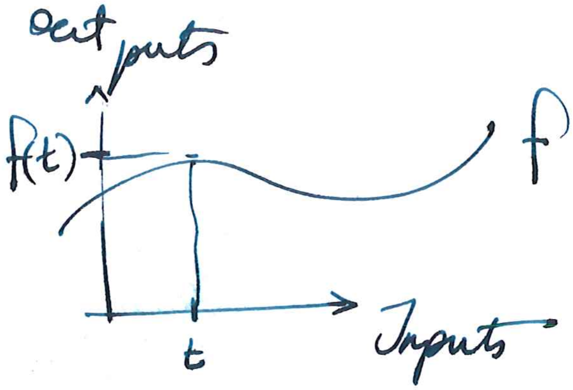
230/231 · <u>Introduction</u> 01/13/2020

<u>Fact</u>: Everything in math is a <u>function</u>, in that it takes Inputs and turns them into outputs.

<u>Notation</u>: $f : \text{Inputs} \longrightarrow \text{Outputs}$

$t \longmapsto f(t)$



We will be ambiguous about f & $f(t)$.

<u>Other Names</u>: function, map, mapping, correspondence (arrow), morphism, operator, functional, transform, functor, ...

· Two functions are <u>equal</u> if they have the same designated Inputs, same designated Outputs, and for the same input they produce the same output.

$\cancel{F(\mathbb{R},\mathbb{R})}$ S : Input set

T : Output set

$F(S,T)$: set of all functions $f : S \to T$.

$\cancel{C^0(S,T)}$ $F(\mathbb{R},\mathbb{R})$

$C^0(\mathbb{R},\mathbb{R})$: continuous

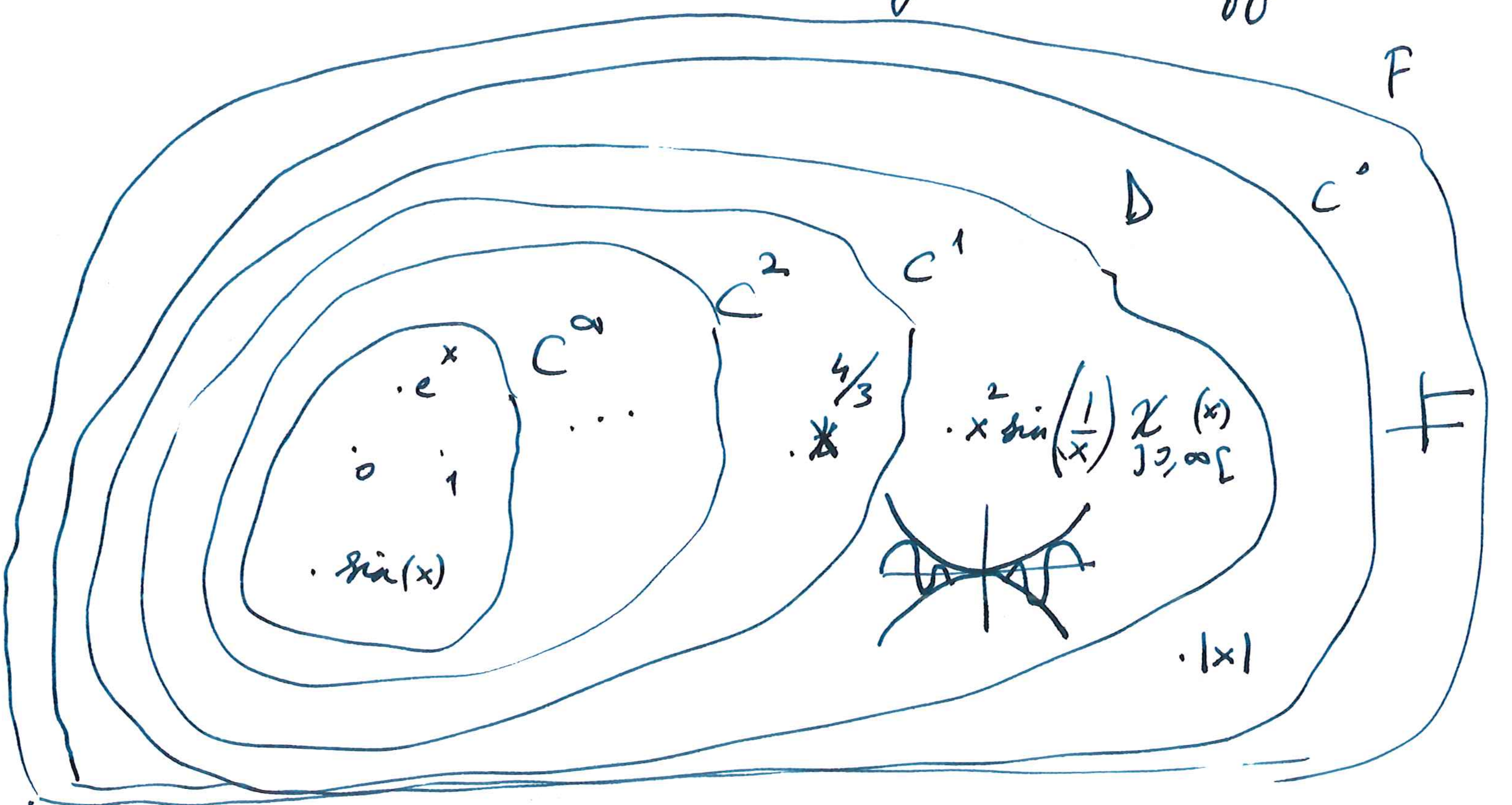
$D(\mathbb{R},\mathbb{R})$: differentiable

$C^1(\mathbb{R},\mathbb{R})$: continuously differentiable

$C^2(\mathbb{R},\mathbb{R})$: twice cont. diff

$\vdots$

$C^\infty(\mathbb{R},\mathbb{R})$: ∞ many times differentiable



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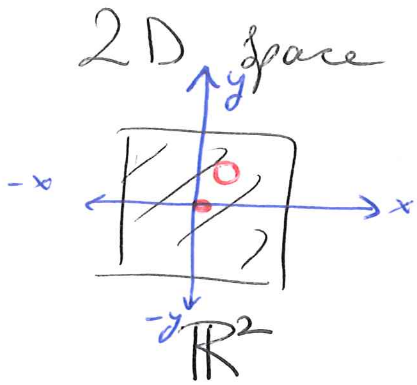
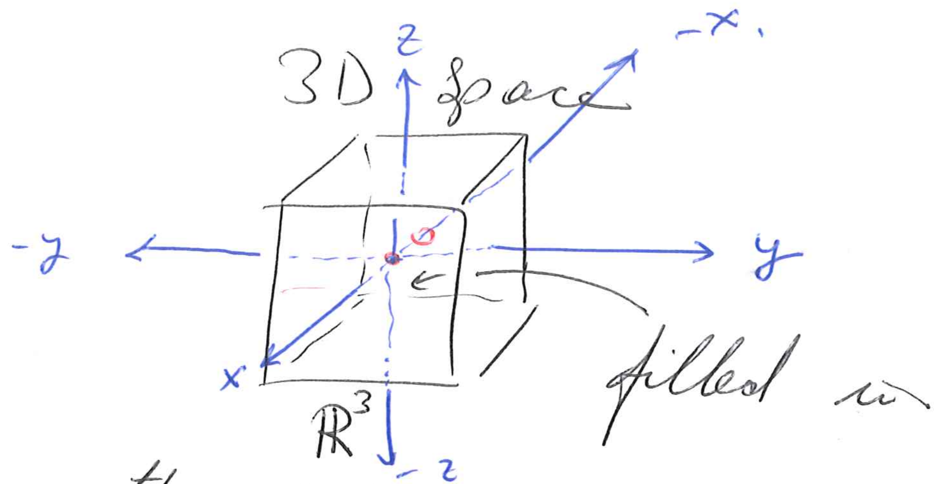
Vectors & Geometry in ~~2D~~ 3D space.

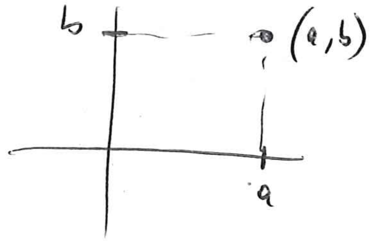


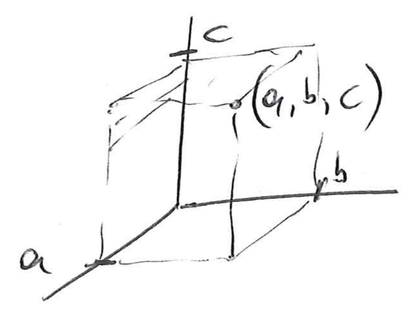


Pick a point as the origin, O
Draw pairwise perpendicular axes.

~~Label points/vectors entri~~

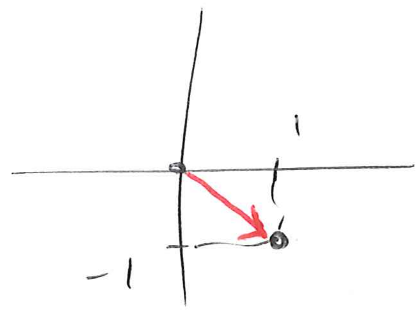
Any point/vector is represented by a pair (in 2D) or a triple (in 3D) numbers.





• What is the difference / relation between points & vectors? In the presence of an origin they are interchangeable, both are denoted by a triples of numbers.

Ex: $(1,-1) \in \mathbb{R}^2$.



Technically, points $\leftrightarrow$ positions
vector $\leftrightarrow$ velocity

"tangent"

phase / state space $\longrightarrow$ $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ $\quad (p,v) = (p_1, p_2, p_3, v_1, v_2, v_3)$

$$\big\downarrow$$

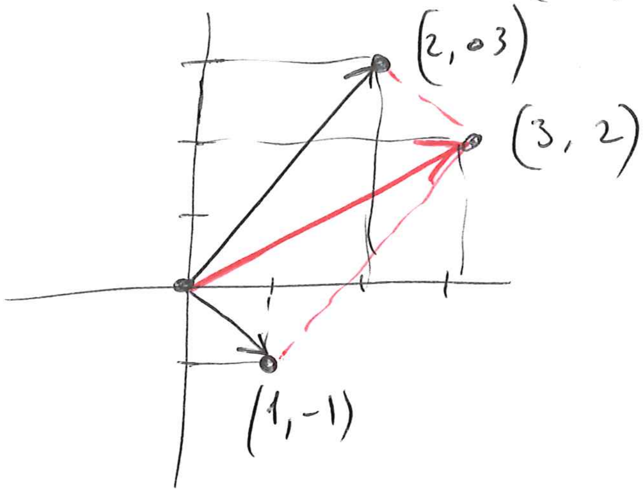
configuration space $\longrightarrow$ $\mathbb{R}^3$ $\quad p = (p_1, p_2, p_3)$

vectors live in the second copy of $\mathbb{R}^3$ in the phase space, while points live in the first copy of $\mathbb{R}^3$.

One can add two vectors & scale a vector by a real number, both entrywise.

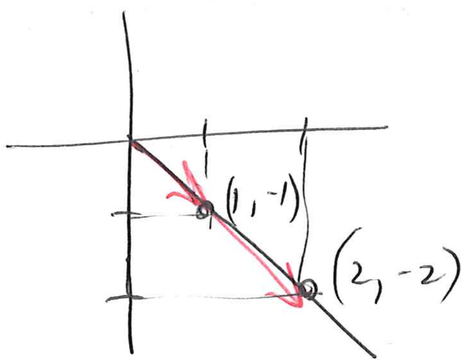
Ex: $(1,-1)+(2,3)=(1+2,-1+3)$

$=(3,2)$.



Complete to a parallelogram.

$$2\cdot(1,-1)=(2\cdot1,\ 2\cdot(-1))=(2,-2).$$

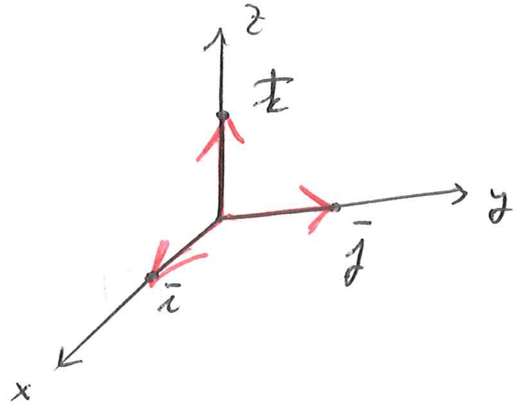


- Standard basis vectors (ONB).

$\bar{\imath} = (1,0,0)$

$\bar{\jmath} = (0,1,0)$

$\bar{k} = (0,0,1)$.

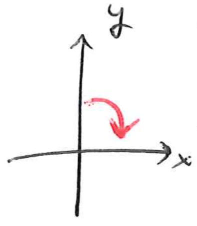


$$\boxed{(a,b,c) = a\bar{\imath} + b\bar{\jmath} + c\bar{k}.}$$

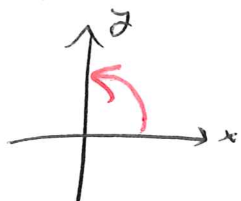
Ex: $(1,-1,5) = \bar{\imath} - \bar{\jmath} + 5\bar{k}$.

- Orientation.

2D space.

