## WM – Ch. 12 : Amenable Groups :

> $G \leq SL(\ell, \mathbb{R})$ be semisimple with finitely many connected components, $\Gamma \leq G$ be a lattice.

### §12.1 : Definition :

**Def. (12.1.3)** : Let $H$ be a Lie group. $H$ is <u>amenable</u> if $\forall$ locally convex topological vector space $V$, $\forall$ continuous action $\alpha : H \longrightarrow \mathrm{End}(V)$, $\forall\, S \subseteq V$, $\exists\, s \in S$ :

$$S \text{ is compact, convex, and } H\text{-invariant } \underset{(\alpha(H,S) \subseteq S)}{} \Rightarrow \alpha(H,s) = \{s\}.$$

**Rem. (12.1.4)** :

(i) We assume that all locally convex topological spaces are Hausdorff.

(ii) In applications, $V$ of Def. (12.1.3) is chosen as the dual of a separable Banach space endowed with weak-$*$ topology.

In this case every compact convex subset $S \subseteq V$ is second countable, and hence metrizable.

assuming this special metrizable case

In fact, [redacted] would result in a definition of equal strength. $\boxed{\text{EX } 12.3.\#17}$

(if we consider only second countable groups, that is)

## §12.2. Examples :

- Abelian groups, and in particular cyclic groups, are amenable.
- Compact groups are amenable.
- Extensions of amenable groups are amenable, whence solvable groups are amenable.
- Closed subgroups of amenable groups are amenable.
- Nonabelian free groups are not amenable.
- $SL(2,\mathbb{R})$ is not amenable.

**Group Extensions & Solvability:**

- $G$ is solvable if it can be obtained by a finite sequence of extensions of abelian groups $\Leftrightarrow \exists n: G^{(n)} = 1$, where $G^{(1)} := G,\ G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$
- $G, H, K \in \mathrm{Obj}(\mathcal{G}rp)$. $G$ is an extension of $K$ by $H$ if there is an ses $1 \to K \xrightarrow{\beta} G \xrightarrow{\alpha} H \to 1$, $K \cong \ker(\alpha)$, $H \cong \mathrm{coker}(\beta)$

**Prop. (12.2.1)** : Cyclic groups are amenable.

**Cor (12.2.3) (Kakutani–Markov Fixed Point Theorem)** : Every abelian group is amenable.

**Prop (12.2.4)** : Compact groups are amenable.

**Prop (12.2.6)** : $\forall N \in \mathcal{P}_s(H) \cap \tau^c(H)$ : $N$ and $H/N$ are amenable $\Rightarrow H$ is amenable. $\left(1 \longrightarrow N \hookrightarrow H \xrightarrow{\text{can}} H/N \longrightarrow 1\right)$

(ie. amenable extensions of amenable groups are amenable)

Cor. (12.2.7):

(i) Every solvable group is amenable.

(ii) $\forall N \in \mathcal{P}_{\trianglelefteq}(H)$: if $N$ is solvable and $H/N$ is compact, then $H$ is amenable. $\left(1 \to N \hookrightarrow H \xrightarrow{\text{con}} H/N \to 1\right)$

(i.e., compact extensions of solvable groups are amenable)

Prop (12.2.8): $\forall K \in \mathcal{P}_{\leq}(H) \cap \tau^{c}(H)$: if $H$ is amenable, then so is $K$.

## §12.3. Characterizations:

Thm (12.3.1): Let $H$ be a Lie group. Then TFAE:

AME1

(i) $H$ is amenable, i.e., $\forall V, \forall \alpha : H \to \mathrm{End}(V), \forall S \subseteq V, \exists s \in S$: $\alpha(H,s) = \{s\}$, where $V$ is a locally convex topological vector space, $\alpha$ is a continuous action, $S$ is compact convex and $H$-invariant.

AME2

(ii) $\forall$ compact metrizable topological space $X$, $\forall$ continuous $\alpha : H \to \mathrm{End}(X)$, $\exists \mu \in \mathrm{Prob}(X), \forall h \in H$:

$\alpha(h)_* \mu = \mu$

AME3

(iii) $\exists$ a left-invariant mean $\lambda$ on $C_b^0(H,\mathbb{R})$.
$\to \lambda(\chi_H) = 1$
$\to$ $\mathrm{im}(f) \subseteq [0,\infty[ \Rightarrow \lambda(f) \subseteq [0,\infty[$.

AME4

(iv) $\exists$ left-invariant finitely-additive map $\rho : \mathcal{L}(H) \to [0,1]$ with:

(Lebesgue measurable subsets of $H$.)

$\to \rho(\emptyset) = 0$

$\to \forall \{L_k\}_{k=1}^n \subseteq \mathcal{L}(H)$. $\rho\left(\biguplus_{k=1}^n L_k\right) = \sum_{k=1}^n \rho(L_k)$.

$\to$: $\rho(H) = 1$ (probability)

$\to \rho \ll \mathrm{Haar}_H$ (i.e., $\forall L \in \mathcal{L}(H)$ : $\mathrm{Haar}_H(L) = 0 \Rightarrow \rho(L) = 0$)

AME5

(v) The left regular representation of $H$ on $L^2(H,\mathbb{R})$ has almost invariant vectors.

$L : H \times H \to H$
$(h,x) \mapsto hx$

$\tilde{L} : H \times L^2(H,\mathbb{R}) \to L^2(H,\mathbb{R})$
$(h, \varphi) \mapsto \boxed{\begin{array}{l} L(h^{-1},\cdot)^{*}\varphi \\ x \mapsto \varphi(h^{-1}x) \end{array}}$

$\pi_{reg} : H \to \mathcal{U}(L^2(H,\mathbb{R}))$
$h \mapsto \tilde{L}(h,\cdot)$

is the left-regular representation of $H$ on $L^2(H,\mathbb{R})$.

Let $(V, \|\cdot\|)$ be a normed vector space, $\alpha : H \times V \to V$ be an action. Then $\alpha$ has almost invariant vectors if $\forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists v(\varepsilon, K) \in V : \|v\| = 1$ and $\sup_{k \in K} \|\alpha(k,v) - v\| < \varepsilon$.

$v = v(\varepsilon, K)$ in this case is $(\varepsilon, K)$-invariant.

($\Leftrightarrow \forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists v = v(\varepsilon,K) \in V$: $\|v\| = 1$ & $\alpha(K,v) \subseteq B_\varepsilon(v)$.)

$\boxed{\text{AME6}}$
(vi) $\exists$ a Følner sequence $\{F_n\}_n \subseteq \mathcal{B}(H)$.

$\mathcal{F} := \{F_n\}_n \subseteq \mathcal{B}(H)$ is a Følner sequence if

→ $\forall n$: $0 < \mathrm{Haar}_H(F_n) < \infty$, and

→ $\forall K \in \mathcal{K}(H)$: $\displaystyle \lim_{n\to\infty} \sup_{k \in K} \frac{\mathrm{Haar}_H(F_n \,\Delta\, k F_n)}{\mathrm{Haar}_H(F_n)} = 0.$

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• More on...

H is amenable $\iff$ $\boxed{\text{AME2}}$ $\boxed{\forall X, \forall \alpha : H \to \mathrm{End}(X)\ \exists \mu \in \mathrm{Prob}(X);\ \alpha(H)_* \mu = \{\mu\}, \text{ where } X \text{ is compact metrizable, } \alpha \text{ is a continuous action.}}$

• Since $X$ is compact,

$\mathrm{Prob}(X) \subseteq \underbrace{\overline{B}_1\left( C^0(X, \mathbb{R})^*\right)}_{\text{closed unit ball} \subseteq C^0(X,\mathbb{R})^*}$ by Riesz Representation Theorem (Thm. (B.6.10))

$\Rightarrow$ $\mathrm{Prob}(X)$ is compact by Banach-Alaoglu Theorem (Prop. (B.7.4))

Ex. (12.3.2) : In the setting of $\boxed{\text{AME2}}$,

$\mathrm{Prob}(X)$ is a compact convex $H$-invariant subset of the locally convex topological vector space $C^0(X, \mathbb{R})^*$.

• More on ...

H is amenable $\Leftrightarrow$ $\boxed{\text{AME2}}$ : $\boxed{\exists \text{ left-invariant mean } \lambda \text{ on } C_b^0(H,\mathbb{R})}$

Def (12.3.6): Let $V \leq L^\infty(H,\mathbb{R})$ (eg.: $V := C_b^0(H,\mathbb{R})$), and suppose $\chi_H \in V$. Then $\lambda : V \longrightarrow \mathbb{R}$ is a mean if

→ $\lambda(\alpha v_1 + v_2) = \alpha\,\lambda(v_1) + \lambda(v_2)$ (linear)

→ $\lambda(\chi_H) = 1$ (normalized)

→ $\mathrm{im}(f) \subseteq [0,\infty[ \implies \lambda(f) \in [0,\infty[$ (positive)

Rem. (12.3.7). If $\lambda$ is a mean, then $\|\lambda\| = 1$, and hence [redacted] $\lambda \in C^0 \cap \mathrm{Lin}(V,\mathbb{R}) \doteq V^*$. $\boxed{\text{EX 12.3.\#8}}$

Ex. (12.3.8): Let $\varphi \in L^1(H,\mathbb{R})$ : $\|\varphi\| = 1$ [redacted] $V \leq L^\infty(H,\mathbb{R})$ be any subspace containing $\chi_H$. Then

$$\lambda_\varphi : V \longrightarrow \mathbb{R}$$
$$v \longmapsto \int_H v\,|\varphi|\, d\,\underbrace{\mathrm{Haar}_H}_{\substack{\text{(left-Haar}\\ \text{on } H)}}$$

is a mean.

$\boxed{\text{EX 12.3.\#12}}$

$\{\lambda_\varphi \mid \varphi \in L^1(H,\mathbb{R}) : \|\varphi\| = 1\}$ is weakly* dense in the set of all means.

