PDE's and Fourier Series:

§ 10.1:

- The (differential) operator

$$\Delta: C^2(\mathbb{R}^d, \mathbb{R}) \longrightarrow C^0(\mathbb{R}^d, \mathbb{R})$$

$$f(x_1, x_2, \dots, x_d) \longmapsto \sum_{k=1}^{d} \partial_{x_k}^2 f(x_1, x_2, \dots, x_d)$$

$$\left(= \partial_{x_1}^2 f(x_1, x_2, \dots, x_d) + \partial_{x_2}^2 f(x_1, x_2, \dots, x_d) + \dots + \partial_{x_d}^2 f(x_1, x_2, \dots, x_d) \right)$$

is called the Laplacian. ("Differential" means that it involves derivatives.) $(\Delta = \nabla \cdot \nabla = \nabla^2)$

SW: (i) Δ is a linear operator

(ii) So is $-\Delta$.

(iii) The set of linear operators between two linear spaces (of functions) is a linear space.

- Δ constitutes the "spatial" parts of the

heat operator $\partial_t - \Delta$ and the

wave operator $\partial_t^2 - \Delta$.

- We know by now that identifying the eigenpairs of a linear operator is crucial for understanding the operator. Thus we would like to find the eigenpairs of $-\Delta$, ie, pairs (λ, f) where $\lambda \in \mathbb{R}$, $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ (to avoid technicalities) : $f \neq 0$, $-\Delta f = \lambda f$

$$\Leftrightarrow \boxed{\Delta f + \lambda f = 0} \quad (*)$$

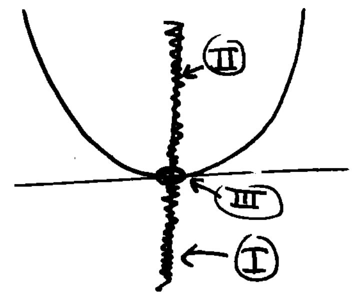
- Let's start easy and consider the case when the "spatial" dimension $d = 1$. Then $\Delta = \partial_x^2$, and $(*)$ reduces to:

$$\boxed{\partial_x^2 f(x) + \lambda f(x) = 0}$$

In particular, this is an ODE and we know how to deal with it.

$$(*) \Leftrightarrow \partial_x \begin{pmatrix} f(x) \\ \partial_x f(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}_{=: A_\lambda} \begin{pmatrix} f(x) \\ \partial_x f(x) \end{pmatrix} \Leftrightarrow \boxed{\partial_x Y(x) = A_\lambda Y(x)} \quad (**)$$

$tr(A_\lambda) = 0$, $\det(A_\lambda) = \lambda \Rightarrow$



(I) $\lambda < 0 \Rightarrow \lambda_1 = -\sqrt{-\lambda} < 0 < \sqrt{-\lambda} \; (\Rightarrow -\lambda_1 = \lambda_2)$

$\Rightarrow$ the gen. sol. of (*) is:

$$\boxed{y(x) = c_1 \, e^{-\sqrt{-\lambda}\,x} \begin{pmatrix} 1 \\ -\sqrt{-\lambda} \end{pmatrix} + c_2 \, e^{\sqrt{-\lambda}\,x} \begin{pmatrix} 1 \\ \sqrt{-\lambda} \end{pmatrix}}$$

(ie., if $\lambda < 0$ and at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $(\lambda, \; c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{\lambda}x})$ is an eigenpair of $-\Delta$.)

SW: A more convenient way of writing these is by using the hyperbolic trigonometric functions:

$$\left.\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \end{aligned}\right\} \Rightarrow \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow \cosh(\theta) := \cos(i\theta) = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh(\theta) := \frac{1}{i}\sin(i\theta) = \frac{e^{\theta} - e^{-\theta}}{2}.$$

Use the hyperbolic trigonometric functions to write the results above in a more convenient way.

(II) $\lambda > 0 \Rightarrow \lambda_1 = i\sqrt{\lambda} = \overline{\lambda_2}$ $\left(\Rightarrow -\lambda_1 = \lambda_2\right)$.

$\Rightarrow$ the gen. sol. of $\left(\begin{smallmatrix}*\\*\end{smallmatrix}\right)$ is:

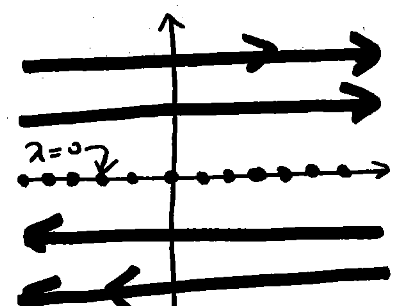
$$\boxed{Y(x) = c_1 \begin{pmatrix} \cos(\sqrt{\lambda}\,x) \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\,x \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{\lambda}\,x) \\ \sqrt{\lambda}\cos(\sqrt{\lambda}\,x) \end{pmatrix}}$$

(ie., if $\lambda > 0$ and at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $\left(\lambda,\ c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)\right)$ is an eigenpair of $-\Delta$.)

(III) $\lambda = 0 \Rightarrow \lambda_1 = 0 = \lambda_2$ $\left(\Rightarrow -\lambda_1 = \lambda_2\right)$

$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so A_0 is in canonical form (improper node, stable)

phase portrait:



(x is "time", not the horizontal axis.)

$\Rightarrow$ the gen. sol. of $\left(\begin{smallmatrix}*\\*\end{smallmatrix}\right)$ is:

$$\boxed{Y(x) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} x \\ 1 \end{pmatrix}}$$

(ie, if at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $(0,\ c_1 + c_2 x)$ is an eigenpair of $-\Delta$.)

• Thus we have the complete list of eigenpairs of $-\Delta$. Let's call this list the <u>spectral data</u> of $-\Delta$:

<u>Region in the map</u>	<u>Eigenvalue</u>	<u>Eigenfunction</u>
(I)	$\lambda < 0$	any lin. combo of $e^{-\sqrt{-\lambda}x}$ and $e^{\sqrt{-\lambda}x}$ (or of $\cosh(\sqrt{-\lambda}x)$ and $\sinh(\sqrt{-\lambda}x)$)
(II)	$\lambda > 0$	any lin. combo of $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$
(III)	$\lambda = 0$	any lin. combo of 1 (the function that is constantly 1) and x

• A (<u>two-point</u>) <u>boundary value problem</u> (BVP) is a triple

$$\left(\text{diff. eq.},\ \begin{matrix}\text{boundary}\\ \text{datum at}\\ x_0\end{matrix},\ \begin{matrix}\text{boundary}\\ \text{datum at}\\ x_1\end{matrix}\right),$$

where $x_0 \neq x_1$. Geometrically speaking specifying a boundary datum corresponds to specifying a line in the phase space.

- As opposed to IVP's, BVP's with even the "nicest" differential equations may fail to have a unique solution.

- A BVP with a homogeneous differential equation and vanishing boundary data (i.e., $y(x_0) = 0 = y(x_1)$) is called <u>homogeneous</u>.

- If the diff. eq. of a BVP is of the form $\Delta y + \lambda y = 0$, then the eigenpairs of $-\Delta$ that satisfy the boundary conditions are also called <u>the eigenpairs of the BVP</u> by proxy.

<u>Ex</u>:

$$\boxed{\begin{aligned} &\Delta y(x) + 4y(x) = 0 \\ &y(0) = -2 \\ &y(\pi/4) = 10 \end{aligned}}$$

$\lambda = 4 > 0 \Rightarrow$ II

$$\Rightarrow \boxed{Y(x) = c_1 \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}}$$

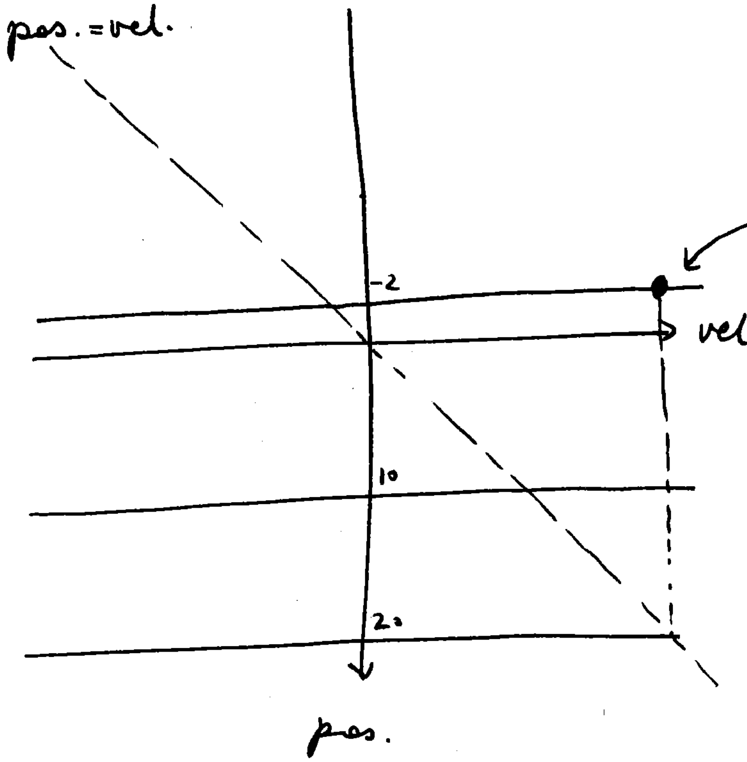
is the gen. sol. (of the ODE).

$$\begin{pmatrix} -2 \\ \partial_x y(0) \end{pmatrix} = Y(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(0)}{2} \end{aligned}$$

$$\begin{pmatrix} 10 \\ \partial_x y(\pi/4) \end{pmatrix} = Y(\pi/4) = c_1 \begin{pmatrix} 0 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} c_2 &= 10 \\ c_1 &= -\frac{\partial_x y(0)}{2} \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} y(x) = &-2\cos(2x) \\ &+ 10\sin(2x) \end{aligned}}$$

is the unique sol.



The trajectory of the unique solution is the unique ellipse passing through this point.

Ex:
$$\boxed{\begin{aligned} &\Delta y(x) + 4y(x) = 0 \\ &y(0) = -2 \\ &y(2\pi) = -2 \end{aligned}}$$

$\lambda = 4 > 0 \Rightarrow$ II

$$\Rightarrow \boxed{Y(x) = c_1 \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}}$$

is the gen. sol. (of the ODE).

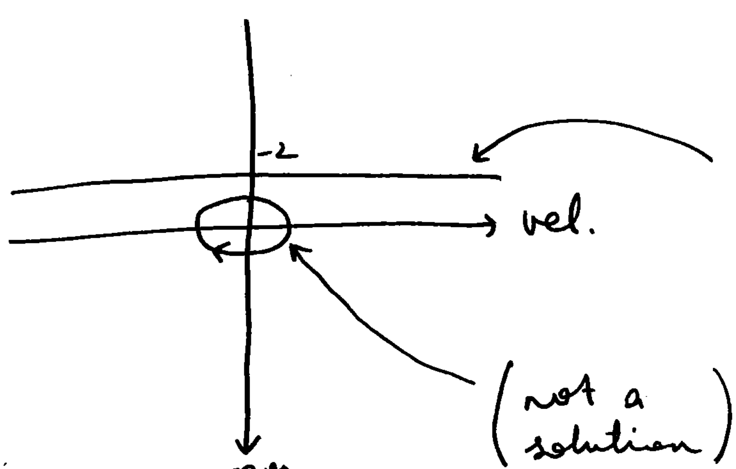
$$\begin{pmatrix} -2 \\ \partial_x y(0) \end{pmatrix} = Y(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(0)}{2} \end{aligned}$$

$$\begin{pmatrix} -2 \\ \partial_x y(2\pi) \end{pmatrix} = Y(2\pi) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(2\pi)}{2} \end{aligned}$$

$\Rightarrow$ For any $c_2 \in \mathbb{R}$,

$$\boxed{y(x) = -2\cos(2x) + c_2 \sin(2x)}$$

is a sol.



Any ellipse that hits this line (at least once) represents a solution.

Ex:
$$\boxed{\begin{array}{l}\Delta y(x) + 25\,y(x) = 0\\ \partial_x y(0) = 6\\ \partial_x y(\pi) = -9\end{array}}$$

$\lambda = 25 > 0 \Rightarrow$ (II)

$$\Rightarrow \boxed{Y(x) = c_1\begin{pmatrix}\cos(5x)\\ -5\sin(5x)\end{pmatrix} + c_2\begin{pmatrix}\sin(5x)\\ 5\cos(5x)\end{pmatrix}}$$

is the gen. sol. (of the ODE).

$$\left.\begin{array}{l}\begin{pmatrix}y(0)\\ 6\end{pmatrix} = Y(0) = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\5\end{pmatrix} \Rightarrow \begin{array}{l}c_1 = y(0)\\ c_2 = 6/5\end{array}\\ \begin{pmatrix}y(\pi)\\ -9\end{pmatrix} = Y(\pi) = c_1\begin{pmatrix}-1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\-5\end{pmatrix} \Rightarrow \begin{array}{l}c_1 = -y(\pi)\\ c_2 = 9/5\end{array}\end{array}\right\} \begin{array}{l}\Rightarrow 6/5 = c_2 = 9/5,\ \lightning\\ \Rightarrow \text{The BVP has no solutions.}\end{array}$$

SW: (i) Let $L > 0$, $\lambda \in \mathbb{R}$, and consider the BVP

$$\boxed{\begin{array}{l}\Delta y(x) + \lambda y(x) = 0\\ y(0) = 0 = y(L)\end{array}}$$

. Find all solutions.

(ii) Do the same with

$$\boxed{\begin{array}{l}\Delta y(x) + \lambda y(x) = 0\\ \partial_x y(0) = 0 = \partial_x y(L)\end{array}}\ .$$

(iii) Do the same with

$$\boxed{\begin{array}{l}\Delta y(x) + \lambda y(x) = 0\\ \partial_x y(0) = 0 = y(L)\end{array}} \quad \text{and} \quad \boxed{\begin{array}{l}\Delta y(x) + \lambda y(x) = 0\\ y(0) = 0 = \partial_x y(L)\end{array}}\ .$$

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§ 10.4 :

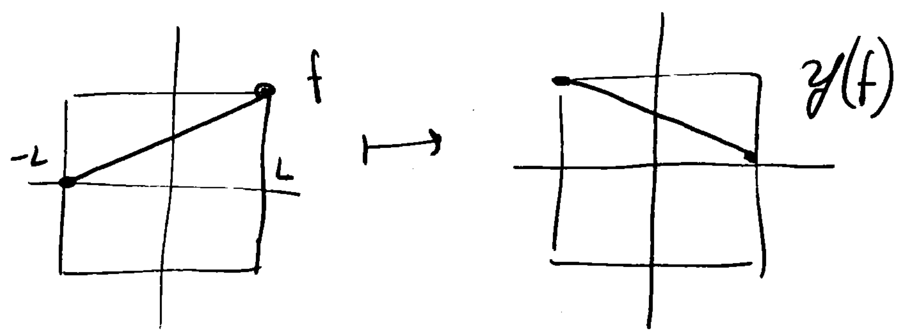
- Fix $L>0$ and put $I := [-L, L]$ or $]-L, L[$

(Thus I is an interval centered at 0 with the property $x \in I \Leftrightarrow -x \in I$).

Define the "reflection along the y-axis" operator

$$\mathcal{Y} : F(I, \mathbb{R}) \longrightarrow F(I, \mathbb{R})$$
$$\left[\begin{array}{l} I \xrightarrow{f} \mathbb{R} \\ x \longmapsto f(x) \end{array}\right] \longmapsto \left[\begin{array}{l} I \xrightarrow{\mathcal{Y}(f)} \mathbb{R} \\ x \longmapsto f(-x) \end{array}\right]$$

($F(I, \mathbb{R})$ is the linear space of all functions $I \to \mathbb{R}$)



SW: • $\mathcal{Y}$ is a linear operator.

• It is also <u>multiplicative</u>, ie.,

$$\mathcal{Y}(fg) = \mathcal{Y}(f)\, \mathcal{Y}(g).$$

• Applying $\mathcal{Y}$ twice is the same as doing nothing at all : $\mathcal{Y} \circ \mathcal{Y}(f) = f$.

(In other words, $\mathcal{Y}^{-1} = \mathcal{Y}$.)

- Let's identify the eigenvalues of $\mathcal{Y}$.

$\underbrace{\mathcal{Y}(f) = \lambda f}_{\left(\Leftrightarrow \text{ For any } x \in I: \; f(-x) = \lambda f(x)\right)}$, $\lambda \in \mathbb{R}$, $f: I \to \mathbb{R}$ is such that there is at least one $x_0 \in I$: $f(x_0) \neq 0$.

If $x_0 = 0$, $0 \neq f(0) = f(-0) = \lambda f(0)$

$$\Rightarrow (\lambda - 1) \underbrace{f(0)}_{\neq 0} = 0 \quad \Rightarrow \lambda = 1.$$

If $x_0 \neq 0$, $\left.\begin{array}{l} f(-x_0) = \lambda f(x_0) \\ f(x_0) = \lambda f(-x_0) \end{array}\right\} \Rightarrow f(x_0) = \lambda^2 f(x_0)$

$$\Rightarrow (\lambda^2 - 1) \underbrace{f(x_0)}_{\neq 0} = 0$$

$$\Rightarrow \lambda = \pm 1.$$

Thus $\mathcal{Y}$ can not have an eigenvalue different than ± 1. These two are eigenvalues of $\mathcal{Y}$ because we can find eigenfunctions for both of them, eg.

$f_1(x) := 1$, $\Rightarrow$ $\mathcal{Y}(f_1) = 1 \cdot f_1$

$f_{-1}(x) := x$ $\qquad \mathcal{Y}(f_{-1}) = (-1) \cdot f_{-1}$

Thus $\mathcal{Y}$ has precisely two eigenvalues: 1 and -1.

Rem: Earlier we looked at another operator, namely $-\Delta = -\partial_x^2$, and we discovered that it has infinitely many eigenvalues. In fact any real number is an eigenvalue of $-\Delta$.

Operator	# of (distinct) eigenvalues
$A \in Mat(2\times 2, \mathbb{R})$ (or $T_A : \binom{x}{y} \mapsto A\binom{x}{y}$)	1 or 2
$\mathcal{Y}$	2
$-\Delta$	∞

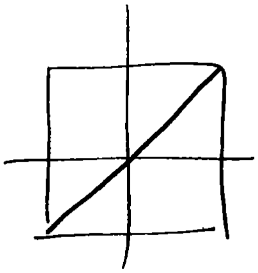
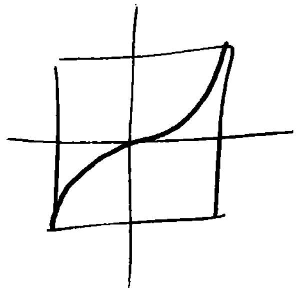
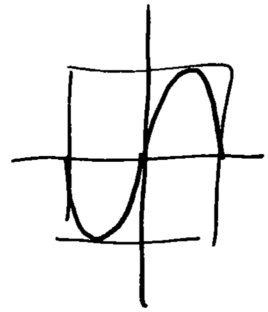
- $f \in F(I, \mathbb{R})$ is called <u>even</u> if it is an eigenfunction associated to 1

$$\left(\text{ie., } f \text{ is even} \Leftrightarrow f(-x) = f(x)\right)$$

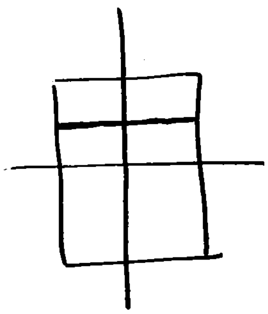
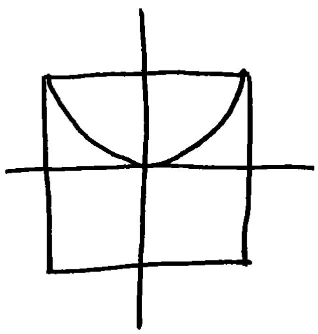
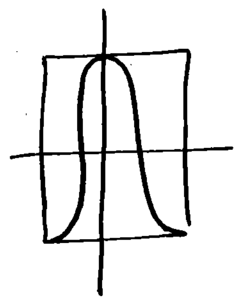
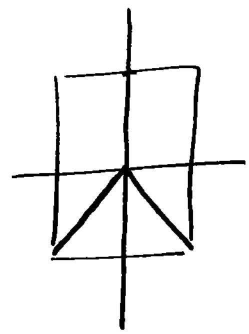
$f \in F(I, \mathbb{R})$ is called <u>odd</u> if it is an eigenfunction associated to -1

$$\left(\text{ie., } f \text{ is odd} \Leftrightarrow f(-x) = -f(x)\right)$$

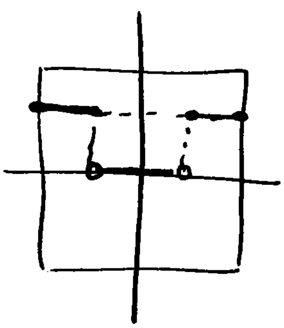
Ex: x^{2n+1} ($n \geq 1$ integer), $\sin(\omega x)$ ($\omega > 0$) are odd:

 ,  ,  , ...