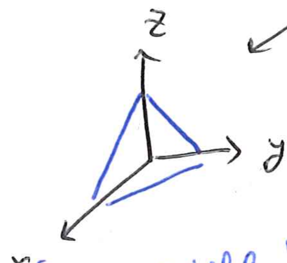


clockwise (Left hand orientation)



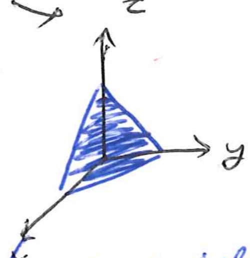
counterclockwise (Right hand orientation).

3D space.



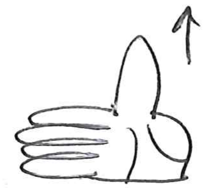
(origin visible) (look up)

LHO.



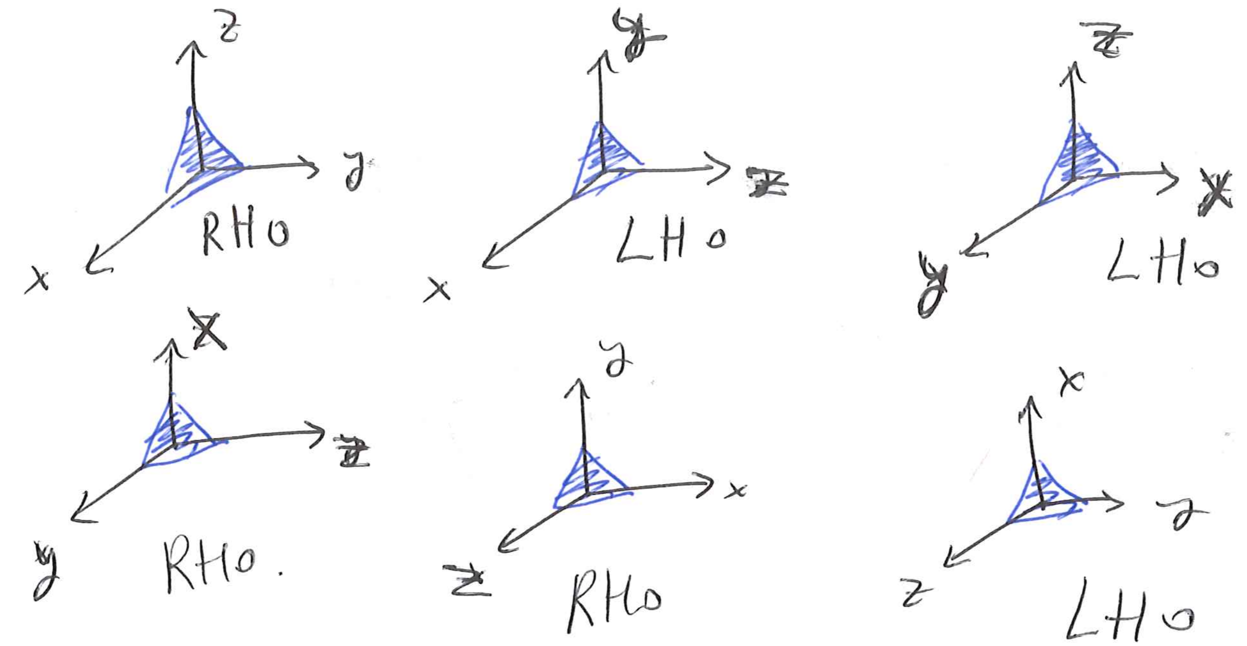
(origin invisible) (look down)

RHO

RHO: $\bar{\imath} \leftarrow$  $\uparrow \bar{k}$, Close RH from $\bar{\imath}$ towards $\bar{\jmath}$. Thumb points to $\bar{k}$.

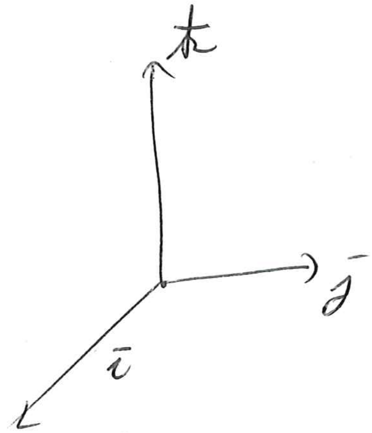
We shall use RHO.

SW: Which ones are RHO?
Which ones are LHO?

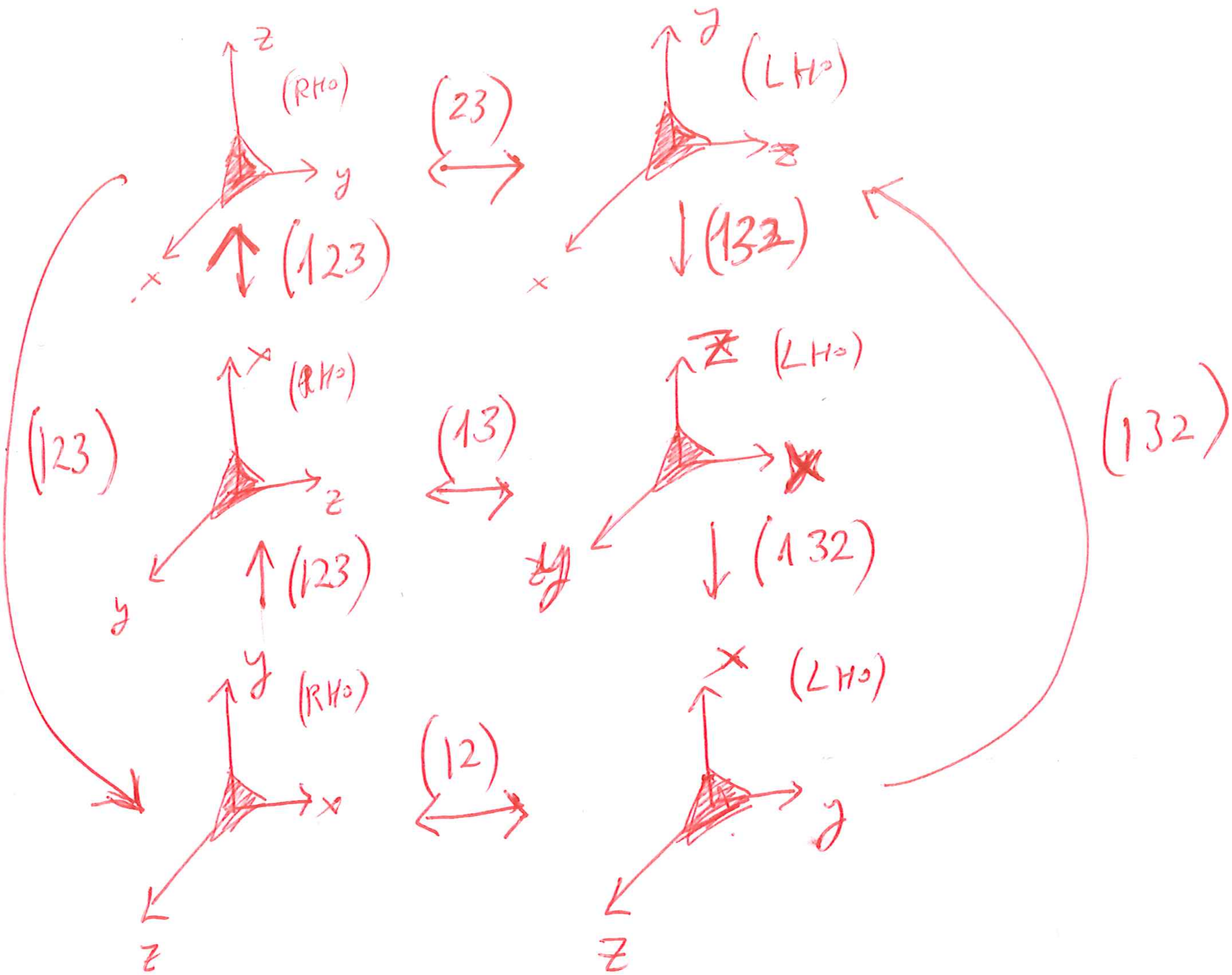


- A multiplication table based on RHO (for cross product ×).

×	0	$\bar{i}$	$\bar{j}$	$\bar{k}$
0	0	0	0	0
$\bar{i}$	0	0	$\bar{k}$	$-\bar{j}$
$\bar{j}$	0	$-\bar{k}$	0	$\bar{i}$
$\bar{k}$	0	$\bar{j}$	$-\bar{i}$	0



Close RH from ~~entry fr~~ [row entry] towards [column entry]. ~~Outco Whe~~ The direction in which the thumb points is [row entry] × [column entry].



$$(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

<u>SW</u> : 2D.

$$(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

• Further structures on $\mathbb{R}^3$:

dot / inner product (angles)

$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$(u, v) \longmapsto u \cdot v$

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

$\xrightarrow{\ (1)\ }$ / $\xleftarrow{\ (2)\ }$

~~norm (length)~~

norm / magnitude (length)

$\mathbb{R}^3 \to \mathbb{R}$

$u \longmapsto |u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

(4) $\uparrow$ $\downarrow$ (3)

distance / metric (length)

$\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$

$(u, v) \longmapsto \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

$d(u,v) =$

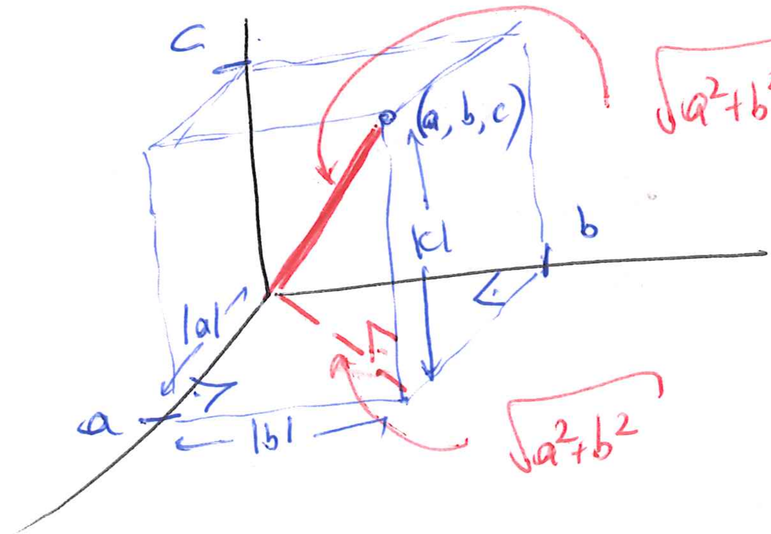
(1) $|u| = \sqrt{u \cdot u}$

(2) $u \cdot v = \frac{1}{2}\left(|u+v|^2 - |u|^2 - |v|^2\right)$

(3) $d(u,v) = |u - v|$

(4) $|u| = d(u, 0)$.

• We start with distance. $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $(p,q) \mapsto d(p,q)$.

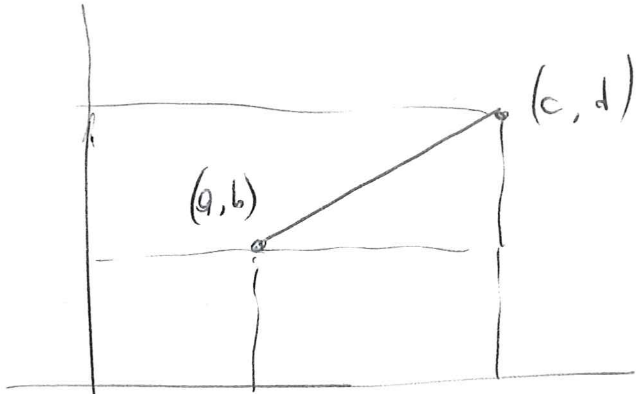


$\sqrt{a^2+b^2+c^2} = |(a, b, c)|$

$= d((a, b, c), (0,0,0))$

$\sqrt{a^2+b^2}$

SW :

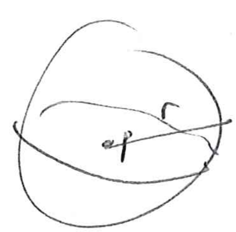


Verify.

Ex : $d\left((2,-1,7),(1,-3,5)\right) = \sqrt{1^2+2^2+2^2} = 3.$

- $p \in \mathbb{R}^3$, $r \geq 0$.

sphere with center p radius r :



$$S(p,r) = \{ q \in \mathbb{R}^3 \mid d(p,q) = r \}$$

$$(p_1-q_1)^2 + (p_2-q_2)^2 + (p_3-q_3)^2 = r^2$$

open ball w/ center p rad r :

$$B(p,r) = \{ q \in \mathbb{R}^3 \mid d(p,q) < r \} \qquad \underset{\text{"boundary of"}}{\partial} B(p,r) = S(p,r).$$

closed ball w/ center p rad r :

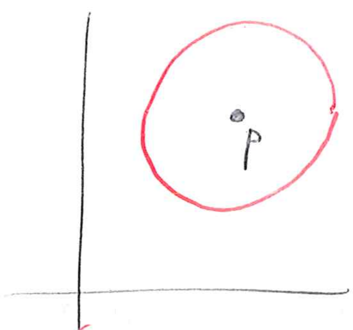
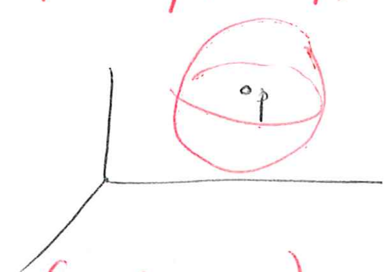
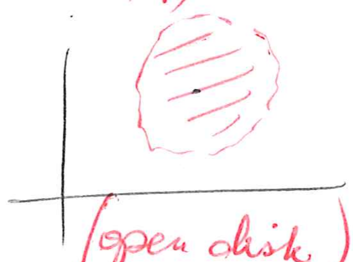
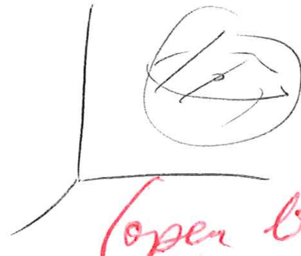
$$\overline{B(p,r)} = \{ q \in \mathbb{R}^3 \mid d(p,q) \leq r \}.$$

SW : Write down open/closed ball inequalities.