Rem. (12.3.10): • If $H$ is amenable $\boxed{\text{AME1}}$, then there is a left-invariant mean on $\mathcal{L}^\infty(H,\mathbb{R})$. $\boxed{\text{EX 12.3.\#14}}$

Thus we have a 'new' criterion for amenability:

$\boxed{\text{AME1}} \Rightarrow \boxed{\text{AME3'}}$ : There is a left-invariant mean on $\mathcal{L}^\infty(H,\mathbb{R})$.

$\boxed{\text{AME1}} \Leftrightarrow \boxed{\text{AME3}}$, $\boxed{\text{AME3'}} \Rightarrow \boxed{\text{AME3}}$

• Also, if $H$ is amenable $\boxed{\text{AME1}}$, then there is a mean on $\mathcal{L}^\infty(H,\mathbb{R})$ that is both left- and right-invariant. $\boxed{\text{EX 12.3\#16}}$

• More on...

$H$ is amenable $\Leftrightarrow$ $\boxed{\text{AME4}}$ : $\exists$ left-invariant finitely-additive probability 'measure' that is absolutely continuous with respect to the left Haar measure on $H$

$\rho : \mathcal{L}(H) \longrightarrow \mathbb{R}$.

Obs: $\mathrm{Prob}(X) \Leftrightarrow C^0(X,\mathbb{R})^*$

$\underbrace{\mathrm{FinProb}(X)}_{\substack{\text{finitely-additive} \\ \text{probability 'measure's}}} \Leftrightarrow \mathcal{L}^\infty(X,\mathbb{R})^*$

• More on ...

H is amenable $\Leftrightarrow$ $\boxed{\text{AMES}}$ $\boxed{\pi_{reg}: H \to \mathcal{U}(\mathcal{L}^2(H,\mathbb{R})) \text{ has almost invariant vectors}}$

Def. (12.3.14): Let $(V, \|\cdot\|)$ be a normed vector space, $\alpha: H \to \mathrm{End}(V)$ be an action. $\alpha$ has almost invariant vectors if

$$\forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists v = v(\varepsilon, K) \in V : \|v\| = 1 \text{ \& } \alpha(K)(v) \subseteq B_\varepsilon(v).$$

In this case $v$ is called $(\varepsilon, K)$-invariant.

Ex (12.3.16): Consider $\pi_{reg}: H \longrightarrow \mathcal{U}(\mathcal{L}^2(H,\mathbb{R}))$.

(i) If $H$ is a compact Lie group, then $\chi_H \in \mathcal{L}^2(H,\mathbb{R})$, whence $\mathcal{L}^2(H,\mathbb{R})$ has an $H$-invariant unit vector.

(namely, $\chi_H$ itself).

(ii) If $H = \mathbb{R}$, then $\mathcal{L}^2(H,\mathbb{R})$ has no nonzero $H$-invariant vectors $\boxed{\text{EX 12.3.\#22}}$, though it has almost invariant vectors.

(Q: Is this true for any other (unitary) representation?)

$\forall \varepsilon > 0, \forall K \in \mathcal{K}(\mathbb{R}), \exists N \in \mathbb{Z}_{>0}$ : $K \subseteq [-N, N]$ and $2/\sqrt{N} < \varepsilon$. Put $\varphi := \frac{1}{N} \chi_{[0, N^2]}$. Then $\varphi$ is $(\varepsilon, K)$-invariant for $\pi_{reg}$.

Rem. (12.3.17): • $L^2(H, \mathbb{R})$ has almost invariant vectors iff $L^1(H, \mathbb{R})$ has almost invariant vectors, thus we have yet another criterion for amenability: $\boxed{\text{EX } 12.3.\#23}$

$\boxed{\text{AME1}} \Updownarrow \boxed{\text{AME5}} \Updownarrow \boxed{\text{AME5'}}$ $\boxed{\pi : H \longrightarrow \text{End}(L^1(H, \mathbb{R})) \text{ has almost invariant vectors}}$

• In fact, any $p \in [1, \infty[$ will do $\boxed{\text{EX } 12.3.\#24}$

$\boxed{\text{AME1}} \Longleftrightarrow \boxed{\text{AME5'}}$ $\boxed{\exists p \in [1, \infty[ : \pi : H \to \text{End}(L^p(H, \mathbb{R})) \text{ has almost invariant vectors}}$

• More on ...

H is amenable $\Leftrightarrow$ $\boxed{\text{AME } 6}$ $\boxed{\text{There is a Følner sequence for H.}}$

<u>Def. (12.3.19)</u>: $\mathcal{F} := \{F_n\}_n \subseteq \mathcal{B}(H)$ is a <u>Følner sequence</u>

if

→ $\forall n: 0 < \mu(F_n) < \infty$

→ $\forall K \in \mathcal{K}(H): \lim\limits_{n \to \infty} \sup\limits_{k \in K} \dfrac{\mu(F_n \Delta k F_n)}{\mu(F_n)} = 0$,

where $\mu = \text{Haar}_H$.

<u>Ex. (12.3.21)</u>:

(i) $\forall n: F_n := B_n(0) \Rightarrow \mathcal{F} := \{F_n\}_n$ is a Følner sequence in $\mathbb{R}^\ell$. $\boxed{\text{EX } 12.3.\#29}$

(ii). $F_2 = \langle a, b \rangle$ has no Følner sequences. $\boxed{\text{EX } 12.4.\#2}$

## §12.4. Nonamenable Groups:

**Prop. (12.4.1)**: Nonabelian free groups are not amenable.

**Cor. (12.4.2)**: Let $H$ be a discrete group. If $H$ contains a nonabelian free group, then $H$ is not amenable.

**Rem (12.4.3)**:

(i) The converse of Cor. (12.4.2), dubbed "von Neumann's Conjecture" is false.

(ii) In Cor. (12.4.2), the discreteness assumption is necessary, eg. $SO(3)$ is compact, hence amenable, but by the Tits Alternative, it contains nonabelian free groups as well.

> **Thm (4.9.1) (Tits Alternative)**: If $\Lambda \leq SL(\ell, \mathbb{R})$, then either $\Lambda$ contains a nonabelian free group, or it has a finite-indexed solvable subgroup.

(iii) The nonamenability of nonabelian free subgroups of $SO(3)$ is related to the Banach–Tarski Paradox.

(iv) If $H$ contains a closed nonabelian free subgroup, then $H$ is not amenable (for any Lie group $H$).

**Prop. (12.4.4)**: $SL(2,\mathbb{R})$ is not amenable.

**Obs.**: This $SL(2,\mathbb{R})$ is an example of a connected nonamenable Lie group.

**Prop. (12.4.5)**: If a connected and semisimple Lie group $G$ is not compact, then it is not amenable.

**Prop (12.4.7) (Classification of Connected, Amenable Lie groups)**: A connected Lie group $H$ is amenable

$\iff \exists N \in \mathcal{P}_s(H) \cap \tau^c(H) \cap \mathrm{Conn}(H)$: $N$ is solvable & $H/N$ is compact.

$$\left(1 \to N \hookrightarrow H \longrightarrow H/N \to 1\right)$$

## §12.5. Closed Subgroups of Amenable Groups:

- Let $\Lambda \in \mathcal{P}_{\trianglelefteq}(H) \cap \tau(H)$, $V$ be a locally convex topological vector space, $\alpha : H \to \mathrm{End}(V)$ be a continuous action, $S \subseteq V$ be a compact convex $H$-invariant set.

$\varphi \in L^\infty(H, S)$ is <u>essentially</u> <u>$\Lambda$-</u> <u>equivariant</u> if

$$\forall \lambda \in \Lambda, \forall h \in_{ae} H : \varphi(\lambda h) = \alpha(\lambda, \varphi(h)).$$

$$\left(
\begin{tikzcd}
\Lambda \times H \arrow[r, "mul"] \arrow[d, "id_\Lambda \times \varphi"'] & H \arrow[d, "\varphi"] \\
\Lambda \times \text{[redacted]} S \arrow[r, "\alpha"] & \text{[redacted]} S
\end{tikzcd}
\qquad
\begin{tikzcd}
(\lambda, h) \arrow[r, mapsto] \arrow[d, mapsto] & \lambda h \arrow[d, mapsto] \\
 & \varphi(\lambda h) \\
(\lambda, \varphi(h)) \arrow[r, mapsto] & \alpha(\lambda, \varphi(h))
\end{tikzcd}
\right)$$

$$L^\infty_\Lambda(H, S) := \left\{ \varphi \in L^\infty(H, S) \;\middle|\; \begin{array}{l} \varphi \text{ is essentially} \\ \Lambda\text{-invariant} \end{array} \right\}$$

<u>Ex. (12.5.2)</u>: (1) If $H$ is discrete, $L^\infty(H, S) = S^H$ is compact by Tychonoff's Theorem.

(ii) If $H$ is any Lie group and $S := \overline{B_1(0)} \subseteq \mathbb{C}$, then $L^\infty(H,S) = B_1(L^\infty(H,\mathbb{R}))$, and so is weak-* compact by the Banach-Alaoglu Theorem.

<u>Lem. (12.5.3)</u>: Let $H$ be a Lie group. Then

- $\Lambda \in \mathcal{P}_\leq(H) \cap \mathcal{T}^c(H)$,
- $V \in \mathrm{Obj}(\overline{\mathrm{Vct}\text{-}\mathbb{R}})$,
- $\alpha : H \to \mathrm{End}(V)$ a continuous action,
- $S \subseteq V$ compact convex $H$-invariant
- $\exists$ separable Banach space $B$: $S \subseteq B^*$

$\Rightarrow$ $L^\infty(H,S)$ and $L^\infty_\Lambda(H,S)$ are nonempty compact, convex and $H$-invariant (with respect to some action of $H$)

$\boxed{P := \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \leq G := SL_3(\mathbb{R})}$

## <u>§ 12.6. Equivariant Maps $G/P \to \mathrm{Prob}(X)$</u>:

<u>Prop. (12.6.1) (Furstenberg's Lemma)</u>: Let $G \leq SL(\ell,\mathbb{R})$ be a semisimple group with finitely many components, $\Gamma \leq G$ be a lattice. If $\exists P \in \mathcal{P}_\leq(G) \cap \mathcal{T}^c(G)$ : $P$ is amenable, and $\exists$ compact metric space $X$ and a continuous action $\alpha : \Gamma \to \mathrm{End}(X)$ then $\exists$ essentially $\Lambda$-equivariant Borel measurable $\varphi : G/P \to \mathrm{Prob}(X)$.