Ex: 1, x^{2n} ($n \geq 1$ integer), $\cos(\omega x)$ ($\omega > 0$) are even:

 ,  ,  ,  ,

$-|x|$

 , ...

$$f(x) = \chi_{[-L,-\frac{L}{2}]}(x) + \chi_{[\frac{L}{2},L]}(x).$$

SW: (i) If $f: I \to \mathbb{R}$ is odd, then $f(0) = 0$

(but not vice versa)

(ii) The set of all even functions $I \to \mathbb{R}$ $F_e(I, \mathbb{R})$

is a linear space. So is the set of all odd functions $I \to \mathbb{R}$. $F_o(I, \mathbb{R})$

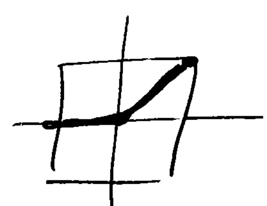
(iii) If $f, g \in F_e(I, \mathbb{R})$, then $fg \in F_e(I, \mathbb{R})$

If $f, g \in F_o(I, \mathbb{R})$, then $fg \in F_e(I, \mathbb{R})$

If $f \in F_e(I, \mathbb{R})$, $g \in F_o(I, \mathbb{R})$, then $fg \in F_o(I, \mathbb{R})$

(iv) If $f \in F_e(I, \mathbb{R})$ and $f \in F_o(I, \mathbb{R})$, then $f(x) = 0$ for any $x \in I$.

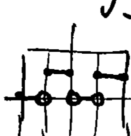
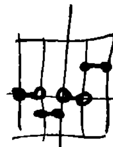
(ie. the only function that is both even and odd is the one that is constantly 0.)

(v) There are functions $f: I \to \mathbb{R}$ that are neither even nor odd. (eg. )

(vi) If $f \in F_e(I, \mathbb{R})$, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx = 2\int_{-L}^{0} f(x)\,dx$

If $f \in F_o(I, \mathbb{R})$, then $\int_{-L}^{L} f(x)\,dx = 0$.

(and piecewise continuous)

(but not vice versa, eg.  $\leftarrow \chi_{[-\frac{1}{2},0]}(x) + \chi_{[\frac{1}{2},1]}(x)$;  $\leftarrow -\chi_{[-\frac{1}{2},0]}(x) + \chi_{[\frac{1}{2},1]}(x)$

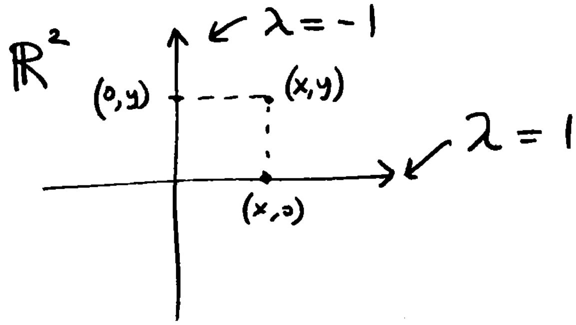
• Put $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in Mat(2\times 2, \mathbb{R})$. Then

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{Y} : F(I, \mathbb{R}) \longrightarrow F(I, \mathbb{R})$$
$$(x,y) \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad\qquad f(x) \longmapsto f(-x)$$

are spectrally very similar: both have 1 and -1 as their only eigenvalues. (Also $A^2 = I$, so $A^{-1} = A$).

For A, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector associated to 1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector associated to -1.

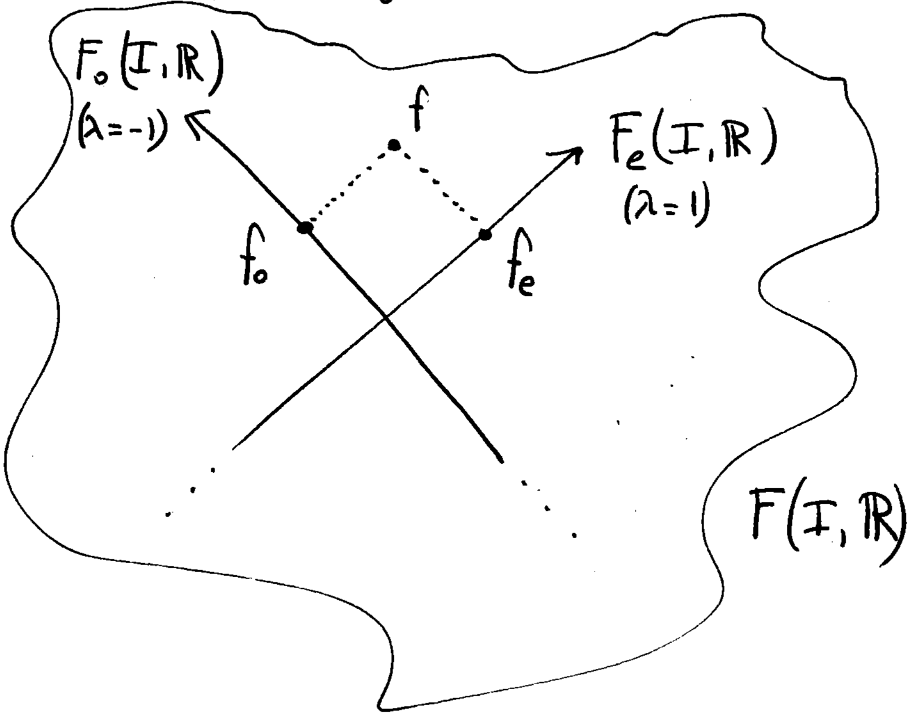
Thus any point $\begin{pmatrix} x \\ 0 \end{pmatrix}$ on the x-axis solves $A\begin{pmatrix} x \\ 0 \end{pmatrix} = 1\begin{pmatrix} x \\ 0 \end{pmatrix}$ and any point $\begin{pmatrix} 0 \\ y \end{pmatrix}$ on the y-axis solves $A\begin{pmatrix} 0 \\ y \end{pmatrix} = (-1)\begin{pmatrix} 0 \\ y \end{pmatrix}$:



Any point $\begin{pmatrix} x \\ y \end{pmatrix}$ on the plane can be written as the sum of an eigenvector associated to 1 and an eigenvector associated to -1:

$$(x,y) = (x,0) + (0,y).$$

Likewise for $\mathcal{Y}$ we have:



(This is a caritature, where each point represents a function $f: I \to \mathbb{R}$)

The natural question now is whether or not the caricature has some truth to it. More precisely, is it the case that any function $f: I \to \mathbb{R}$ can be written as the sum of an even function $f_e: I \to \mathbb{R}$ and an odd function $f_o: I \to \mathbb{R}$:

$$f = f_e + f_o \qquad ?$$

The answer is: yes. Define the "projections"

$$\begin{aligned} \mathcal{P}_e : F(I,\mathbb{R}) &\longrightarrow F_e(I,\mathbb{R}) \\ f &\longmapsto f_e = \tfrac{1}{2}\left(f + \mathcal{Y}(f)\right) \\ \mathcal{P}_o : F(I,\mathbb{R}) &\longrightarrow F_o(I,\mathbb{R}) \\ f &\longmapsto f_o = \tfrac{1}{2}\left(f - \mathcal{Y}(f)\right) \end{aligned}$$

For these to be well-defined, we need to verify that for any $f \in F(I,\mathbb{R})$: $\mathcal{P}_e(f)$ is even and $\mathcal{P}_o(f)$ is odd.

(<u>SW</u>: $\mathcal{P}_e$ and $\mathcal{P}_o$ are linear.)

$$\mathcal{Y} \circ \mathcal{P}_e(f)(x) = \mathcal{P}_e(f)(-x) = \frac{1}{2}\left(f(-x) + \mathcal{Y}(f)(-x)\right) = \frac{1}{2}\left(f(-x) + f(x)\right) = \mathcal{P}_e(f)(x)$$

$$\Rightarrow \mathcal{Y}\left(\mathcal{P}_e(f)\right) = 1 \cdot \mathcal{P}_e(f), \checkmark$$

$$\left(\text{or: } \mathcal{Y}\left(\mathcal{P}_e(f)\right) = \mathcal{Y}\left(\frac{1}{2}\left(f + \mathcal{Y}(f)\right)\right) \underset{\substack{\uparrow \\ \mathcal{Y} \text{ is linear}}}{=} \frac{1}{2}\left(\mathcal{Y}(f) + \mathcal{Y}^2(f)\right) = \mathcal{P}_e(f). \right)$$

$$\mathcal{Y} \circ \mathcal{P}_o(f)(x) = \mathcal{P}_o(f)(-x) = \frac{1}{2}\left(f(-x) - \mathcal{Y}(f)(-x)\right) = \frac{1}{2}\left(f(-x) - f(x)\right)$$

$$= -\frac{1}{2}\left(f(x) - f(-x)\right) = -\frac{1}{2}\left(f(x) - \mathcal{Y}(f)(x)\right) = -\mathcal{P}_o(f)(x)$$

$$\Rightarrow \mathcal{Y}\left(\mathcal{P}_o(f)\right) = (-1) \cdot \mathcal{P}_o(f), \checkmark$$

$$\left(\text{or: } \mathcal{Y}\left(\mathcal{P}_o(f)\right) = \mathcal{Y}\left(\frac{1}{2}\left(f - \mathcal{Y}(f)\right)\right) \underset{\substack{\uparrow \\ \mathcal{Y} \text{ is linear}}}{=} \frac{1}{2}\left(\mathcal{Y}(f) - \mathcal{Y}^2(f)\right) = -\mathcal{P}_o(f) \right)$$

Thus for any $f \in F(I,\mathbb{R})$: $f_e = \mathcal{P}_e(f)$ is even and $f_o = \mathcal{P}_o(f)$ is odd. Also their sum give f back:

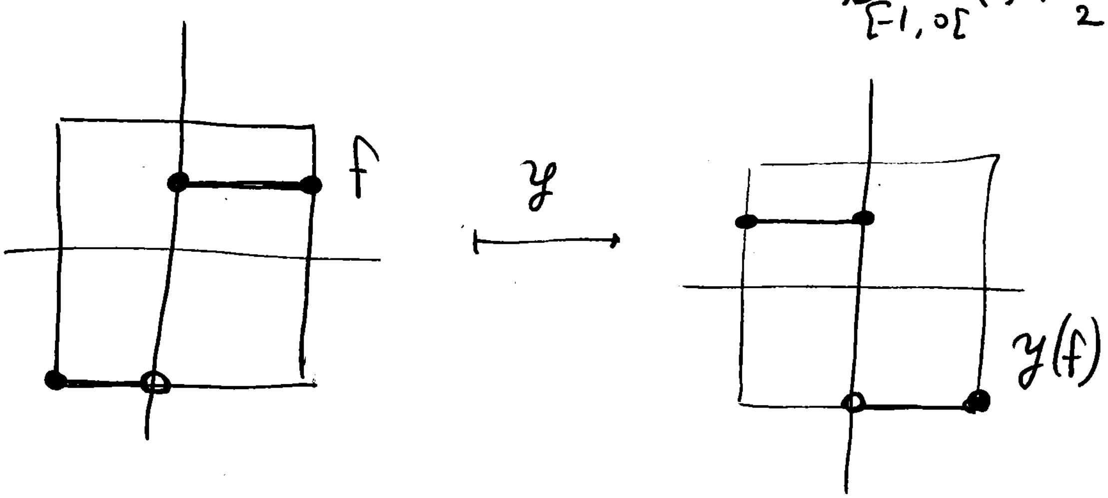
$$f_e + f_o = \frac{1}{2}\left(f + \mathcal{Y}(f)\right) + \frac{1}{2}\left(f - \mathcal{Y}(f)\right) = f.$$

SW : • This even / odd decomposition is unique, ie., if $g \in F_e(I,\mathbb{R})$ and $h \in F_o(I,\mathbb{R})$ are such that $g+h = f$, then $g = f_e$ and $h = f_o$.

• $\mathcal{P}_e \circ \mathcal{P}_e = \mathcal{P}_e$, $\mathcal{P}_o \circ \mathcal{P}_o = \mathcal{P}_o$.

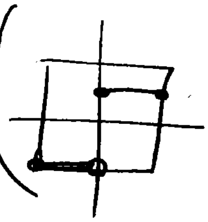
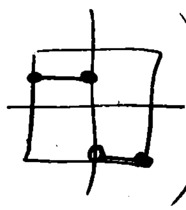
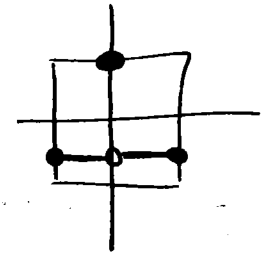
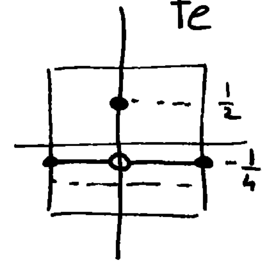
Ex :

$$f : [-1,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto -\chi_{[-1,0[}(x) + \frac{1}{2}\chi_{[0,1]}(x)$$



(f is neither even nor odd.)

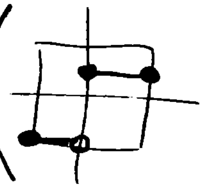
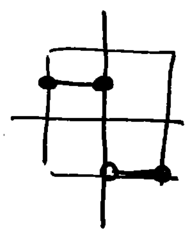
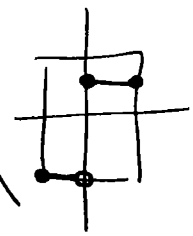
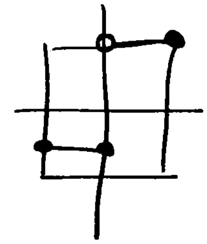
$\gamma_e(f) = \frac{1}{2}\left(f + y(f)\right)$

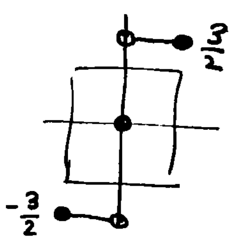
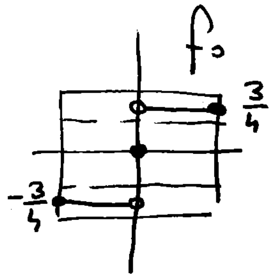
$= \frac{1}{2}\Bigg($  $+$  $\Bigg) = \frac{1}{2}$  $=$ 

f_e

$= -\frac{1}{4}\,\chi_{[-1,0[} + \frac{1}{2}\,\chi_{\{0\}} - \frac{1}{4}\,\chi_{]0,1]}.$

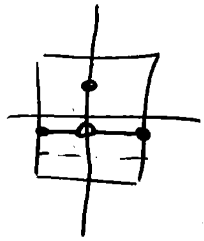
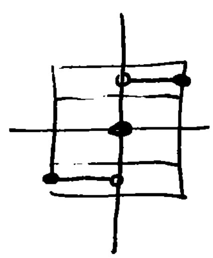
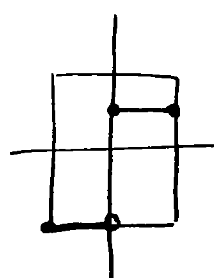
$\gamma_o(f) = \frac{1}{2}\left(f - y(f)\right)$

$= \frac{1}{2}\Bigg($  $-$  $\Bigg) = \frac{1}{2}\Bigg($  $+$  $\Bigg)$

$= \frac{1}{2}$  $=$ 

f_o

$= -\frac{3}{4}\,\chi_{[-1,0[} + \frac{3}{2}\,\chi_{]0,1]}.$

$f_e + f_o =$  $+$  $=$ 

f

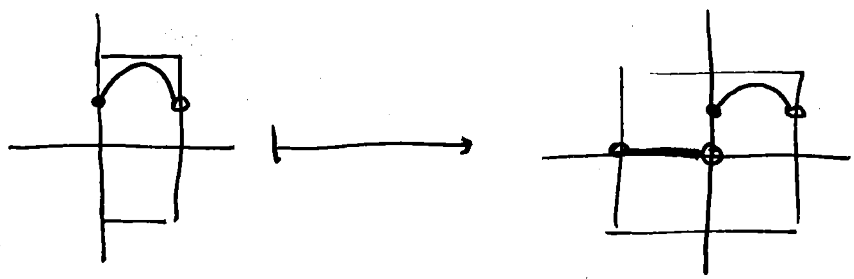
, ✓.

Ex: $\left.\begin{aligned} \cosh(\theta) &= \frac{e^{\theta} + e^{-\theta}}{2} \\ \sinh(\theta) &= \frac{e^{\theta} - e^{-\theta}}{2} \end{aligned}\right\} \Rightarrow e^{\theta} = \underbrace{\cosh(\theta)}_{\text{even}} + \underbrace{\sinh(\theta)}_{\text{odd}}.$

• So far we were dealing with functions that are defined on intervals centered at 0 that are symmetric. Using indicator functions we can apply the machinery we developed to functions defined on arbitrary examples, eg.,

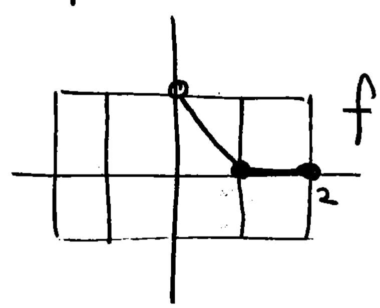
$$F([0,L[,\mathbb{R}) \longrightarrow F(]-L,L[,\mathbb{R})$$

$$f(x) \longmapsto \tilde{f}(x) = f(x)\,\chi_{[0,L[}(x)$$

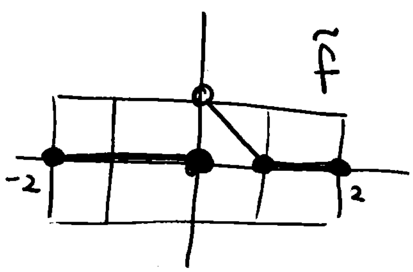


<u>Ex</u>: $f :]0,2] \longrightarrow \mathbb{R}$

$$x \longmapsto (1-x)\,\chi_{]0,1]}(x)$$



$\Rightarrow$



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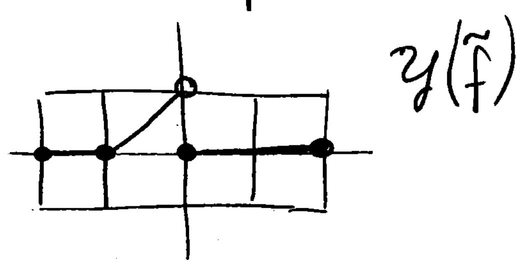
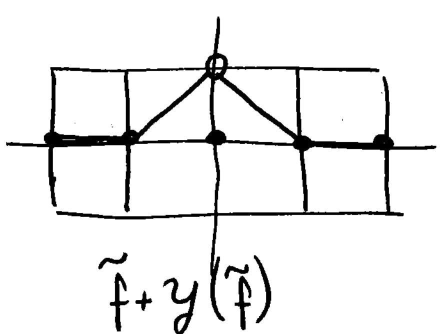
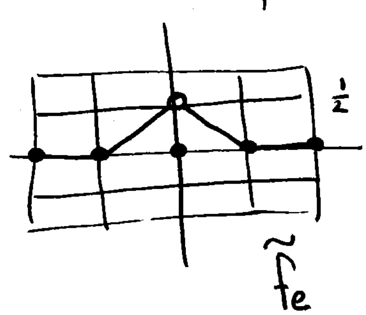
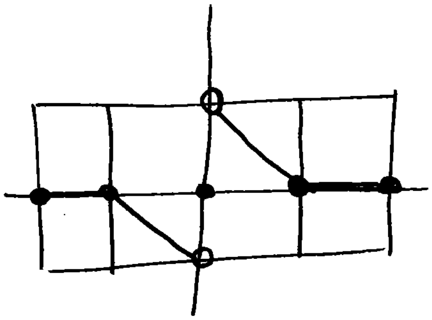