Sphere / Open Ball / Closed Ball

~~Formula Description 1D 2D 3D~~

~~$d\,(x,y,z),p$~~

1D	2D	3D
$\lvert x-p\rvert = r$ $x = p \mp r.$  $p-r$ p $p+r$ (two points)	$(x-p_1)^2 + (x-p_2)^2 = r^2$  (circle)	$(x-p_1)^2 + (x-p_2)^2 + (x-p_3)^2 = r^2$  (sphere)
$\lvert x-p\rvert < r$ $(p-r, p+r)$  (open interval)	$d((x,y),p) < r$  (open disk)	$d((x,y,z),p) < r$  (open ball)
$\lvert x-p\rvert \le r$ $[p-r, p+r]$  (closed interval)	$d((x,y),p) \le r$  (closed disk)	$d((x,y,z),p) \le r$  (closed ball)

• $\mathbb{R}[x,y]$ = set of polynomials in the variables x & y with real coeff.

$\mathbb{R}[x,y,z]$ = set of polys in x, y, z w/ real coeff.

~~A polynomial~~ A subset/ $S \subseteq \mathbb{R}^2$ or $S \subseteq \mathbb{R}^3$
subspace

"subset of"

is algebraic if it is the zero locus of some $\wedge$ poly. $P(x,y,z) = 0$.
(finitely many).

• spheres are algebraic:

$S(p,r)$ is cut out by: $P(x,y,z) = (x-p_1)^2 + (y-p_2)^2 + (z-p_3)^2 - r^2$,

which is a degree two polynomial.

A general degree two poly $P(x,y,z) \in \mathbb{R}[x,y,z]$:

$$\begin{aligned} P(x,y,z) = {} & a_{200}x^2 + a_{020}y^2 + a_{002}z^2 \\ & + a_{110}xy + a_{101}xz + a_{011}yz \\ & + a_{100}x + a_{010}y + a_{001}z \\ & + a_{000}. \end{aligned}$$

$$\begin{aligned} P(x,y,z) = {} & 1x^2 + 1y^2 + 1z^2 \\ & + 0xy + 0xz + 0yz \\ & + (-2p_1)x + (-2p_2)y + (-2p_3)z \\ & + (p_1^2 + p_2^2 + p_3^3 - r^2) \end{aligned}$$

Ex: $P(x,y,z) = x^2+y^2+z^2+4x-6y+2z+6$.

Then $P=0$ is a sphere?

center? radius?

$$P(x,y,z) = (x^2+4x+4)-4+(y^2-6y+9)-9$$
$$+(z^2+2z+1)-1+6$$
$$= (x+2)^2+(y-3)^2+(z+1)^2 - 8$$
$$= (x-(-2))^2+(y-3)^2+(z-(-1))^2 - (2\sqrt{2})^2.$$

$$\boxed{\begin{array}{l} \text{center} = (-2,3,-1) \\ \text{rad.} = 2\sqrt{2}. \end{array}}$$

Circles are algebraic:

<u>2D space</u>

SW.

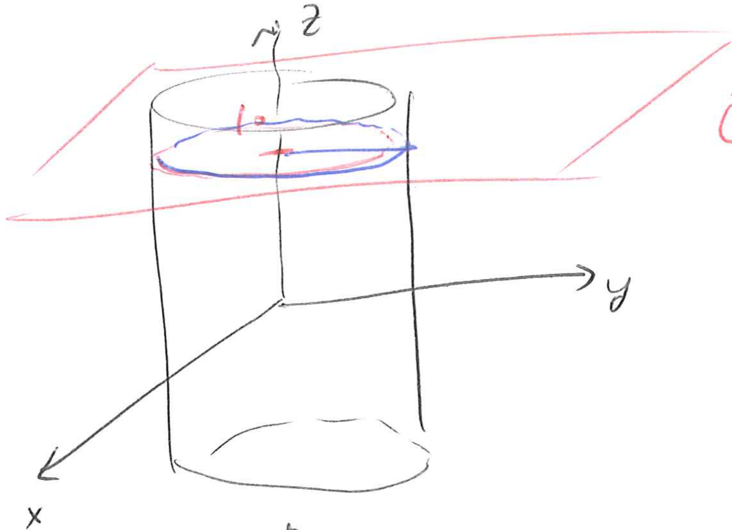
<u>3D space</u>:

Ex: $P(x,y,z) = x^2+y^2-1$

$Q(x,y,z) = z-10.$

Look at all (x,y,z) s.t.

$P=0=Q$ simultaneously



$Q = 0.$

$P = 0.$

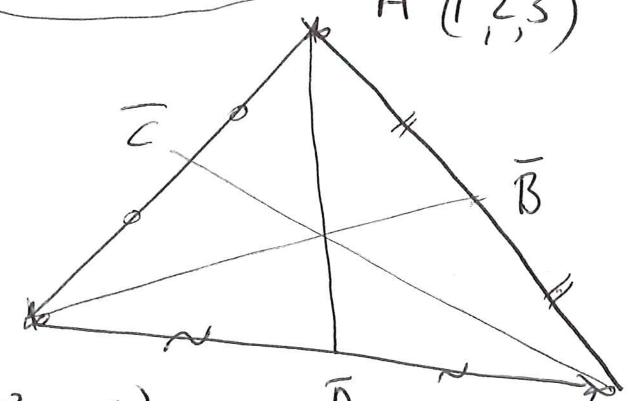
SW:

(i) $P(x, y, z) = x^2+y^2+z^2-r^2$

$Q(x, y, z) = z - k$, where $r, k \in \mathbb{R}$ are parameters. What are the conditions on r, k s.t. they cut out a circle $\subseteq \mathbb{R}^3$?

(ii) $A(-1,5,3)$, $B(6,2,-2)$.
Eq. for points equidistant to A & B? Is the ~~surface~~ space algebraic? What does it look like?

(iii)



$d(A, \bar{A}) = ?$

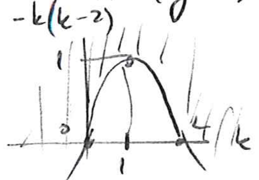
$d(B, \bar{B}) = ?$

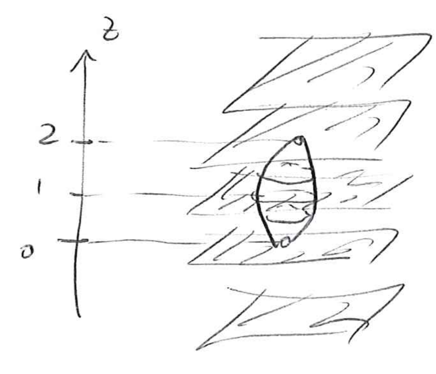
$d(C, \bar{C}) = ?$

(iv) Find an example of a non-alg. subset $\subseteq \mathbb{R}^3$.
(Hint: use trigs, cube roots etc.)

Ex: Sketch $x^2+y^2+z^2 > 2z$.

$z = k$: $x^2+y^2 > -k^2+2k = -k(k-2)$





- Next we have the norm. $\mathbb{R}^3 \longrightarrow \mathbb{R}$

$$u \longmapsto \sqrt{u_1^2+u_2^2+u_3^2} = |u| = d(u,0).$$

__Ex:__ $v = \bar{\imath} + 2\bar{\jmath} - 3k = (1,2,-3)$

$$|v| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{1+4+9} = \sqrt{14}.$$

__Ex.__ $|\bar{\imath}| = |\bar{\jmath}| = |k| = 1$, $|0| = 0$

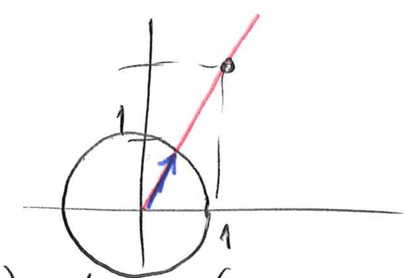
vector/point   number.

- $v \in \mathbb{R}^3$ is __unit__ if $|v| = 1$.

If $|v| \neq 0$, then $\frac{v}{|v|} = \frac{1}{|v|} \cdot v$ is __the__ unit vector in the direction of v.

__Ex:__ $v = (1,2)$. $|v| = \sqrt{1+4} = \sqrt{5}$

$$\frac{v}{|v|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$



__Ex:__ Vector carrying $(0,6,-1)$ to $(3,4,4)$?

$v = (3,-2,5)$

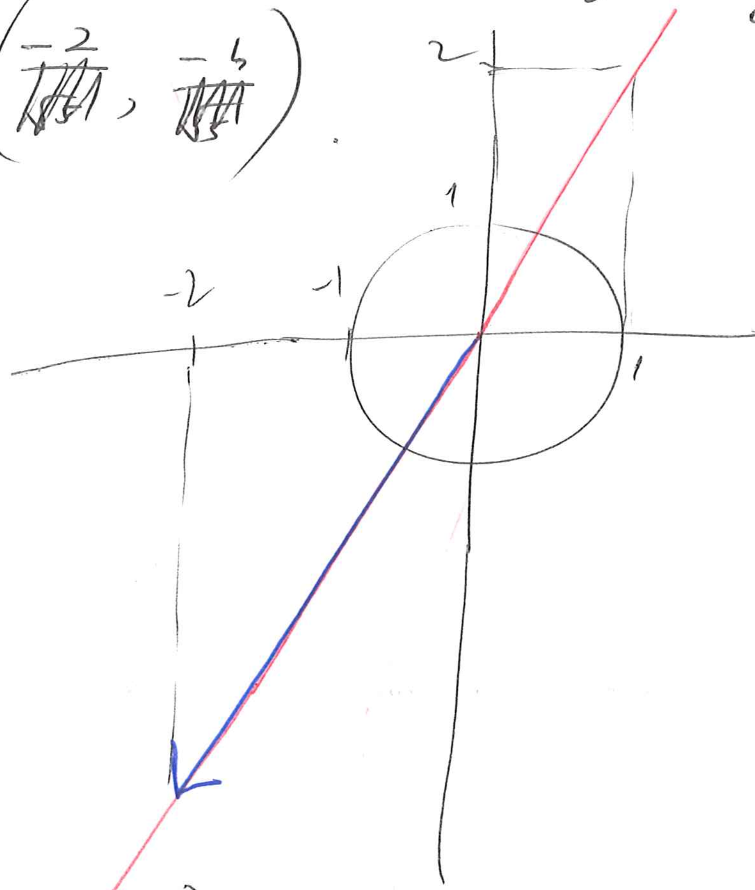
$|v| = \sqrt{9+4+25} = \sqrt{38}$.

$$(0,6,-1) \xrightarrow{v} (3,4,4)$$

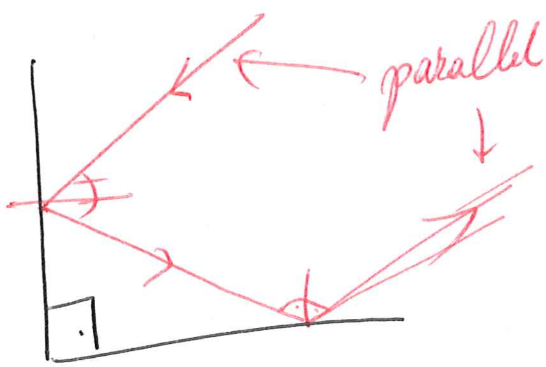
Ex: $v = (1, 2)$. vector in the opposite direction to v with twice the length/magnitude?

$|v| = \sqrt{5}$

$$-2\sqrt{5}\,\frac{v}{|v|} = \left(\frac{-2}{\cancel{\sqrt{5}}}, \frac{-4}{\cancel{\sqrt{5}}}\right).$$



SW: (Corner reflector).



in 3D ?

