230/231 02/11/2020

Multivariable Functions.

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $(n = 2, 3, \ldots)$
  $x = (x_1, x_2, \ldots, x_n) \mapsto (f_1(x), f_2(x), \ldots f_m(x))$ $(m = 1, 2, \ldots)$

or more generally, $f: D \to \mathbb{R}^m$, where $D \subseteq \mathbb{R}^n$ is a subset.

$$\mathrm{dom}(f) = \{x \in \mathbb{R}^n \mid f(x) \text{ is well-defined}\}$$
(no infinities etc.)

$$\mathrm{im}(f) = \underset{\vec{f}}{\mathrm{ran}}(f) = \{f(x) \in \mathbb{R}^m \mid x \in \mathbb{R}^n\} = \vec{f}(D)$$

$\Gamma_f = \{(x, f(x)) \mid x \in D\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is the graph of f.

<u>SW</u>: - Formulate a vertical line test for shapes in $\mathbb{R}^3$

- Which quadric surfaces can be seen ~~written~~ as the graph of a function?

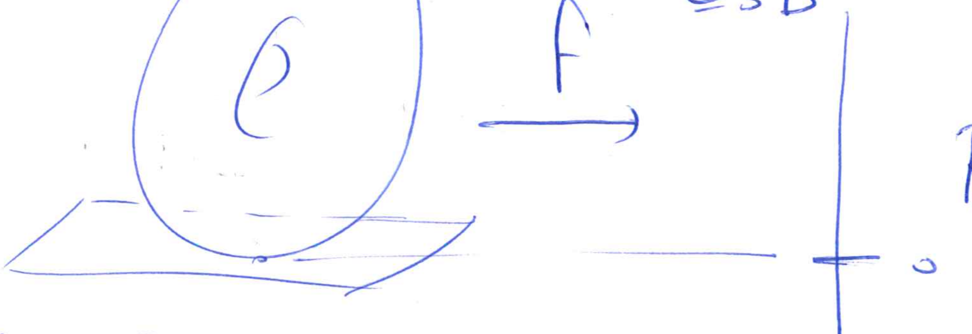
- $x^2+y^2+z^2=1$ cannot be the graph of a function

- For $k \in \mathbb{R}^m$ fixed, $\cancel{\{f=k\}}$

$$\overleftarrow{f}(k) = \{f=k\} = \{x \in \mathbb{R}^n \mid f(x)=k\}$$

is the k-level set of f

SW: D = surface of a donut $\subseteq$ 3D



$f : D \to \mathbb{R}$
$x \mapsto$ height.

at which $k \in \mathbb{R}$ values does the shape of $\overleftarrow{f}(k)$ change dramatically?

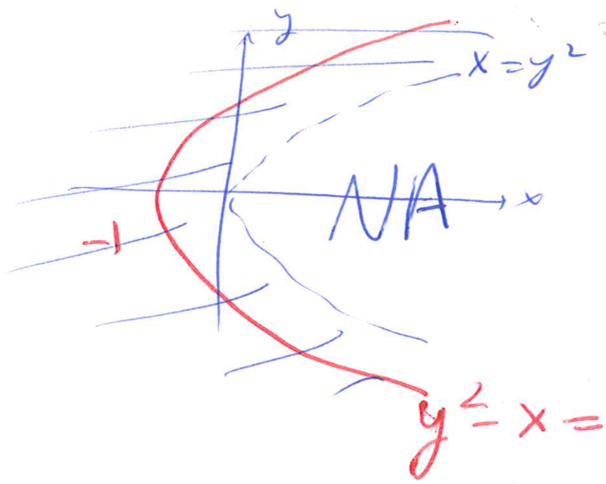
Notation: $f : \mathbb{R}^n \rightharpoonup \mathbb{R}^m$ "partially defined function".

means $\operatorname{dom}(f) \subseteq \mathbb{R}^n$

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto x \ln(y^2 - x).$$

$y^2 - x > 0 \Rightarrow \mathrm{dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid y^2 > x\}$.



SW: graph?

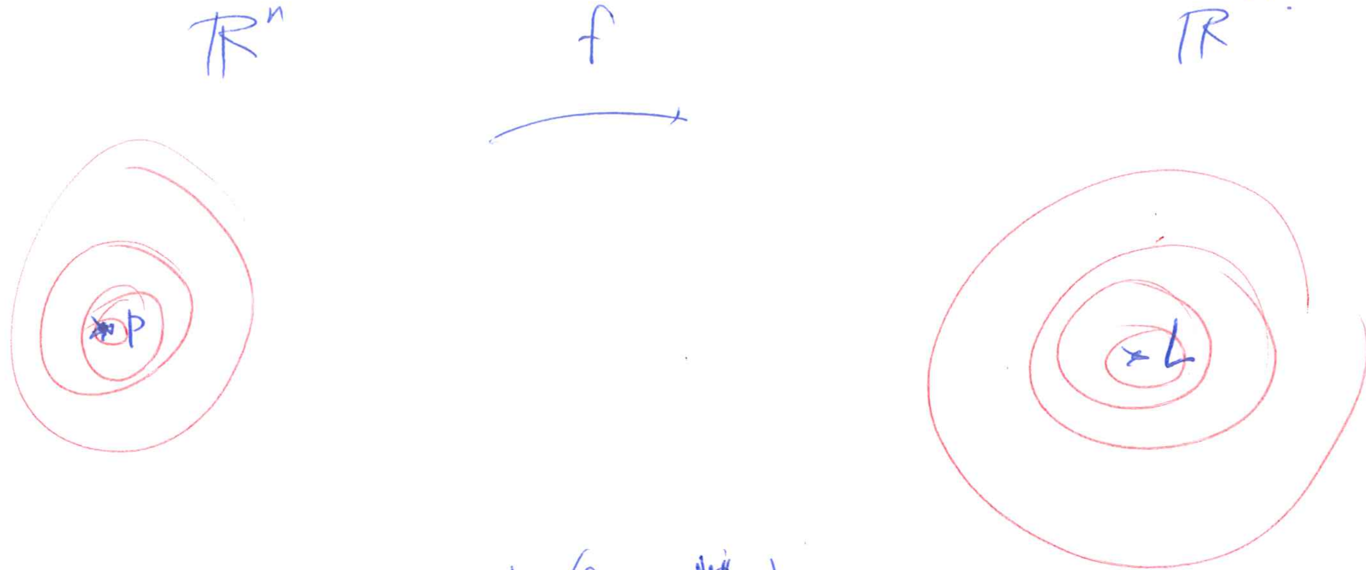
- For $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ single variable calculus applies without modification.

For $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ this is not so.

<u>Def</u> (Limit): Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathbb{R}^n$, $L \in \mathbb{R}^m$: $\boxed{\lim_{x \to p} f(x) = L}$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}^n : d(x,p) < \delta \Rightarrow d(f(x), L) < \varepsilon.$$

$$\forall \varepsilon > 0, \exists \delta > 0 : \vec{f}(B(p,\delta)) \subseteq B(L, \varepsilon).$$



$$d(x,p) < \delta_0 \Rightarrow d(f(x), L) < 1$$

$$d(x,p) < \delta_1 < \delta_0 \Rightarrow d(f(x), L) < \frac{1}{2} < 1$$

$$d(x,p) < \delta_2 < \delta_1 < \delta_0 \Rightarrow d(f(x), L) < \frac{1}{4} < \frac{1}{2} < 1$$

$$d(x,p) < \delta_3 < \delta_2 < \delta_1 < \delta_0 \Rightarrow d(f(x), L) < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1$$

$$\vdots$$

$$\boxed{d(x,p) < \delta_k \Rightarrow d(f(x), L) < \frac{1}{2^k}}$$

How much you can relax (δ_k) depends on how accurate you want to be $\left(\frac{1}{2^k}\right)$.

Def: (Continuity).

$f : \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathbb{R}^n$.

f is cts at p if $\boxed{\lim_{x \to p} f(x) = f(p)}$.

f is cts if it is continuous everywhere.

$C^0(\mathbb{R}^n, \mathbb{R}^m)$: set of all continuous functions $\mathbb{R}^n \to \mathbb{R}^m$

$C^0(D, \mathbb{R}^m)$: set of all cts. functions $D \to \mathbb{R}^m$

SW: $C^0(D, \mathbb{R}^m)$ is a vector space.

Obs: Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathbb{R}^n$.

(i) If f is cts at p, then

for any $\gamma : \mathbb{R} \xrightarrow{C^0} \mathbb{R}^n$: $\gamma(0) = p$:

$\lim_{t \to 0} f \circ \gamma(t) = f(p)$. (In particular for any line γ passing through p.)

(ii) But not vice versa

Ex: $f : \mathbb{R}^2 \longmapsto \mathbb{R}$

$$(x,y) \longmapsto \frac{x^2-y^2}{x^2+y^2}.$$

$$\text{dom}(f) = \{(x,y) \neq (0,0)\}.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$\gamma_1(t) = (t,t) \Rightarrow \lim_{t\to 0} f\circ\gamma_1(t) = \lim_{t\to 0} \frac{0}{2t^2} = \lim_{t\to 0} 0 = 0.$

$\gamma_2(t) = (t,-t) \Rightarrow \lim_{t\to 0} f\circ\gamma_2(t) = \lim_{t\to 0} 0 = 0.$ $\longrightarrow$ lim DNE.

$\gamma_3(t) = (t,0) \Rightarrow \lim_{t\to 0} f\circ\gamma_3(t) = \lim_{t\to 0} \frac{t^2}{t^2} = \lim_{t\to 0} 1 = 1.$

$\gamma_4(t) = (0,t) \Rightarrow \lim_{t\to 0} f\circ\gamma_4(t) = \lim_{t\to 0} \frac{-t^2}{t^2} = -1.$

Ex: $f : \mathbb{R}^2 \longmapsto \mathbb{R}$ $\quad$ $\text{dom}(f) = \{(x,y) \neq (0,0)\}.$

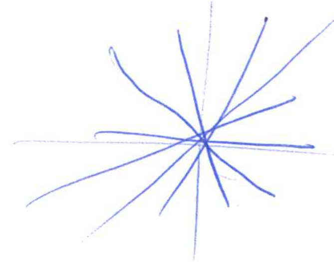
$$(x,y) \longmapsto \frac{xy^2}{x^2+y^4}.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$a_1 = 1 \quad a_2 = 2$

$m_1 = 1 \quad m_2 = 2$

$$\frac{a_1}{2m_1} + \frac{a_2}{2m_2} = \frac{1}{2} + \frac{2}{4} = 1 \text{ DNE}$$



$$\gamma_1(t) = (t, mt)$$

$$\lim_{t\to 0} f\circ\gamma_1(t) = \lim_{t\to 0} \frac{t m^2 t^2}{t^2 + m^4 t^4}$$

$$= \lim_{t\to 0} \frac{m^2 t^3}{m^4 t^4 + t^2} = \frac{1}{m^2} \lim_{t\to 0} \frac{1}{t + \frac{1}{t}} = 0.$$

$$\gamma_2(t) = (t^2, t^2).$$

$$\lim_{t\to 0} f\circ\gamma_2(t) = \lim_{t\to 0} \frac{t^2 t^4}{t^4 + t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$ lim DNE.

Q: If $f\circ\gamma$ is cts at 0 $\forall\, \gamma : \mathbb{R}_0 \to \mathrm{Dom}(f)_p$ with all ~~coeff~~ entries polys of degree ≤ 2, then does this imply that f is cts at p?

• Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $3 \geq n, m \geq 1$, $p \in \mathbb{R}^n$.

$\boxed{\text{I}}$ f is continuous at p

$\boxed{\text{IV}}$ $\lim_{x \to p} f(x)$ exists

$\boxed{\text{II}}$ For any continuous $\gamma: \mathbb{R} \to \mathbb{R}^n$, $\gamma(0) = p$: $f \circ \gamma$ is continuous at 0.

$\boxed{\text{V}}$ For any continuous $\gamma: \mathbb{R} \to \mathbb{R}^n$, $\gamma(0) = p$: $\lim_{t \to 0} f \circ \gamma(t)$ exists

$\boxed{\text{III}}$ For any $v \in \mathbb{R}^n$: $f(p + tv)$ is continuous at 0

$\boxed{\text{VI}}$ For any $v \in \mathbb{R}^n$: $\lim_{t \to 0} f(p + tv)$ exists.

$$
\begin{tikzcd}
\text{I} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{IV} \arrow[d, Rightarrow] \\
\text{II} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{V} \arrow[d, Rightarrow] \\
\text{III} \arrow[r, Rightarrow] & \text{VI}
\end{tikzcd}
$$

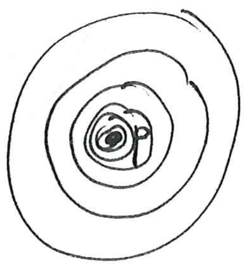
But not vice versa.

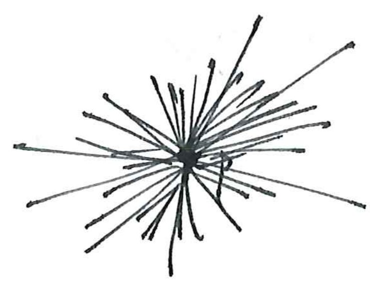
SW :

$$\begin{tikzcd} \neg \mathrm{I} & \neg \mathrm{IV} \arrow[l] \\ \neg \mathrm{II} \arrow[u, Rightarrow] & \neg \mathrm{V} \arrow[l] \arrow[u, Rightarrow] \\ \neg \mathrm{III} \arrow[u, Rightarrow] & \neg \mathrm{VI} \arrow[l] \arrow[u, Rightarrow] \end{tikzcd}$$

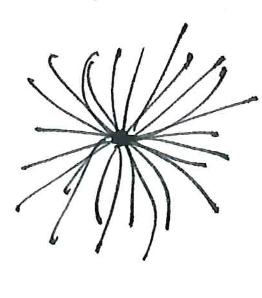
$\neg \mathrm{VI}$: For some $v \in \mathbb{R}^n$:

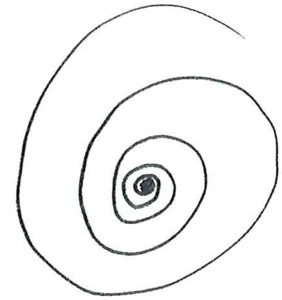
$$\lim_{t \to 0} f(p + tv) \quad \text{does not exist.}$$





still not enough.





Ex: $f(x,y) = y^x$.

$\gamma_{a,m}(t) = (t, at^m)$ $\quad a > 0 \quad m \in \mathbb{Z}_{>0}$.