$\Rightarrow$



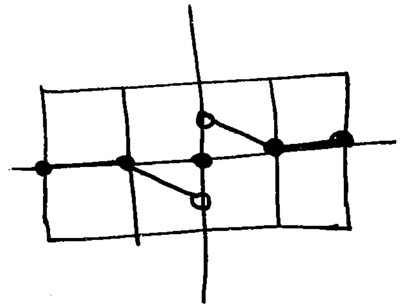
$\Rightarrow$





$\tilde{f} - y(\tilde{f})$

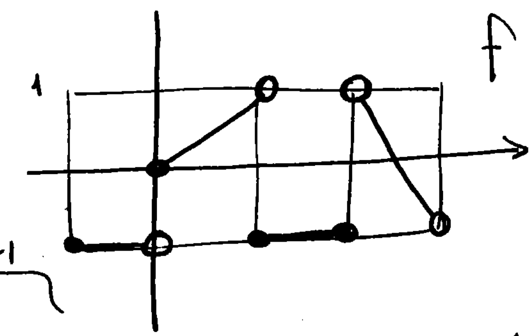
$\Rightarrow$



$\tilde{f}_o$

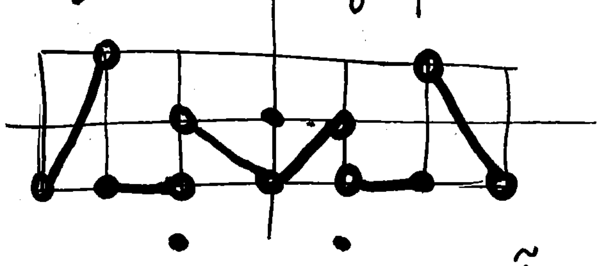
$\tilde{f} + y(\tilde{f})$ is called the <u>even (periodic) extension</u> of f, and $\tilde{f} - y(\tilde{f})$ is called the <u>odd (periodic) extension</u> of f. ("(periodic)" will make sense later)

<u>SW</u>: Find the even and odd (periodic) extensions of $f : [-1, 3[\longrightarrow \mathbb{R}$

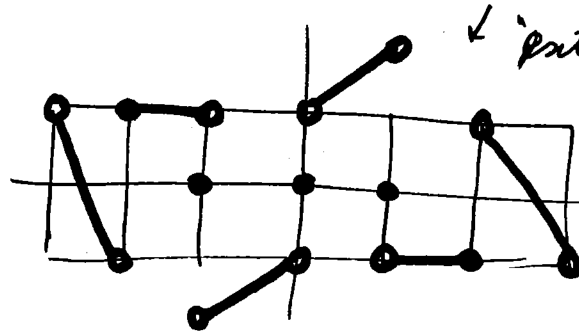


$$f : x \mapsto -\chi_{[-1,0[}(x) + x\,\chi_{[0,1[}(x) - \chi_{[1,2]}(x) + (-2x+5)\cdot\chi_{[2,3[}(x)$$

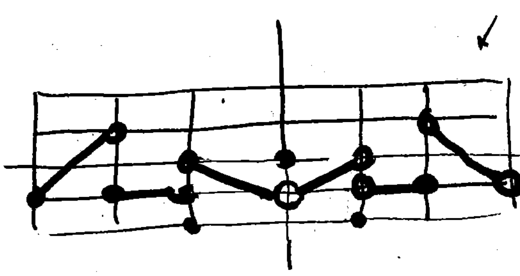
even (periodic) "extension" of f



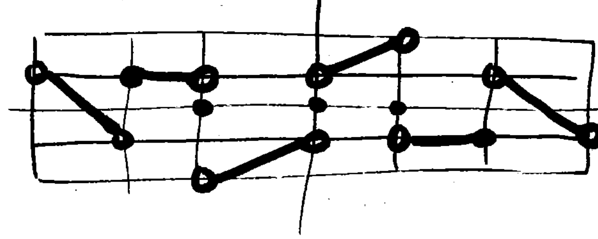
odd (periodic) "extension" of f



$\tilde{f}_e$



$\tilde{f}_o$



§10.2:

- Let $T>0$. $f:\mathbb{R}\to\mathbb{R}$ is a <u>periodic function</u> with <u>period T</u> (or: <u>T-periodic</u>) if

$$f(x-T)=f(x) \quad \left(\Leftrightarrow \quad f(x)=f(x+T)\right)$$

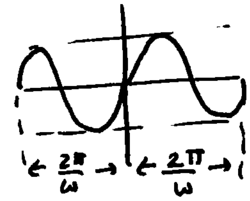
<u>Rem</u>: A T-periodic function is the "same" as a function $\left]-\frac{T}{2},\frac{T}{2}\right] \to \mathbb{R}$.

<u>SW</u>: Reformulate this definition using the "shift" operator S_T (which first needs to be defined).

- If f is T-periodic, then it is also nT-periodic for any $n \in \{1,2,\dots\}$.

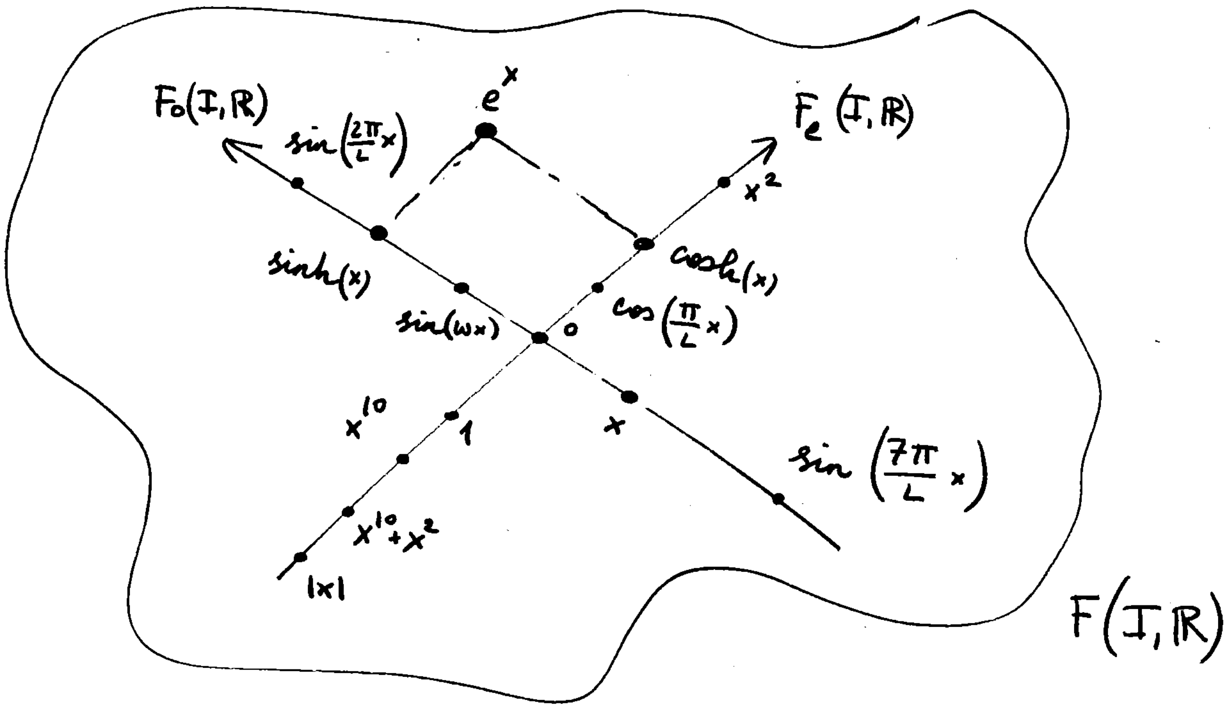
- If f is periodic, then the smallest $T>0$ for which it is T-periodic is called the <u>fundamental period</u> of f.

<u>SW</u>:
- There are periodic functions that have no fundamental period, e.g. $f(x)=1$.
- If $\omega>0$, then both $\sin(\omega x)$ and $\cos(\omega x)$ are periodic with fundamental period $\frac{2\pi}{\omega}$:



- If $\omega>0$, $e^{i\omega\theta}$ is periodic with fundamental period $\frac{2\pi}{\omega}$.

• Fix $L > 0$ and take $I := [-L, L]$ or $]-L, L[$ as before. We would like to make our earlier caricature more realistic by understanding the "shapes" of $F_e(I,\mathbb{R})$ and $F_o(I,\mathbb{R})$. Recall that earlier we represented both of these linear spaces as lines:



It would be very optimistic to expect that all these functions line up like this. However we can still quantify the "norm" of a function (ie., how "far away" it is from 0) and the "angle" between two functions by adapting the static (2-dimensional, say) versions of these to these function spaces.

Define the <u>inner product</u> (or <u>dot product</u>) on $\mathbb{R}^2$ by:

$$\begin{aligned} \mathrm{INN}: \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ ((x,y),(z,t)) &\longmapsto xz+yt =: \langle (x,y),(z,t) \rangle = (x,y) \bullet (z,t) \end{aligned}$$

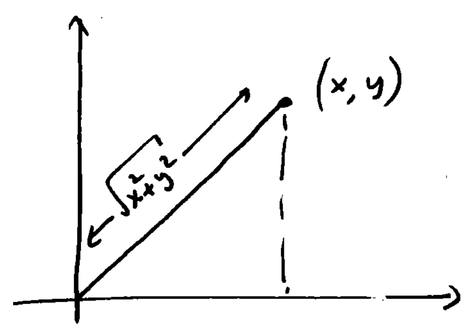
or in matrix notation:

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix} \right\rangle = (x\ y)\begin{pmatrix} z \\ t \end{pmatrix} = xz+yt$$

<u>SW</u>: Describe $\mathrm{MUL}: \mathrm{Mat}(2\times 2,\mathbb{R}) \times \mathrm{Mat}(2\times 2,\mathbb{R}) \to \mathrm{Mat}(2\times 2,\mathbb{R})$ in terms of $\mathrm{INN}: \mathrm{Mat}(2\times 1,\mathbb{R}) \times \mathrm{Mat}(2\times 1,\mathbb{R}) \to \mathrm{Mat}(1\times 1,\mathbb{R})$.

- Observe that the inner product of a vector by itself is the square of its distance from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = (x\ y)\begin{pmatrix} x \\ y \end{pmatrix} = x^2+y^2.$$



We call $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := \sqrt{\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle}$

the <u>norm</u> of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

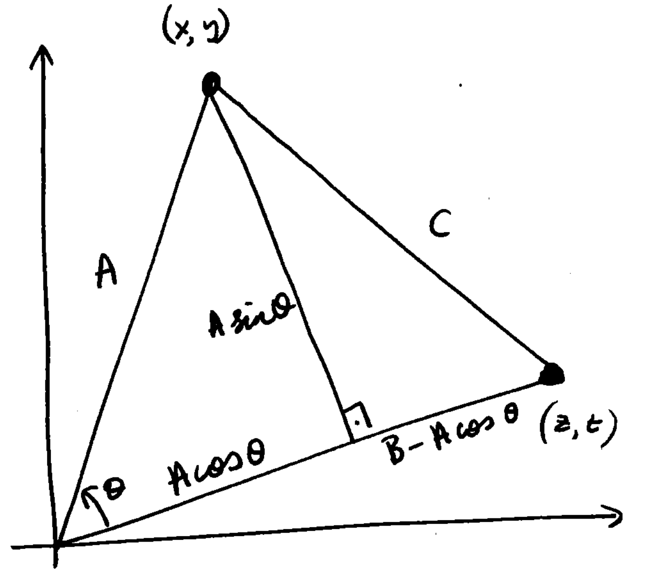
<u>SW</u>: If $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix} \in \mathbb{R}^2$, $d\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right) := \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} z \\ t \end{pmatrix} \right\|$ gives the <u>distance</u> between $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} z \\ t \end{pmatrix}$.

SW: Let $(x,y), (z,t) \in \mathbb{R}^2$. Then

$$\langle (x,y),(z,t)\rangle = \|(x,y)\| \|(z,t)\| \cos\theta,$$

where θ is <u>the angle</u> between $(x,y), (z,t)$

(Actually, one of the angles, but $\cos(360-\theta) = \cos\theta$ anyway)



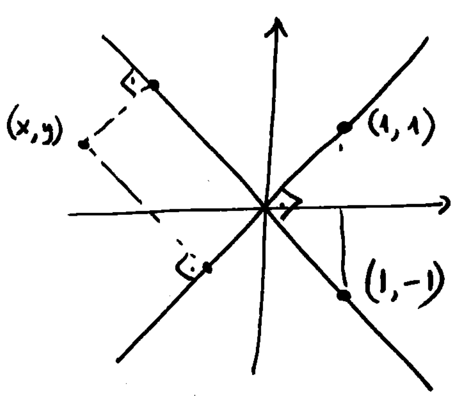
$A := \|(x,y)\|$ $\quad$ $C := d((x,y),(z,t))$

$B := \|(z,t)\|$

- $(x,y), (z,t) \in \mathbb{R}^2$, $(x,y) \neq (0,0) \neq (z,t)$ are <u>orthogonal</u> if $\langle (x,y),(z,t)\rangle = 0$.

<u>Ex</u>: $(1,1), (1,-1) \in \mathbb{R}^2$.

$\Rightarrow \langle (1,1),(1,-1)\rangle = 0 \Rightarrow (1,1)$ and $(1,-1)$ are orthogonal.



If $(x,y) \in \mathbb{R}^2$ is arbitrary,

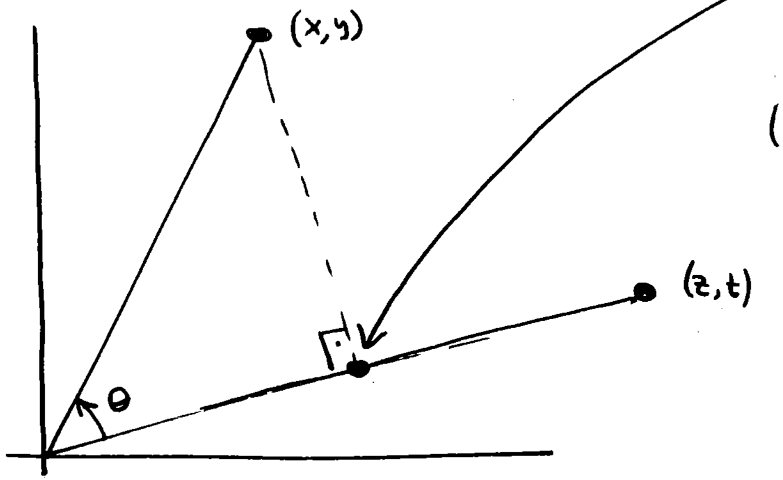
$\langle (x,y),(1,1)\rangle = x+y$, $\|(1,1)\| = \sqrt{2}$

$\langle (x,y),(1,-1)\rangle = x-y$, $\|(1,-1)\| = \sqrt{2}$.

$$\frac{\langle (x,y),(1,1)\rangle}{\|(1,1)\|^2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{\langle (x,y),(1,-1)\rangle}{\|(1,-1)\|^2}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{x+y}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{x-y}{2}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}x+y+x-y\\x+y-x+y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow \boxed{\begin{pmatrix}x\\y\end{pmatrix} = \frac{\langle (x,y),(1,1)\rangle}{\langle (1,1),(1,1)\rangle}\begin{pmatrix}1\\1\end{pmatrix} + \frac{\langle (x,y),(1,-1)\rangle}{\langle (1,-1),(1,-1)\rangle}\begin{pmatrix}1\\-1\end{pmatrix}}$$

SW: (i) Let $(x,y), (z,t) \in \mathbb{R}$. Then this point (ie., the orthogonal projection of (x,y) onto the line cut out by (z,t)) is:



$$\frac{\langle (x,y), (z,t) \rangle}{\langle (z,t), (z,t) \rangle} \begin{pmatrix} z \\ t \end{pmatrix}.$$

(ii) Let $(v_1, v_2), (w_1, w_2) \in \mathbb{R}^2$ be orthogonal (ie., $\langle (v_1,v_2), (w_1,w_2) \rangle = 0$). $(v_1, v_2) \neq (0,0) \neq (w_1, w_2)$.

Then for any $(x,y) \in \mathbb{R}^2$:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\langle (x,y), (v_1,v_2) \rangle}{\langle (v_1,v_2), (v_1,v_2) \rangle} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \frac{\langle (x,y), (w_1,w_2) \rangle}{\langle (w_1,w_2), (w_1,w_2) \rangle} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}$$

- The point of all this is that orthogonal sets are nifty coordinate systems for linear spaces.

- For the matrix replica $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathcal{Y}$:

$$\langle (1,0), (0,1) \rangle = 0 \quad , \langle (1,0), (1,0) \rangle = 1 \quad , \langle (0,1), (0,1) \rangle = 1$$

$$\langle (x,y), (1,0) \rangle = x \quad , \quad \langle (x,y), (0,1) \rangle = y.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\langle (x,y), (1,0) \rangle}{\langle (1,0), (1,0) \rangle} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\langle (x,y), (0,1) \rangle}{\langle (0,1), (0,1) \rangle} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{x}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{y}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

SW: Just because a matrix has distinct eigenvalues, it doesn't mean that the associated eigenvectors are orthogonal, eg.

$\begin{pmatrix} 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \end{pmatrix}$

• It is time to adapt the notion of an inner product to the function space $F(I, \mathbb{R})$. If we fix $(z,t) \in \mathbb{R}^2$ (say, for instance, because we would like to project vectors orthogonally onto the line cut out by it), then "taking inner product against (z,t)" becomes a function

$$\begin{aligned} \langle \bullet, (z,t) \rangle : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \langle (x,y), (z,t) \rangle \end{aligned}$$

<u>SW</u>: $\langle \bullet, (z,t) \rangle$ is linear.

Inspired by this (and also recalling that earlier we mentioned that a function f can be interpreted as a functional as "integrate against f"), we define the <u>inner product</u> for functions as:

$$\begin{aligned} \mathrm{INN} : R(I, \mathbb{R}) \times R(I, \mathbb{R}) &\longrightarrow \mathbb{R} \\ (f, g) &\longmapsto \langle f, g \rangle := \int_{-L}^{L} f(x) g(x) \, dx \end{aligned}$$

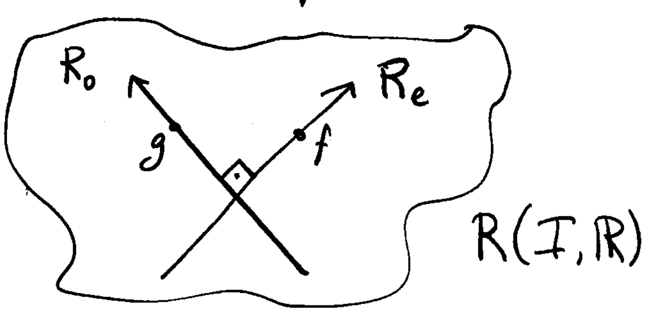
In order for this to work we only allow those functions $f : I \to \mathbb{R}$ for which "$\int_{-L}^{L} f(x) dx$" makes sense, i.e., those functions that are Riemann-integrable (hence the letter R).

For our purposes we may think of $R(I, \mathbb{R})$ as the linear space of all bounded piecewise continuous functions $I \to \mathbb{R}$. Everything we discovered about $\mathcal{Y}$ holds if we replace $F(I, \mathbb{R})$ with $R(I, \mathbb{R})$ (the benefit of this replacement being that now integration is admissible). All the terminology from the static case carries over to the dynamic case.