- $P$ is usually taken to be a minimal parabolic subgroup.

## §12.1:

- $H$ be a Lie group. It is amenable if

$\boxed{\text{AMF1}}$ $\forall$ v.s. $V$, $\forall$ cts $\alpha : H \curvearrowright V$, $\forall$ compact,

locally convex top. v.s. [redacted]

convex nonempty $H$-invariant $S \subseteq V$,

$$\exists s \in S : \alpha(H, s) = \{s\}.$$

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12.1.#1: Every finite group is amenable.

$H = \{h_0 = e, h_1, \dots, h_n\}$.

$V$ be l.c.t.v.s, $\alpha : H \curvearrowright V$, $S \subseteq V$ be compact convex nonempty $H$-inv.

pick $s \in S$. [redacted] $\Rightarrow s = \alpha(h_0, s), \alpha(h_1, s), \dots, \alpha(h_n, s) \in S$.

$\Rightarrow \frac{1}{n+1} \sum_{k=0}^{n} \alpha(h_k, s) \in S$. since $S$ is convex.

$$\alpha\left(h, \frac{1}{n+1} \sum_{k=0}^{n} \alpha(h_k, s)\right) = \frac{1}{n+1} \sum_{k=0}^{n} \alpha\Big(h_k, \underbrace{\alpha(h, s)}_{\in S}\Big) \in S. \checkmark$$

<u>12.1.#2</u> : $H$ amenable, $N \trianglelefteq H$ closed
$\Rightarrow H/N$ amenable.

$V$ be a l.c.t.v.s, $\alpha : H/N \curvearrowright V$ be cts,
$\emptyset \neq S \subseteq V$ be compact convex $H/N$-inv.

Lift $\alpha$ : $\tilde{\alpha} : H \times V \longrightarrow V$
$(h, v) \longmapsto \alpha(hN, v)$.

$\tilde{\alpha}(e, v) = \alpha(eN, v) = v$.

$\tilde{\alpha}(h_1 h_2, v) = \alpha(h_1 h_2 N, v) = \alpha(h_1 N h_2 N, v)$
$= \alpha(h_1 N, \alpha(h_2 N, v))$

$\tilde{\alpha}$ is comp of $\alpha$ and canonical map $\Rightarrow \tilde{\alpha}$ cts.

[redacted] $\tilde{\alpha}(h, s) = \alpha(hN, s) \in S$. ; [redacted] $H$ amenable

$\Rightarrow \exists\, s \in S : \tilde{\alpha}(H, s) = \{s\}$
$\tilde{\alpha}(H, s) = \alpha(H/N, s)$ , ✓.

12.1.#3: $H_1$ be amenable, $\varphi: H_1 \longrightarrow H_2$ be cts with $\overline{\mathrm{im}(\varphi)} = H_2$. Then $H_2$ is [redacted] amenable.

$V$ be l.c.t.v.s., $\emptyset \neq S \subseteq$ [redacted] be cpt, conv. $H_2$-inv.

$\alpha: H_2 \times V \to V$
be cts

$$\tilde{\alpha} := \varphi^* \alpha : \quad H_1 \times V \longrightarrow V$$
$$(h_1, v) \longmapsto \alpha(\varphi(h_1), v).$$

$$\tilde{\alpha}(h_1 h_1', v) = \alpha(\varphi(h_1 h_1'), v) = \alpha(\varphi(h_1)\varphi(h_1'), v)$$
$$= \alpha(\varphi(h_1), \alpha(\varphi(h_1'), v)) = \tilde{\alpha}(h_1, \tilde{\alpha}(h_1', v)).$$

$$\tilde{\alpha}(h_1, s) = \alpha(\varphi(h_1), s) \in S.$$

$$\Rightarrow \exists\, s \in S \,;\, \underset{\substack{\| \\ \alpha(\varphi(H_1), s)}}{\tilde{\alpha}(H_1, s)} = \{s\}$$

__Claim__: In fact $\alpha(H_2, s) = \{s\}$.

$h_2 \in H_2$, $\{h_n\}_n \subseteq H_1$: $\varphi(h_n) \to h_2$ (which exists since $\overline{\mathrm{im}(\varphi)} = H_2$. $\alpha$ cts $\Rightarrow s = \alpha(\varphi(h_n), s) \to \alpha(h_2, s)$
$\Rightarrow \alpha(h_2, s) = s$, ✓.

## §12.2:

**Prop. (12.2.1):** Cyclic groups are amenable.

**Pf:** $H = \langle T \rangle$. Let $V$ be a l.c.t.v.s [redacted]

$\alpha : H \curvearrowright V$ be cts, $\emptyset \neq S \subseteq V$ be compact convex and $H$-inv.

[redacted]

$$\text{Define } A : S \times \overset{\mathbb{Z}_{\geq 0}}{\overset{\shortparallel}{\mathbb{N}}} \longrightarrow S$$

$$(s, n) \longmapsto \frac{1}{n+1} \sum_{k=0}^{n} \alpha(T^k)(s).$$

$S$ is compact $\Rightarrow \forall s \in S, \exists\, n(s) \subseteq n, \exists\, a(s) \in S$:

$$A(s, n(s)) \longrightarrow a(s)$$

**Claim:** $\forall s \in S$: $a(s)$ is a fixed point of $\alpha$

$\boxed{\text{EX 12.2.\#1}}$

$$\cancel{\| T(a(s)) - a(s) \|}$$

$$\cancel{= \| T(a(s)) - T(A(s, a(s))) + T(A(s, a(s))) - a(s) \|}$$

$$= \| \alpha(T, a) - a \|$$

$$= \| \alpha(T, a) - \alpha(T, A(s,n)) + \alpha(T, A(s,n)) - a \|$$

$$\le \underbrace{\| \alpha(T,a) - \alpha(T, A(s,n)) \|} + \underbrace{\| \alpha(T, A(s,n)) - a \|}$$

$A(s,n) \to a$

$\Rightarrow \alpha(T, A(s,n)) \to \alpha(T, a)$

$$\alpha\left(T, \frac{1}{n+1} \sum_{k=0}^{n+1} \alpha(T^k, s)\right) - a$$

$$= \frac{1}{n+1} \sum_{k=0}^{n+1} \alpha(T^{k+1}, s) - a$$

$$= \frac{\alpha(T, s) + \alpha(T^2, s) + \dots + \alpha(T^{n+1}, s) + \alpha(T^{n+2}, s)}{n+1} - a$$

$$= \frac{1}{n+1} \sum_{k=0}^{n+1} \alpha(T^k, s) + \frac{\alpha(T^{n+2}, s) - \alpha(T^0, s)}{n+1} - a$$

$\|\alpha(T,a) - a\|$

$\le \|\alpha(T,a) - \alpha(T, A(s,n))\|$

$+ \|\alpha(T, A(s,n)) - a\|$

$$\le \underbrace{\|\dots\|}_{\to 0} + \underbrace{\left\|\frac{1}{n+1}\sum_{k=0}^{n+1}\alpha(T^k,s) - a\right\|}_{\to 0} + \underbrace{\frac{\|\alpha(T^{n+2},s) - \alpha(T^n,s)\|}{n+1}}_{\to 0}$$

$\to 0$ ✓.

$$\Rightarrow \boxed{\alpha(T, a(s)) = a(s)} \;✓.$$

<u>Cor (12.2.3)</u>: (Kakutani-Markov FPT)

Every abelian group is amenable. $\blacksquare$

<u>Pf</u>: Define

H be abelian, $\alpha: H \curvearrowright V$, $S \subseteq V$ be cpt conv. $H$-inv.

Zim

$$A: H \times \mathbb{N} \xrightarrow{``\mathbb{Z}_{\ge 0}"} S^S$$

$$(h,n) \longmapsto \boxed{A(h,n): s \mapsto \frac{1}{n+1}\sum_{k=0}^{n}\alpha(h^k,s)}$$

A is well-def. since $S$ is convex.

~~Prop. (12.2.4): If H is compact then~~

**Obs:** $A(h,0)(s) = \alpha(h^0, s) = s.$

Let $\mathcal{A} := \left\langle \left\{ A(h,n) \,\middle|\, \begin{smallmatrix} h\in H \\ n \geq 0 \end{smallmatrix} \right\} \right\rangle$ be the monoid generated by $A(h,n)$'s.

$S$ is cpt $\Rightarrow$ $\forall a \in \mathcal{A}$: $a(S) \subseteq S$ is cpt.

**Obs:** $a_1, \dots, a_n \in \mathcal{A} \Rightarrow a_1 a_2 \cdots a_n (S) \underset{\substack{\uparrow \\ H \text{ abel}}}{\subseteq} \bigcap_{k=1}^{n} a_k(S)$

**Claim:** $\bigcap_{a \in \mathcal{A}} a(S) \neq \emptyset$.

Consider $\{a(S) \in \mathcal{P}(S) \mid a \in \mathcal{A}\}$. By the above obs., $\forall$ finite $\mathcal{A}' \subseteq \mathcal{A}$: $\bigcap_{a\in\mathcal{A}'} a(S) \neq \emptyset$, so that $\{a(S) \in \mathcal{P}(S) \mid a \in \mathcal{A}\}$ has the finite intersection property, and $\{a(S) \subseteq \mathcal{P}(S) \mid a \in \mathcal{A}\} \subseteq \mathcal{K}(S) \subseteq \tau^c(S)$.

Then by $\boxed{\text{Folland, Prop. 4.21 p. 128}}$, since $S$ is compact, $\bigcap\limits_{a \in A} a(S) \neq \emptyset$.

