03/03/2020.

Integrals in 2D & 3D.

- $\oint$. $a<b$. $f: \mathbb{R} \to \mathbb{R}$. $\int_a^b f(x)\,dx$.

$$\int_b^a f(x)\,dx = ?$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

integration in the sense of differential forms; Orientation does matter.

(Stokes, Green, ... later)

$$\int_b^a f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_b^a f(x)\,dx = \int_{[b,a]} f(x)\,dx = 0$$

$$[b,a] = \{x \in \mathbb{R} \mid b \leq x,\ a \geq x\} = \emptyset.$$

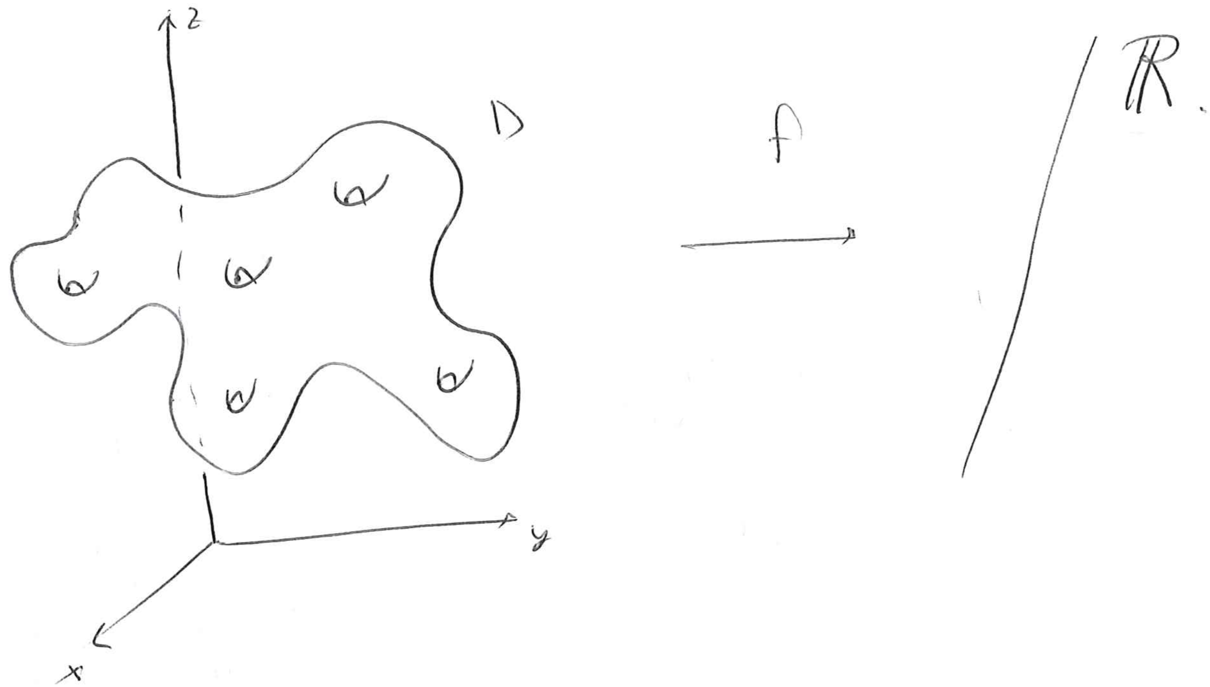
Möbius
Klein bottle

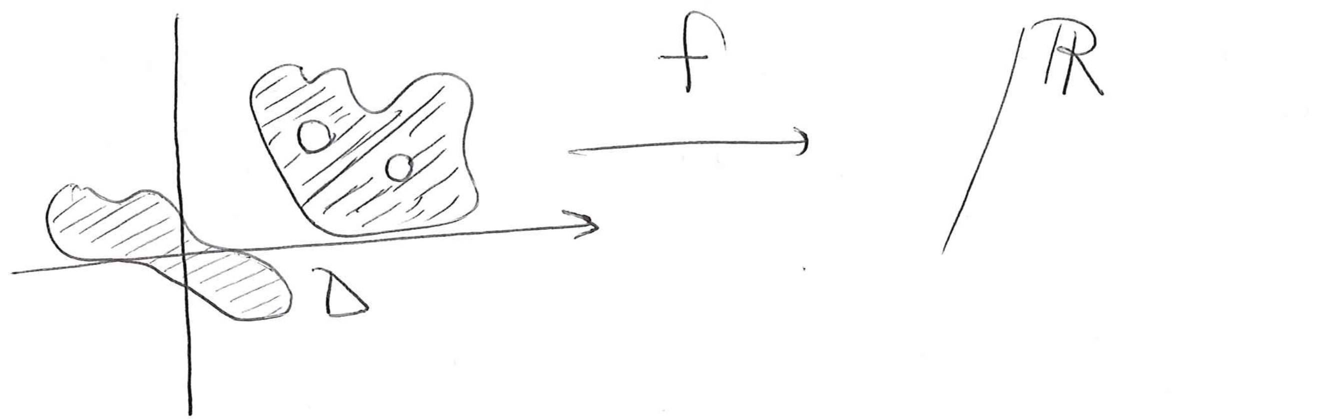
integration in the sense of densities; orientation/ orientability does not matter.

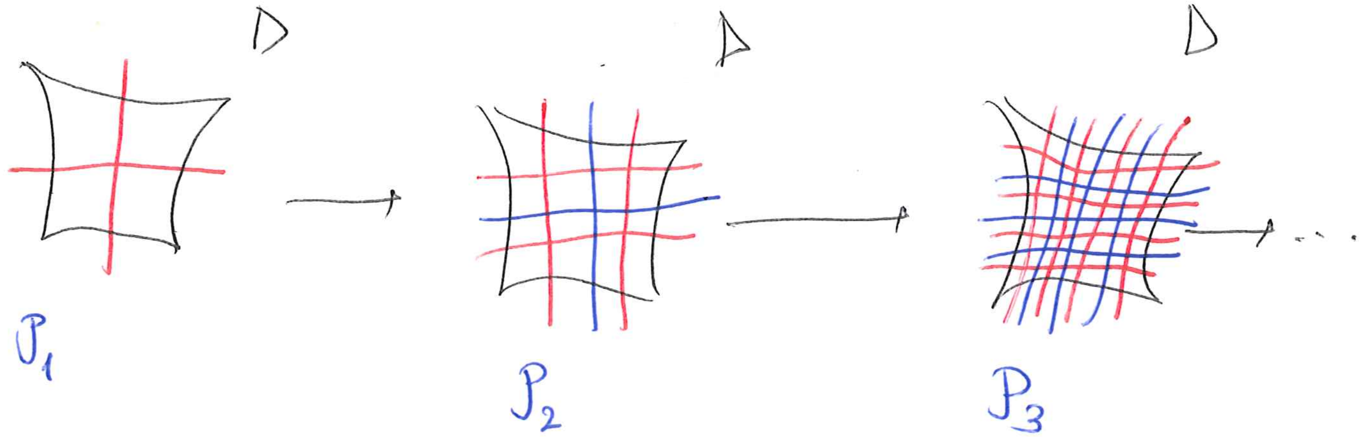
(coordinate change).

arclength
surface area

- $f: \mathbb{R}^3 \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$.







$\mathcal{P}$: finite partition of D.

$$D = \biguplus_{P \in \mathcal{P}} P \qquad P : \text{cell.}$$

$\forall P \in \mathcal{P}$: $(x,y,z)_P \in P$,

$$\lim_{\operatorname{diam}(\mathcal{P}) \to 0} \sum_{P \in \mathcal{P}} f(x_P, y_P, z_P) =: \int_D f(x,y,z)\, dV$$

$$= \iiint_D f(x,y,z)\, dV$$

$$= \int_D f\, dV = \int_D f(x,y,z)\, d(x,y,z)$$

$\int_D f(x,y)\, dA$ in $f: \mathbb{R}^2 \to \mathbb{R}$ case.

- $f: \mathbb{R}^3 \to \mathbb{R}$ is integrable (in the sense of Rieman) if lim does not depend on the seq. $\mathcal{P}_n$ of partitions and sample values.

• Continuous functions are integrable.

