Smooth Ergodic Theory of Higher Rank Abelian Actions

Title: Smooth Ergodic Theory of Higher Rank Abelian Actions
Speaker: Alp
Abstract: We'll introduce differentiable dynamical systems with multidimensional abelian time. In this setting Oseledets Theorem provides existence of Lyapunov exponents that are better thought as linear maps, as opposed to simply numbers for dynamical systems with one dimensional time. We will discuss the suspension construction in this context, and how the arrangements of Lyapunov hyperplanes relate to entropy.

by Alp Uzman

Plan:

~~1~~) Smooth Ergodic Theory of $\mathbb{R}^k \curvearrowright M$

$(k \geq 2)$

~~1~~/2: Suspension.

2) Oseledets Thm.

3) Maximal Rank Assumption

1) M C^∞ compact manifold.

$\mathrm{Diff}^{1+}(M)$ $\qquad k \geq 1$.

<u>action</u> : $\alpha : \mathbb{R}^k \longrightarrow \mathrm{Diff}^{1+}(M)$

group hom.

$$\alpha : \mathbb{R}^k \times M \to M \quad C^{1+}$$

Assume α is locally free

i.e., $\forall x \in M$: $\mathbb{R}^k_x = \{t \in \mathbb{R}^k \mid \alpha_t(x) = x\}$

$\leq \mathbb{R}^k$ is discrete.

$\iff \forall x \in M, \ \exists U_x \in \mathcal{N}(0)$:

$$\alpha : U_x \times \{x\} \hookrightarrow M.$$

$\to \exists$ orbit foliation $\mathrm{Orb}(\alpha)$

$\mathcal{A}(\mathbb{R}^k \curvearrowright M)$ = space of all such actions.

The smooth ergodic theory (with no potential) of $\mathbb{R}^k \curvearrowright M$ is:

$$\mathcal{E}(\mathbb{R}^k \curvearrowright M) = \left\{ (\alpha, \mu) \;\middle|\; \begin{array}{l} \alpha \in \mathcal{A}(\mathbb{R}^k \curvearrowright M) \\ \mu \in \mathrm{Prob}(M; \alpha) \end{array} \right\}$$

$$\begin{tikzcd}
 & \mathcal{E}(\mathbb{R}^k \curvearrowright M) \arrow[dl, "A"'] \arrow[dr, "P"] \arrow[r, "\text{thermo}^{\mu}"] & \mathbb{R}_{\geq 0} \\
\mathcal{A}(\mathbb{R}^k \curvearrowright M) & & \mathrm{Prob}(M).
\end{tikzcd}$$

entropy

μ is α. inv. if

$\forall t \in \mathbb{R}^k$: $\quad \vec{\alpha}_t(\mu) = \mu$.

$\vec{F}$: covariant functor. $\quad f_*$

$\overleftarrow{F}$: contravariant $\quad f^*$

Suspension! $\alpha : \mathbb{Z}^k \curvearrowright M$. $k \geq 1$

$$\gamma^\alpha : \mathbb{Z}^k \curvearrowright (\mathbb{R}^k \times M)$$

$$\gamma^\alpha_t (r, x) = (r - t, \alpha_t(x)).$$

$$\mathbb{R}^k \otimes_\alpha M = (\mathbb{R}^k \times M) / \gamma^\alpha.$$

$$\ell_\cdot : \mathbb{R}^k \curvearrowright (\mathbb{R}^k \times M)$$

$$\ell_t(r, x) = (t + r, x)$$

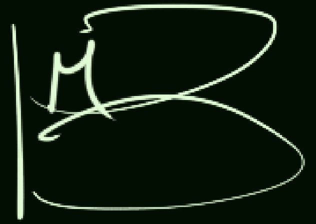
$$\Rightarrow \quad h^\alpha_\cdot : \mathbb{R}^k \curvearrowright (\mathbb{R}^k \otimes_\alpha M)$$

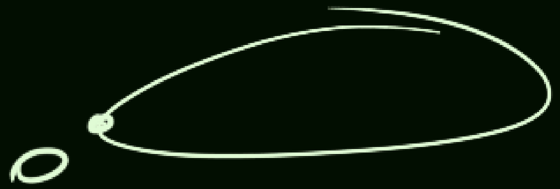
$$h^\alpha_t [r, x] = [t + r, x]$$

$$\begin{tikzcd}
\mathbb{R}^k \otimes_\alpha M \arrow[d, "\pi^\alpha"'] & \mathbb{R}^k \times M \arrow[l] \arrow[d] \\
\mathbb{T}^k & \mathbb{R}^k \arrow[l]
\end{tikzcd}$$

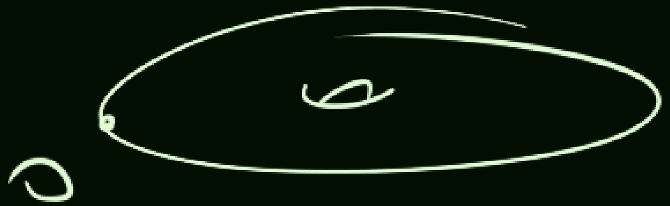
$\Rightarrow \quad M \to \mathbb{R}^k \otimes_\alpha M \xrightarrow{\pi^\alpha} \mathbb{T}^k \quad C^{1+}$

fibré lisse fibration...