$\Rightarrow \lim_{\substack{t \to 0 \\ t > 0}} f \circ \gamma(t)$ exists

$f \circ \gamma(t) = (at^m)^t$

$\log(f \circ \gamma(t)) = t \log(at^m)$

$$\lim_{\substack{t \to 0 \\ t > 0}} t \log(at^m) = \lim_{\substack{t \to 0 \\ t > 0}} \frac{\log(at^m)}{(1/t)} \overset{\frac{-\infty}{\infty}}{=}$$

$$= \lim_{\substack{t \to 0 \\ t > 0}} \frac{\frac{mat^{m-1}}{at^m}}{-\frac{1}{t^2}} = (-m) \cdot \lim_{\substack{t \to 0 \\ t > 0}} t = 0.$$

$$\Rightarrow \boxed{\lim_{\substack{t \to 0 \\ t > 0}} f \circ \gamma_{a,m}(t) = 1 \quad \forall a, m.}$$

$$\rho_a(t) = \left(t, e^{-a/t}\right) \qquad \cancel{a>0}. \quad a \in \mathbb{R}.$$

$$f \circ \rho_a(t) = \left(e^{-a/t}\right)^t = e^{-a}$$

$$\boxed{\lim_{\substack{t \to 0 \\ t > 0}} f \circ \rho_a(t) = e^{-a}.}$$

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \longmapsto \frac{3x^2y}{x^2+y^2}.$$

$$\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$\gamma(t) = (t,0)$. $\lim_{t\to 0} f\circ\gamma(t) = \lim_{t\to 0} \frac{0}{t^2} = 0$.

candidate.

$$|f(x,y)| = \left|\frac{3x^2y}{x^2+y^2}\right| \le 3|y| = 3\sqrt{y^2} \le 3\,|(x,y)| \longrightarrow 0$$

✓.

Obs: · Polys are continuous

· Rationals $\left(\frac{\text{poly}}{\text{poly}}\right)$ are continuous where they are defined.

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ Is cts at $(1,0)$?

$$(x,y) \longmapsto \frac{xy^2+2}{x^2+y^4+3}$$

Yes, $\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0) = \frac{2}{4} = \frac{1}{2}$.

~~• If f is continuous at p, then $\forall\, \gamma : \mathbb{R} \xrightarrow{C^0} \mathrm{dom}(f) : \gamma(0) = p$, $\lim_{t\to 0} f\circ\gamma(t) = f(p)$.~~

SW: $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$x \longmapsto (f_1(x), f_2(x), \ldots, f_m(x))$$

If f is cts at p, then so are all its component functions $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$.

Partial Derivatives / Directional Derivatives:

• $f : \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathbb{R}^n$.

$\gamma_{\bar{i}} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$t \longmapsto (t,0,0) + p = p + t\,\bar{i}$$

$\gamma_{\bar{j}} : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$t \longmapsto (0,t,0) + p = p + t\,\bar{j}$$

$\gamma_{\bar{k}} : \mathbb{R} \to \mathbb{R}^n$

$$t \longmapsto (0,0,t) + p = p + t\,\bar{k}$$

$$\partial_x f(p) := \partial_1 f(p) := \frac{d}{dt}\left(f\circ\gamma_1\right)(0) = \lim_{t\to 0} \frac{f(p+t\bar{\imath}) - f(p)}{t}$$

$= f_x(p) = \frac{\partial}{\partial x} f(p)$

is the <u>partial der.</u> of f w/r/t x

or w/r/t first variable.

$$\partial_y f(p) := \partial_2 f(p) := \frac{d}{dt}\left(f\circ\gamma_2\right)(0) = \lim_{t\to 0} \frac{f(p+t\bar{\jmath}) - f(p)}{t}$$

$= f_y(p)$

<u>second var.</u>

$$\partial_z f(p) = \partial_3 f(p) := \frac{d}{dt}\left(f\circ\gamma_3\right)(0) = \lim_{t\to 0} \frac{f(p+tk) - f(p)}{t}$$

third var.

$|u| = 1$, $\gamma_u : \mathbb{R} \to \mathbb{R}^n$

$t \mapsto p + tu.$

$$\partial_u f(p) = D_u f(p) = \frac{d}{dt}\left(f\circ\gamma_u\right)(0)$$

$$= \lim_{t\to 0} \frac{f\circ\gamma_u(t) - f\circ\gamma_u(0)}{t} = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}.$$

<u>SW</u>: Formulate higher order derivatives $\partial_x\partial_y\partial_z\partial_y f$

$\partial_x^2 f$ ...

Rule: To find ~~the~~ a partial der. freeze the other constants and take ~~derivati~~ single var. derivative

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \mapsto x^3 + x^2 y^4 - 2y^2.$$

$$\partial_1 f(x,y) = 3x^2 + 2xy^4$$

$$\partial_1 \partial_1 f(x,y) = \partial_1^2 f(x,y) = 6x + 2y^4$$

$$\partial_1^3 f(x,y) = 6$$

$$\partial_1^4 f(x,y) = 0.$$

$$\partial_2 f(x,y) = 4x^2y^3 - 4y$$

$$\partial_2^2 f(x,y) = 12x^2y^2 - 4$$

$$\partial_2^3 f(x,y) = 24x^2y$$

$$\partial_2^4 f(x,y) = 24x^2$$

$$\partial_2^5 f(x,y) = 0.$$

$$\partial_2 \partial_1 f(x,y) = 8xy^3$$

SW: Compute all $\partial_{i_1}\partial_{i_2}\partial_{i_3}\partial_{i_4}\partial_{i_5} f$.

$$i_1, i_2, i_3, i_4, i_5 \in \{1, 2\}.$$

Ex: $z = f(x,y)$

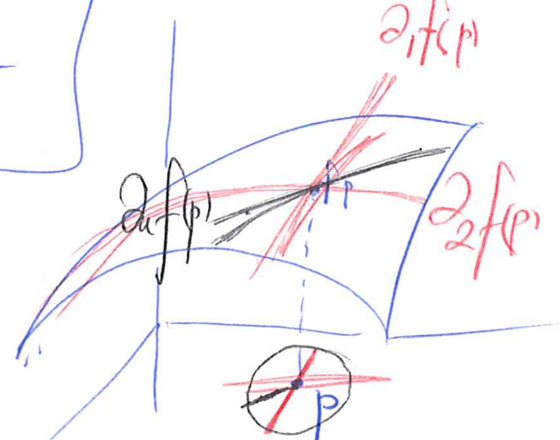
$$x^3 + y^3 + z^3 + 6xyz = 1.$$

$\partial_1 f = ?$ $\quad \underset{\text{sw}}{\partial_2 f = ?}$ $\quad \underset{\text{sw}}{\partial_1 \partial_2 f = ?}$

$$3x^2 + 0 + 3z^2 \cdot \partial_1 f + 6yz + 6xy\, \partial_1 f = 0$$

$$\partial_1 f \cdot (6xy + 3z^2) = -(3x^2 + 6yz)$$

$$\boxed{\partial_1 f(x,y) = -\frac{3x^2 + 6y f(x,y)}{6xy + 3(f(x,y))^2}}$$

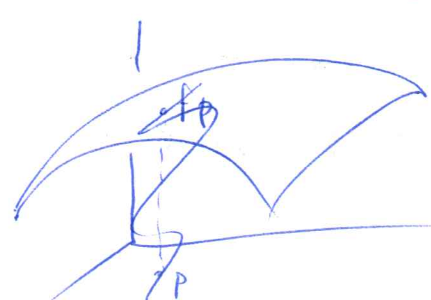


• If $f: \mathbb{R}^2 \to \mathbb{R}$,

$\partial_1 f(p)$ is the slope of the graph of f along $\vec{i}$ at p.

$\partial_2 f(p)$ ———— " ———— along $\vec{j}$

$\partial_u f(p)$ is the slope along u.



$C^1(\mathbb{R}^n, \mathbb{R}^m)$ – set of all functions $\mathbb{R}^n \to \mathbb{R}^m$ with all partials existing and continuous.

__SW__: $C^1(\mathbb{R}^n, \mathbb{R}^m)$ is a linear space

$$f \in C^1(\mathbb{R}^n, \mathbb{R}^m) \Leftrightarrow \partial_1 f, \partial_2 f \in C^0(\mathbb{R}^n, \mathbb{R}^m).$$

$$\partial_u f(p) = \lim_{t \to 0} \frac{f(p+tu) - f(p)}{t}$$

$u = u_1 \vec{i} + u_2 \vec{j}$

$|u| = 1.$

$$= \lim_{t \to 0} \frac{f(p_1 + tu_1, p_2 + tu_2) - f(p_1, p_2)}{t}$$

$\Updownarrow$

$\partial_u f, \partial_v f \in C^0(\mathbb{R}^n, \mathbb{R}^m)$ for any two non-parallel vectors u, v.

__Clairaut's Theorem:__

If f is C^2, $\partial_1 \partial_2 f = \partial_2 \partial_1 f$.

Ex : $f(x,y) = \begin{cases} xy \dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

$f: \mathbb{R}^2 \to \mathbb{R}$

- $\mathrm{dom}(f) = \mathbb{R}^2$.

$f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}.$

- $\lim\limits_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} (f(0,0)=0)$ ~~cand.~~

$x^2 - y^2 \le x^2 \le x^2+y^2 \Rightarrow \dfrac{x^2-y^2}{x^2+y^2} \le 1.$

$\Rightarrow |f(x,y)| = |xy| \left|\dfrac{x^2-y^2}{x^2+y^2}\right| \le |xy| \longrightarrow 0.$ ✓

$\Rightarrow f$ is continuous at $(0,0)$.

- $\partial_y f(a,0) = a \qquad \partial_x f(0,b) = -b.$

$\cancel{\partial_x f = y \dfrac{x^2-y^2}{x^2+y^2}}$ $\quad \partial_x f = \dfrac{(3x^2y - y^3)\cdot(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$

$= \dfrac{(3x^4y + 3x^2y^3 - x^2y^3 - y^5) - (2x^4y - 2x^2y^3)}{(x^2+y^2)^2} = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$

$\partial_x f(0,b) = \dfrac{-b^5}{b^4} = -b.$

$\partial_y f = \dfrac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2} = \dfrac{(x^5 + x^3y^2 - 3x^3y^2 - 3xy^4) - (2x^3y^2 - 2xy^4)}{(x^2+y^2)^2}$

$= \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \Rightarrow \partial_y f(a,0) = \dfrac{a^5}{a^4} = a.$

$$\partial_y\left(\partial_x f(0,y)\right) = \partial_y(-y) = -1.$$

$$\partial_x\left(\partial_y f(x,0)\right) = \partial_x(x) = 1.$$

$$\Rightarrow \partial_y\partial_x f(0,0) = -1 \neq 1 = \partial_x\partial_y f(0,0).$$

SW: Compute $\partial_y\partial_x f(x,y)$ & $\partial_x\partial_y f(x,y)$, then (i) set $(x,y)=(0,0)$ in both of them.

(ii) $\lim\limits_{(x,y)\to(0,0)} \partial_y\partial_x f(x,y) \overset{?}{=} \partial_y\partial_x f(0,0)$?

$\lim\limits_{(x,y)\to(0,0)} \partial_x\partial_y f(x,y) \overset{?}{=} \partial_x\partial_y f(0,0)$?

Ex: $f(x,y) = x\left(x^2+y^2\right)^{-3/2} e^{\sin(x^2y)}$. $f:\mathbb{R}^2\to\mathbb{R}$. $\mathrm{dom}(f)=\mathbb{R}^2$.

$\partial_x f(1,0) = ?$

Can set $y=0$ before ∂_x. $f(x,0) = x\left(x^2\right)^{-3/2} = x^{-2}$.

$$\partial_x f(x,0) = -2x^{-3}.$$

$$\boxed{\partial_x f(1,0) = -2.}$$

Clairaut's Theorem:

Euler	1721
Clairaut	1740
Lagrange	1797
Cauchy	1823
Schwarz	1873 ★

Proof of Clairaut's theorem.

$f:\mathbb{R}^2\to\mathbb{R}$.

$(p_1,p_2)=p\in\mathbb{R}^2$.

$\partial_x\partial_y f$ & $\partial_y\partial_x f$ exists and are continuous near p.

$$F:\mathbb{R}\to\mathbb{R}$$
$$t\mapsto f(p_1+t,p_2+t)-f(p_1,p_2+t)-f(p_1+t,p_2)+f(p_1,p_2).$$

- $g_1:\mathbb{R}\to\mathbb{R}$
$$x\mapsto f(x,p_2+t)-f(x,p_2).$$
$$\dot g_1(x)=\partial_1 f(x,p_2+t)-\partial_1 f(x,p_2).$$

$g_2:\mathbb{R}\to\mathbb{R}$
$$y\mapsto f(p_1+t,y)-f(p_1,y).$$
$$\dot g_2(x)=\partial_2 f(p_1+t,y)-\partial_2 f(p_1,y).$$

$$g_1(p_1+t)-g_1(p_1)=f(p_1+t,p_2+t)-f(p_1+t,p_2)-f(p_1,p_2+t)-f(p_1,p_2)=F(t).$$

$$g_2(p_2+t)-g_2(p_2)=F(t).$$

$$\Rightarrow \boxed{g_1(p_1+t)-g_1(p_1)=F(t)=g_2(p_2+t)-g_2(p_2).}$$

MVT (to g_1,g_2): $\forall t,\ \exists c_t^1\in\,]p_1,p_1+t[\ :\ \dot g_1(c_t^1)=\dfrac{g_1(p_1+t)-g_1(p_1)}{t}$

$\exists c_t^2\in\,]p_2,p_2+t[\ :\ \dot g_2(c_t^2)=\dfrac{g_2(p_2+t)-g_2(p_2)}{t}.$

$$\Rightarrow t\left(\partial_1 f(c_t^1,p_2+t)-\partial_1 f(c_t^1,p_2)\right)=F(t)$$
$$=t\left(\partial_2 f(p_1+t,c_t^2)-\partial_2 f(p_1,c_t^2)\right)$$

$$\Rightarrow \partial_1 f(c_t^1, p_2+t) - \partial_1 f(c_t^1, p_2) = \frac{B(t)}{t} = \partial_2 f(p_1+t, c_t^2) - \partial_2 f(p_1, c_t^2)$$

