Integrability of Center Problem

Outline:

~~Introduction~~: The Problem of Integrability of the Center Distribution of a PH diffeo.

⓪ Intro: Two Lines of Attack: $[\cdot,\cdot]$ or smoothness of E^c ← (which is in gen. guaranteed to be only Hölder)

① Construction of the Algebraic Example of Borel & Smale.

② Non-Integrability is an Open Property via the Result of Brin, Burago, Ivanov.

③ Weak Integrability.

⟶ its Uses in Pugh-Shub Conj.

Q: Can you tweak the parameters to make it nonvolume preserving?

$$\begin{aligned}
L &\longrightarrow L \\
X &\longmapsto \lambda_1 X \\
Y &\longmapsto \lambda_2 Y \\
Z &\longmapsto \lambda_1\lambda_2 Z \\
A &\longmapsto \mu_1 A \\
B &\longmapsto \mu_2 B \\
C &\longmapsto \mu_1\mu_2 C.
\end{aligned}$$

$\lambda_1 = \lambda_2 > 1.$

$\mu_1 = \mu_2 = (\lambda_1)^{-2}$

λ

λ^2

λ^3

μ^0

μ^2

$\mu^3.$

The Problem of the Integrability of the Center Distribution of a Partially Hyperbolic Diffeomorphism:

§0. Introduction:

- Let M be a smooth closed Rie. manifold, $f \in \mathrm{Diff}^1(M)$.

We have induced bounded linear operators $\vec{f}$ and $\overleftarrow{f}$ on $\Gamma^0(TM)$ that are the inverses of each other:

$$\begin{aligned}\vec{f} : \Gamma^0(TM) &\longrightarrow \Gamma^0(TM)\\ X &\longmapsto \left[y \mapsto df(f^{-1}(y))\, X(f^{-1}(y))\right]\end{aligned} \qquad \begin{aligned}\overleftarrow{f} : \Gamma^0(TM) &\longrightarrow \Gamma^0(TM)\\ Y &\longmapsto \left[x \mapsto df^{-1}(f(x))\, Y(f(x))\right]\end{aligned}$$

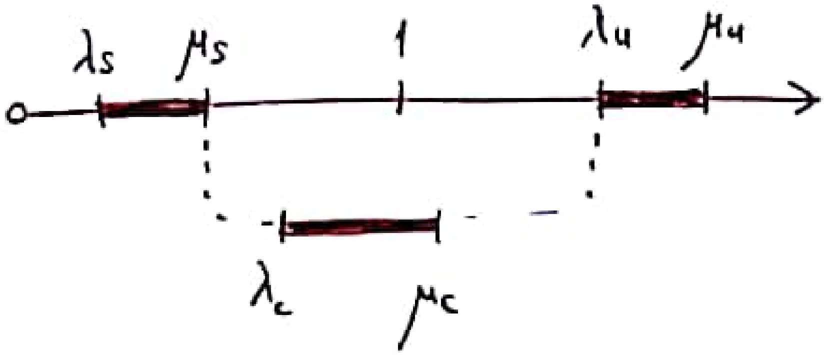
$$\begin{tikzcd} TM \arrow[r, "df"] & TM \\ M \arrow[u, "X"] \arrow[r, "f"] & M \arrow[u, "\vec{f}(X) = df \circ X \circ f^{-1}"'] \end{tikzcd} \qquad \begin{aligned}\overleftarrow{f}(Y) &= df^{-1} \circ Y \circ f\\ &= \overrightarrow{(f^{-1})}(Y)\\ &= (\vec{f})^{-1}(Y)\end{aligned} \qquad \begin{tikzcd} TM \arrow[r, "df"] & TM \\ M \arrow[u] \arrow[r, "f"] & M \arrow[u, "Y"'] \end{tikzcd}$$

$$\vec{f}_{\mathbb{C}} := \vec{f} \otimes \mathbb{C} : \Gamma^0(TM) \otimes \mathbb{C} \longrightarrow \Gamma^0(TM) \otimes \mathbb{C}.$$

The spectrum of $\vec{f}_{\mathbb{C}}$ is the <u>Mather</u> spectrum of f.

• f is *partially hyperbolic* if

$$\mathrm{spec}(f) \subseteq \left([\lambda_s,\mu_s] \uplus [\lambda_c,\mu_c] \uplus [\lambda_u,\mu_u] \right) \times S^1$$



Then we have a df-invariant splitting

$$TM = E_f^s \oplus E_f^c \oplus E_f^u,$$

Brin & Pesin [BrPe74]

which, by a theorem by ~~Brin [Bri01]~~, consists of Hölder distributions ($E_f^s \oplus E_f^c$ and $E_f^c \oplus E_f^u$ are Hölder as well).

• A distribution $E \subseteq TM$ is *integrable* if

"C^0-foliation with C^1 leaves".

$\exists$ a foliation $\mathcal{F}_E$ (C^0 along transverse, C^1 along leaves),

$\forall x \in M$: $T_x \mathcal{F}_E(x) = E(x)$.

By the Stable Manifold Theorem [HPS77], E_f^s and E_f^u are always integrable, to $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$.

HHU - Book:

- A <u>codimension-p C^r-foliation with C^s-leaves</u> of M^n is an atlas $\{(U, \varphi_U)\}_{U \in \mathcal{U}}$ such that

(i) $\exists$ locally compact $T \subseteq \mathbb{R}^p$, $\forall U \in \mathcal{U}$: ~~$\varphi_U : \mathbb{R}^{n-p} \times T$~~

$\varphi_U : U \longrightarrow \mathbb{R}^{n-p} \times T$ is a homeomorphism.

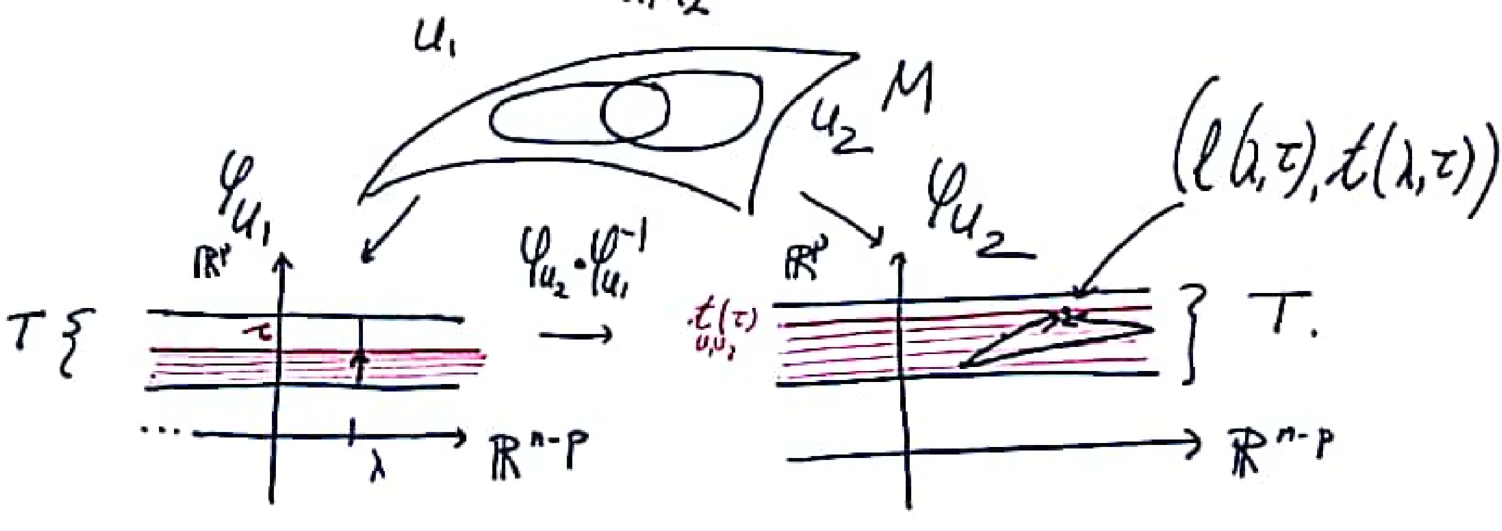
(ii) $\forall U_1, U_2 \in \mathcal{U}$:

$\varphi_{U_2} \circ \varphi_{U_1}^{-1} : \mathbb{R}^{n-p} \times T \longrightarrow \mathbb{R}^{n-p} \times T$ is of the form

$$\varphi_{U_2} \circ \varphi_{U_1}^{-1} : (\lambda, \tau) \longmapsto \left(\ell_{U_1 U_2}(\lambda, \tau), t_{U_1 U_2}(\tau) \right),$$

where $\forall \tau \in T$: $\ell_{U_1 U_2}(\cdot, \tau) \in C^s(\mathbb{R}^{n-p}, \mathbb{R}^{n-p})$

$\forall \lambda \in \mathbb{R}^{n-p}$: $\ell_{U_1 U_2}(\lambda, \cdot) \in C^r(T, \mathbb{R}^{n-p})$



• A natural followup would be to establish the integrability of E_f^c. To detect possible obstructions to integrability, here is a classical theorem: [Law74]

<u>Theorem (Clebsch-Deahna-Frobenius)</u>: Let $E \subseteq TM$ be a C^1 distribution. Then

<u>E is uniquely integrable</u> if $\exists$ a C^1 fol. $\mathcal{F}_E$ with C^1 leaves: $\forall x \in M \cdot T_x \mathcal{F}_E(x) = E(x)$ and $\forall \gamma \in C^1(\mathbb{R}, M)$, $\forall t \in \mathbb{R}$: $\dot\gamma(t) \in E(\gamma(t)) \to \gamma(\mathbb{R}) \subseteq \mathcal{F}_E(\gamma(t))$.

$$
\begin{tikzcd}
E \text{ is uniquely integrable} \arrow[r, Rightarrow] & E \text{ is integrable} \arrow[dr, Rightarrow] & \\
E \text{ is involutive} \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & & E \text{ is weakly integrable} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

(with matching rank).

