

$F(t) = ?$

$$F(t) = m\,a(t) = m\,\ddot{r}(t) = 10 \cdot \frac{d}{dt}(-\sin t, \cos t)$$

$$= 10(-\cos t, -\sin t)\left[kg\, m/s^2\right].$$

$$= -10\, r(t)\ \boxed{\left[kg\, m/s^2\right]}$$

Ex: projectile w/ mass: $m > 0$

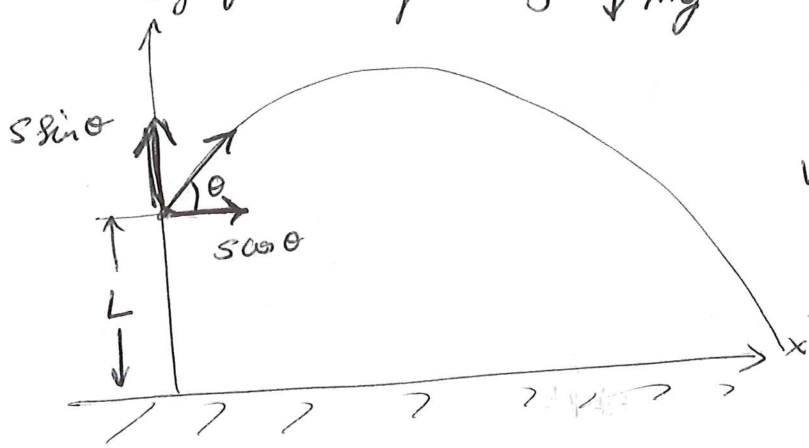
is fired w/ muzzle speed: $s > 0$.

~~above the ground level~~

initial ~~ground~~ height $= L \geq 0$.

With angle of elevation: $\theta \in [0, \pi/2]$.

acting force = gravity $\downarrow mg$




$r(0) = (0, L)$

$v(0) = (s\cos\theta,\ s\sin\theta)$

$$F(t) = m\,a(t)$$

$$\cancel{F(t) = (0, -mg) = (m\,a_1(t),\ m\,a_2(t))}$$

$$\cancel{\Rightarrow a_1(t) = 0. \Rightarrow v_1(t) = C_1}$$

$$\cancel{C_1 = v_1(0) = s\cos\theta}$$

$$\cancel{\Rightarrow \boxed{v_1(t) = s\cos\theta}}$$

$$\cancel{a_2(t) = -g \Rightarrow v_2(t) = -gt + C_2}$$

$$\cancel{s\sin\theta = v_2(0) = C_2}$$

$$\cancel{\Rightarrow \boxed{v_2(t) = -gt + s\sin\theta}}$$

$$a(t) = \frac{1}{m} F(t) = \frac{1}{m}(0, -mg) = (0, -g)$$

$$\Rightarrow v(t) - v(0) = (0, -gt)$$

$$\Rightarrow v(t) = v(0) + (0, -gt) = (s\cos\theta, s\sin\theta - gt)$$

$$\Rightarrow r(t) - r(0) = t\,v(0) + \left(0, -\frac{g t^2}{2}\right)$$

$$\Rightarrow r(t) = r(0) + t\,v(0) + \left(0, \frac{-g}{2} t^2\right)$$

$$= (0, L) + (t s\cos\theta, t s\sin\theta) + \left(0, \frac{-g}{2} t^2\right)$$

$$= \left(t s\cos\theta, L + t s\sin\theta - \frac{g}{2} t^2\right)$$

~~Other parameters fixed, what value of $\theta$ maximizes the distance?~~

Projectile hits the ground when $r(t) = (a, 0)$, $a > 0$.

$$\boxed{\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}}$$

$$0 = L + t s\sin\theta - \frac{g}{2} t^2$$

$$= \left(-\frac{g}{2}\right) t^2 + (s\sin\theta)\, t + L$$

$$t_{\mp} = \frac{-s\sin\theta \mp \sqrt{(s\sin\theta)^2 + 4\frac{g}{2}L}}{-g}$$

$$t_{\mp} = \frac{s \sin\theta \pm \sqrt{s^2 \sin^2\theta + 2gL}}{g}$$

$$\Rightarrow \boxed{t_{+} = \frac{s \sin\theta + \sqrt{s^2 \sin^2\theta + 2gL}}{g}}$$

time of hitting the ground.