- If $f \in R_e(I, \mathbb{R})$ and $g \in R_o(I, \mathbb{R})$ (so that f is an even bounded p.w. continuous function and g is an odd bounded p.w. continuous function), then

$$\langle f, g \rangle = \int_{-L}^{L} \underbrace{f(x) g(x)}_{\text{odd}} \, dx = 0.$$

Thus even and odd functions are orthogonal:

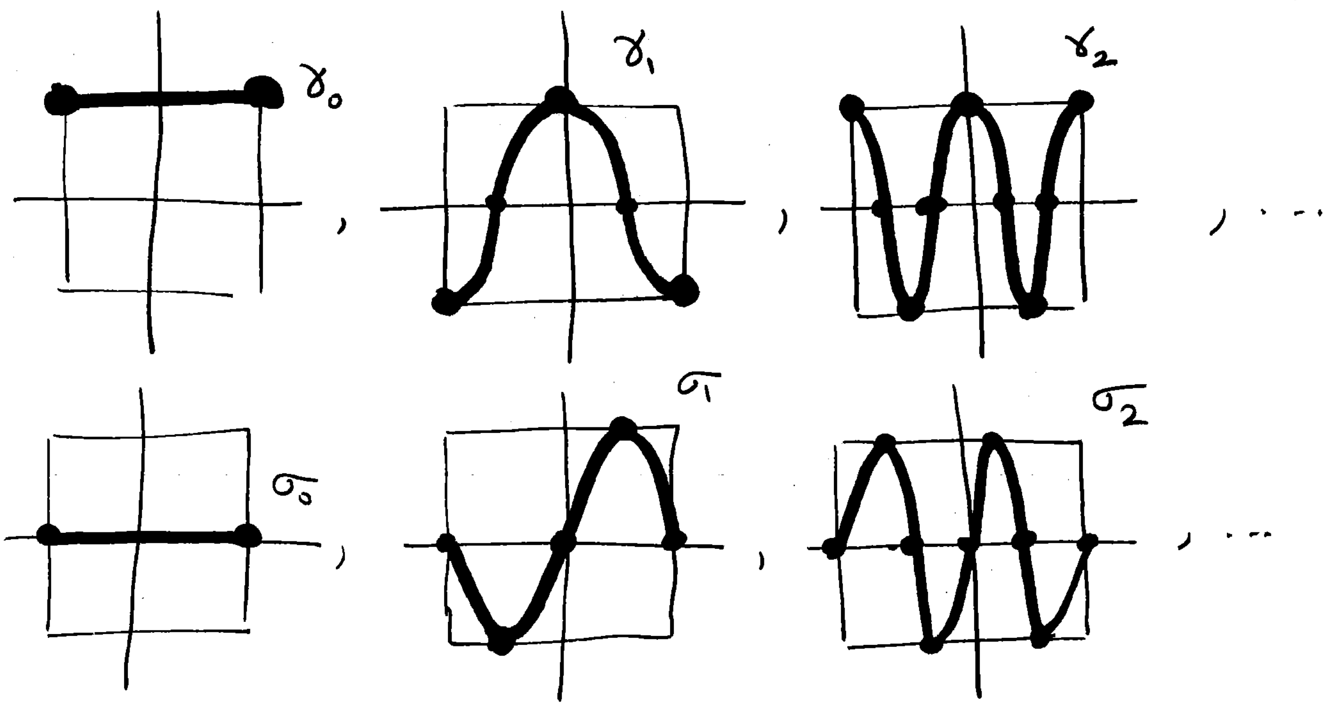


SW: Pythagorean

$\|e^x\|^2 = \|\cosh(x)\|^2 + \|\sinh(x)\|^2$

• Define (for brevity) for all $n \in \mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$:

$$\mathcal{E}_n(x) := e^{i\frac{n\pi}{L}x}, \quad \gamma_n(x) := \cos\left(\frac{n\pi}{L}x\right), \quad \sigma_n(x) := \sin\left(\frac{n\pi}{L}x\right)$$



Here are the standard formulas:

$$\mathcal{E}_n = \gamma_n + i\sigma_n, \quad \gamma_{-n} = \gamma_n, \quad \gamma_n = \frac{1}{2}(\mathcal{E}_n + \mathcal{E}_{-n}), \quad \mathcal{E}_0 = 1 = \gamma_0$$

$$\mathcal{E}_{-n} = \gamma_n - i\sigma_n, \quad \sigma_{-n} = -\sigma_n, \quad \sigma_n = \frac{1}{2i}(\mathcal{E}_n - \mathcal{E}_{-n}), \quad \sigma_0 = 0.$$

$$\mathcal{E}_{n+m} = \mathcal{E}_n\mathcal{E}_m, \quad \gamma_{n+m} = \gamma_n\gamma_m - \sigma_n\sigma_m, \quad \sigma_{n+m} = \gamma_n\sigma_m + \sigma_n\gamma_m.$$

• $\gamma_0, \gamma_1, \gamma_2, \ldots \in R_e(I, \mathbb{R})$ and $\sigma_1, \sigma_2, \ldots \in R_0(I, \mathbb{R})$. Shortly we (you) will verify that $\{\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \sigma_3, \gamma_3, \ldots\}$ is an orthogonal set, and as a result we can use these trigonometric functions to upgrade our caricature to a higher resolution picture.

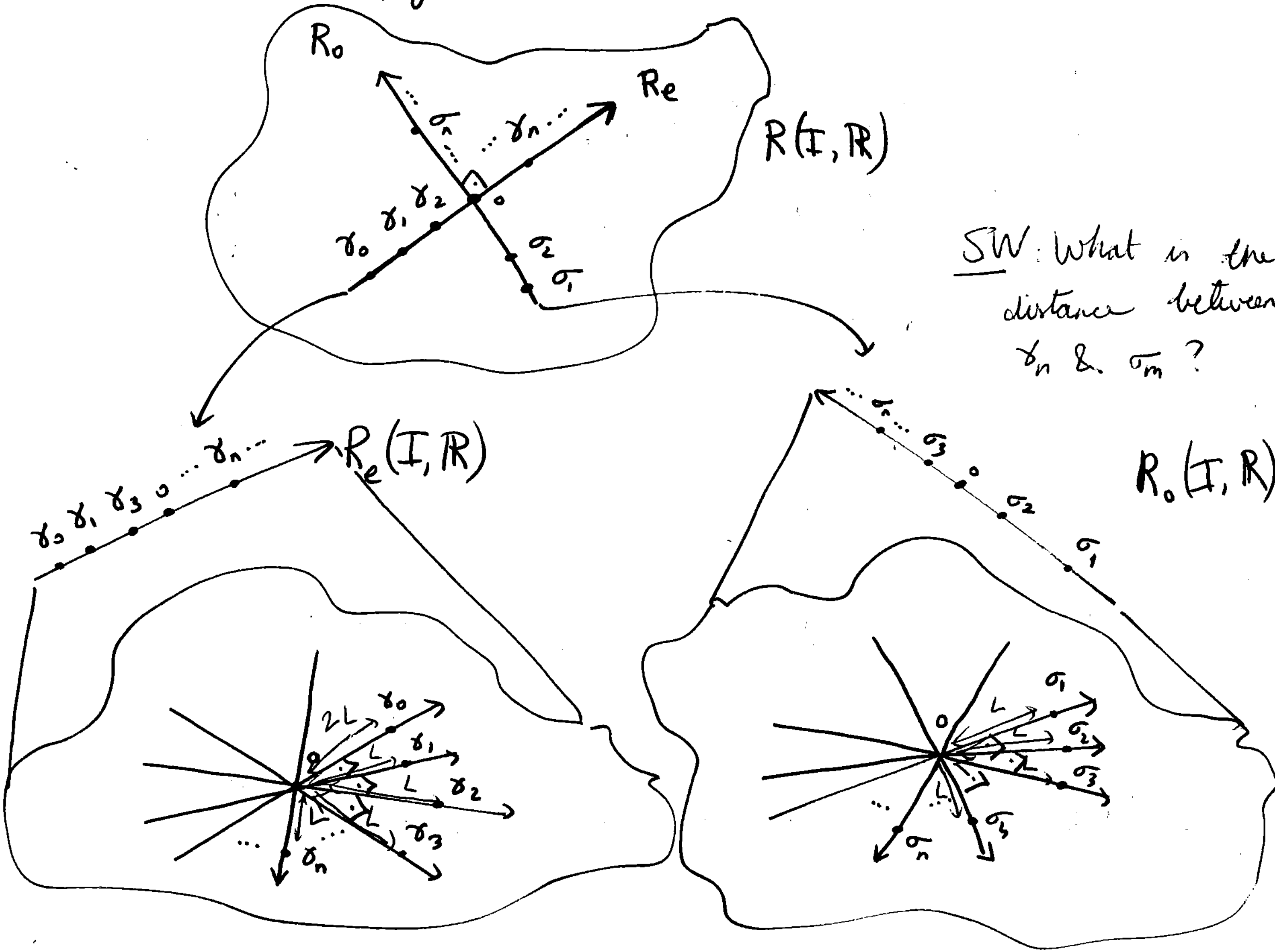
SW (i) Verify:

$$\int_{-L}^{L} \mathcal{E}_n(x)\,dx = \begin{Bmatrix} 2L, \text{ if } & n=0 \\ 0, \text{ if } & n\neq 0 \end{Bmatrix}, \quad \langle \gamma_n, \sigma_m \rangle = 0,$$

$$\langle \gamma_n, \gamma_m \rangle = \begin{Bmatrix} 2L, \text{ if } & n=m=0 \\ L, \text{ if } & n=m\neq 0 \\ 0, \text{ if } & n\neq m \end{Bmatrix}, \quad \langle \sigma_n, \sigma_m \rangle = \begin{Bmatrix} L, \text{ if } & n=m \\ 0, \text{ if } & n\neq m \end{Bmatrix}$$

(ii) Use indicator functions to write the RHS's.

- We now can upgrade our caricature:



SW: What is the distance between γ_n & σ_m?

Thus $R_e(I, \mathbb{R})$ is a linear space with infinitely many coordinate axes that are orthogonal to eachother; and similarly for $R_o(I, \mathbb{R})$. What is more, any coordinate axis in $R_e(I, \mathbb{R})$ is perpendicular to any coordinate axis in $R_o(I, \mathbb{R})$.

What is more surprising is that the list

$$\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \sigma_3, \gamma_3, \dots, \sigma_n, \gamma_n, \dots$$

misses no coordinate axis of $R(I, \mathbb{R})$!

In other words, $\{\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \dots, \gamma_n, \sigma_{n+1}, \dots\}$ is a <u>complete orthogonal set</u>.

(This last statement we'll take for granted.)

- In the static case, we saw that if $\{V_1, \dots, V_d\} \subseteq \mathbb{R}^d$ is an orthogonal set then

$$\text{for any } X \in \mathbb{R}^d: \quad X = \sum_{k=1}^{d} \frac{\langle X, V_k \rangle}{\langle V_k, V_k \rangle} V_k .$$

A similar statement holds for the dynamic case, except since now we have infinitely many coordinates the sum may fail to be finite.

Def: Let $f \in R(\mathbb{R},\mathbb{R})$ be $2L$-periodic (or, equivalently, $f \in R(I,\mathbb{R})$). Put

for all $n \in \mathbb{Z}$: $e_n := \dfrac{\langle f, \mathcal{E}_{-n}\rangle}{\langle \mathcal{E}_n, \mathcal{E}_{-n}\rangle} = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\, e^{-\frac{in\pi}{L}}\, dx$,

$$c_0 := 2\,\frac{\langle f, \gamma_0\rangle}{\langle \gamma_0, \gamma_0\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

for all $n \geq 1$: $c_n := \dfrac{\langle f, \gamma_n\rangle}{\langle \gamma_n, \gamma_n\rangle} = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$,

$$s_n := \frac{\langle f, \sigma_n\rangle}{\langle \sigma_n, \sigma_n\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Then

$$\mathcal{F}_{\mathbb{C}}(f)(x) := \sum_{n\in\mathbb{Z}} \frac{\langle f, \mathcal{E}_{-n}\rangle}{\langle \mathcal{E}_n, \mathcal{E}_{-n}\rangle}\mathcal{E}_n(x) = \sum_{n\in\mathbb{Z}} e_n\,\mathcal{E}_n(x) = \boxed{\sum_{n\in\mathbb{Z}} e_n\, e^{i\frac{n\pi}{L}x}}$$ and

$$\mathcal{F}_{\mathbb{R}}(f)(x) := \sum_{n\geq 0}\frac{\langle f, \gamma_n\rangle}{\langle \gamma_n, \gamma_n\rangle}\gamma_n(x) + \sum_{n\geq 1}\frac{\langle f, \sigma_n\rangle}{\langle \sigma_n, \sigma_n\rangle}\sigma_n(x) = \frac{c_0}{2} + \sum_{n\geq 1} c_n\gamma_n(x) + \sum_{n\geq 1} s_n\sigma_n(x)$$

$$= \boxed{\frac{c_0}{2} + \sum_{n\geq 1} c_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n\geq 1} s_n \sin\left(\frac{n\pi}{L}x\right)}$$ are called

the <u>complex</u> and <u>real</u> <u>Fourier series</u> of f, respectively.

e_n's are the <u>complex Fourier coefficients</u> of f and c_n's and s_n's are the <u>real Fourier coefficients</u> of f.

SW: (i) Consider $\boxed{\sum_{n \in \mathbb{Z}} \tilde{e}_n \varepsilon_n}$ (*) and

$$\boxed{\frac{\tilde{c}_0}{2} + \sum_{n \geq 1} \tilde{c}_n \gamma_n + \sum_{n \geq 1} \tilde{s}_n \sigma_n} \quad (**)$$

Verify the conversion formulas:

$$\boxed{(*) = (**)} \Leftrightarrow \boxed{\begin{array}{l} \tilde{c}_0 = 2\tilde{e}_0 \\ \tilde{c}_n = \tilde{e}_n + \tilde{e}_{-n} \\ \tilde{s}_n = i(\tilde{e}_n - \tilde{e}_{-n}) \end{array}} \Leftrightarrow \boxed{\begin{array}{l} \tilde{e}_0 = \frac{\tilde{c}_0}{2} \\ \tilde{e}_n = \begin{cases} \frac{1}{2}(\tilde{c}_n - i s_n), \text{ if } n \geq 1 \\ \frac{1}{2}(\tilde{c}_{-n} + i s_{-n}), \text{ if } n \leq -1 \end{cases} \end{array}}$$

(ii) Using these formulas, derive the real Fourier series of f from its complex Fourier series (and vice versa).

(iii) If $f \in R_e(I, \mathbb{R})$, then for any $n \geq 1$: $s_n = 0$.

If $f \in R_o(I, \mathbb{R})$, then for any $n \geq 0$: $c_n = 0$.

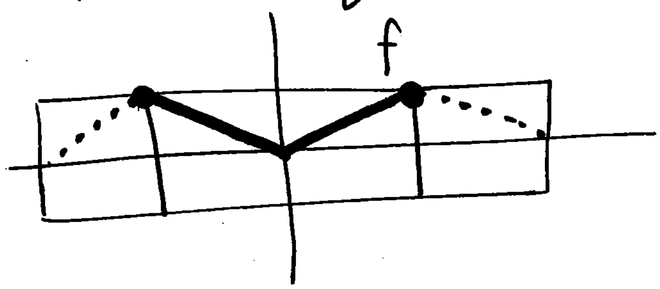
If $f \in R(I, \mathbb{R})$, then

$$\underbrace{\frac{c_0}{2} + \sum_{n \geq 1} c_n \gamma_n}_{\substack{\text{real Fourier} \\ \text{series of } f_e = r_e(f)}} + \underbrace{\sum_{n \geq 1} s_n \sigma_n}_{\substack{\text{real Fourier} \\ \text{series of } f_o = r_o(f).}} \quad \left(\begin{array}{c}\text{real Fourier} \\ \text{series of } f\end{array}\right)$$

- Observe that for now we are keeping a function f and its Fourier series separate.

<u>Ex</u>: Find the (real) Fourier coefficients of

$$f : [-2,2] \longrightarrow \mathbb{R}$$
$$x \longmapsto |x|$$



f is even $\Rightarrow$ $s_n = 0$.

$$\frac{c_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \underset{\substack{\uparrow \\ L=2}}{=} \frac{1}{4}\, 2\int_0^2 |x|\,dx = \frac{1}{2}\int_0^2 x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^2 = \frac{1}{2}\,2 = 1.$$

$(n \geq 1)$ $$c_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}\int_{-2}^{2} \underbrace{|x|\cos\left(\frac{n\pi}{2}x\right)}_{\text{even}}dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx \underset{\uparrow}{=} \frac{2}{n\pi}\left[x\sin\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx$$

$$\begin{pmatrix} u = x & dv = \cos\left(\frac{n\pi}{2}x\right) \\ du = dx & v = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right) \end{pmatrix}$$

$$= \frac{2}{n\pi}\left(2\underbrace{\sin(n\pi) - 0}_{=0}\right) + \left(\frac{2}{n\pi}\right)^2\left[\cos\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 = \left(\frac{2}{n\pi}\right)^2\left(\cos(n\pi) - 1\right)$$

$$= \begin{cases} -2\left(\frac{2}{n\pi}\right)^2, & \text{if } \cos(n\pi) = -1 \\ 0, & \text{if } \cos(n\pi) = 1 \end{cases} = \begin{cases} -\frac{8}{(n\pi)^2}, & \text{if } n\pi = \pi, 3\pi, 5\pi, \dots \\ 0, & \text{if } n\pi = 2\pi, 4\pi, \dots \end{cases}$$

$$= \begin{cases} -\frac{8}{\pi^2} \cdot \frac{1}{n^2}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

$\Rightarrow$ The Fourier series of f is:

$$\frac{c_0}{2} + \sum_{n \geq 1} c_n \gamma_n(x) + \sum_{n \geq 1} \underbrace{s_n}_{=0} \sigma_n(x)$$

$$= 1 - \frac{8}{\pi^2} \sum_{\substack{n \geq 1 \\ n: \text{odd}}} \frac{1}{n^2} \gamma_n(x) = \boxed{1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2} x\right)}.$$

$$= \mathcal{F}_{\mathbb{R}}(f)(x)$$

Obs: If $f(x_0) = \mathcal{F}_{\mathbb{R}}(f)(x_0)$ for $x_0 = 0$, we would have:

$$0 = f(x_0) = \mathcal{F}_{\mathbb{R}}(f)(x_0) = 1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n \geq 0} \frac{1}{(2n+1)^2} \Rightarrow \boxed{\pi = \sqrt{8 \sum_{n \geq 0} \frac{1}{(2n+1)^2}}}$$

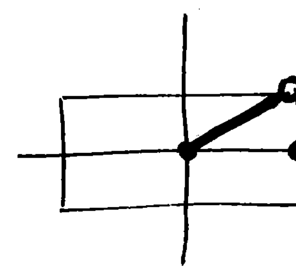
(This is the case, but we'll find this out later.)

SW: (i) Compute the coefficients of $\mathcal{F}_{\mathbb{C}}(f)$.

(ii) Does $\sqrt{8 \sum_{n=0}^{N} \frac{1}{(2n+1)^2}}$ approximate π at all (for large N)?

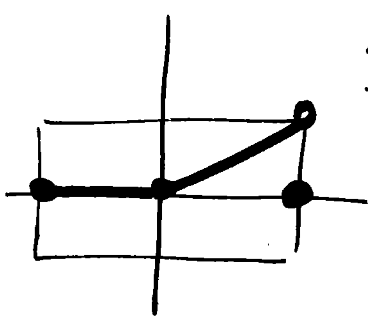
Ex: Find the (real) Fourier coefficients of

$$f : [0,2] \longrightarrow \mathbb{R}$$
$$x \longmapsto x\chi_{[0,2[}(x) = \left\{ \begin{array}{ll} x, \text{ if } & 0 \le x < 2 \\ 0, \text{ if } & x = 2 \end{array} \right\}.$$



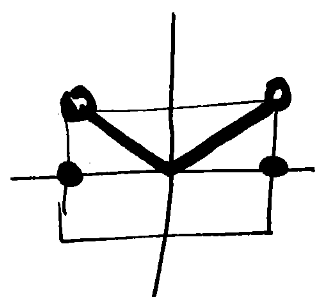
f

We first need a function defined on a symmetric interval centered at 0.

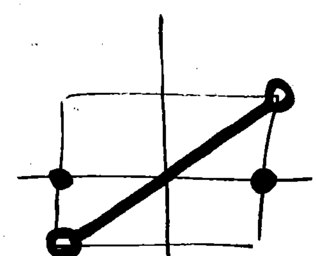


$\tilde{f}$

$= \frac{1}{2}$

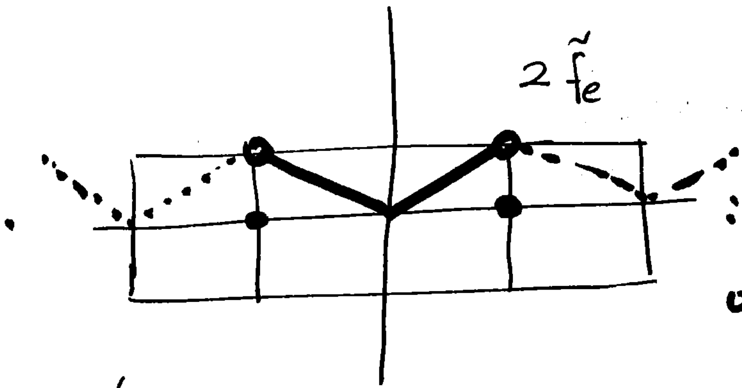


$+ \frac{1}{2}$

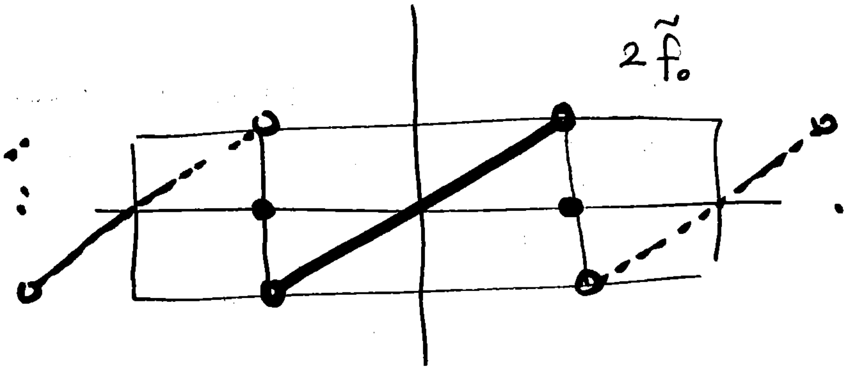


$$\underbrace{\tilde{f} + \mathcal{Y}(\tilde{f})}_{\tilde{f}_e = \mathcal{P}_e(\tilde{f})} \qquad \underbrace{\tilde{f} - \mathcal{Y}(\tilde{f})}_{\tilde{f}_o = \mathcal{P}_o(\tilde{f})}$$

Recall: $\tilde{f} + \mathcal{Y}(\tilde{f})$ is the even periodic extension of f and $\tilde{f} - \mathcal{Y}(\tilde{f})$ is the odd periodic extension of f.