Here (i) E is <u>weakly integrable</u> if

$$\forall x \in M,\ \exists \text{ a } C^1 \text{ submanifold } (F_{E,x}, \varphi_{E,x}) : x \in \operatorname{im}(\varphi_{E,x}),$$
$$\forall y \in \operatorname{im}(\varphi_{E,x}) : T_y \operatorname{im}(\varphi_{E,x}) = E(y)$$

("submanifold" means:

$$
\begin{tikzcd}
TF \arrow[r, tail, "d\varphi"] \arrow[d] & TM \arrow[d] \\
F \arrow[r, tail, "\varphi"] & M
\end{tikzcd}
$$

)

(ii) E is <u>involutive</u> if $\Gamma(E)$ is closed under the Lie bracket, which is defined by

$$[\cdot,\cdot]:\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)$$
$$(X,Y)\longmapsto\Big[z\longmapsto\big[f\longmapsto\big[X_z(Y(f))-Y_z(X(f))\big]\big]\Big].$$

Thus we have two potential obstructions (there may be more!)

(1) E_f^c may fail to be involutive.

(2) E_f^c may fail to be C^1. (although there are DCF-thms for less regular objects)

- There is essentially one family of explicit examples due to Armand Borel & Steven Smale [Sma 67] where due to violation (1) E_f^c fails to be integrable. Said family is of algebraic origin, and they also constitute the first example of nontoral Anosov diffeomorphisms.

$$X : \mathbb{R}^2 \longrightarrow T\mathbb{R}^2$$
$$(x,y) \longmapsto (1, 3y^{2/3})_{(x,y)},$$

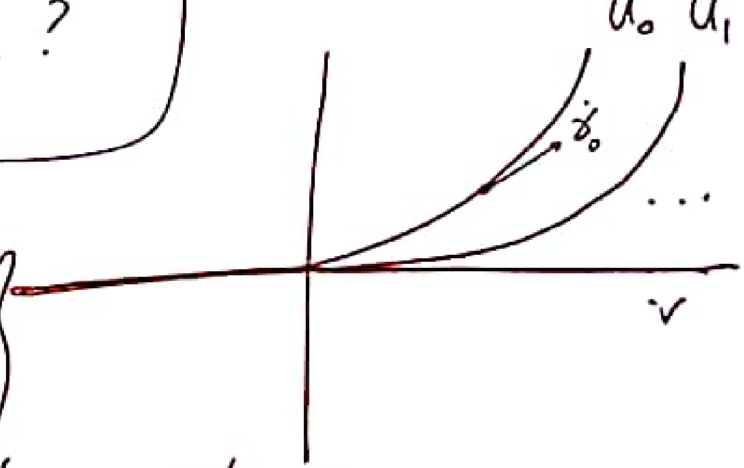
$$\boxed{\dot{x} = 3x^{2/3}}$$

$$E := \langle X \rangle.$$

$\boxed{\begin{array}{l} \underline{Q}: \text{Is there a fol. } \mathcal{F}_E : \\ \forall \gamma : \mathbb{R} \to M, \left[\forall t \in \mathbb{R} : \dot{\gamma}(t) \in E(\gamma(t))\right] \\ \Rightarrow \vec{\gamma}(\mathbb{R}) \subseteq \mathcal{F}_E(\gamma(0)).\ ? \end{array}}$

$\forall c \in \mathbb{R}_{\geq 0}$:

$$u_c(t) = \begin{cases} 0, & \text{if } t \le c \\ (t-c)^3, & \text{if } t > c \end{cases}.$$



$$v(t) = 0.$$

$$= \begin{cases} (1, 3(t-c)^2), & \text{if } t > c \\ (1, 0), & \text{if } t \le c \end{cases}.$$

$$\dot{u}_c(t) = 3(t-c)^2$$

$$E(\gamma_c(t)) \cancel{=}$$

$$\cancel{E(u_c(t))}$$

$$= E(t, u_c(t))$$

a $\gamma_c : \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t, u_c(t)).$

$$= (1, 3(u_c(t))^{2/3})$$

$$\cancel{\dot{x}_c(t) = (1, 3(t-c)^2)}$$

$$\dot{\gamma}_c(t) = \begin{cases} (1, 3(t-c)^2), & \text{if } t > c \\ (1, 0), & \text{if } t \le c \end{cases}.$$

$\forall c \in \mathbb{R}_{>0}$:

$\forall t \in \mathbb{R}$: $\dot{\gamma}_c(t) \in E(\gamma_c(t))$.

$\Rightarrow$ ⚡. $\boxed{E \text{ is not uniquely integrable!}}$

Q. Is E weakly int.? ✓ ~~(gerst. frei)~~

Q. Is E integrable? ✓.

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as $n_1 \to \infty$, $n_2 - n_1 \to \infty$, $n_3 - n_2 \to \infty$ . . .

$n_k - n_{k-1} \to \infty$.

Rokhlin problem : 1-mixing $\Longrightarrow$ k mixing ?

$$\partial_t u(t) = 3(u(t))^{2/3}$$

$$\partial_t (u(t))^{1/3} = \frac{1}{3} (u(t))^{-2/3} \partial_t u(t) = 1.$$

$$(u(t))^{1/3} = t + c \qquad c \in \mathbb{R}.$$

$$(u(0))^{1/3} = 0 + c. \qquad c = (u(0))^{1/3}.$$

$$u(t) = (t+c)^3.$$

$$\boxed{\begin{array}{l} \forall c \in \mathbb{R}: \ u_c(t) = (t+c)^3, \text{ where } (u(0))^{1/3} = c. \\ \qquad \text{is a sol. of } \ \dot{u} = 3(u)^{2/3}. \end{array}}$$

$$\forall c \in \mathbb{R} : \gamma_c : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$t \longmapsto (t, u_c(t))$$

$$\gamma_c(t) = (t, (t+c)^3)$$

$$\dot{\gamma}_c(t) = (1, 3(t+c)^2) = \left(1, 3(u_c(t))^{\frac{2}{3}}\right).$$

$$X : \mathbb{R}^2 \longrightarrow T\mathbb{R}^2$$

$$(x,y) \longmapsto (1, 3(y)^{2/3})$$

Q: Are $\{\gamma_c \mid c \in \mathbb{R}\}$ disjoint?

$\gamma_{c_1}(t_1) = \gamma_{c_2}(t_2)$

$\rightarrow \boxed{t_1 = t_2}$ $\quad (t_1+c_1)^3 = (t_1+c_2)^3$

$t_1 + c_1 = t_1 + c_2$

$\boxed{c_1 = c_2}$ ✓

Q: $\mathbb{R}^2 \overset{?}{=} \biguplus_{c \in \mathbb{R}} \vec{\gamma}_c(\mathbb{R})$?

