Arithmeticity for Smooth Maximal Rank Positive Entropy Actions of $\mathbb{R}^k$

PhD Dissertation Defense

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Short Take-Away

A nonuniform measure rigidity result:

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- certain ("maximal rank positive entropy" - MRPE) but fairly general higher rank abelian actions of $\mathbb{R}^k$ via diffeomorphisms ($k \geq 2$ commuting vector fields) are in fact suspensions of $\mathbb{Z}^k$ actions on $(k+1)$-dimensional tori via commuting hyperbolic infranilmanifold automorphisms ("homomorphism" but may not fix the "origin").

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A nonuniform measure rigidity result:

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- "rigidity" $=$ algebra in hiding.
- "measure" $=$ start with measure theoretical properties, entropy
- "nonuniform" $=$ Pesin theory is involved

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Algebra to be Recovered

Three algebraic ideas.

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Ex from Brown et al - Entropy, Lyapunov exponents, and rigidity of group actions

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

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In $\mathrm{SL}(3,\mathbb{Z})$, $AB = BA$, generate a faithful action $\gamma_\bullet : \mathbb{Z}^2 \curvearrowright \mathbb{T}^3$, $\gamma_{(n,m)}(x) = A^n B^m(x)$ ("no rank one factor").
γ is "linear Cartan" (acts ergodically w/r/t Haar, or equivalently [not so for one matrix!], hyperbolic matrices).

Algebra to be Recovered

Simultaneous diagonalization gives:

$$A \sim \mathrm{diag}(l^1(A), l^2(A), l^3(A)), \quad B \sim \mathrm{diag}(l^1(B), l^2(B), l^3(B))$$

$$0 < l^3(A) < l^2(A) < 1 < l^1(A), \quad 0 < l^2(B) < 1 < l^3(B) < l^1(B)$$

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associated splitting: $\mathbb{R}^3 = E^1 \oplus E^2 \oplus E^3$
A contracts $S(A) = E^2 \oplus E^3$, expands $U(A) = E^1$ at an exponential rate ($=$ log of associated eigenvalue $=$ Lyap exponent).
B contracts $S(B) = E^2$, expands $U(B) = E^1 \oplus E^3$ at an exponential rate.
Stables and unstables intersect but don't coincide. This is good; can distinguish directions via dynamics.

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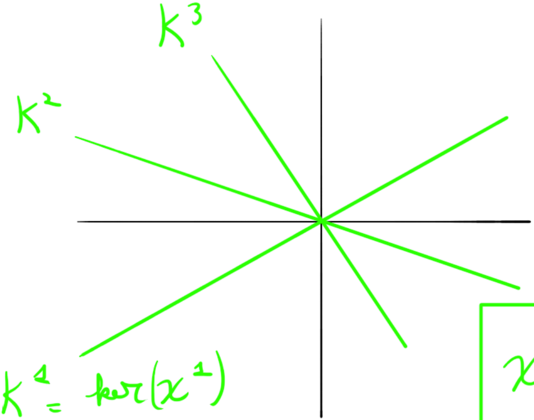
Stables and unstables intersect but don't coincide. This is good; can distinguish directions via dynamics.

Algebra to be Recovered

$\chi^i : (n, m) \mapsto \log(l^i(A^n B^m))$ is linear ("ith Lyapunov exponent"); it's kernel $K^i =$ "ith Lyapunov hyperplane" (but...)
"can distinguish directions" $=$ Lyap hyperplanes in general position $=$ eigenspaces are intersections of codimension-1 stables.



$$x^i : \mathbb{Z}^2 \to \mathbb{R}$$

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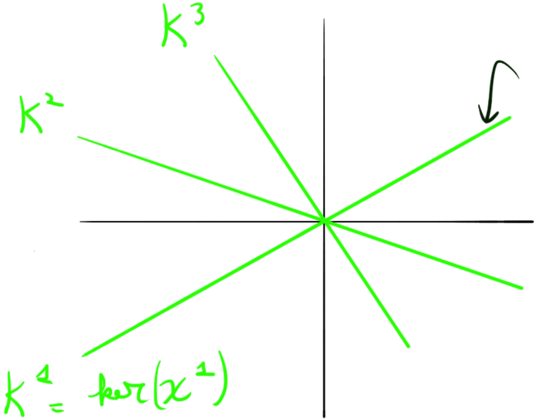
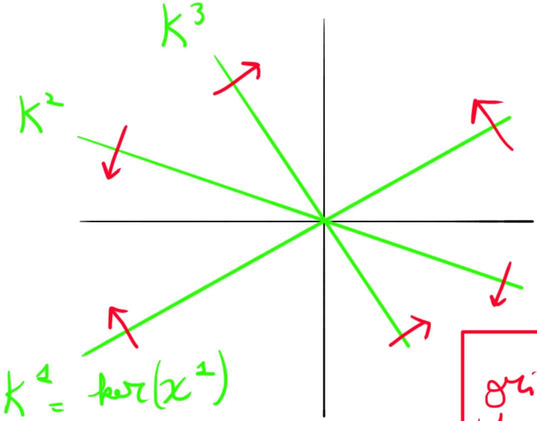
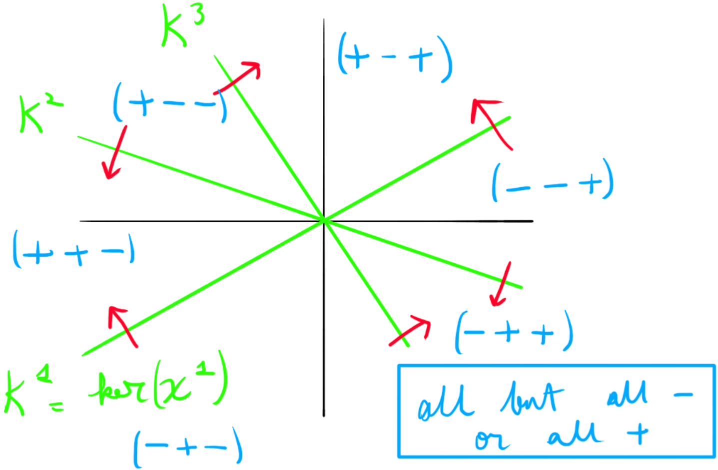
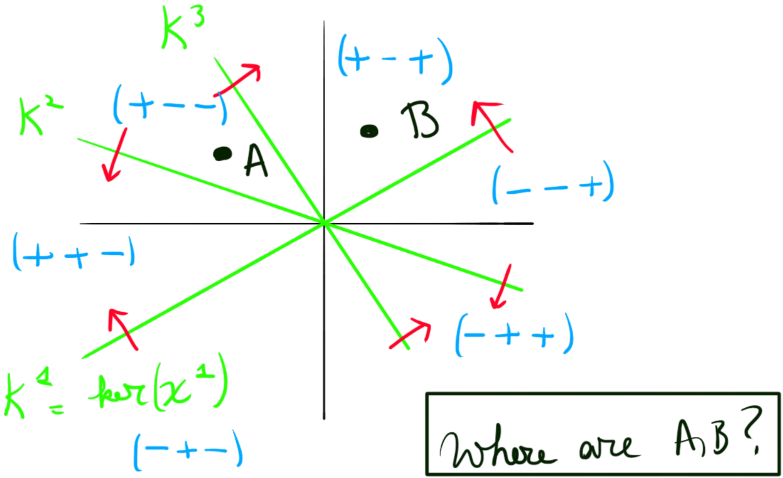