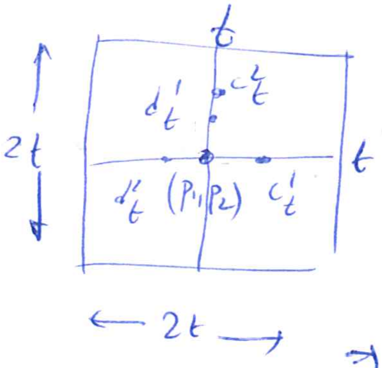
MVT to $\left(\begin{array}{l}\partial_1 f(c_t^1, -)\\ \& \ \partial_2 f(-, c_t^2)\end{array}\right)$

$$\exists\, d_t^1 \in \,]p_2, p_2+t[\; : \partial_2 \partial_1 f(c_t^1, d_t^1) = \frac{\partial_1 f(c_t^1, p_2+t) - \partial_1 f(c_t^1, p_2)}{t}$$

$$\exists\, d_t^2 \in \,]p_1, p_1+t[\; : \partial_1 \partial_2 f(d_t^2, c_t^2) = \frac{\partial_2 f(p_1+t, c_t^2) - \partial_2 f(p_1, c_t^2)}{t}.$$

$$\Rightarrow t\, \partial_2 \partial_1 f(c_t^1, d_t^1) = \frac{B(t)}{t} = t\, \partial_1 \partial_2 f(d_t^2, c_t^2)$$

$$\Rightarrow \partial_2 \partial_1 f(c_t^1, d_t^1) = \frac{B(t)}{t^2} = \partial_1 \partial_2 f(d_t^2, c_t^2)$$



$$\cancel{\Rightarrow \lim_{t\to 0} \partial_2 \partial_1 f(c_t^1, d_t^1) = \lim_{t \to 0} \frac{B(t)}{t^2}}$$

$$\Rightarrow \partial_2 \partial_1 f(c_t^1, d_t^1) = \partial_1 \partial_2 f(d_t^2, c_t^2)$$

$$\Rightarrow \lim_{t\to 0} \partial_2 \partial_1 f(c_t^1, d_t^1) = \cancel{\partial_1} \lim_{t\to 0} f(d_t^2, c_t^2)$$

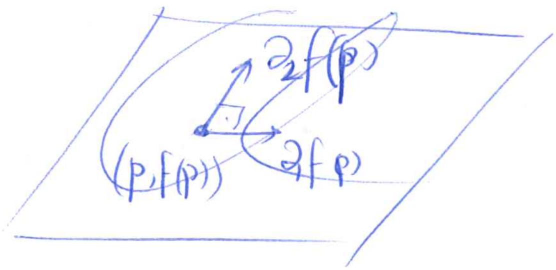
$$\partial_2 \partial_1 f(p_1, p_2) \qquad\qquad \partial_1 \partial_2 f(p_1, p_2)$$

C^2

Tangent Planes & Linear Approximations

$f:\mathbb{R}^2 \to \mathbb{R}$ $\qquad C^1$

$S = \Gamma_f \subseteq \mathbb{R}^3$. $\qquad p=(p_1,p_2)\in\mathbb{R}^2$.



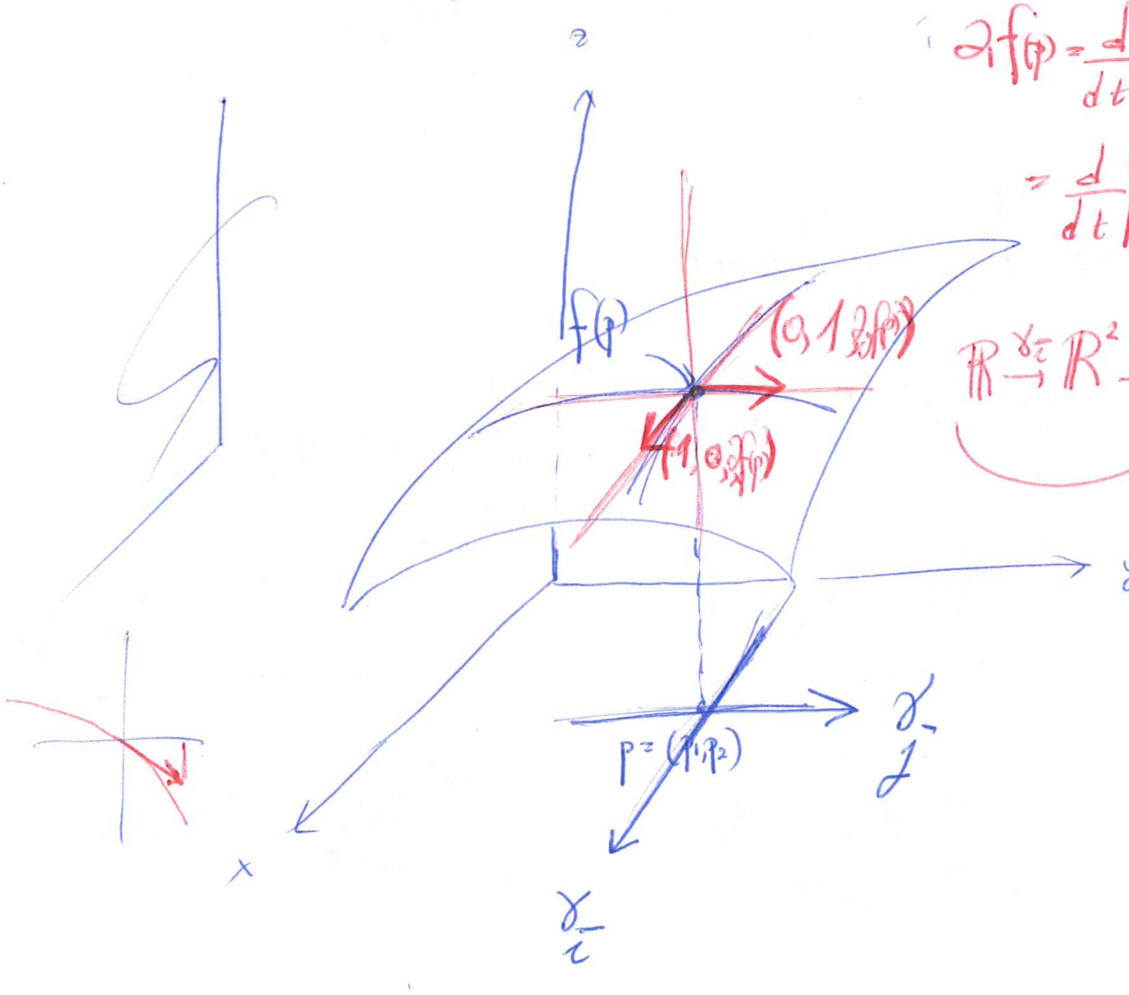
$\cancel{CN = \partial f(p) \times \partial_2 f(p)}$.

$\cancel{= \tau_1 \times (a_1, b_1) \times a_2}$

$$\partial f(p) = \frac{d}{dt}(f\circ\gamma_{\tau})(0)$$

$$= \frac{d}{dt}\Big|_{t=0} f(p+t\tau)$$

$$\mathbb{R} \xrightarrow{\gamma_\tau} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$



$$N = (1, 0, \partial_x f(p)) \times (0, 1, \partial_y f(p))$$

$$= (-\partial_x f(p), -\partial_y f(p), 1)$$

$$0 = N \cdot \left[(x,y,z) - (p_1, p_2, f(p_1,p_2))\right].$$

$$= -\partial_x f(p)(x-p_1) - \partial_y f(p)(y-p_2) + (z - f(p_1,p_2)).$$

$$\boxed{z - f(p_1,p_2) = \partial_x f(p)(x-p_1) + \partial_y f(p)(y-p_2).}$$

$\nabla f = \mathrm{grad}(f) : \mathbb{R}^2 \to \mathbb{R}^2$

$p \mapsto (\partial_x f(p), \partial_y f(p)).$

gradient vector field of f.

$$\boxed{z - f(p) = \mathrm{grad}(f) \cdot ((x,y) - p)}$$

Sw: Formulate all this for $f : \mathbb{R}^n \to \mathbb{R}^1$, $n \in \{1,2,3\}$.

Ex: Tangent plane to $2x^2 + y^2 - z = 0$ at $(1,1,3)$?

$f(x,y) = 2x^2 + y^2$. $\quad f(1,1) = 3$

$\partial_x f(x,y) = 4x \quad \partial_x f(1,1) = 4$

$\partial_y f(x,y) = 2y. \quad \partial_y f(1,1) = 2.$

$\Rightarrow z - 3 = 4(x-1) + 2(y-1)$

$z - 3 = 4x + 2y - 6$

$\boxed{4x + 2y - z - 3 = 0}$

$z = 4x + 2y - 3$ lin. app.

$$f\circ\gamma_{\vec{i}} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto f(1+t, 1) = 2(t+1)^2+1 = 2(t^2+2t+1)+1$$
$$= 2t^2+4t+3.$$
$$f\circ\gamma_{\vec{j}} : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto f(1, 1+t) = 2+(1+t)^2 = t^2+2t+3$$

$$\partial_x f(1,1) = \frac{d}{dt}(f\circ\gamma_{\vec{i}})(0) = [4t+4]\big|_{t=0} = 4. \quad \begin{pmatrix}\text{slope}\\ \text{along } x\end{pmatrix}$$

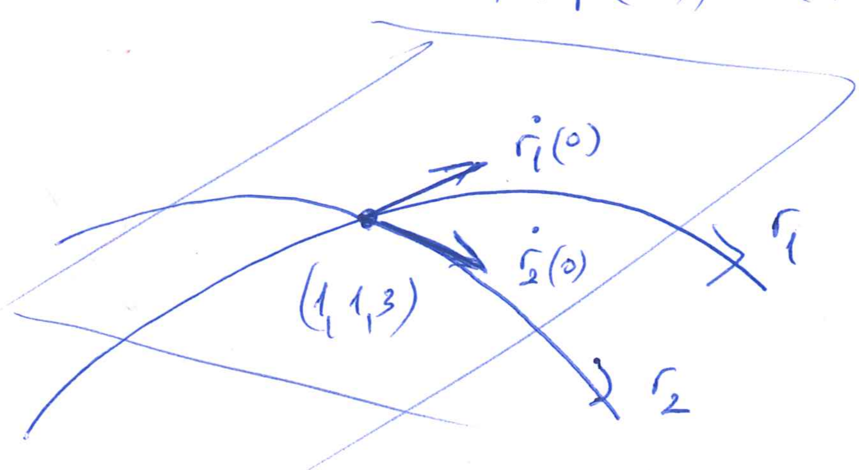
$$\partial_y f(1,1) = \frac{d}{dt}(f\circ\gamma_{\vec{j}})(0) = [2t+2]\big|_{t=0} = 2. \quad \begin{pmatrix}\text{slope}\\ \text{along } y\end{pmatrix}.$$

In 3D:

$$r_1(t) = (1+t, 1, f(1+t, 1))$$
$$r_2(t) = (1, 1+t, f(1, 1+t)).$$

$$\dot{r}_1(0) = (1, 0, \partial_x f(1,1)) = (1, 0, 4)$$
$$\dot{r}_2(0) = (0, 1, \partial_y f(1,1)) = (0, 1, 2)$$

$$r_1(0) = (1, 1, f(1,1)) = (1, 1, 3) = r_2(0).$$



SW: verify.

Ex: $f(x,y) = xe^{xy}$ at $(1,0)$.

- $f(1,0) = 1$

$\partial_x f(x,y) = e^{xy} + xe^{xy}y$, $\partial_x f(1,0) = 1$

$\partial_y f(x,y) = xe^{xy}x$, $\partial_y f(1,0) = 1$.

$$z - 1 = 1 \cdot (x-1) + 1 \cdot (y-0)$$
$$= x + y - 1.$$

$$\boxed{z = x+y}.$$

- Differentiability

$f: \mathbb{R} \to \mathbb{R}$, $p \in \mathbb{R}$. f is diff. at p if

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h} = f'(p) \quad \text{exists.}$$

$$\Leftrightarrow \boxed{\lim_{h \to 0} \frac{f(p+h) - [f(p) + f'(p)h]}{h} = 0}.$$

• $n, m \in \{1, 2, 3\}$. $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u>

if $\cancel{\Lambda(cv+w) = c\Lambda(v) + \Lambda(w)}$

$$\Lambda(v+w) = \Lambda(v) + \Lambda(w) \quad \&$$

$$\Lambda(cv) = c\,\Lambda(v)$$

for any $c \in \mathbb{R}$ &
for any $v, w \in \mathbb{R}^n$.

• $\Lambda : \mathbb{R} \to \mathbb{R}$ linear $\Rightarrow \Lambda(x) = \Lambda(x \cdot 1) = x\,\Lambda(1)$

$\Rightarrow \Lambda$ is multiplication by $\Lambda(1)$.

• $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ linear $\Rightarrow \Lambda(x,y) = \Lambda(x\bar{\imath} + y\bar{\jmath})$

$= x\,\Lambda(\bar{\imath}) + y\,\Lambda(\bar{\jmath}) = \big(\Lambda(\bar{\imath}), \Lambda(\bar{\jmath})\big) \cdot (x,y)$

$\Rightarrow \Lambda$ is dot product by $\big(\Lambda(\bar{\imath}), \Lambda(\bar{\jmath})\big)$.

• $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ linear $\Rightarrow \Lambda(x,y) = x\,\Lambda(\bar{\imath}) + y\,\Lambda(\bar{\jmath})$

$$\left.\begin{array}{l} \Lambda(\bar{\imath}) = (\Lambda_{11}, \Lambda_{21}) \\ \Lambda(\bar{\jmath}) = (\Lambda_{12}, \Lambda_{22}) \end{array}\right\} \quad = x(\Lambda_{11}, \Lambda_{21}) + y(\Lambda_{12}, \Lambda_{22}) = (\Lambda_{11}x + \Lambda_{12}y, \Lambda_{21}x + \Lambda_{22}y)$$

$$= \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$

- $f: \mathbb{R}^n \to \mathbb{R}^m$. $p \in \mathbb{R}^n$. f is <u>differentiable</u> at p if there is a linear ~~A~~ $\Lambda: \mathbb{R}^n \to \mathbb{R}^m$ ("the derivative of f at p"):

$$\lim_{h \to 0} \frac{\left| f(p+h) - \left[f(p) + \Lambda(h) \right] \right|}{|h|} = 0.$$

$$\boxed{Df(p) = f'(p) = T_p f = \Lambda.}$$

<u>Thm</u>: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff. at $p \in \mathbb{R}^n$, then there is a unique linear $\Lambda: \mathbb{R}^n \to \mathbb{R}^m$:

$$0 = \lim_{h \to 0} \frac{\left| f(p+h) - \left[f(p) + \Lambda(h) \right] \right|}{|h|}$$

<u>Pf</u>: Suppose $\displaystyle\lim_{h \to 0} \frac{\left| f(p+h) - \left(f(p) + \Lambda(h) \right) \right|}{|h|} = 0 = \lim_{h \to 0} \frac{\left| f(p+h) - \left(f(p) + \tilde{\Lambda}(h) \right) \right|}{|h|}$

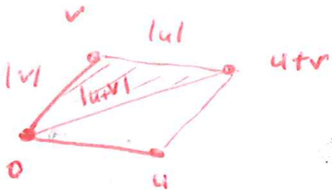
$\Lambda, \tilde{\Lambda}: \mathbb{R}^n \to \mathbb{R}^m$ linear. We claim that $\Lambda = \tilde{\Lambda}$.

$$0 \le \frac{|(\Lambda - \tilde{\Lambda})(h)|}{|h|} = \frac{\left| -\left[f(p+h) - (f(p) + \Lambda(h)) \right] + \left[f(p+h) - (f(p) - \tilde{\Lambda}(h)) \right] \right|}{|h|}$$

$$\leq \underbrace{\frac{\left| f(p+h) - \left(f(p) + \Lambda(h)\right) \right|}{|h|}}_{\to 0} + \underbrace{\frac{\left| f(p+h) - \left(f(p) + \tilde{\Lambda}(h)\right) \right|}{|h|}}_{\to 0}$$

SW:

$|u+v| \leq |u| + |v|$



$$\Rightarrow 0 \leq \lim_{h \to 0} \frac{\left|(\Lambda - \tilde{\Lambda})(h)\right|}{|h|} \leq 0.$$

$$\Rightarrow \lim_{h \to 0} \frac{\left|(\Lambda - \tilde{\Lambda})(h)\right|}{|h|} = 0.$$

$u \in \mathbb{R}^n$, $|u| = 1$

$\ominus \; d = f \lim_{h \to 0}$

$$\Rightarrow 0 = \lim_{t \to 0} \frac{\left|(\Lambda - \tilde{\Lambda})(tu)\right|}{|tu|} = \lim_{t \to 0} \frac{\cancel{|t|}\left|(\Lambda - \tilde{\Lambda})(u)\right|}{\cancel{|t|}\underbrace{|u|}_{=1}}$$

$$= \lim_{t \to 0} \left|(\Lambda - \tilde{\Lambda})(u)\right| = (\Lambda - \tilde{\Lambda})(u)$$

$$\Rightarrow \boxed{\Lambda(u) = \tilde{\Lambda}(u)}$$

$v \in \mathbb{R}^n$.

