§ 10.1 : (2)

- The (differential) operator

$$\Delta : C^2(\mathbb{R}^d, \mathbb{R}) \longrightarrow C^0(\mathbb{R}^d, \mathbb{R})$$

$$f(x_1, x_2, \dots, x_d) \longmapsto \sum_{k=1}^{d} \partial_{x_k}^2 f(x_1, x_2, \dots, x_d)$$

$$\left(= \partial_{x_1}^2 f(x_1, x_2, \dots, x_d) + \partial_{x_2}^2 f(x_1, x_2, \dots, x_d) + \dots + \partial_{x_d}^2 f(x_1, x_2, \dots, x_d) \right)$$

is called the Laplacian. ("Differential" means that it involves derivatives.) $(\Delta = \nabla \cdot \nabla = \nabla^2)$

SW: (i) Δ is a linear operator

(ii) So is $-\Delta$.

(iii) The set of linear operators between two linear spaces (of functions) is a linear space.

- Δ constitutes the "spatial" parts of the

heat operator $\partial_t - \Delta$ and the

wave operator $\partial_t^2 - \Delta$.

- We know by now that identifying the eigenpairs of a linear operator is crucial for understanding the operator. Thus we would like to find the eigenpairs of $-\Delta$, ie, pairs (λ, f) where $\lambda \in \mathbb{R}$, $f \in C^{\infty}(\mathbb{R}^d, \mathbb{R})$ (to avoid technicalities) : $f \neq 0$, $-\Delta f = \lambda f$

$$\Leftrightarrow \quad \boxed{\Delta f + \lambda f = 0} \quad (*)$$

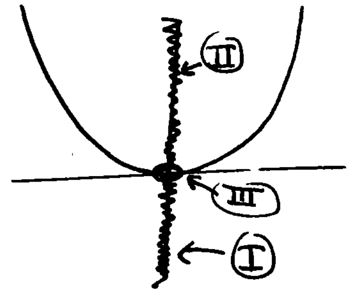
- Let's start easy and consider the case when the "spatial" dimension $d = 1$. Then $\Delta = \partial_x^2$, and $(*)$ reduces to:

$$\boxed{\partial_x^2 f(x) + \lambda f(x) = 0}$$

In particular, this is an ODE and we know how to deal with it.

$$(*) \Leftrightarrow \partial_x \begin{pmatrix} f(x) \\ \partial_x f(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}_{=: A_\lambda} \begin{pmatrix} f(x) \\ \partial_x f(x) \end{pmatrix} \Leftrightarrow \boxed{\partial_x Y(x) = A_\lambda Y(x)} \quad (**)$$

$tr(A_\lambda) = 0, \quad det(A_\lambda) = \lambda \Rightarrow$



(I) $\lambda < 0 \Rightarrow \lambda_1 = -\sqrt{-\lambda} < 0 < \sqrt{-\lambda} \; (\Rightarrow -\lambda_1 = \lambda_2)$

$\Rightarrow$ the gen. sol. of (*) is:

$$\boxed{Y(x) = c_1 \, e^{-\sqrt{-\lambda}\,x} \begin{pmatrix} 1 \\ -\sqrt{-\lambda} \end{pmatrix} + c_2 \, e^{\sqrt{-\lambda}\,x} \begin{pmatrix} 1 \\ \sqrt{-\lambda} \end{pmatrix}}$$

(ie., if $\lambda < 0$ and at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $(\lambda, \; c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x})$ is an eigenpair of $-\Delta$.)

SW: A more convenient way of writing these is by using the hyperbolic trigonometric functions:

$$\left.\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \end{aligned}\right\} \Rightarrow \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow \cosh(\theta) := \cos(i\theta) = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh(\theta) := \frac{1}{i}\sin(i\theta) = \frac{e^{\theta} - e^{-\theta}}{2}.$$

Use the hyperbolic trigonometric functions to write the results above in a more convenient way.

(II) $\lambda > 0 \Rightarrow \lambda_1 = i\sqrt{\lambda} = \overline{\lambda_2}$ $(\Rightarrow -\lambda_1 = \lambda_2)$.

$\Rightarrow$ The gen. sol. of $(**)$ is:

$$\boxed{Y(x) = c_1 \begin{pmatrix} \cos(\sqrt{\lambda}\,x) \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\,x \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{\lambda}\,x) \\ \sqrt{\lambda}\cos(\sqrt{\lambda}\,x) \end{pmatrix}}$$

(ie., if $\lambda > 0$ and at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $(\lambda,\ c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x))$ is an eigenpair of $-\Delta$.)

(III) $\lambda = 0 \Rightarrow \lambda_1 = 0 = \lambda_2$ $(\Rightarrow -\lambda_1 = \lambda_2)$

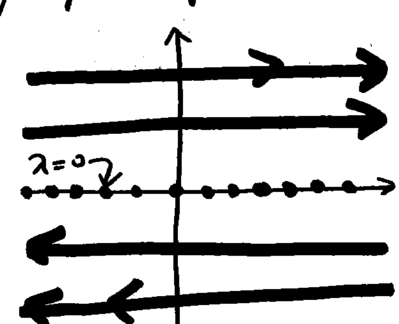
$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so A_0 is in canonical form (improper node, stable)

$\Rightarrow$ The gen. sol. of $(**)$ is:

[redacted] $$\boxed{Y(x) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} x \\ 1 \end{pmatrix}}$$

(ie, if at least one of $c_1, c_2 \in \mathbb{R}$ is nonzero, then $(0, c_1 + c_2 x)$ is an eigenpair of $-\Delta$.)

(phase portrait:



(x is "time", not the horizontal axis.))

- Thus we have the complete list of eigenpairs of $-\Delta$. Let's call this list the <u>spectral data</u> of $-\Delta$:

<u>Region in the map</u>	<u>Eigenvalue</u>	<u>Eigenfunction</u>
I	$\lambda < 0$	any lin. combo of $e^{-\sqrt{-\lambda}x}$ and $e^{\sqrt{-\lambda}x}$ (or of $\cosh(\sqrt{-\lambda}x)$ and $\sinh(\sqrt{-\lambda}x)$)
II	$\lambda > 0$	any lin. combo of $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$
III	$\lambda = 0$	any lin. combo of 1 (the function that is constantly 1) and x

- A (<u>two-point</u>) <u>boundary value problem</u> (BVP) is a triple

$$\left(\text{diff. eq.}, \begin{array}{c}\text{boundary}\\ \text{datum at}\\ x_0\end{array}, \begin{array}{c}\text{boundary}\\ \text{datum at}\\ x_1\end{array}\right),$$

where $x_0 \neq x_1$. Geometrically speaking specifying a boundary datum corresponds to specifying a line in the phase space.

- As opposed to IVP's, BVP's with even the "nicest" differential equations may fail to have a unique solution.

- A BVP with a homogeneous differential equation and vanishing boundary data (ie., $y(x_0) = 0 = y(x_1)$) is called <u>homogeneous</u>.

- If the diff. eq. of a BVP is of the form $\Delta y + \lambda y = 0$, then the eigenpairs of $-\Delta$ that satisfy the boundary conditions are also called <u>the eigenpairs of the BVP</u> by proxy.

<u>Ex</u>:

$$\boxed{\begin{aligned} &\Delta y(x) + 4y(x) = 0 \\ &y(0) = -2 \\ &y(\pi/4) = 10 \end{aligned}}$$

$\lambda = 4 > 0 \Rightarrow$ II

$$\Rightarrow \boxed{Y(x) = c_1 \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}}$$

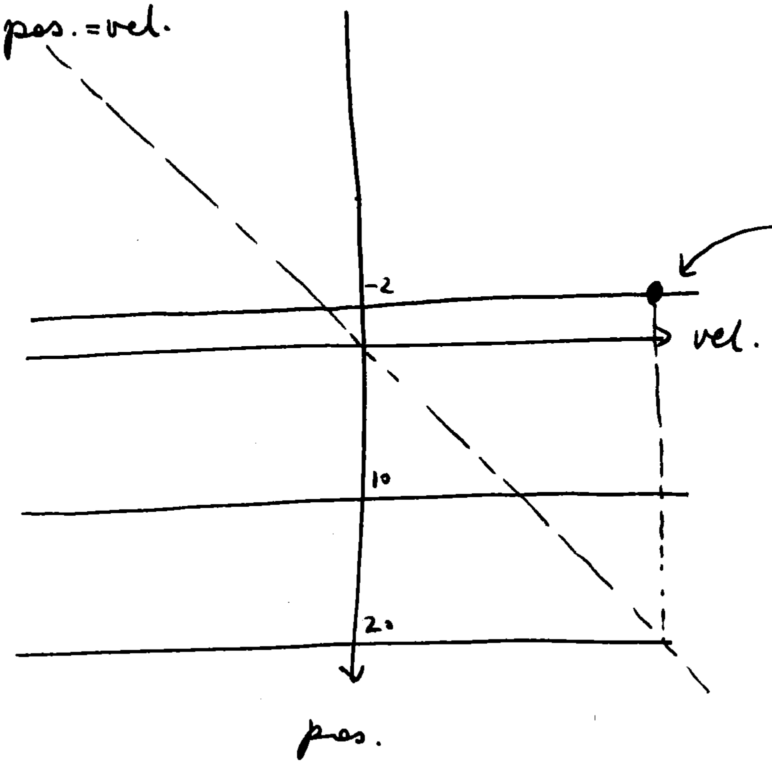
is the gen. sol. (of the ODE).

$$\begin{pmatrix} -2 \\ \partial_x y(0) \end{pmatrix} = Y(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(0)}{2} \end{aligned}$$

$$\begin{pmatrix} 10 \\ \partial_x y(\pi/4) \end{pmatrix} = Y(\pi/4) = c_1 \begin{pmatrix} 0 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} c_2 &= 10 \\ c_1 &= -\frac{\partial_x y(0)}{2} \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} y(x) = &-2\cos(2x) \\ &+ 10\sin(2x) \end{aligned}}$$

is the unique sol.



The trajectory of the unique solution is the unique ellipse passing through this point.

<u>Ex</u>:
$$\boxed{\begin{aligned} &\Delta y(x) + 4y(x) = 0 \\ &y(0) = -2 \\ &y(2\pi) = -2 \end{aligned}}$$

$\lambda = 4 > 0 \Rightarrow$ II

$$\Rightarrow \boxed{Y(x) = c_1 \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}}$$

is the gen. sol. (of the ODE).

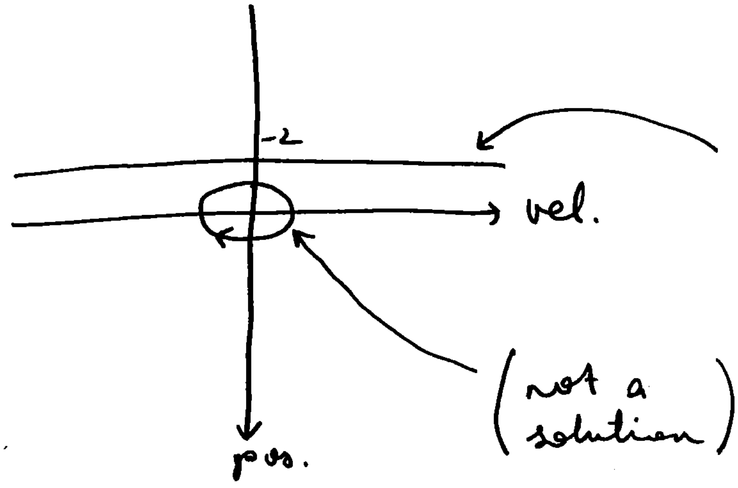
$$\begin{pmatrix} -2 \\ \partial_x y(0) \end{pmatrix} = Y(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(0)}{2} \end{aligned}$$

$$\begin{pmatrix} -2 \\ \partial_x y(2\pi) \end{pmatrix} = Y(2\pi) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= \frac{\partial_x y(2\pi)}{2} \end{aligned}$$

$\Rightarrow$ For any $c_2 \in \mathbb{R}$,

$$\boxed{y(x) = -2\cos(2x) + c_2 \sin(2x)}$$

is a sol.



Any ellipse that hits this line (at least once) represents a solution.