Third we look at dot products $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$(u, v) \mapsto u \cdot v$

$= u_1 v_1 + u_2 v_2 + u_3 v_3$

Ex: $(-1, 7, 4) \cdot (6, 2, -\frac{1}{2}) = -6 + 14 - 2 = 6.$

$(\bar{\imath} + 2\bar{\jmath} - 3k) \cdot (2\bar{\jmath} - k) = 4 + 3 = 7.$

SW:

$\cdot$	$0 = (0,0,0)$	$\bar{\imath}$	$\bar{\jmath}$	k
$(0,0,0) = 0$	0	0	0	0 ← number
$\bar{\imath}$	0	1	0	0
$\bar{\jmath}$	0	0	1	0
k	0	0	0	1

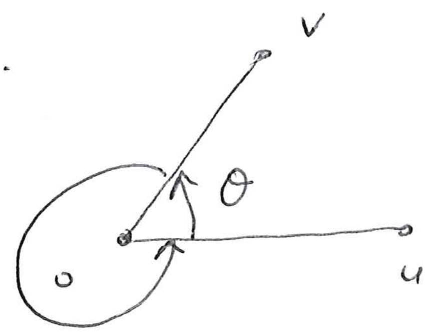
and extend bilinearly

$l : \mathbb{R}^3 \to \mathbb{R}$ linear

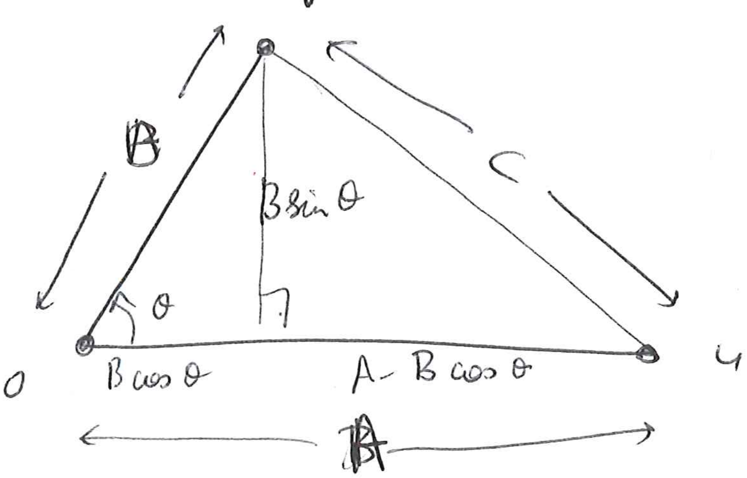
$b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ bilinear.

Thm: $u \cdot v = |u|\,|v| \cos\theta,$

where $\theta = \measuredangle(u, v).$



Pf:



$A = |u|$

$B = |v|$

$C = |u - v|.$

$$c^2 = (B\sin\theta)^2 + (A - B\cos\theta)^2$$

$$LHS = c^2 = |u-v|^2 = (u-v)\cdot(u-v)$$

$$= |u|^2 + |v|^2 - 2\, u\cdot v$$

$$RHS = \underbrace{\underbrace{B^2\sin^2\theta} + A^2 + \underbrace{B^2\cos^2\theta}}_{=B^2} - 2AB\cos\theta$$

$$= A^2 + B^2 - 2AB\cos\theta = |u|^2 + |v|^2 - 2|u||v|\cos\theta$$

$$\Rightarrow u\cdot v = |u||v|\cos\theta.$$

<u>Def</u>: $u \perp v$ (<u>orthogonal</u>) if
$u \neq 0,\ v \neq 0.$
$u\cdot v = 0$

<u>Ex</u>: $(2\vec{i} + 2\vec{j} - \vec{k}) \perp (5\vec{i} - 4\vec{j} + 2\vec{k})$

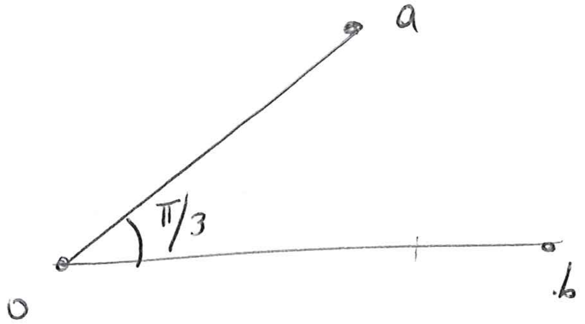
$(2, 2, -1)\cdot(5, -4, 2)$ ~~Stw~~

$= 10 - 8 - 2 = 0.$ ✓

Ex: $|a| = 4$, $|b| = 6$,

$\measuredangle(a, b) = \pi/3$

$\Rightarrow a \circ b = ?$



$$a \circ b = |a|\,|b| \underbrace{\cos\left(\frac{\pi}{3}\right)}_{=\frac{1}{2}}$$

$$= 4 \cdot 6 \cdot \frac{1}{2} = 12.$$

Ex: Find all $b \in \mathbb{R}$:

$\measuredangle(\underbrace{(2,1,-1)}_{=u}, \underbrace{(1,b,0)}_{=v}) = 45^\circ$

$(2,1,-1) \circ (1,b,0) = |u|\,|v| \frac{1}{\sqrt{2}}$

$2 + b = \sqrt{4+1+1}\,\sqrt{1+b^2}/\sqrt{2}$

$2 + b = \sqrt{3(1+b^2)} \geq 0$ $\quad$ $b \geq -2$

$b^2 + 4b + 4 = 3b^2 + 3$

$2b^2 - 4b - 1 = 0$, $\quad (b-1)^2 - \frac{3}{2} = 0$

$b^2 - 2b - \frac{1}{2} = 0$ $\quad$ $\boxed{b = 1 \mp \sqrt{\frac{3}{2}}}$

$4b - 4 = 0$

$b = 1$

Projections: $u, v \in \mathbb{R}^3$, $u \neq 0$.

$u \circ v = |u|\,|v| \cos\theta$

$|v| \cos\theta = \frac{u \circ v}{|u|}$

$\boxed{\text{proj}_u(v) \text{ (vector projection)}}$

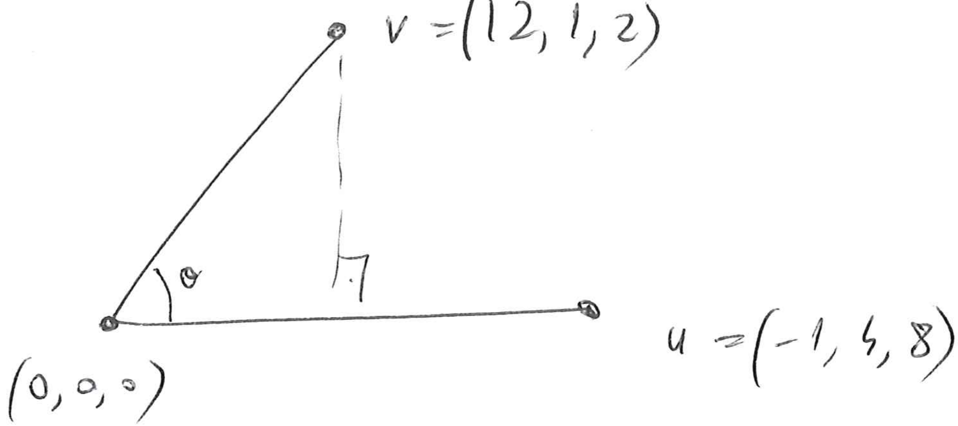
$\boxed{\text{comp}_u(v) = |v| \cos\theta = \frac{u \circ v}{|u|} \text{ (scalar projection)}}$

$\cancel{|v| \cos\theta \frac{u}{|u|} = \text{proj}_u(v)}$.

$$\boxed{\text{proj}_u(v) = \frac{|v| \cos\theta}{|u|}\, u = \frac{u \circ v}{|u|^2}\, u = \frac{u \circ v}{u \circ u}\, u.}$$

Ex: $u = (-1, 4, 8)$ $\quad v = (12, 1, 2)$

$\text{comp}_u(v) = ?$ $\quad \text{proj}_u(v) = ?$



$u \cdot v = (-1, 4, 8) \cdot (12, 1, 2) = -12 + 4 + 16 = 8$

$|u|^2 = 1 + 16 + 64 = 81 \Rightarrow |u| = 9$

$|v|^2 = 144 + 1 + 4 = 149 \Rightarrow |v| = \sqrt{149}$

$$8 = 9 \cdot \sqrt{149} \cos\theta$$

$$\text{comp}_u(v) = \frac{u \cdot v}{|u|} = \frac{8}{9}.$$

$$\text{proj}_u(v) = \frac{8}{9^2} \cdot u = \frac{8}{81}(-1, 4, 8).$$

$$= \left(-\frac{8}{81}, \frac{32}{81}, \frac{64}{81}\right).$$

SW: (i) Switch the roles of u & v.

(ii) When is $\text{comp}_u(v) = \text{comp}_v(u)$?

(iii) When is $\text{proj}_u(v) = \text{proj}_v(u)$?

· Yet another structure on $\mathbb{R}^3$: cross product.

We'll define it via the RHR table & bilinearity.

~~Def: Let $n, m, p \in \{1, 2, 3\}$.~~

~~(i) A function~~

~~Def: Let V_1, V_2, W be three linear~~

Def: (i) A set V is a <u>linear space / vector space</u> if addition & scalar multiplication are defined on it.

Ex: $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots$ are all linear spaces.

So are $F(\mathbb{R}, \mathbb{R})$, $C^0(\mathbb{R}, \mathbb{R})$, $C^q(\mathbb{R}, \mathbb{R})$ ...

Def: Let V, W be linear spaces. A function $l: V \to W$ is <u>linear</u> if

$l(v_1 + v_2) = l(v_1) + l(v_2)$ and

$l(cv) = c\,l(v)$ for all $c \in \mathbb{R}$,
$v_1, v_2 \in V$.

Ex: ~~Let~~ Fix $\lambda \in \mathbb{R}$, $\ell : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$. Then ℓ is
$$t \longmapsto \lambda t$$

linear. $\ell(t_1+t_2) = \lambda(t_1+t_2) = \lambda t_1 + \lambda t_2 = \ell(t_1)+\ell(t_2)$
$$\ell(ct) = \lambda(ct) = c(\lambda t) = c\,\ell(t).$$

~~If $\ell : \mathbb{R}^1 \to \mathbb{R}^1$ is~~ SW: Any linear $\ell : \mathbb{R}^1 \to \mathbb{R}^1$ is ~~really~~ of this form.

Ex: Any linear $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form $\ell : (x,y) \mapsto (ax+by,\, cx+dy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Ex: $\partial : C^\infty(\mathbb{R},\mathbb{R}) \longrightarrow C^\infty(\mathbb{R},\mathbb{R})$ is linear.
$$f \longmapsto f'$$

Def: Let V_1, V_2, W be linear spaces. A function $b : V_1 \times V_2 \longrightarrow W$ is **bilinear** if it is linear in ~~each~~ both coordinates separately.

Ex: $m : \mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ is bilinear.
$$(t,s) \mapsto ts$$

~~Ex~~ SW: $C^0(\mathbb{R},\mathbb{R}) \times C^0(\mathbb{R},\mathbb{R}) \longrightarrow \mathbb{R}$

$$(f, g) \longmapsto \int_0^1 f(t)\, g(t)\, dt.$$

is bilinear.

Recall:

$\times$	$0=(0,0,0)$	$\bar{i}$	$\bar{j}$	k
0	0	0	0	0
$\bar{i}$	0	0	k	$-\bar{j}$
$\bar{j}$	0	$-k$	0	$\bar{i}$
k	0	$\bar{j}$	$-\bar{i}$	0

Also $(a,b,c) = a\,\bar{i} + b\,\bar{j} + c\,k$.

Extend $\times$ = cross product bilinearly: (pedestrian def. of $\times$)

cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$(u, v) \longmapsto u \times v = (u_1, u_2, u_3) \times (v_1, v_2, v_3)$$

$$= (u_1 \bar{i} + u_2 \bar{j} + u_3 k) \times (v_1 \bar{i} + v_2 \bar{j} + v_3 k).$$

[illegible]

$$= u_1 v_1 \overset{=0}{\bar{i} \times \bar{i}} + u_1 v_2\, \bar{i} \times \bar{j} + u_1 v_3\, \bar{i} \times k$$

$$+ u_2 v_1\, \bar{j} \times \bar{i} + u_2 v_2 \overset{=0}{\bar{j} \times \bar{j}} + u_2 v_3\, \bar{j} \times k$$

$$+ u_3 v_1\, k \times \bar{i} + u_3 v_2\, k \times \bar{j} + u_3 v_3 \underset{=0}{k \times k}$$

$$= (u_1v_2 - u_2v_1)\, \underbrace{\bar{i}\times\bar{j}}_{=k} + (u_2v_3 - u_3v_2)\, \underbrace{\bar{j}\times k}_{=\bar{i}} + (u_3v_1 - u_1v_3)\, \underbrace{k\times\bar{i}}_{=\bar{j}}$$

$$= (u_2v_3 - u_3v_2)\,\bar{i} + (u_3v_1 - u_1v_3)\,\bar{j} + (u_1v_2 - u_2v_1)\,k.$$

$$= \boxed{(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) = u\times v.}$$

<u>Ex</u>: $u = (1,3,4)$, $v = (2,7,-5)$. $u\times v = ?$

$$u\times v = (1,3,4)\times(2,7,-5) = (\bar{i} + 3\bar{j} + 4k)\times(2\bar{i} + 7\bar{j} - 5k)$$

~~$(u\times v)\cdot u =$~~

$$= (7-6)\,\bar{i}\times\bar{j} + (5+8)\,k\times\bar{i} + (-15-28)\,\bar{j}\times k$$

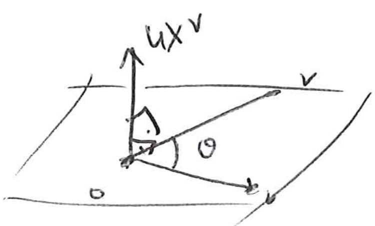
$$= k + 13\bar{j} - 43\bar{i} = (-43, 13, 1)$$

$(u\times v)\cdot u = (-43,13,1)\cdot(1,3,4) = -43+39+4 = 0.$
$(u\times v)\cdot v = (-43,13,1)\cdot(2,7,-5) = -86+91-5 = 0.$

<u>SW</u>: (i) $v\times u = -(u\times v)$.