E^1 is frozen (= isometry) along this.

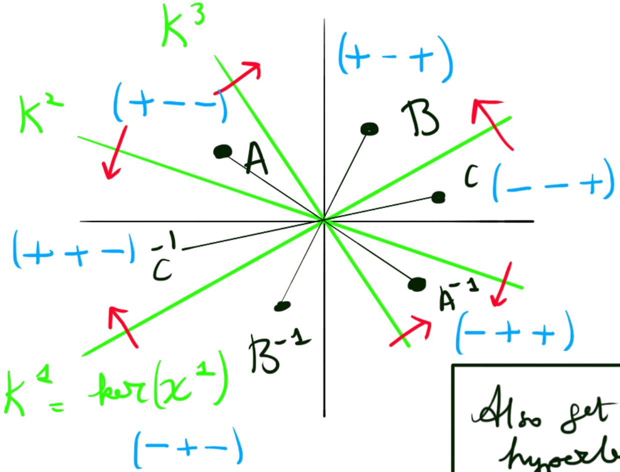


$\boxed{\text{orient the Lyap hyperplanes}}$

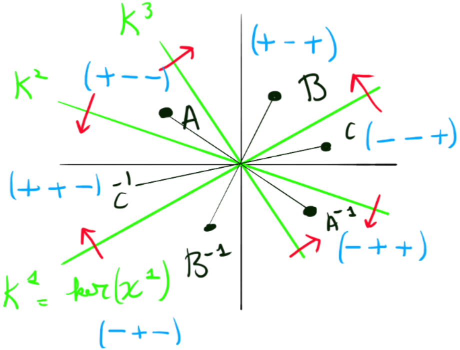

K^3
$(+-+)$
K^2
$(+--)$
$(--+)$
$(++-)$
$(-++)$
$K^4 = \ker(x^1)$
$(-+-)$
all but all $-$
or all $+$



K^3
$(+-+)$
K^2
$(+--)$
B
A
$(--+)$
$(++-)$
$(-++)$
$K^4 = \ker(x^1)$
$(-+-)$
Where are A, B?



$\boxed{\text{Also get } C = A^n B^m, \text{ hyperbolic.}}$



$$
\boxed{
\begin{aligned}
E^1 &= S(C) \cap S(B^{-1}) \\
E^2 &= S(A) \cap S(C) \\
E^3 &= S(A) \cap S(B^{-1})
\end{aligned}
}
$$

More Rigorous Statement of Result

Main Theorem [Vague]

Let $\mathbb{R}^2$ act on M^5 by diffeos. If positive entropy and can distinguish directions, then can find A, B as above so that the $\mathbb{R}^2$ action is isomorphic to suspension of $A - B$ action.

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Let $\mathbb{R}^2$ act on M^5 by diffeos. If positive entropy and can distinguish directions, then can find A, B as above so that the $\mathbb{R}^2$ action is isomorphic to suspension of $A - B$ action.

Now the algebraic ideas, continuing from $A - B$ example.

First Algebraic Idea: Suspensions are Solv.

All eigenvalues are positive, so $A - B$ action γ embeds into an action $\widetilde{\gamma} : \mathbb{R}^2 \to \mathrm{GL}(3, \mathbb{R})$ (essentially $A^p B^q$ with fractional powers).

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Semidirect product: Take product of $\mathbb{R}^2$ and $\mathbb{R}^3$, but twist via $\widetilde{\gamma}$.
Gives a five dim solvable group $G = \mathbb{R}^2 \ltimes \mathbb{R}^3$, $\mathbb{R}^3$ normal, $\mathbb{R}^2$ factor.
G modulo integer points $\Gamma = \mathbb{Z}^2 \ltimes \mathbb{Z}^3$ is 3-torus bundle over 2-torus, so solvmanifold.