Folland, Prop. 4.21 on p. 128 : A topological space $X$ is compact iff $\forall \mathcal{F} \subseteq \tau^c(X)$ : $\mathcal{F}$ has FIP $\Rightarrow \bigcap \mathcal{F} \neq \emptyset$.

($\mathcal{F}$ has FIP if $\forall$ finite $\mathcal{F}' \subseteq \mathcal{F}$ : $\bigcap \mathcal{F}' \neq \emptyset$.)

Pf: [redacted] Let $\mathcal{F} \subseteq \tau^c(X)$ have FIP, $\mathcal{U} := \{U \mid U^c \in \mathcal{F}\}$

$\bigcup \mathcal{U} \neq X \Leftrightarrow \bigcap \mathcal{F} \neq \emptyset$,

$\mathcal{U}$ has a finite subcover $\Leftrightarrow$ $\mathcal{F}$ does not have the FIP.

Claim: $\forall s \in \bigcap\limits_{a \in A} a(S)$ : $s$ is $H$-inv.

$$s \in \bigcap_{a \in A} a(S) \Rightarrow \forall a \in A : s \in a(S)$$

$$\Rightarrow \forall h \in H, \forall n \in \mathbb{Z}_{\geq 0}, \exists s_{h,n} \in S : s = A(h,n)(s_{h,n})$$

$$\| \alpha(h,s) - s \| = \| \alpha(h, A(h,n)(s_{h,n})) - A(h,n)(s_{h,n}) \|$$

$$= \left\| \alpha\left(h, \frac{1}{n+1}\sum_{k=0}^{n} \alpha(h^k, s_{h,n})\right) - \frac{1}{n+1}\sum_{k=0}^{n} \alpha(h^k, s_{h,n}) \right\|$$

$$= \frac{\left\| \begin{array}{l} \alpha(h, s_{h,n}) + \alpha(h^2, s_{h,n}) + \dots + \alpha(h^{n+1}, s_{h,n}) \\ - s_{h,n} - \alpha(h, s_{h,n}) - \dots - \alpha(h^n, s_{h,n}) \end{array} \right\|}{n+1}$$

$$= \frac{\| \alpha(h^{n+1}, s_{h,n}) - s_{h,n} \|}{n+1} \leq \frac{\mathrm{diam}(S)}{n+1} \longrightarrow 0, \checkmark.$$

$\boxed{\text{EX 12.2 \#5}}$ (Alternative Proof for KMFPT)

$\forall h \in H: \; S_h := \{ s \in S \mid \alpha(h,s) = s \}$.

By Prop 12.2.1, $\forall h \in H: \; S_h \neq \emptyset$.

$\forall h \in H: \; S_h$ is compact and convex, and

$\forall h_1, h_2 \in H: \; S_{h_1}$ is $h_2$-invariant

$\Rightarrow$ By Prop. 12.2.1, [redacted] and induction,

$\forall \{h_1, \dots, h_n\} \subseteq H: \; \bigcap_{k=1}^{n} S_{h_k} \neq \emptyset$.

Thus $\{S_h \mid h \in H\} \subseteq \mathcal{K}(S) \subseteq \tau^c(S)$ has FIP. $S$ is compact

$\Rightarrow \bigcap_{h \in H} S_h \neq \emptyset$ by (Folland, Prop. 4.21). $\checkmark$.

**Prop. (12.2.4)**: Compact groups are [redacted] amenable.

**Pf**: $H$ be cpt, $V$ be L.C.T.V.S, $S \subseteq V$ be nonempty cpt, cvx, $H$-inv. under some cts $\alpha: H \curvearrowright V$.

Pick $s \in S$, set
$$f_s : H \longrightarrow S, \quad h \longmapsto \alpha(h, s),$$

$$\bar{s} := f_{s_*}\mu(S), \text{ where } \mu = \mathrm{Haar}_H.$$

**Barycenter Construction on Compact Convex Sets (Zimmer, p. 61)**:
Let $S$ be a compact convex set in a locally convex linear space $V$.

- If $\mu$ is an atomic measure on $S$, ie, $\exists \{s_k\}_{k=1}^{n} \subseteq S$, $\exists \{c_k\}_{k=1}^{n} \subseteq [0,1]$, $\sum_{k=1}^{n} c_k = 1$, $\mu = \sum_{k=1}^{n} c_k \delta_{s_k}$, then $\beta(\mu) := \sum_{k=1}^{n} c_k s_k \in S$ is the barycenter of $\mu$.
- Atomic measures are dense in $\mathrm{Prob}(S)$, whence we have a barycenter map $\beta: \mathrm{Prob}(S) \longrightarrow S$.
- $\forall \lambda \in V^*$, $\forall \mu \in \mathrm{Prob}(S)$: $\lambda(\beta(\mu)) = \int_S \lambda(\sigma)\, d\mu(\sigma)$.
- $\forall \mu \in \mathrm{Prob}(S)$: $\beta(\mu) = \int_S \sigma \, d\mu(\sigma)$.

- If $T : S \to S$ is linear (affine), then $\forall \mu \in \mathcal{P}rob(S)$:

$$\beta(T_*\mu) = T(\beta(\mu))$$

$$\begin{tikzcd}
\mathcal{P}rob(S) \arrow[r, "T_*"] \arrow[d, "\beta"'] & \mathcal{P}rob(S) \arrow[d, "\beta"] \\
S \arrow[r, "T"'] & S
\end{tikzcd}$$

$$\begin{array}{ccc}
\mu = \sum c_t \delta_{s_t} & \overset{T_*}{\longmapsto} & T_*\left(\sum c_t \delta_{s_t}\right) \\
 & & \parallel \\
\Big\downarrow \beta & & \sum c_t T_* \delta_{s_t} \\
 & & \parallel \\
 & & \sum c_t \delta_{T(s_t)} \\
 & & \Big\downarrow \beta \\
\sum c_t s_t & \overset{T}{\longmapsto} & \sum c_t T(s_t)
\end{array}$$

$\rightsquigarrow$ then by the density of atomic measures.

$$\bar{s} = f_{s_*}\mu(S) = \underbrace{\int_S \sigma \, d f_{s_*}\mu(\sigma)}_{= \beta(f_{s_*}\mu)} = \int_{f_s^{-1}(S)} f_s(h) \, d\mu(h)$$

$$= \int_H \alpha(h,s) \, d\mu(h) \in S,$$

$\forall h \in H$: $\alpha(h,\cdot)$ is linear $\Rightarrow \alpha(h,\bar{s}) = \alpha(h, \beta(f_{s_*}\mu))$

$$= \beta(\alpha(h,\cdot)_* f_{s_*}\mu) \underset{\circledast}{=} \beta(f_{s_*}\mu) = \bar{s}, \checkmark$$

$\circledast$ $\alpha(h,\cdot)_* f_{s_*}\mu(B) = (\alpha(h,\cdot)\circ f_s)_*\mu(B) = \mu((\alpha(h,\cdot)\circ f_s)^{-1}(B))$

$= \mu(\{g\in H \mid \alpha(h,\cdot)\circ f_s(g)\in B\}) = \mu\{g\in H \mid \alpha(hg,s)\in B\}$

$= \mu(\{h^{-1}g\in H \mid \alpha(g,s)\in B\}) \overset{(*)}{=} \mu(\{g\in H \mid \alpha(g,s)\in B\}) = f_{s_*}\mu(B)$

$(*)$ ($\mu$ is Haar, hence is $H$-inv.)

**Prop.(12.2.6)**: Amenable extensions of amenable groups are amenable, i.e.,

if $\varphi: G\to H$ is a continuous surjective homomorphism, then if $K:=\ker\varphi$ and $H$ are amenable, then so is $G$.

(in fact iff; Zimmer, Prop. 4.1.6.(b))

Pf:

$$1 \longrightarrow K \hookrightarrow G \xrightarrow{\varphi} H \longrightarrow 1 \text{ ses.}$$

Let $V$ be a L.C.T.V.S, $\alpha: G\curvearrowright V$ be a cts action, $S\subseteq V$ be a nonempty compact convex $G$-inv. subset. $S_K := \{s\in S \mid \alpha(K,s)=\{s\}\}$. $K$ is amenable, so $S_K\neq\emptyset$. Also $S_K$ is compact and convex.

$s \in S_K,\ g \in G \Rightarrow \alpha(K, \alpha(g,s)) = \alpha(Kg, s)$

$\overset{K \trianglelefteq G}{=} \alpha(gK, s) = \alpha(g, \underbrace{\alpha(K,s)}_{=s}) = \alpha(g,s)$

$\Rightarrow S_K \subseteq S$ is $G$-inv.

$\Rightarrow \tilde{\alpha} : G/K \times S_K \longrightarrow S_K$ is an action
$\quad (gK, s) \longmapsto \alpha(g,s)$

$g_1 K = g_2 K \Rightarrow g_2^{-1} g_1 \in K \Rightarrow \alpha(g_2^{-1} g_1, s) = s$

$\Rightarrow \alpha(g_1, s) = \alpha(g_2, s)$ ✓.

$H \cong G/K$ is amenable $\Rightarrow S_{K_H} = \{ s \in S_K \mid \tilde{\alpha}(H,s) = \{s\} \} \subseteq S_K$ is nonempty.

$\Rightarrow \exists\, s \in S: \ \alpha(K,s) = \{s\}$

$\tilde{\alpha}(G/K, s) = \alpha(G,s) = \{s\}$

$\Rightarrow G$ amenable. ✓

**Cor. (12.2.7):**

(i) Solvable groups are amenable.

(ii) Compact extensions of solvable groups are amenable.

Pf: (ii) Solvables are amen by (i), compacts are amen by Prop. (12.2.4). $\Rightarrow$ by Prop (12.2.6) we are done, ✓.