$v = 0 \Rightarrow \Lambda(v) = 0 = \tilde{\Lambda}(v)$.

$v \neq 0 \Rightarrow \left|\frac{v}{|v|}\right| = 1$

$$\Rightarrow \Lambda\left(\frac{v}{|v|}\right) = \tilde{\Lambda}\left(\frac{v}{|v|}\right)$$

$$\frac{1}{|v|}\Lambda(v) \qquad \frac{1}{|v|}\tilde{\Lambda}(v).$$

✓.

Chain Rule:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & \mathbb{R}^k \\
& \mathbb{R}^m \arrow[ur, "g"'] &
\end{tikzcd}$$

$p \in \mathbb{R}^n$.

$q = g(p) \in \mathbb{R}^m$.

f diff. at p.

g diff at q.

Then $g \circ f$ diff. at p, and

$$\boxed{D(g \circ f)(p) = D(g)(f(p)) \circ D(f)(p).}$$

$$\boxed{(g \circ f)'(p) = g'(f(p)) \circ f'(p).}$$

Chain Rule (Stewart):

(1)

$$\begin{tikzcd}
\mathbb{R} \arrow[rr, "t \quad\quad z"] \arrow[dr, "(g,h)"'] & & \mathbb{R} \\
& \mathbb{R}^2 \arrow[ur, "f"'] \ (x,y) &
\end{tikzcd}$$

$\cancel{t \mapsto (g(t), h(t))}$

$$\begin{tikzcd}
t \arrow[rr, mapsto] \arrow[d, mapsto] & & f(g(t),h(t)) \\
(g(t),h(t)) & (x,y) \arrow[r, mapsto] & f(x,y)
\end{tikzcd}$$

$$(x,y) \longmapsto f(x,y) = z$$

$$\mathbb{R} \xrightarrow{(g,h)} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$

$$t \longmapsto (g(t), h(t))$$

$$\begin{tikzcd}
& z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
& t &
\end{tikzcd}$$

$$\Rightarrow \boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt}}$$

$\cancel{\frac{dz}{dt}(t_0) =}$

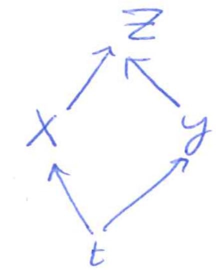
$$\boxed{\frac{dz}{dt}(t_0) = \frac{\partial f}{\partial x}(g(t_0), h(t_0)) \cdot \frac{dg}{dt}(t_0) + \frac{\partial f}{\partial y}(g(t_0), h(t_0)) \cdot \frac{dh}{dt}(t_0).}$$

Chain Rule (Stewart):

① $\cancel{(g,h): \mathbb{R} \to \mathbb{R}^2}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "{(g,h)}"] \arrow[rr, bend left, "\Phi"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \\[-20pt] t & (x,y) & \{z\} \end{tikzcd}$$

$$\Phi(t) = f(g(t), h(t)).$$

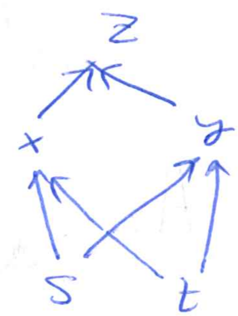


$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\boxed{\frac{d\Phi}{dt}(t_0) = \frac{\partial f}{\partial x}\big(g(t_0), h(t_0)\big) \cdot \frac{dg}{dt}(t_0) + \frac{\partial f}{\partial y}\big(g(t_0), h(t_0)\big) \cdot \frac{dh}{dt}(t_0).}$$

② $$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "{(g,h)}"] \arrow[rr, bend right, "\Phi"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R} \\[-20pt] (s,t) & (x,y) & z. \end{tikzcd}$$

$$\Phi(s,t) = f(g(s,t), h(s,t)).$$



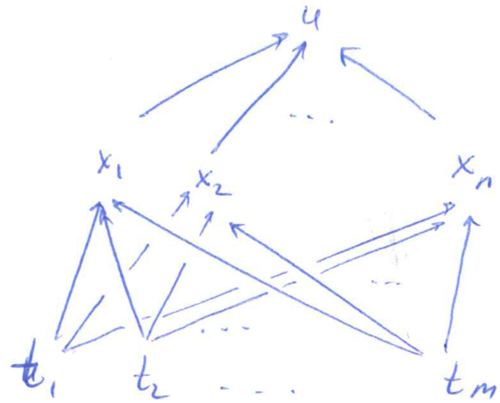
∂

$$\boxed{\begin{aligned} &\cdot\ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &\cdot\ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \end{aligned}}$$

$$\frac{\partial \Phi}{\partial s_{/t}}(s_0,t_0) = \frac{\partial f}{\partial x}\big(g(s_0,t_0), h(s_0,t_0)\big) \cdot \frac{\partial g}{\partial s_{/t}}(s_0,t_0) + \frac{\partial f}{\partial y}\big(g(s_0,t_0), h(s_0,t_0)\big) \cdot \frac{\partial h}{\partial s_{/t}}(s_0,t_0)$$

③ $\mathbb{R}^m \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}$

$(t_1, t_2, \dots, t_m) \qquad (x_1, x_2, \dots, x_n) \qquad u$



$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial u}{\partial t_m} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

<u>Ex</u> : $\begin{cases} z = x^2 y + 3xy^4 \\ x = \sin(2t) \\ y = \cos(t) \end{cases}$ $\qquad \dfrac{dz}{dt}(0) = ?$

$$\begin{tikzcd} & z & \\ x \arrow[ur] & & y \arrow[ul] \\ & t \arrow[ul] \arrow[ur] & \end{tikzcd}$$

0 x, y 1, t 0

$$\frac{dz}{dt}(0) = \frac{\partial z}{\partial x}(0,1) \cdot \frac{dx}{dt}(0) + \frac{\partial z}{\partial y}(0,1) \cdot \frac{dy}{dt}(0).$$

$\dfrac{\partial z}{\partial x} = 2xy + 3y^4, \quad \dfrac{\partial z}{\partial x}(0,1) = 3$

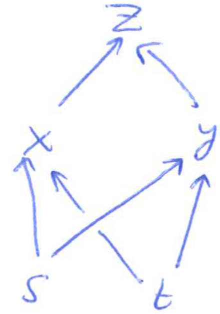
$\dfrac{\partial z}{\partial y} = x^2 + 12xy^3, \quad \dfrac{\partial z}{\partial y}(0,1) = 0$

$\dfrac{dx}{dt} = 2\cos(2t), \quad \dfrac{dx}{dt}(0) = 2$

$\dfrac{dy}{dt} = -\sin(t), \quad \dfrac{dy}{dt}(0) = 0.$

$$\frac{dz}{dt}(0) = 3 \cdot 2 + 0 \cdot 0 = 6.$$

Ex : $\begin{cases} z = e^x \sin y \\ x = st^2 \\ y = s^2 t \end{cases}$ $\qquad \dfrac{\partial z}{\partial s} = ? \qquad \dfrac{\partial z}{\partial t} = ?$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= e^x \sin y \\ \frac{\partial z}{\partial y} &= e^x \cos y \\ \frac{\partial x}{\partial s} &= t^2 \\ \frac{\partial y}{\partial s} &= 2st. \end{aligned}\right\} \quad \begin{aligned} \frac{\partial z}{\partial s} &= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st \\ &= \boxed{e^{st^2} \sin(s^2 t) \cdot t^2 + e^{st^2} \cdot \cos(s^2 t) \cdot 2st} \end{aligned}$$

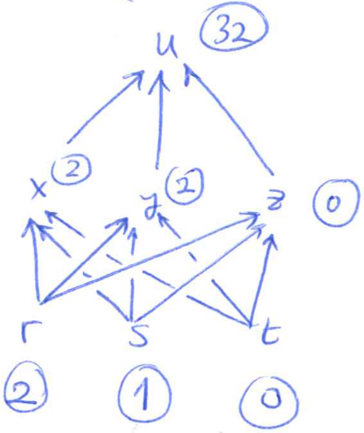
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\left.\begin{aligned} \frac{\partial x}{\partial t} &= 2st \\ \frac{\partial y}{\partial t} &= s^2 \end{aligned}\right\} \quad \begin{aligned} \frac{\partial z}{\partial t} &= e^x \sin y \cdot 2st + e^x \cos y \cdot s^2 \\ &= \boxed{e^{st^2} \sin(s^2 t)\, 2st + e^{st^2} \cos(s^2 t)\, s^2} \end{aligned}$$

Ex: $\begin{cases} u = x^4 y + y^2 z^3 \\ x = r s e^{t} \\ y = r s^2 e^{-t} \\ z = r^2 s \sin t \end{cases}$

$$\frac{\partial u}{\partial s}(r_0, s_0, t_0) = ?$$

$$(r_0, s_0, t_0) = (2, 1, 0).$$



$$\frac{\partial u}{\partial s}(2,1,0) = \frac{\partial u}{\partial x}(2,2,0)\cdot\frac{\partial x}{\partial s}(2,1,0) + \frac{\partial u}{\partial y}(2,2,0)\cdot\frac{\partial y}{\partial s}(2,1,0) + \frac{\partial u}{\partial z}(2,2,0)\,\frac{\partial z}{\partial s}(2,1,0)$$

$\frac{\partial u}{\partial x} = 4x^3 y$, $\quad \frac{\partial u}{\partial x}(2,2,0) = 4\cdot 2^3\, 2 = 64$

$\frac{\partial u}{\partial y} = x^4 + 2yz^3$, $\quad \frac{\partial u}{\partial y}(2,2,0) = 16$

$\frac{\partial u}{\partial z} = 3y^2 z^2$, $\quad \frac{\partial u}{\partial z}(2,2,0) = 0$

$\frac{\partial x}{\partial s} = r e^{t}$, $\quad \frac{\partial x}{\partial s}(2,1,0) = 2$

$\frac{\partial y}{\partial s} = 2 r s e^{-t}$, $\quad \frac{\partial y}{\partial s}(2,1,0) = 4$

$\frac{\partial z}{\partial s} = r^2 \sin(t)$, $\quad \frac{\partial z}{\partial s}(2,1,0) = 0.$

$$\to \frac{\partial u}{\partial s}(2,1,0) = 64\cdot 2 + 16\cdot 4 + 0\cdot 0$$

$$= 128 + 64 = 192.$$

Ex: $g(s,t) = f(s^2 - t^2,\ t^2 - s^2)$ satisfies

$$t\,\frac{\partial g}{\partial s} + s\,\frac{\partial g}{\partial t} = 0 \quad ?$$

$\cancel{z = g(s,t)}$

$$\left\{\begin{aligned} x &= s^2 - t^2 \\ y &= t^2 - s^2 \\ z &= g(s,t) = f(x,y) \end{aligned}\right\}$$

$$\begin{tikzcd}
 & z & \\
x \arrow[ur] & & y \arrow[ul] \\
 & s \arrow[ul] \arrow[ur] & t \arrow[u] \arrow[ull, bend left=60]
\end{tikzcd}$$

$$\frac{\partial g}{\partial s} = \frac{\partial z}{\partial s}, \quad \frac{\partial g}{\partial t} = \frac{\partial z}{\partial t}. \qquad \boxed{t\,\frac{\partial z}{\partial s} + s\,\frac{\partial z}{\partial t} = 0 \quad ?}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \partial_x f \cdot 2s + \partial_y f \cdot (-2s).$$

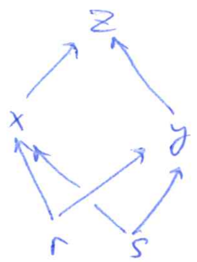
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \partial_x f \cdot (-2t) + \partial_y f \cdot (2t).$$

$$t \cdot \frac{\partial z}{\partial s} + s\,\frac{\partial z}{\partial t} = 2st \cdot \partial_x f - 2st \cdot \partial_y f - 2st\,\partial_x f + 2st\,\partial_y f = 0$$

✓.