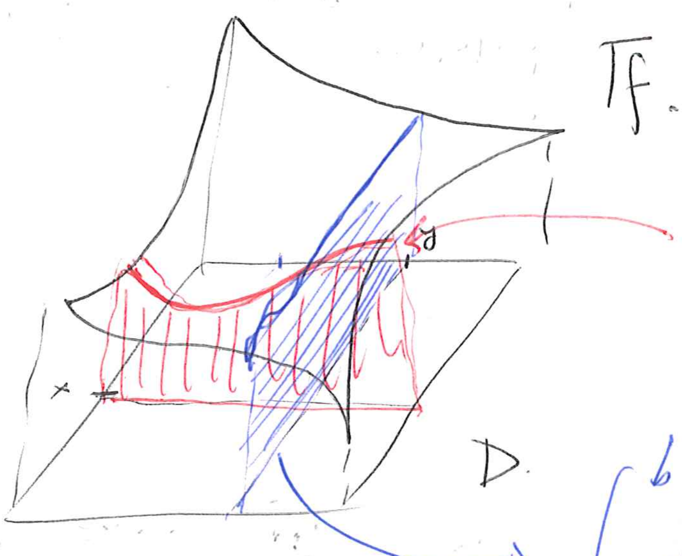
Fubini's Theorem (Rectangles/Boxes)

(i) $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$

$= [a,b] \times [c,d]$

$f : \mathbb{R}^2 \to \mathbb{R}$ integrable.

$$\Rightarrow \int_R f\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.$$



$\int_c^d f(x,y)\, dy.$ (function of x)

$\int_a^b f(x,y)\, dx$ (function of y).

$$\boxed{dA = dx \wedge dy}$$

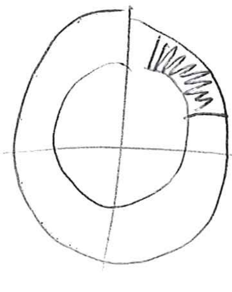
(ii) Polar coordinates:

$$\boxed{dA = r\, dr \wedge d\theta}$$

$x = r\cos\theta$

$y = r\sin\theta.$

$f : \mathbb{R}^2 \to \mathbb{R}.$



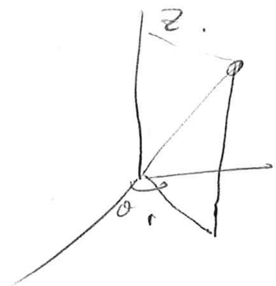
(iii) Cylindrical Coordinates

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z.$



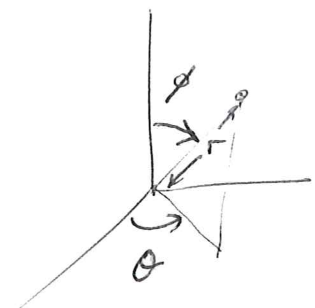
(iv) Spherical Coordinates.

$$\boxed{dV = r^2 \sin\phi\,dr\,d\theta\,d\phi}$$

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

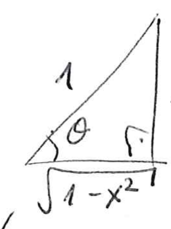
$z = r\cos\phi$



Ex: $R = [-1,1]\times[-2,2]$.

$f(x,y) = \sqrt{1-x^2}$.

$$\int_R f\,dA = \left(\int_{-2}^{2} dy\right)\cdot\left(\int_{-1}^{1}\sqrt{1-x^2}\,dx\right)$$



$$\left(\begin{array}{c}\sqrt{1-x^2} = \cos\theta.\\ x = \sin\theta.\\ dx = \cos\theta\,d\theta\end{array}\right)$$

$-1 \le x \le 1$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= 4\int_{-\pi/2}^{\pi/2} \sin\theta\cos\theta\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sin(2\theta)\,d\theta = -2\left[\frac{+\cos(2\theta)}{2}\right]\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \cos(2\theta)\Big|_{-\pi/2}^{\pi/2} = \cos(\pi) - \cos(-\pi) = -2.$$