Ex: $$\boxed{\begin{aligned} &\Delta y(x) + 25 y(x) = 0 \\ &\partial_x y(0) = 6 \\ &\partial_x y(\pi) = -9 \end{aligned}}$$ $\lambda = 25 > 0 \Rightarrow$ (II)

$$\Rightarrow \boxed{Y(x) = c_1 \begin{pmatrix} \cos(5x) \\ -5\sin(5x) \end{pmatrix} + c_2 \begin{pmatrix} \sin(5x) \\ 5\cos(5x) \end{pmatrix}}$$

is the gen. sol. (of the ODE).

$$\left.\begin{aligned} \begin{pmatrix} y(0) \\ 6 \end{pmatrix} = Y(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} &\Rightarrow \begin{aligned} c_1 &= y(0) \\ c_2 &= 6/5 \end{aligned} \\ \begin{pmatrix} y(\pi) \\ -9 \end{pmatrix} = Y(\pi) = c_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -5 \end{pmatrix} &\Rightarrow \begin{aligned} c_1 &= -y(\pi) \\ c_2 &= 9/5 \end{aligned} \end{aligned}\right\} \Rightarrow 6/5 = c_2 = 9/5, \text{↯}$$

$\Rightarrow$ The BVP has no solutions.

SW: (i) Let $L > 0$, $\lambda \in \mathbb{R}$, and consider the BVP

$$\boxed{\begin{aligned} &\Delta y(x) + \lambda y(x) = 0 \\ &y(0) = 0 = y(L) \end{aligned}}$$

. Find all solutions.

(ii) Do the same with

$$\boxed{\begin{aligned} &\Delta y(x) + \lambda y(x) = 0 \\ &\partial_x y(0) = 0 = \partial_x y(L) \end{aligned}}$$

.

(iii) Do the same with

$$\boxed{\begin{aligned} &\Delta y(x) + \lambda y(x) = 0 \\ &\partial_x y(0) = 0 = y(L) \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} &\Delta y(x) + \lambda y(x) = 0 \\ &y(0) = 0 = \partial_x y(L) \end{aligned}}$$

.

§ 10.4 : (1.5)

- Fix $L > 0$ and put $I := [-L, L]$ or $]-L, L[$

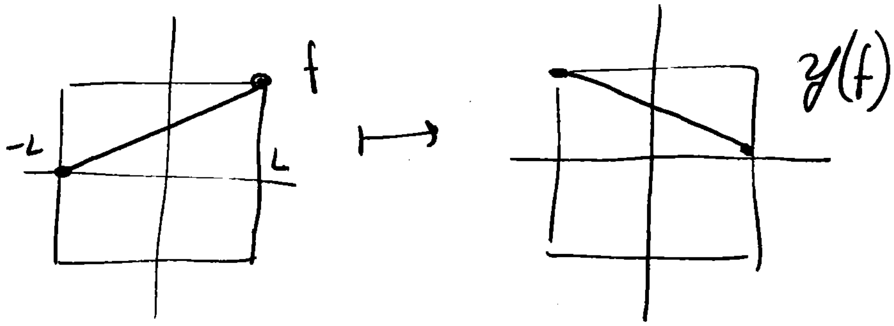
(Thus I is an interval centered at 0 with the property $x \in I \Leftrightarrow -x \in I$).

Define the "reflection along the y-axis" operator

$$\mathcal{Y} : F(I, \mathbb{R}) \longrightarrow F(I, \mathbb{R})$$

$$\left[\begin{array}{l} I \xrightarrow{f} \mathbb{R} \\ x \longmapsto f(x) \end{array} \right] \longmapsto \left[\begin{array}{l} I \xrightarrow{\mathcal{Y}(f)} \mathbb{R} \\ x \longmapsto f(-x) \end{array} \right]$$

($F(I, \mathbb{R})$ is the linear space of all functions $I \to \mathbb{R}$)



<u>SW</u>: • $\mathcal{Y}$ is a linear operator.

• It is also <u>multiplicative</u>, ie.,

$$\mathcal{Y}(fg) = \mathcal{Y}(f)\, \mathcal{Y}(g).$$

• Applying $\mathcal{Y}$ twice is the same as doing nothing at all: $\mathcal{Y} \circ \mathcal{Y}(f) = f$.

(In other words, $\mathcal{Y}^{-1} = \mathcal{Y}$.)

• Let's identify the eigenvalues of $\mathcal{Y}$.

$$\underbrace{\mathcal{Y}(f) = \lambda f}_{\left(\Leftrightarrow \text{ For any } x \in I: \; f(-x) = \lambda f(x)\right)}, \quad \lambda \in \mathbb{R}, \quad f : I \to \mathbb{R} \text{ is such that there is at least one } x_0 \in I : f(x_0) \neq 0.$$

If $x_0 = 0$, $0 \neq f(0) = f(-0) = \lambda f(0)$

$$\Rightarrow (\lambda - 1) \underbrace{f(0)}_{\neq 0} = 0 \Rightarrow \lambda = 1.$$

If $x_0 \neq 0$,
$$\left.\begin{array}{l} f(-x_0) = \lambda f(x_0) \\ f(x_0) = \lambda f(-x_0) \end{array}\right\} \Rightarrow f(x_0) = \lambda^2 f(x_0)$$

$$\Rightarrow (\lambda^2 - 1) \underbrace{f(x_0)}_{\neq 0} = 0$$

$$\Rightarrow \lambda = \pm 1.$$

Thus $\mathcal{Y}$ can not have an eigenvalue different than ± 1. These two are eigenvalues of $\mathcal{Y}$ because we can find eigenfunctions for both of them, eg.

$$f_1(x) := 1, \quad \Rightarrow \quad \mathcal{Y}(f_1) = 1 \cdot f_1$$
$$f_{-1}(x) := x \qquad\qquad \mathcal{Y}(f_{-1}) = (-1) \cdot f_{-1}$$

Thus $\mathcal{Y}$ has precisely two eigenvalues: 1 and -1.

Rem: Earlier we looked at another operator, namely $-\Delta = -\partial_x^2$, and we discovered that it has infinitely many eigenvalues. In fact any real number is an eigenvalue of $-\Delta$.

Operator	# of (distinct) eigenvalues
$A \in Mat(2\times 2, \mathbb{R})$ (or $T_A : \binom{x}{y} \mapsto A\binom{x}{y}$)	1 or 2
$\mathcal{Y}$	2
$-\Delta$	∞

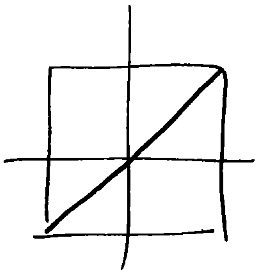
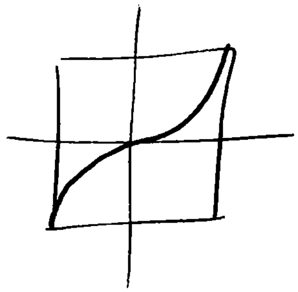
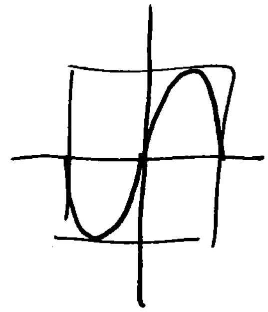
- $f \in F(I, \mathbb{R})$ is called <u>even</u> if it is an eigenfunction associated to 1

$$\left(\text{ie., } f \text{ is even} \Leftrightarrow f(-x) = f(x)\right)$$

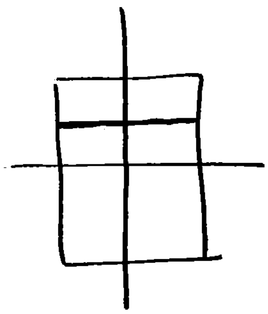
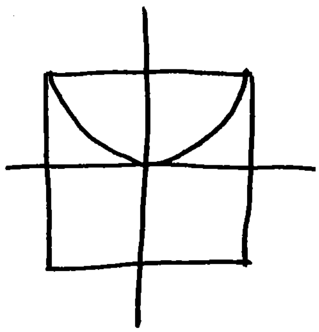
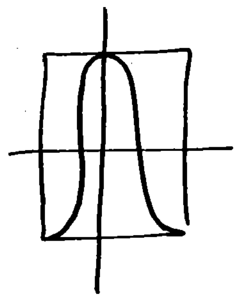
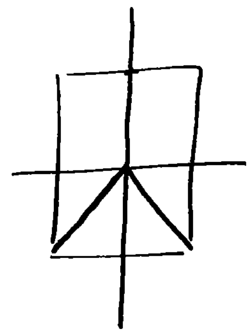
$f \in F(I, \mathbb{R})$ is called <u>odd</u> if it is an eigenfunction associated to -1

$$\left(\text{ie., } f \text{ is odd} \Leftrightarrow f(-x) = -f(x)\right)$$

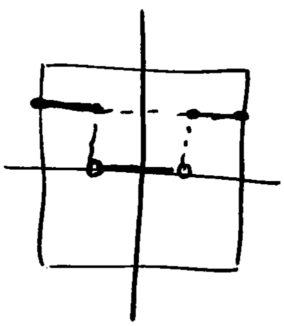
Ex: x^{2n+1} ($n \geq 1$ integer), $\sin(\omega x)$ ($\omega > 0$) are odd:

 ,  ,  , ...

Ex: 1, x^{2n} ($n \geq 1$ integer), $\cos(\omega x)$ ($\omega > 0$) are even:

 ,  ,  ,  ,

$-|x|$

 , ...

$$f(x) = \chi_{[-L, -\frac{L}{2}]}(x) + \chi_{[\frac{L}{2}, L]}(x).$$

SW: (i) If $f: I \to \mathbb{R}$ is odd, then $f(0) = 0$
(but not vice versa)

(ii) The set $F_e(I,\mathbb{R})$ of all even functions $I \to \mathbb{R}$ is a linear space. So is the set of all odd functions $I \to \mathbb{R}$. $F_o(I,\mathbb{R})$

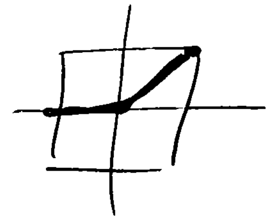
(iii) If $f, g \in F_e(I,\mathbb{R})$, then $fg \in F_e(I,\mathbb{R})$

If $f, g \in F_o(I,\mathbb{R})$, then $fg \in F_e(I,\mathbb{R})$

If $f \in F_e(I,\mathbb{R})$, $g \in F_o(I,\mathbb{R})$, then $fg \in F_o(I,\mathbb{R})$

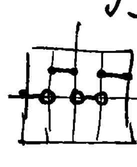
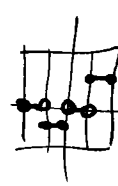
(iv) If $f \in F_e(I,\mathbb{R})$ and $f \in F_o(I,\mathbb{R})$, then $f(x) = 0$ for any $x \in I$.

(ie. the only function that is both even and odd is the one that is constantly 0.)

(v) There are functions $f: I \to \mathbb{R}$ that are neither even nor odd. (eg. )

(vi) If $f \in F_e(I,\mathbb{R})$, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx = 2\int_{-L}^0 f(x)\,dx$

If $f \in F_o(I,\mathbb{R})$ (and piecewise continuous), then $\int_{-L}^{L} f(x)\,dx = 0$.

(but not vice versa, eg.  $\leftarrow \chi_{[-\frac{1}{2},0]}(x) + \chi_{[\frac{1}{2},1]}(x)$;  $\leftarrow -\chi_{[-\frac{1}{2},0]}(x) + \chi_{[\frac{1}{2},1]}(x)$)

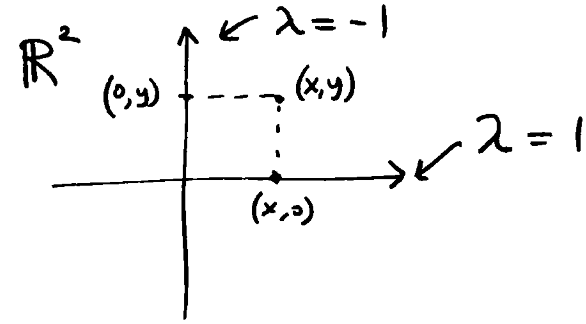
• Put $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in Mat(2\times 2, \mathbb{R})$. Then

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{and} \qquad \mathcal{Y} : F(I, \mathbb{R}) \longrightarrow F(I, \mathbb{R})$$
$$(x,y) \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad\qquad f(x) \longmapsto f(-x)$$

are spectrally very similar: both have 1 and -1 as their only eigenvalues. (Also $A^2 = I$, so $A^{-1} = A$).