$a = t_{+}\, s \cos\theta$ = distance traveled until hitting the ground.

m is irrelevant.

SW: · Max height? · speed when the projectile hits the ground?

· ~~Find the combo of s, L~~

· Fix $L > 0$, $s > 0$.

Find ~~the combo of s~~ $\theta$ that maximizes:

→ speed ~~of~~ when hitting the ground

~~→ max h~~

→ distance travelled

## Arc Length:

- Let $\gamma : [a,b] \to \mathbb{R}^3$ be diff. Then its <u>arc length</u> is defined as:

$$AL(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt.$$

The <u>arc length function</u> is:

$$\begin{aligned} h_\gamma : [a,b] &\to \mathbb{R} \\ t &\longmapsto \int_a^t |\dot{\gamma}(\tau)|\, d\tau \end{aligned}$$

$h_\gamma(a) = 0, \quad h_\gamma(b) = AL(\gamma).$

<u>Ex</u>: $\gamma : [0, 2\pi] \to \mathbb{R}^3$

$$t \longmapsto (\cos t, \sin t, t).$$

$\dot{\gamma}(t) = (-\sin t, \cos t, 1)$

$|\dot{\gamma}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$

$\Rightarrow h_\gamma : [0, 2\pi] \to \mathbb{R}, \quad h_\gamma(t) = \int_0^t \sqrt{2}\, d\tau = t\sqrt{2}.$

$AL(\gamma) = h_\gamma(2\pi) = 2\pi\sqrt{2}.$

# Parametrization by Arc Length

· Suppose ~~$\gamma:\mathbb{R}\to$~~ $\gamma:[a,b]\to\mathbb{R}^3$

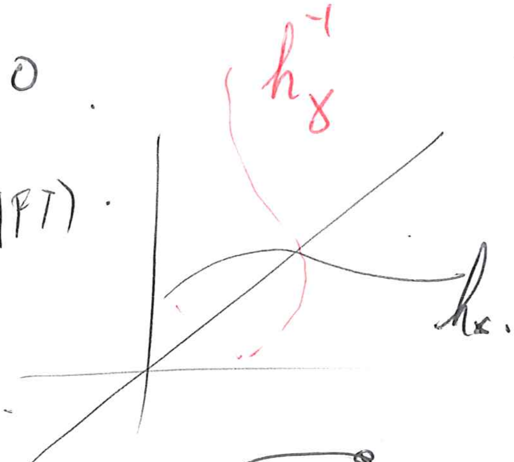
is regular (i.e. $\dot\gamma(t)\neq 0$).

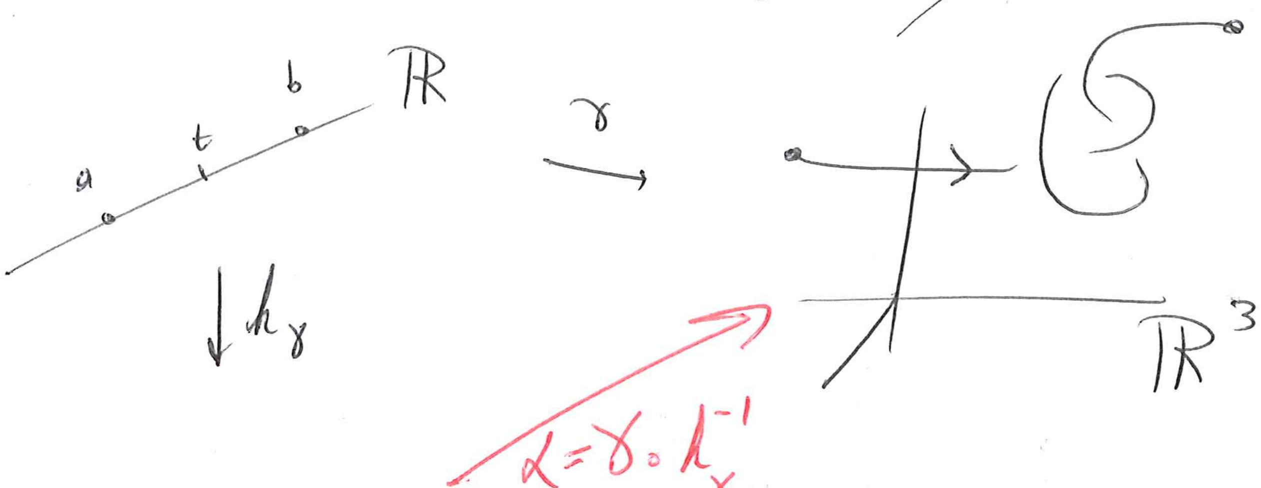
Then $h_\gamma(t)=\int_a^t|\dot\gamma(\tau)|\,d\tau$