(ii) $(u\times v)\cdot u = 0 = (u\times v)\cdot v$

Thus $(u\times v)\perp u$, $(u\times v)\perp v$



<u>L.A. def. of $\times$.</u>

For $u, v \in \mathbb{R}^3$, $u\times v \in \mathbb{R}^3$ is the unique vector s.t. $\forall w \in \mathbb{R}^3$:

$$(u\times v)\cdot w = \det\begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$

· Another way to keep the formula for $\times$ in mind.

$$\mathrm{Mat}(3\times 3, \mathbb{R}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \;\middle|\; a_{ij} \in \mathbb{R} \right\}.$$

$$\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathrm{Mat}(3\times 3, \mathbb{R}) \cong \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

$$(u, v, w) \longmapsto \begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}$$

$$\begin{pmatrix} - & u & - \\ - & w & - \\ - & w & - \end{pmatrix} \longleftarrow\!\!\shortmid\; (u, v, w)$$

$$\det : \mathrm{Mat}(3\times 3, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix} \qquad \begin{array}{c} - \qquad\qquad + \\ \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \\ A \;\; B \;\; C \\ D \;\; E \;\; F \end{array} \longmapsto \begin{array}{l} AEI + DHC + GBF \\ - CEG - FHA - IBD. \end{array}$$

Rule of Sarrus.

$$\det \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} = A(EI - FH) - D(BI - HC) + G(BF - CE)$$

$$= C(DH - EG) - F(AH - BG) + I(AE - BD)$$

$$u \times v = \det\begin{pmatrix} | & | & \bar{i} \\ u & v & \bar{j} \\ | & | & \bar{k} \end{pmatrix}$$

$$\begin{matrix} u_1 & v_1 & \bar{i} \\ u_2 & v_2 & \bar{j} \end{matrix}$$

$$= u_1 v_2 \bar{k} + u_2 v_3 \bar{i} + u_3 v_1 \bar{j} - u_3 v_2 \bar{i} - u_1 v_3 \bar{j} - u_2 v_1 \bar{k}$$

$$= \bar{i}\,(u_2 v_3 - u_3 v_2) - \bar{j}\,(u_1 v_3 - u_3 v_1) + \bar{k}\,(u_1 v_2 - u_2 v_1)$$

- To interpret $\times$ geometrically we need the following blackbox from linear algebra:

$$\underbrace{(u \times v) \cdot w}_{\text{scalar triple product}} = \det(u, v, w) = \text{(signed) volume of the parallelepiped defined by } 0, u, v, w \text{ as vertices.}$$

. Yet another algebraic structure on $\mathbb{R}^3$: cross product

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(u, v) \longmapsto u \times v.$$

Def($\times$) $\boxed{\text{Pedestrian}}$ distribute $\times$ over scalar mult. & add., use the table for $\times$.

$\boxed{LA}$ $u \times v$ is the unique vector such that for any $w \in \mathbb{R}^3$: $\underbrace{(u \times v) \cdot w}_{=\text{scalar triple product}} = \det(u, v, w)$ signed

Def (det): ~~det~~ $\det(u,v,w)$ is the volume of the parallelepiped determined by $0, u, v, w$

Formula. (Rule of Sarrus)
$$\det \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} = AEI + DHC + GBF - CEG - FHA - IBD.$$

$$\begin{matrix} A & B & C \\ D & E & F \end{matrix}$$

$$u \times v = (u_1 \bar{i} + u_2 \bar{j} + u_3 \bar{k}) \times (v_1 \bar{i} + v_2 \bar{j} + v_3 \bar{k}) = \ldots$$

$$= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$$

Abbreviation. (Actually doesn't make sense)

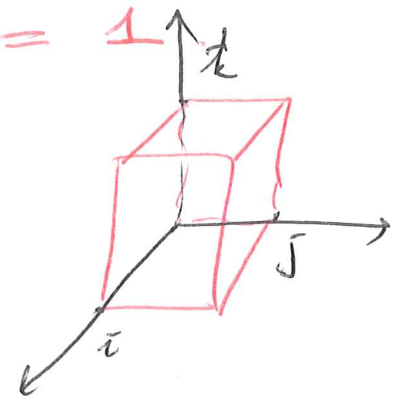
$$u \times v = \det \begin{pmatrix} | & | & \bar{i} \\ u & v & \bar{j} \\ | & | & \bar{k} \end{pmatrix} = \det \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ - & u & - \\ - & v & - \end{pmatrix}$$

SW: Verify

Ex: $(\bar{i} \times \bar{j}) \cdot \bar{k} = \bar{k} \cdot \bar{k} = 1$

$\shortparallel$

Vol of unit cube. ✓.



~~Obs: $u \parallel v$ i.e. $\exists c \in \mathbb{R} : v = cu$ $\Leftrightarrow u \times v = 0$~~

parallel

Obs: $u \parallel v$ (i.e. $\exists c \in \mathbb{R} : v = cu$)

Thm: $\boxed{|u \times v| = |u|\,|v| \sin\theta, \quad \theta = \measuredangle(u,v)}$ $\Leftrightarrow u \times v = 0$.

= area of the parallelogram det. by o, u, v.

Obs: $|u \times v|^2 = (u \times v) \cdot (u \times v) = \det(u, v, u \times v)$

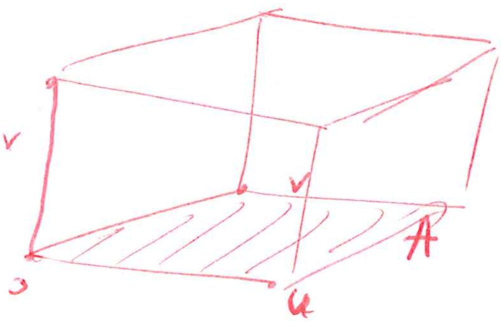
= signed vol. of parallelogram determined by $o, u, v, u \times v$

$u \times v \perp u$ & $u \times v \perp v$ $\Rightarrow$

$\Rightarrow u \times v \perp$ [plane determined by o, u, v]

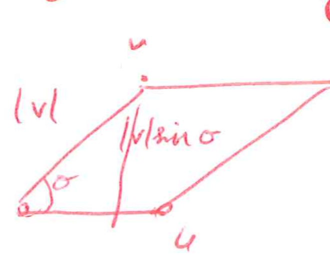
$= |u \times v| \cdot A$

$= |u \times v|\,|u|\,|v| \sin\theta$



If $u \parallel v$, $u \times v = 0$,
$\sin\theta = 0$

$\Rightarrow \boxed{|u \times v| = |u|\,|v| \sin\theta}$ ✓.

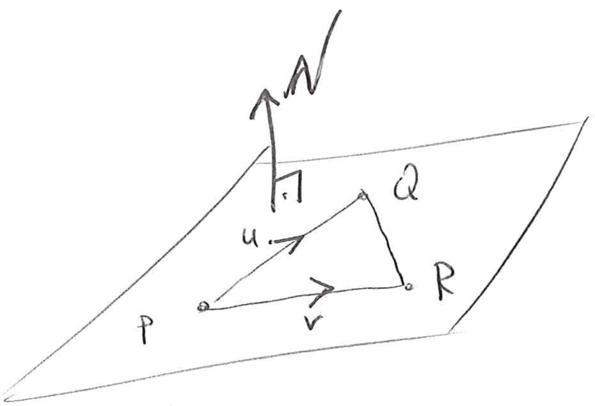


$A = |u \times v|$.

If $u \nparallel v$, $\boxed{|u \times v| = |u|\,|v| \sin\theta}$

Ex: Find a vector n $\perp$ to the plane containing

$P(1,4,6)$, $Q(-2,5,-1)$, $R(1,-1,1)$.



$u = (-3, 1, -7)$

$v = (0, -5, -5)$.

$$u \times v = (-3,1,-7) \times (0,-5,-5)$$

$$= (15)\underbrace{\bar{i} \times \bar{j}}_{=\bar{k}} + (-5-35)\underbrace{\bar{j} \times \bar{k}}_{=\bar{i}} + (-15)\underbrace{\bar{k} \times \bar{i}}_{=\bar{j}}$$

$$= \boxed{(-40, -15, 15) = N.}$$

SW: Do it with another base point.

Ex: Area of $\widehat{PQR}$?

$$= \frac{|u \times v|}{2} = \frac{|N|}{2} = \frac{\sqrt{40^2 + 15^2 + 15^2}}{2} = \frac{5\sqrt{82}}{2}.$$

SW: Do it with another base point.

Ex: $\left.\begin{array}{l} u = (1, 4, -7) \\ v = (2, -1, 4) \\ w = (0, -9, 18) \end{array}\right\}$ coplanar?

u, v, w coplanar $\Leftrightarrow$ $(u \times v) \cdot w = 0$.

$$(u \times v) \cdot w = \det(u, v, w) = \det \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & -9 \\ -7 & 4 & 18 \end{pmatrix}$$

$$\begin{array}{ccc} 1 & 2 & 0 \\ 4 & -1 & -9 \end{array}$$

$$\cancel{= (-18 - 14 \cdot 9) - (42 - 36 + 8 \cdot 18)}$$

$$\cancel{= (-18 - 126) - (42 - 36 + 144)}$$

$$\cancel{= 42 + 36 - 144 -}$$

$$= \det \begin{pmatrix} 1 & 2 & 0 \\ 4 & -1 & -9 \\ -7 & 4 & 18 \end{pmatrix} = (-18 + 14 \cdot 9) - (-36 + 8 \cdot 18)$$

$$\begin{array}{ccc} 1 & 2 & 0 \\ 4 & -1 & -9 \end{array} \qquad = 136 - 18 + 36 - 144$$

$$= 172 - 172 = 0$$

$\Rightarrow$ coplanar ✓.

$\underline{SW}$: (i) $(u \times v) \cdot w = (v \times w) \cdot u = (w \times u) \cdot v$

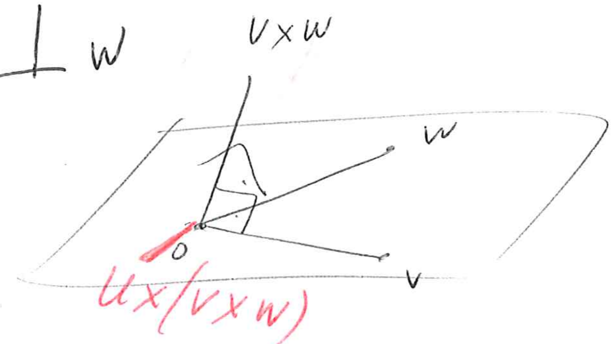
(ii) $\underbrace{u \times (v \times w)}_{\text{vector triple product}} = (u \cdot w)v - (u \cdot v)w$

(iii) (Jacobi) $\quad u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$

$$[(u \cdot w)v - (u \cdot v)w] + [(v \cdot u)w - (v \cdot w)u] + [(w \cdot v)u - (w \cdot u)v] = 0.$$

Pf. $v \times w \perp v, \quad v \times w \perp w$

$u \times (v \times w) \perp (v \times w)$



$\Rightarrow \exists k, l \in \mathbb{R}$:

$u \times (v \times w) = kv + lw$

$(u \times (v \times w)) \cdot u = (kv + lw) \cdot u = k\, v \cdot u + l\, w \cdot u$

$\|$

$\det(u, v \times w, u)$

$\|$

0

$-k\, v \cdot u = l\, w \cdot u.$

$\cancel{k = \alpha l\, w \cdot u}$ $\quad \Rightarrow k = \lambda\, w \cdot u$

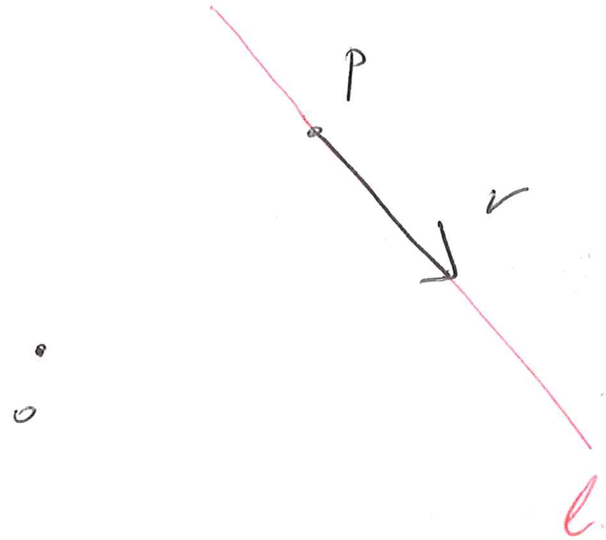
$l = -\lambda\, v \cdot u$

$\exists \lambda \in \mathbb{R}$

$u \times (v \times w) = kv + lw$

$= \lambda [(u \cdot w)v - (u \cdot v)w] \ldots$

Equations of Lines $\subseteq \mathbb{R}^3$:



$p \in \mathbb{R}^3$ point on the line l

$v \in \mathbb{R}^3$ vector collinear with the line l.

$$\boxed{\begin{aligned} \vec{l} : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto p + tv \end{aligned}}$$

~~parametrization~~ / vector equation of l.