(i) $G$ is <u>solvable</u> if $\exists n: G^{(n)} = 1$, where $G^{(1)} = G$, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$.

$$1 \to G^{(n)}/G^{(n+1)} \to G^{(n)} \to G^{(n+1)} \to 1 \qquad G^{(n-1)}, \quad G^{(n+1)} = 1$$

<u>by induction</u>:

$$1 \to G^{(n-1)}/G^{(n)} \to G^{(n-1)} \to G^{(n)} \to 1, \quad G^{(n)} = 1$$

$$1 \to G^{(n-2)}/G^{(n-1)} \to G^{(n-2)} \to G^{(n-1)} \to 1$$

$$\vdots$$

$$1 \to G^{(2)}/G^{(3)} \to G^{(2)} \to G^{(3)} \to 1$$

$$1 \to G^{(1)}/G^{(2)} \to G^{(1)} \to G^{(2)} \to 0, \quad G^{(1)} = G$$

$G^{(n-1)}$ is abelian $\Rightarrow$ amenable $\to \frac{G^{(n-2)}}{G^{(n-1)}}$ & $G^{(n-1)}$ are ab. $\Rightarrow$ amen $\Rightarrow G^{(n-2)}$ amen.

$\vdots$

$\underline{\text{Ex}}$: $\begin{pmatrix} \mathbb{R} & & \mathbb{R} \\ & \ddots & \\ 0 & & \mathbb{R} \end{pmatrix} \leq SL(n, \mathbb{R})$ is solvable,(*)

hence amenable. (Zimmer, p. 62)

. If $G$ is a connected semisimple Lie group and $P \leq G$ is a minimal parabolic subgroup, then $P$ is a compact extension of a solvable group, and hence is amenable.

(*) $\underline{\text{For } n=3}$: $\boxed{\text{EX-12.2.\#14}}$

$$1 \to \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \mathbb{R} & 0 & 0 \\ 0 & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} \end{pmatrix} \to 1$$

$$1 \to \underset{\substack{\cong \\ \mathbb{R}}}{\begin{pmatrix} 1 & 0 & \mathbb{R} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \hookrightarrow \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \twoheadrightarrow \mathbb{R}^2 \to 1$$

$\underline{\text{For } n=4}$:

$$1 \to \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 0 & \mathbb{R} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} \mathbb{R} & 0 & 0 & 0 \\ 0 & \mathbb{R} & 0 & 0 \\ 0 & 0 & \mathbb{R} & 0 \\ 0 & 0 & 0 & \mathbb{R} \end{pmatrix} \to 1$$

$$1 \to \begin{pmatrix} 1 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \hookrightarrow \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} \twoheadrightarrow \mathbb{R}^3 \to 1$$

$$1 \to \underset{\substack{\cong \\ \mathbb{R}}}{\begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \hookrightarrow \begin{pmatrix} 1 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \twoheadrightarrow \mathbb{R}^2 \to 1$$

$\boxed{\text{EX. 12.2. \#12}}$ If $\Lambda \leq H$ is a lattice and $\Lambda$ is amenable, then so is $H$.

The proof of this is very similar to that of Prop. (12.2.4).

Zimmer, Prop. 4.1.11, p. 63: If $\Gamma \subseteq G$ is a closed cocompact subgroup, then $G$ is amenable iff $\Gamma$ is amenable.

Pf: [redacted]

Let $V$ be a l.c.t.v.s. $\alpha : H \curvearrowright V$ be cts, $S \subseteq V$ be a nonempty compact convex $H$-inv. set.

$\Rightarrow$ $S$ is also $\Lambda$-inv. $\Lambda$ is amen

$\Rightarrow$ $\exists\, s \in S$ $\alpha(\Lambda, s) = \{s\}$.

Define $f_s : H/\Lambda \longrightarrow S$, $h\Lambda \longmapsto \alpha(h, s)$. $\quad h_1\Lambda = h_2\Lambda \Rightarrow h_2^{-1}h_1 \in \Lambda \Rightarrow \alpha(h_2^{-1}h_1, s) = s$ ✓

By Prop. 4.1.3, $\exists$ $G$-inv. Borel prob. $\nu$ on $H/\Lambda$.

$$\Rightarrow \bar{s} := \beta(f_{s*}\nu) = f_{s*}\nu(S) \in S$$

is [redacted] $H$-inv., ✓.

## §12.3 :

**Prop. 12.3.5:** $\boxed{\text{AME1}} \Leftrightarrow \boxed{\text{AME2}}$

$\boxed{\text{AME1}}$ $\forall$ l.c.l.v.s. $V$, $\forall$ $C^0$ $\alpha: H \curvearrowright V$, $\forall$ nonempty compact convex $H$-inv. $S \subseteq V$, $\exists\, s \in S$: $\alpha(H, s) = \{s\}$.

$\boxed{\text{AME2}}$ $\forall$ compact metrizable top. sp. $X$, $\forall$ $C^0$ $\alpha: H \curvearrowright X$, $\exists \mu \in Prob(X)$: $\alpha(H, \cdot)_* \mu = \{\mu\}$.

Pf:

($\Rightarrow$) If $X$ is compact metrizable, by Riesz-Rep. & Banach-Alaoglu, $Prob(X) \subseteq B_1(C^0(X, \mathbb{R})^*)$ is compact, convex.

[redacted]

$\alpha: H \times X \xrightarrow{C^0} X$ is also given.

$\Rightarrow \tilde{\alpha}: H \times Prob(X) \longrightarrow Prob(X)$ is cts with respect to weak-*-top on $Prob(X)$.

$(h, \mu) \longmapsto \boxed{\begin{array}{l}\alpha(h, \cdot)_* \mu : \\ B \mapsto \mu(\alpha(h^{-1}, B))\end{array}}$

$\boxed{\text{EX. 12.3. \#2}}$ (needs $H$ to be second countable ?) c.f. (Zimmer p.59)

$\Rightarrow$ By $\boxed{\text{AME1}}$, $\exists \mu \in Prob(X)$, $\tilde{\alpha}(H, \mu) = \{\mu\}$, i.e., $\mu$ is $H$-inv., ✓.

($\Leftarrow$) Let $V$ be a l.c.l.v.s, $\alpha : H \overset{c^0}{\curvearrowright} V$, $S \subseteq V$ be non-$\emptyset$, cpt, conv., $H$-inv.

Wlog we may assume $S$ is metrizable.

By $\boxed{\text{AME2}}$, $\exists$ $H$-inv. [redacted] $\mu \in \mathrm{Prob}(S)$.

Since $S$ is cpt & conv.,

$$\beta(\mu) = \int_S \sigma \, d\mu(\sigma) \in S$$

as in the proof of Prop. 12.2.4,

since $\alpha$ is an affine action

$$\forall h \in H: \ \alpha(h, \beta(\mu)) = \beta\left(\alpha(h, \cdot)_* \mu\right) \underset{\substack{\uparrow \\ \mu \ H\text{-inv.}}}{=} \beta(\mu)$$

$\boxed{\text{EX. 12. 3. \#2}}$ $\alpha : H \times X \longrightarrow X$ $C^0$

$\tilde{\alpha} : H \times Prob(X) \longrightarrow Prob(X)$ $C^0$ ?

$(h, \mu) \longmapsto \alpha(h, \cdot)_* \mu$

$$\Leftrightarrow \boxed{\begin{array}{l} h_n \to h \overset{?}{\Longrightarrow} \tilde{\alpha}(h_n, \mu) \xrightarrow{w*} \tilde{\alpha}(h, \mu), \text{ and} \\ \mu_n \xrightarrow{w*} \mu \overset{?}{\Longrightarrow} \tilde{\alpha}(h, \mu_n) \xrightarrow{w*} \tilde{\alpha}(h, \mu) \end{array}}$$

- $h_n \to h \Rightarrow \alpha(h_n^{-1}, b) \to \alpha(h, b)$ since $\alpha$ is cts, $X$ cpt. & DCT

$\Rightarrow \mu(\alpha(h_n^{-1}, B)) \longrightarrow \mu(\alpha(h^{-1}, B))$ for Borel $B \subseteq X$.

$\Rightarrow$ By approximating a measurable function by simple measurable functions,

$\tilde{\alpha}(h_n, \mu) \xrightarrow{w*} \tilde{\alpha}(h, \mu)$ ✓.

- $\mu_n \xrightarrow{w*} \mu$. $\tilde{\alpha}(h, \mu_n)(B) = \mu_n(\alpha(h^{-1}, B)) \longrightarrow \mu(\alpha(h^{-1}, B))$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \tilde{\alpha}(h, \mu)(B)$$

$\Rightarrow \tilde{\alpha}(h, \mu_n) \xrightarrow{w*} \tilde{\alpha}(h, \mu)$ on Borels.

$\Rightarrow \tilde{\alpha}(h, \mu_n) \xrightarrow{w*} \tilde{\alpha}(h, \mu)$. ✓

(but where is second countability of $H$ is needed?)

**Def. (12.3.6)**: Let $V \leq \mathcal{L}^\infty(H,\mathbb{R})$ (eg., $V := C_b^0(H,\mathbb{R})$) be such that $\chi_H \in V$. Then $\lambda : V \longrightarrow \mathbb{R}$ is a mean if

(i) $\lambda(cf+g) = c\lambda(f) + \lambda(g)$

(ii) $\lambda(\chi_H) = 1$,

(iii) $\forall f \in V : im(f) \subseteq [0,\infty[ \Rightarrow \lambda(f) \geq 0$.

(Let $Mean(V)$ denote the set of means on $V$.)

**Rem. (12.3.7)**: $Mean(V) \subseteq C^0 \cap Lin(V,\mathbb{R}) = V^*$, because $\forall \lambda \in Mean(V) : \|\lambda\| = 1$. $\boxed{\text{EX. 12.3.\#8}}$

$\boxed{\text{EX. 12.3.\#8}}$ $\|\lambda\| \stackrel{def}{=} \sup\limits_{\substack{f \in \mathcal{L}^\infty(V,\mathbb{R}) \\ \|f\| \leq 1}} |\lambda(f)| \geq |\lambda(\chi_H)| = 1$.

$\Rightarrow \|\lambda\| \geq 1$.