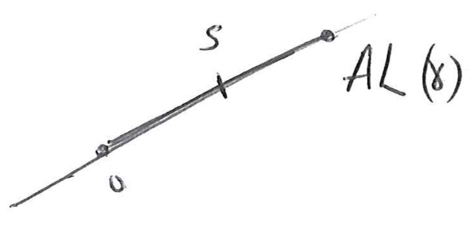
$\Rightarrow \frac{d}{dt}h_\gamma(t)=|\dot\gamma(t)|\neq 0$.

$\Rightarrow h_\gamma$ is invertible (IFT).

$s=h_\gamma(t) \Leftrightarrow t=h_\gamma^{-1}(s)$.




$\gamma\circ h_\gamma^{-1}:[0, AL(\gamma)]\to\mathbb{R}^3$

is the reparametrization of $\gamma$ by arclength.

$$\frac{d}{ds}\alpha(s) = \frac{d}{ds}\left(\gamma \circ h_\gamma^{-1}\right)(s)$$

$$= \frac{d}{dt}\gamma\left(h_\gamma^{-1}(s)\right) \cdot \frac{d}{ds} h_\gamma^{-1}(s)$$

$$= \dot\gamma\left(\underbrace{h_\gamma^{-1}(s)}_{=t}\right) \cdot \frac{d}{ds} h_\gamma^{-1}(s)$$

$$= \dot\gamma(t) \frac{1}{|\dot\gamma(t)|}$$

$$\Rightarrow \left|\frac{d}{ds}\alpha(s)\right| = 1.$$

$$\Rightarrow h_\alpha(s) = \int_0^s \left|\frac{d}{ds}\alpha(\sigma)\right| d\sigma$$

$$= \int_0^s 1 \, d\sigma = s.$$

$$t = h_\gamma \circ h_\gamma^{-1}(t)$$

$$\cancel{1 = \frac{dh_\gamma}{dt} \frac{dh_\gamma}{}}$$

$$1 = \frac{d(h_\gamma^{-1})}{ds}(s) \frac{dh_\gamma}{dt}(t)$$

$$= \frac{d(h_\gamma^{-1})}{ds}(s) |\dot\gamma(t)|$$

$$\Rightarrow \frac{d(h_\gamma^{-1})}{ds}(s) = \frac{1}{|\dot\gamma(t)|}$$

$$= \frac{1}{}$$

Ex: Arclength par. of $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $t \mapsto (\cos t, \sin t, t)$ ?

$h_\gamma(t) = t\sqrt{2} = s$ $\qquad$ $\dot\gamma \neq 0 \Rightarrow \gamma$ is regular.

$h_\gamma : [0, 2\pi\sqrt{2}] \to \mathbb{R}$. $\qquad t = h_\gamma^{-1}(s) = \frac{s}{\sqrt{2}}$. $\qquad \alpha(s) = \gamma\left(\frac{s}{\sqrt{2}}\right)$

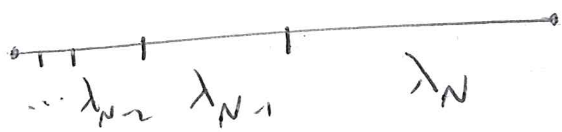
$$\alpha(s) = \gamma\left(\frac{s}{\sqrt{2}}\right) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right).$$

SW: Let ~~$\gamma = \gamma_1 + \gamma_2 +$~~

$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_N$ be

a polygonal path.

