Handout for MATH 036: Study Questions for "Translatory, Rotational, And Related Symmetries" (*Symmetry*, Chapter 2)

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(i) Can one identify a space with its identity mapping?

(ii) Let S_1, S_2 and T be three one-to-one mappings. If we denote "applying first S_1 and then S_2" by $S_2 \circ S_1$ (read "S_2 composed with S_1") and by T^{-1} the inverse of T, then on p. 42 Weyl states that $(S_1 \circ S_2)^{-1} = S_2^{-1} \circ S_1^{-1}$. For instance, to reverse "first wearing socks, then wearing shoes", you have to first take off the shoes, and then take of the socks, i.e.

$$\begin{aligned}\text{[first wearing socks, then wearing shoes]}^{-1} &= (\text{[wearing shoes]} \circ \text{[wearing socks]})^{-1} \\ = \text{[wearing shoes]}^{-1} \circ \text{[wearing socks]} &= \text{[taking shoes off]} \circ \text{[taking socks off]}.\end{aligned}$$

Argue that the same order reversing would occur when we applied any number of one-to-one mappings one after the other, viz., if $S_1, \ldots, S_N$ are N one-to-one mappings, then

$$(S_N \circ \cdots \circ S_1)^{-1} = S_1^{-1} \circ \cdots \circ S_N^{-1}.$$

This phenomenon of order reversing is called **contravariance**.

(iii) What does Weyl mean by the word "space"?

(iv) On pp. 42-43 Weyl defines the group of automorphisms of a space, namely, if X is a space, then the set $\mathrm{Aut}\,(X)$ of all automorphisms of X satisfies the following three conditions:

(a) The identity mapping id_X of X is in $\mathrm{Aut}\,(X)$.
(b) If S is in $\mathrm{Aut}\,(X)$, then so is its inverse S^{-1}.
(c) If S and T are in $\mathrm{Aut}\,(X)$, then so is their composition $S \circ T$[1].

Check that the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of all integer numbers has as its automorphism group the set of all mappings "add n", i.e.,

$$\mathrm{Aut}\,(\mathbb{Z}) = \{\ldots, \text{“add } -1\text{”}, \text{“add } 0\text{”}, \text{“add } 1\text{”}, \ldots\}.$$

Find $\mathrm{id}_{\mathbb{Z}}$, the inverse of "add n", and the composition "add n" $\circ$ "add m". Observe that the transformation "add 1" is nothing but shifting the space $\mathbb{Z}$ to the right by one unit.

(v) What is the difference between proper and improper congruences? (Hint: The modern terminology is "orientation-preserving" for "proper" and "orientation-reversing" for "improper").

(vi) Find spaces for which there are more than two categories of congruences. (Hint: What is the space we are working on when we apply proper and improper transformations?)

(vii) What is an infinite rapport?

[1] There is one more rule that Weyl does not list, which goes like this: if P, Q, R are in $\mathrm{Aut}\,(X)$, then $(P \circ Q) \circ R = P \circ (Q \circ R)$.

(viii) Research Leonardo da Vinci's list of symmetry groups of the plane, which consists of all finite cyclic groups and all dihedral groups. (Hint: `http://www-history.mcs.st-and.ac.uk/~john/geometry/Lectures/L8.html` and p. 65 of Weyl.)

(ix) On p. 67 Weyl states that "[i]t seems that the origin of the magic power ascribed to these patterns lies in their startling incomplete symmetry—rotations without reflections". Which patterns is he talking about? What is the automorphism group of said patterns? Research the different contexts said patterns were used throughout history.

(x) Research "phyllotaxis". (Hint: If you attended the seminar I told you about in the beginning of the semester, it was about this.)

(xi) What is the Fibonacci sequence? Observe that this idea can be generalized quite easily, since in order to construct such a "nice" sequence all we need to do is to determine the general rule and two seeds from which we can start applying the formula. For instance, we could take $s_{n+2} = 2s_{n+1} + s_n$ as the general rule and $s_0 = -2, s_1 = 3$ as the seeds. Then we have the following sequence:

$$s_0 = -2, s_1 = 3, s_2 = 4, s_3 = 11, s_4 = 26, \ldots$$

Observe that we could have done this iteration not only in forward time, but also in backward time. Indeed, since $s_{n+2} = 2s_{n+1} + s_n$, we also know that $s_{n+2} - 2s_{n+1} = s_n$, or, shifting the indices by two, $s_n - 2s_{n-1} = s_{n-2}$. Thus in fact we also have the information on the history of this sequence:

$$\ldots, s_{-3} = 39, s_{-2} = -16, s_{-1} = 7, s_0 = -2, s_1 = 3, s_2 = 4, s_3 = 11, s_4 = 26, \ldots$$

In the same vain, find the past of the Fibonacci sequence.

(xii) List all the Platonic solids. Find the numbers of their vertices, edges, and faces. For instance, the cube is a Platonic solid. It has $V_{\text{cube}} = 8$ vertices, $E_{\text{cube}} = 12$ edges and $F_{\text{cube}} = 6$ faces. In particular, we have

$$V_{\text{cube}} - E_{\text{cube}} + F_{\text{cube}} = 8 - 12 + 6 = 2.$$

Show, by computing, that the same arithmetic operation would end up with a 2 for any Platonic solid.