University of Utah Spring 2024

MATH 2270-002
Final Exam Questions

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May 1, 2024, 8:00 AM - 10:00 AM

Surname:	KEY
First Name:	
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Before turning the page make sure to read and sign the exam policy document, distributed separately.

1. **[35 points]** Consider the following system of linear equations:

$$x_1 + 3x_2 + 4x_3 + 5x_4 = 0$$
$$2x_1 + 6x_2 + 9x_3 + 5x_4 = 0$$

(a) **[10 points]** Solve the system.

augmented matrix:

$$\begin{pmatrix} 1 & 3 & 4 & 5 & 0 \\ 2 & 6 & 9 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 & 5 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 & 25 & 0 \\ 0 & 0 & 1 & -5 & 0 \end{pmatrix}$$

reduced echelon

$$\Rightarrow \boxed{\begin{aligned} x_1 &= -3x_2 - 25x_4 \\ x_2 &\ \text{free} \\ x_3 &= 5x_4 \\ x_4 &\ \text{free} \end{aligned}}$$

(b) **[10 points]** What is the dimension of the space of solutions?

2 free variables

$$\Rightarrow \boxed{\dim = 2}$$

(c) **[10 points]** Find a basis for the space of solutions.

$$x_2=1,\ x_4=0 \Rightarrow \boxed{\beta_1=(-3,1,0,0)}$$

$$x_2=0,\ x_4=1 \Rightarrow \boxed{\beta_2=(-25,0,5,0)}$$

$$\Rightarrow \boxed{\beta=(\beta_1,\beta_2) \text{ basis}}$$

(d) **[5 points]** Find a[redacted] basis for the orthogonal complement of the space of solutions.

$$A=\begin{pmatrix}1&3&4&5\\2&6&9&5\end{pmatrix} \Rightarrow \text{Nul}(A)=S=\text{space of solutions.}$$

$$\text{nullity}(A)=2 \Rightarrow \text{rank}(A)=\text{rank}(A^T)=2.$$

$$\Rightarrow S^\perp=\text{Nul}(A)^\perp=\text{Col}(A^T). \Rightarrow \boxed{\text{rows of } A \text{ is a basis for } S^\perp}$$

2. **[15 points]** Consider the following two matrices:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix}$$

Compute [redacted] the matrix $B - 2A$.

$$B - 2A = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix} - 2\begin{pmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 2 \\ -8 & 6 & -4 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3 & -5 & 3 \\ -7 & 2 & -7 \end{pmatrix}}$$

3. **[25 points]** Consider the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) **[10 points]** Is A diagonalizable?

Yes by spec thm, since $A^T = A$.

(b) **[5 points]** Compute the trace and determinant of A.

$\boxed{tr(A) = 5}$

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (4+0+0) - (2+1+0)$$

$$= \boxed{1}$$

(c) **[5 points]** Compute the rank and nullity of A^T.

A^T inv. $\Rightarrow$ $\boxed{\text{rank}(A) = 3, \text{ nullity}(A) = 0.}$

$\det(A^T) = \det(A)$

(d) **[5 points]** Is A invertible? If yes, [redacted] compute the [redacted] determinant of A^{-1}.

Yes, because $\det(A) = 1 \neq 0$.

$$\det(A^{-1}) = \frac{1}{\det(A)} = \boxed{1}$$

4. **[15 points]** Consider the following matrix:

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}.$$

(a) **[10 points]** Compute the singular values of A.

$$A^TA = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$$

$$0 = \begin{vmatrix} \lambda - 17 & -8 \\ -8 & \lambda - 17 \end{vmatrix} = (\lambda - 17)^2 - 8^2$$

$$= (\lambda - 17 - 8)(\lambda - 17 + 8) = (\lambda - 25)(\lambda - 9)$$

$$\lambda_1 = 25 > \lambda_2 = 9$$

$$\Rightarrow \boxed{\begin{array}{l} \text{singular values:} \\ \sigma_1 = 5 > \sigma_2 = 3 \end{array}}$$

(b) **[5 points]** ~~Compute~~ Find an orthonormal basis of $\mathbb{R}^2$ consisting of right singular vectors of A.

Need: eigenvectors of $A^TA = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$.

Distinct real eigenvalues $\Rightarrow$ by spec thm $\perp$.