If $\|f\|_{\mathcal{L}^\infty} \leq 1$, since $\|f\|_{\mathcal{L}^\infty} = \text{esssup}|f|$, $|f| \leq 1$ ae

$\Leftrightarrow -1 \leq f \leq 1$ ae $\Leftrightarrow 0 \leq \chi_H - f,\ f + \chi_H$ ae

$\Rightarrow 0 \leq \lambda(\chi_H - f),\ \lambda(f+\chi_H) \Rightarrow -\lambda(\chi_H) \leq \lambda(f) \leq \lambda(\chi_H)$

$\Rightarrow |\lambda(f)| \leq |\lambda(\chi_H)| = 1 \Rightarrow \|\lambda\| \leq 1 \Rightarrow \|\lambda\| = 1$.

$\lambda$ is by def linear, hence is cts linear, ✓.

Ex. (12.3.8): $P(H) := \{\varphi \in L^1(H,\mathbb{R}) \mid \varphi \geq 0, \|\varphi\| = 1\}$.

then $\forall \varphi \in P(H)$: $\lambda_\varphi : L^\infty(H,\mathbb{R}) \longrightarrow \mathbb{R}$ is a mean.

$$f \longmapsto \langle f, \varphi \rangle$$

$\boxed{\text{EX. 12.3.\#12}}$ $\{\lambda_\varphi \mid \varphi \in P(H)\}$ is weakly dense in $Mean(L^\infty(H,\mathbb{R}))$

Prop. 7.2.3 (Zimmer, p. 133):

(i) $Mean(V)$ is a nonempty closed convex subspace of the unit ball in $V^*$, where $V^*$ has the weak-$*$-top.

(ii) $\{\lambda_\varphi \mid \varphi \in P(H)\}$ is weak-$*$-dense in $Mean(V)$.

Pf: (i) • $\{ev_v \mid v \in V\} \subseteq Mean(V) \Rightarrow Mean(V) \neq \emptyset$.

• $(t\lambda_1 + (1-t)\lambda_2)(\chi_H) = t\,\lambda_1(\chi_H) + (1-t)\,\lambda_2(\chi_H) = 1$.

For $f \geq 0$, $(t\lambda_1 + (1-t)\lambda_2)(f) = t\,\lambda_1(f) + (1-t)\,\lambda_2(f) \subseteq [\lambda_1(f), \lambda_2(f)] \subseteq [0,\infty[$

$\Rightarrow Mean(V)$ is convex.

• $\{\lambda_n\}_n \subseteq Mean(V)$, $\lambda \in V^*$; $\lambda_n \xrightarrow{*} \lambda$

$\Rightarrow 1 = \lambda_n(\chi_H) \longrightarrow \lambda(\chi_H) \Rightarrow \lambda(\chi_H) = 1$.

For $f \geq 0$, $0 \leq \lambda_n(f) \longrightarrow \lambda(f) \Rightarrow \lambda(f) \geq 0$.

$\Rightarrow Mean(V)$ is closed.

• $Mean(V) \subseteq B_1(V^*)$ by Rem (12.3.7).

(ii) Suppose $\{\lambda_\varphi \mid \varphi \in P(H)\}$ is not weak-*-dense in $Mean(V)$.

$\Rightarrow \exists \lambda \in Mean(V), \exists \varepsilon > 0, \exists v \in V$:

$$\{\lambda_\varphi \mid \varphi \in P(H)\} \subseteq \{\mu \in Mean(V) \mid |\mu(v) - \lambda(v)| \geq \varepsilon\}.$$

$\Rightarrow \exists \lambda \in Mean(V), \exists \varepsilon > 0, \exists v \in V, \forall \varphi \in P(H)$:

$$|\lambda_\varphi(v) - \lambda(v)| \geq \varepsilon.$$

("by Hahn-Banach"?)

$\Rightarrow \lambda(v) \geq \varepsilon + \lambda_\varphi(v)$

$\Rightarrow \lambda(v) > \lambda_\varphi(v), \quad \forall \varphi \in P(H)$

$\Rightarrow \lambda(v) > \sup_{\varphi \in P(H)} \lambda_\varphi(v) \overset{\circledast}{=} \operatorname{esssup}(v)$

(Why does this contradict $f \geq 0 \Rightarrow \lambda(f) \geq 0$ ?)

$\circledast$ Folland, p. 189, Thm. 6.14: Let $p$ and $q$ be conjugate exponents, $g \in L^0$, $\forall s \in S$: $gs \in L^1$; and $M_q(g) = \sup_{\substack{s \in S \\ \|s\|_p = 1}} \left|\int gs\right| < \infty$, and $S_g = \{x \mid g(x) \neq 0\}$ is $\sigma$-finite. Then $g \in L^q$, $M_q(g) = \|g\|_q$.

**Prop. (12.3.9)** : $\boxed{\text{AME1}} \Rightarrow \boxed{\text{AME3}}$

$\boxed{\text{AME1}}$ $\forall$ l.c.t.v.s. $V$, $\forall$ $C^0$ $\alpha : H \curvearrowright V$, $\forall$ nonempty, cpt, conv., $H$-inv. $S \subseteq V$, $\exists\, s \in S$ : $\alpha(H, s) = \{s\}$.

$\boxed{\text{AME3}}$ $\exists\, \lambda \in \mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$, $\forall h \in H$ : $\underbrace{\alpha(h^{-1}, \cdot)^* \lambda = \lambda}_{\text{left-invariance}}$.

Pf: • $\mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$ is nonempty convex and $L$-inv., where $L : H \times \mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right) \to \mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$

$$(h, \lambda) \longmapsto \alpha(h^{-1}, \cdot)^* \lambda .$$

$\boxed{\text{EX. 12.3. \#13}}$ $\leftarrow$ $\boxed{\text{Prop. (7.2.3) of Zimmer, (a)}}$

• $\mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$ is a weak-$*$-closed subset of the unit ball in $C_b^0(H, \mathbb{R})^*$.

$\boxed{\text{EX. 12.3. \#14}}$

$\Rightarrow$ By Banach–Alaoglu $\mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$ is compact.

$\Rightarrow$ $\boxed{\text{AME1}}$ gives $\lambda \in \mathrm{Mean}\left(C_b^0(H, \mathbb{R})\right)$ :

$$\alpha(H^{-1}, \cdot)^* \lambda = \lambda, \checkmark$$

Prop (12.3.11): $\boxed{\text{AME3}} \Rightarrow \boxed{\text{AME2}}$

$\boxed{\text{AME3}}$ $\exists$ L-inv. $\lambda \in \mathrm{Mean}\,(C_b^0(H,\mathbb{R}))$.

$\boxed{\text{AME2}}$ $\forall$ compact metrizable $X$, $\forall$ $C^0$ $\alpha : H \curvearrowright X$, $\exists$ $H$-inv. $\mu \in \mathrm{Prob}(X)$.

Pf: Consider a cts $\alpha : H \curvearrowright X$.

Pick $x \in X$

$$\begin{aligned} \varphi_x : C^0(X,\mathbb{R}) &\longrightarrow C_b^0(H,\mathbb{R}) \\ f &\longmapsto \boxed{h \mapsto f(\alpha(h,x))} \end{aligned}$$

$\varphi_x$ is $H$-equivariant.

$$\begin{tikzcd} H \times C^0(X,\mathbb{R}) \arrow[r] \arrow[d] & C^0(X,\mathbb{R}) \arrow[d] \\ H \times C_b^0(H,\mathbb{R}) \arrow[r] & C_b^0(H,\mathbb{R}) \end{tikzcd}
\qquad
\begin{tikzcd} (h,f) \arrow[r, mapsto] \arrow[d, mapsto] & f(\alpha(h^{-1}, \cdot)) \arrow[d, mapsto] \\ & f(\alpha(h^{-1}, \alpha(\cdot, x))) \arrow[d, phantom, "\shortparallel"] \\ (h, f(\alpha(\cdot, x))) \arrow[r, mapsto] & f(\alpha(h^{-1}\cdot, x)) \end{tikzcd}$$

$\Rightarrow$ Any $L$-inv. $\lambda \in Mean(C^0_L(H,\mathbb{R}))$ induces an $H$-invariant $\varphi_x^* \lambda \in Mean(C^0(X,\mathbb{R}))$, which, by RRT, can be represented by an $H$-inv. $\mu \in \mathcal{P}rob(X)$. $\boxed{\text{EX. 12.3. \#15}}$

($\lambda$ is a mean $\Rightarrow \mu(X)=1$).

$\boxed{\text{EX. 12.3. \#15}}$

$\varphi_x : C^0(X,\mathbb{R}) \longrightarrow C^0_b(H,\mathbb{R})$

$\varphi_x^* : Mean(C^0_b(H,\mathbb{R})) \longrightarrow Mean(C^0(X,\mathbb{R}))$

$\lambda \longmapsto \boxed{f \mapsto \lambda(\varphi_x(f))}$

✓.

**Def (12. 3.14)**: Let $(V, \|\cdot\|)$ be a normed vector space, $\alpha : H \curvearrowright V$. $\alpha$ has <u>almost invariant vectors</u> if $\forall \varepsilon > 0, \forall K \in \mathcal{K}(H), \exists\, v = v(\varepsilon, K) \in V$ :

$$\|v\| = 1 \quad \& \quad \sup_{k \in K} \|\alpha(k, v) - v\| < \varepsilon.$$

$$\left(\Leftrightarrow \alpha(K, v) \subseteq B_\varepsilon(v)\right).$$

In this case $v = v(\varepsilon, K)$ is $(\varepsilon, K)$-invariant.

- $\pi_{reg} : H \longrightarrow \mathcal{U}(L^2(H, \mathbb{R}))$ is the
  $h \longmapsto \boxed{\varphi \mapsto \alpha(h^{-1}, \cdot)^* \varphi}$

<u>left-regular rep.</u> of $H$.

**Ex. (12. 3.16)**:

(i) If $H$ is compact, then $\chi_H \in L^2(H, \mathbb{R})$,

$\pi(h)\chi_H = \chi_H \Rightarrow \chi_H$ is an $H$-inv. unit vector.