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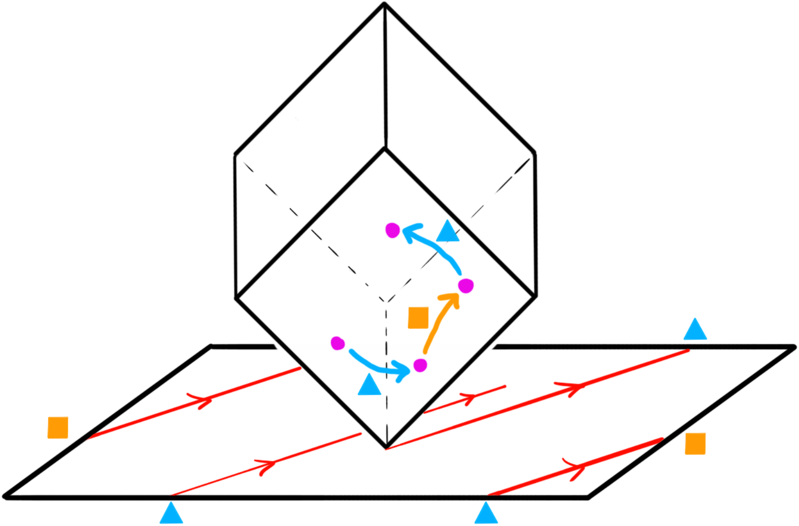
G modulo integer points $\Gamma = \mathbb{Z}^2 \ltimes \mathbb{Z}^3$ is 3-torus bundle over 2-torus, so solvmanifold.
3-torus $\sim$ vertical $\sim$ space crystal.
2-torus $\sim$ horizontal $\sim$ time crystal.
G carries another action $\mathbb{R}^2 \to \mathrm{Aff}(G)$ ("affine" means diffeo with constant derivative), $t = (t_1, t_2)$ acts as left multiplication by $(t, 0)$, descends to G mod integer points.
When t is with integer entries (and only when), this action preserves the 3-torus, and coincides with the original $A - B$ action. So translation action on solv is suspension.



Time Crystals (if Asked)

Name from Contemporary Physics
Frank Wilczek, Norman Yao, Chetan Nayak [various definitions, no-go theorems, claims to reality, physicality]
Roughly they mean something like a quantum period doubling phenomenon ("there appears a time crystal if there is a period doubling") in Floquet paradigm (periodically forced).

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Roughly they mean something like a quantum period doubling phenomenon ("there appears a time crystal if there is a period doubling") in Floquet paradigm (periodically forced).
I use "time crystal" in a different sense: when one can section off some spatial dimensions as a representation of the intrinsic clock of the system ("time crystal" in physics is to my "time crystal" what Pugh closing is to Anosov closing: "Anosov Closing" is "Anosov Always-Already-Closed").
But also, Totoki-Gurevich (also Ornstein, Smorodinsky, Katok,...) as a mathematical precedent to my "time crystal": special flow over Bernoulli is K iff no time crystal.
There are many things to be done in the context of the mathematics of time crystals in the sense of WYC!!!!
But also the Verjovsky problem is a time crystal (in my sense) problem.

Second Algebraic Idea: Homoclinic points.

Consider points x in the 3-torus that are asymptotic to 0 under the $A-B$ action ("homoclinic points"): $|n|+|m|$ large implies $A^nB^m(x) \approx 0$.

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Can define "homoclinic group" to be the symmetries of the universal covering map to begin with.
If there is a covering map of some sort, it has a group of symmetries.

Third Algebraic Idea: Affine Holonomies.

Consider a rectangular prism, and two opposite corners of it p, q.
Can move from p to q along edges in any order I want.
Translate the prism in 3-space to tile it.
Any motion along edges determine holonomies between two parallel affine hyperplanes, by "component substitution" w/r/t appropriately chosen coordinates.
There are many choices, but they all cohere.

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Tie this to AI2: if x is homoclinic to 0, it determines a piece of the *global* stable manifold (=wrapped-around eigenspace) and a piece of the *global* unstable manifold. Complete these to an obligue rectangular prism. The point opposite to 0 is new.

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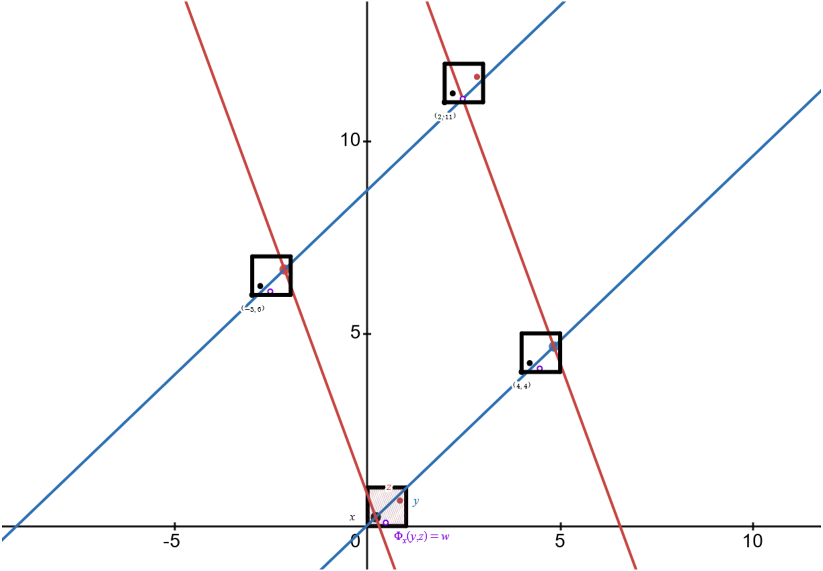
This is a covering map of some sort.