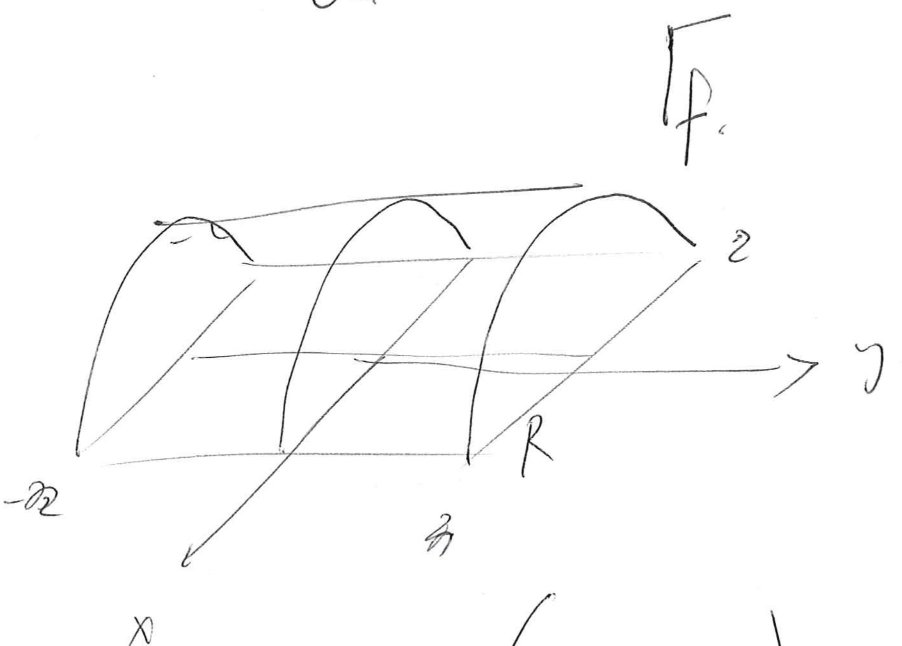
$$= 4 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = 4 \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2} \, d\theta$$

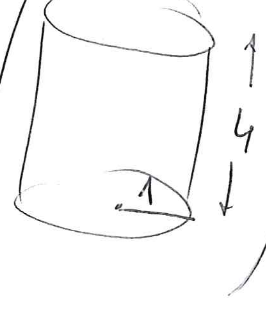
$$= 2 \left[\theta - \frac{\sin(2\theta)}{2}\right]\Bigg|_{-\pi/2}^{\pi/2}$$

$$\boxed{\begin{array}{l} \cos(2\theta) = \cos^2\theta - \sin^2\theta \\ 1 = \cos^2\theta + \sin^2\theta. \end{array}}$$

$$= 2 \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi.$$

Or:



$$\int_R f \, dA = \frac{1}{2} \text{ vol.}\Bigg($$ 

$$\Bigg) = \frac{1}{2} \pi \cdot 4 = 2\pi.$$

Ex: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$.

$$= \int_0^3 x^2 \left(\int_1^2 y\,dy\right) dx$$

$$= \left(\int_1^2 y\,dy\right) \cdot \left(\int_0^3 x^2\,dx\right)$$

$$= \left[\frac{y^2}{2}\right]\Bigg|_{y=1}^{2} \cdot \left[\frac{x^3}{3}\right]\Bigg|_{x=0}^{3}$$

$$= \left(\frac{4}{2} - \frac{1}{2}\right) \cdot \frac{27}{3} = \frac{27}{2}.$$

Ex: $R = [0,2] \times [1,2]$ $\quad f(x,y) = x - 3y^2$.

$$\int_R f\,dA = \int_0^2 \left(\int_1^2 x - 3y^2\,dy\right) dx$$

$$= \int_0^2 \left[xy - y^3\right]\Big|_{y=1}^{2} dx = \int_0^2 \left[(2x-8) - (x-1)\right] dx$$

$$= \int_0^2 (x-7)\,dx = \left[\frac{x^2}{2} - 7x\right]\Bigg|_{x=0}^{2} = 2 - 14 = -12.$$

Ex: Vol. of solid bounded by

$x^2+2y^2+z=16$, $x=2$, $y=2$ and

the coordinate planes.

$$f(x,y) = 16 - x^2 - 2y^2.$$

$$R = [0,2]\times[0,2].$$

$$\int_R f\,dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - 2xy^2\right]\Big|_{x=0}^{2} dy$$