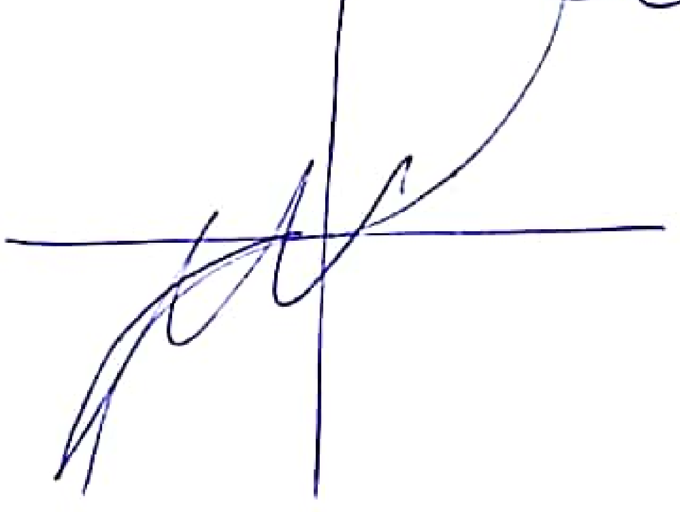
Fie $(x_0, y_0) \in \mathbb{R}^2$. $\quad (x_0, y_0) \in$ RHS $\Leftrightarrow \exists! c \in \mathbb{R}$:

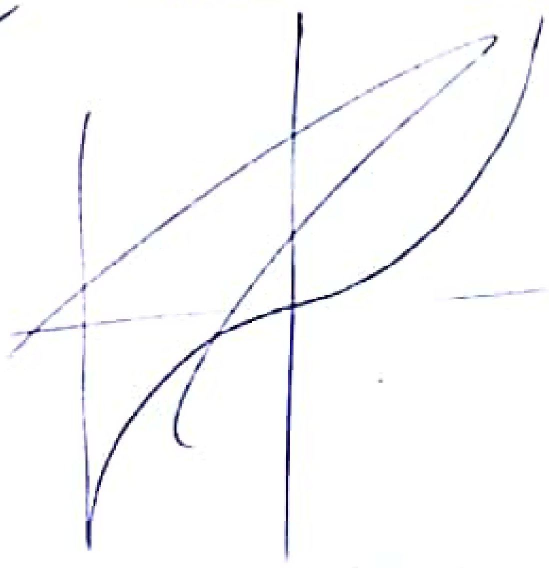
$(x_0, y_0) \in \vec{\gamma}_c(\mathbb{R}) = \{(t, (t+c)^3) \mid t \in \mathbb{R}\}$

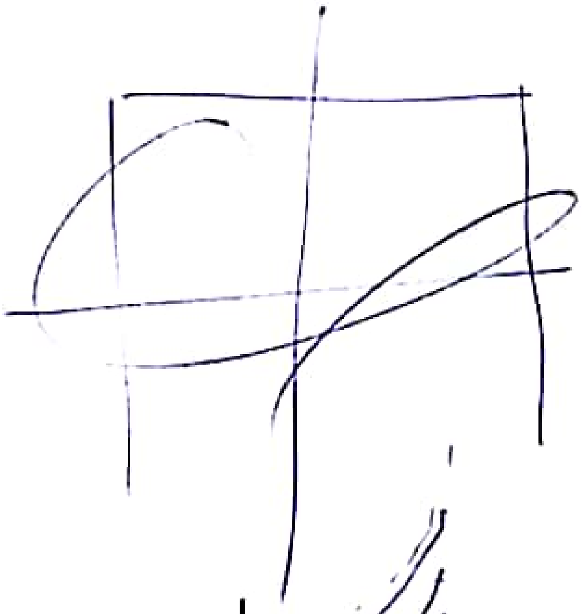
$x_0 = t$

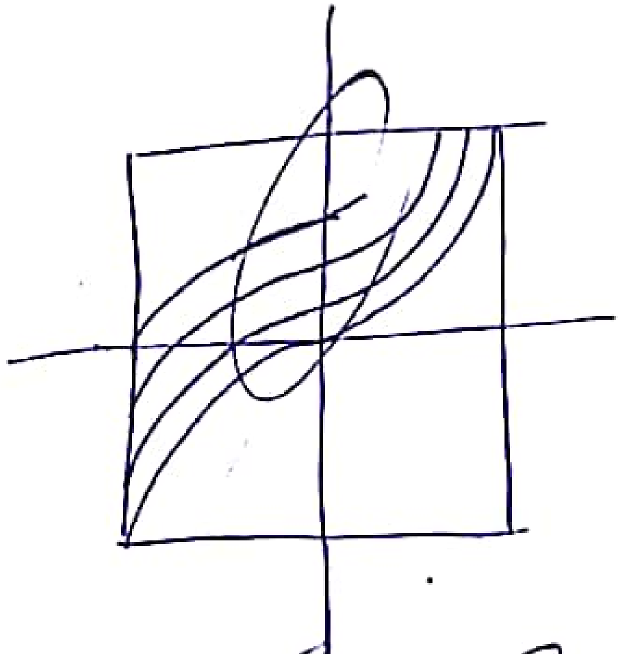
$y_0 = (t+c)^3$. $\quad y_0 = (x_0+c)^3 \quad \boxed{c := (y_0)^{1/3} - x_0}$ ✓

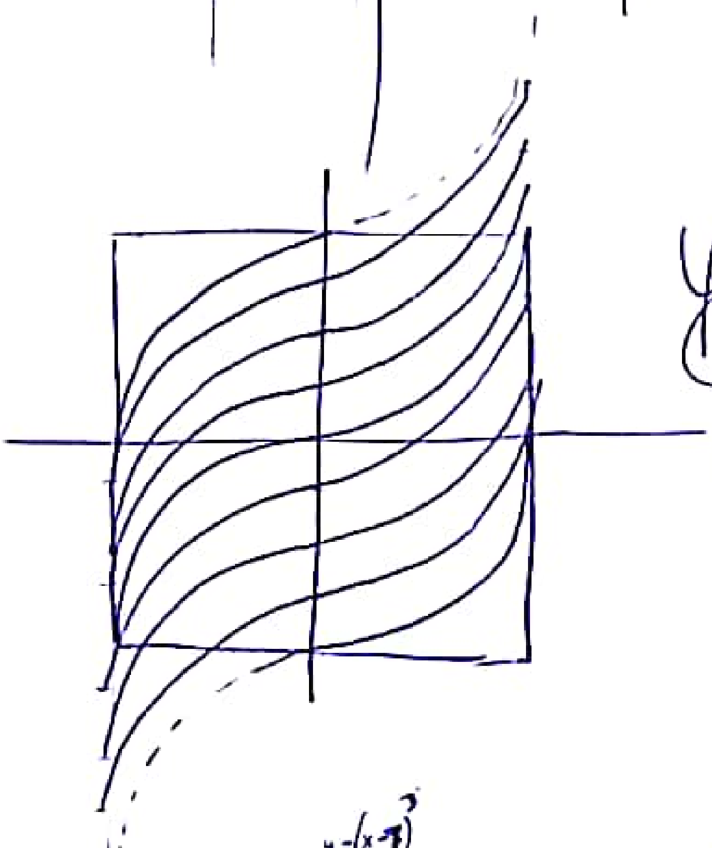
$\cancel{c := \sqrt[3]{y_0}}$







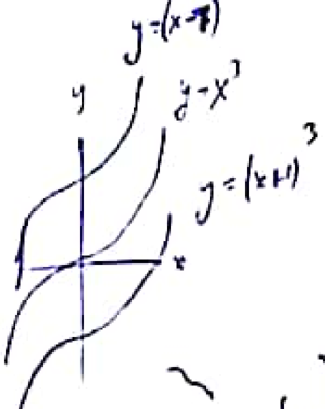




$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \times \mathbb{R}^1$$
$$(x, y) \longmapsto (x, y^{1/3}).$$

homeo.

$$\mathbb{R}^1 \times \{\tau^3\} = \varphi(\mathbb{R}^1 \times \{\tau\})$$
$$\{\lambda\} \times \mathbb{R}^1 = \varphi(\{\lambda\} \times \mathbb{R}^1)$$



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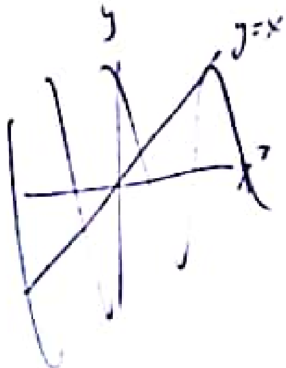
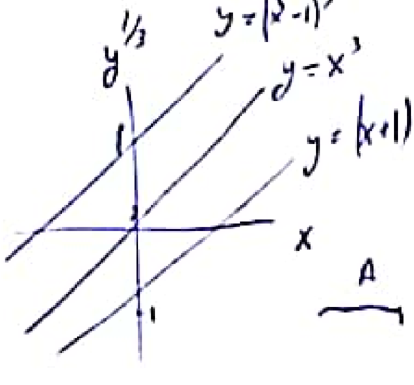
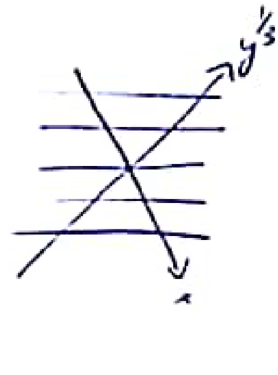
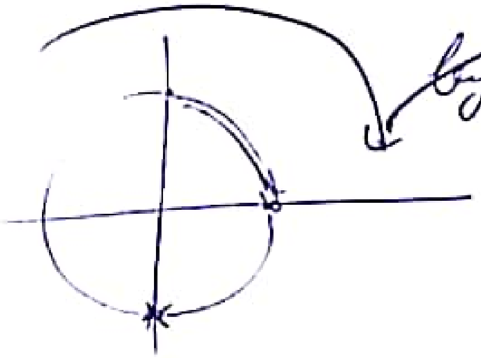




$\xrightarrow{A}$

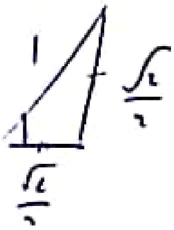


$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$



by 90.