(ii) $\pi_{reg} : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{R})$ has no nonzero inv. vectors:

If $\varphi : \mathbb{R} \to \mathbb{R}$ is $\pi$-inv., then $\boxed{\text{EX. 12.3. \#22}}$

$\forall t: \ \varphi(x) = \varphi(-t+x)$ ae.

$\Rightarrow \varphi(x) = \text{const.}$ ae.

$\int |\varphi|^2 < \infty \Rightarrow \varphi = 0$ ae.

But it has almost-inv-vectors:

(ZIM) Let $\varepsilon > 0$, $K = [a,b] \subseteq \mathbb{R}$

$\Rightarrow$ Take $[c,d]$ large enough, and consider

$C \chi_{[c,d]}$ where $C$ is to be chosen wisely.

Let $\varepsilon > 0$, $K \in \mathcal{K}(\mathbb{R})$, pick $N$: $K \subseteq [-N, N]$

& $2/\sqrt{N} < \varepsilon$. Put $\varphi := \frac{1}{N} \chi_{[0,N^2]}$. $\|\varphi\|_2 = 1$,

$$\pi(t,\varphi)(x) = \varphi(-t+x) = \frac{1}{N}\chi_{[0,N^2]}(-t+x)$$

[redacted] $$= \frac{1}{N}\chi_{[t,N^2+t]}(x).$$

$$\Rightarrow \|\pi(t,\varphi)-\varphi\|_2^2 = \int_0^t + \int_t^{N^2} + \int_{N^2}^{N^2+t} = \frac{2|t|}{N^2} \leq \frac{2}{N} < \varepsilon^2$$

$$\Rightarrow \|\pi(t,\varphi)-\varphi\|_2 < \varepsilon.$$

- Similar results hold for the left-regular representations of $\mathbb{Z}^n$ or $\mathbb{R}^n$.

- The basic discovery of Kazhdan that led to the formulation of property (T) was that many semisimple groups and their lattice subgroups do not have representations with this type of behavior.

## Rem. (12.3.17):

(i) $L^2(H,\mathbb{R})$ has almost-inv. vectors

$\Leftrightarrow L^1(H,\mathbb{R})$ has almost-inv. vectors $\boxed{\text{EX. 12.3.\#23}}$

(ii) $L^2(H,\mathbb{R})$ has almost-inv. vectors

$\Leftrightarrow \exists p \in [1,\infty[ \, : \, L^p(H,\mathbb{R})$ has almost-inv. vectors.

$\boxed{\text{EX. 12.3.\#24}}$

$\boxed{\text{EX. 12.3.\#23}}$

Consider
$$\alpha : H \times L^0(H,\mathbb{R}) \longrightarrow L^0(H,\mathbb{R})$$
$$(h, \varphi) \longmapsto \boxed{x \mapsto \varphi(h^{-1}x)}.$$

($\Rightarrow$) Let $\varepsilon>0$, $K \in \mathcal{K}(H)$, $\varphi \in L^2(H,\mathbb{R})$ be a unit $(\varepsilon, K)$-inv. vector.

[redacted]

Put $\psi := \varphi^2 \Rightarrow \psi \in L^1(H,\mathbb{R}), \|\psi\|_1 = 1$.

For $k \in K$,

$$\int_H |\psi(k^{-1}x) - \psi(x)|\,dx = \int_H |\varphi(k^{-1}x)^2 - \varphi(x)^2|\,dx \le \left(\int_H |\varphi(k^{-1}x) - \varphi(x)|^2 dx\right)^{1/2} \underbrace{\left(\int_H |\varphi(k^{-1}x) + \varphi(x)|^2 dx\right)^{1/2}}_{\le 2\|\varphi\|_2 = 2} < \varepsilon, \checkmark$$

($\Leftarrow$) Let $\varepsilon>0$, $K \in \mathcal{K}(H)$, $\varphi \in L^1(H,\mathbb{R})$ be a unit $(\varepsilon, K)$-inv. vec.

$\psi := |\varphi|^{1/2} \in L^2(H,\mathbb{R})$, $\|\psi\|_2 = 1$, $\displaystyle\int |\psi(k^{-1}x) - \psi(x)|^2 dx \underset{?}{\le} \int (\psi(k^{-1}x)^2 - \psi(x)^2)\,dx$

$$= \int |\varphi(k^{-1}x) - \varphi(x)|\,dx < \varepsilon, \checkmark$$

$t \mapsto t^2$ convex

(Hulanicki Thm.)

**Prop. (12.3.18)**: $\boxed{\text{AME3}} \iff \boxed{\text{AME5}}$.

$\boxed{\text{AME3}}$ $\exists$ left-inv. $\lambda \in \text{Mean}\,(C_b^0(H,\mathbb{R}))$

$\boxed{\text{AME5}}$ $\pi_{reg}$ has almost-invariant vectors. (in $L^2(H,\mathbb{R})$).

(Sketch)
**Pf**: By Rem. (12.3.17).(i), we can instead show $\boxed{\text{AME 3}} \iff \boxed{\text{AME 5}'}$.

$\boxed{\text{AME5}'}$ $\pi_{reg} : H \times L^1(H,\mathbb{R}) \longrightarrow L^1(H,\mathbb{R})$ has almost-invariant vectors.

($\Rightarrow$) By Prop. 7.2.3 of Zimmer,

$\{\lambda_\varphi \mid \varphi \in P(H)\}$ is weak-$*$-dense in $\text{Mean}\,(C_b^0(H,\mathbb{R}))$.

$\Rightarrow$ $\exists\, \varphi \in P(H) = \{\varphi \in L^1(H,\mathbb{R}) \mid \varphi \geq 0, \|\varphi\|_1 = 1\}$:

$\lambda_\varphi \overset{*}{\approx} \lambda$, where $\lambda$ is [redacted] a left-inv. mean.

"vectors close to $\varphi$ will be almost invariant"

( $\boxed{\text{EX 12.3.\#26}}$ for the correct proof in the case when $H$ is discrete! )

$\boxed{\text{Zimmer pp. 133-139}}$

$(\Leftarrow)$ Thm: $\boxed{\text{AMES}} \Leftrightarrow \exists \{\varphi_n\}_n \subseteq P(H)$:

$\| \pi_{reg}(t, \varphi_n) - \varphi_n \|_1 \xrightarrow{u.} 0$ on compacta.

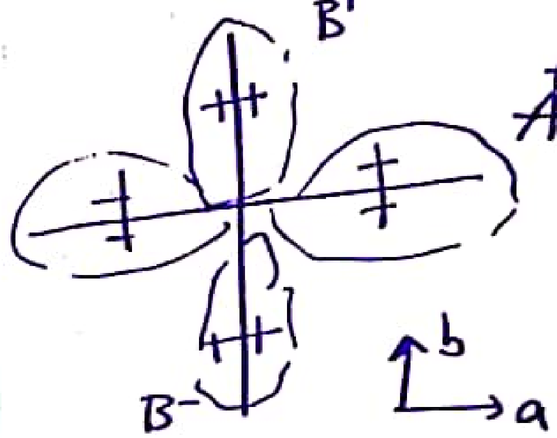
- $Mean(L^\infty(H, \mathbb{R}))$ is compact (by Banach-Alaoglu)

$\Rightarrow \exists N \subseteq \mathbb{N} : {}^{\lambda}\varphi_n \xrightarrow[n \in N]{} m \in Mean(L^\infty(H, \mathbb{R}))$.

Claim: $m$ is [redacted] left-inv. $\boxed{\text{EX. } 12.3.\#25}$

## §12.4:

**Prop (12.4.1):** Nonabelian free groups are non-amenable.




Pf. $\boxed{\text{Case}: F_2 = \langle a, b\rangle.}$

· Using $\boxed{\text{AME3}}$ : (From Tao's blog)

Suppose $\exists$ left-inv. [redacted] $\lambda \in \text{Mean}(C_b^0(F_2, \mathbb{R}))$

$A^+ =$ all words beginning by a

$A^- =$ —— // —————— $a^{-1}$

$B^+ =$ —— // —————— $b$

$B^- =$ —— // —————— $b^{-1}$

Obs: $B^+ \subseteq (a^{-1}A^+) \setminus A^+ = B^+ \uplus B^-$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{mean}}}{0 \leq} \lambda(\chi_{B^+}) \leq \lambda(a^{-1}A^+ \setminus A^+) = \lambda(a^{-1}A^+) - \lambda(A^+) \underset{\substack{\uparrow \\ \text{by inv.}}}{=} 0$$

$\Rightarrow \lambda(\chi_{B^+}) = 0.$

Similarly,

$$B^- \subseteq (a^{-1}A^+) \setminus A^+ = B^+ \uplus B^-$$

$$A^+ \subseteq (b^{-1}B^+) \setminus B^+ = A^+ \uplus A^-$$

$$A^- \subseteq (b^{-1}B^+) \setminus B^+ = A^+ \uplus A^-.$$

$$\Rightarrow 1 = \lambda(\chi_{F_2}) = \lambda(\chi_{A^+} + \chi_{A^-} + \chi_{B^+} + \chi_{B^-})$$

$$= \lambda(\chi_{A^+}) + \lambda(\chi_{A^-}) + \lambda(\chi_{B^+}) + \lambda(\chi_{B^+}).$$

$= 0$, $\lightning$.

(This proof generalizes to $F_n$.)

---

<u>Cor.(12.4.2)</u>: If $H$ is discrete and it contains a nonabelian free group, then it is not amenable.

<u>Pf</u>: $F_n$ is not amenable; and by Prop(12.2.8) closed subgroups of amenable groups are amenable.