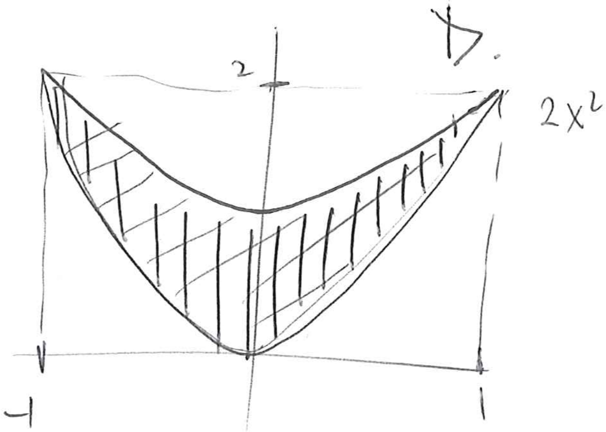
$$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2\right) dy = \int_0^2 \left(\frac{88}{3} - 4y^2\right) dy$$

$$= \left[\frac{88}{3}y - \frac{4}{3}y^3\right]\Big|_0^2 = \frac{176}{3} - \frac{4\cdot 8}{3} = \frac{176-32}{3}$$

$$= \frac{144}{3} = 48.$$

Ex: $D = \{(x,y) \in \mathbb{R}^2 \mid 2x^2 \le y \le 1+x^2\}$

$f(x,y) = x + 2y$.



$2x^2 = 1+x^2$

$x^2 = 1$

$\Rightarrow x = \mp 1$.

$$\int_R f\,dA = \int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} f\,dy \right) dx$$

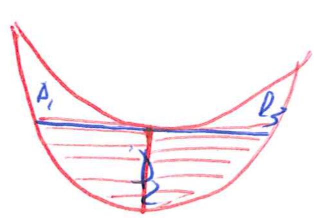
$$= \int_{-1}^{1} \left[xy + y^2 \right] \Big|_{y=2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left[\left(x(1+x^2) + (1+x^2)^2 \right) - \left(x\, 2x^2 + 4x^4 \right) \right] dx$$

$$= \int_{-1}^{1} \left[\left(x + x^3 + x^4 + 2x^2 + 1 \right) - \left(2x^3 + 4x^4 \right) \right] dx$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1 \right) dx \underset{SW}{=} \frac{32}{15}.$$

SW.

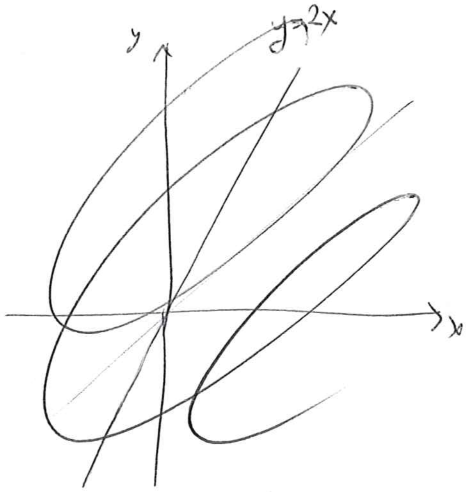


Ex: Vol. of solid under $z=x^2+y^2$,

above $D \subseteq \mathbb{R}^2$ bounded by

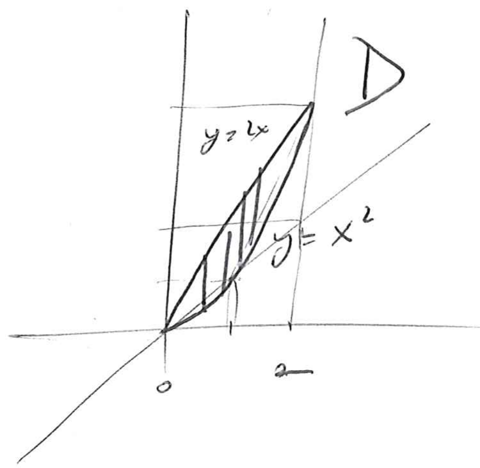
$y=2x$ & $y=x^2$.

$$f(x,y)=x^2+y^2.$$



$$2x = x^2$$

$$\Rightarrow x=0 \quad \text{or} \quad x=2.$$



$$\iint_D f\,dA=\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$$

$$=\int_0^4\int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy \quad \text{SW.}$$

$$=\int_0^2 \left[x^2y+\frac{y^3}{3}\right]\Bigg|_{y=x^2}^{2x} dx = \int_0^2\left[\left(x^2\cdot 2x+\frac{(2x)^3}{3}\right)-\left(x^2\cdot x^2+\frac{x^6}{3}\right)\right]dx$$

$$=\int_0^2\left[\left(2x^3+\frac{8}{3}x^3\right)-\left(x^4+\frac{x^6}{3}\right)\right]dx=\int_0^2\left(-\frac{x^6}{3}-x^4+\frac{14}{3}x^3\right)dx=\frac{216}{35}.$$

SW

$\underline{Ex}$: $f(x,y) = xy$

D region bounded by $y = x-1$ &
$y^2 = 2x+6$.