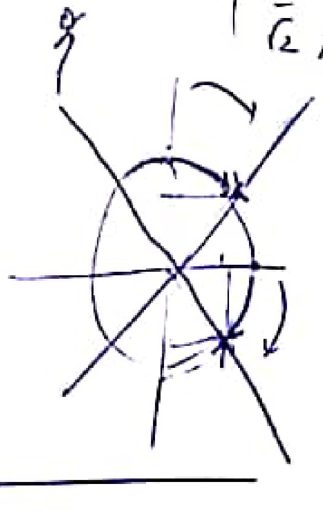
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$A : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad A : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\frac{1}{2} + \frac{1}{2} = 1.$



$1 < \sqrt{2}$

$\frac{1}{\sqrt{2}} < 1.$

$\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} =$

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \times \mathbb{R}^1$$
$$(x, y) \longmapsto \cancel{\tfrac{1}{\sqrt{2}}}\,\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y^{1/3} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + y^{1/3} \\ -x + y^{1/3} \end{pmatrix}.$$

foliation chart.

$$\overset{\leftarrow}{\varphi}\left(\mathbb{R}^1 \times \{\tau\}\right) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -x + y^{1/3} = \tau \right\}.$$

$$= \left\{ (x,y) \in \mathbb{R}^3 \;\middle|\; y = (x+\tau)^3 \right\}$$

$$\Rightarrow \cancel{\mathcal{O}}_{F\tau} = \overrightarrow{\gamma_\tau}(\mathbb{R}).$$

There is only one chart, namely $(\mathbb{R}^2, \varphi)$, so

$$\varphi_{U_2} \circ \varphi_{U_1}^{-1} = id_{\mathbb{R}^2} : \mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1 \times \mathbb{R}^1$$

$$(\lambda, \tau) \longmapsto \left(\ell(\lambda,\tau) = \lambda,\ t(\lambda) = \tau\right)$$

$$\forall \tau \in \mathbb{R}^1 : \ell(\cdot, \tau) = id_{\mathbb{R}^1} \in C^\omega(\mathbb{R}^1, \mathbb{R}^1).$$

$$\forall \lambda \in \mathbb{R}^1 : \ell(\lambda, \cdot) = \lambda \in C^\omega(\mathbb{R}^1, \mathbb{R}^1).$$

$\Rightarrow \mathcal{F}_E := \left\{ \overrightarrow{\gamma_c}(\mathbb{R}) \;\middle|\; c \in \mathbb{R} \right\}$ is a codim $=1$ C^ω-fol. with C^ω-leaves.

$\Rightarrow F$ is integrable but not uniquely integrable.

Q: An example of ~~a~~ a weakly integrable but not integrable ~~f~~ distribution?

· E is WI (top. dist.) if $\forall x \in M$, $\exists$ C^1 subman. (F_x, i_x):

$$i_x(F_x) \ni x, \quad \forall y \in i_x(F_x): T_y\big(i_x(F_x)\big) = E(y)$$

· E^k is integrable if $\exists$ a dimension-k, C^0 foliation with C^1 leaves $\mathcal{F}_E$:

$$\forall x \in M: T_x \mathcal{F}_F(x) = E(x).$$

Ex (Lebesgue?): $X: \mathbb{R}^3 \to T\mathbb{R}^3$ (WI but not I)

$(x,y,z) \longmapsto (1, f(y,z), 0)$,

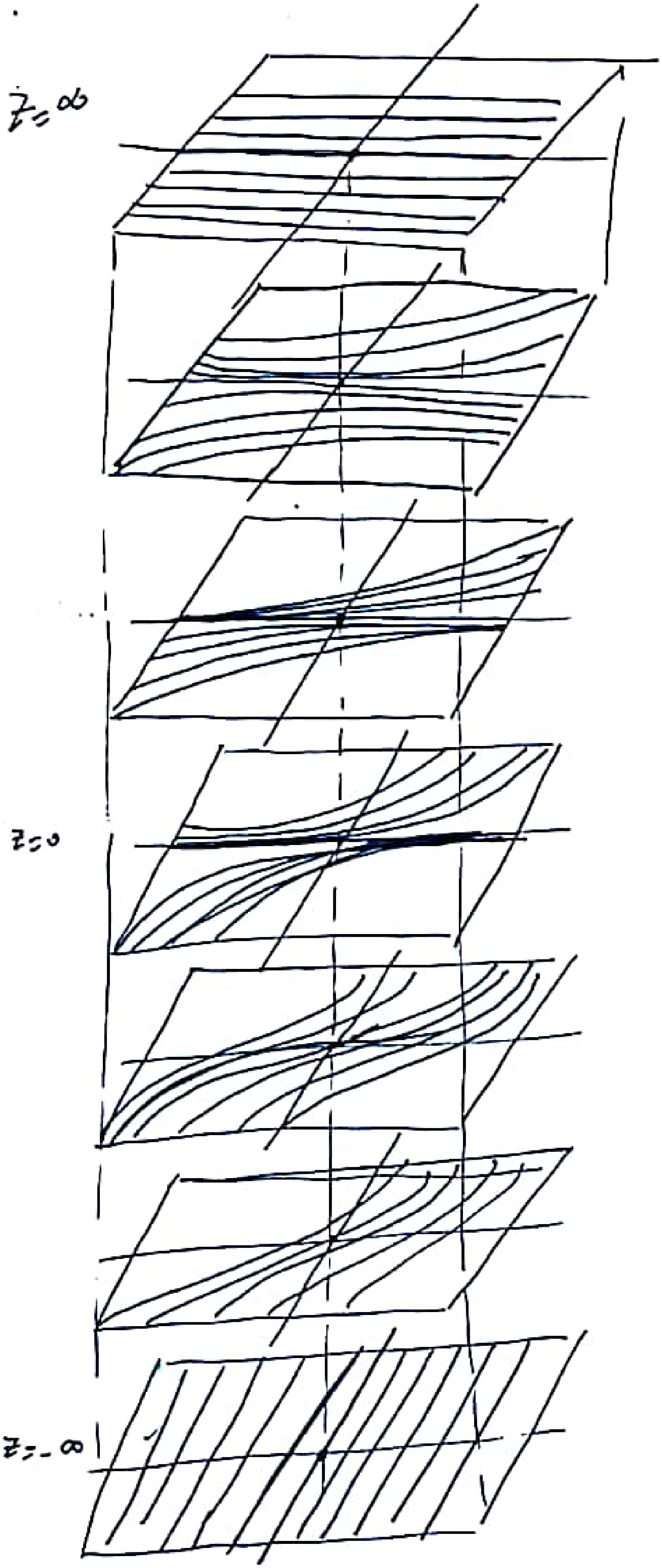
$f(y,z) := (y^2+z^2)^{1/3} - (z^2)^{1/3}\chi_{[0,\infty[}(z)$

Ex: $X: \mathbb{R}^2 \to T\mathbb{R}^2$

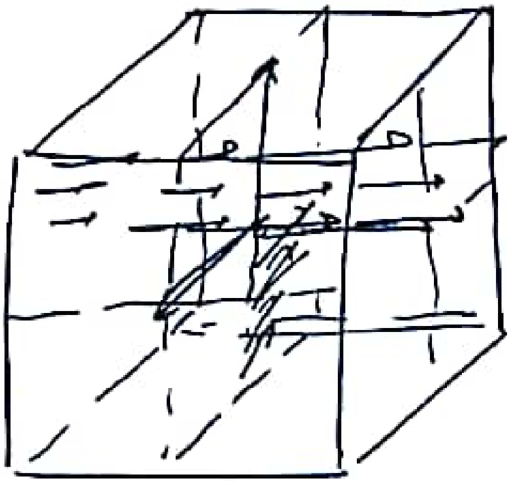
$(x,y) \mapsto (1, f(y))$

$f(y) = \chi_{]-\infty,0[}(y) - \chi_{[0,\infty[}(y)$.