**Prop. (12.4.4)**: $SL(2,\mathbb{R})$ is not amenable.

Pf: Consider

$$\alpha : SL(2,\mathbb{R}) \times \overbrace{(\mathbb{R} \times \{\infty\})}^{\cong S^1} \longrightarrow \overbrace{(\mathbb{R} \times \{\infty\})}^{\cong S^1}$$

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x \right) \longmapsto \begin{cases} \frac{ax+b}{cx+d}, & \text{if } x \neq -\frac{d}{c},\ c \neq 0 \\ \infty, & \text{if } x = -\frac{d}{c},\ c \neq 0 \\ \infty, & \text{if } x = \infty,\ c = 0 \\ \frac{a}{c}, & \text{if } x = \infty,\ c \neq 0. \end{cases}$$

$\alpha$ is a transitive action, and

$$\mathrm{Stab}_\alpha(0) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \,\middle|\, \alpha\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 \right) = 0 \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, b = 0,\ d \neq 0 \right\}.$$

$$= \underbrace{\begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}}_{=:P} \subseteq SL(2,\mathbb{R}).$$

• By Ex. (8.4.4) (1), $P \leq SL(2,\mathbb{R})$ is a
(p. 184)
minimal parabolic subgroup.

By Cor. (8.4.11), $SL(2,\mathbb{R})/P$ is compact.
(p. 187)

• Borel Density theorem implies that there is no $SL(2,\mathbb{R})$-inv. prob. measure on $SL(2,\mathbb{R})/P$

$\boxed{\text{EX. 4.6.\#2}}$

$\Rightarrow G = SL(2,\mathbb{R})$ is not amenable, via $\boxed{\text{AME2}}$, and,

$\beta : SL(2,\mathbb{R}) \times G/P \to G/P$.
$(g, hP) \mapsto ghP$.

• Actually: $SL(n,\mathbb{R})$ for $n \geq 2$ is not amenable.

Lemma (Furstenberg) 3.2.1 [Zimmer p. 39]: $\{[g_m]\}_m \subseteq PGL(n,\mathbb{R})$,

$\mu, \nu \in \mathcal{P}rob(\mathbb{P}^{n-1}(\mathbb{R}))$, $\mu \cdot [g_m] \longrightarrow \nu$.

Then either (i) $\{[g_m]\}_m$ is bounded, or

(ii) $\exists\, V, W \leq \mathbb{R}^n$ (linear), $1 \leq \dim V, \dim W \leq n-1$:

$\nu$ is supported on $[V] \cup [W]$.

This Furstenberg's lemma shows that there are no invariant measures on $\mathbb{P}^{n-1}(\mathbb{R})$ under the $SL(n,\mathbb{R})$ action. (Zimmer, p. 62).

**Prop. (12.4.5)**: Let $H$ be a connected semisimple Lie group. If $H$ is amenable, then it is compact.

**Pf**: By Cor. 4.9.2 (p.72), non compact groups contain nonabelian free groups, which are closed, which are not amenable by Prop (12.4.1). Then by Prop. (12.2.8) $H$ cannot be amenable.

**Prop. (12.4.7)**: A connected Lie group $H$ is amenable iff

$\exists$ normal closed connected $N \trianglelefteq H$ :

[redacted]

$N$ is solvable &

$H/N$ is compact.

**Pf**: $(\Leftarrow)$ Cor (12.2.7).(ii).

($\Rightarrow$) • By the structure theory of Lie groups $\exists$ connected closed normal solvable $R \trianglelefteq H$ such that $H/R$ is semisimple.

• Since $H$ is amenable, by $\boxed{\text{EX. 12.1. \#2}}$ $H/R$ is amenable.

$\Rightarrow$ By Prop. (12.4.5) $H/R$ is compact, ✓

——.

## §12.5:

In this section our aim is to prove:

**Prop. (12.2.8)**: If $H$ is amenable, then so is any of its closed subgroups.

- Let $\Lambda \leq H$ be a closed subgroup.

$V$ be a l.c.t.v.s, $S \subseteq V$ be a nonempty cpt, conv. [redacted] $\Lambda$-inv. set for some $C^0$ action $\alpha : \Lambda \curvearrowright V$.

$S$ cpt $\Rightarrow L^0(H,S) = L^\infty(H,S)$

$\varphi \in L^0(H, S)$ is essentially $\Lambda$-equivariant if $\forall \lambda \in \Lambda, \forall h \in_{ae} H : \varphi(\lambda^{-1} h) = \lambda^{-1} \varphi(h)$

$\left( \Leftrightarrow \forall \lambda \in \Lambda : \varphi(\lambda^{-1} \cdot) =_{ae} \lambda^{-1} \varphi. \right)$.

$L^0_\Lambda(H, S)$ denotes all essentially $\Lambda$-equiv. $\varphi \in L^0(H, S)$.

Ex. (12.5.2):

(i) If $H$ is discrete, then $\mathcal{L}^{0}(H,S) = S^{H}$, and so by Tychonoff's Thm. $\mathcal{L}^{0}(H,S)$ is compact.

(ii) If $S := B_1(\mathbb{C})$, then

$$\mathcal{L}^{0}(H,S) = B_1(\mathcal{L}^{\infty}(H,\mathbb{R})),$$

whence by Banach-Alaoglu $\mathcal{L}^{0}(H,S)$ is weak-$*$-compact.

Lem. (12.5.3): Let $\Lambda \leq H$ be closed, $B$ be a separable Banach space, $\alpha : H \curvearrowright B^{*}$ be cts, $S \subseteq B^{*}$ be nonempty compact convex $H$-inv. Then

[redacted] $\mathcal{L}^{0}(H,S)$ and $\mathcal{L}^{0}_{\Lambda}(H,S)$ are nonempty compact convex $H$-inv.

Pf: • $\mathcal{L}^1(H,B)^* = \mathcal{L}^\infty(H,B^*)$ has a weak-*-top.

$S \subseteq B^* \Rightarrow \underset{\underset{\mathcal{L}^\infty(H,S)}{\shortparallel}}{\mathcal{L}^\infty(H,S)} \subseteq \mathcal{L}^\infty(H,B^*)$

$\mathcal{L}^\infty(H,S)$ is closed bounded & convex

in $\mathcal{L}^\infty(H,B^*)$

$\Rightarrow$ By Banach-Alaoglu $\mathcal{L}^\infty(H,S)$ is

weak-*-compact,

$R: H \times \mathcal{L}^\infty(H,S) \longrightarrow \mathcal{L}^\infty(H,S)$ is cts

$(h, \varphi) \longmapsto \boxed{x \mapsto \varphi(xh)}$

• Similarly for $\mathcal{L}^\infty_\Lambda(H,S) = \mathcal{L}^\infty_\Lambda(H,S)$.

Pf of Prop. 12.2.8: Let $H$ be amenable, $\Lambda \leq H$ be closed, We'll show that $\Lambda$ is amenable via $\boxed{\text{AME 2}}$.

So let $X$ be compact metrizable, $\alpha : \Lambda \curvearrowright X$ be continuous.

$$B := C_b^0(X, \mathbb{R}) \Rightarrow \overset{S:=}{\mathrm{Prob}(X)} \subseteq B^*$$

$\mathrm{Prob}(X)$ is nonempty $\overset{\text{weak-}*\text{-}}{\text{compact}}$ convex and H-invariant.

$$\left(\begin{array}{l} \tilde{\alpha} : \Lambda \times \mathrm{Prob}(X) \to \mathrm{Prob}(X) \\ (\lambda, \mu) \mapsto \boxed{\mu(\lambda B) \leftarrow B} \\ \vdots \\ H \times \mathrm{Prob}(X) \to \mathrm{Prob}(X) \ ?? \end{array}\right)$$

$\Rightarrow$ By Lem (12.5.3), $L^\infty_\Lambda(\mathbf{H}, \mathrm{Prob}(X))$ is nonempty compact convex H-invariant with respect to R. action of H.

$$R : H \times L^\infty_\Lambda(H, \mathrm{Prob}(X)) \to L^\infty_\Lambda(H, \mathrm{Prob}(X))$$
$$(h, \varphi) \longmapsto [x \mapsto \varphi(xh)]$$

Since $H$ is amenable, by $\boxed{\text{AME1}}$,

$$\exists \varphi \in L^\infty_\Lambda(H, \mathcal{P}rob(X)) :$$

$$\left.\begin{aligned} &\forall h \in H : \varphi(xh) =_x \varphi(x) \\ &\forall \lambda \in \Lambda : \varphi(\lambda^{-1}x) =_x \text{[redacted]}\ \tilde{\alpha}(\lambda^{-1}, \varphi(x)). \end{aligned}\right\} (*)$$

<u>Claim</u>: $\forall x \in H$ for which $(*)$ holds, $\varphi(x)$ is a $\Lambda$-inv. [redacted] probability measure on $X$.

## §12.6 :

**Prop.** (12.6.1) (Furstenberg's Lemma) : Let $G$ be a semisimple Lie group with finitely many connected components, $\Gamma \leq G$ be a lattice, $A \leq G$ be a closed amenable subgroup, $X$ be compact metrizable, $\alpha : \Gamma \curvearrowright X$ be continuous.

Then $L^\circ_\Gamma(G/A, \mathrm{Prob}(X)) \neq \emptyset$.

**Proof** : $\cdot$ $\mathrm{Prob}(X)$ is nonempty compact convex $\Gamma$-inv $[\overset{?}{\Rightarrow} H\text{-inv}]$

$\Rightarrow$ By Lem 12.5.3, $L^\circ_\Gamma(G$[redacted]$, \mathrm{Prob}(X))$ is nonempty compact convex $G$-inv.

$\Rightarrow L^\circ_\Gamma(G, \mathrm{Prob}(X))$ is also $A$-inv.

By $\boxed{\text{AME 1}}$, $\exists$ $A$-inv $\varphi \in L^\circ_\Gamma(G, \mathrm{Prob}(X))$

$\Rightarrow \exists \tilde{\varphi} \in L^{\circ}_{\Gamma}(G/A, \mathcal{P}rob(X))$

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d] & \mathcal{P}rob(X) \\ G/A \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

✓.

<u>Ex</u>: $G := SL(3, \mathbb{R})$

$A := \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \subseteq G.$

$\alpha : \Gamma \curvearrowright X.$ $C^{\circ}$