For A, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector associated to 1 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector associated to -1.

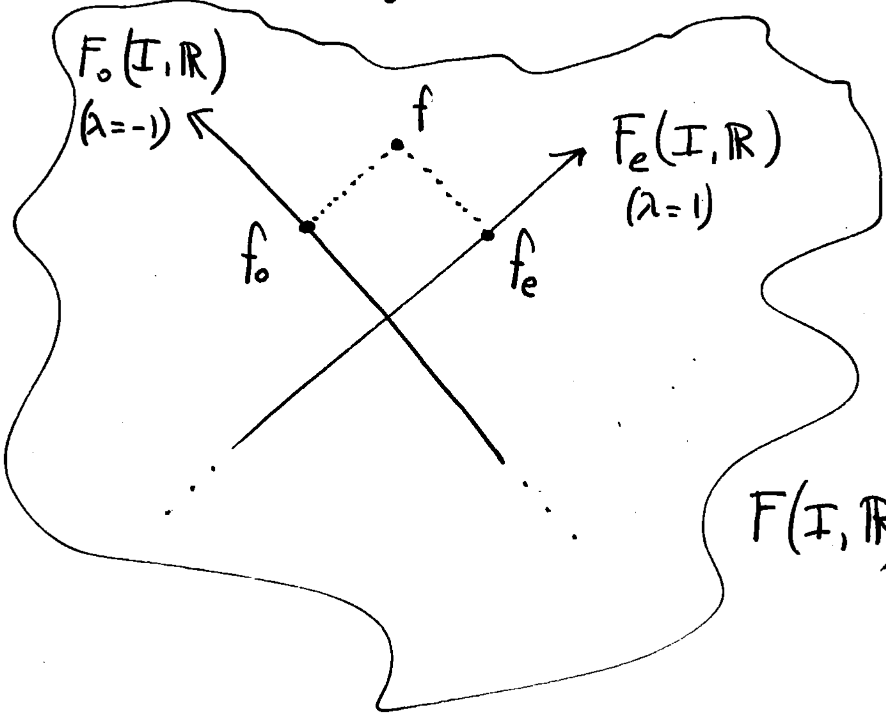
Thus any point $\begin{pmatrix} x \\ 0 \end{pmatrix}$ on the x-axis solves $A\begin{pmatrix} x \\ 0 \end{pmatrix} = 1\begin{pmatrix} x \\ 0 \end{pmatrix}$ and any point $\begin{pmatrix} 0 \\ y \end{pmatrix}$ on the y-axis solves $A\begin{pmatrix} 0 \\ y \end{pmatrix} = (-1)\begin{pmatrix} 0 \\ y \end{pmatrix}$:



Any point $\begin{pmatrix} x \\ y \end{pmatrix}$ on the plane can be written as the sum of an eigenvector associated to 1 and an eigenvector associated to -1:

$$(x,y) = (x,0) + (0,y).$$

Likewise for $\mathcal{Y}$ we have:



(This is a caricature, where each point represents a function $f: I \to \mathbb{R}$)

The natural question now is whether or not the caricature has some truth to it. More precisely, is it the case that any function $f: I \to \mathbb{R}$ can be written as the sum of an even function $f_e: I \to \mathbb{R}$ and an odd function $f_o: I \to \mathbb{R}$:

$$f = f_e + f_o \qquad ?$$

The answer is: yes. Define the "projections"

$$\begin{aligned} \mathcal{P}_e : F(I,\mathbb{R}) &\longrightarrow F_e(I,\mathbb{R}) \\ f &\longmapsto f_e = \tfrac{1}{2}\left(f + \mathcal{Y}(f)\right) \\ \mathcal{P}_o : F(I,\mathbb{R}) &\longrightarrow F_o(I,\mathbb{R}) \\ f &\longmapsto f_o = \tfrac{1}{2}\left(f - \mathcal{Y}(f)\right) \end{aligned}$$

For these to be well-defined, we need to verify that for any $f \in F(I,\mathbb{R})$: $\mathcal{P}_e(f)$ is even and $\mathcal{P}_o(f)$ is odd:

(SW: $\mathcal{P}_e$ and $\mathcal{P}_o$ are linear.)

$$\mathcal{Y} \circ \mathcal{P}_e(f)(x) = \mathcal{P}_e(f)(-x) = \frac{1}{2}\left(f(-x) + \mathcal{Y}(f)(-x)\right) = \frac{1}{2}\left(f(-x) + f(x)\right) = \mathcal{P}_e(f)(x)$$

$$\Rightarrow \mathcal{Y}\left(\mathcal{P}_e(f)\right) = 1 \cdot \mathcal{P}_e(f), \checkmark$$

$$\left(\text{or: } \mathcal{Y}\left(\mathcal{P}_e(f)\right) = \mathcal{Y}\left(\frac{1}{2}\left(f + \mathcal{Y}(f)\right)\right) \underset{\substack{\uparrow \\ \mathcal{Y} \text{ is linear}}}{=} \frac{1}{2}\left(\mathcal{Y}(f) + \mathcal{Y}^2(f)\right) = \mathcal{P}_e(f). \right)$$

$$\begin{aligned} \mathcal{Y} \circ \mathcal{P}_o(f)(x) &= \mathcal{P}_o(f)(-x) = \frac{1}{2}\left(f(-x) - \mathcal{Y}(f)(-x)\right) = \frac{1}{2}\left(f(-x) - f(x)\right) \\ &= -\frac{1}{2}\left(f(x) - f(-x)\right) = -\frac{1}{2}\left(f(x) - \mathcal{Y}(f)(x)\right) = -\mathcal{P}_o(f)(x) \end{aligned}$$

$$\Rightarrow \mathcal{Y}\left(\mathcal{P}_o(f)\right) = (-1) \cdot \mathcal{P}_o(f), \checkmark$$

$$\left(\text{or: } \mathcal{Y}\left(\mathcal{P}_o(f)\right) = \mathcal{Y}\left(\frac{1}{2}\left(f - \mathcal{Y}(f)\right)\right) \underset{\substack{\uparrow \\ \mathcal{Y} \text{ is linear}}}{=} \frac{1}{2}\left(\mathcal{Y}(f) - \mathcal{Y}^2(f)\right) = -\mathcal{P}_o(f) \right)$$

Thus for any $f \in F(I, \mathbb{R})$: $f_e = \mathcal{P}_e(f)$ is even and $f_o = \mathcal{P}_o(f)$ is odd. Also their sum give f back:

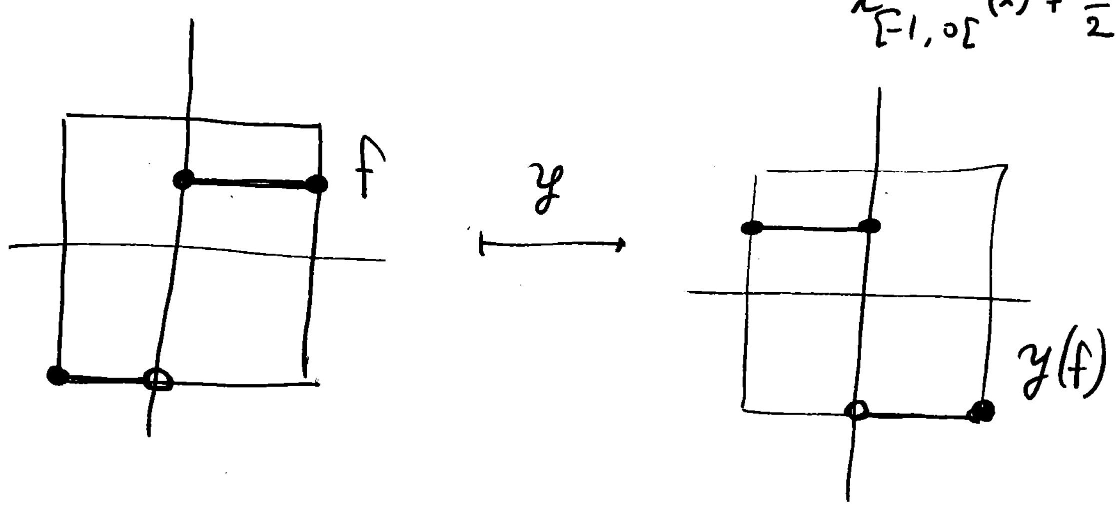
$$f_e + f_o = \frac{1}{2}\left(f + \mathcal{Y}(f)\right) + \frac{1}{2}\left(f - \mathcal{Y}(f)\right) = f.$$

SW : • This even / odd decomposition is unique, ie., if $g \in F_e(I, \mathbb{R})$ and $h \in F_o(I, \mathbb{R})$ are such that $g + h = f$, then $g = f_e$ and $h = f_o$.

• $\mathcal{P}_e \circ \mathcal{P}_e = \mathcal{P}_e$, $\mathcal{P}_o \circ \mathcal{P}_o = \mathcal{P}_o$.

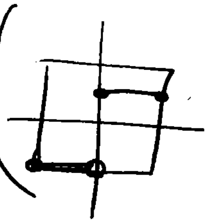
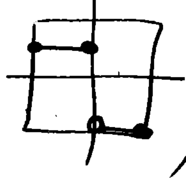
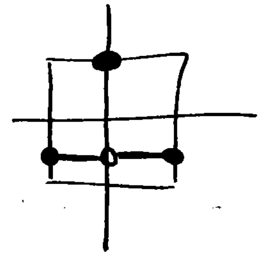
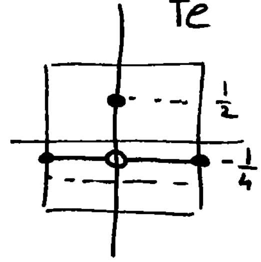
Exe : [redacted] $f : [-1,1] \longrightarrow \mathbb{R}$

$$x \longmapsto -\chi_{[-1,0[}(x) + \frac{1}{2}\chi_{[0,1]}(x)$$



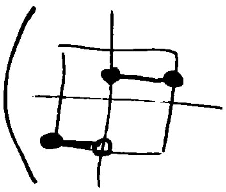
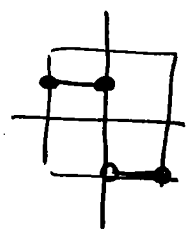
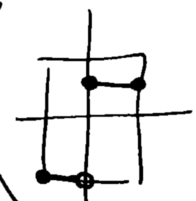
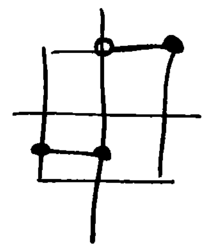
(f is neither even nor odd.)

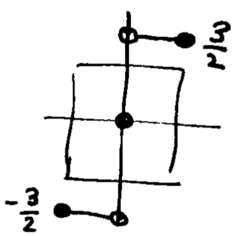
$\varphi_e(f) = \frac{1}{2}\left(f + y(f)\right)$

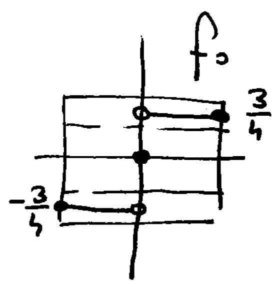
$= \frac{1}{2}\Big($  $+$  $\Big) = \frac{1}{2}$  $=$ 

$= -\frac{1}{4}\chi_{[-1,0[} + \frac{1}{2}\chi_{\{0\}} - \frac{1}{4}\chi_{]0,1]}$.

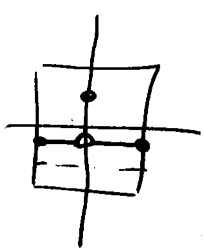
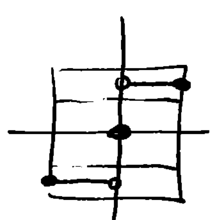
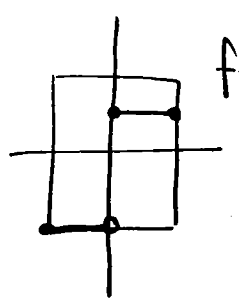
$\varphi_o(f) = \frac{1}{2}\left(f - y(f)\right)$

$= \frac{1}{2}\Big($  $-$  $\Big) = \frac{1}{2}\Big($  $+$  $\Big)$

$= \frac{1}{2}$ 

$=$ 

$= -\frac{3}{4}\chi_{[-1,0[} + \frac{3}{2}\chi_{]0,1]}$.