Not WI.



z=∞
z=0
z=-∞



§1. Construction of the Algebraic Example of Borel & Smale:

- Let $G_1 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}$, $G_2 := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$

be two copies of the Heisenberg group (which is the unique (up to isomorphism) 3 dimensional nilpotent simply connected Lie groups).

Their Lie algebras are

$$L_1 := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}, \; L_2 := \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$$

with the exponential map (which is a diffeomorphism)

$$\begin{aligned} \exp : L_1 &\longrightarrow G_1 \\ (x, y, z) &\longmapsto \left(x, y, z + \tfrac{xy}{2}\right) \end{aligned}$$

(and similarly for $L_2 \to G_2$).

Define the generators of L_1 & L_2 by:

$X := (1,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$Y := (0,1,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$Z := [X, Y] = (0,0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(and $A, B, C = [A, B]$ for L_2.

(Here $[E,F] = EF - FE$.))

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \qquad \Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}.$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy')$$

$$(x,y,z)\cdot(-x,-y,-z+xy) = (x-x,\ y-y,\ z-z+xy-xy) = (0,0,0)$$

$$(x,y,z)^{-1} = (-x,-y,-z+xy).$$

$$N = {}_{\Gamma}\backslash^{H}. \qquad (x_1,y_1,z_1) \sim (x_2,y_2,z_2) \Leftrightarrow (x_1,y_1,z_1)(-x_2,-y_2,-z_2+x_2y_2)$$

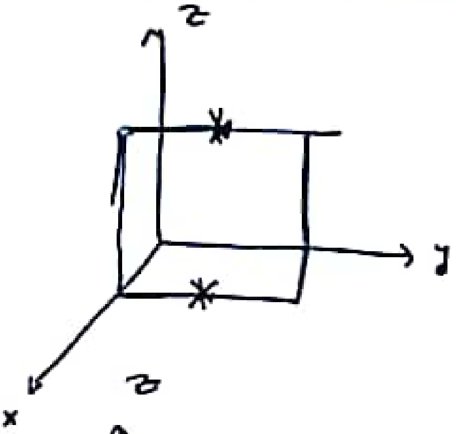
$$\Leftrightarrow (x_1-x_2,\ y_1-y_2,\ z_1-z_2+x_2y_2-x_1y_2) \in \mathbb{Z}^3$$

$$\Leftrightarrow \begin{array}{l} x_1-x_2 \in \mathbb{Z} \\ y_1-y_2 \in \mathbb{Z} \\ z_1-z_2+y_2(x_2-x_1) \in \mathbb{Z}. \end{array}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & S^1 \arrow[r] & {}_{\Gamma}\backslash^{H} \arrow[r] & \mathbb{T}^2 \arrow[r] & 0.
\end{tikzcd}$$

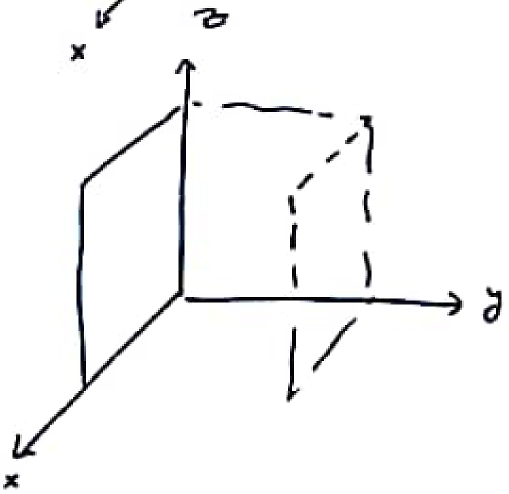
$$\boxed{(x_1,y_1,z_1)\sim(x_2,y_2,z_2) \iff \begin{array}{l} x_1-x_2\in\mathbb{Z} \\ y_1-y_2\in\mathbb{Z} \\ z_1-z_2+y_2(x_2-x_1)\in\mathbb{Z}. \end{array}}$$

- $x_2=x_1$.



$$\boxed{\begin{array}{l} y_1-y_2\in\mathbb{Z} \\ z_1-z_2\in\mathbb{Z}. \end{array}}$$

- $y_2=0$.



$$\boxed{\begin{array}{l} x_1-x_2\in\mathbb{Z} \\ y_1\in\mathbb{Z} \\ z_1-z_2\in\mathbb{Z}. \end{array}}$$

$\cancel{(x,y,z)\in\mathbb{T}^2\times S^1}$, $\cancel{(x,y,z)\sim(x,y,\bar z)}$ $\cancel{\Rightarrow z\quad \bar z + y}$

$(0,y_1,z_1)\sim(1,y_2,z_2)$

$\iff y_1-y_2\in\mathbb{Z}$

$z_1-z_2+y_2\in\mathbb{Z}.$

$\cancel{\Leftrightarrow (x_1,y_1,z_1)\sim\left(x_1+\frac{1}{2},\ y_1+\frac{1}{2},\ z_2\right)}$

$\Rightarrow$

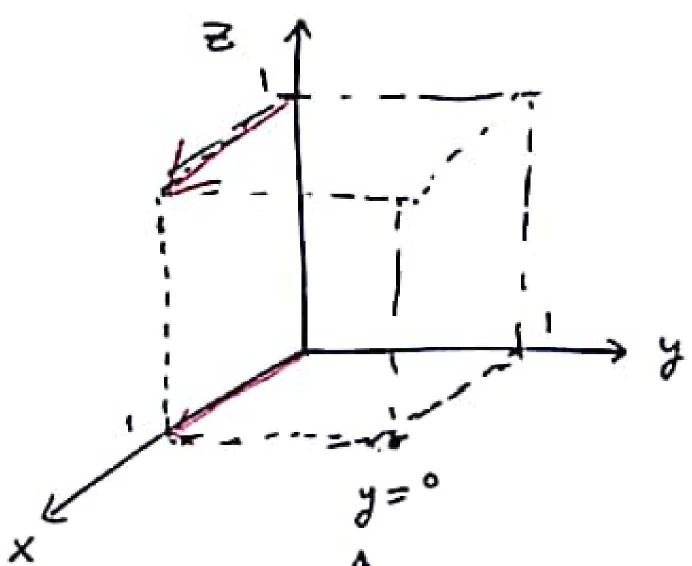
$\cancel{(x_1,y_1,z_1)\sim x_1+1,}$

$\cancel{(x,y,z)\sim\left(x+1,\frac{1}{2},z\right)\Leftrightarrow}$

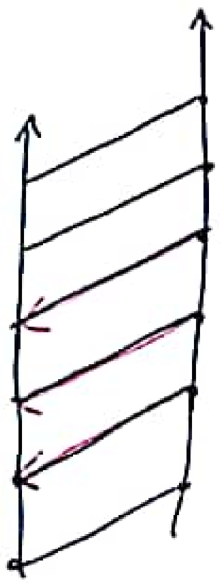
$(x,y,z)\sim\left(x+1,\frac{1}{2},\bar z\right)$

$\iff y-\frac{1}{2}\in\mathbb{Z}$

$z-\bar z+\frac{1}{2}\in\mathbb{Z}.$



$y = 0$



$$(0, y, z_1) \sim (0, y, z_2)$$

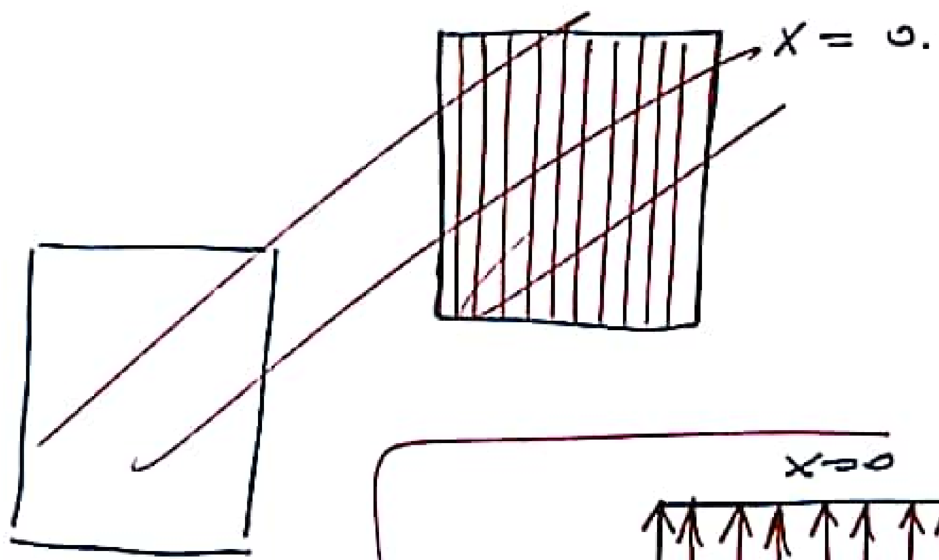
$$\Leftrightarrow \boxed{z_1 - z_2 + y \in \mathbb{Z}}$$