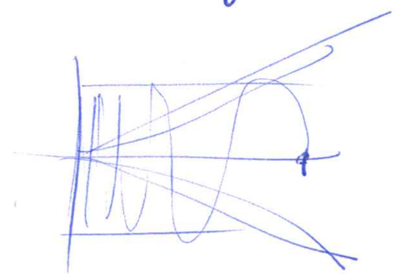
Parametrize by arc length.



Let $\gamma : [a,b] \to \mathbb{R}^3$ be regular (i.e., $\dot{\gamma} \neq 0$).

$$h_\gamma : [a,b] \to \cancel{\mathbb{R}}\ [0,\infty]. \qquad \text{is its } \underline{\text{arclength function}}$$
$$t \mapsto \int_a^t |\dot{\gamma}(\tau)|\, d\tau$$

$h_\gamma(b) = \int_a^b |\dot{\gamma}(t)|\, dt = AL(\gamma)$ is the <u>arclength</u> of $\gamma$.



<u>Ex</u>: $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$$t \mapsto \begin{cases} \left(t, t^2 \sin \frac{1}{t}\right), & \text{if } t \neq 0 \\ (0,0), & \text{if } t = 0 \end{cases}.$$

$$t \mapsto \begin{cases} \left(2\pi - t, (2\pi - t) \sin\left(\frac{1}{2\pi - t}\right)\right), & \text{if } t \neq 2\pi \\ (0,0) & \text{if } t = 2\pi \end{cases}$$

$$\dot{\gamma}(t) = \left(-1, -1 \sin\left(\frac{1}{2\pi - t}\right) + (2\pi - t)\cos\left(\frac{1}{2\pi - t}\right) \cdot \frac{+1}{(2\pi - t)^2}\right)$$
(for $t \neq 2\pi$.)

$$= \left(-1, -\sin\left(\frac{1}{2\pi - t}\right) + \frac{1}{2\pi - t}\cos\left(\frac{1}{2\pi - t}\right)\right).$$

$$|\dot{\gamma}(t)|^2 = 1 + \sin^2\left(\frac{1}{2\pi - t}\right) + \frac{1}{(2\pi - t)^2}\cos^2\left(\frac{1}{2\pi - t}\right) + \frac{2}{2\pi - t}\sin\left(\frac{1}{2\pi - t}\right)\cos\left(\frac{1}{2\pi - t}\right)$$

$$= 1 + \sin^2\left(\frac{1}{2\pi - t}\right) + \frac{1}{(2\pi - t)^2}\cos^2\left(\frac{1}{2\pi - t}\right) + \frac{\sin\left(\frac{2}{2\pi - t}\right)}{2\pi - t}$$

$$|\dot{\gamma}(t)| = \sqrt{1 + \sin^2\left(\frac{1}{2\pi - t}\right) + \frac{1}{(2\pi - t)^2}\cos^2\left(\frac{1}{2\pi - t}\right) + \frac{1}{2\pi - t}\sin\left(\frac{2}{2\pi - t}\right)}$$

$\cdot (X, d) \in \underline{\mathcal{Met}}$, $\gamma \in C^0(I, X)$.

$$\mathrm{Par}([a,b]) = \left\{ \frac{\{p_0, p_1, \dots, p_n\} \subseteq \mathcal{P}([a,b])}{a = p_0 \leq p_1 \leq \dots \leq p_{n-1} \leq p_n = b} \right\}.$$
$n \in \mathbb{Z}_{\geq 0}$

~~sup~~ $d$

$$\sup \left\{ \sum_{k=1}^{n} \overleftarrow{\gamma}(d)(p_k, p_{k-1}) \;\middle|\; \{p_0, p_1, \dots, p_n\} \in \mathrm{Par}([a,b]) \right\}$$