10
5
-5
0
5
10
(2, 11)
(−3, 6)
(4, 4)
z
y
x
$\Phi_x(y,z) = w$

Three Algebraic Ideas Again

- AI1: Suspensions are Solv.
- AI2: Homoclinic points.
- AI3: Affine Holonomies.

Three Algebraic Ideas Again

- AI1: Suspensions are Solv.
- AI2: Homoclinic points.
- AI3: Affine Holonomies.

In my proof: AI3 is the workhorse. AI1+AI2 is the endgame.
Main strategy: Do Pesin theory + entropy theory to unlock more algebra.
Follow Katok - FRH (they do $\mathbb{Z}^k$ and get abelian instead of solvable; get finite time crystal). Worry about orbit directions.

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Matsumoto's Theorem: Uniformly Hyperbolic, C^∞ Category

Matsumoto Thm

M closed oriented, $k \in \mathbb{Z}_{\geq 2}$, $\alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}_+^\infty(M)$. If

- α is locally free,
- $\dim(M) = 2k+1$,
- $TM = O \oplus \bigoplus_{i \in \overline{k+1}} E^i$, O orbit, E^i line bundle with $k+1$ many compatible normally uniformly hyperbolic ("Anosov elements"),

then there is

- an affine Cartan action $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{Aff}_+(\mathbb{T}^{k+1})$, and
- $\kappa \in \mathrm{GL}(k, \mathbb{R})^\circ$

such that

$$\exists \Phi_\alpha : \alpha \xrightarrow{\cong_{C^\infty}} \hbar_\kappa^\gamma,$$

$\hbar_\kappa^\gamma$ suspension of γ with a constant time change κ.

Same picture as before. A few ($= k+1$ for $\mathbb{R}^k$) well positioned Anosov elements + structural stability = many Anosov elements.
done in 8 pages (nice read, if after HPS + Plante).

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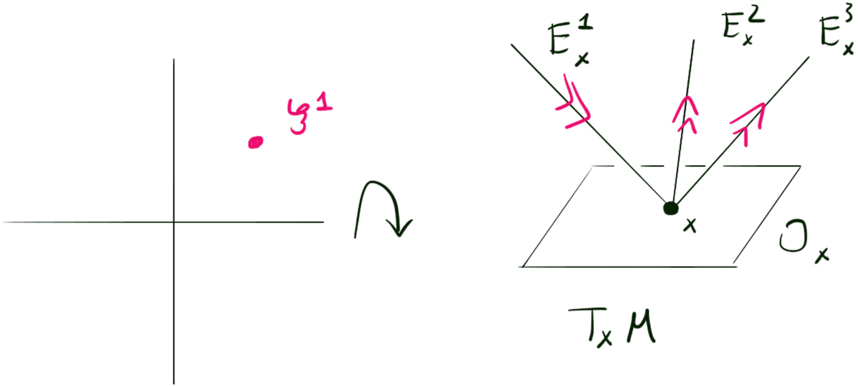
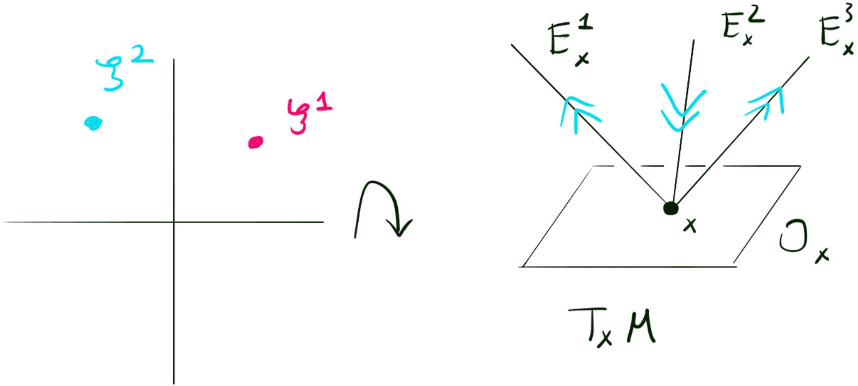
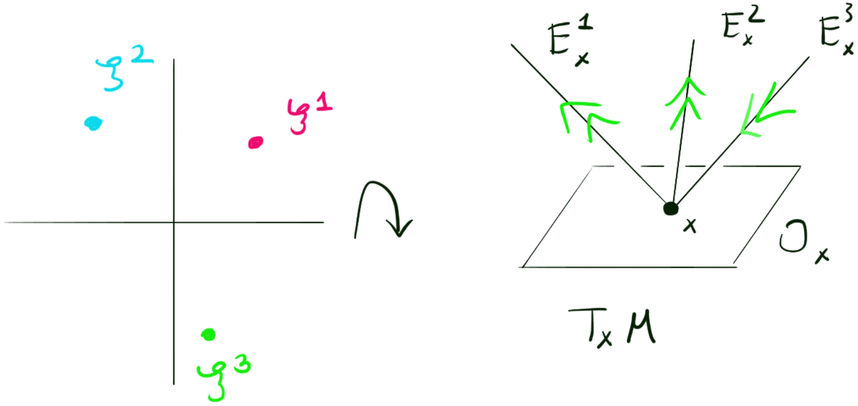
After FRH-Wang; Conjecture

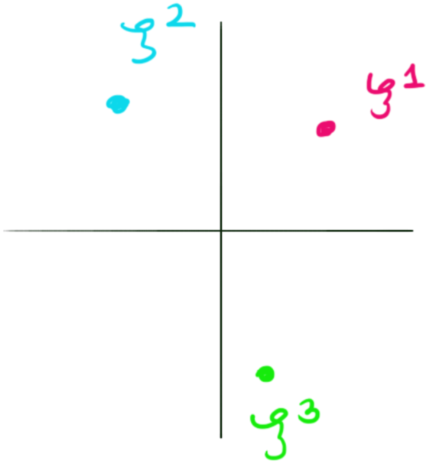
One Anosov is enough, even in the nonuniformly hyperbolic version.