$y = \frac{1}{2}$ $\qquad z_1 - z_2 + \frac{1}{2} \in \mathbb{Z}.$

$$z_2 = z_1 + \frac{1}{2}.$$

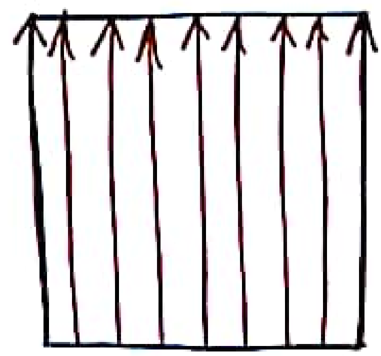
$y = t.$

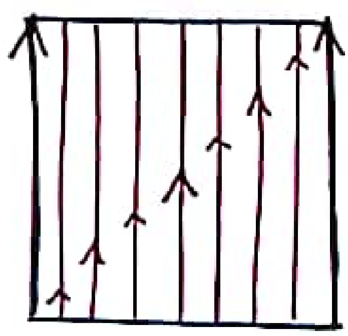
$$z_2 = z_1 + t.$$



X=1

x=0





x=1

· <u>Homomorphisms of $\mathfrak{h}$</u> :

$l : \mathfrak{h} \to \mathfrak{h}$ linear & bracket preserving.

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h} \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

$$z \longmapsto (0,0,z)$$

$$(x,y,z) \longmapsto (x,y). \qquad Z(\mathfrak{h}) = \{(0,0,z) \mid z \in \mathbb{R}\}$$

$$l(0,0,z_0) = (\bar{x}, \bar{y}, \bar{z}).$$

$$[(\bar{x},\bar{y},\bar{z}),(x,y,z)] = (0,0,\ \bar{x}y - x\bar{y})$$

$$l : Z(\mathfrak{h}) \to Z(\mathfrak{h}).$$

$$\varphi(0,0,z_0) \quad \varphi(x,y,z) \qquad \_\_ \ \_\_ .$$

$$\left(\begin{array}{c|c|c} m_{11} & m_{12} & 0 \\ \hline m_{21} & m_{22} & 0 \\ \hline m_{31} & m_{32} & m_{33}. \end{array}\right) \qquad m_{33} = m_{11} m_{22} - m_{12} m_{21}$$

$$\left(x, y, z - \frac{xy}{2}\right) \longmapsto \begin{pmatrix} m_{11} x + m_{12} y \\ m_{11} x + m_{12} y \\ m_{31} x + m_{32} y + m_{33}\left(z - \frac{xy}{2}\right) \end{pmatrix}$$

$$\downarrow$$

$$\begin{matrix} \uparrow \\ (x,y,z) \end{matrix} \qquad \begin{pmatrix} m_{11} x + m_{12} y \\ m_{21} x + m_{22} y \\ m_{31} x + m_{32} y + m_{33}\left(z - \frac{xy}{2}\right) + \frac{1}{2}(m_{11} x + m_{12} y)(m_{21} x + m_{22} y) \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \lambda^{\alpha} & 0 & 0 \\ 0 & \lambda^{\beta} & 0 \\ \hline 0 & 0 & \lambda^{\alpha+\beta} \end{array}\right) \cdot \longmapsto \begin{pmatrix} \lambda^{\alpha} x \\ \lambda^{\beta} y \\ \lambda^{\alpha+\beta}\left(z - \frac{xy}{2}\right) + \frac{1}{2} \lambda^{\alpha+\beta} xy \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{\alpha} x \\ \lambda^{\beta} y \\ \lambda^{\alpha+\beta} z \end{pmatrix}.$$

$(x,y,z)^{-1} = (-x,-y,-z+xy)$

- $\boxed{Z(\mathcal{H}) = [\mathcal{H},\mathcal{H}].}$

$(\subseteq)$ $(x,y,z)\,(a,b,c)\,(x,y,z)^{-1}\,(a,b,c)^{-1}$

$= (x+a,\, y+b,\, z+c+xb)\,(a+x,\, b+y,\, c+z+ay)^{-1}$

$= (x+a,\, y+b,\, z+c+xb)\,\big(-(a+x), -(b+y), -c-z-ay+(a+x)(b+y)\big)$

$= (0, 0, \cancel{z}+\cancel{c}+xb-\cancel{z}-\cancel{c}-ay+\cancel{ab}+\cancel{ay}+\cancel{xb}+\cancel{xy})$

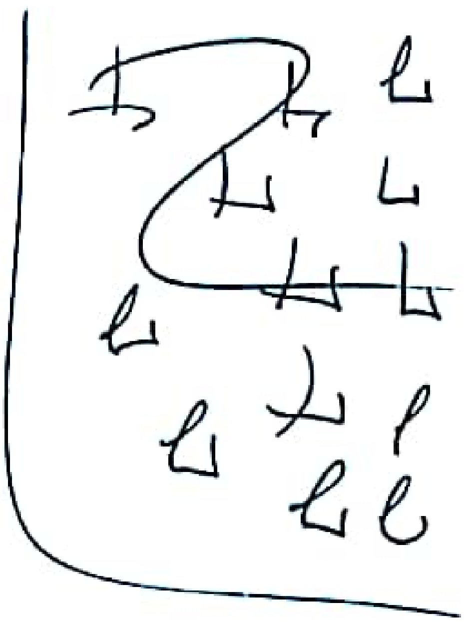
$= (0, 0, xb-ay)$

$[(x,y,z),(a,b,c)] = 0$

$\Longleftrightarrow \boxed{xb-ya=0}$

$\boxed{Z(\mathcal{H}) = \{(0,0,z)\}}$

$$\boxed{Z(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}]}$$



$$[(x\,y\,z), (a\,b\,c)] = \cancel{xb - ya.}$$
$$(0, 0, xb - ya).$$

$$Z(\mathfrak{h}) = \{(0,0,z)\}. \ldots$$

$$l : \mathfrak{h} \to \mathfrak{h} \Rightarrow \vec{l}(Z(\mathfrak{h})) \subseteq Z(\mathfrak{h}).$$

$$l(0,0,z_0) = (\bar{x}_0, \bar{y}, \bar{z})$$

$$[l(0,0,z_0), (x,y,z)] = \bar{x}y - \bar{y}x.$$

$$(0,0,z_0) = [(a_1, b_1, c_1), (a_2, b_2, c_2)].$$

$$[l(0,0,z_0), (x\,y\,z)] = [l[(a_1 b_1 c_1), (a_2 b_2 c_2)], (x\,y\,z)]$$

$$= [[l(a_1 b_1 c_1), l(a_2 b_2 c_2)], (x, y, z)] = 0.$$

$$\ell = \left(\begin{array}{cc|c} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ \hline m_{31} & m_{32} & m_{33} \end{array}\right) \xleftarrow{} A \qquad \ell[\cdot,\cdot] = [\ell\cdot,\ell\cdot]$$

$$LHS = m_{33}(xb - ya).$$

$$\cancel{RHS = A(x\ y\ z)} \quad RHS = [\ell(x\ y\ z), \ell(a\ b\ c)]$$

$$=$$

$$(m_{11}x + m_{12}y)(m_{21}a + m_{22}b) - (m_{21}x + m_{22}y)(m_{11}a + m_{12}b)$$

$$= \cancel{m_{11}m_{21}xa} + m_{11}m_{22}xb + m_{12}m_{21}ya + \cancel{m_{12}m_{22}yb}$$
$$- m_{11}m_{22}ya - m_{12}m_{21}xb$$

$$= m_{11}m_{22}(xb - ya) - m_{12}m_{21}(xb - ya)$$

$$= (m_{11}m_{22} - m_{12}m_{21})(xb - ya)$$

$a \in \frac{1}{2}\mathbb{Z}$, $|a|>1$, λ be a root of

$$X^2+2aX+1 \in \mathbb{Z}[X].$$

(thus $\lambda = -a \pm \sqrt{a^2-1}$).