$p = (p_1, p_2, p_3)$

$v = (v_1, v_2, v_3)$

$$\vec{l}(t) = (x(t), y(t), z(t))$$

$$\rightarrow \boxed{\begin{aligned} x(t) &\overset{\mathbb{R} \to \mathbb{R}}{=} p_1 + tv_1 \\ y(t) &= p_2 + tv_2 \\ z(t) &= p_3 + tv_3 \end{aligned}}$$

parametric equations of l

$$\boxed{t = \frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3} \quad (\text{for } v_1 \neq 0;\ v_2 \neq 0;\ v_3 \neq 0.)}$$

symmetric equations of l.

If e.g. $v_1 = 0$ replace $\frac{x - p_1}{v_1} = t$ with $x = p_1$.

Ex: l passes through $(5,1,3)$
parallel to $(\vec{i}+4\vec{j}-2\vec{k})$.

$\mathbb{R} \to \mathbb{R}^3$

$$\boxed{r(t) = (5,1,3) + t(1,4,-2)}$$ vector eq.

$$\boxed{\begin{aligned} x(t) &= 5+t \\ y(t) &= 1+4t \\ z(t) &= 3-2t \end{aligned}}$$ par. eq.

$\left(\begin{array}{l} \vec{r} :]-1,1[\to \mathbb{R}^3 \\ t \longmapsto (5,1,3) + \tan(t)(1,4,-2) \\ \text{works also.} \end{array} \right)$

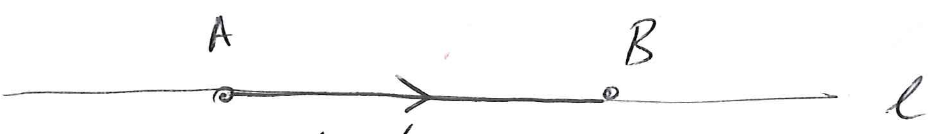
$$\boxed{t = x-5 = \frac{y-1}{4} = \frac{z-3}{-2}}$$ sym eq.

Ex: Does it pass through 0?

$(0,0,0) = r(t) \Rightarrow t = -4$

$1 - 16 \neq 0$ ↯. No.

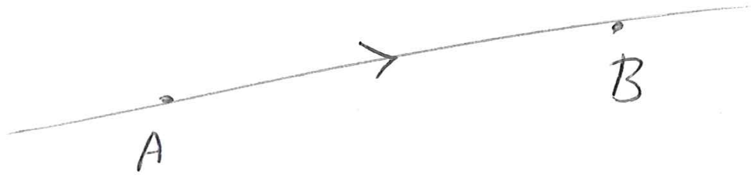
Ex: l that passes through $A(2,4,-3)$ &
$B(3,-1,1)$?



$u = (1,-5,4)$

$$\boxed{r(t) = (2,4,-3) + t(1,-5,4).}$$

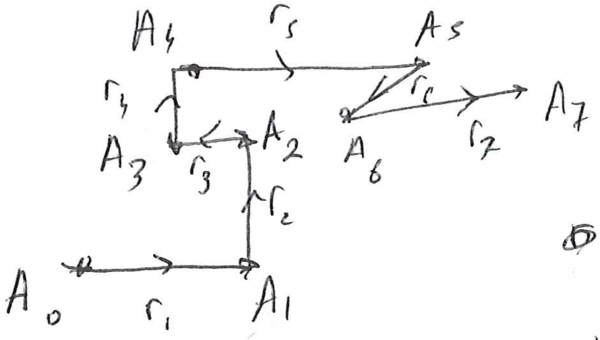
Ex: Eq. for line segment from A to B?



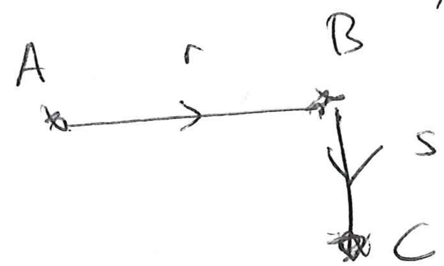
$$r : [0,1] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (1-t)A + tB$$

$$r(t) = (1-t)(2,4,-3) + t(1,-5,4)$$

SW: Write a parametrization for a ~~poly~~ piecewise linear / polygonal curve.



$r, s : [0,1] \to \mathbb{R}^3$



$$r * s : [0,1] \to \mathbb{R}^3$$
$$t \mapsto \begin{cases} r(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ s(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

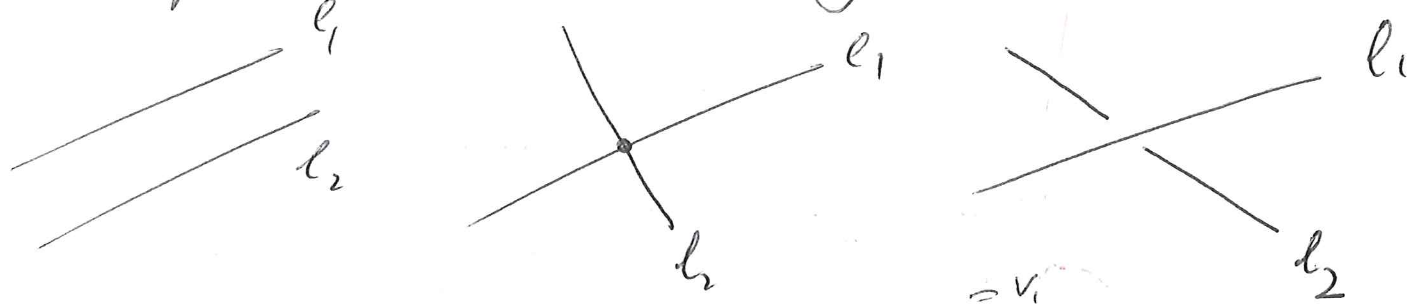
concatenation

Ex: $l_1 : r_1(t) = (1+t, -2+3t, 4-t)$

later

$l_2 : r_2(s) = (2s^3, 3+s^3, -3+4s^3)$

parallel? intersecting? skew?



$$r_1(t) = (1,-2,4) + t\overbrace{(1,3,-1)}^{=v_1}$$

$$r_2(s) = (0,3,-3) + s^3\underbrace{(2,3,4)}_{=v_2}$$

$$\tilde{r}_2(s) = (0,3,-3) + s(2,3,4)$$

$$r_1(t) = \tilde{r}_2(s) \Rightarrow \begin{array}{l} 1+t = 2s \to t = 2s-1 \\ -2+3t = 3+3s \to 3t = 5+3s \end{array}$$

$$\Rightarrow 6s-3 = 5+3s$$

$$3s = 8 \quad s = \frac{8}{3} \quad t = \frac{13}{3}$$

$$4-t \overset{?}{=} -3+4s$$

not intersecting $\Leftarrow$ $\frac{-1}{3} \neq \frac{23}{3}$ ↯.

$l_1 \parallel l_2 \iff$ there is $c \in \mathbb{R}$:

$$v_2 = c v_1$$

$$(2,3,4) = (c, 3c, -c)$$

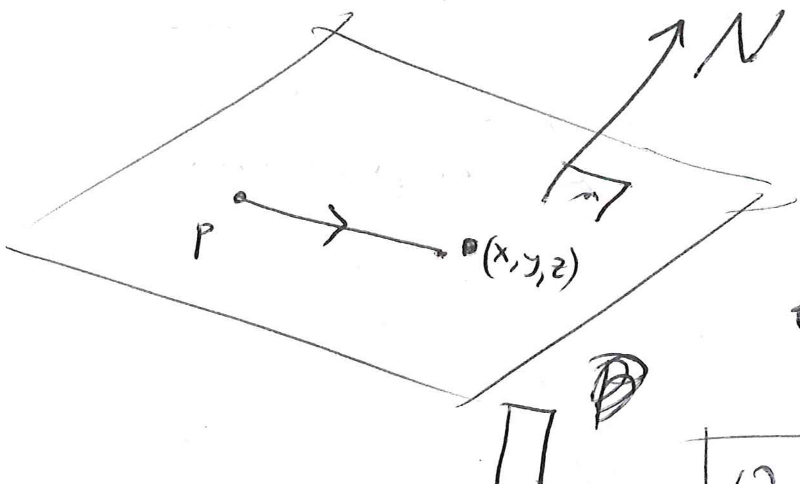
$\Rightarrow c = 2,\ c = -4$, ↯. (SW: Or compute $v_1 \times v_2$.)

$\Rightarrow$ not parallel $\Rightarrow$ skew.

Equations of Planes $\subseteq \mathbb{R}^3$:

$p \in \mathbb{R}^3$ point on the plane Π

$N \in \mathbb{R}^3$ vector normal to the plane.



$\cancel{\mathbb{R}^2 \to \mathbb{R}^3}$

$\cancel{(s,t) \mapsto}$

$$\boxed{0 = N \cdot \big((x,y,z) - p\big)}$$

vector eq. of a plane.

$N = (n_1, n_2, n_3)$

$p = (p_1, p_2, p_3)$

$$\boxed{0 = n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3)}$$

~~sta~~ scalar eq. of a plane

Ex: Plane through $(2,4,-1)$ w/ $N = (2,3,4)$? x, y, z intercepts?

vector eq.: $0 = N \cdot ((x,y,z) - (2,4,-1))$

$$\boxed{0 = (2,3,4) \cdot [(x,y,z) - (2,4,-1)]}$$

scalar eq. $0 = 2(x-2) + 3(y-4) + 4(z+1)$

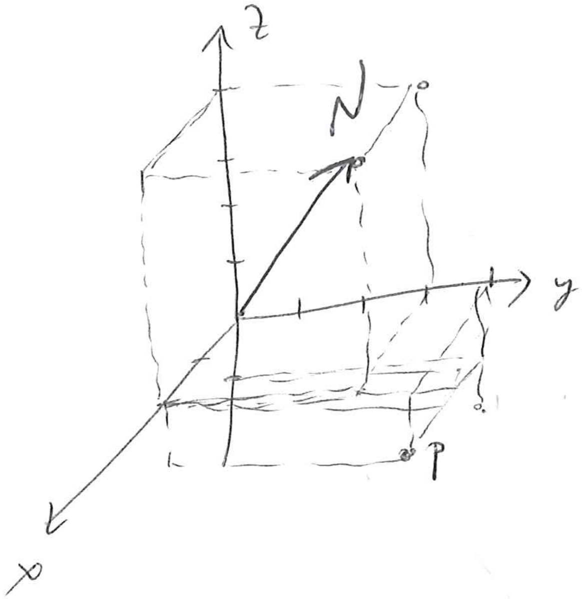
$$\boxed{2x + 3y + 4z - 12 = 0.}$$

$P(x,y,z) = 2x + 3y + 4z - 12 \in \mathbb{R}[x,y,z]$ degree 1 poly

$$\boxed{P(x,y,z) = 0}$$

(nonhomog.) linear eq.

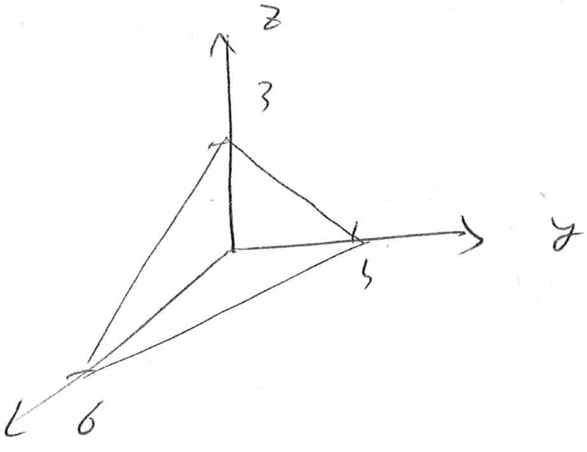
SW: Generic degree 1 poly $\in \mathbb{R}[x,y,z]$? What is its zero locus?



$x = 0 = y \Rightarrow z = 3$

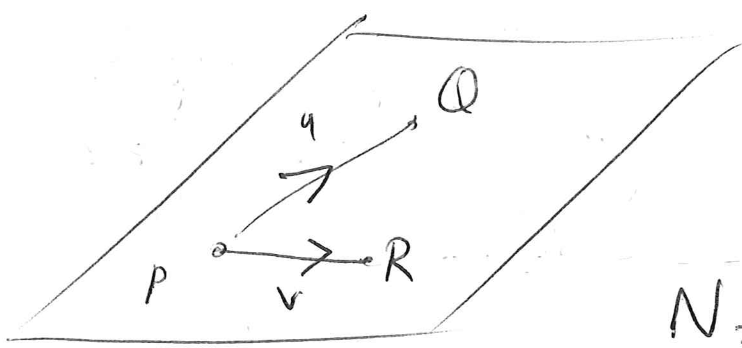
$x = 0 = z \Rightarrow y = 4$

$y = 0 = z \Rightarrow x = 6.$



<u>Ex</u>: Plane passing through

$P(1,3,2)$, $Q(3,-1,6)$, $R(5,2,0)$



$u = (2,-4,4)$

$v = (4,-1,-2).$

$N = u \times v.$

$= (8+4, \quad 4+16, \quad -2+16)$

$= (12, 20, 14)$

$0 = 12(x-1) + 20(y-3) + 14(z-2).$

$0 = 12x + 20y + 14z - (12+60+28) = 12x + 20y + 14z - 100.$

$\boxed{0 = 12x + 20y + 14z - 100}$

Ex: $r(t) = (2+3t, -4t, 5+t)$ line

$4x+5y-2z=18$ plane

hit or miss?

$$18 = 4(2+3t) + 5(-4t) - 2(5+t)$$
$$= 8 + 12t - 20t - 10 - 2t$$
$$= -10t - 2 \Rightarrow 10t = -20$$
$$\Rightarrow \boxed{t = -2}$$

$\Rightarrow$ hit at $r(-2) = (-4, 8, 3)$.