Will do.

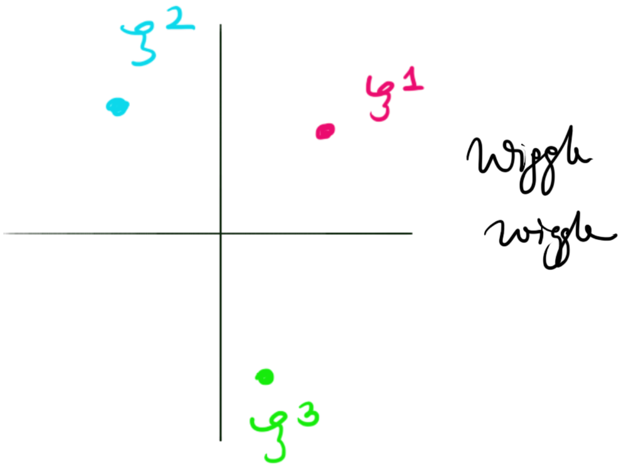

g^1
E_x^1
E_x^2
E_x^3
x
$\mathcal{O}_x$
$T_x M$

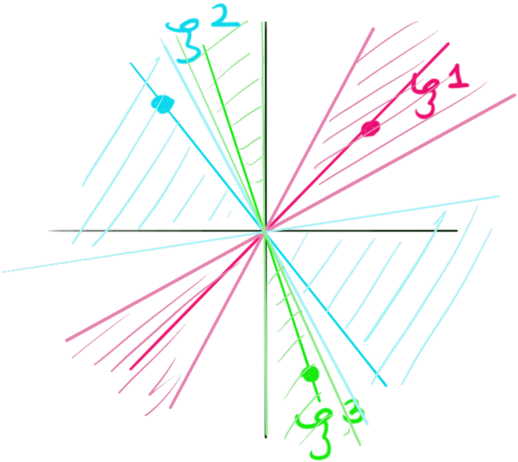

ξ^2
ξ^1
E_x^1
E_x^2
E_x^3
x
$\mathcal{O}_x$
$T_x M$


ξ^2
ξ^1
ξ^3
E_x^1
E_x^2
E_x^3
x
$\mathcal{O}_x$
$T_x M$



Abelian subgroup of $\mathrm{Diff}^\infty(M)$.


ξ²
ξ¹
ξ³
wiggle
wiggle



lots of Anosov.

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Rigorous Statement of Main Theorem; Hypotheses

In the nonuniformly hyperbolic setting. The big hypothesis is the positive entropy hypothesis.

Main Theorem

Let $\alpha_\bullet : \mathbb{R}^k \to \mathrm{Diff}^r(M)$ be a C^r action ($r = (1, \theta), \theta \in]0, 1[$ local Hölder estimates) on a compact manifold M, μ be a Borel probability measure invariant under α. If (μ, α) is an MRPES, that is,

- $k \in \mathbb{Z}_{\geq 2}$ and $\dim(M) = 2k + 1$,
- (μ, α) is locally free and ergodic,
- The system (μ, α) has exactly $k + 1$ distinct Lyapunov hyperplanes, and
- $\forall t \in \mathbb{R}^k \setminus 0 : \mathfrak{e}_{(\mu,\alpha)}(t) = \mathrm{ent}_\mu(\alpha_t) > 0$, ...

Rigorous Statement of Main Theorem; Conclusions Part 1

Main Theorem

... then there is

- an affine Cartan action $\gamma_\bullet : \mathbb{Z}^k \to \mathrm{Aff}(T^{k+1})$, where T^{k+1} is the torus or the $\pm$-infratorus of dimension $k+1$, and
- a $\kappa \in \mathrm{GL}(k, \mathbb{R})^\circ$

such that

$$\exists \Phi_{(\mu,\alpha)} : (\mu, \alpha) \xrightarrow{\cong_{\mathrm{Meas}}} (\mathrm{haar}_{\mathbb{T}^k} \otimes^\gamma \mathrm{haar}_{T^{k+1}}, \hbar^\gamma_\kappa),$$

where $\hbar^\gamma_\kappa$ is the suspension of γ with a constant time change κ. ...

Main Theorem

... Furthermore,

- The restriction of the measure theoretical isomorphism $\Phi_{(\mu,\alpha)}$ to any global stable manifold of any Weyl chamber of (μ, α) is C^r, and
- For any $\theta' \in]0, \theta[$ there is a open subset $U_{\theta'} \subseteq M$ with $\mu(M \setminus U_{\theta'}) < \theta'$ and the measure theoretical isomorphism $\Phi_{(\mu,\alpha)}$ extends to a $C^{r-\theta'}$ injective immersion on $U_{\theta'}$.