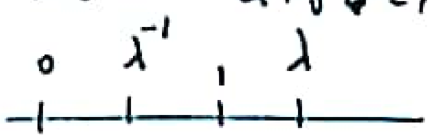
$\alpha, \beta \in \mathbb{Z}$.

$$\boxed{\begin{aligned}&\left(-a+\sqrt{a^2-1}\right)\left(-a-\sqrt{a^2-1}\right)\\ &= (-a)^2-(a^2-1) = 1.\\ &\Rightarrow -a-\sqrt{a^2-1} = \left(-a+\sqrt{a^2-1}\right)^{-1}\end{aligned}}$$

$$f := f_{\lambda,\alpha,\beta} : L \longrightarrow L$$

$$X_1 \longmapsto \lambda^{\alpha} X_1$$
$$X_2 \longmapsto \lambda^{\beta} X_2$$
$$A_1 \longmapsto \lambda^{-\alpha} A_1$$
$$A_2 \longmapsto \lambda^{-\beta} A_2.$$

$\lambda = -a+\sqrt{a^2-1}$!



$f_{\lambda,\alpha,\beta}$ is PH if

$\alpha \neq 0$ or $\beta \neq 0$.

$f_{\lambda,\alpha,\beta}$ is UH if

$\alpha \neq 0$, $\beta \neq 0$ and $\alpha+\beta \neq 0$.

$f_{\lambda,\alpha,\beta}$ is PH but not UH if

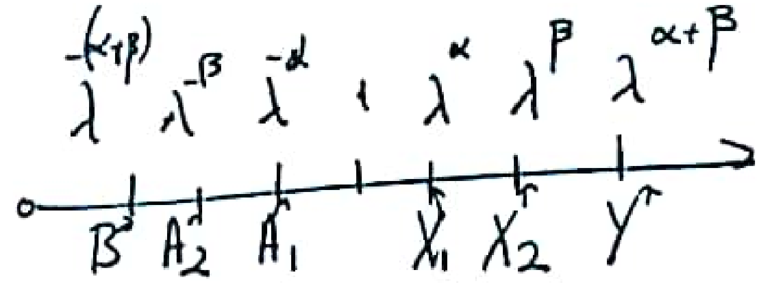
$[\alpha = 0 \quad \beta \neq 0]$

or

$[\alpha \neq 0 \quad \beta = 0]$

or

$[\alpha \neq 0 \neq \beta \quad \cancel{\text{and}} \quad \alpha+\beta = 0]$.

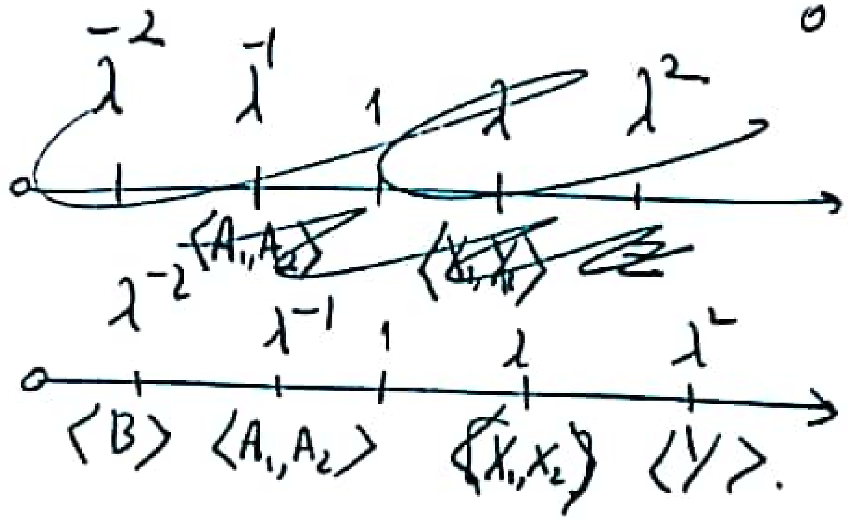


~~wlog~~

~~$0 < \alpha$~~

Wlog

$0 < \alpha = \beta.$



$L = \mathfrak{S} \oplus \mathfrak{C} \oplus \mathfrak{U}$

$\mathfrak{S} = \langle B\rangle$, $\mathfrak{C} = \langle A_1, A_2, X_1, X_2\rangle$, $\mathfrak{U} = \langle Y\rangle$

$\mathfrak{C}$ is not integrable by the bracket condition.

or $L = \mathfrak{S}' \oplus \mathfrak{C}' \oplus \mathfrak{U}'$

$\mathfrak{S}' = \langle B, A_1, A_2\rangle$, $\mathfrak{C}' = \langle X_1, X_2\rangle$, $\mathfrak{U}' = \langle Y\rangle$

or ~~$L = \mathfrak{S}'' \oplus \mathfrak{C}'' \oplus$~~ $L = S'' \oplus C'' \oplus U''$

$S'' = \langle B, A_1, A_2\rangle$, $C'' = 0$, $U'' = \langle X_1, X_2, Y\rangle$.

$$\begin{tikzcd}
L \arrow[r, "\ell"] \arrow[d, "\exp"'] & L \arrow[d, "\exp"] \\
G \arrow[r, "f"] & G
\end{tikzcd}
\qquad
\ell = \ell_{\lambda,\alpha,\beta} = \left(\begin{array}{ccc|ccc}
\lambda^{\alpha} & 0 & 0 & & & \\
0 & \lambda^{\beta} & 0 & & O & \\
0 & 0 & \lambda^{\alpha+\beta} & & & \\
\hline
 & & & \lambda^{-\alpha} & 0 & 0 \\
 & O & & 0 & \lambda^{-\beta} & 0 \\
 & & & 0 & 0 & \lambda^{-\alpha-\beta}
\end{array}\right)$$

$$\exp : (x,y,z,a,b,c) \longmapsto \left(x, y, z + \frac{xy}{2}, a, b, c + \frac{ab}{2}\right).$$

$$\log : (x,y,z,a,b,c) \longmapsto \left(x, y, z - \frac{xy}{2}, a, b, c - \frac{ab}{2}\right)$$

$$\begin{tikzcd}
\begin{pmatrix} x \\ y \\ z - \frac{xy}{2} \\ a \\ b \\ c - \frac{ab}{2} \end{pmatrix} \arrow[r, mapsto] & \left(\lambda^{\alpha}x, \lambda^{\beta}y, \lambda^{\alpha+\beta}\left(z - \frac{xy}{2}\right), \lambda^{-\alpha}a, \lambda^{-\beta}b, \lambda^{-(\alpha+\beta)}\left(c - \frac{ab}{2}\right)\right) \arrow[d, mapsto] \\
\begin{pmatrix} x \\ y \\ z \\ a \\ b \\ c \end{pmatrix} \arrow[u, mapsto] \arrow[r, mapsto, "f"] & \left(\lambda^{\alpha}x, \lambda^{\beta}y, \lambda^{\alpha+\beta}\left(z - \frac{xy}{2}\right) + \frac{\lambda^{\alpha+\beta}xy}{2}, \lambda^{-\alpha}a, \lambda^{-\beta}b, \lambda^{-(\alpha+\beta)}\left(c - \frac{ab}{2}\right) + \frac{\lambda^{-(\alpha+\beta)}ab}{2}\right)
\end{tikzcd}$$

$$= \left(\lambda^{\alpha}x, \lambda^{\beta}y, \lambda^{\alpha+\beta}z, \lambda^{-\alpha}a, \lambda^{-\beta}b, \lambda^{-(\alpha+\beta)}c\right).$$

$$\boxed{f_{\lambda,\alpha,\beta}(x,y,z,a,b,c) = \left(\lambda^{\alpha}x, \lambda^{\beta}y, \lambda^{\alpha+\beta}z, \lambda^{-\alpha}a, \lambda^{-\beta}b, \lambda^{-(\alpha+\beta)}c\right)}$$

$a \in \frac{1}{2}\mathbb{Z}$, $\lambda = -a \pm \sqrt{a^2-1}$

(a root of $X^2 + 2aX + 1 \in \mathbb{Z}[X]$.)

$\alpha, \beta \in \mathbb{Z}$

$$\ell = \ell_{\lambda,\alpha,\beta} : L \longrightarrow L$$
$$X_1 \longmapsto \lambda^{\alpha} X_1$$
$$Y_1 \longmapsto \lambda^{\beta} Y_1$$
$$X_2 \longmapsto \lambda^{-\alpha} X_2$$
$$Y_2 \longmapsto \lambda^{-\beta} Y_2.$$

$$\ell = \left(\begin{array}{ccc|ccc} \lambda^{\alpha} & 0 & 0 & & & \\ 0 & \lambda^{\beta} & 0 & & 0 & \\ 0 & 0 & \lambda^{\alpha+\beta} & & & \\ \hline & & & \lambda^{-\alpha} & 0 & 0 \\ & 0 & & 0 & \lambda^{-\beta} & 0 \\ & & & 0 & 0 & \lambda^{-(\alpha+\beta)} \end{array}\right).$$

, f be the associated automorphism of G.

$$\mathbb{B} := \{ \underbrace{(100100)}_{B_1},\ \underbrace{\sqrt{a^2-1}\,(100-100)}_{B_2},\ \underbrace{(010010)}_{B_3},\ \underbrace{\sqrt{a^2-1}\,(0100-10)}_{B_4},\ \underbrace{(001001)}_{B_5},\ \underbrace{\sqrt{a^2-1}\,(00100-1)}_{B_6} \}.$$

$[B_1, B_3] = B_5 \qquad [B_2, B_3] = B_6$

$[B_1, B_4] = B_6 \qquad [B_2, B_4] = (a^2-1)\, B_{\cancel{5}}.$

all others are 0. (or -)

~~$\ell : B_1 \longmapsto (\lambda^\alpha \; 0 \; 0 \; \lambda^{-\alpha}\, e_2)$~~

$K :=$ integer lattice generated by $\mathbb{B}$.