$f_e + f_o =$  $+$  $=$ 

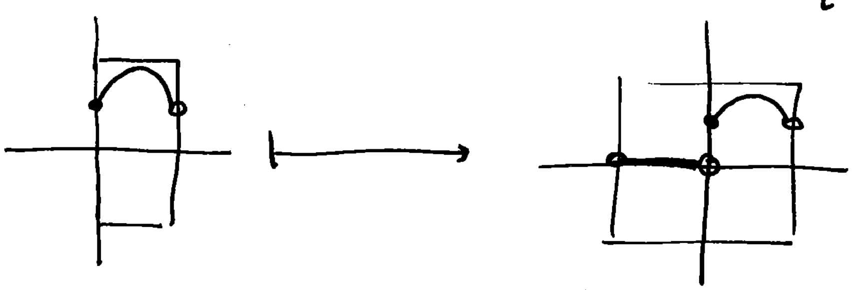
, ✓.

Ex: $\left.\begin{aligned} \cosh(\theta) &= \frac{e^{\theta} + e^{-\theta}}{2} \\ \sinh(\theta) &= \frac{e^{\theta} - e^{-\theta}}{2} \end{aligned}\right\} \Rightarrow e^{\theta} = \underbrace{\cosh(\theta)}_{\text{even}} + \underbrace{\sinh(\theta)}_{\text{odd}}.$

• So far we were dealing with functions that are defined on intervals centered at 0 that are symmetric. Using indicator functions we can apply the machinery we developed to functions defined on arbitrary examples, eg.,

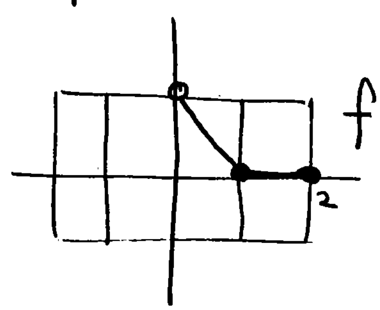
$$F([0,L[,\mathbb{R}) \longrightarrow F(]-L,L[,\mathbb{R})$$

$$f(x) \longmapsto \tilde{f}(x) = f(x)\,\chi_{[0,L[}(x)$$

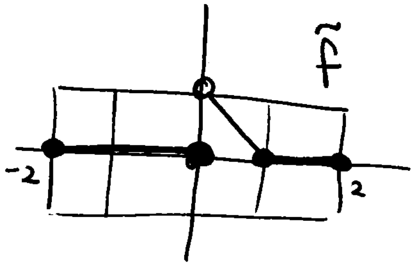


<u>Ex</u>: $f :]0,2] \longrightarrow \mathbb{R}$

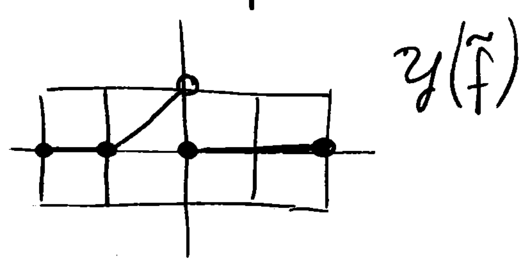
$x \longmapsto (1-x)\,\chi_{]0,1]}(x)$



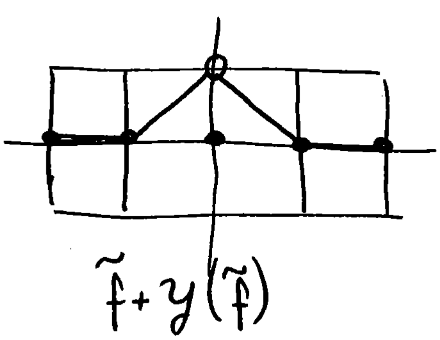
$\Rightarrow$



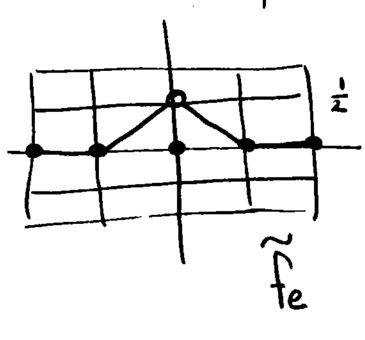
,

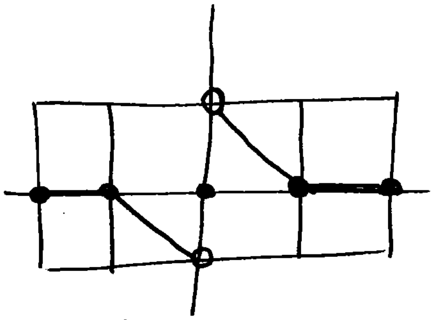


$\Rightarrow$



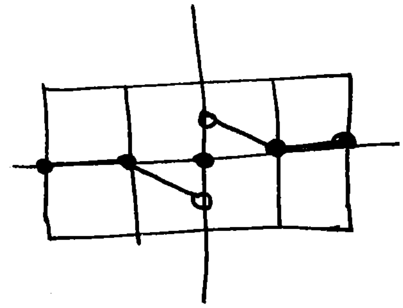
$\Rightarrow$





$\tilde f - \mathcal{Y}(\tilde f)$

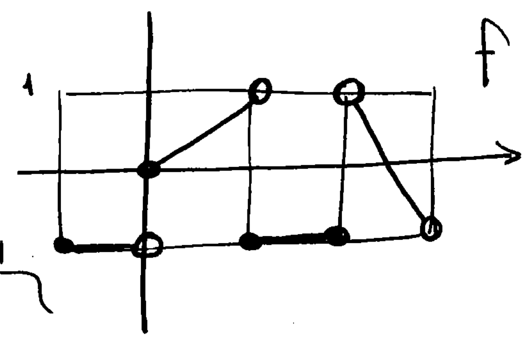
$\Rightarrow$



$\tilde f_o$

$\tilde f + \mathcal{Y}(\tilde f)$ is called the <u>even (periodic) extension</u> of f, and $\tilde f - \mathcal{Y}(\tilde f)$ is called the <u>odd (periodic) extension</u> of f. ("(periodic)" will make sense later)

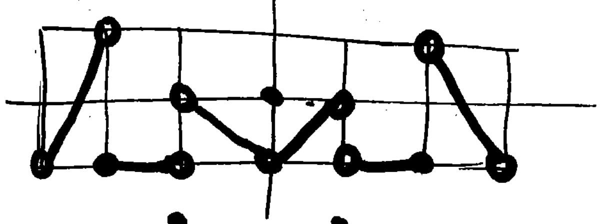
<u>SW</u>: Find the even and odd (periodic) extensions of $f : [-1,3[\longrightarrow \mathbb{R}$



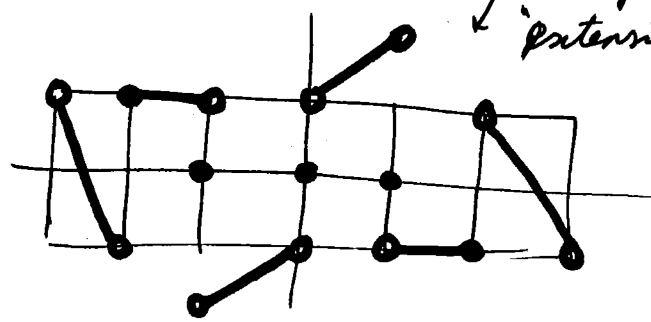
f

$$f : x \mapsto -\chi_{[-1,0[}(x) + x\,\chi_{[0,1[}(x) - \chi_{[1,2]}(x) + (-2x+5)\cdot\chi_{[2,3[}(x) \; -1$$

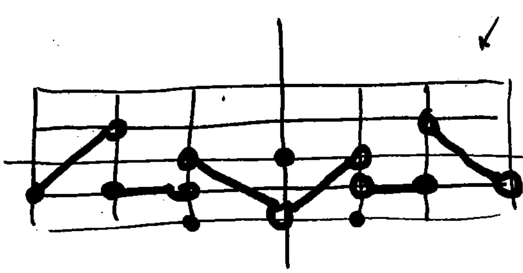
even (periodic) "extension" of f



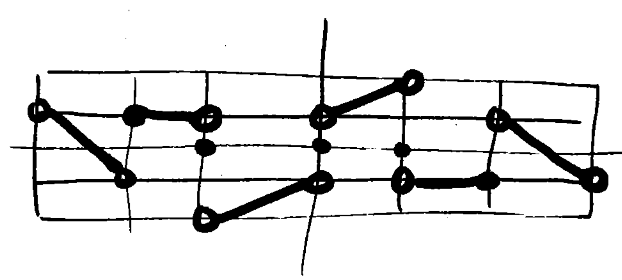
odd (periodic) "extension" of f



$\tilde f_e$



$\tilde f_o$



§10.2: (2)

- Let $T > 0$. $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period T if (or: T-periodic)

$$f(x - T) = f(x) \quad \left(\Leftrightarrow \quad f(x) = f(x+T) \right)$$

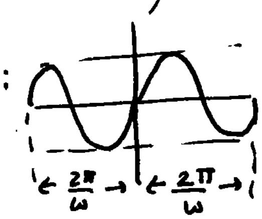
Rem: A T-periodic function is the "same" as a function $\left]-\frac{T}{2}, \frac{T}{2}\right] \to \mathbb{R}$.

SW: Reformulate this definition using the "shift" operator S_T (which first needs to be defined).

- If f is T-periodic, then it is also nT-periodic for any $n \in \{1, 2, \dots\}$.

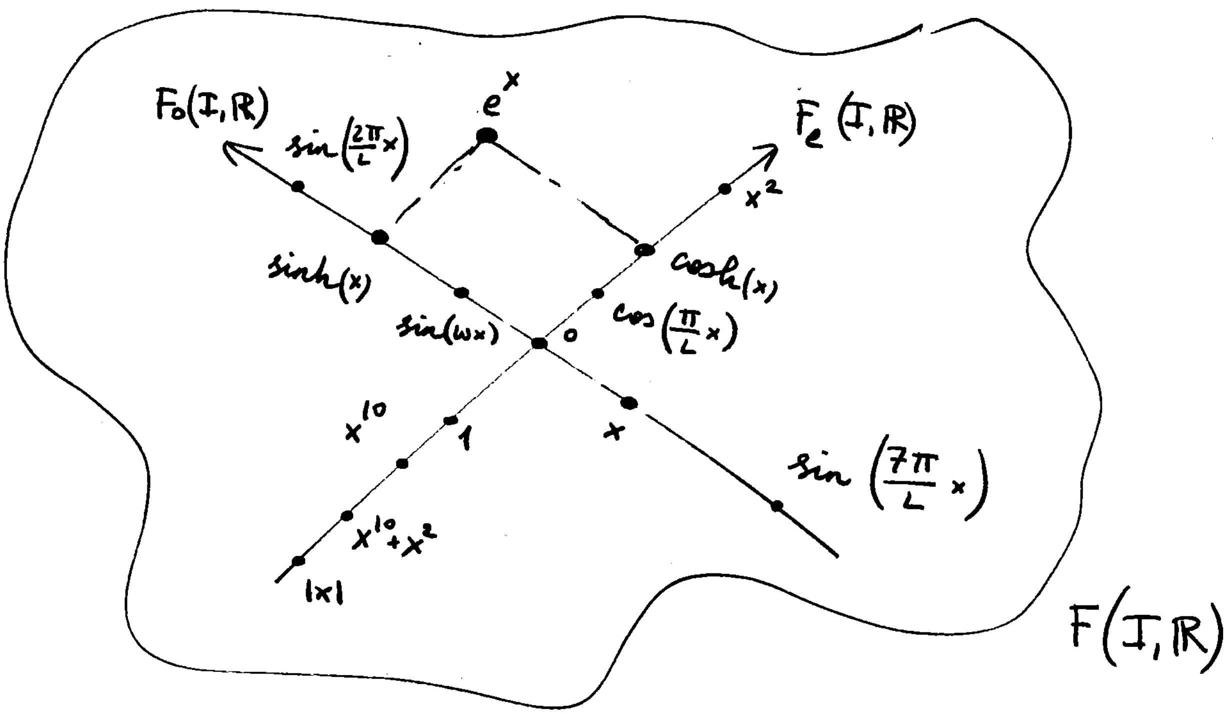
- If f is periodic, then the smallest $T > 0$ for which it is T-periodic is called the fundamental period of f.