Farming Corollaries

Many corollaries follow immediately; e.g. two big ones:

[Cor]

Any $\mathbb{R}^k$ MRPES is the suspension of a $\mathbb{Z}^k$ MRPES up to a measure theoretical isomorphism and a constant linear time change. In particular, the answer to Problem 4 of Kalinin-Katok-FRH NMR is negative up to $\overline{\text{Meas}}$-isomorphism.

[Cor]

The Jacobian cocycle of any $\mathbb{R}^k$ MRPES along any Lyapunov æ-foliation is cohomologous to a cocycle constant in space with measurable transfer that is C^r along any Lyapunov æ-foliation. In particular, this solves Conjecture 1 of KKRH for the $\mathbb{R}^k$ case.

Farming Corollaries Cont.

Lyapunov exponents of any MRPES are logs of algebraic numbers; hence so are entropies (by the entropy formula) ("arithmeticity"), there are exactly countably many hyperplanes of non-ergodic elements, time crystals are obstructions to K-property; so get dimension bounds on manifold etc..

Maximal Rank Positive Entropy Assumptions

There are relations between components of the hypotheses; e.g. one can instead say "Lyapunov hyperplanes are in gen pos" + there is at least one time-t with positive entropy.
"locally free" is redundant, have "essentially free".
Restatements in terms of Fried entropy; $\mathfrak{e}_{(\mu,\alpha)} : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ is a seminorm (Abramov, Hu); MRPE means norm. ...

Fried Entropy

Fried entropy = inversely proportional to volume of the unit ball w/r/t $\mathfrak{e}_{(\mu,\alpha)}$.
If $A \in GL(d, \mathbb{Z})$, then (any kind of) entropy of A is the sum of logs of mods of eigenvalues outside the unit circle in $\mathbb{C}$, counted with (algebraic) multiplicity:

$$\mathrm{topent}(A) = \mathrm{ent}_{\mathrm{haar}_{\mathbb{T}^d}}(A) = \sum_{\lambda \in \mathrm{Spec}(A)} \mathrm{AM}(\lambda) \max(\log(|\lambda|), 0) \in \mathbb{R}_{\geq 0}.$$

(Arov-Berg entropy formula; Sinai did it before for A hyperbolic and 2×2.)

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Steps of the Proof

- KKRH NMR: μ absolutely continuous w/r/t Lebesgue class, in particular conditionals are nice.
- Oseledets: $TM =_\mu O \oplus \bigoplus_\chi L^\chi$, O orbit, L^χ line æ-bundle with Lyap exponent $= \chi$ (vectors in L_x^χ grow/decay like $\chi(t)$ under α_t). Chamber $=$ connected component of times not in any Lyap hyperplane.
- Pesin: L^χ æ-integrates to $\mathcal{L}^\chi$.
- Unique C^r global affine manifold structures $\Lambda_x^\chi : L_x^\chi \to \mathcal{L}_x^\chi$ on *global* æleaves of $\mathcal{L}^\chi$ (comes from the standard telescoping product $=$ RN-derivative of holonomy $=$ density of conditionals of SRB $=$ PCF $=$ ...); α is affine diffeo w/r/t these structures.
- Assemble to get C^r affine manifold structures Σ^C, Υ^C on leaves of stable and unstable æ-foliation $\mathcal{S}^C$, $\mathcal{U}^C$ of any chamber C. $T\mathcal{S}^C = S^C = \bigoplus_{(\chi,C)=-} L^\chi$, $\mathcal{U}^C = \mathcal{S}^{-C}$.

Steps of the Proof Part 2

- *Global* holonomies $\mathcal{U}^C_{y\leftarrow x} : \mathcal{OS}^C_x \to \mathcal{OS}^C_y$, C^r affine diffeo, defined *everywhere* on æleaf, preserve conditionals (up to a normalization)
- Measurable covering map: $\Phi_x : T_xM \to M$, take $v = (v^o, v^s, v^u) \in_{æ} T_xM$, apply the affine manifold structures to obtain points $y = \Sigma^C_x(v^s) \in \mathcal{S}^C_x$, $z = \Gamma\Upsilon^C_x(v^o, v^u) \in \mathcal{OU}^C_x$. Then $\Phi_x(v) = \mathcal{S}^C_{y\leftarrow x}(z)$. Independent of all choices involved, except x and v. Measure theoretically étale, image has full μ-measure.
- Homoclinic group $\mathfrak{H}_x$: set of affine isomorphisms $A : T_xM \to T_xM$ such that $\Phi_x \circ A =_{æ} \Phi_x$.
- Identify a specific, explicit finite-index subgroup of $\mathfrak{H}_x$ as $\mathbb{Z}^k \ltimes \mathbb{Z}^{k+1}$. The index is at most 2, not due to the first component.
- $T_xM/\mathfrak{H}_x \cong M$ gives isomorphism of (μ, α) and suspension of affine Cartan.
- For smoothness use de la Llave's Journé.