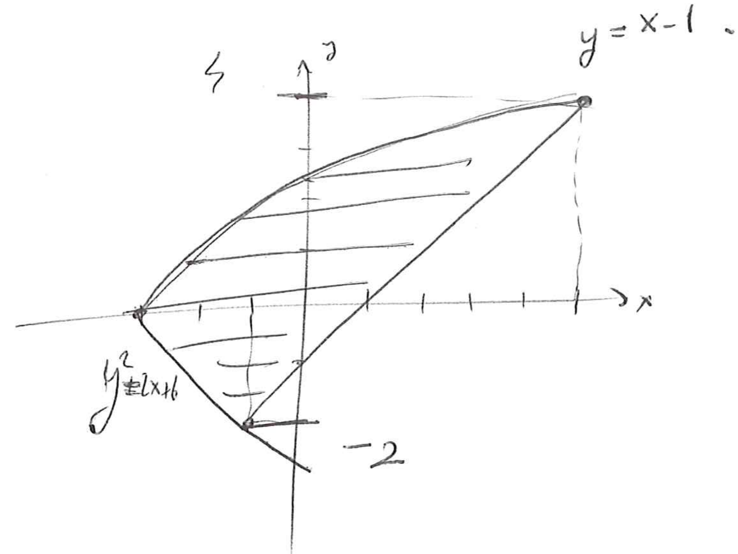
$x^2-2x+1 = 2x+6$

$x^2-4x-5 = 0$

$(x-5)(x+1) = 0.$

$x = 5$ or $x = -1$

$y = 4$ or $y = -2$.



$$\iint_D f\,dA = \int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} f(x,y)\,dx\,dy.$$

$$= \int_{-2}^{4}\left[y\left(\int_{\frac{y^2-6}{2}}^{y+1} x\,dx\right)\right]dy$$

$$= \int_{-2}^{4} y\left[\frac{x^2}{2}\right]\Bigg|_{x=\frac{y^2-6}{2}}^{y+1} dy = \int_{-2}^{4}\left[\frac{y}{2}\left((y+1)^2 - \left(\frac{y^2-6}{2}\right)^2\right)\right]dy$$

$$= \int_{-2}^{4}\frac{y}{2}\left((y^2+2y+1) - \frac{y^4-12y^2+36}{4}\right)dy = 36$$

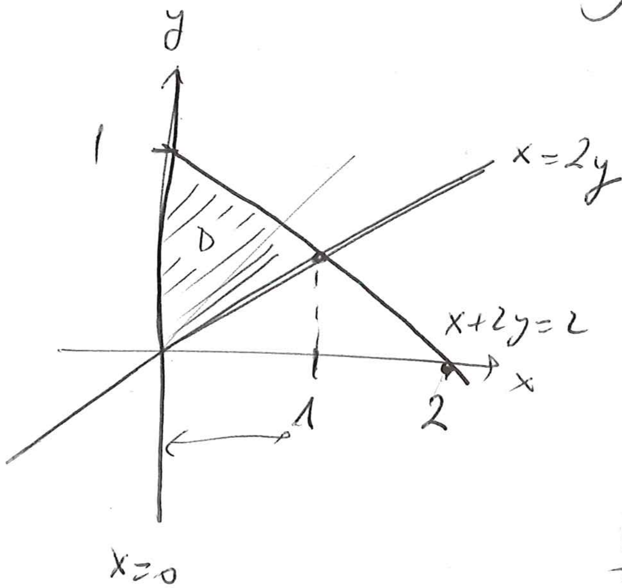
sw.