(even periodic extension of f)



(odd periodic extension of f).

First let's compute the coefficients of $\mathcal{F}_{\mathbb{R}}(\tilde{f})$.

$$c_0 = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x)\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^2 = 1.$$

$$(n\geq 1)\quad c_n = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x)\,\gamma_n(x)\,dx = \frac{1}{2}\int_0^2 x \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\cdot\begin{Bmatrix} \frac{-8}{\pi^2}\cdot\frac{1}{n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{Bmatrix}.$$

$$(n\geq 1)\quad s_n = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x)\,\sigma_n(x)\,dx = \frac{1}{2}\int_0^2 x \sin\left(\frac{n\pi}{2}x\right)dx \qquad \begin{pmatrix} u = x & dv = \sin\left(\frac{n\pi}{2}x\right)dx \\ du = dx & \\ v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right) & \end{pmatrix}$$

$$= \frac{1}{2}\left(-\frac{2}{n\pi}\left[x\cos\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi}{2}x\right)dx\right)$$

$$= \left(\frac{-1}{n\pi}\right) 2\cos(n\pi) + \frac{1}{n\pi}\,\frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2}\underbrace{\sin(n\pi)}_{=0} = \frac{-2}{n\pi}\cos(n\pi) = \begin{Bmatrix} \frac{-2}{n\pi}, & \text{if } n = 2,4,\dots \\ \frac{2}{n\pi}, & \text{if } n = 1,3,\dots \end{Bmatrix}$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}\left(\tilde{f}\right)(x) = \frac{1}{2} + \left(\frac{-4}{\pi^2}\right) \sum_{\substack{n\geq 1\\ n:\text{odd}}} \frac{1}{n^2}\gamma_n(x) + \left(-\frac{2}{\pi}\right)\sum_{n\geq 1} \frac{(-1)^n}{n}\sigma_n(x)$$

$$= \frac{1}{2} + \left(\frac{-4}{\pi^2}\right)\sum_{n\geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right) + \left(\frac{-2}{\pi}\right)\left(\sum_{\substack{n\geq 1\\ n:\text{odd}}}\frac{(-1)^n}{n}\sigma_n(x)\right)$$

$$+ \left(\frac{-2}{\pi}\right)\left(\sum_{\substack{n\geq 1\\ n:\text{even}}}\frac{(-1)^n}{n}\sigma_n(x)\right)$$

$$= \frac{1}{2} + \left(\frac{-4}{\pi^2}\right)\sum_{n\geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right) + \frac{2}{\pi}\sum_{n\geq 0}\frac{1}{(2n+1)}\sin\left(\frac{(2n+1)\pi}{2}x\right)$$

$$- \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{2n}\sin\left(\frac{2n\pi}{2}x\right)$$

$$= \boxed{\underbrace{\frac{1}{2} + \left(\frac{-4}{\pi^2}\right)\sum_{n\geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}_{=\mathcal{F}_{\mathbb{R}}\left(\tilde{f}_e\right)} + \underbrace{\frac{2}{\pi}\sum_{n\geq 0}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi}{2}x\right) - \frac{1}{\pi}\sum_{n\geq 1}\frac{1}{n}\sin(n\pi x)}_{=\mathcal{F}_{\mathbb{R}}\left(\tilde{f}_o\right)}}$$

Onto the coeff.s of $\mathcal{F}_{\mathbb{R}}(\tilde{f}_e)$:

$$c_0 = \frac{1}{2}\int_{-2}^{2} \tilde{f}_e(x)\,dx = \int_0^2 \frac{x}{2}\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^2 = 1$$

$$(n \geq 1) \quad c_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\gamma_n(x)}_{\text{even}}\,dx = \int_0^2 \frac{x}{2}\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \begin{cases} -\frac{4}{\pi^2}\cdot\frac{1}{n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}.$$

$$n \geq 1 \quad s_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\sigma_n(x)}_{\text{odd}}\,dx = 0.$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}(\tilde{f}_e)(x) = \frac{1}{2} + \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(\frac{-4}{\pi^2}\cdot\frac{1}{n^2}\right)\gamma_n(x)$$

$$= \boxed{\frac{1}{2} + \left(-\frac{4}{\pi^2}\right)\sum_{n \geq 0} \frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}.$$

$$\Rightarrow \boxed{\mathcal{F}_{\mathbb{R}}(2\tilde{f}_e)(x) = 1 + \left(-\frac{8}{\pi^2}\right)\sum_{n \geq 0}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}$$

is the Fourier series of the even periodic extension of f.

Finally let's look at $\mathcal{F}(\tilde{f}_o)$.

$\tilde{f}_o$ is odd $\Rightarrow$ $c_n = 0$.

$$(n \geq 1) \quad s_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_o(x)\, \sigma_n(x)}_{\text{even}}\, dx = \int_0^2 \tilde{f}_o(x)\, \sigma_n(x)\, dx$$

$$= \int_0^2 \frac{x}{2} \sin\left(\frac{n\pi}{2} x\right) dx = \begin{cases} \dfrac{-2}{n\pi}, & \text{if } n = 2, 4, \dots \\ \dfrac{2}{n\pi}, & \text{if } n = 1, 3, \dots \end{cases}$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}(\tilde{f}_o)(x) = \sum_{n \geq 1} \left(\frac{-2}{\pi}\right) \frac{(-1)^n}{n} \sigma_n(x)$$

$$= \left(-\frac{2}{\pi}\right)\left(\sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{(-1)^n}{n} \sigma_n(x) + \sum_{\substack{n \geq 1 \\ n:\text{even}}} \frac{(-1)^n}{n} \sigma_n(x) \right)$$

$$= \left(+\frac{2}{\pi}\right) \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{2n} \sin\left(\frac{2n\pi}{2} x\right)$$

$$= \boxed{\frac{2}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin(n\pi x)}$$

$$\Rightarrow \boxed{\mathcal{F}_{\mathbb{R}}(2\tilde{f}_o)(x) = \frac{4}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin(n\pi x)}$$

is the Fourier series of the odd periodic extension of f.

<u>SW</u>: (i) Compute the coefficients of $\mathcal{F}_{\mathbb{C}}(\tilde{f})$.

(ii) $\mathcal{F}_{\mathbb{R}}(\tilde{f}) = \mathcal{F}_{\mathbb{R}}(\tilde{f}_e) + \mathcal{F}_{\mathbb{R}}(\tilde{f}_o) = \frac{1}{2}\,\mathcal{F}_{\mathbb{R}}(2\tilde{f}_e) + \frac{1}{2}\,\mathcal{F}_{\mathbb{R}}(2\tilde{f}_o)$

(iii) $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{C}}$ are both linear.

(iv) How do $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}} \circ \mathcal{Y}$ relate to eachother?

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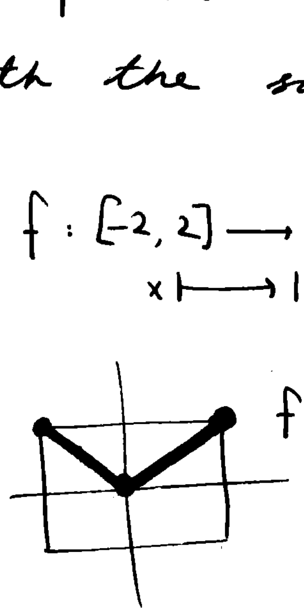
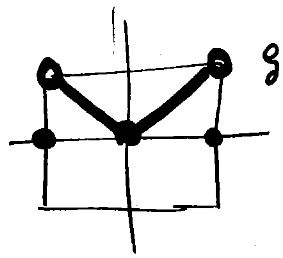
§10.3 :

- So far we have encountered two distinct functions with the same real Fourier series:

$$f : [-2,2] \longrightarrow \mathbb{R}, \quad x \longmapsto |x| \qquad , \qquad g : [-2,2] \longrightarrow \mathbb{R}, \quad x \longmapsto |x| \, \chi_{]-2,2[}(x).$$





$$\boxed{\mathcal{F}(f)(x) = \mathcal{F}(g)(x) = 1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2} \gamma_{2n+1}(x)}$$

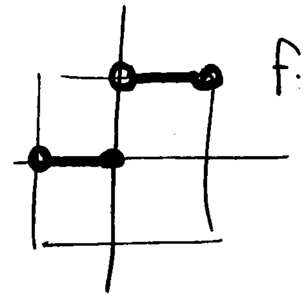
This indicates that the Fourier series can not be faithful to both of them for each $x \in [-2,2]$. Since f and g differ only at two points, these two points are the usual suspects. Still, we have the next best thing (which we'll take for granted):

Fourier Convergence Theorem : Let $L > 0$, $I := [-L, L]$ or $]-L, L[$, $f \in R(I, \mathbb{R})$ (is C_{pw} and) have pw. continuous $\partial_x f$ (well-defined everywhere except at finitely many points). Then for all $x \in I$:

$$\mathcal{F}_{\mathbb{R}}(f)(x) = \frac{1}{2}\left(\lim_{\substack{h \to 0 \\ h > 0}} f(x+h) + \lim_{\substack{h \to 0 \\ h < 0}} f(x+h) \right).$$

Ex: Find the Fourier series of $f :]-1,1[\longrightarrow \mathbb{R}$, $x \longmapsto \chi_{]0,1[}(x)$

$$c_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 dx = 1.$$



$$(n \geq 1) \quad c_n = \int_{-1}^{1} f(x)\gamma_n(x)\,dx = \int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\left[\sin(n\pi x)\right]\Big|_0^1 = 0.$$

$$(n \geq 1) \quad s_n = \int_{-1}^{1} f(x)\,\sigma_n(x) = \int_0^1 \sin(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}\left[\cos(n\pi x)\right]\Big|_0^1 = -\frac{1}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \left\{\begin{array}{ll} \frac{2}{n\pi} & , \text{if } \cos(n\pi) = -1 \\ 0 & , \text{if } \cos(n\pi) = 1 \end{array}\right\} = \left\{\begin{array}{ll} \frac{2}{n\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{array}\right\}$$

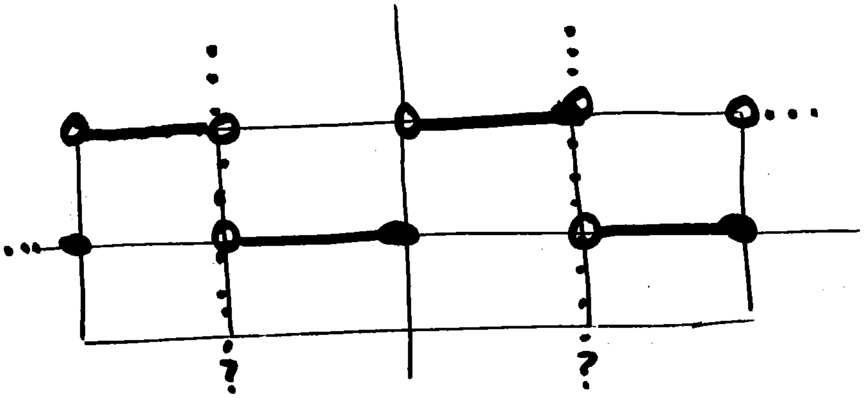
$$\Rightarrow \mathcal{F}_{\mathbb{R}}(f)(x) = \frac{1}{2} + \sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{2}{n\pi}\,\sigma_n(x)$$

$$= \boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{n \geq 0} \frac{1}{2n+1}\sin((2n+1)\pi x)}$$

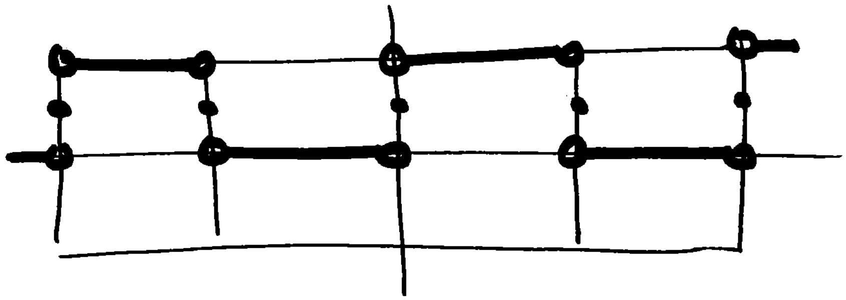
$$\mathcal{F}_{\mathbb{R}}(f)(1) = \frac{1}{2} = \frac{1}{2}\left(\lim_{\substack{h \to 0 \\ h > 0}} f(x+h) + \lim_{\substack{h \to 0 \\ h < 0}} f(x+h)\right)$$

(Here we consider f to be periodic.)

$$\mathcal{F}_{\mathbb{R}}(f)(0) = \frac{1}{2}.$$



f



$\mathcal{F}_{\mathbb{R}}(f)$

<u>SW</u> : (i) Find $\mathcal{F}_{\mathbb{C}}(f)$.

(ii) Prove that

$$\boxed{\pi = 4 \sum_{n \geq 0} \frac{(-1)^n}{2n+1}}.$$

---

§10.5:

- Up until now the differential equations that we dealt with involved derivatives with respect to not more than one variable. Accordingly our unknown functions were single variable.
- It turns out many phenomena are way too intricate to admit a mathematical model with a single variable. This leads us to consider equations whose unknowns are multivariable functions. Such equations are forced to involve derivatives with respect to more than one variable, whence they are called <u>partial differential equations</u> (<u>PDE</u>).
- We'll focus on those PDE's that can be dealt with using ODE methods, together with a method that allows us to disentangle (certain) PDE's into ODE's (called the <u>method of separation of variables</u>, <u>or eigenfunction decomposition</u>, <u>or disentanglement</u>).

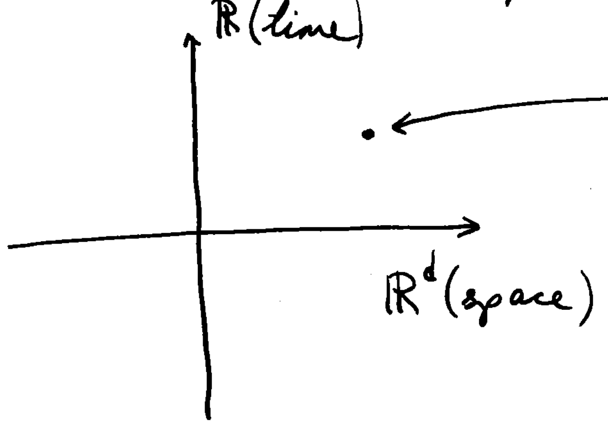
• Our new unknown functions will typically be of the form

$$u : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x_1, x_2, \dots, x_d, t) \longmapsto u(x_1, x_2, \dots, x_d, t),$$

where $x_1, x_2, \dots, x_d$ are called the "space" coordinates and t is called the "time" coordinate. Mathematics does not distinguish $\mathbb{R}^d \times \mathbb{R}$ and $\mathbb{R}^{d+1}$. Indeed, both of these symbols represent the set of $(d+1)$-tuples of real numbers. In fact, forgetting the primordiality (and tyranny) of "time" will be very convenient.

Yet, since we are still trying to understand physical phenomena, the "time" coordinate should be kept separate from the "space" coordinates.

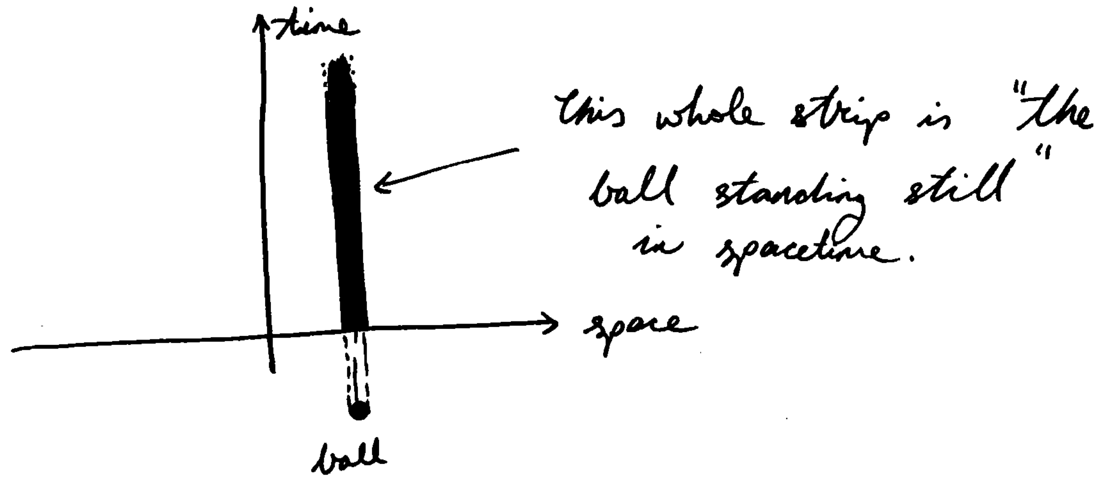
The notion of "spacetime" provides a compromise between these two positions:



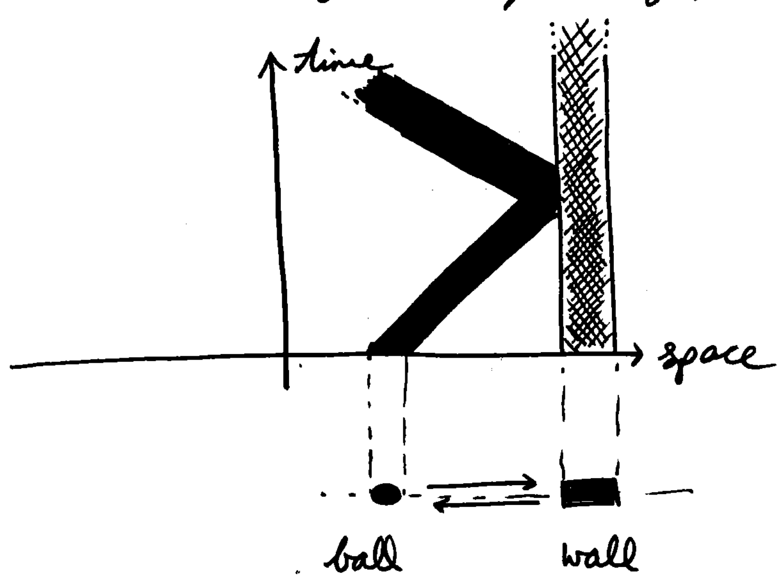
Points of the spacetime $\mathbb{R}^d \times \mathbb{R}$ are called "herenow"'s or "therethen"'s.

Thus we'll take the "time" coordinate to be on equal footing with the "space" coordinates (until it is time to interpret the mathematical results physically).

Here is how a ball standing still looks like in spacetime:



Here is how a ball bouncing off a wall looks like in spacetime (no bouncing angle, no gravity, no friction):

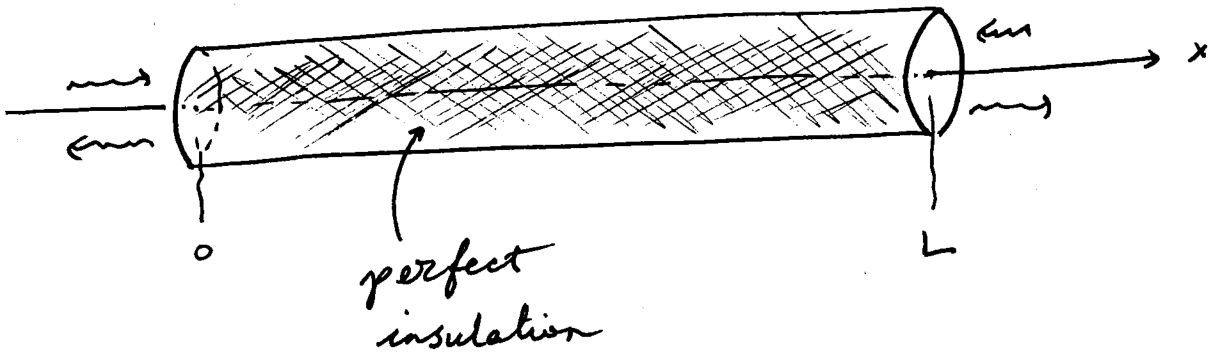


SW: (Derivation of the Heat Conduction Equation Through a Uniform Medium for $d=1$)

(This equation is based on the conception of heat as something that can flow as an incompressible fluid throughout a region of space occupied by a (uniform) substance.)