$\underline{Ex}$: $\begin{cases} z = f(x,y) \\ x = r^2+s^2 \\ y = 2rs \end{cases}$ $\qquad \dfrac{\partial z}{\partial r} = ?$ $\qquad (f \in C^2)$.

$$\frac{\partial^2 z}{\partial r^2} = ?$$



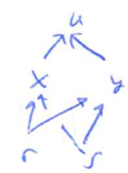
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \boxed{\partial_x f \cdot 2r + \partial_y f \cdot 2s}$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2r \cdot \partial_x f + 2s \cdot \partial_y f\right)$$

$$= 2\,\partial_x f + 2r\,\frac{\partial}{\partial r}\big(\overbrace{\partial_x f}^{=u}\big) + 2s\,\frac{\partial}{\partial r}\big(\overbrace{\partial_y f}^{=v}\big)$$

$\partial_x f = u \qquad \partial_y f = v$



$$= 2\,\partial_x f + 2r\,\frac{\partial u}{\partial r} + 2s \cdot \frac{\partial v}{\partial r}$$

$$= 2\,\partial_x f + 2r\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + 2s \cdot \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)$$

$$= 2\,\partial_x f + 2r\left(\partial_x^2 f \cdot 2r + \partial_y\partial_x f \cdot 2s\right) + 2s\left(\partial_x\partial_y f \cdot 2r + \partial_y^2 f \cdot 2s\right)$$

$$= \boxed{2\,\partial_x f + 4r^2 \cdot \partial_x^2 f + 4rs \cdot \partial_y\partial_x f + 4rs \cdot \partial_x\partial_y f + 4s^2\,\partial_y^2 f.}$$

Ex: $\boxed{x^3+y^3=6xy}$ $\rightarrow$ $\dfrac{dy}{dx}=?$

$\cancel{z=F(x,y)=x^3+y^3-6xy=0}$

$$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$(3x^2-6y)=(6x-3y^2)\frac{dy}{dx}$$

$$\boxed{\frac{x^2-2y}{2x-y^2}=\frac{dy}{dx}}$$

Ex: $\boxed{x^3+y^3+z^3+6xyz=1}$ $\qquad \dfrac{\partial z}{\partial x}=?$

$z=f(x,y)$ $\quad$ y & x independent.

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

$$(3x^2+6yz)+(3z^2+6xy)\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}=-\frac{3x^2+6yz}{3z^2+6xy} \Rightarrow \boxed{\frac{\partial z}{\partial x}=-\frac{x^2+2yz}{z^2+2xy}}$$

If $y=g(x)$, $z=f(x,y)$.

$$0=3x^2+3y^2\frac{dy}{dx}+6yz+6x\frac{\partial}{\partial x}(yz)$$

$$=3x^2+3y^2\frac{dy}{dx}+6yz+6x\left(\frac{dy}{dx}\cdot z+y\cdot\frac{\partial z}{\partial x}\right)\ \ldots$$

Gradient Vector Field & Directional Derivatives.

- $f: \mathbb{R}^n \to \mathbb{R}$, $n \in \{1,2,3\}$ $\quad p \in \mathbb{R}^n$.

$$\mathrm{grad}(f) = \nabla f : \mathbb{R}^n \to \mathbb{R}^n$$
$$p = (p_1, p_2, \dots, p_n) \mapsto (\partial_1 f(p), \partial_2 f(p), \dots, \partial_n f(p))$$

is the <u>gradient vector field</u> of f.

<u>SW</u>:

- $\mathrm{grad} : C^r(\mathbb{R}^n, \mathbb{R}) \to C^{r-1}(\mathbb{R}^n, \mathbb{R}^n)$.

is a linear operator.

- $\mathrm{grad}(fg) = (\partial_1(fg), \partial_2(fg), \dots, \partial_n(fg))$

$$= (\partial_1 f \cdot g + f \cdot \partial_1 g, \partial_2 f \cdot g + f\, \partial_2 g, \dots, \partial_n f \cdot g + f \cdot \partial_n g)$$
$$= (\partial_1 f, \partial_2 f, \dots, \partial_n f) \cdot g + f\,(\partial_1 g, \partial_2 g, \dots, \partial_n g)$$
$$= \mathrm{grad}(f) \cdot g + f\, \mathrm{grad}(g).$$

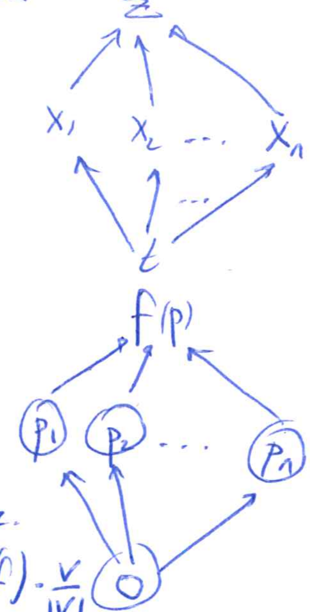
$$\mathrm{grad}(fg)(p) = \mathrm{grad}(f)(p) \cdot g(p) + f(p) \cdot \mathrm{grad}(g)(p)$$

Obs: (1) $u \in \mathbb{R}^n$, $|u| = 1$.

$$\partial_u f(p) = \frac{d}{dt} f(p+tu)\Big|_{t=0}$$

$$= \frac{d}{dt} f(p_1+tu_1, p_2+tu_2, \dots, p_n+tu_n)\Big|_{t=0}$$

$$= \sum_{i=1}^{n} \partial_i f(p) \cdot u_i = \mathrm{grad}(f)(p) \cdot u.$$



(2) $v \in \mathbb{R}^n$. $\cancel{\frac{d}{dt} f(p+tv)\Big|_{t=0}}$

$v \in \mathbb{R}^n$, $v \neq 0$. Dir. der. along v $= \partial_{\frac{v}{|v|}} f = \mathrm{grad}(f) \cdot \frac{v}{|v|} = \frac{1}{|v|}\, \mathrm{grad}(f) \cdot v = \frac{1}{|v|} \partial_v f$.

(3) $u \in \mathbb{R}^n$, $|u| = 1$.

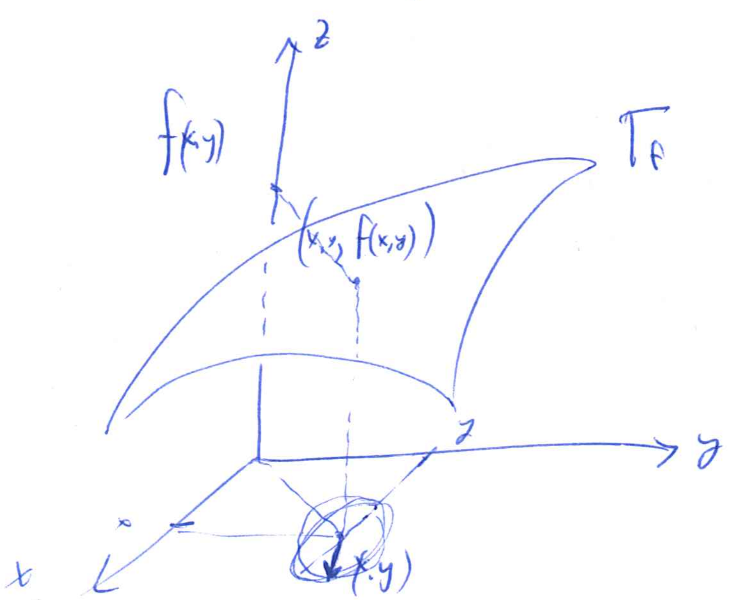
$$\partial_u f = \mathrm{grad}(f) \cdot u = |\mathrm{grad}(f)|\underbrace{|u|}_{=1} \cos(\measuredangle(\mathrm{grad}(f), u))$$

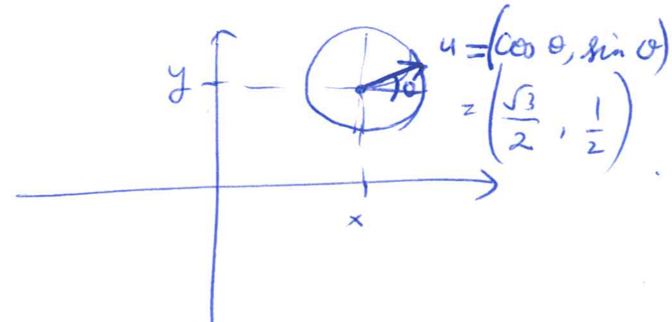
is maximized when $u \,/\!/\, \mathrm{grad}(f)$ & minimized.

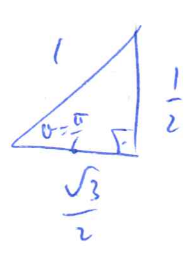
Ex: $f(x,y) = x^3 - 3xy + 4y^2$.

u : unit vector given by $\theta = \pi/6$

$\partial_u f(x,y) = ?$ $\quad \partial_u f(1,2) = ?$







$$\mathrm{grad}(f)(x,y) = (\partial_1 f(x,y),\ \partial_2 f(x,y))$$

$$= (3x^2 - 3y,\ -3x + 8y).$$

$$\partial_u f(x,y) = \mathrm{grad}(f)(x,y) \cdot u = (3x^2 - 3y,\ -3\cancel{x} + 8y) \cdot \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}}{2} x^2 - \frac{3\sqrt{3}}{2} y - \frac{3}{2} x + 4y.$$

$$\frac{d}{dt}\left(f((x,y) + tu)\right)\Big|_{t=0} = ? \leftarrow \text{SW.}$$

Ex: $f(x,y,z) = x \sin(yz)$.

(i) $\mathrm{grad}(f) = ?$

(ii) $v = (1,2,-1)$, $p = (1,3,0)$ dir. der. of f at p along v?

$$\mathrm{grad}(f) = (\sin(yz),\ xz\cos(yz),\ xy\cos(yz))$$

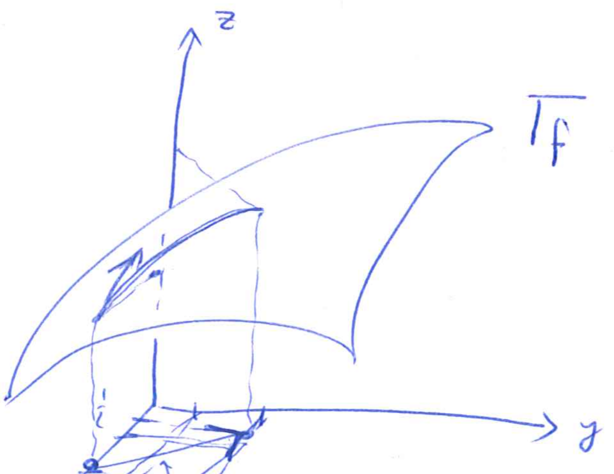
$$\partial_{\frac{v}{|v|}} f(p) = \mathrm{grad}(f)(p) \cdot \frac{v}{|v|} = \frac{1}{\sqrt{6}} (0,0,3) \cdot (1,2,-1)$$

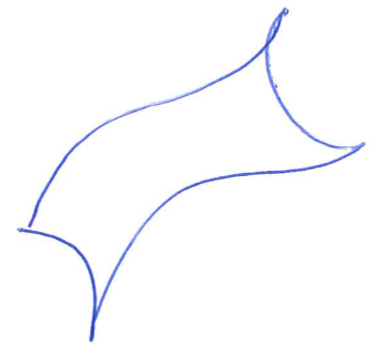
$$= \frac{1}{\sqrt{6}} \cdot (-3) = \boxed{\frac{-3}{\sqrt{6}}}.$$

Ex: $f(x,y) = x e^y$.

(i) Rate of change of f at $(2,0)$ toward $\left(\frac{1}{2}, 2\right)$?

(ii) Max. rate of the change ?





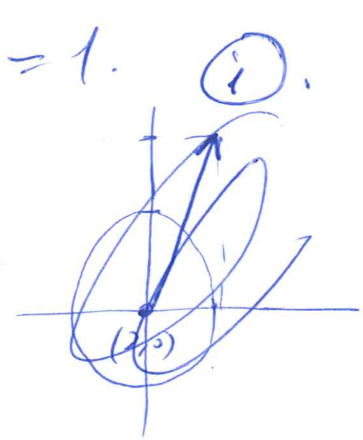
x $\quad v = \left(-\frac{3}{2}, 2\right) \qquad |v| = \sqrt{\frac{9}{4}+4} = \frac{5}{2}$

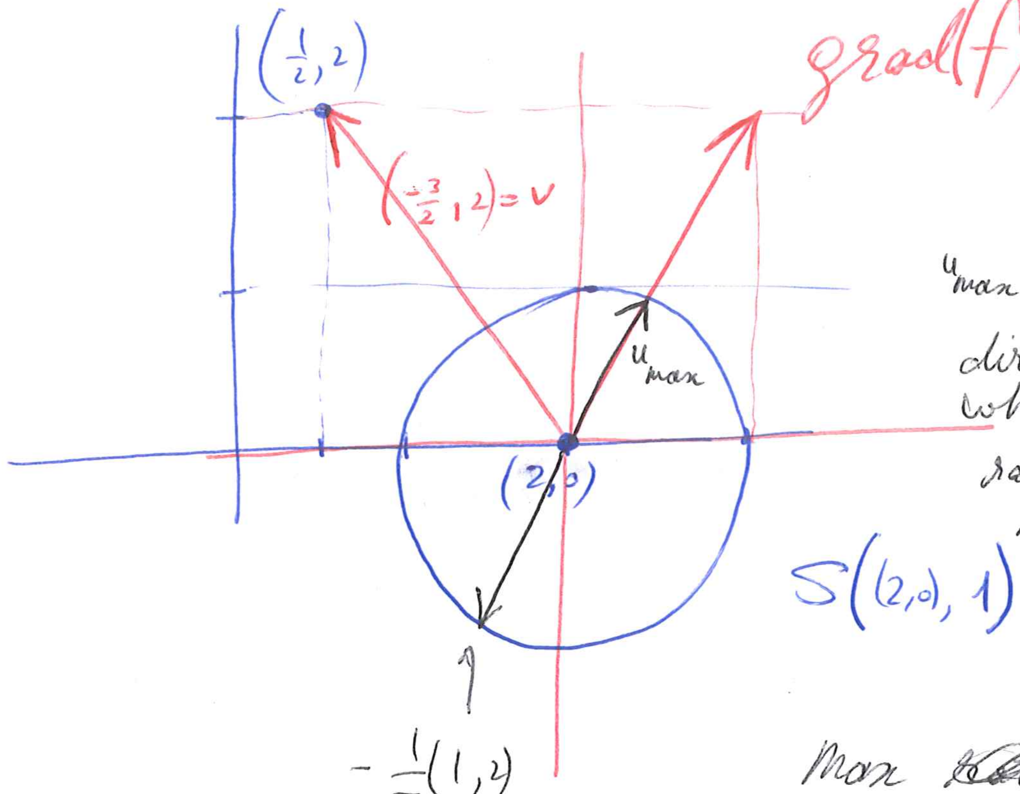
$\mathrm{grad}(f) = (e^y, x e^y)$.

$\partial_{\frac{v}{|v|}} f_p = \frac{1}{|v|} \mathrm{grad}(f)(p) \cdot v = \frac{2}{5}(1,2) \cdot \left(-\frac{3}{2}, 2\right)$

$= \frac{2}{5}\left(-\frac{3}{2}+4\right) = \frac{2}{5} \cdot \frac{5}{2} = 1.$ (i).

(ii) $\mathrm{grad}(f)(2,0) = (1,2)$





decrease is maximized

$u_{max} = \frac{1}{\sqrt{5}}(1,2)$
direction along which (the increase in) rate of change is maximized.