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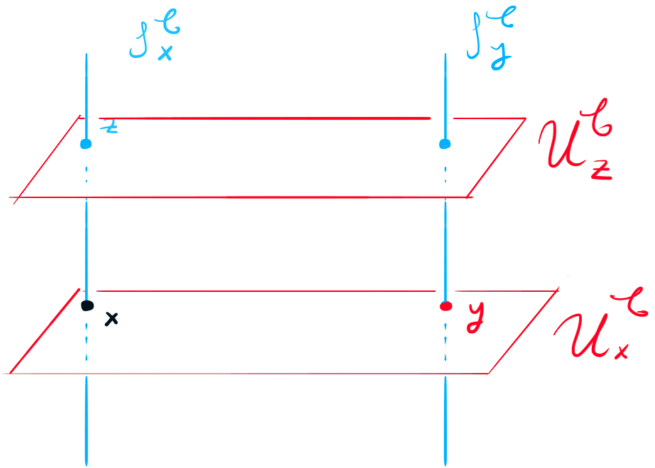
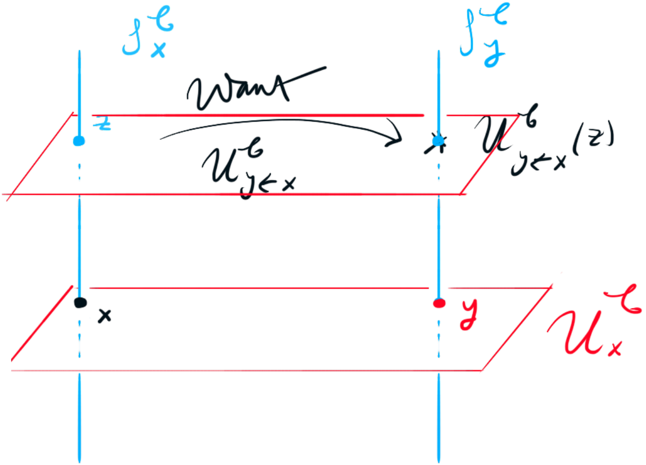
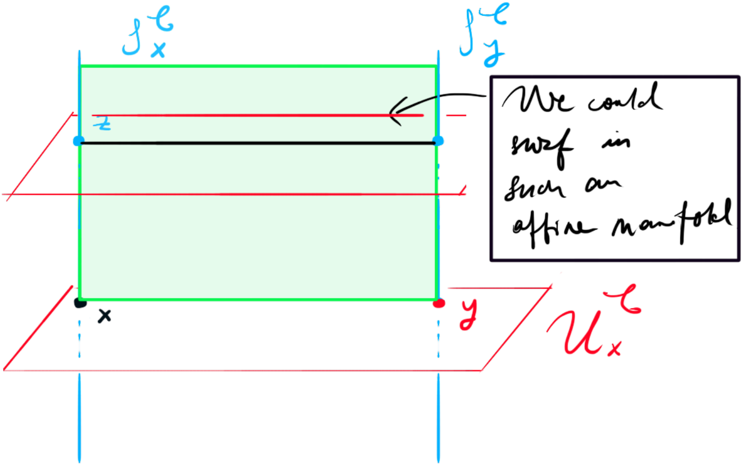
One Ingredient: Global Affine Holonomies

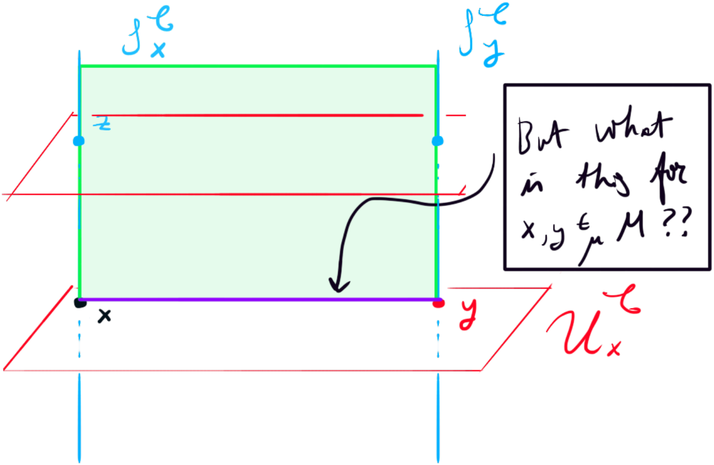
An instance of paying Pesin theory to get algebra.
Let C be a chamber such that $S^C = L^\chi$ is a line æ-bundle, so $U^C = \oplus_{\rho \neq \chi} L^\rho$ is codimension $(k+)1$.
$\mathcal{S}^C$ stable æ-foliation, $\mathcal{U}^C$ unstable æ-foliation, $\mathcal{OS}^C$ orbit-stable æ -foliation.
Want: global affine holonomy $\mathcal{U}^C_{y \leftarrow x} : \mathcal{OS}^C_x \to \mathcal{OS}^C_y$ for $x, y \in_\mu M$, $y \in \mathcal{U}^C_x$.
So once x, y are fixed (in a full measure set) with $y \in \mathcal{U}^C_x$, for *any* point z in $\mathcal{OS}^C_x$, want $\mathcal{U}^C_{y \leftarrow x}(z)$ well defined, $\mathcal{U}^C_{y \leftarrow x}(z) \in \mathcal{U}^C_z \cap \mathcal{OS}^C_y$, and that $\mathcal{U}^C_{y \leftarrow x}$ is a C^r diagonal-affine diffeo.
For illustration forget the orbit directions.

