University of Utah Spring 2024

MATH 2270-002 PSet 2 Specification

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1 Background

This problem set delves into two central concepts in linear algebra: linear independence and the interpretation of matrices as linear transformations.

We start by exploring linear independence, a concept that is in some ways complementary to linear span. While linear span concerns "accessibility" — the places that can be reached using a given set of vectors — linear independence focuses on "non-redundancy". It examines whether removing a vector from a set of vectors limits the reach or span that could otherwise be achieved with the full set.

The problems on matrices shifts focus from their role in linear equations to their geometric significance as linear transformations. This perspective reveals matrices as dynamic mathematical objects influencing geometric space.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

1. Determine if the sets of vectors are linearly independent. Here we used the so-called roster notation to talk about sets and develop more familiarity with the notation, but in practice it is also common to just list vectors without using $\{\cdots\}$ and ask if "the vectors are linearly independent". Note that some of these require no calculations whatsoever, if one uses the heuristic

$$\boxed{\text{"linearly independent"} \ \sim \ \text{"no redundant vectors"}}$$

(a)

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 6 \end{pmatrix} \right\}$$

(b)

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix} \right\}$$

(c)

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(d)

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(e)

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(f)

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -8 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \right\}$$

(g)

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -8 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 41 \\ -37 \\ 1/463 \end{pmatrix} \right\}$$

2. Consider the following three vectors:

$$v_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -10 \\ h \end{pmatrix}.$$

Here one of the entries of the third vector v_3 is left as a parameter; for this reason one could also call $v_3 = v_3(h)$ a family of vectors parameterized by h.

(a) Find all values of the parameter h such that $v_3 \in \text{Span}\{v_1, v_2\}$.

(b) Find all values of h such that $\{v_1, v_2, v_3\}$ is linearly independent.

3. Consider the following three vectors:

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -6 \\ 7 \\ -3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 8 \\ h \\ 4 \end{pmatrix}.$$

Find all values of h such that $\text{Span}\{v_1, v_2, v_3\} \neq \mathbb{R}^3$.

4. Consider the following three 2×2 matrices:

I:

$$A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

II:

$$A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

III:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Interpret each of these matrices as a linear transformation of the plane, and for each of these matrices, perform the following tasks:

(a) Use a coordinate system to plot

$$u = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

and their images under the linear transformation.

(b) Describe geometrically what the linear transformation does to each vector $x = (x_1, x_2) \in \mathbb{R}^2$.

5. In each part of this problem you are given a certain geometric description as to how a linear transformation operates; your job is to write down the corresponding matrix.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(e_1) = (2, 1, 2, 1)$ and $T(e_2) = (-5, 2, 0, 0)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (in the counterclockwise direction).

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps e_1 into $e_1 - 2e_2$ but leaves the vector e_2 unchanged.

   (d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radians and then reflects points through the horizontal x_1-axis.

   (e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms e_2 into $e_2 - 3e_1$ (leaving e_1 unchanged) and then reflects points through the line $x_2 = -x_1$.

   (f) $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2-axis and then rotates points $3\pi/2$ radians.

6. Consider the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \qquad y_2 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

   and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps e_1

into y_1 and maps e_2 into y_2. Compute the images of $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ under T.

7. Consider the vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -9 \end{pmatrix},$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps x into $x_1 v_1 + x_2 v_2$. What is the corresponding matrix of T?

8. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin by a certain angle. What is the angle of that rotation?

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2).$$

   (a) Write T in matrix form.
   (b) Find x such that $T(x) = (-1, 4, 9)$.
   (c) Find the set of all those $y \in \mathbb{R}^3$ such that there is no $x \in \mathbb{R}^2$ with $T(x) = y$.

10. Describe the possible reduced echelon forms of the matrix for a linear transformation T. Use the symbol ★ for an anonymous number that is possibly zero.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

  > Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

  Here are some guides regarding prompt engineering:

  - https://platform.openai.com/docs/guides/prompt-engineering
  - https://developers.google.com/machine-learning/resources/prompt-eng
  - https://www.ibm.com/topics/prompt-engineering
  - https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/sQeXMrg7to1TnDmD9.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866301,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on January 26, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.