University of Utah Spring 2024

MATH 2270-002 PSet 4 Specification

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Subject to Change; Last Updated: 2024-03-11 13:10:29-06:00

1 Background

This problem set is about the concepts of linear subspaces, affine transformations, and block matrices. It begins by examining linear subspaces, specifically focusing on identifying a basis for a subspace and investigating special subspaces associated with a matrix——its nullspace and columnspace. The notions of rank and nullity of a matrix are also related concepts, as rank is the dimension of the columnspace and nullity is the dimension of the nullspace. Yet another problem establishes another connection between linear subspaces and linear transformations.

The problems then transition to affine transformations of $\mathbb{R}^2$ and $\mathbb{R}^3$. Even though affine transformations are not linear transformations, one can still store them in matrices, using the block matrix formalism. The last problem delves into block matrix algebra, highlighting the Schur complement, a concept with wide-ranging applications.

A Technical Matter: Ordered v Unordered Basis While a "basis" for a linear subspace is technically an ordered list of vectors that are linearly independent and span the subspace, it is commonly thought of as a set without specific order in practice. This distinction, though subtle, is critical in certain contexts. For instance

$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right), \quad \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right) \in \mathbb{R}^2 \times \mathbb{R}^2$$

are two distinct bases of $\mathbb{R}^2$.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the following sets of vectors:

I:

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 7 \\ 0 \\ -5 \end{pmatrix}$$

II:

$$\begin{pmatrix} 3 \\ -8 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix}$$

III:

$$\begin{pmatrix} 1 \\ -6 \\ -7 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}$$

IV:

$$\begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 9 \\ -6 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix}$$

V:

$$\begin{pmatrix} 1 \\ -1 \\ -2 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -3 \\ -1 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ -6 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 4 \\ -7 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -8 \\ 9 \\ -5 \end{pmatrix}$$

For each of the sets of vectors, perform the following tasks:

(a) Find a basis for their span.

(b) Compute the dimension of their span.

2. Find a basis for the space of solutions of the following homogeneous system of linear equations:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 5x_4 - x_5 &= 0 \\ 2x_1 + 7x_2 + 4x_3 + 11x_4 + 2x_5 &= 0 \\ 2x_1 + 6x_2 + 5x_3 + 12x_4 - 7x_5 &= 0 \end{aligned}$$

3. Consider the following matrix:

$$A = \begin{pmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{pmatrix}$$

Perform the following tasks:

(a) Find an invertible matrix L such that $A = LU$, where U is the reduced echelon form of A.

(b) Find a basis for each of the following subspaces:

i. Columnspace of U
ii. Nullspace of U
iii. Columnspace of L
iv. Nullspace of L
v. Columnspace of A
vi. Nullspace of A
vii. Columnspace of A^T
viii. Nullspace of A^T

(c) Compute the rank and nullity of each of U, L, A and A^T.

Challenge Find another invertible matrix R such that $A = LMR$, where M is a matrix in block matrix form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. One way to find such an R is to further reduce U using "elementary column operations": $L^{-1}A = U$, hence $U^T = A^T(L^{-1})^T$, which one can then apply row reduction to reduce further.

4. This problem is about the rank and nullity of 3×5 matrices, and how taking transposes affect these numbers. In one of the problems in problem set 1 you were asked to list all possible echelon forms for 3×5 matrices; you may use your prior work in this problem. As a sanity check, there should be exactly 26 distinct forms in your list.

   (a) List all possible reduced echelon forms for 3×5 matrices. For each such form, state the rank and nullity of the matrix.

   (b) List all possible reduced echelon forms for 5×3 matrices; in your list there should be exactly 8 distinct forms. For each such form, state the rank and nullity of the matrix.

   (c) For any 3×5 reduced echelon form U, compute the rank and nullity of U^T. Note that it is likely that U^T will not be in reduced echelon form.

5. In this problem we consider not only linear transformations but arbitrary functions between the vector spaces $\mathbb{R}^n \to \mathbb{R}^m$. For instance $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x^2 + y, \sin(y), e^{xy})$ is a function but is not a linear transformation, since $f(0, 0) = (0, 0, 1)$[1].

   To each function $f : \mathbb{R}^n \to \mathbb{R}^m$, linear or not, one can associate a certain subset of $\mathbb{R}^n \times \mathbb{R}^m$, namely its graph: by definition the

[1] Note that due to the axiomatic definition of a linear transformation, any linear transformation is forced to take the zero vector to the zero vector: $F(0) = F(0 + 0) = F(0) + F(0) \implies F(0) = 0$.

graph of f is the set of all those pairs of vectors (x, y) such that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $y = f(x)$. More concisely one can write

$$\text{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x)\}.$$

(a) Verify that any affine transformation $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = ax + b$ for some numbers a and b.

(b) Consider an anonymous affine transformation $f : \mathbb{R} \to \mathbb{R}$; say $f(x) = ax + b$. Sketch graph(f) as a subset of $\mathbb{R}^2$.

(c) Describe the set of all those affine transformations whose graph is a linear subspace of $\mathbb{R}^2$.