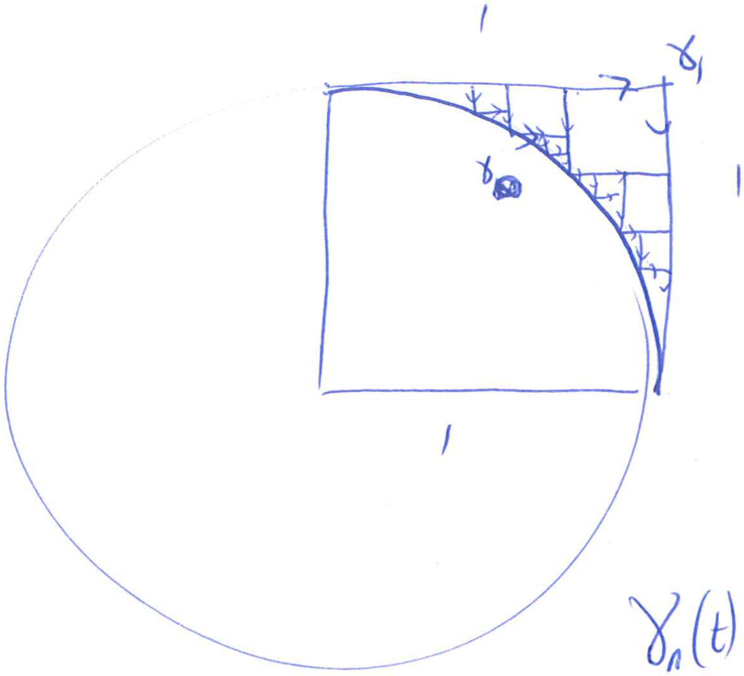
$=: \mathrm{Length}(\gamma)$.

$\mathrm{Length} : C^0(I, X) \longrightarrow [0, \infty]$.

$\mathcal{L}$ is a lower semi-continuous functional on $C^0([a,b], X)$ w/r/t pointwise/uniform convergence.

$\forall$ rectifiable $\{\gamma_n\}_n$ : $\gamma_n(t) \to \gamma(t)$ $\forall t$,

then $\liminf_{n \to \infty} \mathcal{L}(\gamma_n) \geq \mathcal{L}(\gamma)$.

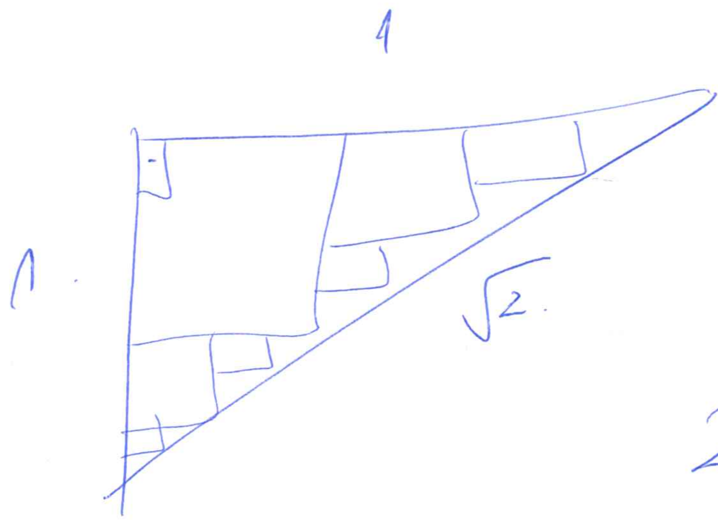



$$\frac{2\pi \, 1}{4} = \boxed{\frac{\pi}{2}}$$

$$\mathcal{L}(\gamma_0) = \pi/2$$

$$\mathcal{L}(\gamma_n) = 2.$$

$$\gamma_n(t) \longrightarrow \gamma_0(t), \quad \forall t.$$




$$2 = \sqrt{2}$$

$$\sqrt{2}\,(\sqrt{2}-1) = 0.$$

$$\sqrt{2} = 0 \quad \text{or} \quad \sqrt{2} = 1.$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{2 = 0} \qquad\qquad \boxed{2 = 1}.$$

# Curvature of a curve in 3D.

- Let $\gamma : [a,b] \to \mathbb{R}^3$ be regular. Its curvature

is $$\kappa : [a,b] \to \mathbb{R}$$
$$t \longmapsto \frac{|\dot{T}(t)|}{|\dot{\gamma}(t)|}.$$

Ex: $\gamma(t) = p + tv$.

$$\dot{\gamma}(t) = v \to |\dot{\gamma}(t)| = |v|$$

$$T(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \frac{v}{|v|}$$

$$\dot{T}(t) = 0. \Rightarrow \kappa(t) = 0. \Rightarrow$$ (lines don't bend).