SW: - There are periodic functions that have no fundamental period, e.g. $f(x) = 1$.
  - If $\omega > 0$, then both $\sin(\omega x)$ and $\cos(\omega x)$ are periodic with fundamental period $\frac{2\pi}{\omega}$:



  - If $\omega > 0$, $e^{i\omega\theta}$ is periodic with fundamental period $\frac{2\pi}{\omega}$.

• Fix $L>0$ and take $I := [-L, L]$ or $]-L, L[$ as before. We would like to make our earlier caricature more realistic by understanding the "shapes" of $F_e(I,\mathbb{R})$ and $F_o(I,\mathbb{R})$. Recall that earlier we represented both of these linear spaces as lines:



It would be very optimistic to expect that all these functions line up like this. However we can still quantify the "norm" of a function (ie., how "far away" it is from 0) and the "angle" between two functions by adapting the static (2-dimensional, say) versions of these to these function spaces.

Define the <u>inner product</u> (or <u>dot product</u>) on $\mathbb{R}^2$ by:

$$\begin{aligned} INN : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ ((x,y),(z,t)) &\longmapsto xz + yt =: \langle (x,y),(z,t) \rangle = (x,y) \bullet (z,t) \end{aligned}$$

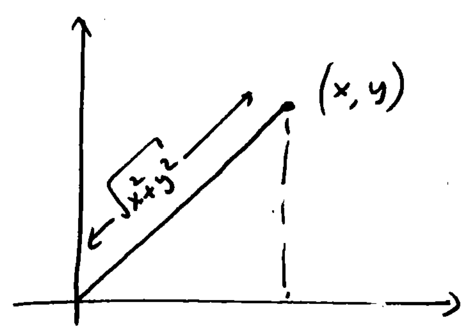
or in matrix notation:

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix} \right\rangle = (x\ y)\begin{pmatrix} z \\ t \end{pmatrix} = xz + yt$$

<u>SW</u>: Describe $MUL : Mat(2\times 2, \mathbb{R}) \times Mat(2\times 2, \mathbb{R}) \to Mat(2\times 2, \mathbb{R})$ in terms of $INN : Mat(2\times 1, \mathbb{R}) \times Mat(2\times 1, \mathbb{R}) \to Mat(1\times 1, \mathbb{R})$.

- Observe that the inner product of a vector by itself is the square of its distance from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

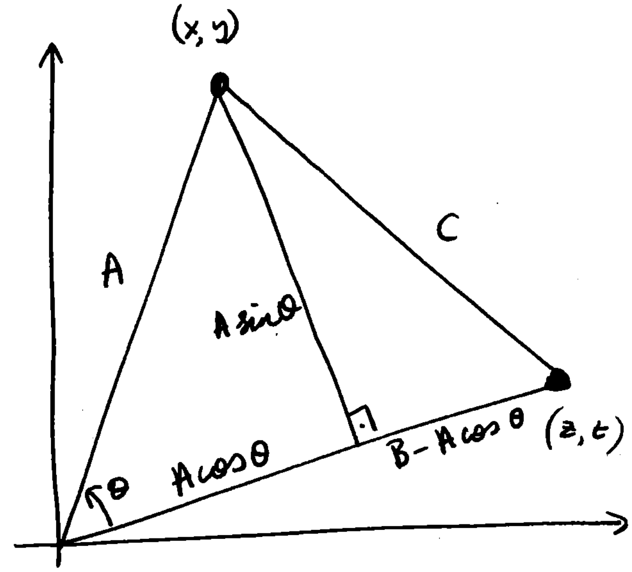
$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = (x\ y)\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2.$$



We call $\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := \sqrt{\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle}$

the <u>norm</u> of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

<u>SW</u>: If $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix} \in \mathbb{R}^2$, $d\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right) := \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} z \\ t \end{pmatrix} \right\|$ gives the <u>distance</u> between $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} z \\ t \end{pmatrix}$.



$A := \|(x,y)\|$ $\quad C := d((x,y),(z,t))$.

$B := \|(z,t)\|$

<u>SW</u>: Let $(x,y), (z,t) \in \mathbb{R}^2$. Then

$\langle (x,y), (z,t) \rangle = \|(x,y)\| \|(z,t)\| \cos\theta$,

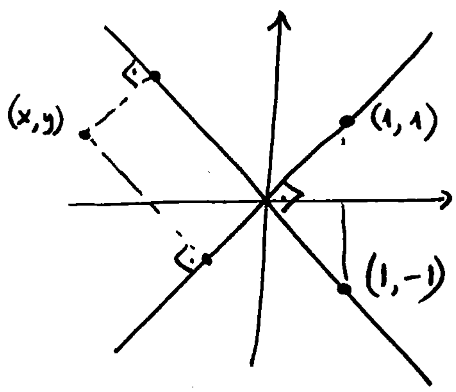
where θ is <u>the angle</u> between (x,y), (z,t)

(Actually, one of the angles, but $\cos(360-\theta) = \cos\theta$ anyway)

- $(x,y), (z,t) \in \mathbb{R}^2$ are <u>orthogonal</u> if $\langle (x,y), (z,t) \rangle = 0$.

<u>Ex</u>: $(1,1), (1,-1) \in \mathbb{R}^2$.

$\Rightarrow \langle (1,1), (1,-1) \rangle = 0 \Rightarrow (1,1)$ and $(1,-1)$ are orthogonal.



If $(x,y) \in \mathbb{R}^2$ is arbitrary,

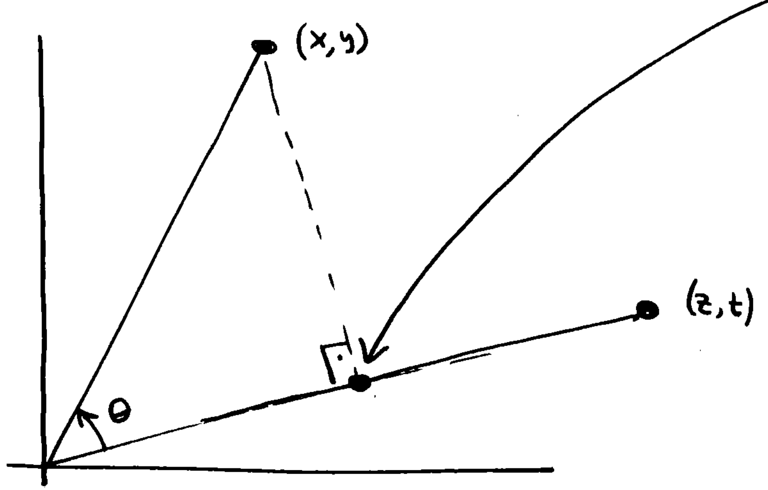
$\langle (x,y), (1,1) \rangle = x+y$, $\|(1,1)\| = \sqrt{2}$

$\langle (x,y), (1,-1) \rangle = x-y$, $\|(1,-1)\| = \sqrt{2}$.

$$\frac{\langle (x,y),(1,1) \rangle}{\|(1,1)\|^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\langle (x,y),(1,-1) \rangle}{\|(1,-1)\|^2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{x+y}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{x-y}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} x+y+x-y \\ x+y-x+y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\langle (x,y),(1,1) \rangle}{\langle (1,1),(1,1) \rangle} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\langle (x,y),(1,-1) \rangle}{\langle (1,-1),(1,-1) \rangle} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

SW: (i) Let $(x,y), (z,t) \in \mathbb{R}$. Then this point (ie., the orthogonal projection of (x,y) onto the line cut out by (z,t)) is:



$$\frac{\langle (x,y), (z,t) \rangle}{\langle (z,t), (z,t) \rangle} \begin{pmatrix} z \\ t \end{pmatrix}.$$

(ii) Let $(v_1,v_2), (w_1,w_2) \in \mathbb{R}^2$ be orthogonal (ie., $\langle (v_1,v_2),(w_1,w_2) \rangle = 0$). Then for any $(x,y) \in \mathbb{R}^2$:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\langle (x,y),(v_1,v_2) \rangle}{\langle (v_1,v_2),(v_1,v_2) \rangle} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \frac{\langle (x,y),(w_1,w_2) \rangle}{\langle (w_1,w_2),(w_1,w_2) \rangle} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}$$

- The point of all this is that orthogonal sets are nifty coordinate systems for linear spaces.

- For the matrix replica $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\mathcal{Y}$:

$$\langle (1,0),(0,1) \rangle = 0, \quad \langle (1,0),(1,0) \rangle = 1, \quad \langle (0,1),(0,1) \rangle = 1$$

$$\langle (x,y),(1,0) \rangle = x, \quad \langle (x,y),(0,1) \rangle = y.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\langle (x,y),(1,0) \rangle}{\langle (1,0),(1,0) \rangle} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\langle (x,y),(0,1) \rangle}{\langle (0,1),(0,1) \rangle} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{x}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{y}{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

SW: Just because a matrix has distinct eigenvalues, it doesn't mean that the associated eigenvectors are orthogonal, eg. $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

• It is time to adapt the notion of an inner product to the function space $F(I,\mathbb{R})$. If we fix $(z,t)\in\mathbb{R}^2$ (say, for instance, because we would like to project vectors orthogonally onto the line cut out by it), then "taking inner product against (z,t)" becomes a function

$$\begin{aligned}\langle \bullet, (z,t)\rangle : \mathbb{R}^2 &\longrightarrow \mathbb{R}\\ (x,y) &\longmapsto \langle (x,y),(z,t)\rangle\end{aligned}$$

<u>SW</u>: $\langle \bullet, (z,t)\rangle$ is linear.

Inspired by this (and also recalling that earlier we mentioned that a function f can be interpreted as a functional as "integrate against f"), we define the <u>inner product</u> for functions as:

$$\begin{aligned}\mathrm{INN} : R(I,\mathbb{R})\times R(I,\mathbb{R}) &\longrightarrow \mathbb{R}\\ (f,\ g) &\longmapsto \langle f,g\rangle := \int_{-L}^{L} f(x)g(x)\,dx\end{aligned}$$

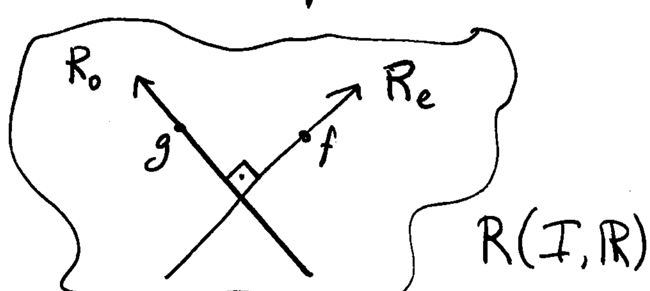
In order for this to work we only allow those functions $f: I\to\mathbb{R}$ for which "$\int_{-L}^{L} f(x)dx$" makes sense, i.e., those functions that are Riemann-integrable (hence the letter R).

For our purposes we may think of $R(I, \mathbb{R})$ as the linear space of all bounded piecewise continuous functions $I \to \mathbb{R}$. Everything we discovered about $\mathcal{Y}$ holds if we replace $F(I, \mathbb{R})$ with $R(I, \mathbb{R})$ (the benefit of this replacement being that now integration is admissible). All the terminology from the static case carries over to the dynamic case.