- For Π_1, Π_2 two planes, $\underline{\Pi_1 // \Pi_2}$ if their normals are parallel.

Ex: $P(x,y,z) = x+y+z-1$ $\Pi_P = \{P=0\}$

$Q(x,y,z) = x-2y+3z-1$ $\Pi_Q = \{Q=0\}$.

parallel?

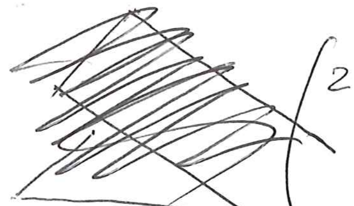
angle between? (SW).

If not parallel, intersection?

$N_p = (1,1,1)$ $\qquad N_Q = (1,-2,3)$.

$(1,1,1) = c(1,-2,3) \Rightarrow c = 1$ ↯.

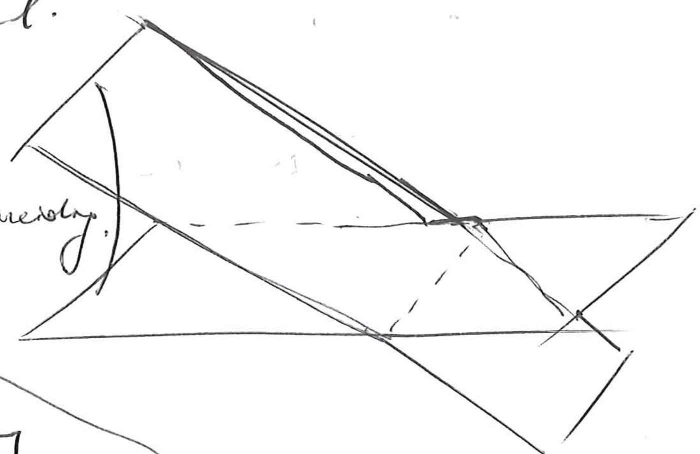
Not parallel.

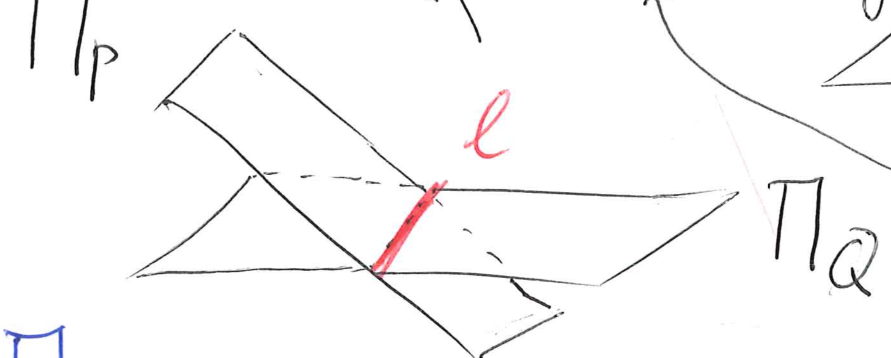


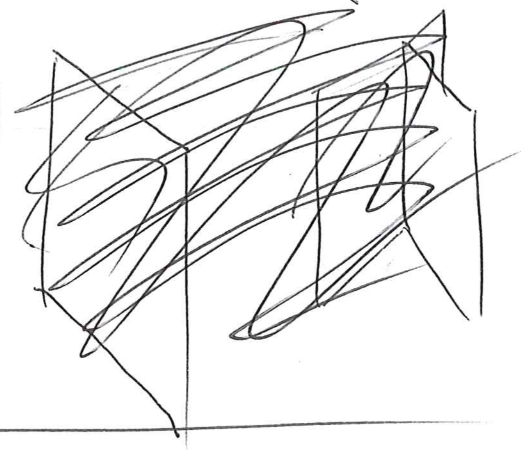
$2 = \sqrt{3}\sqrt{14}\cos(\theta)$

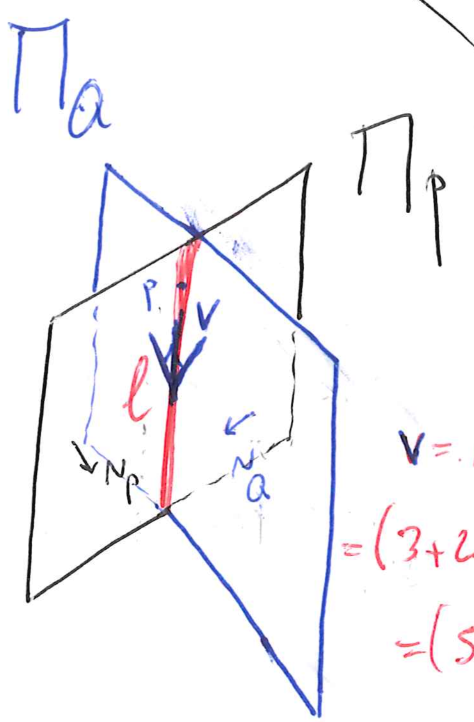
$\theta \neq 0$, $\theta = \pi/2$

$\Rightarrow$ not correctly.









$v = N_p \times N_a$

$= (3+2, 1-3, -2-1)$

$= (5, -2, -3)$

$N_p \cdot N_Q = 1 - 2 + 3 = 2$.

$|N_p| = \sqrt{3}$ $\quad |N_Q| = \sqrt{14}$

$\theta = \arccos\left(\frac{2}{\sqrt{42}}\right)$.

Also need a point $\in \Pi_P \cap \Pi_Q$.

$a+b+c=1$ $\quad a=1 \Rightarrow b+c=0 \quad c=-b=0$

$a-2b+3c=1$ $\quad -2b+3c=0$

$(a,b,c) = (1,0,0)$ works.

$$\Rightarrow \boxed{\ell(t)=(1,0,0)+t(5,-2,-3)}$$

SW: Solve $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x+y+z=1$

$x-2y+3z=1$.

Sol set is ℓ.

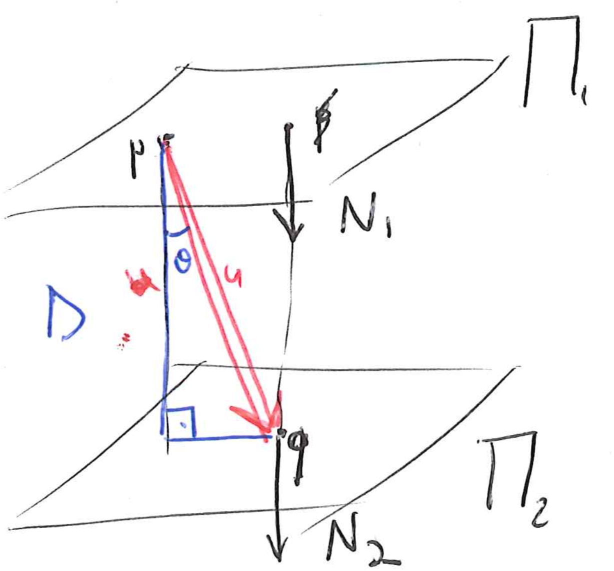
Formula for $d(p,\Pi)$: $p=(p_1,p_2,p_3)$

$\Pi: ax+by+cz+d=0$

$$D=\frac{|ap_1+bp_2+cp_3+d|}{\sqrt{a^2+b^2+c^2}}$$

Ex: distance between $\begin{cases} 10x+2y-2z=5 \} = \Pi_1 \\ 5x+y-z=1 \} = \Pi_2 \end{cases}$

~~$P_1(x,y,z)$~~ $\left.\begin{array}{l} N_1=(10,2,-2)=2N_2 \ ? \\ N_2=(5,1,-1) \end{array}\right\} \Rightarrow \Pi_1 \parallel \Pi_2$.



$q=(0,0,-1)$ will do

$q \in \Pi_2$.

$p=(0,0,-\frac{5}{2}) \in \Pi_1$

$u=(0,0,\frac{3}{2})$

$$D=|u|\cos\theta = \frac{|u| \ u\cdot N_1}{|u|\,|N_1|}=\frac{u\cdot N_1}{|N_1|}$$

$$|u|\cos\theta = \frac{u\cdot N_1}{|N_1|} = \frac{-3}{\sqrt{108}} \Rightarrow D = \frac{3}{\sqrt{108}}$$

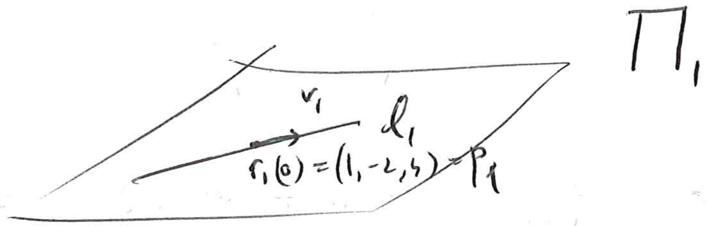
Ex: $r_1(t) = (1+t, -2+3t, 4-t) = \overbrace{(1,-2,4)}^{=p_1} + t\overbrace{(1,3,-1)}^{=v_1}$

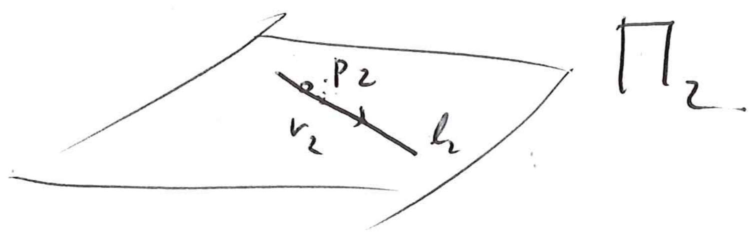
$r_2(s) = (2s, 3+s, -3+4s) = \underbrace{(0,3,-3)}_{=p_2} + s\underbrace{(2,1,4)}_{=v_2}$

distance between?

from before: skew.

$\rightarrow$





$$N = v_1\times v_2 = (12+1,\ -2-4,\ 1-6)$$
$$= (13, -6, -5)$$

230/231 01/27/2020

Quadric Surfaces in 3D:

Def: A quadric / quadratic surface in 3D is the zero locus of an (irreducible) polynomial $P(x,y,z) \in \mathbb{R}[x,y,z]$ of degree 2.

Quadric Surfaces

Cylinders

~~C1 $x^2+y^2-1=0$ elliptic cylinder~~

~~C2 $x^2+y^2+1=0$ imaginary elliptic cyl.~~

7 (+2 imaginary)

Ellipsoid

1 (+1 imaginary)

+ + +

all degree 2

Paraboloids

2

+ + −

One degree 1

Hyperboloids

3 (+1 imaginary)

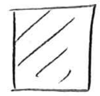
+ + −

all degree 2.

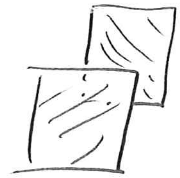
Complete list up to rotation & translation.

Cylinders (7) : $\left(X = \frac{x}{a},\ Y = \frac{y}{b},\ Z = \frac{z}{c} \quad a,b,c \in \mathbb{R},\ a \neq 0,\ b \neq 0,\ c \neq 0. \right)$

$\boxed{C1}$ Plane : $X^2 = 0.$



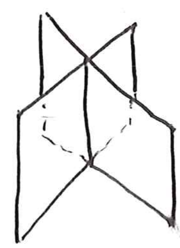
$\boxed{C2}$ Pair of parallel planes : $X^2 - 1 = 0.$



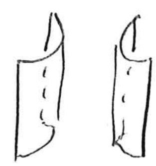
$\boxed{C3}$ Parabolic cylinder : $\cancel{X^2 = Y = 0}$ $\cancel{Y^2 - X = 0}$.