Consider a cylindrical rod made of a uniform material of length $L>0$ with density $\rho>0$.

Suppose the rod is perfectly insulated along its curved surface so that heat can enter or leave only at the ends. Also suppose the cross-section of the rod has such a small area that the nonnegligible heat flow is along a one-dimensional axis:



The unknown function of the heat equation is temperature $u: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$, i.e., $u(x,t)$ denotes the temperature at the herenow (x,t). Then the total thermal energy (ie., heat) E contained in the rod at time t is:

$$E(t) = \int_0^L s\rho \, u(x,t)\, dx ,$$

where $s>0$ is a physical constant called the specific heat of the material the rod is made of.

Fourier's Law of Heat Conduction (which is an empirical law) dictates that heat flows from hot to cold regions proportionately to the difference in temperature.

(i) Using Fourier's Law, show that

$$\partial_t E(t) = c \;\; \partial_x u(L,t) - c\, \partial_x u(0,t),$$

where $c>0$ is another physical constant called the heat conductivity of the material the rod is made of.

(ii) Deduce that

$$0 = \int_0^L \left(\partial_t u(x,t) - \frac{c}{s\rho} \Delta u(x,t) \right) dx.$$

$k := \frac{c}{s\rho} > 0$ is called the <u>thermal diffusivity</u> of the material the rod is made of. Consequently it does not depend on $L > 0$.

$$\Rightarrow \ 0 = \partial_L \left(\int_0^L \left(\partial_t u(x,t) - k \Delta u(x,t) \right) dx \right)$$

$$= \partial_t u(x,t) - k \Delta u(x,t)$$

$$\Rightarrow \quad \boxed{\partial_t u(x,t) - k \Delta u(x,t) = 0}. \qquad \text{(heat eq.)}$$

• A solution $u(x,t)$ of

$$\boxed{\partial_t u(x,t) - k\,\Delta u(x,t) = 0} \quad (*) \qquad \begin{pmatrix} k>0 \\ \Delta = \partial_x^2 \end{pmatrix}$$

is called <u>disentangled</u> (or <u>separated</u>) if it is not the constantly zero function and there are two functions $\pi : \mathbb{R}^1 \to \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ such that

$$u(x,t) = \pi(x)\,\tau(t).$$

Here the <u>space component</u> $\pi : \mathbb{R}^1 \to \mathbb{R}$ of u depends only on the "space" coordinate and the <u>time component</u> $\tau : \mathbb{R} \to \mathbb{R}$ of u depends only on the "time" coordinate.

If $u(x,t) = \pi(x)\,\tau(t)$ is a disentangled sol. of $\circledast$, then there are $x_0 \in \mathbb{R}^1$, $t_0 \in \mathbb{R}$:

$$\pi(x_0) \neq 0 \quad , \quad \tau(t_0) \neq 0 .$$

$$\circledast \Rightarrow \quad \partial_t \left(\pi(x)\,\tau(t)\right) - k\,\Delta\left(\pi(x)\,\tau(t)\right) = 0$$

$$\Rightarrow \left(-\Delta\,\pi(x)\right)\left(k\,\tau(t)\right) = \left(-\partial_t\,\tau(t)\right)\,\pi(x)$$

$$\Rightarrow \quad -\Delta\,\pi(x) = \underbrace{\left.\left[\frac{-\partial_t\,\tau(t)}{k\,\tau(t)}\right]\right|_{t=t_0}}_{=:\lambda} \quad \pi(x)$$

$=: \lambda$ (this is well-defined because $\tau(t_0) \neq 0 \neq k$.)

$$\Rightarrow \quad -\Delta\,\pi(x) = \lambda\,\pi(x) \quad \Leftrightarrow \quad (\lambda, \pi(x)) \text{ is an eigenpair of } -\Delta.$$

$$\Rightarrow \left(\lambda\,\pi(x)\right)\left(k\,\tau(t)\right) = \left(-\partial_t\,\tau(t)\right)\,\pi(x)$$

$$\Rightarrow \quad \lambda\,\pi(x_0)\; k\;\tau(t) = -\partial_t\,\tau(t)\;\pi(x_0)$$

$$\Rightarrow \quad \partial_t\,\tau(t) = -k\lambda\,\tau(t) \quad \Leftrightarrow \quad (-k\lambda, \tau(t)) \text{ is an eigenpair of } \partial_t .$$

$(\pi(x_0) \neq 0)$

Thus $(*)$ <u>disentangles</u> into

$$\boxed{\begin{aligned} u(x,t) &= \pi(x)\,\tau(t) \\ -\Delta\pi(x) &= \lambda\,\pi(x) \\ \partial_t \tau(t) &= -k\lambda\tau t \end{aligned}} \quad (**)\,.$$

<u>Abbreviated Version</u>:

$$u = \pi\tau \Rightarrow (*) \Leftrightarrow \pi\dot{\tau} - k\ddot{\pi}\tau = 0$$

$$\Leftrightarrow (-\ddot{\pi})(k\tau) = (\pi)(-\dot{\tau})$$

$$\Leftrightarrow \frac{-\ddot{\pi}}{\pi} = \frac{-\dot{\tau}}{k\tau} = \lambda \text{ is constant}$$

$$\Rightarrow \boxed{\begin{aligned} u &= \pi\tau \\ -\ddot{\pi} &= \lambda\pi \\ \dot{\tau} &= -k\lambda\tau \end{aligned}}$$

and $(**)$ is called a <u>disentanglement</u> of $(*)$. Observe that λ in $(**)$ is a new parameter that is not fixed.

$$(**) \Rightarrow \pi(x) = \left\{ \begin{array}{ll} c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}, & \text{if } \lambda < 0 \\ c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), & \text{if } \lambda > 0 \\ c_1 + c_2 x & , \text{if } \lambda = 0 \end{array} \right\}, \quad \tau(t) = d\, e^{-k\lambda t}$$

$$\Rightarrow u(x,t) = \left\{ \begin{array}{ll} \left(d_1 e^{-\sqrt{-\lambda}x} + d_2 e^{\sqrt{-\lambda}x}\right) e^{-k\lambda t} & , \text{if } \lambda < 0 \\ \left(d_1 \cos(\sqrt{\lambda}x) + d_2 \sin(\sqrt{\lambda}x)\right) e^{-k\lambda t} & , \text{if } \lambda > 0 \\ (d_1 + d_2 x) & , \text{if } \lambda = 0 \end{array} \right\}$$

is the <u>general disentangled solution</u> of $(*)$.

A PDE that has a disentanglement is called <u>disentangleable</u> (or <u>separable</u>).

<u>Ex</u>: Disentangle $\boxed{x^2 \Delta u(x,t) - t^2 \partial_t^2 u(x,t) = 0}$ $(*)$.

$$u = \pi \tau \Rightarrow (*) \Leftrightarrow (x^2 \ddot{\pi})(\tau) - (\pi)(t^2 \ddot{\tau}) = 0$$

$$\Leftrightarrow \frac{x^2 \ddot{\pi}}{\pi} = \frac{t^2 \ddot{\tau}}{\tau} =: \lambda \in \mathbb{R}$$

$$\Leftrightarrow \boxed{\begin{aligned} &u(x,t) = \pi(x)\,\tau(t) \\ &x^2 \Delta \pi(x) = \lambda \pi(x) \\ &t^2 \partial_t^2 \tau(t) = \lambda \tau(t) \end{aligned}}$$

λ is called the <u>disentanglement constant</u> (or the <u>separation constant</u>)

<u>SW</u>: Let n be a nonnegative integer, $\alpha > 0$, and consider

$$\boxed{\partial_t^n u(x,y,z,t) - \alpha \Delta u(x,y,z,t) = 0} \quad (*) \qquad (\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2)$$

(i) Using $u(x,y,z,t) = \pi(x,y,z)\,\tau(t)$, disentangle $(*)$ into

$$\boxed{\begin{aligned} &u(x,y,z,t) = \pi(x,y,z)\,\tau(t) \\ &-\Delta \pi(x,y,z) = \lambda_0 \pi(x,y,z) \\ &-\partial_t^n \tau(t) = \alpha \lambda_0 \tau(t) \end{aligned}} \quad (**)$$

(ii) Using $\pi(x,y,z) = p(x)\,q(y)\,r(z)$, disentangle $(**)$ into

$$\boxed{\begin{aligned} &u(x,y,z,t) = p(x)\,q(y)\,r(z)\,\tau(t) \\ &-\partial_x^2 p(x) = \lambda_1 p(x) \\ &-\partial_y^2 q(y) = \lambda_2 q(y) \\ &-\partial_z^2 r(z) = (\lambda_0 - \lambda_1 - \lambda_2)\, r(z) \\ &-\partial_t^n \tau(t) = \alpha \lambda_0 \tau(t) \end{aligned}}$$

(iii) Generalize to $\Delta = \sum_{k=1}^{d} \partial_{x_k}^2$.

Ex: Allegedly $\boxed{\Delta u(x,y) + \partial_x \partial_y u(x,y) = 0}$ (⊛) $(\Delta = \partial_x^2 + \partial_y^2)$

does not admit a disentanglement. But consider:

(i) $u(x,y) = p(x)\, q(y)$, $p(x_0) \neq 0 \neq q(y_0)$

$$\Rightarrow \quad (\circledast) \Leftrightarrow \ddot{p}\, q + p\, \ddot{q} + \dot{p}\, \dot{q} = 0 \Rightarrow \underbrace{\frac{\ddot{p}(x_0)}{p(x_0)}}_{=:\lambda_4} + \underbrace{\frac{\dot{p}(x_0)}{p(x_0)}}_{=:\lambda_3} \underbrace{\frac{\dot{q}(y_0)}{q(y_0)}}_{=:\lambda_1} + \underbrace{\frac{\ddot{q}(y_0)}{q(y_0)}}_{=:\lambda_2} = 0$$

$$\Leftrightarrow \quad \lambda_1 \lambda_3 + \lambda_2 + \lambda_4 = 0.$$

$$\Rightarrow \quad \boxed{\begin{aligned} & u(x,y) = p(x)\, q(y) \\ & \ddot{p}(x) + \lambda_1 \dot{p}(x) + \lambda_2 p(x) = 0 \\ & \ddot{q}(y) + \lambda_3 \dot{q}(y) + \lambda_4 q(y) = 0 \\ & \lambda_1 \lambda_3 + \lambda_2 + \lambda_4 = 0 \end{aligned}}$$

is a disintegration of (⊛) with three parameters.

(ii) $$0 = \frac{\ddot{p}}{p} + \frac{\dot{p}}{p}\frac{\dot{q}}{q} + \frac{\ddot{q}}{q} \Rightarrow 0 = \partial_{xy}\left(\frac{\ddot{p}}{p} + \frac{\dot{p}}{p}\frac{\dot{q}}{q} + \frac{\ddot{q}}{q}\right)$$

$$= \partial_x \left(\frac{\dot{p}}{p}\, \partial_y\left(\frac{\dot{q}}{q}\right) + \partial_y\left(\frac{\ddot{q}}{q}\right)\right) = \partial_x\left(\frac{\dot{p}}{p}\right) \partial_y\left(\frac{\dot{q}}{q}\right)$$

$$\Rightarrow \partial_x\left(\frac{\dot{p}}{p}\right) = 0 \quad \text{or} \quad \partial_y\left(\frac{\dot{q}}{q}\right) = 0$$

$\Rightarrow \dfrac{\dot{p}}{p} =: \mu_3$ is a constant or $\dfrac{\dot{q}}{q} =: \mu_1$ is a constant

If $\frac{\dot{p}}{p} = \mu_3$, then $\frac{\ddot{p}}{p} = -\frac{\mu_3 \dot{q} + \ddot{q}}{q} =: \mu_4$ is a constant

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} \dot{p} = \mu_3 p \\ \ddot{p} = \mu_4 p \end{array}\right\} \mu_4 = \mu_3^2 \\ \ddot{q} + \mu_3 \dot{q} = -\mu_4 q \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} u(x,y) = p(x)\, q(y) \\ \dot{p}(x) = \mu_3\, p(x) \\ \ddot{q}(y) + \mu_3\, \dot{q}(y) = -\mu_3^2\, q(y) \end{array}}$$

If $\frac{\dot{q}}{q} = \mu_1$, then $\frac{\ddot{q}}{q} = -\frac{\mu_1 \dot{p} + \ddot{p}}{p} =: \mu_2$ is a constant

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} \dot{q} = \mu_1 q \\ \ddot{q} = \mu_2 q \end{array}\right\} \mu_2 = \mu_1^2 \\ \ddot{p} + \mu_1 \dot{p} = -\mu_2 p \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} u(x,y) = p(x)\, q(y) \\ \dot{q}(y) = \mu_1\, q(y) \\ \ddot{p}(y) + \mu_1\, \dot{p}(y) = -\mu_1^2\, p(y) \end{array}}$$

$$\Rightarrow \boxed{\begin{array}{lcl} u(x,y) = p(x)\, q(y) & & \\ \dot{p}(x) - \mu_3 p(x) = 0 & \text{or} & \dot{q}(y) - \mu_1\, q(y) = 0 \\ \ddot{q}(y) + \mu_3\, \dot{q}(y) + \mu_3^2\, q(y) = 0 & & \ddot{p}(y) + \mu_1\, \dot{p}(y) + \mu_1^2\, p(y) = 0 \end{array}}$$

is another disintegration of (*) with two parameters (and two cases).

- We will typically encounter a PDE as part of an initial / boundary value problem (IBVP), which is a triple of the form

$$\left(\text{PDE}, \begin{array}{l}\text{boundary conditions}\\ \text{in terms of the}\\ \text{"space" coordinates}\end{array}, \begin{array}{l}\text{initial data}\\ \text{in terms of the}\\ \text{"time" coordinate}\end{array}\right)$$

- The method of disentanglement for solving homogeneous IBVP's (ie., boundary conditions = 0.) goes like this:

  (i) Disentangle the PDE.

  (ii) Use the boundary conditions to detect the relevant disentangled solutions.

  (iii) Any entangled solution satisfying the boundary condition is the limit of a linear combination of disentangled solutions (ie., by taking infinite sums of disentangled solutions we can obtain any solution) (This we'll take for granted.).

  (iv) Determine the coefficients for (iii) by looking at the Fourier coefficients of the initial data.

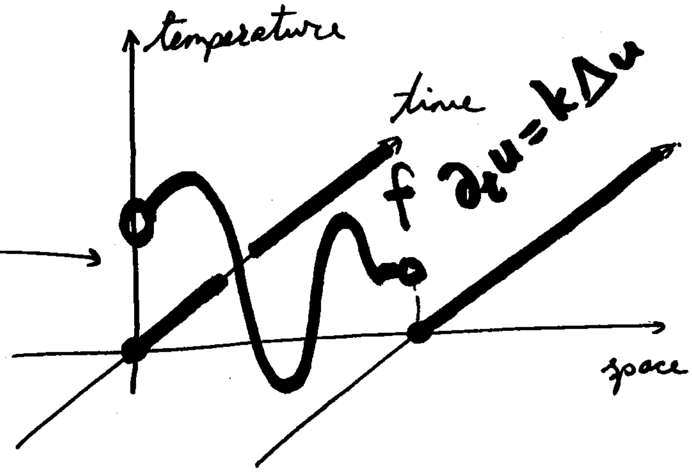
Ex : (Homogeneous Heat Conduction Problem for $d=1$)

Let $k>0$, $L>0$, $f \in R(]0,L[,\mathbb{R})$ be piecewise smooth.

Consider

$$(*)\quad \boxed{\begin{array}{lll} \partial_t u(x,t) - k\Delta u(x,t) = 0 & , \text{ for } (x,t) \in]0,L[\times]0,\infty[\\ u(0,t) = 0 = u(L,t) & , \text{ for } t \in [0,\infty[\\ u(x,0) = f(x) & , \text{ for } x \in]0,L[\end{array}} \quad \begin{array}{l} \text{(PDE)} \\ \text{(BC)} \\ \text{(ID)} \end{array}$$

Geometrically, we are trying to find that surface which fits into this frame whose concavity in the "space" direction is proportional to its slope in the "time" direction.



Physically, the boundary conditions mean that the ends of the rod are kept at constant zero temperature (but quite possibly there is still heat flow in and out of the rod at the ends). The initial data $f(x)$ represents the initial temperature distribution on the rod (except the ends).

$\circledast$ disentangles into: $\boxed{\begin{aligned} u(x,t) &= \pi(x)\,\tau(t) \\ -\Delta \pi(x) &= \lambda\, \pi(x) \\ \partial_t \tau(t) &= -k\lambda\, \tau(t) \end{aligned}}$

(BC) disentangles into: $\pi(0)\,\tau(t) = 0 = \pi(L)\,\tau(t)$.

There is a $t_0 \geq 0$: $\tau(t_0) \neq 0$

$\Rightarrow \boxed{\pi(0) = 0 = \pi(L)}$ (see the SW.(i) at the end of §10.1)

<u>Relevant eigenpairs of $-\Delta$:</u>

(I) $\lambda < 0 \Rightarrow \quad \pi(x) = c_1 e^{-\sqrt{-\lambda}\,x} + c_2 e^{\sqrt{-\lambda}\,x}$

$\Rightarrow \; 0 = c_1 + c_2, \quad 0 = c_1 e^{-\sqrt{-\lambda}\,L} + c_2 e^{\sqrt{\lambda}\,L}$

$\Rightarrow \; c_2 = -c_1, \quad 0 = c_1 \Big(\underbrace{e^{-\sqrt{-\lambda}\,L} - e^{\sqrt{-\lambda}\,L}}_{\neq 0} \Big)$

$\Rightarrow c_1 = 0 = c_2$

$\Rightarrow$ no relevant eigenpairs.

(II) $\lambda > 0 \Rightarrow \quad \pi(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$

$\Rightarrow \; 0 = c_1, \quad 0 = c_1 \cos(\sqrt{\lambda}\,L) + c_2 \sin(\sqrt{\lambda}\,L)$

$\Rightarrow \; 0 = c_2 \sin(\sqrt{\lambda}\,L)$.

$c_2 \neq 0 \Leftrightarrow \sin(\sqrt{\lambda}\,L) = 0 \Leftrightarrow \sqrt{\lambda}\,L = \begin{matrix} \pi, 3\pi, \dots, (2n+1)\pi, \dots \\ 2\pi, 4\pi, \dots, (2n)\pi, \dots \end{matrix}$

$\Rightarrow$ For any $n \geq 1$: $\left(\left(\frac{n\pi}{L}\right)^2, \sin\left(\frac{n\pi}{L}x\right) \right)$ is a relevant eigenpair.

(III) $\lambda = 0 \Rightarrow \Pi(x) = c_1 + c_2 x$

$$\Rightarrow 0 = c_1 , \quad 0 = c_1 + c_2 L \quad \Rightarrow c_2 = 0$$

$\Rightarrow$ no relevent eigenpairs.

$\Rightarrow$ The relevant disentanged solutions are:

$$\boxed{\text{For any } n \geq 1: \quad \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$\Rightarrow$ Any solution is of the form

$$u(x,t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

where the coefficients $b_1, b_2, \dots, b_n, \dots$ are yet to be determined.