Ex: $\gamma(t) = (r\cos(t), r\sin(t))$

$$\dot{\gamma}(t) = (-r\sin(t), r\cos(t)) \qquad |\dot{\gamma}(t)| = r.$$

$$T(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = (-\sin(t), \cos(t))$$

$$\dot{T}(t) = (-\cos(t), -\sin(t)) \qquad |\dot{T}(t)| = 1$$

$$\kappa(t) = \frac{|\dot{T}(t)|}{|\dot{\gamma}(t)|} = \frac{1}{r}.$$

- circles bend at a constant rate.
- bigger circles bend less.
- lines are circles w/ $\infty$ radius.

$\underline{Ex}$: $\gamma(t) = (t, t^2, t^3)$.

$$\dot\gamma(t) = (1, 2t, 3t^2)$$

$$|\dot\gamma(t)| = \sqrt{1 + 4t^2 + 9t^4}.$$

$$T(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|} = \frac{1}{\sqrt{1+4t^2+9t^4}} (1, 2t, 3t^2).$$

$$\dot T(t) = \left(\frac{-1}{2}\right) (1+4t^2+9t^4)^{-3/2} \cdot (8t + 36t^3) (1, 2t, 3t^2)$$

$$+ \frac{1}{\sqrt{1+4t^2+9t^4}} (0, 2, 6t)$$

$$= \frac{-4t - 18t^3}{\left(\sqrt{1+4t^2+9t^4}\right)^3} (1, 2t, 3t^2) + \frac{1}{\sqrt{1+4t^2+9t^4}} (0, 2, 6t)$$

$$= \frac{-4t - 18t^3}{|\dot\gamma(t)|^3} (1, 2t, 3t^2) + \frac{1}{|\dot\gamma(t)|} (0, 2, 6t)$$

$$= \frac{1}{|\dot\gamma(t)|} \left( \frac{-(4t+18)}{|\dot\gamma(t)|^2} (1, 2t, 3t^2) + (0, 2, 6t) \right)$$

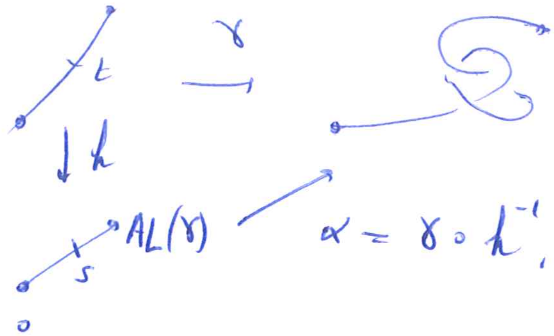
$\cancel{K(t) = \dot T(t)}$

? ? . . . .

**Thm:** $\gamma : [a,b] \to \mathbb{R}^3$ be reg. (and twice diff.)

$$\kappa(t) = \frac{|\dot\gamma(t) \times \ddot\gamma(t)|}{|\dot\gamma(t)|^3}.$$

**Pf:**




$\alpha = \gamma \circ h^{-1}$.

$\dot h(t) = |\dot\gamma(t)|$

$$\kappa(t) = \frac{|\dot T(t)|}{|\dot\gamma(t)|} \qquad T(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|} = \frac{\dot\gamma(t)}{\dot h(t)} \Rightarrow \begin{array}{l}\dot\gamma(t) = \dot h(t)\, T(t) \\ \ddot\gamma(t) = \ddot h T + \dot h \dot T\end{array}$$

$$\Rightarrow |\dot T(t)| = \kappa(t)\, |\dot\gamma(t)| \qquad \Rightarrow \cancel{\dot T(t) = \frac{\ddot\gamma(t)\dot h(t) - \dot\gamma(t)\ddot h(t)}{(\dot h(t))^2}}$$

$$= \kappa(t)\, \dot h(t)$$

$$\dot\gamma(t) \times \ddot\gamma(t) = \left(\dot h(t)\, T(t)\right) \times \left(\ddot h T + \dot h \dot T\right)$$

$$= \dot h \ddot h \underbrace{T \times T}_{=0} + (\dot h)^2\, T \times \dot T$$

$\theta = \measuredangle(T, \dot T)$

$$= (\dot h)^2\, T \times \dot T$$

$$\Rightarrow |\dot\gamma \times \ddot\gamma| = (\dot h)^2 |T \times \dot T| = (\dot h)^2 |T|\,|\dot T| \sin(\theta)$$

$$= (\dot h)^2 |\dot T| = (\dot h)^3 \kappa(t) \rightarrow \kappa(t) = \frac{|\dot\gamma \times \ddot\gamma|}{\dot h^3} = \frac{|\dot\gamma \times \ddot\gamma|}{|\dot\gamma|^3}$$