$X^2 - Y = 0.$

$\boxed{C4}$ ~~Plane~~ Pair of intersecting planes : $X^2 - Y^2 = 0.$



$\boxed{C5}$ Hyperbolic Cylinder : $X^2 - Y^2 - 1 = 0$



$\boxed{C6}$ Line : $X^2 + Y^2 = 0$

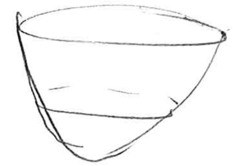
$\boxed{C7}$ Elliptic Cylinder : $X^2 + Y^2 - 1 = 0.$

Ellipsoid (1) :

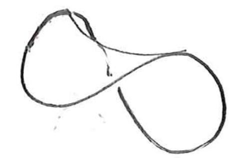
$\boxed{E1}$ Ellipsoid : $X^2 + Y^2 + Z^2 - 1 = 0$

Paraboloids (2):

$\boxed{P1}$ Elliptic paraboloid: $X^2+Y^2-Z=0$

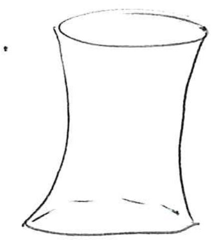


$\boxed{P2}$ Hyperbolic paraboloid: $X^2-Y^2-Z=0.$
(saddle = pringle)

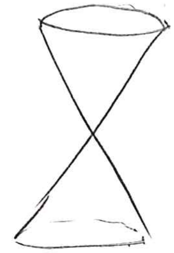


Hyperboloids (3):

$\boxed{H1}$ Hyperboloid of one sheet: $X^2+Y^2-Z^2-1=0.$



$\boxed{H2}$ Cone: $X^2+Y^2-Z^2=0$

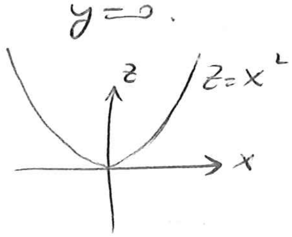


$\boxed{H3}$ Hyperboloid of two sheets: $X^2+Y^2-Z^2+1=0.$

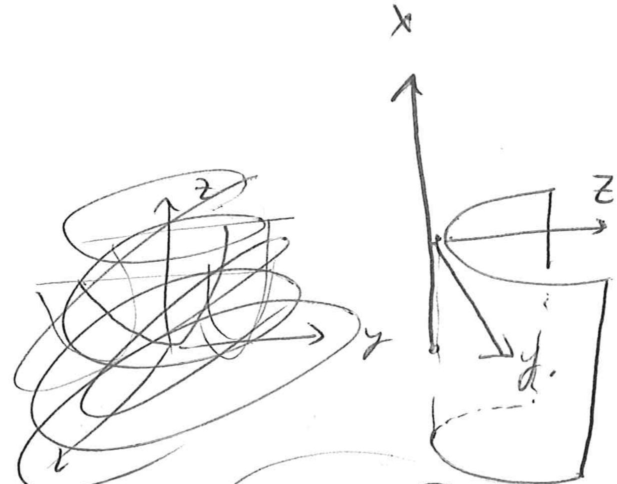
Ex: $\boxed{z - x^2 = 0.}$

y : free

$y = 0.$



$\rightarrow$



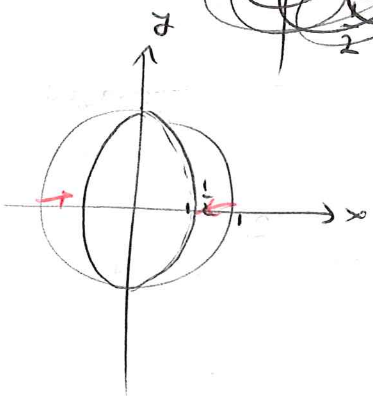
par. cyl.

Ex: $\boxed{4x^2+y^2=1}$

$(2x)^2+y^2=1$

z : free.

$z=0.$



$X = 2x$

$Y = y.$

(ell. cyl)

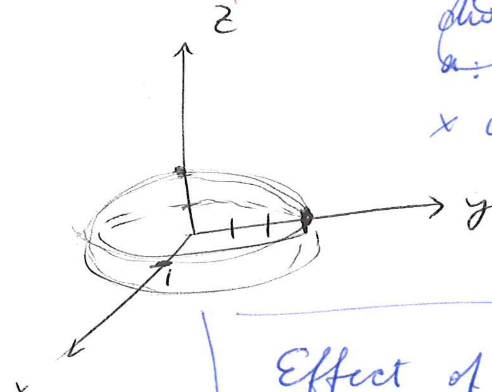
$X^2+Y^2=1.$

Effect of dividing x by a: scale x coordinate by $\frac{1}{a}$

Ex: $\boxed{x^2+\frac{y^2}{9}+\frac{z^2}{4}=1}$

$$x^2+\left(\frac{y}{3}\right)^2+\left(\frac{z}{2}\right)^2=1.$$

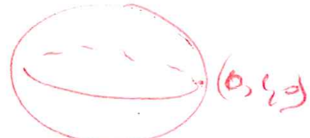
$\rightarrow$ ellipsoid.



$X = x$ $\quad Z = z/2$ $\quad X^2+Y^2+Z^2=1.$

$Y = y/3$ $\quad Y = (0,1,0)$

$\rightarrow y = (0,3,0)$



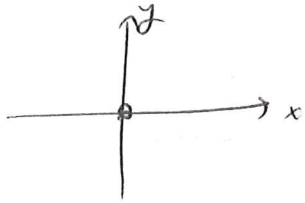
Effect of dividing x by $a>1$: stretch the x coordinate by a.

Ex: $\boxed{z = 4x^2 + y^2}$

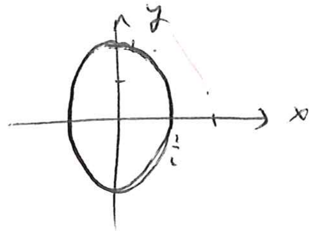
$(2x)^2 + y^2 - z = 0.$

- Effect of dividing x by $a>1$: stretch the x axis by a.
- Effect of multiplying x by $a>1$: squeeze ~~by~~ the x axis by $1/a$.

$z = 0.$



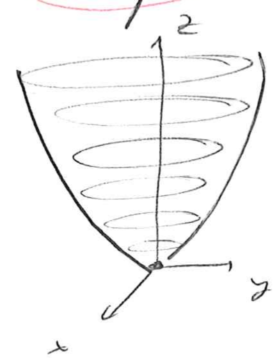
$z = 1$



$z = -1$

$\emptyset.$

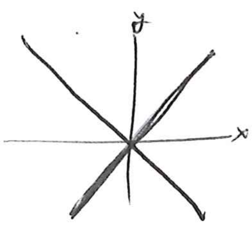
ell. par.



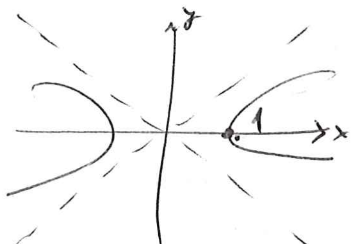
Ex: $\boxed{z = y^2 - x^2}$

$z = (y-x)(y+x)$

$z = 0.$



$z = 1$

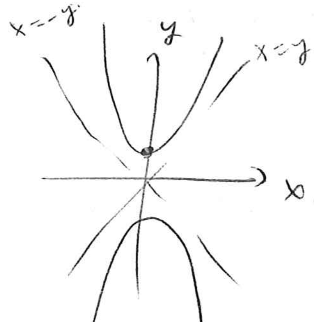


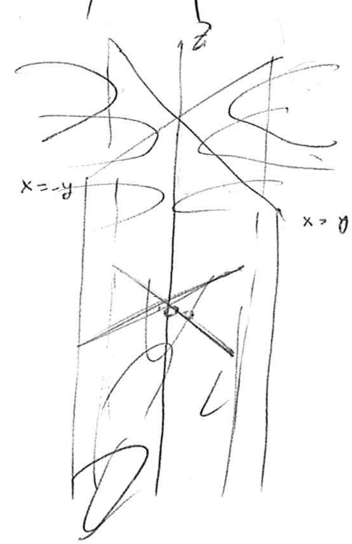
$(y-x) > 0$

$(y+x) > 0$

$\Rightarrow y > x > -y$

$z = -1$





pringle

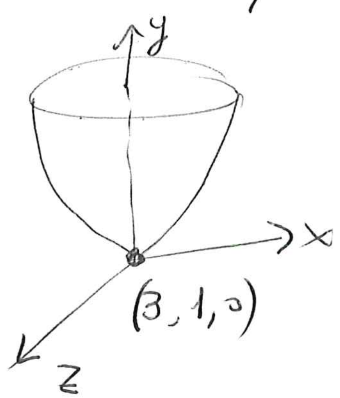
Ex: $\boxed{x^2+2z^2-6x-y+10=0}$

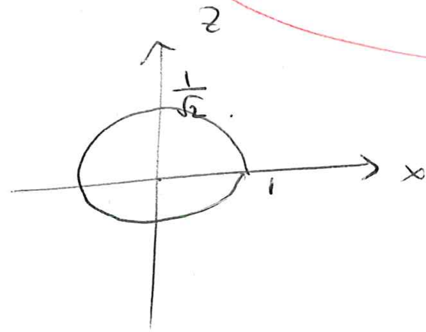
$$(x-3)^2-9+(\sqrt{2}z)^2-y+10$$

$$= (x-3)^2+(\sqrt{2}z)^2-(y-1) = 0.$$

$X = x-3, \quad Y = y-1, \quad Z = \sqrt{2}z$

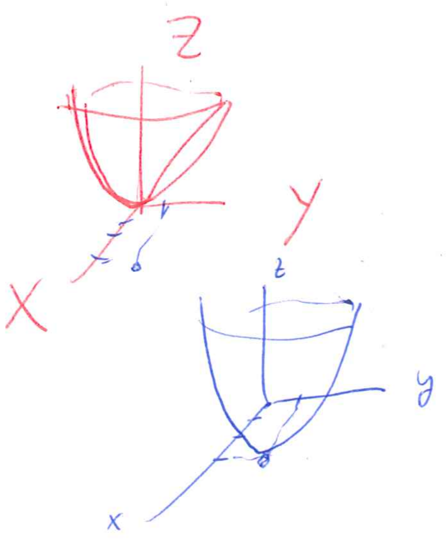
$X^2+Z^2-Y = 0. \quad \to$ elliptic paraboloid.

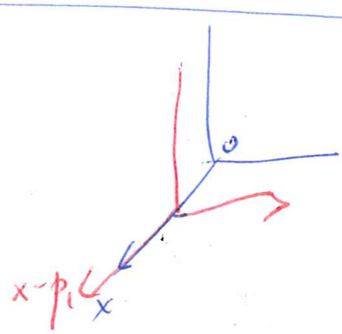




- Effect of subtracting $p_1 > 0$ from x: translate the ~~shape~~ origin in the $+x$ axis by p_1 units.

$x = X+3.$





SW: Verify the shapes in the list of canonical forms.

$\underline{Ex}$: $x^2 + 2xy - 2xz = 1$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \left(\underbrace{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} ax+by+cz \\ bx+dy+ez \\ cx+ey+fz \end{pmatrix} = ax^2 + bxy + cxz + bxy + dy^2 + eyz + cxz + ezy + fz^2$$

$$= \cancel{ax \quad +} = ax^2 + dy^2 + fz^2 + 2b\,xy + 2c\,xz + 2e\,yz$$

$\cancel{a=1 \quad d=0 \quad f=0 \quad b=1 \quad e=}$

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$$char_A(\lambda) = det(A - \lambda I) = det \begin{pmatrix} 1-\lambda & 1 & -1 \\ 1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{pmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (-\lambda - \lambda)$$

$$\begin{matrix} 1-\lambda & 1 & -1 \\ 1 & -\lambda & 0 \end{matrix}$$

$$= \lambda^2(1-\lambda) + 2\lambda$$

$$= -\lambda^3 + \lambda^2 + 2\lambda$$

$$= -\lambda(\lambda^2 - \lambda - 2)$$

$$= -\lambda(\lambda-2)(\lambda+1)$$

$\Rightarrow$ eigenvalues of A: $-1, 0, 2$.

$\lambda_1 = -1.$

$$0 = (A - \lambda_1) v_1 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} v_1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_1 = \frac{1}{\sqrt{3}} v_1$$

$\lambda_2 = 0.$

$$0 = A \cdot v_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} v_2 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}} v_2$$

$\lambda_3 = 2$

$$0 = (A - \lambda_3) v_3 = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix} v_3 \Rightarrow v_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{6}} v_3$$

SW: $v_1 \cdot v_2 = 0$

$v_1 \cdot v_3 = 0$

$v_2 \cdot v_3 = 0$

~~$R = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$ orthogonal~~

$$R := \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \text{ orthogonal} \Rightarrow R^{-1} = R^t \quad \cancel{x \cdot (Ax) = 1}$$

$\cancel{Az}$

$$AR = \begin{pmatrix} | & | & | \\ Au_1 & Au_2 & Au_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \lambda_3 u_3 \\ | & | & | \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}}\underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}_{= D}.$$

$AR = RD.$

$A = RDR^{-1}$

$$1 = X \cdot (AX) = X \cdot (RDR^{-1}X)$$

$$= (R^t X) \cdot (DR^{-1}X) = \underbrace{(R^{-1}X)}_{=Y} \cdot (D \underbrace{(R^{-1}X)}_{=Y})$$

$$= Y \cdot (DY). \Rightarrow 1 = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 + \lambda_3 \tilde{z}^2$$
$$= -\tilde{x}^2 + 2\tilde{z}^2 \Rightarrow \cancel{1 - \tilde{x}^2 + 2\tilde{z}^2 = 0}$$

$$\boxed{\tilde{x}^2 - 2\tilde{z}^2 - 1 = 0}$$

$\Rightarrow$ hyperbolic cylinder.

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = Y = R^{-1}X = R^t X = \begin{pmatrix} - u_1 - \\ - u_2 - \\ - u_3 - \end{pmatrix} \begin{pmatrix} | \\ X \\ | \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \cdot X \\ u_2 \cdot X \\ u_3 \cdot X \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{3}} + \frac{-y}{\sqrt{3}} + \frac{z}{\sqrt{3}} \\ \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{2}} \\ \frac{-2x}{\sqrt{6}} - \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{6}} \end{pmatrix}$$

SW: Verify.

$$\boxed{\begin{aligned} \tilde{x} &= \frac{x}{\sqrt{3}} - \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} \\ \tilde{y} &= \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{2}} \\ \tilde{z} &= \frac{-2x}{\sqrt{6}} - \frac{y}{\sqrt{6}} + \frac{z}{\sqrt{6}}. \end{aligned}}$$

SW: Do

$x^2 + 2xy - 2xz + x + z = 1.$