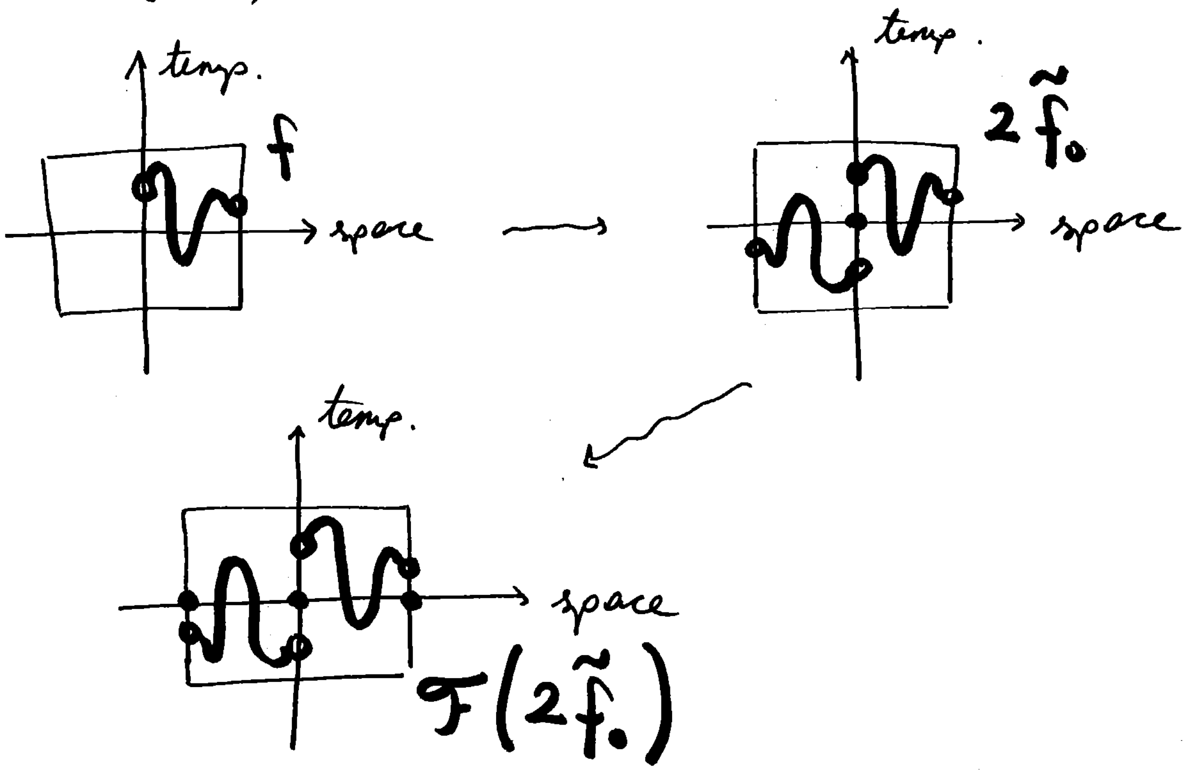
If $u(x,t) = \sum_{n \geq 1} b_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ solves (*),

then $$f(x) = u(x,0) = \sum_{n \geq 1} b_n \sigma_n(x).$$

Observe that since $f :]0,L[\to \mathbb{R}$ is piecewise smooth, by the Fourier Convergence Theorem

$$\text{for all } x \in [0,L] : \mathcal{F}(2\tilde{f}_0)(x) = f(x) \chi_{]0,L[}(x)$$

where $2\tilde{f}_0$ is the odd periodic extension of f onto $]-L, L[$



$$\mathcal{F}(2\tilde{f}_0)(x) = \frac{c_0}{2} + \sum_{n \geq 1} c_n \gamma_n(x) + \sum_{n \geq 1} s_n \sigma_n(x).$$

$(n \geq 0)$ $\quad c_n = 0$ because $2\tilde{f}_0$ is odd.

$(n \geq 1)$ $\quad s_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} \underbrace{2\tilde{f}_0(x)\, \sigma_n(x)}_{\text{even}}\, dx = \dfrac{2}{L} \displaystyle\int_0^L 2\tilde{f}_0(x)\, \sigma_n(x)\, dx$

$$= \frac{2}{L} \int_0^L f(x)\, \sigma_n(x)\, dx$$

$\left(\begin{array}{l}\text{for } 0 < x < L, \\ 2\tilde{f}_0(x) = f(x)\end{array}\right)$

$\Rightarrow$ for all $x \in [0, L]$: $f(x)\chi_{]0,L[}(x) = \mathcal{F}(2\tilde{f}_0)(x) = \sum_{n \geq 1} s_n \sigma_n(x)$,

where $s_n = \frac{2}{L} \int_0^L f(x) \sigma_n(x)\, dx$.

$\Rightarrow \sum_{n\geq 1} b_n \sigma_n(x) = u(x,0) = f(x) = \sum_{n \geq 1} s_n \sigma_n(x)$

$\Rightarrow$ Picking $b_n := s_n$ produces the solution of $\circledast$:

$$\boxed{u(x,t) = \sum_{n\geq 1} s_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad \text{where } s_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx}$$

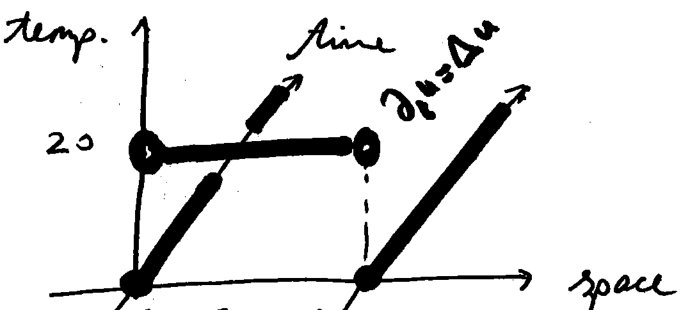
Observe that $\lim_{t\to\infty} u(x,t) = 0$.

Ex: $k := 1 \left[\frac{cm^2}{s}\right]$, $L := 50$ [cm], $f :]0,50[\to \mathbb{R}$, $x \mapsto 20$ [°C],

Solve

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &u(0,t) = 0 = u(50,t) \\ &u(x,0) = 20. \end{aligned}}$$

$$u(x,t) = \sum_{n\geq 1} s_n \sin\left(\frac{n\pi}{50}x\right) e^{-\left(\frac{n\pi}{50}\right)^2 t}$$



$$s_n = \frac{2}{L}\int_0^L f(x)\sigma_n(x)\,dx = \frac{4}{5}\int_0^{50} \sin\left(\frac{n\pi}{50}x\right)dx$$

$$= \frac{4}{5}\left(\frac{-50}{n\pi}\right)\left[\cos\left(\frac{n\pi}{50}x\right)\right]\Big|_0^{50} = \left(\frac{-40}{n\pi}\right)\left(\cos(n\pi) - 1\right)$$

$$= \frac{80}{n\pi}\chi_{2\mathbb{Z}+1}(n). \quad \checkmark$$

SW : (i) Replace the boundary condition of ⊛ with "$\partial_x u(0,t) = 0 = \partial_x u(L,t)$ for $t \in [0,\infty[$", then find the solution $u(x,t)$. Physically this new boundary condition means that the ends of the rod are isolated as well. Also show that

$$\lim_{t\to\infty} u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx = \text{average of the initial data}.$$

(ii) Replace the boundary condition of ⊛ with "$\partial_x u(0,t) = 0 = u(0,t)$ for $t \in [0,\infty[$", then find the solution $u(x,t)$. Interpret this new boundary condition physically. Find $\lim_{t\to\infty} u(x,t)$.

(iv) Solve ⊛ with $k := \frac{i}{2} = \frac{\sqrt{-1}}{2}$. Interpret physically. $\partial_t u(x,t) - \frac{i}{2}\Delta u(x,t) = 0$ is the free <u>Schrödinger eq.</u> for $d=1$.

(iii) Solve

$$\boxed{\begin{aligned} &\partial_t u(x,t) - k\,\Delta u(x,t), \text{ for } (x,t) \in\,]-L,L[\times]0,\infty[\\ &u(-L,t) - u(L,t) = 0 = \partial_x u(-L,t) - \partial_x u(L,t), \text{ for } t \in [0,\infty[\\ &u(x,0) = f(x), \text{ for } x \in\,]-L,L[\end{aligned}}\,,$$

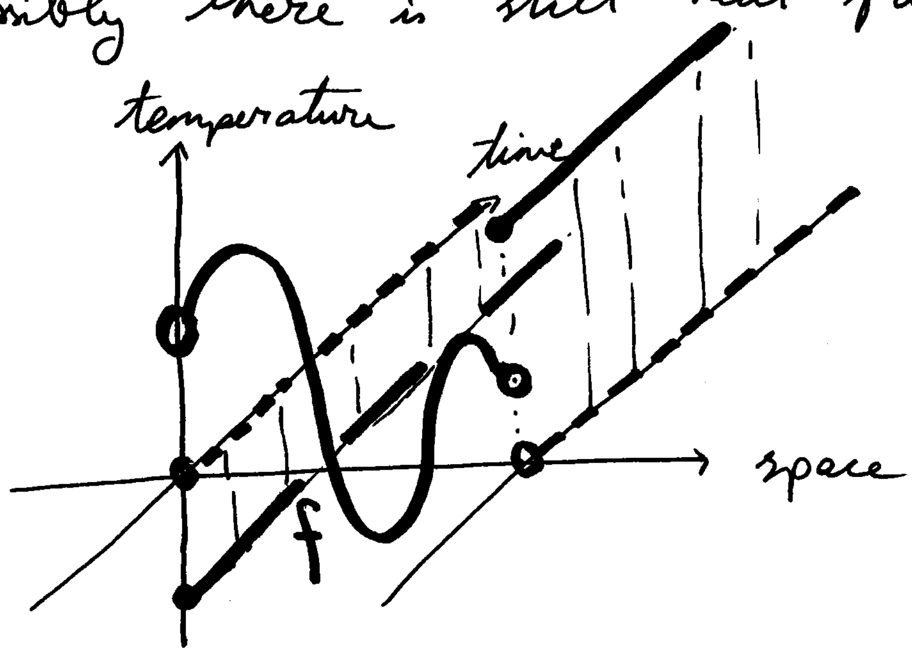
where $L>0$, $k>0$, $f \in R(]-L,L[,\mathbb{R})$ is pw. smooth. Interpret physically.

§10.6 :

Ex: (Nonhomogeneous Heat Conduction Problem for $d=1$)
Let $k>0$, $L>0$, $T_0, T_L \in \mathbb{R}$, $f \in R(]0,L[,\mathbb{R})$ be pw. smooth. Consider

$$(*)\quad \boxed{\begin{array}{lll} \partial_t u(x,t) - k\Delta u(x,t) = 0 & , \text{ for } (x,t) \in]0,L[\times]0,\infty[\\ u(0,t) - T_0 = 0 = u(L,t) - T_L & , \text{ for } t \in [0,\infty[\\ u(x,0) = f(x) & , \text{ for } x \in]0,L[\end{array}} \quad \begin{array}{l} (PDE) \\ (BC) \\ (ID) \end{array}$$

The boundary conditions now mean that the left end of the rod is kept at the temperature T_0 and the right end is kept at T_L (again quite possibly there is still heat flow at the ends).



Recall the general disentangled solution of the heat equation:

$$u(x,t) = \begin{cases} \left(d_1 e^{-\sqrt{-\lambda}\,x} + d_2 e^{\sqrt{-\lambda}\,x}\right) e^{-k\lambda t} & , \text{ if } \lambda < 0 \\ \left(d_1 \cos(\sqrt{\lambda}x) + d_2 \sin(\sqrt{\lambda}\,x)\right) e^{-k\lambda t} & , \text{ if } \lambda > 0 \\ d_1 + d_2 x & , \text{ if } \lambda = 0. \end{cases}$$

We do not have an external source of heat (ie. forcing), whence we would expect that no disentangled solution with $\lambda < 0$ will be relevant to the IBVP. When the boundary conditions were homogeneous we had also eliminated the disentangled solutions with $\lambda = 0$ (except possibly constant ones, eg. when $\partial_x u(0,t) = 0 = \partial_x u(L,t)$). Since now the boundary conditions are (quite possibly) not homogeneous, we have:

$$\lambda = 0 \Rightarrow \pi(x) = d_1 + d_2 x$$

$$\Rightarrow \left.\begin{array}{l} T_0 = \pi(0) = d_1 \\ T_L = \pi(L) = d_1 + d_2 L \end{array}\right\} \quad T_L - T_0 = d_2 L \Rightarrow d_2 = \frac{T_L - T_0}{L}$$

$\Rightarrow$ $T_0 + \dfrac{T_L - T_0}{L} x$ is a relevant disentangled solution.

Thus the general solution of (*) should be of the form

$$u(x,t) = \underbrace{\left(T_0 + \frac{T_L - T_0}{L} x \right)}_{=: u_E(x)} + \underbrace{\int_{\substack{]0,\infty[\\ \lambda: \text{ relevant}}} \left(a(\lambda) \cos(\sqrt{\lambda} x) + b(\lambda) \sin(\sqrt{\lambda} x) \right) e^{-k\lambda t} \, d\lambda}_{=: v(x,t)} ,$$

where $a, b : \{ \lambda \in]0,\infty[\mid \lambda \text{ is relevant} \} \to \mathbb{R}$ are coefficients yet to be determined.

$$\Rightarrow \lim_{t \to \infty} v(x,t) = 0$$

$$\Rightarrow \lim_{t \to \infty} u(x,t) = \lim_{t \to \infty} \left(u_E(x) + v(x,t) \right) = u_E(x)$$

u_E is called the <u>equilibrium solution</u> of the IBVP (or <u>steady-state</u>) (it is constant in time) and $v(x,t)$ is called the <u>transient solution</u> of the IBVP.

(Recall the periodically forced harmonic oscillator.)

To determine u_E we did not use f, consequently it is unreasonable to expect that u_E solves the whole IBVP. Likewise since $v(x,t)$ decays exponentially fast in time, unless $T_0 = 0 = T_L$ it won't solve the whole IBVP. But we have:

$$\partial_t u_E(x) - k\Delta u_E(x) = -k\,\partial_x^2\left(T_0 + \frac{T_L - T_0}{L}x\right) = 0$$

$$\partial_t v(x,t) - k\Delta v(x,t) = \partial_t\big(u(x,t) - u_E(x)\big) - k\Delta\big(u(x,t) - u_E(x)\big)$$

$$= \big(\partial_t u(x,t) - k\Delta u(x,t)\big) - \big(\partial_t u_E(x) - k\Delta u_E(x)\big) = 0.$$

$$u_E(0) - T_0 = 0 = u_E(L) - T_L$$

$$v(0,t) = u(0,t) - u_E(0) = T_0 - T_0 = 0$$

$$v(L,t) = u(L,t) - u_E(L) = T_L - T_L = 0.$$

$$v(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x)$$

$\Rightarrow$ If $u(x,t) = u_E(x) + v(x,t)$ solves (*), then

$u_E(x)$ solves: and $v(x,t)$ solves:

$$\boxed{\begin{aligned} &\underbrace{\partial_t u_E(x)}_{=0} - k\Delta u_E(x) = 0 \\ &u_E(0) = T_0 \\ &u_E(L) = T_L \end{aligned}} \quad (**)$$

$$\boxed{\begin{aligned} &\partial_t v(x,t) - k\Delta v(x,t) = 0 \\ &v(0,t) = 0 = v(L,t) \\ &v(x,0) = f(x) - u_E(x) \end{aligned}} \quad (***)$$

SW: Show that the converse holds as well, i.e., if $p(x)$ solves $(**)$ and $q(x,t)$ solves $(***)$ (with "u_E" replaced with "p"), then $r(x,t) := p(x)+q(x,t)$ solves $(*)$ and $r_E = p$.

Observe that $(***)$ is a homogeneous IBVP whose solution we already discovered:

$$\boxed{\begin{aligned} v(x,t) &= \sum_{n\geq 1} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \\ b_n &= \frac{2}{L}\int_0^L \left(f(x)-u_E(x)\right)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}}$$

$$\Rightarrow \boxed{\begin{aligned} u(x,t) = u_E(x) + v(x,t) &= \left(T_0 + \frac{T_L-T_0}{L}x\right) + \sum_{n\geq 1} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \\ b_n &= \frac{2}{L}\int_0^L \left(f(x)-u_E(x)\right)\sigma_n(x)\,dx. \end{aligned}}$$

is the solution of $(*)$.

SW: (i) Make this more explicit by using the fact that $u_E(x) = T_0 + \frac{T_L-T_0}{L}x$ in b_n's.

(ii) Find the eq. sol.s of all SW's at the end of §10.5.

Ex: Find the eq. sol. $u_E(x)$ of:

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &\partial_x u(0,t) - 8 = 0 = u(10,t) - 100 \\ &u(x,0) = 5x + 27 \end{aligned}} \longrightarrow \boxed{\begin{aligned} -\Delta u_E(x) &= 0 \\ \partial_x u_E(0) &= 8 \\ u_E(10) &= 100 \end{aligned}}$$

(This is irrelevant to u_E.)

$$\Rightarrow \left.\begin{aligned} u_E(x) &= a + bx \\ \partial_x u_E(x) &= b \end{aligned}\right\} \Rightarrow 8 = b, \quad 100 = a + 10b$$

$$\Rightarrow a = 20 \Rightarrow \boxed{u_E(x) = 20 + 8x.}$$

Ex: If $u(x,t)$ solves:

$$\boxed{\begin{aligned} &\partial_t u(x,t) - \Delta u(x,t) = 0 \\ &\partial_x u(0,t) - 30 = 0 = \partial_x u(10,t) - 10 \\ &u(x,0) = x^2 \end{aligned}}, \quad \text{find } \lim_{t\to\infty} u(x,t).$$

$\lim_{t\to\infty} u(x,t) = u_E(x)$ and

$u_E(x)$ solves:

$$\boxed{\begin{aligned} \Delta u_E(x) &= 0 \\ \partial_x u_E(0) &= 30 \\ \partial_x u(10) &= 10 \end{aligned}}$$

$$\Rightarrow u_E(x) = a + bx$$

$$\partial_x u_E(x) = b$$

$$\left.\begin{aligned} \Rightarrow 30 &= \partial_x u_E(0) = b \\ 10 &= \partial_x u_E(10) = b \end{aligned}\right\} \Rightarrow b = 10 = 30$$

$\lightning$.

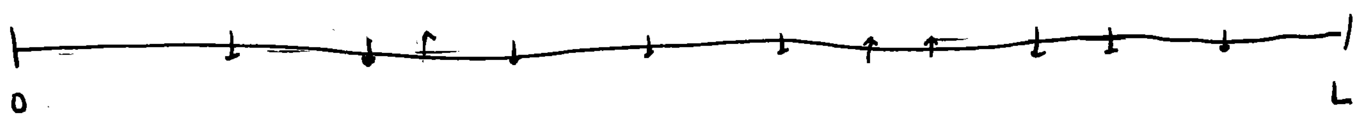
$\Rightarrow$ The limit does not exist.

SW: Can $u(x,t)$ exist?

§10.7 :

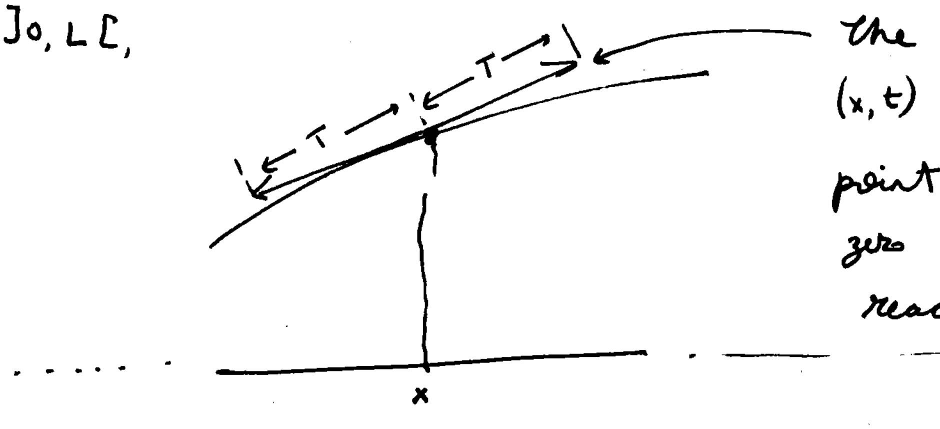
SW: (Derivation of the Wave Equation for a Uniform Medium with Small Vibrations for $d=1$)

Consider a perfectly elastic and flexible string of length $L>0$ and density $\rho>0$ stretched out. Suppose the string undergoes small transverse vibrations and remains in a plane:



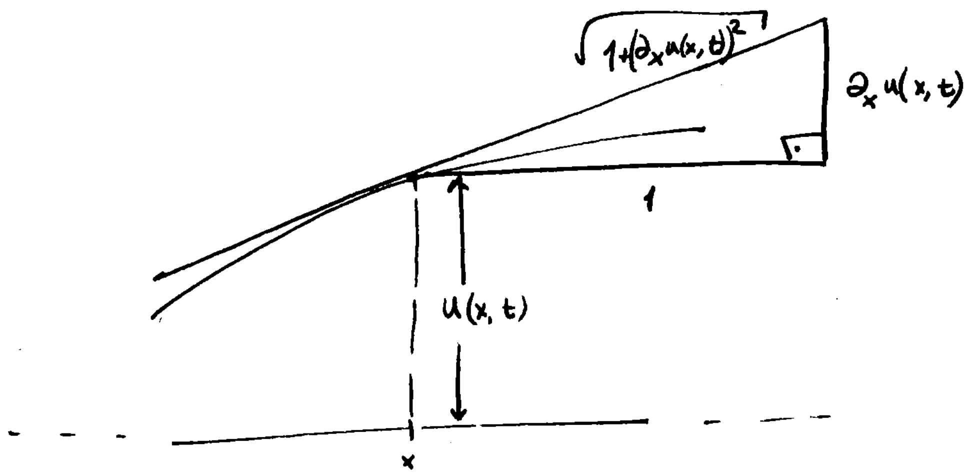
The only force acting on the string is the tension force. We assume that its magnitude is a constant $T>0$. Since the string is flexible the tension force is always tangent to the string:

If $x \in]0,L[$,

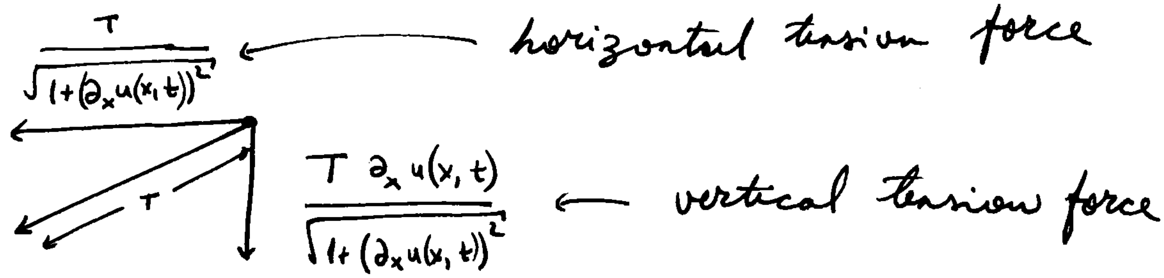


the tension force at (x,t) acting on the point marked as x. is zero by the action-reaction principle.