- If $f \in R_e(I, \mathbb{R})$ and $g \in R_o(I, \mathbb{R})$ (so that f is an even bounded p.w. continuous function and g is an odd bounded p.w. continuous function), then

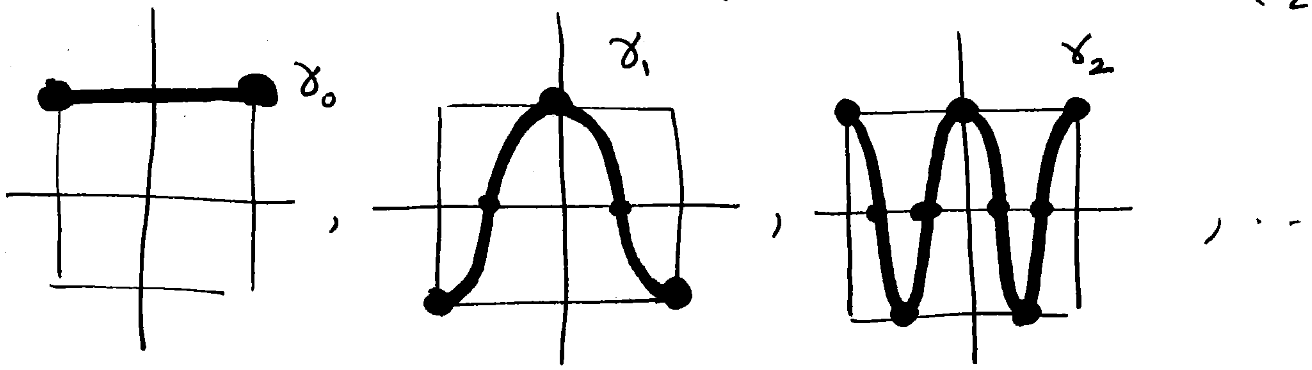
$$\langle f, g \rangle = \int_{-L}^{L} \underbrace{f(x) g(x)}_{\text{odd}} \, dx = 0.$$

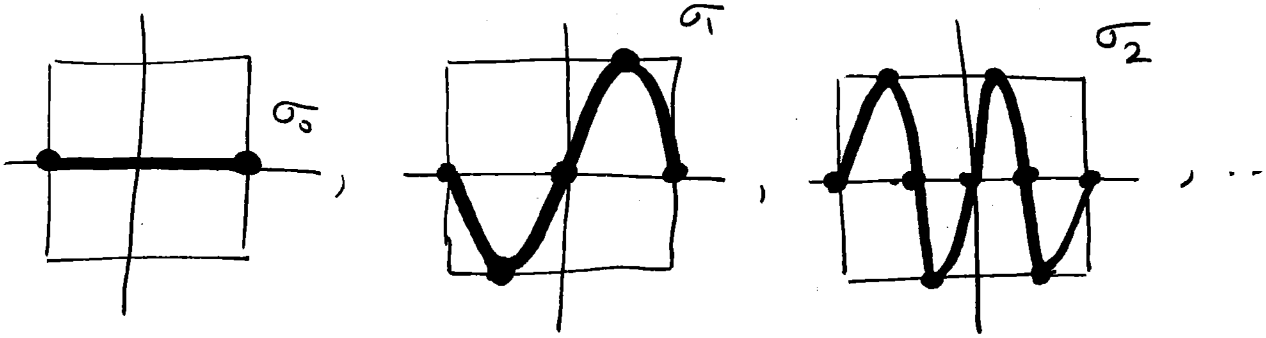
Thus even and odd functions are orthogonal:



• Define (for brevity) for all $n \in \mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$:

$$\mathcal{E}_n(x) := e^{i\frac{n\pi}{L}x}, \quad \gamma_n(x) := \cos\left(\frac{n\pi}{L}x\right), \quad \sigma_n(x) := \sin\left(\frac{n\pi}{L}x\right)$$





Here are the standard formulas:

$$\mathcal{E}_n = \gamma_n + i\sigma_n, \quad \gamma_{-n} = \gamma_n, \quad \gamma_n = \frac{1}{2}(\mathcal{E}_n + \mathcal{E}_{-n}), \quad \mathcal{E}_0 = 1 = \gamma_0$$

$$\mathcal{E}_{-n} = \gamma_n - i\sigma_n, \quad \sigma_{-n} = -\sigma_n, \quad \sigma_n = \frac{1}{2i}(\mathcal{E}_n - \mathcal{E}_{-n}), \quad \sigma_0 = 0.$$

$$\mathcal{E}_{n+m} = \mathcal{E}_n \mathcal{E}_m, \quad \gamma_{n+m} = \gamma_n\gamma_m - \sigma_n\sigma_m, \quad \sigma_{n+m} = \gamma_n\sigma_m + \sigma_n\gamma_m.$$

• $\gamma_0, \gamma_1, \gamma_2, \dots \in R_e(I, \mathbb{R})$ and $\sigma_1, \sigma_2, \dots \in R_o(I, \mathbb{R})$. Shortly we (you) will verify that $\{\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \sigma_3, \gamma_3, \dots\}$ [redacted] is an orthogonal set, and as a result we can use these trigonometric functions to upgrade our caricature to a higher resolution picture.

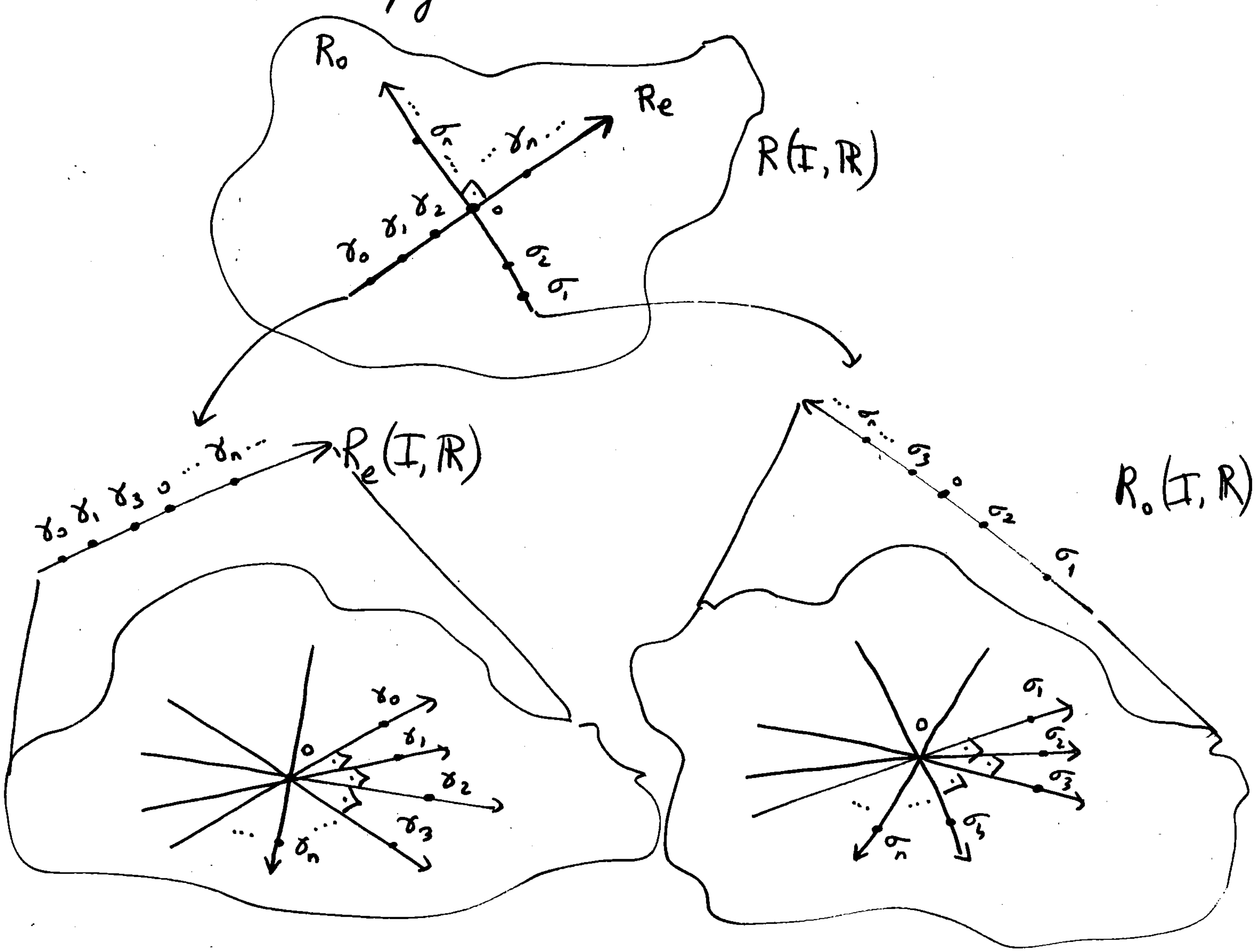
SW (i) Verify:

$$\int_{-L}^{L} \mathcal{E}_n(x)\,dx = \begin{cases} 2L, \text{ if } & n=0 \\ 0, \text{ if } & n\neq 0 \end{cases}, \quad \langle \gamma_n, \sigma_m \rangle = 0.,$$

$$\langle \gamma_n, \gamma_m \rangle = \begin{cases} 2L, \text{ if } & n=m=0 \\ L, \text{ if } & n=m\neq 0 \\ 0, \text{ if } & n\neq m \end{cases}, \quad \langle \sigma_n, \sigma_m \rangle = \begin{cases} L, \text{ if } & n=m \\ 0, \text{ if } & n\neq m \end{cases}$$

(ii) Use indicator functions to write the RHS's.

- We now can upgrade our caricature:



Thus $R_e(I, \mathbb{R})$ is a linear space with infinitely many coordinate axes that are orthogonal to eachother; and similarly for $R_o(I, \mathbb{R})$. What is more, any coordinate axis in $R_e(I, \mathbb{R})$ is perpendicular to any coordinate axis in $R_o(I, \mathbb{R})$.

What is more surprising is that the list

$$\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \sigma_3, \gamma_3, \ldots, \sigma_n, \gamma_n, \ldots$$

misses no coordinate axis of $R(I, \mathbb{R})$!

In other words, $\{\gamma_0, \sigma_1, \gamma_1, \sigma_2, \gamma_2, \ldots, \gamma_n, \sigma_{n+1}, \ldots\}$ is a <u>complete orthogonal set</u>.

(This last statement we'll take for granted.)

- In the static case, we saw that if $\{V_1, \ldots, V_d\} \subseteq \mathbb{R}^d$ is an orthogonal set then

$$\text{for any } X \in \mathbb{R}^d: \quad X = \sum_{k=1}^{d} \frac{\langle X, V_k \rangle}{\langle V_k, V_k \rangle} V_k .$$

A similar statement holds for [redacted] the dynamic case, except since now we have infinitely many coordinates the sum may fail to be finite.

Def: Let $f \in R(\mathbb{R},\mathbb{R})$ be $2L$-periodic (or, equivalently, $f \in R(I,\mathbb{R})$). Put

for all $n \in \mathbb{Z}$: $$e_n := \frac{\langle f, \mathcal{E}_{-n}\rangle}{\langle \mathcal{E}_n, \mathcal{E}_{-n}\rangle} = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi}{L}}\, dx,$$

$$c_0 := 2\,\frac{\langle f, \gamma_0\rangle}{\langle \gamma_0, \gamma_0\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\,dx,$$

for all $n \geq 1$: $$c_n := \frac{\langle f, \gamma_n\rangle}{\langle \gamma_n, \gamma_n\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx,$$

$$s_n := \frac{\langle f, \sigma_n\rangle}{\langle \sigma_n, \sigma_n\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Then

$$\mathcal{F}_{\mathbb{C}}(f)(x) := \sum_{n\in\mathbb{Z}} \frac{\langle f, \mathcal{E}_{-n}\rangle}{\langle \mathcal{E}_n, \mathcal{E}_{-n}\rangle}\mathcal{E}_n(x) = \sum_{n\in\mathbb{Z}} e_n\,\mathcal{E}_n(x) = \boxed{\sum_{n\in\mathbb{Z}} e_n\, e^{i\frac{n\pi}{L}x}}$$ and

$$\mathcal{F}_{\mathbb{R}}(f)(x) := \sum_{n\geq 0}\frac{\langle f, \gamma_n\rangle}{\langle \gamma_n, \gamma_n\rangle}\gamma_n(x) + \sum_{n\geq 1}\frac{\langle f, \sigma_n\rangle}{\langle \sigma_n, \sigma_n\rangle}\sigma_n(x) = \frac{c_0}{2} + \sum_{n\geq 1} c_n\gamma_n(x) + \sum_{n\geq 1} s_n\sigma_n(x)$$

$$= \boxed{\frac{c_0}{2} + \sum_{n\geq 1} c_n\cos\left(\frac{n\pi}{L}x\right) + \sum_{n\geq 1} s_n\sin\left(\frac{n\pi}{L}x\right)}$$ are called

the <u>complex</u> and <u>real</u> <u>Fourier series</u> of f.

e_n's are the <u>complex Fourier coefficients</u> of f and c_n's and s_n's are the <u>real Fourier coefficients</u> of f.