$\Downarrow$

Max ~~rate~~ increase:

$= \partial_{u_{max}}$

$\partial_u f(2,0) = (grad(f)(2,0)) \cdot u$

$= |grad(f)(2,0)|\,\underbrace{|u|}_{=1} \cos(\theta)$

$\leq |grad(f)(2,0)| = \sqrt{5}.$

Ex: $T(x,y,z) = \dfrac{80}{1+x^2+2y^2+3z^2}$ [°C] temperature at (x,y,z) [m,m,m]

(i) In which direction does the temp increase fastest at $(1,1,-2)$? (ii) What is the max rate of ~~change~~ increase?

$$grad(T) = 80\left(-\frac{2x}{(1+x^2+2y^2+3z^2)^2}, \frac{-4y}{(1+x^2+2y^2+3z^2)^2}, \frac{-6z}{(1+x^2+2y^2+3z^2)^2}\right)$$

$$= \frac{-160}{(1+x^2+2y^2+3z^2)^2} (x, 2y, 3z).$$

$$\mathrm{grad}(T)(1,1,-2) = \frac{-160}{(1+1+2+12)^2} (1, \cancel{4}, -6)$$

$$= \frac{-160}{16.16} (1, 2, -6) = \frac{-10}{16} (1, 2, -6) \quad (i)$$

$$(ii) \left| \mathrm{grad}(T)(1,1,-2) \right| = \frac{10}{16} \cdot \sqrt{1+4+36} = \frac{10}{16}\sqrt{41}.$$

$$\left[{}^{\circ}C/m \right]$$

<u>Tangents ~~Planes~~ to Level Sets</u>

$f : \mathbb{R}^3 \to \mathbb{R}$, $k \in \mathbb{R}$ $\quad S_k = \{ \cancel{F} (x,y,z) = k \}$

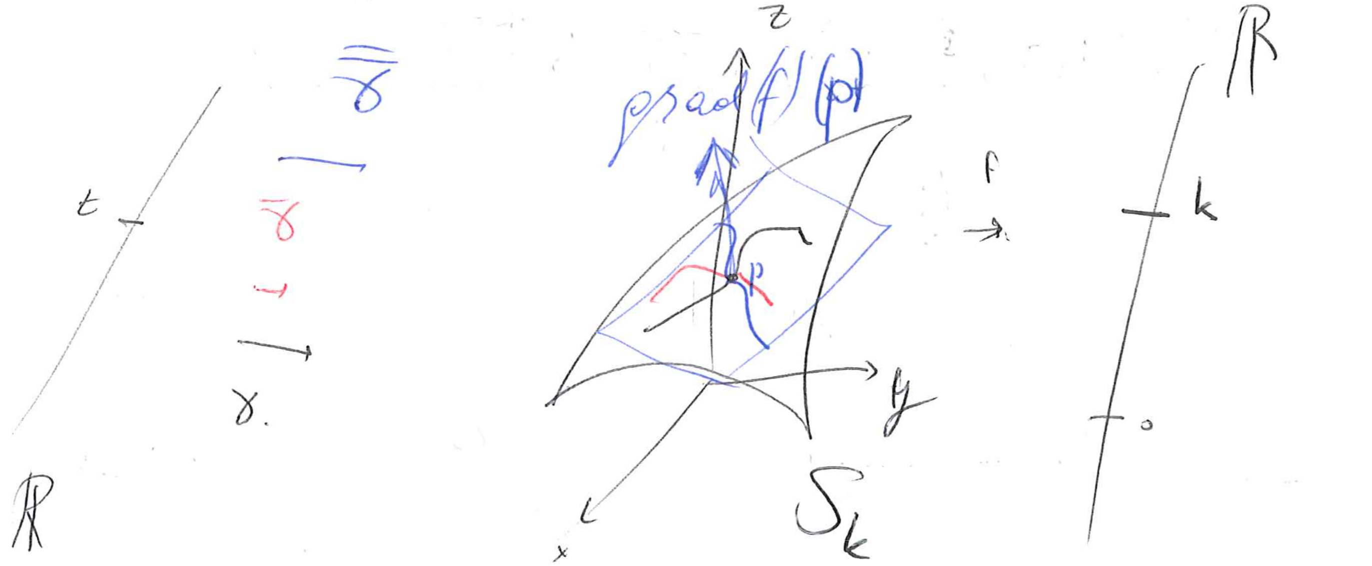
$\gamma : \mathbb{R} \to S_k \quad (\text{i.e. } f \circ \gamma(t) = k).$ $\quad p \in S_k$

$0 \mapsto p$

$$0 = \frac{d}{dt}(f \circ \gamma)(t) = \partial_x f(\gamma(t))\, \dot{\gamma}_1(t) + \partial_y f(\gamma(t))\, \dot{\gamma}_2(t) + \partial_z f(\gamma(t))\, \dot{\gamma}_3(t)$$

$$= \mathrm{grad}(f)(\gamma(t)) \cdot \dot{\gamma}(t) \Rightarrow \boxed{\mathrm{grad}(f)(\gamma(t)) \perp \dot{\gamma}(t).}$$

→ $\mathrm{grad}(f)(\gamma(t))$ is a normal vector to S_k at $\gamma(t)$.



→ eq. of tangent plane to S_k at p:

$$0 = \mathrm{grad}(f)(p) \cdot \big((x,y,z) - p\big)$$

$$\boxed{0 = \partial_x f(p)(x-p_1) + \partial_y f(p)(y-p_2) + \partial_z f(p)(z-p_3).}$$

$T(S_k) \oplus N(S_k) = T(\mathbb{R}^3)$

→ par. ~~eq.~~ of normal line to S_k at p:

$$N(t) = (p, f(p)) + t\, \mathrm{grad}(f)(p).$$

Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $\quad p = (p_1, p_2) \in \mathbb{R}^2$

eq. for tangent plane to Γ_f at p:

$$\boxed{z - f(p_1, p_2) = \partial_x f(p)(x - p_1) + \partial_y f(p)(y - p_2).}$$

from before.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x, y, z) \longmapsto f(x, y) - z$$

~~$x, y, f(x,y)$~~

$$\Rightarrow S_0 = \{F = 0\} = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$$
$$= \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\} = \Gamma_f.$$

$$\operatorname{grad}(F)_{\substack{p \\ (p_1, p_2, p_3)}} = \left(\partial_x f_{(p_1,p_2)}, \partial_y f_{(p_1,p_2)}, -1\right).$$

$(p_1, p_2, p_3) \in S_0$
$\Leftrightarrow p_3 = f(p_1, p_2)$

$$\rightarrow \boxed{0 = \partial_x f(p_1, p_2)(x - p_1) + \partial_y f(p_1, p_2)(y - p_2) - (z - p_3)}$$

<u>normal line</u>: $(p, f(p)) + t\, \operatorname{grad}(F)(p) = N(t)$

$$N(t) = (p, f(p)) + t\left(\partial_x f(p), \partial_y f(p), -1\right)$$
$$= \left(p_1 + t\,\partial_x f(p), p_2 + t\,\partial_y f(p), f(p) - t\right).$$

Ex: $E = \left\{ \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \right\}$. eqs for

$T_p E$, $N_p E$, $p = (-2, 1, -3)$?

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \mapsto \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

$$\Rightarrow E = S_3^f = \{ f = 3 \}.$$

$$\operatorname{grad}(f) = \left(\frac{x}{2}, 2y, \frac{2z}{9} \right)$$

$$\operatorname{grad}(f)(-2, 1, -3) = \left(-1, 2, -\frac{2}{3} \right).$$

$T_p E$: $0 = (-1)(x+2) + 2(y-1) - \frac{2}{3}(z+3)$.

$$= -x - 2 + 2y - 2 - \frac{2}{3} z - 2$$

$$\Rightarrow \boxed{x - 2y + \frac{2}{3} z + 6 = 0}$$

$N_p E$: $N(t) = (-2, 1, -3) + t\left(-1, 2, -\frac{2}{3}\right)$

$$= \boxed{\left(-2 - t, 1 + 2t, -3 - \frac{2}{3} t\right)}$$

$$\Leftrightarrow \boxed{\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}}$$

230/231 02/21/2020.

Optimization / Maximums & Minimums . . .

Def: $f : \mathbb{R}^n \to \mathbb{R}$, $n \in \{1, 2, 3\}$. $f : \mathbb{R}^2 \to \mathbb{R}$.

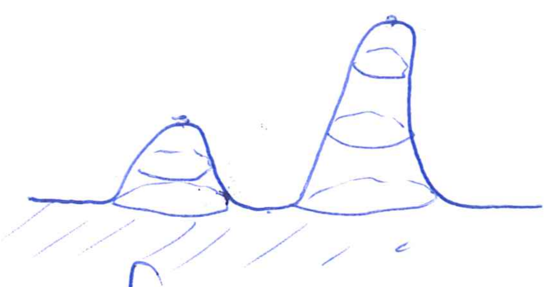
$p \in \mathbb{R}^n$: f has a <u>local maximum</u> at p

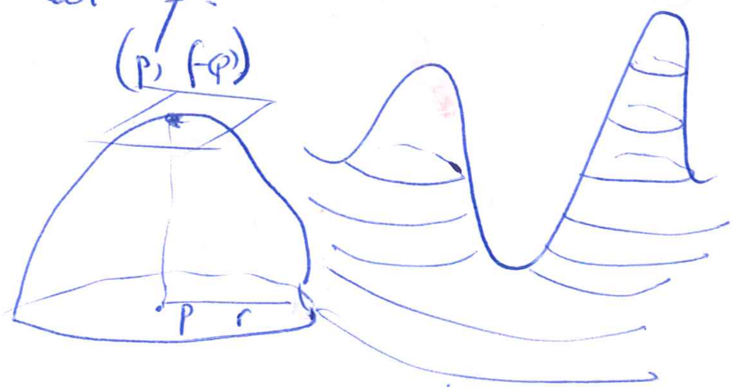
if ~~$f(x,y) < f(p)$~~ for (x,y)

for some $r > 0$: $d((x,y), p) < r$

$$\Rightarrow f(x,y) \le f(p).$$

f has a <u>local minimum</u> at p if $-f$ has a local maximum at p.





f has a <u>global/absolute max.</u> at p if

for any (x,y) $f(x,y) \le f(p)$

f has a <u>global/absolute min</u> at p if

$-f$ has a global/absolute max at p.

Thm: Let $f : \mathbb{R}^n \to \mathbb{R}$. If at $p \in \mathbb{R}^n$ a local max/min occurs ~~at~~ ~~?~~, then any partial derivative that exists at p must vanish.
but not vice versa.

Pf: Say ~~$\partial_i f(p) = \frac{d}{dt} f(p+ti)$~~

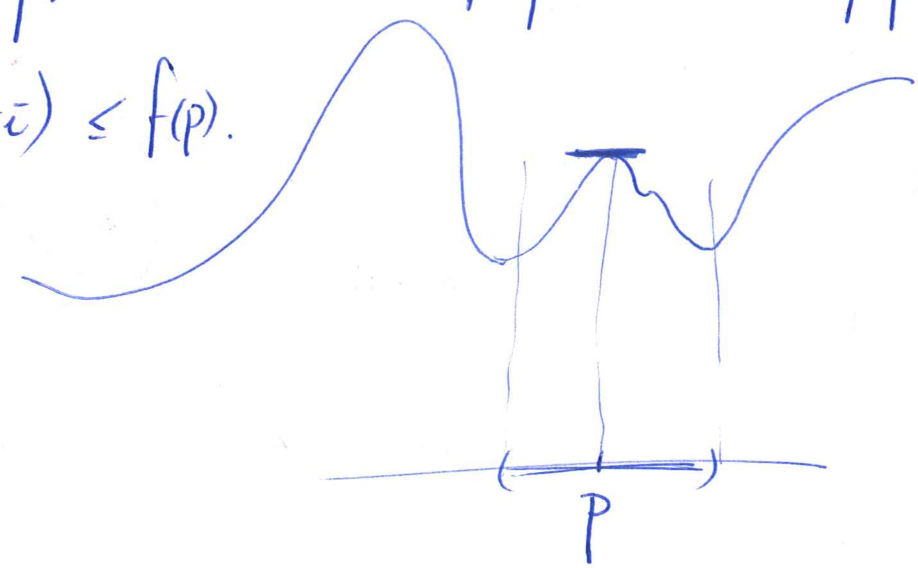
$$\partial_x f(p) = \frac{d}{dt} f(p + t\bar{i})\Big|_{t=0}$$ exists, $f(p)$ is a local max.

~~$\exists r > 0, d(f(x,y)) <$~~

$$\exists r > 0 : d((x,y), p) < r \Rightarrow f(x,y) \leq f(p).$$

$$\Rightarrow d((p + t\bar{i}), p) < r \Rightarrow f(p + t\bar{i}) \leq f(p).$$

$$\Rightarrow |t| < r \Rightarrow f(p + t\bar{i}) \leq f(p).$$



$$\Rightarrow \partial_x f(p) = 0.$$

Ex : $f(x,y) = x^2+y^2-2x-6y+14.$

$\operatorname{grad}(f) = (2x-2,\ 2y-6) = (0,0)$

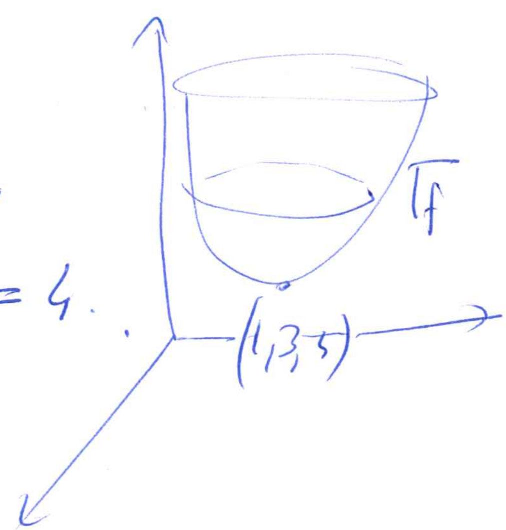
$\Leftrightarrow \boxed{x=1,\ y=3} \to (1,3)$ is a candidate for a local extremum.

$\cancel{\operatorname{grad}(f)(1+\delta_1, 3+\delta_2) = (2\delta_1, 2\delta_2)}$

$f(x,y) = (x-1)^2+(y-3)^2+4$

$\to (1,3)$ minimizes f,

$\min f = f(1,3) = 4.$



__Not vice versa:__

__Ex:__ $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^3$. $\quad f'(0)=0$

but 0 is not an local extremum.