$\Gamma := \exp(K)$.

$\ell(K) = K \quad , \quad \Leftrightarrow \quad f(\Gamma) = \Gamma$

$\Rightarrow \exists!\, \tilde{f} : G/\Gamma \to G/\Gamma$.

$$C = \begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix} \sim \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$$

$$C^\alpha \sim \begin{pmatrix} \lambda^\alpha & 0 \\ 0 & \lambda^{-\alpha} \end{pmatrix}.$$

$$\lambda^2 = \left(-a + \sqrt{a^2-1}\right)^2 = a^2 - 2a\sqrt{a^2-1} + (a^2-1)$$

$$= (2a^2-1) - 2a\sqrt{a^2-1}.$$

$$C^2 = \begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix}\begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix} = \left(\begin{array}{c|c} a^2+a^2-1 & -2a(a^2-1) \\ \hline -2a & -1 \end{array}\right).$$

$\alpha = 1.$ $\qquad\qquad \lambda = -a + \sqrt{a^2-1}$

$$\ell: B_1 \longmapsto (\lambda\ 0\ 0\ \lambda^{-1}\ 0\ 0)$$

$$= (-a+\sqrt{a^2-1},\ 0, 0,\ -a-\sqrt{a^2-1},\ 0, 0)$$

$$= (-a)\ (1\ 0\ 0\ 1\ 0\ 0) + \sqrt{a^2-1}\ (1\ 0\ 0\ -1\ 0\ 0)$$

$$= (-a)\ B_1 + B_2.$$

$$B_2 \longmapsto \sqrt{a^2-1}\ (\lambda\ 0\ 0\ -\lambda^{-1}\ 0\ 0)$$

$$= \sqrt{a^2-1}\ (-a+\sqrt{a^2-1},\ 0, 0,\ a+\sqrt{a^2-1},\ 0, 0)$$

$$= \sqrt{a^2-1}\ \left((-a)\ (1\ 0\ 0\ -1\ 0\ 0) + \sqrt{a^2-1}\ (1\ 0\ 0\ 1\ 0\ 0)\right)$$

$$= (a^2-1)(1\ 0\ 0\ 1\ 0\ 0) - a\sqrt{a^2-1}\ (1\ 0\ 0\ -1\ 0\ 0)$$

$$= (a^2-1)\ B_1 - a\ B_2$$

$$[\ell]_{\mathbb{B}} = \left(\begin{array}{cc|c|c} -a & a^2-1 & & \\ 1 & -a & & \\ \hline & & \cdot & \\ \hline & & & \cdot \end{array}\right)$$

$$\ell_\alpha : \ddot{B}_1 \longmapsto \begin{pmatrix} \lambda^\alpha & 0 & 0 & \lambda^{-\alpha} & 0 & 0 \end{pmatrix}.$$

$$\cancel{-a+\sqrt{a^2-1}}$$

$$\{\lambda, \lambda^{-1}\} = \operatorname{spec}\begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix}$$

$$\{\lambda^\alpha, \lambda^{-\alpha}\} = \operatorname{spec}\left(\begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix}^\alpha\right)$$

$$\det(A^2 - \lambda I) = \begin{vmatrix} a^2-\lambda & b^2 \\ c^2 & d^2-\lambda \end{vmatrix}$$

$$= \lambda^2 - (a^2+d^2)\lambda + a^2 d^2 - b^2 c^2$$

$$\cancel{= a^2+d^2 \pm \left((a^2+d^2)^2 - 4 \right)}$$

$$\det(A^2) = \det(A)^2.$$

$P(X) \in \mathbb{Z}[X]$.
$A \in \mathcal{L}in.(\mathbb{R}^n, \mathbb{R}^n)$.

$$\boxed{\operatorname{spec}(P(A)) = \vec{P}(\operatorname{spec}(A))}$$

$$\ell: B_3 \mapsto (0\ \lambda\ 0\ 0\ \lambda^{-1}\ 0)$$

$$= (0, -a+\sqrt{a^2-1}, 0, 0, -a-\sqrt{a^2-1}, 0)$$

$$= (-a)(010010) + \sqrt{a^2-1}\,(0100{-1}0)$$

$$= (-a)B_3 + B_4$$

$$B_4 \mapsto \sqrt{a^2-1}\,(0\ \lambda\ 0\ 0\ {-\lambda^{-1}}\ 0)$$

$$= \sqrt{a^2-1}\,(0, -a+\sqrt{a^2-1}, 0, 0, a+\sqrt{a^2-1}, 0)$$

$$= \sqrt{a^2-1}\,\left((-a)(0100{-1}0) + \sqrt{a^2-1}\,(010010)\right)$$

$$= (-a)B_4 + (a^2-1)B_3.$$

$$\ell(B_5) = \ell([B_1, B_3]) = [\ell(B_1), \ell(B_3)]$$

$$= [(-a)B_1 + B_2, (-aB_3 + B_4)]$$

$$= [-aB_1, -aB_3 + B_4] + [B_2, -aB_3 + B_4]$$

$$= [-aB_1, -aB_3] + [-aB_1, B_4] + [B_2, -aB_3] + [B_2, B_4] = a^2 B_5 - aB_6 - aB_6 + (a^2-1)B_5$$

$\ell : B_5 \mapsto$ 

$$\ell(B_5) = \ell([B_1, B_3]) = [\ell(B_1), \ell(B_3)]$$
$$= [-aB_1 + B_2, -aB_3 + B_4]$$
$$= a^2[B_1, B_3] - a[B_1, B_4] - a[B_2 B_3] + [B_2 B_4].$$
$$= a^2 B_5 - a B_6 - a B_6 + (a^2-1) B_5$$
$$= (2a^2-1) B_5 - 2a B_6.$$

$$\begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix} \begin{pmatrix} -a & a^2-1 \\ 1 & -a \end{pmatrix} = \left(\begin{array}{c|c} a^2+a^2-1 & -2a(a^2-1) \\ \hline -2a & 2a^2-1 \end{array}\right).$$

$\cancel{\ell : B_6 \mapsto}$

$$\ell(B_6) = \ell([B_1, B_4]) = [\ell(B_1), \ell(B_4)]$$
$$= [-aB_1 + B_2, (a^2-1)B_3 - aB_4]$$
$$= -a(a^2-1)[B_1, B_3] + a^2[B_1, B_4] + (a^2-1)[B_2 B_3] - a[B_2 B_4].$$
$$= -a(a^2-1) B_5 + a^2 B_6 + (a^2-1) B_6 - a(a^2-1) B_5$$
$$= -2a(a^2-1) B_5 + (2a^2-1) B_6$$

$\alpha = 1,\ \beta = 1.\quad \ell : (x\ y\ z\ a\ b\ c) \mapsto (\lambda x,\ \lambda y,\ \lambda^2 z,\ \lambda^{-1} a,\ \lambda^{-1} b,\ \lambda^{-2} c)$

$$\begin{aligned}
\ell : B_1 &\mapsto (-a)\, B_1 + B_2 \\
B_2 &\mapsto (a^2-1)\, B_1 + (-a) B_2 \\
B_3 &\mapsto (-a)\, B_3 + B_4 \\
B_4 &\mapsto (a^2-1)\, B_3 + (-a) B_4 \\
B_5 &\mapsto (2a^2-1) B_5 + (-2a) B_6. \\
B_6 &\mapsto -2a(a^2-1) B_5 + (2a^2-1) B_6
\end{aligned}$$

$\mathbb{B} = \{B_1, \dots, B_6\}.$

$$\Rightarrow [\ell]_{\mathbb{B}} = \left(\begin{array}{cc|cc|cc}
-a & a^2-1 & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\
1 & -a & & & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & -a & a^2-1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\
 & & 1 & -a & & \\
\hline
\multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 2a^2-1 & -2a(a^2-1) \\
 & & & & -2a & 2a^2-1
\end{array}\right).$$

$$\in GL(6, \mathbb{Z}).$$