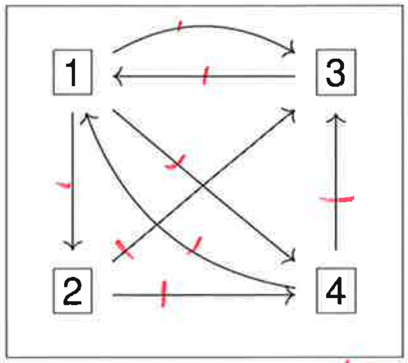
$\boxed{\lambda_1 = 25} \Rightarrow \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix} \Rightarrow \boxed{v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\boxed{\lambda_2 = 9} \Rightarrow \begin{pmatrix} -8 & -8 \\ -8 & -8 \end{pmatrix} \Rightarrow \boxed{v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$\Rightarrow \boxed{(v_1, v_2) \text{ ONB for } \mathbb{R}^2.}$

right singular vectors for A.

5. **[5 points]** Write the associated hyperlink matrix of the following network:



adjacency matrix: $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

outdegree matrix: $D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

hyperlink matrix: $H = AD^{-1}$

$$= \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

6. **[2 points]** This question is about the relation between rank one matrices and outer products of vectors.

(a) **[1 point]** Verify that if $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ are two nonzero vectors, then the $m \times n$ matrix uv^T has rank one.

$v \in \mathbb{R}^n \leftrightarrow v : \mathbb{R} \to \mathbb{R}^n.$

$\leftrightarrow v^T : \mathbb{R}^n \to \mathbb{R}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "uv^T"] \arrow[dr, "v^T"'] & & \mathbb{R}^m \\ & \mathbb{R} \arrow[ur, "u"'] & \end{tikzcd} \Rightarrow \boxed{\begin{array}{l} \text{rank}(uv^T) \\ \le \text{rank}(v^T) = 1 \end{array}}$$

$\boxed{\text{rank}(uv^T) > 0}$ because $uv^T \neq 0$

$\Rightarrow \boxed{\text{rank}(uv^T) = 1}$

(b) **[1 point]** Verify that if A is a rank one $m \times n$ matrix with real entries, then there are two nonzero vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that $A = uv^T$.

permuted LU factorization!

$$\underset{m}{P}\,\underset{m \,\, n}{A} = \underset{m \,\, m}{L}\,\underset{n}{U} \;:\quad \begin{array}{l} P \text{ perm} \\ L \text{ lower unitriangular} \\ U \text{ echelon form.} \end{array}$$

$$\operatorname{rank}(A) = 1 \Rightarrow U = \begin{pmatrix} 0\,0 \cdots 0 & \boxed{\blacksquare} & * \, * \cdots * \\ 0 & & 0 \end{pmatrix}$$

$\boxed{\text{Here } \blacksquare \neq 0.}$ Take r^T to be the first row of $U \Rightarrow v^T \neq 0$, $v \in \mathbb{R}^n$

$$\Rightarrow A = P^{-1}LU = \underbrace{P^{-1}L \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{= u \,\in\, \mathbb{R}^m} r^T$$

$$= uv^T. \quad r.$$

u nonzero because $P^{-1}L$ inv.

7. **[2 points]** Find all real numbers a, b, c such that the three functions

$$\boxed{\underline{\text{Answer}}\text{: need } |a| \neq |b| \neq |c| \neq |a|, \text{ all nonzero.}}$$

$$\sin(at),\ \sin(bt),\ \sin(ct)$$

are linearly independent.

$f_a(t) = \sin(at) \Rightarrow \boxed{f_a' = a g_a}$

$g_a(t) = \cos(at)$

$f_a'' = -a^2 f_a$

Say $\alpha f_a + \beta f_b + \gamma f_c = 0$.