SW: (i) Consider $\boxed{\sum_{n\in\mathbb{Z}} \tilde{e}_n \varepsilon_n}$ (*) and

$$\boxed{\frac{\tilde{c}_0}{2} + \sum_{n\geq 1} \tilde{c}_n \gamma_n + \sum_{n\geq 1} \tilde{s}_n \sigma_n} \quad (**)$$

Verify the conversion formulas:

$$\boxed{(*) = (**)} \Leftrightarrow \boxed{\begin{aligned} \tilde{c}_0 &= 2\tilde{e}_0 \\ \tilde{c}_n &= \tilde{e}_n + \tilde{e}_{-n} \\ \tilde{s}_n &= i(\tilde{e}_n - \tilde{e}_{-n}) \end{aligned}} \Leftrightarrow \boxed{\begin{aligned} \tilde{e}_0 &= \frac{\tilde{c}_0}{2} \\ \tilde{e}_n &= \begin{cases} \frac{1}{2}(\tilde{c}_n - i s_n), & \text{if } n\geq 1 \\ \frac{1}{2}(\tilde{c}_{-n} + i s_{-n}), & \text{if } n\leq -1 \end{cases} \end{aligned}}$$

(ii) Using these formulas, derive the real Fourier series of f from its complex Fourier series (and vice versa).

(iii) If $f \in R_e(I, \mathbb{R})$, then for any $n \geq 1$: $s_n = 0$.

If $f \in R_o(I, \mathbb{R})$, then for any $n \geq 0$: $c_n = 0$.

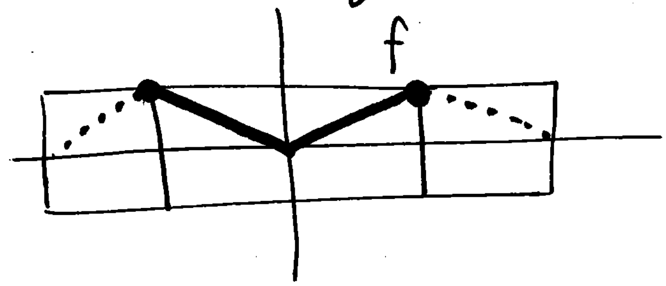
If $f \in R(I, \mathbb{R})$, then

$$\underbrace{\frac{c_0}{2} + \sum_{n\geq 1} c_n \gamma_n}_{\substack{\text{real Fourier} \\ \text{series of } f_e = \gamma_e(f)}} + \underbrace{\sum_{n\geq 1} s_n \sigma_n}_{\substack{\text{real Fourier} \\ \text{series of } f_o = \gamma_o(f).}} \quad \begin{pmatrix}\text{real Fourier} \\ \text{series of } f\end{pmatrix}$$

• Observe that for now we are keeping a function f and its Fourier series separate.

<u>Ex</u>. Find the (real) Fourier coefficients of

$$f : [-2,2] \longrightarrow \mathbb{R}$$
$$x \longmapsto |x|$$



f is even $\Rightarrow$ $s_n = 0$.

$$\frac{c_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \underset{\substack{\uparrow \\ L=2}}{=} \frac{1}{4}\, 2\int_0^{2} |x|\,dx = \frac{1}{2}\int_0^{2} x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^2 = \frac{1}{2}\, 2 = 1.$$

$(n \geq 1)$ $$c_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}\int_{-2}^{2} \underbrace{|x|\cos\left(\frac{n\pi}{2}x\right)}_{\text{even}} dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx \underset{\uparrow}{=} \frac{2}{n\pi}\left[x\sin\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx$$

$$\begin{pmatrix} u = x & dv = \cos\left(\frac{n\pi}{2}x\right) \\ du = dx & v = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right) \end{pmatrix}$$

$$= \frac{2}{n\pi}\left(\underbrace{2\sin(n\pi) - 0}_{=0}\right) + \left(\frac{2}{n\pi}\right)^2\left[\cos\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 = \left(\frac{2}{n\pi}\right)^2\left(\cos(n\pi) - 1\right)$$

$$= \begin{cases} -2\left(\frac{2}{n\pi}\right)^2, & \text{if } \cos(n\pi) = -1 \\ 0, & \text{if } \cos(n\pi) = 1 \end{cases} = \begin{cases} \frac{-8}{(n\pi)^2}, & \text{if } n\pi = \pi, 3\pi, 5\pi, \ldots \\ 0, & \text{if } n\pi = 2\pi, 4\pi, \ldots \end{cases}$$

$$= \begin{cases} -\frac{8}{\pi^2} \cdot \frac{1}{n^2}, & \text{if } n = 1, 3, 5, \ldots \\ 0, & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

$\Rightarrow$ The Fourier series of f is:

$$\frac{c_0}{2} + \sum_{n \geq 1} c_n \gamma_n(x) + \sum_{n \geq 1} \underbrace{s_n}_{=0} \sigma_n(x)$$

$$= 1 - \frac{8}{\pi^2} \sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{1}{n^2} \gamma_n(x) = \boxed{1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2} x\right)} = \mathcal{F}_{\mathbb{R}}(f)(x).$$

Obs: If $f(x_0) = \mathcal{F}_{\mathbb{R}}(f)(x_0)$ for $x_0 = 0$, we would have:

$$0 = f(x_0) = \mathcal{F}_{\mathbb{R}}(f)(x_0) = 1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n \geq 0} \frac{1}{(2n+1)^2} \Rightarrow \boxed{\pi = \sqrt{8 \sum_{n \geq 0} \frac{1}{(2n+1)^2}}}$$

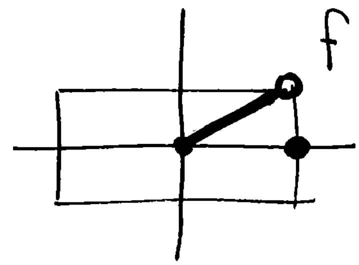
(This is the case, but we'll find this out later.)

SW.: (i) Compute the coefficients of $\mathcal{F}_{\mathbb{C}}(f)$.

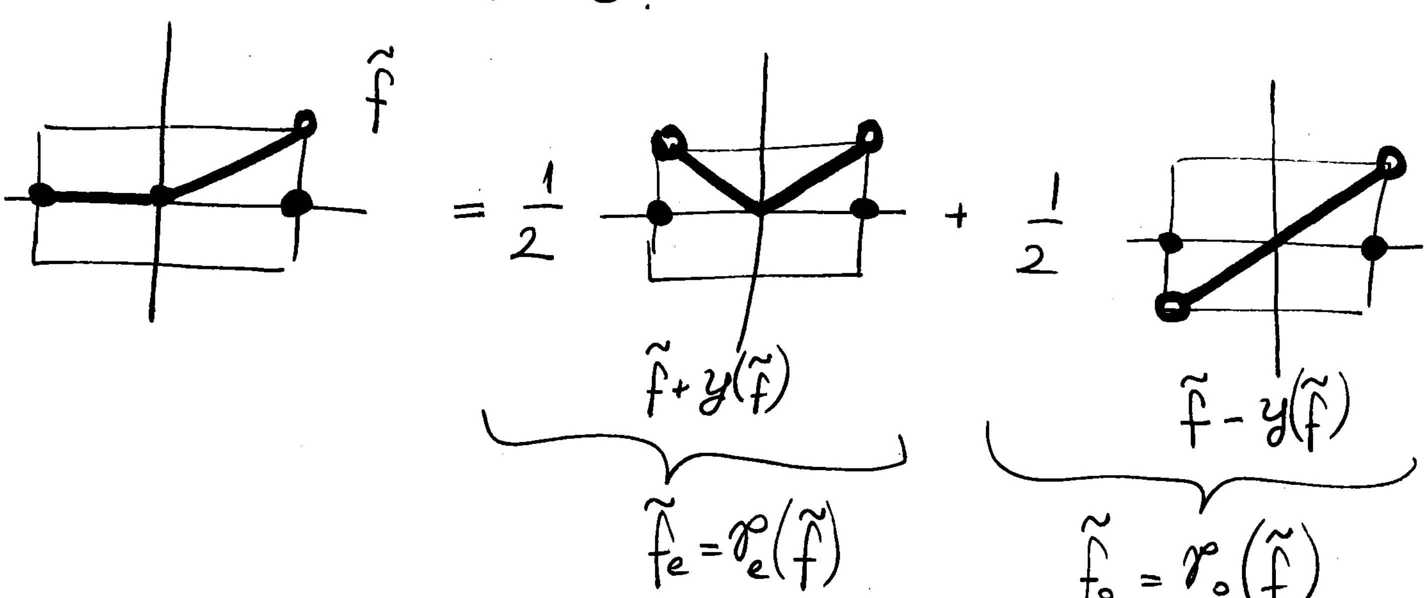
(ii) Does $\sqrt{8 \sum_{n=0}^{N} \frac{1}{(2n+1)^2}}$ approximate π at all (for large N) ?

Ex: Find the (real) Fourier coefficients of

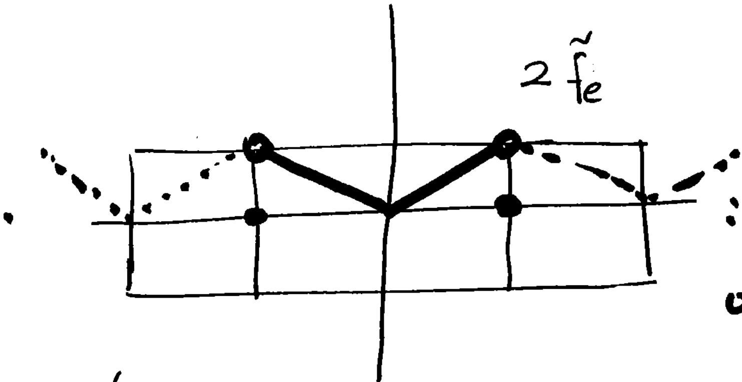
$$f : [0,2] \longrightarrow \mathbb{R}$$
$$x \longmapsto x\chi_{[0,2[}(x) = \begin{cases} x, & \text{if } 0 \le x < 2 \\ 0, & \text{if } x = 2 \end{cases}.$$



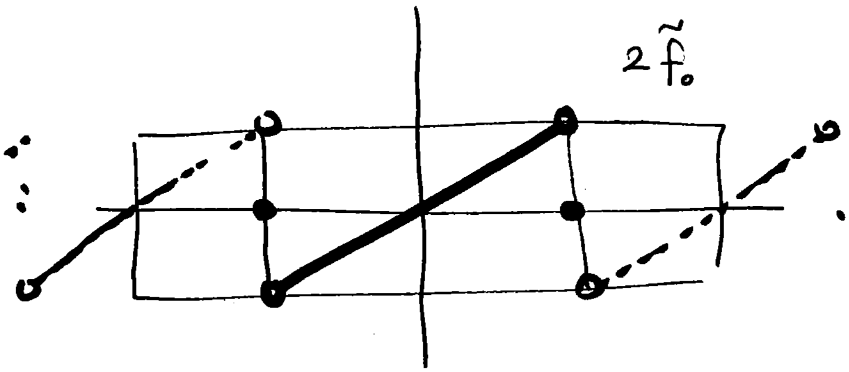
We first need a function defined on a symmetric interval centered at 0.



Recall: $\tilde{f} + \mathcal{Y}(\tilde{f})$ is the even periodic extension of f and $\tilde{f} - \mathcal{Y}(\tilde{f})$ is the odd periodic extension of f.



(even periodic extension of f)



(odd periodic extension of f).