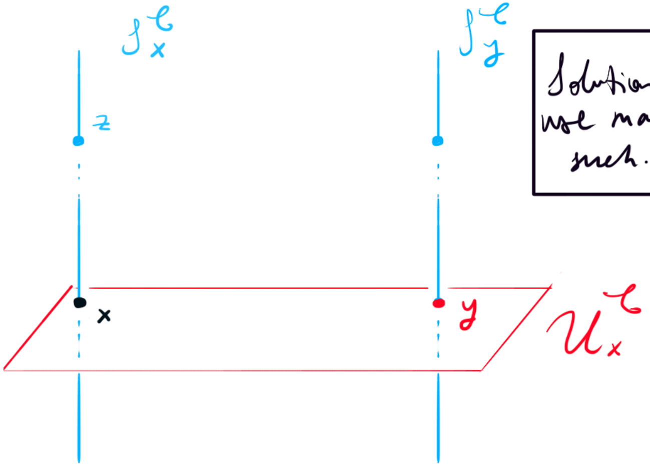


$\mathcal{J}_x^{\mathcal{C}}$
$\mathcal{J}_y^{\mathcal{C}}$
z
$\mathcal{U}_z^{\mathcal{C}}$
x
y
$\mathcal{U}_x^{\mathcal{C}}$

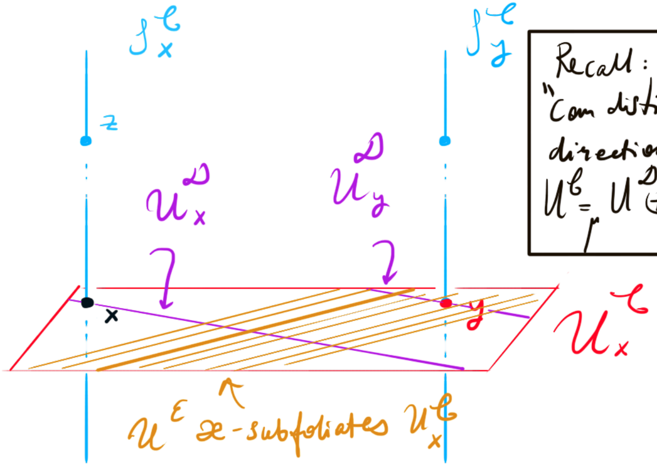

$\mathcal{J}^{\mathcal{C}}_x$
$\mathcal{J}^{\mathcal{C}}_y$
want
z
$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}$
$\mathcal{U}^{\mathcal{C}}_{y\leftarrow x}(z)$
x
y
$\mathcal{U}^{\mathcal{C}}_x$


$\mathcal{J}^{\mathcal{C}}_x$
$\mathcal{J}^{\mathcal{C}}_y$
z
We could surf in such an affine manifold
x
y
$\mathcal{U}^{\mathcal{C}}_x$

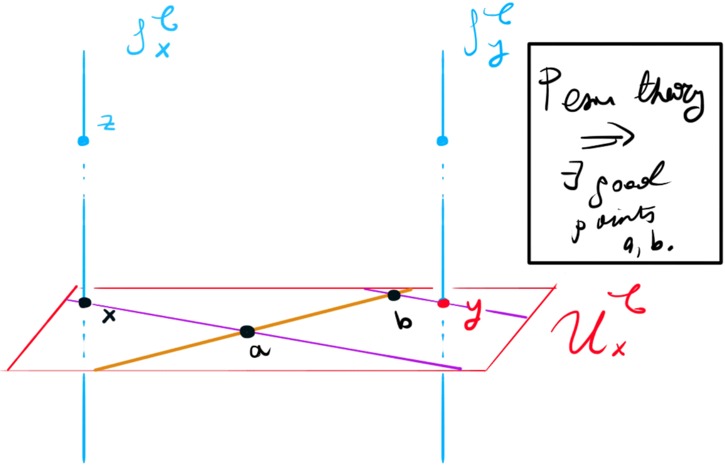

$J^{\mathcal{C}}_x$
$J^{\mathcal{C}}_y$
z
But what is this for $x, y \in_\mu M$??
x
y
$\mathcal{U}^{\mathcal{C}}_x$

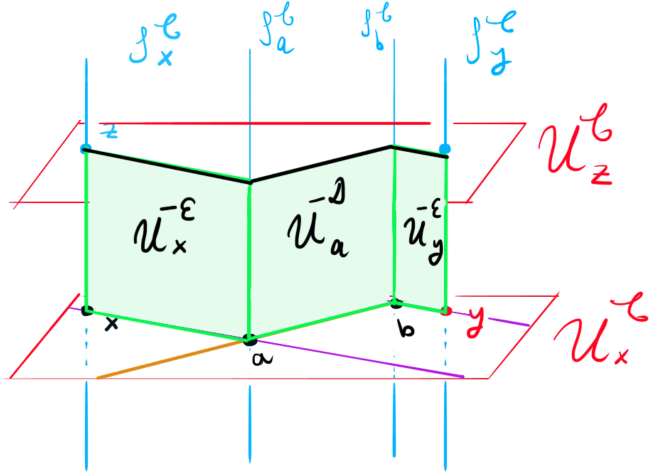


Solution: use many such.



Recall:
"Can distinguish directions"
$\mathcal{U}^{\mathcal{C}} = \mathcal{U}^{\mathcal{D}} \oplus \mathcal{U}^{\mathcal{E}}$


$\mathcal{J}_x^{\mathcal{C}}$
$\mathcal{J}_y^{\mathcal{C}}$
z
x
a
b
y
$\mathcal{U}_x^{\mathcal{C}}$
Perron theory
$\Longrightarrow$
$\exists$ good points a, b.



$\mathcal{J}_x^{\mathcal{C}}$
$\mathcal{J}_a^{\mathcal{C}}$
$\mathcal{J}_b^{\mathcal{C}}$
$\mathcal{J}_y^{\mathcal{C}}$
z
$\mathcal{U}_z^{\mathcal{C}}$
$\mathcal{U}_x^{-\varepsilon}$
$\mathcal{U}_a^{-\delta}$
$\mathcal{U}_y^{-\varepsilon}$
x
a
b
y
$\mathcal{U}_x^{\mathcal{C}}$

Thank you for listening!