__Ex:__ $f:\mathbb{R}^2\to\mathbb{R}$ $\quad \Gamma_f=\{(x,y,z)\mid z=x^2-y^2\}$

$(x,y)\longmapsto x^2-y^2$. $\quad$ hyp. par. = pringle.

$$\left.\begin{aligned}\partial_x f &= 2x\\ \partial_y f &= -2y\end{aligned}\right\}\quad \begin{aligned}\partial_x f(0,0)&=0\\ \partial_y f(0,0)&=0\end{aligned}$$

but neither a local max nor a local min.

Thus points where partials vanish are candidates for local/global extrema.

- $p\in\mathbb{R}^n$ is a __critical point__ of $f:\mathbb{R}^n\to\mathbb{R}$ if

$\mathrm{grad}(f)(p)=0=(0,0,\dots,0)$ or it does not exist.

If $f:\mathbb{R}^2\to\mathbb{R}$ is twice diff., its <u>Hessian</u> is defined to be

$$\begin{aligned} \mathrm{Hess}(f): \mathbb{R}^2 &\to \mathrm{Mat}(2\times 2,\mathbb{R}) \\ p &\mapsto \begin{pmatrix} \partial_x^2 f(p) & \partial_x\partial_y f(p) \\ \partial_y\partial_x f(p) & \partial_y^2 f(p) \end{pmatrix}. \end{aligned}$$

$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = AD-BC.$ $\quad H_f(p) = \det\left(\mathrm{Hess}(f)(p)\right)$

<u>2nd Derivative ~~test~~ test</u>: Let $f:\mathbb{R}^2\to\mathbb{R}$, $p=(p_1,p_2)\in\mathbb{R}^2$, f is C^2 near p. & $\mathrm{grad}(f)(p)=0$. Then

(i) ~~$\det(\mathrm{Hess}(f)(p))$~~ $H_f(p)>0$ & $\partial_x^2 f(p)>0$ $\Rightarrow$ $f(p)$ is a local min.

(ii) $H_f(p)>0$ & $\partial_x^2 f(p)<0$ $\Rightarrow$ $f(p)$ is a local max.

(iii) $H_f(p)<0$ $\Rightarrow$ ~~$f(p)$~~ f has a saddle (ie neither local min nor max)

(iv) $H_f(p)=0$ $\Rightarrow$ inconclusive

Ex: $f(x,y) = x^4 + y^4 - 4xy + 1.$

classify critical points?

$$\operatorname{grad}(f)(x,y) = (4x^3 - 4y,\ 4y^3 - 4x) = (0,0)$$

$$\Leftrightarrow \quad x^3 - y = 0 = y^3 - x.$$

$$\Leftrightarrow \quad y = x^3 \text{ and } x = y^3$$

$$\Rightarrow \quad y = y^9 \Rightarrow y(y^8 - 1) = 0$$

$$\cancel{\Rightarrow y(y+1)y}$$

$$\Rightarrow y(y-1)(y+1)(y^2+1)(y^4+1) = 0.$$

$\Rightarrow y = -1$	$y = 0$	$y = 1$
$\Rightarrow x = -1$	$x = 0$	$x = 1$

critical points.

$$\mathrm{Hess}(f)(x,y) = \left(\begin{array}{c|c} 12x^2 & -4 \\ \hline -4 & 12y^2 \end{array}\right).$$

$$H_f(x,y) = 144x^2y^2 - 16 = 16(9x^2y^2 - 1)$$

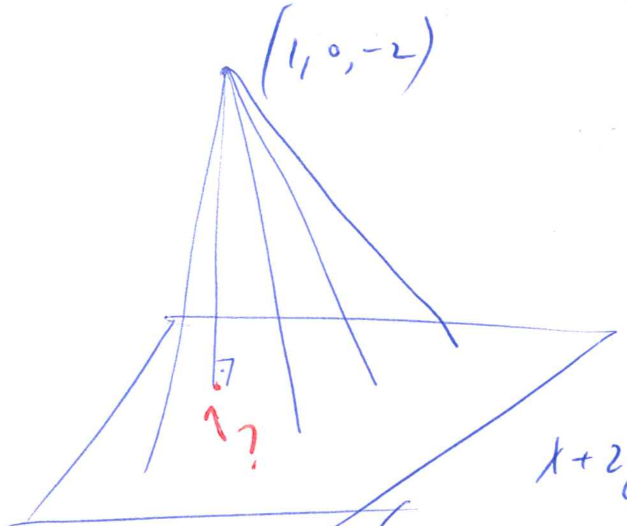
$$\begin{cases} H_f(x,y) > 0, & \text{if } (xy)^2 > 1/3 \\ H_f(x,y) = 0, & \text{if } (xy)^2 = 1/3 \\ H_f(x,y) = 0, & \text{if } (xy)^2 < 1/3 \end{cases}$$

$\rightarrow H_f(1,1) > 0 \Rightarrow \partial_x^2 f(1,1) > 0 \Rightarrow$ local min at $(1,1)$

$H_f(0,0) = -16 < 0 \rightarrow$ $\boxed{\text{saddle at } (0,0)}$

$H_f(-1,-1) > 0$, $\partial_x^2 f(-1,-1) > 0 \rightarrow$ local min.

Ex: distance of the point $(1,0,-2)$ to the plane $x+2y+z=4$.



$$\frac{|1-2-4|}{\sqrt{1^2+2^2+1^2}} = \boxed{\frac{5}{\sqrt{6}}}.$$

$x+2y+z=4$.

$$f(x,y) = \left(d\left((1,0,-2),\ (x,y,4-x-2y)\right)\right)^2$$

$$= (x-1)^2 + y^2 + (x+2y-6)^2.$$

$$= x^2 - 2x + 1 + y^2 + (x+2y)^2 - 12(x+2y) + 36$$

$$= x^2 - 2x + 1 + y^2 + x^2 + 4xy + 4y^2 - 12x - 24y + 36$$

$$= 2x^2 - 14x + 4xy + 5y^2 - 24y + 37$$

$$\operatorname{grad}(f) = \left(2(x-1) + 2(x+2y-6),\ 2y + 4(x+2y-6)\right)$$

$$= (2x - 2 + 2x + 4y - 12,\ 4x + 10y - 24)$$

$$= (4x + 4y - 14,\ 4x + 10y - 24).$$

$$\mathrm{grad}(f)(x,y) = 0$$

$$\Leftrightarrow \quad 4x + 4y - 14 = 0$$
$$4x + 10y - 24 = 0.$$

$$\Rightarrow \quad 6y - 10 = 0$$
$$y = \frac{5}{3} \Rightarrow 4x + \frac{20}{3} - 14 = 0$$
$$x = \frac{42 - 20}{3.4} = \frac{22}{3.4} = \frac{11}{6}.$$

$\Rightarrow \left(\frac{11}{6}, \frac{5}{3}\right)$ is the only crit pt

$$\mathrm{Hess}(f) = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}.$$

$$\left.\begin{array}{l} H_f = 40 - 16 > 0 \\ \partial_x^2 f > 0 \end{array}\right\} \Rightarrow \text{local min at } \left(\frac{11}{6}, \frac{5}{3}\right)$$

$$f\left(\frac{11}{6}, \frac{5}{3}\right) = \left(d\left((1,0,2), \left(\frac{11}{6}, \frac{5}{3}, 6 - \frac{11}{6} - \frac{10}{3}\right)\right)\right)^2$$

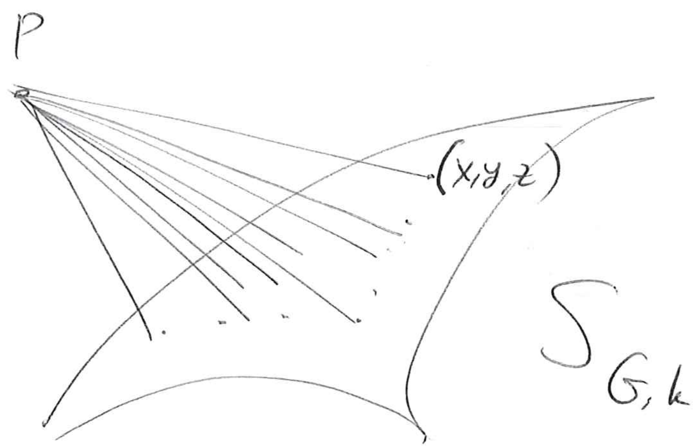
$$= \left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{36}{6} - \frac{11}{6} - \frac{20}{6}\right)^2 = \left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2 = \frac{25 + 100 + 25}{36} = \frac{150}{36}$$

$$\text{dist} = \sqrt{\frac{150}{36}} = \sqrt{\frac{25}{6}} = \boxed{\frac{5}{\sqrt{6}}}$$

SW: $p \in \mathbb{R}^3$, $G: \mathbb{R}^3 \to \mathbb{R}$, $k \in \mathbb{R}$.

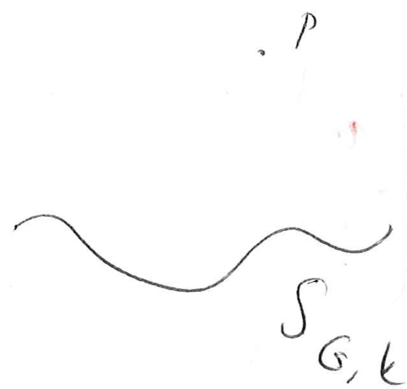
Take e.g. a quadric surface.

Min. distance of p, to $S_{G,k}$?



$$\varphi : S_{G,k} \to \mathbb{R}$$

$$(x,y,z) \longmapsto \left(d((x,y,z),p)\right)^2$$



$$G(x,y,z) = k.$$

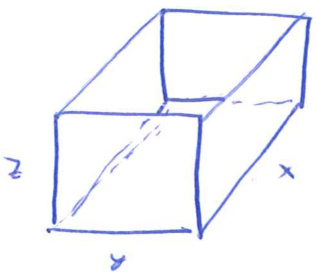
$$\varphi(x,y,z) = (x-p_1)^2 + (y-p_2)^2 + (z-p_3)^2$$

$$\nabla\varphi(x,y,z) = \left(2(x-p_1),\ 2(y-p_2),\ 2(z-p_3)\right)$$

$$\boxed{\begin{aligned} \nabla\varphi(x,y,z) &= \lambda(x,y,z)\,\nabla G(x,y,z) \\ G(x,y,z) &= k \end{aligned}}$$

Ex: rectangular box without lid made from 12 m^2 cardboard.

Maximize volume.



$$12 = 2xz + 2yz + xy.$$

$$= z(2x+2y) + xy.$$

$$z = \frac{12 - xy}{2x+2y}.$$

$$V(x,y) = xyz = xy\,\frac{12-xy}{2x+2y} = \frac{12xy - x^2y^2}{2x+2y}.$$

$$\partial_x V = \frac{(12y - 2xy^2)(2x+2y) - (12xy - x^2y^2)\,2}{(2x+2y)^2}$$

$$= \frac{\cancel{24xy} + 24y^2 - 4x^2y^2 - 4xy^3 - \cancel{24xy} + 2x^2y^2}{(2x+2y)^2}$$

$$= \frac{-4xy^3 - 2x^2y^2 + 24y^2}{(2x+2y)^2} = \frac{2y^2(-2xy - x^2 + 12)}{(2x+2y)^2}.$$

$\partial_y V =$

$$\partial_y V = \frac{(12x - 2x^2y)(2x+2y) - (12xy - x^2y^2)2}{(2x+2y)^2}$$

$$= \frac{24x^2 + \cancel{24xy} - 4x^3y - 4x^2y^2 - \cancel{24xy} + 2x^2y^2}{(2x+2y)^2}$$

$$= \frac{-4x^3y - 2x^2y^2 + 24x^2}{(2x+2y)^2} = \frac{2x^2(-2xy - y^2 + 12)}{(2x+2y)^2}.$$

$$\operatorname{grad}(V)(x,y) = \frac{1}{(2x+2y)^2}\left(2y^2(-2xy - x^2 + 12),\ 2x^2(-2xy - y^2 + 12)\right)$$

$$\operatorname{grad}(V) = 0 \iff \boxed{\begin{array}{l} y^2(-2xy - x^2 + 12) = 0 \\ \& \ x^2(-2xy - y^2 + 12) = 0 \end{array}}$$

$$x, y > 0 \Rightarrow \boxed{\begin{array}{l} -x^2 - 2xy + 12 = 0 \\ y^2 + 2xy - 12 = 0 \end{array}}$$

$x^2 = 2xy - 12 = +y^2.$

$\Rightarrow \boxed{y = x}$ ($y = -x$ does not make sense)

$\Rightarrow$ $-2x^2 + x^2 + 12 = 0$

$12 = 3x^2$

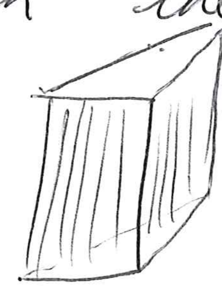
$\boxed{x = 2 = y}$

$V(2,2) = 2 \cdot 2 \cdot \dfrac{12 - 4}{4 + 4} = 4 \cdot \dfrac{8}{8} = 4.$

max volume

SW: Use 2nd Derivative ~~base~~ Test to verify that $4 = V$ is max.

SW: • Do the same with the base a triangle.



• Do the same with the base arbitrary

Ex: ~~Absolute~~ local/global extrema

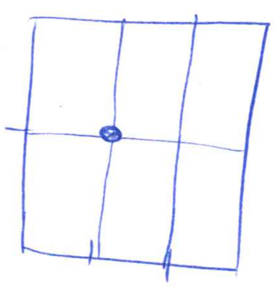
of $f(x,y) = x^2 - 2xy + 2y$ on

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 \le x \le 3 \\ 0 \le y \le 2 \end{array} \right\}$$

$$\operatorname{grad}(f) = (2x - 2y, \; -2x + 2)$$

$$\operatorname{grad}(f) = 0 \iff x = y \;\; \& \;\; x = 1$$

$\Rightarrow$ $(1,1)$ is the only crit. point.