The unknown function of the wave equation is the displacement $u : \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$, i.e., $u(x,t)$ denotes the vertical displacement of the string from its equilibrium position at the herenow (x,t):



$\Rightarrow$



(i) Using Newton's 2nd Law and the fact that there is no lateral motion, show that if $x_0, x_1 \in [0, L] : x_0 < x_1$ then

$$\left[\frac{T}{\sqrt{1+(\partial_x u(x,t))^2}}\right]\Bigg|_{x_0}^{x_1} = 0 \quad \text{and} \quad \left[\frac{T\,\partial_x u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}}\right]\Bigg|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho\,\partial_t^2 u(x,t)\,d$$

(ii) Differentiating the second equality with respect to x_1, deduce that

$$\rho\, \partial_t^2 u(x,t) = T \frac{\partial_x^2 u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}^{\,3}}.$$

Define $c := \sqrt{\frac{T}{\rho}} > 0$. c is called the <u>velocity of propagation of waves along the string</u>.

(or: <u>the standing waves' speed of propagation along the string</u>)

$$\Rightarrow \partial_t^2 u(x,t) - c^2 \frac{\partial_x^2 u(x,t)}{\sqrt{1+(\partial_x u(x,t))^2}^{\,3}} = 0.$$

Since we assumed that the vibrations of the string are small, $\partial_x u(x,t) \approx 0 \Rightarrow \sqrt{1+(\partial_x u(x,t))^2}^{\,3} \approx 1$

$$\Rightarrow \boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0} \quad \text{(wave eq.)}.$$

D'Alembert's Theorem: Let $c \neq 0$. The general solution of

$$\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$$

is

$$\boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)},$$

where $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$ are arbitrary.

Pf: Define $A := \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \in Mat(2\times 2, \mathbb{R})$.

$\det(A) = -2c \neq 0$, so A is invertible with inverse $A^{-1} = \frac{1}{-2c}\begin{pmatrix} -c & -c \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} \end{pmatrix}$

Thus we have a coordinate change for spacetime:

$$T_A : \mathbb{R}^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ t \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x+ct \\ x-ct \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$T_{A^{-1}} = (T_A)^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^1 \times \mathbb{R}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto A^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{\xi+\eta}{2} \\ \frac{\xi-\eta}{2c} \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}.$$

If $u: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$, then $\tilde{u}: \mathbb{R}^2 \to \mathbb{R}$

$$(\xi,\eta) \mapsto u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

is the unique function that fits into

$$\begin{tikzcd}
\mathbb{R}^1 \times \mathbb{R} \arrow[rr, "u"] \arrow[dr, "T_A"'] & & \mathbb{R} \\
& \mathbb{R}^2 \arrow[ur, "\tilde{u}"', very thick] &
\end{tikzcd}$$

SW: Conversely, if $v: \mathbb{R}^2 \to \mathbb{R}$, then $\underset{\sim}{v}: \mathbb{R}^1 \times \mathbb{R} \to \mathbb{R}$

$$(x,t) \mapsto v(x+ct, x-ct)$$

is the unique function that fits into

$$\begin{tikzcd}
\mathbb{R}^1 \times \mathbb{R} \arrow[rr, "\underset{\sim}{v}", very thick] & & \mathbb{R} \\
& \mathbb{R}^2 \arrow[ul, "T_{A^{-1}}"] \arrow[ur, "v"'] &
\end{tikzcd}$$

and $\widetilde{(\underset{\sim}{v})} = v$.

The Chain Rule applied to the second triangle gives:

for any $(\xi,\eta) \in \mathbb{R}^2$: $D_{(\xi,\eta)}(v) = D_{T_{A^{-1}}(\xi,\eta)}(\underset{\sim}{v}) \cdot \underbrace{D_{(\xi,\eta)}(T_{A^{-1}})}_{= A^{-1}}$

$\Rightarrow$ for any $(\xi,\eta) \in \mathbb{R}^2$:

$$\left(\partial_\xi v(\xi,\eta) \quad \partial_\eta v(\xi,\eta)\right)_{1\times 2} = \left(\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \quad \partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)\right)_{1\times 2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} \end{pmatrix}_{2\times 2}$$

$$= \left(\frac{\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2} + \frac{\partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2c} \;\middle|\; \frac{\partial_x \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2} - \frac{\partial_t \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)}{2c} \right)_{1\times 2}$$

$$= \left(\left(\frac{\partial_t + c\partial_x}{2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \;\middle|\; \left(\frac{\partial_t - c\partial_x}{-2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \right)_{1\times 2}$$

$$\Rightarrow \partial_\xi v(\xi,\eta) = \left(\frac{\partial_t + c\partial_x}{2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right) \quad \text{and}$$

$$\partial_\eta v(\xi,\eta) = \left(\frac{\partial_t - c\partial_x}{-2c}\right) \underset{\sim}{v}\left(T_A^{-1}(\xi,\eta)\right)$$

$$\Rightarrow \partial_\xi \tilde{u}(x+ct, x-ct) = \left(\frac{\partial_t + c\partial_x}{2c}\right) u(x,t)$$

$$\partial_\eta \tilde{u}(x+ct, x-ct) = \left(\frac{\partial_t - c\partial_x}{-2c}\right) u(x,t).$$

$$\Rightarrow (\partial_t + c\partial_x)\, u(x,t) = 2c\, \partial_\xi \tilde{u}(x+ct, x-ct)$$

$$(\partial_t - c\partial_x)\, u(x,t) = -2c\, \partial_\eta \tilde{u}(x+ct, x-ct).$$

$\Rightarrow$ If $u(x,t)$ solves $\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$, then

$$0 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u(x,t) = (\partial_t - c\partial_x)\ \underbrace{2c\, \partial_\xi \tilde{u}(x+ct, x-ct)}_{=:\, w(x,t)}$$

$$= 2c\, (\partial_t - c\partial_x)\, w(x,t) = 2c\,(-2c)\, \partial_\eta \tilde{w}(x+ct, x-ct)$$

$$= (-4c^2)\, \partial_\eta \partial_\xi \tilde{u}(x+ct, x-ct) \underset{\begin{pmatrix} \xi := x+ct \\ \eta := x-ct \end{pmatrix}}{\overset{?}{=}} (-4c^2)\, \partial_\eta \partial_\xi \tilde{u}(\xi, \eta)$$

$$\Rightarrow \quad 0 = \partial_\eta \partial_\xi \tilde{u}(\xi, \eta)$$

$$\Rightarrow \quad \partial_\xi \tilde{u}(\xi, \eta) = \varphi_1(\xi)$$

$$\Rightarrow \quad \tilde{u}(\xi, \eta) = \varphi(\xi) + \psi(\eta)$$

$$\Rightarrow \quad u(x,t) = \varphi(x+ct) + \psi(x-ct), \ \checkmark.$$

SW: Conversely, any function $u(x,t)$ of the form

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

solves $\boxed{\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0}$. $\checkmark$

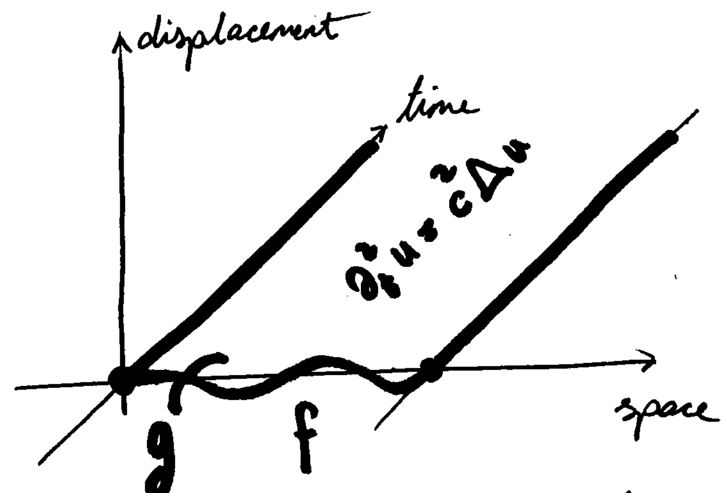
---.

Ex: (Homogeneous Vibration Problem for $d=1$)

Let $c \neq 0$, $L > 0$, $f, g \in C^0([0,L], \mathbb{R})$ be piecewise smooth and $f(0) = 0 = f(L)$, $g(0) = 0 = g(L)$. Consider

$$(*)\quad \boxed{\begin{array}{ll} \partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0, & \text{for } (x,t) \in \,]0,L[\, \times \,]0,\infty[\\ u(0,t) = 0 = u(L,t), & \text{for } t \in [0,\infty[\\ u(x,0) - f(x) = 0 = \partial_t u(x,0) - g(x), & \text{for } x \in [0,L] \end{array}} \quad \begin{array}{l} \text{(PDE)} \\ \text{(BC)} \\ \text{(ID)} \end{array}$$

SW: Give the geometric and physical interpretation of the initial data and the boundary conditions.



By D'Alembert's Theorem we know that the solution of the PDE is of the form

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

for two yet to be determined functions $\varphi, \psi \in C^1(\mathbb{R}, \mathbb{R})$.

Let $f_1, g_1 \in R(\mathbb{R},\mathbb{R})$ be piecewise smooth and extend f, g, respectively (ie., for any $x \in [0,L]$: $f_1(x) = f(x)$ and $g_1(x) = g(x)$, but as opposed to f and g, f_1 and g_1 are defined everywhere).

$\Rightarrow$ (ID) gives : $\varphi(x) + \psi(x) = f_1(x)$ & $c(\dot\varphi(x) - \dot\psi(x)) = g_1(x)$

$$\Rightarrow \frac{d}{dx}(\varphi - \psi)(x) = \frac{1}{c} g_1(x) \Rightarrow (\varphi - \psi)(x) = \frac{1}{c}\int_{-\infty}^{x} g_1(y)dy + D$$

$(D \in \mathbb{R})$

$$\Rightarrow \varphi(x) = \frac{1}{2}\left(f_1(x) + \frac{1}{c}\int_{-\infty}^{x} g_1(y)dy + D \right)$$

$$\psi(x) = \frac{1}{2}\left(f_1(x) - \frac{1}{c}\int_{-\infty}^{x} g_1(y)dy - D \right)$$

$$\Rightarrow u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\left(\int_{-\infty}^{x+ct} g_1(y)dy - \int_{-\infty}^{x-ct} g_1(y)dy\right)$$

$$= \frac{1}{2}\left(f_1(x+ct) + f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(y)dy.$$

$(x - ct \leq x+ct)$

(BC) gives: $\varphi(ct) + \psi(-ct) = 0 = \varphi(L+ct) + \psi(L-ct)$

$$\Rightarrow \quad 0 = \frac{1}{2}\left(f_1(ct) + f_1(-ct) \right) + \frac{1}{2c} \int_{-ct}^{ct} g_1(y)\,dy$$

$$0 = \frac{1}{2}\left(f_1(L+ct) + f_1(L-ct) \right) + \frac{1}{2c} \int_{L-ct}^{L+ct} g_1(y)\,dy,$$

but these are not sufficient to specify f_1 & g_1 (and consequently $u(x,t)$).

Thus we turn to the method of disentanglement:

$$u(x,t) := \pi(x)\,\tau(t) \Rightarrow (*) \Rightarrow \pi\ddot{\tau} - c^2\ddot{\pi}\tau = 0 \Leftrightarrow \frac{-\ddot{\tau}}{c^2\tau} = \frac{-\ddot{\pi}}{\pi} = \lambda$$

$$\Rightarrow \boxed{\begin{array}{l} u(x,t) = \pi(x)\,\tau(t) \\ -\Delta\pi(x) = \lambda\,\pi(x) \\ \pi(0) = 0 = \pi(L) \\ -\partial_t^2 \tau(t) = c^2\lambda\,\tau(t) \end{array}}$$ is a disentanglement of $(*)$.

$\pi(0) = 0 = \pi(L) \Rightarrow$ For any $n \geq 1$: $\left(\left(\frac{n\pi}{L}\right)^2, \sin\left(\frac{n\pi}{L}x\right) \right)$ is a relevant eigenpair of $-\Delta$.

$$\Rightarrow -\partial_t^2 \tau(t) = \underbrace{\left(\frac{cn\pi}{L}\right)^2}_{>0} \tau(t) \Rightarrow \tau(t) = c_1 \cos\left(\frac{cn\pi}{L}t\right) + c_2 \sin\left(\frac{cn\pi}{L}t\right)$$

$\Rightarrow$ The relevant disentangled solutions of ⊛ are:

For any $n \geq 1$:

$$\sin\left(\frac{n\pi}{L}x\right)\cos\left(c\frac{n\pi}{L}t\right) = \sigma_n(x)\,\gamma_n(ct) = \frac{1}{2}\left(\sigma_n(x+ct) + \sigma_n(x-ct)\right)$$

$$\sin\left(\frac{n\pi}{L}x\right)\sin\left(c\frac{n\pi}{L}t\right) = \sigma_n(x)\,\sigma_n(ct) = -\frac{1}{2}\left(\gamma_n(x+ct) - \gamma_n(x-ct)\right)$$

<u>SW</u>: Verify the (nontrivial) equalities above.

If $u(x,t) = \sum_{n\geq 1} a_n\,\sigma_n(x)\,\gamma_n(ct) + \sum_{n\geq 1} b_n\,\sigma_n(x)\,\sigma_n(ct)$ solves ⊛,

then $f(x) = u(x,0) = \sum_{n\geq 1} a_n\,\sigma_n(x)$ and

$$g(x) = \partial_t u(x,0) = \left[\sum_{n\geq 1} b_n \frac{cn\pi}{L}\,\sigma_n(x)\,\gamma_n(ct)\right]\Bigg|_{t=0} = \sum_{n\geq 1} b_n \frac{cn\pi}{L}\,\sigma_n(x).$$

Both f and g are piecewise smooth, so by the Fourier Convergence Theorem

for all $x \in [0,L]$: $\mathcal{F}(2\tilde{f}_o)(x) = f(x)$

and $\mathcal{F}(2\tilde{g}_o)(x) = g(x)$

$$\mathcal{F}\left(2\tilde{f}_o\right)(x) = \frac{c_0^f}{2} + \sum_{n\geq 1} c_n^f \gamma_n(x) + \sum_{n\geq 1} s_n^f \sigma_n(x).$$

$(n\geq 0)$ $c_n^f = 0$ because $2\tilde{f}_o$ is odd.

$(n\geq 1)$ $$s_n^f = \frac{1}{L}\int_{-L}^{L} \underbrace{2\tilde{f}_o(x)\,\sigma_n(x)}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\,\sigma_n(x)\,dx$$

$$\mathcal{F}\left(2\tilde{g}_o\right)(x) = \frac{c_0^g}{2} + \sum_{n\geq 1} c_n^g \gamma_n(x) + \sum_{n\geq 1} s_n^g \sigma_n(x)$$

$(n\geq 0)$ $c_n^g = 0$ because $2\tilde{g}_o$ is odd.

$(n\geq 1)$ $$s_n^g = \frac{1}{L}\int_{-L}^{L} \underbrace{2\tilde{g}_o(x)\,\sigma_n(x)}_{\text{even}}\,dx = \frac{2}{L}\int_0^L g(x)\,\sigma_n(x)\,dx$$

$\Rightarrow$ Picking $a_n := s_n^f$ and $b_n := \frac{L}{cn\pi} s_n^g$ produces the solution of $\circledast$:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n\geq 1} s_n^f \sin\left(\frac{n\pi}{L}x\right)\cos\left(c\frac{n\pi}{L}t\right) + \sum_{n\geq 1}\frac{L}{cn\pi} s_n^g \sin\left(\frac{n\pi}{L}x\right)\sin\left(c\frac{n\pi}{L}t\right), \\ \text{where } s_n^f &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\ s_n^g &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx. \end{aligned}}$$

Alternatively, we could rewrite the general solution as:

$$u(x,t) = \sum_{n\geq 1} a_n \sigma_n(x)\gamma_n(ct) + \sum_{n\geq 1} b_n \sigma_n(x)\sigma_n(ct)$$

$$= \sum_{n\geq 1} a_n \left(\frac{\sigma_n(x+ct)+\sigma_n(x-ct)}{2}\right) + \sum_{n\geq 1} b_n \left(\frac{\gamma_n(x+ct)-\gamma_n(x-ct)}{-2}\right)$$

$$= \frac{1}{2}\left(\sum_{n\geq 1} a_n \sigma_n(x+ct) + \sum_{n\geq 1} a_n \sigma_n(x-ct)\right)$$
$$-\frac{1}{2}\left(\sum_{n\geq 1} b_n \gamma_n(x+ct) - \sum_{n\geq 1} b_n \gamma(x-ct)\right)$$

$$\underset{\uparrow}{=} \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) - \frac{1}{2}\left(h_1(x+ct)-h_1(x-ct)\right).$$

$$\left(\begin{array}{l} f_1 := \sum_{n\geq 1} a_n\sigma_n \\ h_1 := \sum_{n\geq 1} b_n\gamma_n \end{array}\right) \qquad = \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) - \frac{1}{2}\int_{x-ct}^{x+ct} \dot{h}_1(y)\,dy$$

$$\left(\begin{array}{l} \dot{h}_1 = -\sum_{n\geq 1} b_n \frac{n\pi}{L}\sigma_n \\ g_1 := -c\,\dot{h}_1 \end{array}\right) = \frac{1}{2}\left(f_1(x+ct)+f_1(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_1(y)\,dy.$$

$$f(x) = u(x,0) = f_1(x) \overset{\text{def}}{=} \sum_{n\geq 1} a_n\sigma_n(x)$$
$$g(x) = \partial_t u(x,0) = g_1(x) \overset{\text{def}}{=} \sum_{n\geq 1} b_n \frac{cn\pi}{L}\sigma_n(x).$$

$\Rightarrow$ f_1 is the odd periodic extension of f & g_1 is closely related to the odd periodic extension of g (as before).

SW: Verify.

. The point is that D'Alembert's Theorem provides a large class of solutions, but determining which particular solution actually satisfies the boundary conditions as well is hard. On the other hand, the method of entanglement provides solutions in terms of limits of linear combinations of disentangled solutions, but since no disentangled solution decays exponentially fast in time it is not easy to see that the limits in question actually make sense.

SW: Replace the boundary condition of ⊛ with your favorite boundary condition, then find the solution.

Ex: Let $c := 3$, $L := 5$, $f : [0,5] \to \mathbb{R}$

$$x \mapsto 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x)$$

$$g : [0,5] \to \mathbb{R}$$

$$x \longmapsto 0.$$

Solve

$$\boxed{\begin{aligned} &\partial_t^2 u(x,t) - 9\Delta u(x,t) = 0 \\ &u(0,t) = 0 = u(5,t) \\ &u(x,0) = f(x) \\ &\partial_t u(x,0) = g(x) = 0. \end{aligned}}$$

$$f(x) = 4\sin\left(\frac{5\pi}{5}x\right) - \sin\left(\frac{10\pi}{5}x\right) - 3\sin\left(\frac{25\pi}{5}x\right)$$

is odd and equal to its Fourier series:

$$\mathcal{F}(f) = \sum_{n\geq 1} s_n \sigma_n, \text{ where } \quad s_n = \begin{cases} 4, & \text{if } n = 5 \\ -1, & \text{if } n = 10 \\ -3, & \text{if } n = 25 \\ 0, & \text{otherwise} \end{cases}$$

$$\sigma_{\mathcal{F}}(g) = g = 0$$

D'Alembert's Theorem dictates:

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+3t) + f(x-3t)\right)}$$

$$u(0,t) = \frac{1}{2}\left(f(3t) + f(-3t)\right) = 0 \quad (f \text{ is odd})$$

$$u(5,t) = \frac{1}{2}\left(f(5+3t) + f(5-3t)\right) = 0.$$

SW: (i) Verify this

(ii) Solve it via disentanglement.