Ex: $\gamma(t) = (t, t^2, t^3)$

$$\kappa(t) = \frac{|\dot\gamma(t) \times \ddot\gamma(t)|}{|\dot\gamma(t)|^3}$$

$\dot\gamma(t) = (1, 2t, 3t^2)$

$\ddot\gamma(t) = (0, 2, 6t)$

$|\dot\gamma(t)| = \sqrt{1+4t^2+9t^4}$

$$\dot\gamma(t) \times \ddot\gamma(t) = (12t^2 - 6t^2, -6t, 2) = (6t^2, -6t, 2)$$

$$|\dot\gamma(t) \times \ddot\gamma(t)| = \sqrt{36t^4 + 36t^2 + 4}$$

$$\Rightarrow \boxed{\kappa(t) = \sqrt{\frac{36t^4 + 36t^2 + 4}{9t^4 + 4t^2 + 1}}}$$

$$\kappa(0) = 2, \quad \underset{\displaystyle \kappa(-1)}{\overset{\kappa(1)}{\shortparallel}} = \sqrt{\frac{36+36+4}{9+4+1}} = \sqrt{\frac{76}{14}} = \sqrt{\frac{38}{7}}$$

$> 2.$

# T, N, B : (Frenet-Serret Apparatus)

- Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be regular, at least twice diff.

$$T(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$$ is the unit tangent vector

$$N(t) = \frac{\dot{T}(t)}{|\dot{T}(t)|}$$ is the principal normal vector

$$B(t) = T(t) \times N(t)$$ is the binormal vector


$$\kappa(t) = \frac{|\dot{T}(t)|}{|\dot{\gamma}(t)|}$$

$$\kappa N = \frac{\dot{T}}{|\dot{\gamma}|}$$

$$h(t) = \int_0^t |\dot{\gamma}(\tau)| \, d\tau$$

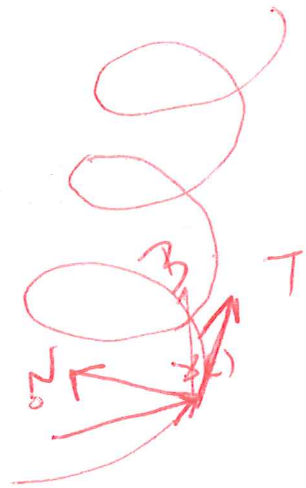
$$\dot{h}(t) = |\dot{\gamma}(t)|$$

$$T(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \frac{\dot{\gamma}}{\dot{h}}$$

$$\dot{T} = \frac{\ddot{\gamma}\dot{h} - \dot{\gamma}\ddot{h}}{(\dot{h})^2}$$

TNB

Ex : $\gamma(t) = (\cos t, \sin t, t)$.




$$\dot\gamma(t) = (-\sin t, \cos t, 1)$$

$$|\dot\gamma(t)| = \sqrt{1+1} = \sqrt{2}.$$

$$T(t) = \frac{1}{\sqrt2}(-\sin t, \cos t, 1)$$

$$\dot T(t) = \frac{1}{\sqrt2}(-\cos t, -\sin t, 0)$$

$$|\dot T(t)| = \frac{1}{\sqrt2}$$

$$N(t) = \frac{\dot T(t)}{|\dot T(t)|} = \cancel{\tfrac{1}{\sqrt2}}\,(-\cos t, -\sin t, 0)$$

$$B(t) = T(t) \times N(t) = \frac{1}{\cancel{2}\sqrt2}\begin{vmatrix} \bar\imath & \bar\jmath & \bar k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\cancel{2}\sqrt2}\left(\sin t\,\bar\imath - \cos t\,\bar\jmath + \bar k\right)$$

$$= \frac{1}{\cancel{2}\sqrt2}(\sin t, -\cos t, 1).$$