University of Utah Spring 2024

MATH 2270-002 PSet 9 Specification

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1 Background

This problem set delves into the concept of orthogonality within the realm of linear algebra, particularly focusing on its applications and implications in the context of $\mathbb{R}^n$. At the heart of this exploration is the inner product of two vectors u and v, denoted as $\langle u, v \rangle$, which is defined as:

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

When $\langle u, v \rangle = 0$, it signifies that vectors u and v are orthogonal, or in geometric terms, perpendicular to each other. This foundational idea of orthogonality extends to the concepts of orthogonal and orthonormal bases for vector subspaces, as well as to orthogonal projections.

Orthogonal projections, while geometric in nature, have profound implications beyond geometry. They provide a framework for understanding how to approach the solution of a system of linear equations in an approximate manner. This involves finding solutions that, while not exact, closely align with the system's constraints.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the following three vectors in $\mathbb{R}^3$:

$$u = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

   and let $S = \text{Span}\{u, v\}$. Perform the following tasks:

   (a) Compute the dimension of S.

   (b) Find an orthonormal basis β of S.

(c) Compute the orthogonal projection $\mathrm{proj}_S(w)$ of w onto S.

(d) Define the matrix B as that matrix whose columns are the basis vectors of the orthonormal basis β of S.

i. Compute the matrices B^TB, BB^T and $(BB^T)^2$.

ii. Compute BB^Tu, BB^Tv and BB^Tw.

iii. Compute the distance between the vectors $\mathrm{proj}_S(w)$ and BB^Tw.

(e) Find another orthonormal basis δ of S such that no basis vector in δ is proportional to a basis vector in β.

(f) Define the matrix D as that matrix whose columns are the basis vectors of the orthonormal basis δ of S.

i. Compute the matrices D^TD, DD^T and $(DD^T)^2$.

ii. Compute DD^Tu, DD^Tv, and DD^Tw.

iii. Compute the distance between the vectors BB^Tw and DD^Tw.

(g) Find an orthonormal basis γ of $S^\perp$.

(h) Compute the orthogonal projection $\mathrm{proj}_{S^\perp}(w)$ of w onto $S^\perp$.

(i) Define the matrix C as that matrix whose columns are the basis vectors of the orthonormal basis γ of $S^\perp$.

i. Compute the matrices C^TC, CC^T and $(CC^T)^2$.

ii. Compute CC^Tu, CC^Tv, and CC^Tw.

iii. Compute the distance between the vectors $\mathrm{proj}_{S^\perp}(w)$ and CC^Tw.

(j) Define the matrix B as that matrix whose columns are the basis vectors of some orthonormal basis of S, and similarly define the matrix C as that matrix whose columns are the basis vectors of some orthonormal basis of $S^\perp$. Compute $BB^T + CC^T$.

(k) Find a 3×3 orthogonal matrix Q such that the image Q(S) of S under Q is the same as S.

2. Consider the following matrix:

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$

Perform the following tasks:

(a) Find an orthonormal basis for the columnspace of A.

(b) Find an orthonormal basis for the nullspace of A.

(c) Find an orthonormal basis for the columnspace of A^T.

(d) Find an orthonormal basis for the nullspace of A^T.

(e) Define the matrix B as that matrix whose columns are the basis vectors of some orthonormal basis of the columnspace of A. Verify that $A = BB^TA$.

(f) Find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

(g) Find a 3×4 matrix U with orthonormal columns and an upper triangular 4×4 matrix V such that $A^T = UV$.

(h) i. Find a vector b that is not in the nullspace of A^T such that the system $Ax = b$ is inconsistent.

ii. Find a vector x that minimizes the distance between Ax and b.

Challenge Let A be an $m \times n$ matrix of rank r.

(a) Verify that

- there is a matrix Q with orthonormal columns, and
- there is a matrix R whose ijth entry is zero for any $i > j$,

such that $A = QR$.

(b) Find all positive integers s such that there is an $m \times s$ matrix Q and an $s \times n$ matrix R with the above properties such that $A = QR$.

Challenge For n a nonnegative integer, let $\mathcal{P}_n$ be the space of polynomials of degree at most n. Let further $\tau = (\tau_0, \tau_1, ..., \tau_n) \in \mathbb{R}^{n+1}$ be a vector .

(a) Verify that $E_\tau : \mathcal{P}_n \to \mathbb{R}^{n+1}$ defined by

$$E_\tau(p) = (p(\tau_0), p(\tau_1), ..., p(\tau_n))$$

is a linear transformation.

(b) Compute the determinant of E_τ.

(c) Verify that E_τ is invertible if and only if entries of the vector τ are pairwise distinct.

(d) Verify that if entries of the vector τ are distinct, then for any $y = (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$, the polynomial $E_\tau^{-1}(y)$ is the unique polynomial of degree at most n that interpolates the points on the plane

$$(t_0, y_0), \quad (t_1, y_1), \quad ..., \quad (t_n, y_n).$$

(e) Define

$$\langle p, q \rangle_\tau = \langle E_\tau(p), E_\tau(q) \rangle_{\mathbb{R}^{n+1}},$$

where the inner product on the RHS is the standard inner product on $\mathbb{R}^{n+1}$. Verify that $\langle \bullet, \bullet \rangle_\tau$ is an inner product on $\mathcal{P}_n$ if and only if entries of the vector τ are pairwise distinct.

(f) Suppose entries of the vector τ are pairwise distinct. Define for each $0 \leq i \leq n$,

$$\beta_{\tau,i}(t) = \prod_{j \neq i} \frac{t - \tau_j}{\tau_i - \tau_j},$$

and put

$$\beta_\tau = (\beta_{\tau,0}, \beta_{\tau,1}, ..., \beta_{\tau,n}).$$

Verify that β_τ is an orthonormal basis relative to the inner product $\langle \bullet, \bullet \rangle_\tau$.

(g) Suppose entries of the vector τ are pairwise distinct. Verify that for any $p \in \mathcal{P}_n$ and for any $0 \leq i \leq n$,

$$\langle p, \beta_{\tau,i} \rangle_\tau = p(\tau_i).$$

(h) For m a nonnegative integer not larger than n, given a vector $y = (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$, describe the set $A_{\tau,y,m}$ of polynomials of degree at most m whose graph approximate best—according to the distance $|\bullet|_\tau$ defined by the inner product $\langle \bullet, \bullet \rangle_\tau$—the following points on the plane:

$$(t_0, y_0), \quad (t_1, y_1), \quad ..., \quad (t_n, y_n).$$

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

  > Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

  Here are some guides regarding prompt engineering:

  - https://platform.openai.com/docs/guides/prompt-engineering
  - https://developers.google.com/machine-learning/resources/prompt-eng
  - https://www.ibm.com/topics/prompt-engineering
  - https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/HVeBMewiKDbNrXes5.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866308,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on April 5, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.