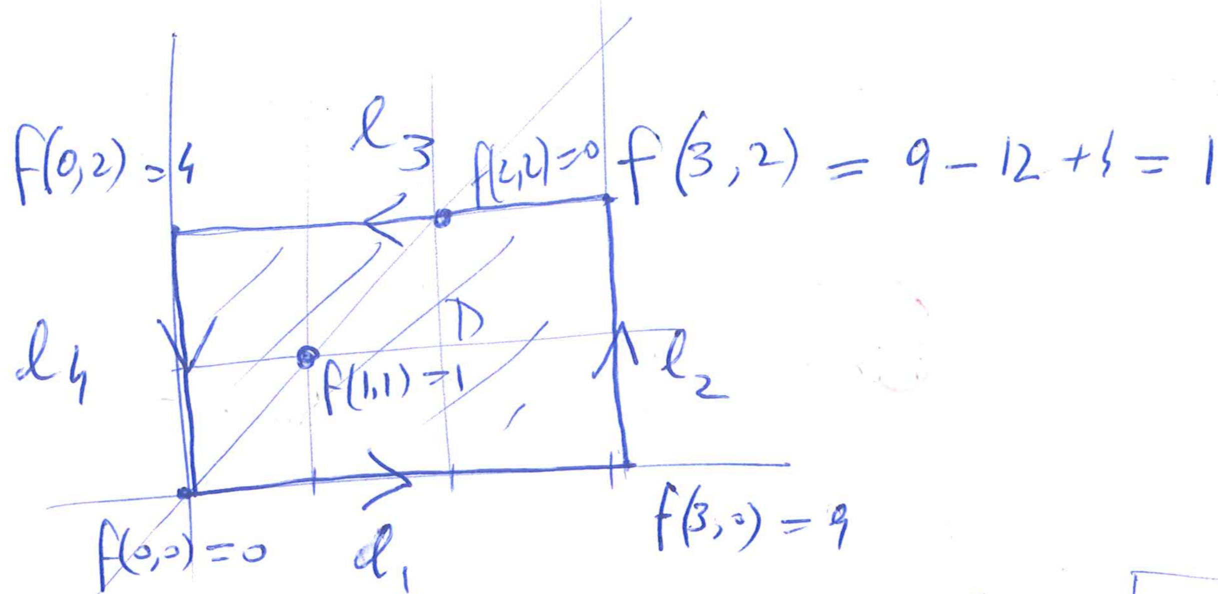
$$\operatorname{Hess}(f) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$$

$$H_f = -4 < 0$$

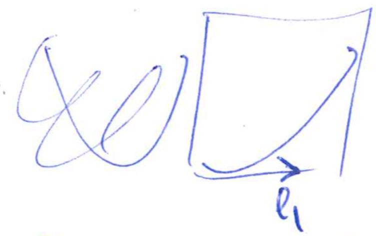
$$\Rightarrow \boxed{\text{saddle at } (1,1)}$$

$f(1,1) = 1$

We need to investigate what happens at the boundary as well.



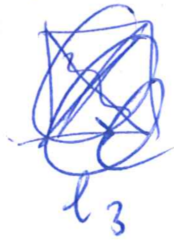
On l_1 : $f(x,0)=x^2$.

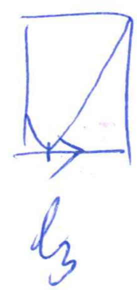


On l_2 : $f(3,y)=9-6y+2y=9-4y$.

On l_3 : $f(x,2)=x^2-4x+4$ $\to (x-2)^2$

On l_4 : $f(0,y)=2y$.





global max $= f(3,0)=9$

global min $= f(0,0)=0 = f(2,2)$.

SW: CalcPlot3D.

The Method of Lagrange Multipliers:

- Let ~~[scribble]~~ $F, G : \mathbb{R}^3 \to \mathbb{R}$, $k \in \mathbb{R}$. Suppose $F, G \in C^1$, and $S_{G,k}$ contains no crit. points of G.

Goal: Optimize F on $S_{G,k}$

Here G is the constraint function.

Obs: If $p \in S_{G,k}$ is a local extremum of F on $S_{G,k}$, then

$$\nabla F(p) \,/\!/\, \nabla G(p).$$

$$\left(\text{ie., } T_p S_{F,F(p)} = T_p S_{G,k}\right).$$

(ie, there is a $\lambda(p) \in \mathbb{R}$: $\nabla F(p) = \lambda(p) \nabla G(p)$.)

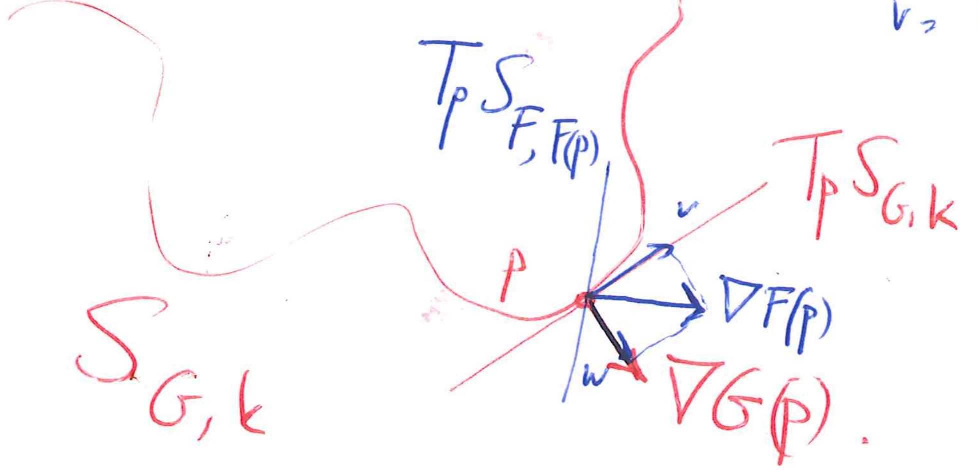
(Lagrange multiplier).

SW:

$w = \text{proj}_{\nabla G(p)} (\nabla F(p))$.

$v = \nabla F(p) - w$.

Show that if $v \neq 0$ (ie, $\nabla F(p) \not\parallel \nabla G(p)$), then $\partial_{\frac{v}{|v|}} F(p) > 0$.



S.O.P (Lagrange Multiplier)

(i) Solve $\boxed{\begin{array}{c}\nabla F(p) = \lambda(p)\nabla G(p).\\ p \in S_{G,k}.\end{array}}$ for p, & $\lambda(p)$.

ie., $\boxed{\begin{array}{l}\partial_x F(p) = \lambda(p)\,\partial_x G(p)\\ \partial_y F(p) = \lambda(p)\,\partial_y G(p)\\ \partial_z F(p) = \lambda(p)\,\partial_z G(p)\\ G(p) = k.\end{array}}$

(ii) Compare

Ex: $12\,m^2$ cardboard $\to$ ~~box~~ rectangular box w/o lid. maximize vol.

$x, y, z > 0$.

$F(x,y,z) = xyz$ (vol., to be optimized).

$G(x,y,z) = xy + 2xz + 2yz$, $k = 12$. constraint

$\nabla F = (yz,\ xz,\ xy)$

$\nabla G = (y+2z,\ x+2z,\ 2x+2y) \neq (0,0,0)$

$p = (a, b, c)$.

SW: Optimize G on $S_{F,12}$. Interpret physically.

$$bc = \lambda(b+2c)$$
$$ac = \lambda(a+2c)$$
$$ab = \lambda(2a+2b)$$
$$ab + 2ac + 2bc = 12$$

$$\cancel{abc = \lambda ab + 2\lambda ac = \lambda ab + 2\lambda bc \quad 2\lambda ac + 2\lambda bc}$$

$$abc = a\lambda(b+2c) = b\lambda(a+2c) = c\lambda(2a+2b)$$

$\lambda = 0 \Rightarrow 0 = bc = ac = ab \Rightarrow 0 = 12$ ↯.

$\Rightarrow \lambda \neq 0$.

$$\Rightarrow a(b+2c) = b(a+2c) = c(2a+2b).$$

$\Rightarrow ac = bc$

$\Rightarrow (a-b)c = 0$. $c \neq 0$ $\Rightarrow a = b$

$a(a+2c) = c(4a)$

$(a \neq 0)$ $a + 2c = 4c$

$a = 2c$.

$\Rightarrow \boxed{a = b = 2c}$

$4c^2 + 4c^2 + 4c^2 = 12$

$\Rightarrow c^2 = 1 \Rightarrow \boxed{c = 1 \Rightarrow a = b = 2.}$

Ex:

$$\left.\begin{array}{l} f(x,y) = x^2 + 2y^2 \\ g(x,y) = x^2 + y^2 \\ k = 1 \end{array}\right\}$$ Optimize f on $S_{g,k}$

$$\nabla f = (2x, 4y)$$
$$\nabla g = (2x, 2y).$$

~~$2x = 4\lambda y$~~

$$\boxed{\begin{array}{l} \cancel{2}x = \lambda \cancel{2} x \\ \cancel{4}2y = \lambda 2 y \\ x^2 + y^2 = 1 \end{array}}$$

$$(1-\lambda)x = 0$$

$\lambda = 1$ or $x = 0$.

$\lambda = 1 \Downarrow y = 0 \Downarrow x = \mp 1$

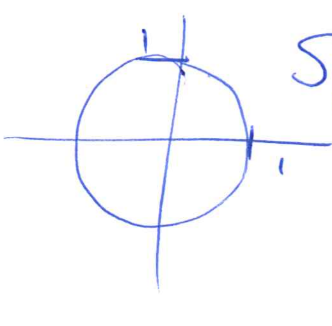
$x = 0 \Downarrow y = \mp 1 \Downarrow \lambda = \mp 2.$

→ ~~Three~~ Four points:

$(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$.

$f(\mp 1, 0) = 1 \to$ min s.

$f(0, \mp 1) = 2. \to$ max s



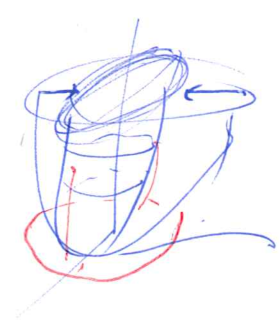
$S_{g,1}.$

$$z = x^2 + (\sqrt{2}y)^2.$$
$$Z = X^2 + Y^2.$$

$Z = z$

$X = x$

$Y = \sqrt{2}y.$ $\quad y = \frac{Y}{\sqrt{2}}$



Optimize

Ex: $f(x,y) = x^2+2y^2 \ (\geq 0)$ on $x^2+y^2 \leq 1$.

$\nabla f = (2x, 4y) = (0,0)$

$\Leftrightarrow (x,y)=0$.

$f(0,0) = 0$.

global min. global max: $f(0,\mp 1) = 2$.

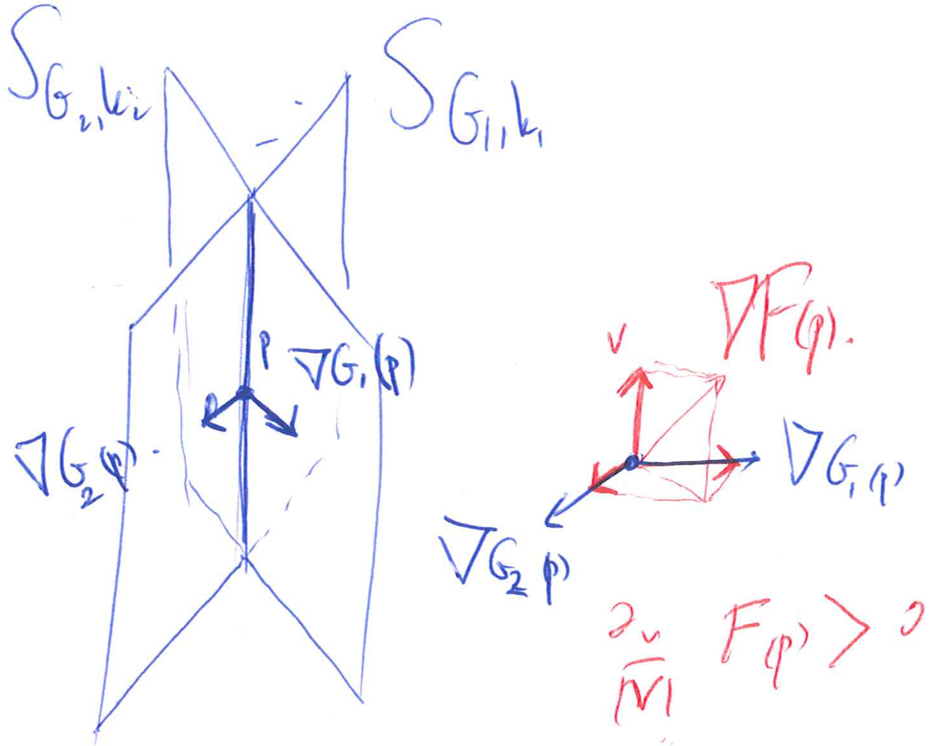
- $F, G_1, G_2 : \mathbb{R}^3 \to \mathbb{R}$, $k_1, k_2 \in \mathbb{R}$, all C^1.

  ~~S_{G_i,k_i}~~ S_{G_i,k_i} contains no crit points of G_i. $i=1,2$.

Goal: Optimize F on the intersection $S_{G_1,k_1} \cap S_{G_2,k_2}$.

Obs: If $p \in S_{G_1,k_1} \cap S_{G_2,k_2}$ is a local extremum of F on $S_{G_1,k_1} \cap S_{G_2,k_2}$, then $\mathrm{proj}_{T_p(S_{G_1,k_1} \cap S_{G_2,k_2})}(\nabla F(p)) = 0$.

$$\cancel{V = \frac{(k - A(R))\,A(R)}{L(\partial R)}}$$



S.O.P. (Lagrange Multipliers)

1) Solve
$$\boxed{\begin{aligned} &\nabla F(p) = \lambda_1 \nabla G_1(p) + \lambda_2 \nabla G_2(p). \\ &p \in S_{G_1,k_1} \cap S_{G_2,k_2}. \end{aligned}}$$

for p, λ_1, λ_2.

(ii) Compare.

Ex: $F(x,y,z) = x+2y+3z$. on

$x-y+z=1$ & $x^2+y^2=1$.

~~g(x,y)~~ $G_1(x,y,z) = x-y+z$, $k_1 = 1$

$G_2(x,y,z) = x^2+y^2$, $k_2 = 1$.

$$\left.\begin{aligned}\nabla F &= (1,2,3)\\ \nabla G_1 &= (1,-1,1)\\ \nabla G_2 &= (2x,2y,0)\end{aligned}\right\} \quad (1,2,3) = \lambda_1 (1,-1,1) + \lambda_2 (2x,2y,0).$$

$\cancel{1 = \lambda_1}$, $\cancel{2}$, $\cancel{3}$

$3 = \lambda_1 + \lambda_2 \cdot 0$.

$\rightarrow \boxed{\lambda_1 = 3}$.

$1 = 3 + \lambda_2 2x$

$2 = -3 + \lambda_2 2y$.

$$\left.\begin{aligned} x &= \frac{-2}{2\lambda_2} = -\frac{1}{\lambda_2} \\ y &= \frac{5}{2\lambda_2} = \frac{5}{2\lambda_2}\end{aligned}\right\} \quad 1 = \frac{1}{\lambda_2^2} + \frac{25}{4\lambda_2^2}$$

(sw).

$\cancel{= \frac{5}{\lambda_2^2} \rightarrow \lambda_2 = \pm\sqrt{5}}$.

$= \frac{29}{4\lambda_2^2}$

$\lambda_2 = \pm\frac{\sqrt{29}}{2} \rightsquigarrow \boxed{\text{max} = 3+\sqrt{29}}$.