First let's compute the coefficients of $\mathcal{F}_{\mathbb{R}}(\tilde{f})$.

$$c_0 = \frac{1}{2}\int_{-2}^{2} \tilde{f}(x)\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^2 = 1.$$

$(n\geq 1)$ $$c_n = \frac{1}{2}\int_{-2}^{2}\tilde{f}(x)\,\gamma_n(x)\,dx = \frac{1}{2}\int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \begin{cases} \dfrac{-8}{\pi^2}\cdot\dfrac{1}{n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}.$$

$(n\geq 1)$ $$s_n = \frac{1}{2}\int_{-2}^{2}\tilde{f}(x)\,\sigma_n(x)\,dx = \frac{1}{2}\int_0^2 x\sin\left(\frac{n\pi}{2}x\right)dx \quad \begin{pmatrix} u=x & dv=\sin\left(\frac{n\pi}{2}x\right)dx \\ du=dx & \\ v=-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right) & \end{pmatrix}$$

$$= \frac{1}{2}\left(-\frac{2}{n\pi}\left[x\cos\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2 + \frac{2}{n\pi}\int_0^2\cos\left(\frac{n\pi}{2}x\right)dx\right)$$

$$= \left(\frac{-1}{n\pi}\right)2\cos(n\pi) + \frac{1}{n\pi}\,\frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}x\right)\right]\Big|_0^2$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2}\underbrace{\sin(n\pi)}_{=0} = \frac{-2}{n\pi}\cos(n\pi) = \begin{cases} \dfrac{-2}{n\pi}, & \text{if } n=2,4,\ldots \\ \dfrac{2}{n\pi}, & \text{if } n=1,3,\ldots \end{cases}$$

$$\Rightarrow \sigma F_{\mathbb{R}}(\tilde{f})(x) = \frac{1}{2} + \left(\frac{-8}{\pi^2}\right) \sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{1}{n^2} \gamma_n(x) + \left(-\frac{2}{\pi}\right) \sum_{n \geq 1} \frac{(-1)^n}{n} \sigma_n(x)$$

$$= \frac{1}{2} + \left(\frac{-8}{\pi^2}\right) \sum_{n \geq 0} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2} x\right) + \left(\frac{-2}{\pi}\right)\left(\sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{(-1)^n}{n} \sigma_n(x)\right)$$

$$+ \left(\frac{-2}{\pi}\right)\left(\sum_{\substack{n \geq 1 \\ n:\text{even}}} \frac{(-1)^n}{n} \sigma_n(x)\right)$$

$$= \frac{1}{2} + \left(\frac{-8}{\pi^2}\right) \sum_{n \geq 0} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2} x\right) + \frac{2}{\pi} \sum_{n \geq 0} \frac{1}{(2n+1)} \sin\left(\frac{(2n+1)\pi}{2} x\right)$$

$$- \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{2n} \sin\left(\frac{2n\pi}{2} x\right)$$

$$= \boxed{\frac{1}{2} + \left(\frac{-8}{\pi^2}\right) \sum_{n \geq 0} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi}{2} x\right) + \frac{2}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin(n\pi x)}$$

Onto the coeff.s of $\mathcal{F}_{\mathbb{R}}(\tilde{f}_e)$:

$$c_0 = \frac{1}{2}\int_{-2}^{2} \tilde{f}_e(x)\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]\Big|_0^2 = 2.$$

$$(n \geq 1) \quad c_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\gamma_n(x)}_{\text{even}}\,dx = \int_0^2 x \cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \left\{ \begin{array}{ll} -\frac{8}{\pi^2}\cdot\frac{1}{n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{array} \right\}.$$

$$n \geq 1 \quad s_n = \frac{1}{2}\int_{-2}^{2} \underbrace{\tilde{f}_e(x)\,\sigma_n(x)}_{\text{odd}}\,dx = 0.$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}(\tilde{f}_e)(x) = 1 + \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \left(\frac{-8}{\pi^2}\cdot\frac{1}{n^2}\right)\gamma_n(x)$$

$$= \boxed{1 + \left(-\frac{8}{\pi^2}\right)\sum_{n\geq 0} \frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}.$$

$$\Rightarrow \boxed{\mathcal{F}_{\mathbb{R}}(2\tilde{f}_e)(x) = 2 + \left(-\frac{16}{\pi^2}\right)\sum_{n\geq 0} \frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right)}$$

is the Fourier series of the even periodic extension of f.

Finally let's look at $\mathcal{F}(\tilde{f}_o)$.

$\tilde{f}_o$ is odd $\Rightarrow$ $c_n = 0$.

$$(n \geq 1) \quad s_n = \frac{1}{2} \int_{-2}^{2} \underbrace{\tilde{f}_o(x)\, \sigma_n(x)}_{\text{even}}\, dx = \int_0^2 \tilde{f}_o(x)\, \sigma_n(x)\, dx$$

$$= \int_0^2 x \sin\left(\frac{n\pi}{2} x\right) dx = \begin{cases} \dfrac{-2}{n\pi}, & \text{if } n = 2, 4, \dots \\[2mm] \dfrac{2}{n\pi}, & \text{if } n = 1, 3, \dots \end{cases}$$

$$\Rightarrow \mathcal{F}_{\mathbb{R}}(\tilde{f}_o)(x) = \sum_{n \geq 1} \left(\frac{-2}{\pi}\right) \frac{(-1)^n}{n} \sigma_n(x)$$

$$= \left(-\frac{2}{\pi}\right)\left(\sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{(-1)^n}{n} \sigma_n(x) + \sum_{\substack{n \geq 1 \\ n:\text{even}}} \frac{(-1)^n}{n} \sigma_n(x) \right)$$

$$= \left(+\frac{2}{\pi}\right) \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{2n} \sin\left(\frac{2n\pi}{2} x\right)$$

$$= \boxed{\frac{2}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{1}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin(n\pi x)}$$

$$\Rightarrow \boxed{\mathcal{F}_{\mathbb{R}}(2\tilde{f}_o)(x) = \frac{4}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{2} x\right) - \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin(n\pi x)}$$

is the Fourier series of the odd periodic extension of f.

<u>SW</u>: (i) Compute the coefficients of $\mathcal{F}_{\mathbb{C}}(\tilde{f})$.

(ii) $\mathcal{F}_{\mathbb{R}}(\tilde{f}) = \mathcal{F}_{\mathbb{R}}(\tilde{f}_e) + \mathcal{F}_{\mathbb{R}}(\tilde{f}_o) = \frac{1}{2}\mathcal{F}_{\mathbb{R}}(2\tilde{f}_e) + \frac{1}{2}\mathcal{F}_{\mathbb{R}}(2\tilde{f}_o)$

(iii) $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{C}}$ are both linear.

(iv) How do $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}} \circ \mathcal{Y}$ relate to eachother?

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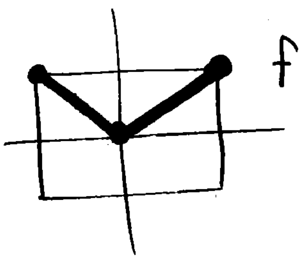
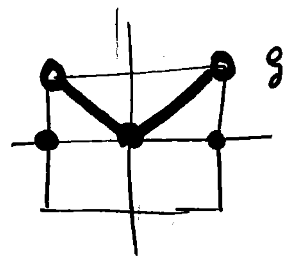
§10.3 : (1)

- So far we have encountered two distinct functions with the same real Fourier series:

$$f : [-2,2] \longrightarrow \mathbb{R}, \quad x \longmapsto |x| \qquad , \qquad g : [-2,2] \longrightarrow \mathbb{R}, \quad x \longmapsto |x|\, \chi_{]-2,2[}(x).$$





$$\boxed{\mathcal{F}(f)(x) = \mathcal{F}(g)(x) = 1 - \frac{8}{\pi^2} \sum_{n \geq 0} \frac{1}{(2n+1)^2}\, \gamma_{2n+1}(x)}$$

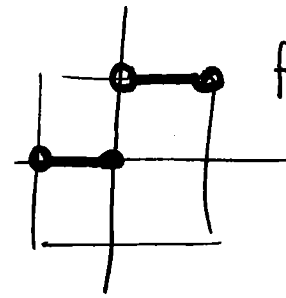
This indicates that the Fourier series can not be faithful to both of them for each $x \in [-2,2]$. Since f and g differ only at two points, these two points are the usual suspects. Still, we have the next best thing (which we'll take for granted):

<u>Fourier Convergence Theorem</u>: Let $L > 0$, $I := [-L, L]$ or $]-L, L[$, $f \in R(I, \mathbb{R})$ have pw. continuous $\partial_x f$. Then

$$\text{for all } x \in I: \quad \mathcal{F}_{\mathbb{R}}(f)(x) = \frac{1}{2}\left(\lim_{\substack{h \to 0 \\ h > 0}} f(x+h) + \lim_{\substack{h \to 0 \\ h < 0}} f(x+h) \right).$$

Ex: Find the Fourier series of $f :]-1,1[\longrightarrow \mathbb{R}$, $x \longmapsto \chi_{]0,1[}(x)$

$$c_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 dx = 1.$$



$$(n \geq 1) \quad c_n = \int_{-1}^{1} f(x)\gamma_n(x)\,dx = \int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\left[\sin(n\pi x)\right]\Big|_0^1 = 0.$$

$$(n \geq 1) \quad s_n = \int_{-1}^{1} f(x)\,\sigma_n(x) = \int_0^1 \sin(n\pi x)\,dx$$

$$= -\frac{1}{n\pi}\left[\cos(n\pi x)\right]\Big|_0^1 = -\frac{1}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \begin{cases} \dfrac{2}{n\pi} & , \text{if } \cos(n\pi) = -1 \\ 0 & , \text{if } \cos(n\pi) = 1 \end{cases} = \begin{cases} \dfrac{2}{n\pi}, & \text{if } n = 1, 3, 5, \dots \\ 0, & \text{if } n = 2, 4, 6, \dots \end{cases}$$

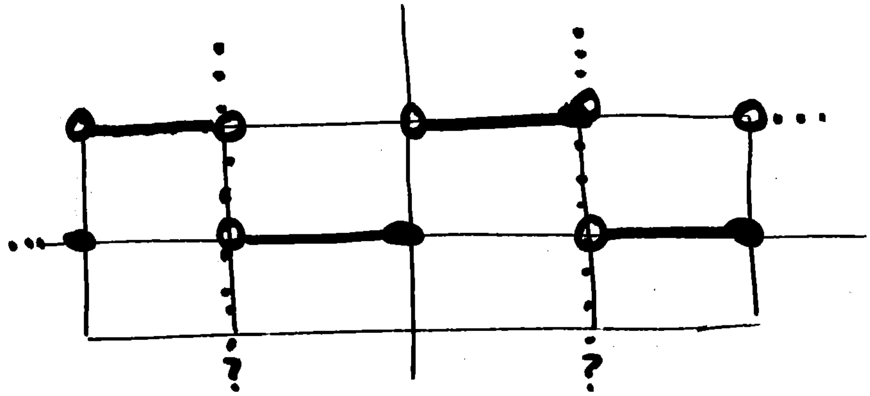
$$\Rightarrow \mathcal{F}_{\mathbb{R}}(f)(x) = \frac{1}{2} + \sum_{\substack{n \geq 1 \\ n:\text{odd}}} \frac{2}{n\pi}\,\sigma_n(x)$$

$$= \boxed{\frac{1}{2} + \frac{2}{\pi} \sum_{n \geq 0} \frac{1}{2n+1} \sin((2n+1)\pi x)}$$

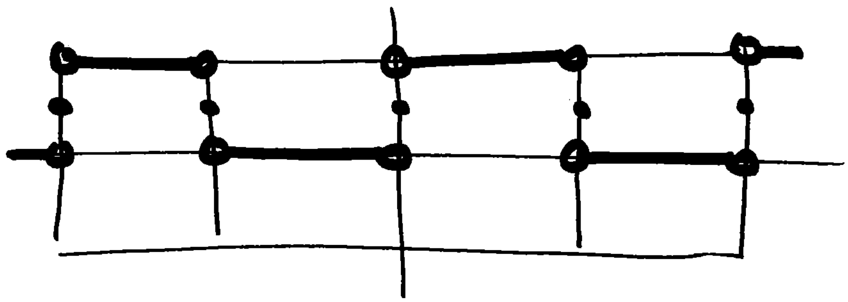
$$\mathcal{F}_{\mathbb{R}}(f)(1) = \frac{1}{2} = \frac{1}{2}\left(\lim_{\substack{h \to 0 \\ h > 0}} f(x+h) + \lim_{\substack{h \to 0 \\ h < 0}} f(x+h)\right)$$

(Here we consider f to be periodic.)

$$\mathcal{F}_{\mathbb{R}}(f)(0) = \frac{1}{2}.$$



f



$\mathcal{F}_{\mathbb{R}}(f)$

<u>SW</u>: (i) Find $\mathcal{F}_{\mathbb{C}}(f)$.

(ii) Prove that

$$\boxed{\pi = 4 \sum_{n \geq 0} \frac{(-1)^n}{2n+1}}$$