Differentiate once, thrice, five times; evaluate at $t=0$

$\boxed{f_a''' = -a^3 g_a}$

$f_a^{iv} = a^4 f_a$

$\boxed{f^{v} = -a^5 g_a.}$

$$\Rightarrow \boxed{\begin{aligned} \alpha a + \beta b + \gamma c &= 0 \\ \alpha a^3 + \beta b^3 + \gamma c^3 &= 0 \\ \alpha a^5 + \beta b^5 + \gamma c^5 &= 0 \end{aligned}} \Rightarrow \underbrace{\begin{pmatrix} a & b & c \\ a^3 & b^3 & c^3 \\ a^5 & b^5 & c^5 \end{pmatrix}}_{A} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\text{Need}}$: A inv. or $\det(A) \neq 0$. Vandermonde.

$$\det(A) = abc \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{vmatrix} = \boxed{\pm abc\,(a^2-b^2)(a^2-c^2)(b^2-c^2) \neq 0}$$

8. **[1 point]** You are given the following scalar multiplication of a six dimensional vector with integer entries by 37:

$$c \quad 37\begin{pmatrix}5492\\11213\\21180\\7804\\4120\\18937\end{pmatrix}=\begin{pmatrix}203204\\414881\\783660\\288748\\152440\\700669\end{pmatrix} = b$$

Storing each digit of each entry of the outcome vector in an entry, one obtains the following 6×6 matrix:

$$A=\begin{pmatrix}2&0&3&2&0&4\\4&1&4&8&8&1\\7&8&3&6&6&0\\2&8&8&7&4&8\\1&5&2&4&4&0\\7&0&0&6&6&9\end{pmatrix}$$

Verify that the determinant of A is an integer multiple of 37.

$$\underbrace{\begin{pmatrix}1&&&&&0\\0&1&&&&\\0&0&1&&&\\0&0&0&1&&\\0&0&0&0&1&\\10^5&10^4&10^3&10^2&10&1\end{pmatrix}}_{L} \qquad A^T=\begin{pmatrix}*\\ \hline b^T\end{pmatrix}$$

$$\underbrace{\det(L)}_{1}\det(A)=\det\begin{pmatrix}*\\ \hline b^T\end{pmatrix}=37\underbrace{\left|\begin{pmatrix}*\\ \hline c^T\end{pmatrix}\right|}_{\text{integer}}$$

9. **[0 points]** For arbitrary vectors $u, v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$, verify the inequality

$$\sum_{i=1}^{p} \frac{|\langle u, v_i \rangle|^2}{\sum_{j=1}^{p} |\langle v_j, v_i \rangle|} \le |u|^2.$$

$$0 \le \left|u - \sum_i c_i v_i\right|^2 = \langle u - \sum_i c_i v_i, u - \sum_i c_i v_i \rangle$$

$$= \langle u, u\rangle - 2\sum_i c_i \langle u, v_i\rangle + \sum_{i,j} c_i c_j \langle v_i, v_j\rangle$$

(†)

$$\le |u|^2 - 2\sum_i c_i \langle u, v_i\rangle + \sum_{i,j} c_i^2 |\langle v_i, v_j\rangle|$$

(††)

$$= |u|^2 - 2\sum_i \frac{|\langle u, v_i\rangle|^2}{\sum_j |\langle v_j, v_i\rangle|} + \sum_{i,j} \frac{|\langle u, v_i\rangle|^2}{\left(\sum_k |\langle v_k, v_i\rangle|\right)^2} |\langle v_i, v_j\rangle|$$

$$= |u|^2 - 2\sum_i \frac{|\langle u, v_i\rangle|^2}{\sum_j |\langle v_j, v_i\rangle|} + \sum_i \frac{|\langle u, v_i\rangle|^2}{\left(\sum_k |\langle v_k, v_i\rangle|\right)^{\cancel{2}}} \cancel{\left(\sum_j |\langle v_i, v_j\rangle|\right)}$$

$$= |u|^2 - 2\sum_i \frac{|\langle u, v_i\rangle|^2}{\sum_j |\langle v_j, v_i\rangle|} + \sum_i \frac{|\langle u, v_i\rangle|^2}{\sum_j |\langle v_j, v_i\rangle|}$$

$$= |u|^2 - \sum_i \frac{|\langle u, v_i\rangle|^2}{\sum_j |\langle v_j, v_i\rangle|}$$ ✓

(†) ↓

$$0 \le (a-b)^2 = a^2 + b^2 - 2ab \Rightarrow ab \le \frac{a^2+b^2}{2}$$

$$\langle p, q\rangle \le |\langle p, q\rangle|$$

$$c_i = \frac{\langle u, v_i\rangle}{\sum_j |\langle v_j, v_i\rangle|}$$

(††)