Ex: Vol. of tetrahedron bounded by

$$\left.\begin{aligned} &x+2y+z=2 \\ &x=2y \\ &x=0 \\ &z=0. \end{aligned}\right\} \mathcal{B}.$$

$\mathcal{B}$

$z=0 \Rightarrow x+2y=2$



$$f(x,y) = 2-x-2y.$$

$$\iint_{\mathcal{D}} f\,dA := \int_0^1 \left(\int_{x/2}^{\frac{2-x}{2}} f(x,y)\,dy \right) dx$$

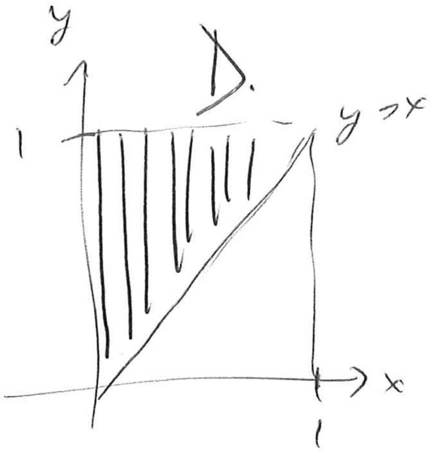
$$= \int_0^1 \left(\int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y)\,dy \right) dx$$

$$= \int_0^1 \left[2y - xy - y^2 \right] \Big|_{y=\frac{x}{2}}^{\frac{2-x}{2}} dx$$

$$= \int_0^1 \left[\left(2\left(\frac{2-x}{2}\right) - x\left(\frac{2-x}{2}\right) - \left(\frac{2-x}{2}\right)^2 \right) - \left(2\cdot\frac{x}{2} - x\frac{x}{2} - \left(\frac{x}{2}\right)^2 \right) \right] dx$$

$$\underset{\text{sw}}{=} \frac{1}{3}.$$

Ex:
$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \int_D f(y)\,dA$$



$$\left\{ \begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array} \right\}$$

$$\left\{ \begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le y \end{array} \right\}.$$

$$= \int_0^1 \int_0^y f(y)\,dx\,dy$$

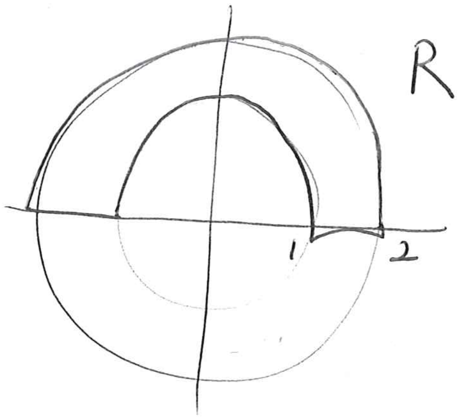
$$= \int_0^1 \left[f(y) \left(\int_0^y dx \right) \right] dy$$

$$= \int_0^1 y \sin(y^2)\,dy =$$

$$= \left[\frac{-\cos(y^2)}{2} \right] \Bigg|_{y=0}^{1} = \frac{-\cos(1)}{2} + \frac{1}{2}.$$

Ex: $f(x,y) = 3x + 4y^2$.

R = region in the upper half plane bounded by $x^2+y^2=1$ & $x^2+y^2=4$.



$1 \le r \le 2$.

$0 \le \theta \le \pi$.

$f(r\cos\theta, r\sin\theta)$

$= r\cos\theta + 4r^2\sin^2\theta$.

$$\int_R f\,dA = \int_0^{\pi}\int_1^2 (r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

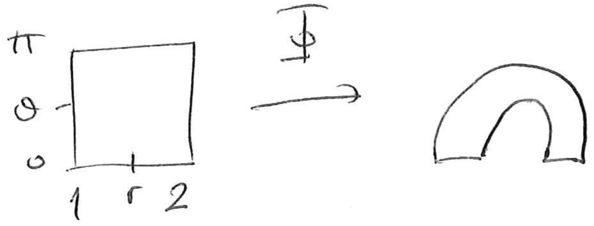
$$= \int_0^{\pi}\int_1^2 \left(r^2\cos\theta + 4r^3\sin^2\theta\right) dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{r^3}{3}\cos\theta + r^4\sin^2\theta\right]\Bigg|_{r=1}^{2} d\theta$$

$$= \int_0^{\pi}\left[\left(\frac{8}{3}\cos\theta + 16\sin^2\theta\right) - \left(\frac{1}{3}\cos\theta + \sin^2\theta\right)\right] d\theta$$

$$= \int_0^{\pi}\left(\frac{7}{3}\cos\theta + 15\sin^2\theta\right) d\theta \underset{\text{SW}}{=} \frac{15\pi}{2}$$

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$.