$k=1$



$k=2$

Ex: $\alpha : \mathbb{Z}^2_{\geq 0} \curvearrowright \Pi$

$$\alpha_{(t_1, t_2)}(x) = 2^{t_1} 3^{t_2} x$$

$\exists \quad \mathbb{R}^k \otimes_\alpha \Pi$

$\to$ 1 dyadic

$\to$ 1 triadic

1 Archimedean

Oseledets Thm: Let $\alpha_\cdot \in \mathcal{Q}(\mathbb{R}^k, \mu^d)$

$\boxed{\text{EX 1}}$ $\exists\, M(\alpha_\cdot) \in \mathcal{B}(\mu^d)$. $\quad \bar{d} = \{1, 2, \ldots, d\}$

$\forall x \in M(\alpha_\cdot)$,

total Lyap. exp. of $\alpha_\cdot$ at x.

$\exists!\, l_x \in \bar{d}$, $\exists!\, X_x(\alpha_\cdot) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^d)$

$\exists!\, \tilde{X}_x(\alpha_\cdot) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^{l_x})$

reduced total Lyap. exp. of $\alpha_\cdot$ at x.

$$\tilde{X}_x(\alpha_\cdot) = \left(x_x^1(\alpha_\cdot), \dots, x_x^{l_x}(\alpha_\cdot) \right)$$

$$X_x(\alpha_\cdot) = \left(X_x^1(\alpha_\cdot), \_\_\_\_, X_x^d(\alpha_\cdot) \right)$$

$$\exists!\, T_x M = \bigoplus_{i \in l_x} L_x\left(x_x^i(\alpha_\cdot)\right)$$

$$\mathrm{LSpec}_x(\alpha.) = \{\chi_x^1(\alpha.), \ldots, \chi_x^{l_x}(\alpha.)\} \subseteq (\mathbb{R}^k)^*$$

Lyap. spec. of $\alpha.$ at x.

$$DM_x(\chi_x^i(\alpha.)) := \dim(L(\chi_x^i(\alpha.))).$$

$\boxed{\text{EST}}$ $\forall v \in L_x(\chi_x^i(\alpha.)) \setminus 0$

$$\lim_{|t|\to\infty} \frac{\ln(\|T_x\alpha_t v\|) - \chi_x^i(t)}{|t|} = 0.$$

w.r.t any C^0 norm on TM.

$$\lim_{|t|\to\infty} \frac{\ln\left(\mathrm{Jac}_x(\alpha_t)\right) - \sum_{i\in I_x} DM_x\left(\chi^i(\alpha_\cdot)\right)\cdot \xi^i_x(\alpha(t))}{|t|}$$

$$= 0.$$

w/r/t any C^0 1-density.

$\boxed{\text{INV}}$ • $\mathcal{M}(\alpha)$ is α. inv.,

$\forall_\mu \in \mathrm{Prob}(\mathcal{M}, \alpha.)$: $\mathcal{M}(\alpha) =_\mu \mathcal{M}$,

$$\ell_\bullet,\ X_\bullet,\ \tilde{X}_\bullet,\ \bigoplus_{i \in \ell_\bullet} L_\bullet\left(x^i_\bullet(\alpha)\right)$$

DM. are all measurable and

α. inv.

. So in the case of

$\mu \in e\mathrm{Prob}(M, \alpha)$, all these

things are invariant.

(except the splitting).

Consider $(\alpha_\cdot, \mu)$ ergodic.

$X(\alpha_\cdot, \mu) : \mathbb{R}^k \longrightarrow \mathbb{R}^d$. linear map.

$$\operatorname{defect}(\alpha_\cdot, \mu) = \dim\left(\ker\left(X(\alpha_\cdot, \mu)\right)\right)$$

$$\operatorname{codefect}(\alpha_\cdot, \mu) = \dim\left(\operatorname{im}\left(X(\alpha_\cdot, \mu)\right)\right).$$

Prop: If f in $\mathcal{K}(\alpha_i, \gamma_i)$ intersects the positive hyperoctant $\mathbb{R}^d_{>0}$, then

Γ is atomic $\Rightarrow$ zero entropy