University of Utah Spring 2024

MATH 2270-002 PSet 3 Specification

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Subject to Change; Last Updated: 2024-03-11 13:10:07-06:00

1 Background

This problem set is centered on algebra with matrices, covering matrix addition, scalar multiplication, transposes, as well as matrix multiplication and commutators. One of the key properties we'll explore is that matrix multiplication, for certain square matrices, exhibits similarities to number multiplication. For instance, some square matrices A have an inverse matrix A^{-1}, defined as a square matrix of the same dimensions as A satisfying the equations

$$AA^{-1} = I = A^{-1}A.$$

While a crucial condition[1] for a square matrix to be invertible will be discussed later in the course, at this stage what is important is to develop familiarity with row reduction techniques to compute the inverses of given matrices, assuming they exist.

A significant point of divergence between number multiplication and matrix multiplication of square matrices is the lack of commutativity in the latter. Generally,

[1] For inverting numbers, aka 1×1 matrices, this condition simply states that the number cannot be zero, as "dividing by zero" is undefined.

$$AB \neq BA,$$

which means it's vital to distinguish between these two products. This difference is characterized by the commutator $[A, B] = AB - BA$, a concept essential for understanding the nuanced behavior of matrix multiplication.

Arithmetic with Complex Numbers Some problems in this problem set involves complex numbers. In case you need a refresher on how to do arithmetic with complex numbers, here is all you need to know for this problem set: A complex number is by definition an expression of the form

$$a + ib,$$

where a and b are two real numbers, and i is the so-called imaginary unit. Thus in a sense one complex number is two real numbers written in a peculiar way. One can then add two complex numbers like so:

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

and multiply two complex numbers like so:

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Note that in particular by plugging in $a = 0$, $b = 1$, $c = 0$, $d = 1$ we have

$$i^2 = (0 + i1)(0 + i1) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1 + i0 = -1,$$

so that one can think of i as a number whose square is -1!

One denotes the set of complex numbers by $\mathbb{C}$. The topics we have covered in this class so far extends to the case when one uses complex numbers instead of real numbers.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the following five matrices:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}, \qquad E = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

Some of the following expressions involving matrix addition, scalar multiplication, matrix multiplication and matrix transpose are syntactic (aka they are "grammatical"), whereas some are not. For each of the following expressions, compute the expression by writing it in one matrix if the expression is syntactic, or state that it is not syntactic otherwise.

(a) $B - 2A$

(b) $\left(\left(\left(A^T\right)^T\right)^T\right)^T$

(c) $B^T + A$

(d) AC

(e) $A^T C$

(f) $\left[A, B^T\right]$

(g) $\left[[C, D], AB^T\right]$

(h) CDC^T

(i) $\frac{C+C^T}{2}$

(j) $3C - E$

(k) CB

(l) EB

(m) $E^T B$

(n) E^TE

(o) E^2

(p) EE^T

2. Let A be an anonymous 2×2 matrix, say

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Verify the Cayley-Hamilton identity by computing both sides and comparing the resulting matrices entry-by-entry:

$$A^2 = (a + d)A - (ad - bc)I$$

where I denotes the 2×2 identity matrix. Note that the Cayley-Hamilton identity displays the square $A^2 = AA$ of any 2×2 matrix A as a certain linear combination of A and I.

Challenge Derive the Cayley-Hamilton identity by solving

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for the numbers α and β in terms of the numbers a, b, c and d. You are not given any information regarding whether or not any of the matrix entries a, b, c, d are nonzero.

3. Consider the family of linear transformations

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

parameterized by θ.

(a) Verify the identity

$$R_\alpha R_\beta = R_{\alpha+\beta} = R_\beta R_\alpha$$

by using trigonometric identities.

(b) Describe each three sides of the identity in the previous part geometrically.

(c) Euler's formula states that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

which suggests that one could more generally store a complex number $z = a + ib$ in a 2×2 matrix with real entries:

$$z = a + ib \rightsquigarrow M(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Verify that indeed storing complex numbers in matrices this way allows one to use matrix addition and matrix multiplication to model complex number addition and complex number multiplication. More concisely, verify that the following two identities hold for z and w complex numbers:

$$M(z) + M(w) = M(z + w),$$
$$M(z)M(w) = M(zw).$$

(d) The complex conjugate $\bar{z}$ of the complex number $z = a + ib$

is by definition

$$\bar{z} = a - ib.$$

Verify that

$$M(\bar{z}) = M(z)^T$$

by computing both sides.

Challenge Find another way of storing complex numbers in square matrices that transforms complex number addition to matrix addition and complex number multiplication to matrix multiplication.

4. Consider the following three matrices:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Fill in the multiplication table by computing the products of matrices. Ensure the matrix from the row is always the first factor. For example, in the bottom left cell, enter the result of the product ZX, with Z from the row and X from the column.

mult.	X	Y	Z
X			
Y			
Z			

(b) Fill in the commutator table by computing the commutators of matrices. Ensure the matrix from the row is always the first entry in the commutator. For example, in the bottom left cell, enter the result of the commutator $[Z, X] = ZX - XZ$, with Z from the row and X from the column.

commutator	X	Y	Z
X			
Y			
Z			

5. Consider the following 2×2 matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here i is the imaginary unit.

(a) Fill in the multiplication table by computing the products of matrices. Ensure the matrix from the row is always the first factor.

mult.	I	σ_1	σ_2	σ_3
I				
σ_1				
σ_2				
σ_3				

(b) Fill in the commutator table by computing the commutators of matrices. Ensure the matrix from the row is always the first entry in the commutator.

comm.	I	σ_1	σ_2	σ_3
I				
σ_1				
σ_2				
σ_3				

6. Consider the following matrix:

$$A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$$

Perform the following tasks:

(a) Compute the (multiplicative) inverse A^{-1} of A.

(b) Verify that $AA^{-1} = I = A^{-1}A$ by multiplying the matrices and comparing the corresponding entries.

(c) Use A^{-1} to solve the system

$$\begin{aligned} 8x_1 + 3x_2 &= 2 \\ 5x_1 + 2x_2 &= -1 \end{aligned}$$

7. Compute the inverses of each of the following matrices:

(a)

$$\begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & 4 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

8. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (6x - 8y, -5x + 7y).$$

   Verify that T is invertible by computing the inverse linear transformation $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

> Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/bvgWW9oMxANZHXZG7.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses
  /694951/assignments/3866302,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on February 2, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.