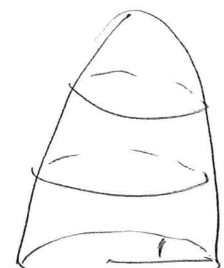


$\boxed{|J_\Phi| = r.}$

Ex: Vol. of solid bounded by

$$z = 0 \quad \& \quad z = 1 - x^2 - y^2.$$

$$0 = 1 - x^2 - y^2 \iff x^2 + y^2 = 1.$$



$$D = \left\{ (r,\theta) \;\middle|\; \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix} \right\}.$$

$$f(x,y) = 1 - x^2 - y^2.$$

$$\cancel{g(r,\theta) = 1 - r^2}.$$

$$\int_R f\,dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \int_0^1 \left(-r^3 + r\right) dr$$

$$= 2\pi \cdot \left[\frac{-r^4}{4} + \frac{r^2}{2} \right]\Bigg|_{r=0}^{1} = 2\pi \left(-\frac{1}{4} + \frac{1}{2} \right) = \frac{\pi}{2}.$$

Ex : Vol. of the solid that lies

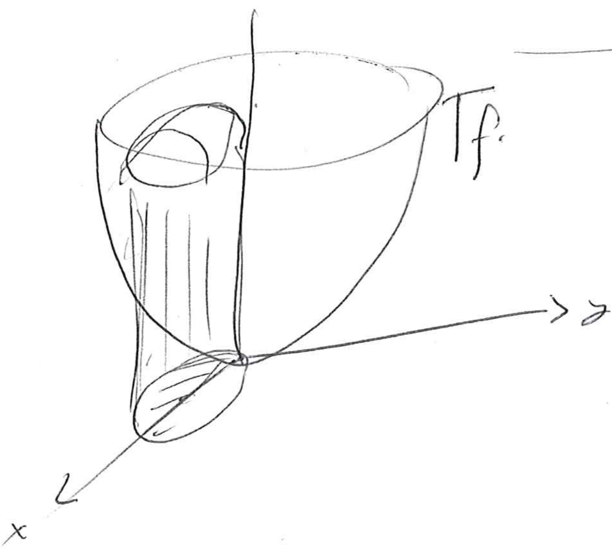
under $z = x^2+y^2$

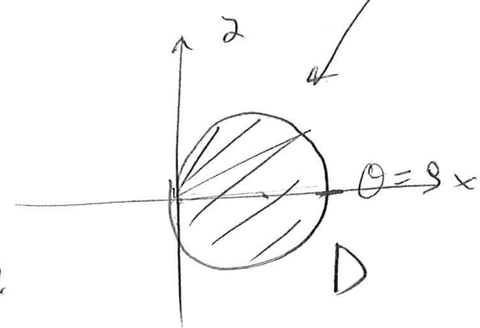
above $z = 0$

inside $x^2+y^2 = 2x$.

$\left((x-1)^2+y^2 = 1.\right)$.

$f(x,y) = x^2+y^2$.





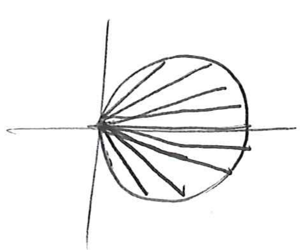
∂D : $\cancel{(x-1)^2+y^2=1}$.

$x = r\cos\theta$
$y = r\sin\theta$

$\cancel{(r\cos\theta-1)^2}$

$r^2 = x^2+y^2 = 2r\cos\theta$

$\Rightarrow \boxed{r = 2\cos\theta}$



(6) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$0 \le r \le 2\cos\theta$$

$$\iint_D f\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]\Bigg|_{r=0}^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4(\cos\theta)^4\,d\theta$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4(\cos^2\theta)^2\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(1+\cos(2\theta)\right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[1 + 2\cos(2\theta) + \cos^2(2\theta)\right] d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right] d\theta \overset{SW}{=} \frac{3\pi}{2}.$$