(d) Give an example of a linear subspace of $\mathbb{R}^2$ that is not the graph of any function $f : \mathbb{R} \to \mathbb{R}$.

(e) Verify using the axiomatic definition of a linear transformation that f being a linear transformation is equivalent to its graph being a linear subspace. More precisely:

  i. If a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then its graph graph(f) is a linear subspace of $\mathbb{R}^{n+m}$.

  ii. If a function $f : \mathbb{R}^n \to \mathbb{R}^m$ fails to be linear, then its graph graph(f) fails to be a linear subspace of $\mathbb{R}^{n+m}$ too.

(f) Say $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Write the graph of f as the image (aka columnspace) of a linear transformation.

6. In each part of this problem you are given a certain geometric description as to how an affine transformation operates on the plane; your job is to write down the corresponding 3×3 matrix using homogeneous coordinates.

(a) Translate by $(-3, 4)$ and then scale the x-coordinate by 0.7 and the y-coordinate by 1.3.

(b) Reflect points through the x-axis, and then rotate 30° about the origin.

(c) Rotate points 30°, and then reflect through the x-axis.

(d) Rotate points through 45° about the point $(3, 7)$.

7. Give the 4×4 matrix that corresponds to the affine transformation of $\mathbb{R}^3$ that first rotates points about the z-axis through an angle of -30°, and then translates by $p = (5, -2, 1)$.

8. Let A, B, C, D be four matrices with the following dimensions:

   - A is $p \times p$,
   - B is $p \times q$,
   - C is $q \times p$,
   - D is $q \times q$.

   Store these four matrices in one $(p + q) \times (p + q)$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

   Much like how one can do block matrix addition and block matrix multiplication, one can also do elementary block row operations and consequently do block row reduction.

   (a) Suppose the matrix A on the top left corner is invertible and abbreviate the matrix $D - CA^{-1}B$ by M/A. $M/A = D - CA^{-1}B$ is called the Schur complement of A with respect to M.

Using elementary block row operations, verify that M block row reduces to the block echelon form

$$\begin{pmatrix} I & A^{-1}B \\ 0 & M/A \end{pmatrix}.$$

(b) Suppose A is invertible and compute the matrices X and Y such that

$$M = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & M/A \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}.$$

(c) Verify that if A is invertible, then the following two formulas hold:

$$\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(M/A),$$
$$\operatorname{nullity}(M) = \operatorname{nullity}(A) + \operatorname{nullity}(M/A).$$

(d) Verify that if A is invertible, then the invertibility of M is equivalent to the invertibility of M/A. More precisely:

i. If A and M are invertible, then so is M/A.

ii. If A and M/A are invertible, then so is M.

(e) Suppose both M and A are invertible. Find a formula for the block inverse of M. Such a formula would display M^{-1} as a block matrix each of whose entries is a matrix in terms of the blocks A, B, C and D:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}.$$

In the case when each block is 1×1, so that M is a 2×2

matrix with real entries, your formula ought to reduce to the formula for inverses of 2×2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Challenge Consider three block matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, L = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}, R = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix}.$$

Suppose the top left blocks A, X, U are all invertible and of the same dimension and the input/output splittings match so that LMR too is a block matrix with top left block XAU. Verify that

$$(LMR)/(XAU) = (L/X)\,(M/A)\,(R/U).$$

Challenge 2 Consider

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

so that M is a block matrix whose top left block A itself is a block matrix. Suppose both A and E are invertible. Verify the Haynsworth quotient formula

$$M/A = (M/E)\,/\,(A/E).$$

Here is one way to verify this formula. First consolidate the two block matrix structures:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & F & B_1 \\ G & H & B_2 \\ C_1 & C_2 & D \end{pmatrix}.$$

Then use the formula from the previous challenge problem with the substitutions

$$X = \begin{pmatrix} I & 0 \\ -GE^{-1} & I \end{pmatrix}, \quad Y = \begin{pmatrix} -C_1E^{-1} & 0 \end{pmatrix}, \quad Z = I,$$
$$U = \begin{pmatrix} I & -E^{-1}F \\ 0 & I \end{pmatrix}, \quad V = \begin{pmatrix} -E^{-1}B_1 \\ 0 \end{pmatrix}, \quad W = I.$$

Challenge 3 One can use the block matrix formalism to store a matrix in a part of itself. This procedure leads to infinite matrices. For instance, one can consider the "infinite dimensional identity matrix"

$$I_\infty = \begin{pmatrix} 1 & 0 \\ 0 & I_\infty \end{pmatrix}.$$

Substituting I_∞ in itself one obtains

$$I_\infty = \begin{pmatrix} 1 & 0 \\ 0 & I_\infty \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_\infty \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider the following four matrices:

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},\ c = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

Fill in the multiplication table by computing the products of matrices. Ensure the matrix from the row is always the first factor.

mult.	I_∞	b	c	d
I_∞				
b				
c				
d				

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff

will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

> Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/Ngqg42yTpgrqvptJ6.

You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866303,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on February 16, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.