University of Utah Spring 2024

MATH 2270-002
Midterm 1 Questions

Instructor: Alp Uzman

February 9 2024, 8:35 AM - 9:25 AM

Surname:

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uNID:

Before turning the page make sure to read and sign the exam policy document, distributed separately.

1. **[65 points]** Consider the following system of linear equations:

$$x_1 + 3x_2 + 8x_3 - x_4 = 0$$
$$x_1 - 3x_2 - 10x_3 + 5x_4 = 0$$
$$x_1 + 4x_2 + 11x_3 - 2x_4 = 0$$

(a) **[10 points]** How many equations are there in the system? How many unknowns (aka variables) are there in the system?

3 equations
4 unknowns.

(b) **[10 points]** Store the unknowns in a column vector x and write the system in matrix form Ax = b.

$$\underbrace{\begin{pmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

(c) **[10 points]** Write the augmented matrix that corresponds to the system.

$$(A\,b) = \begin{pmatrix} 1 & 3 & 8 & -1 & 0 \\ 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \end{pmatrix}$$

(d) **[10 points]** Perform elementary row operations and compute an echelon form of the augmented matrix. Clearly state the elementary row operations you are applying in each step.

$$\begin{pmatrix} 1 & 3 & 8 & -1 & 0 \\ 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \end{pmatrix} \xrightarrow{E_1} \begin{pmatrix} 1 & 3 & 8 & -1 & 0 \\ 0 & -6 & 18 & 6 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{pmatrix} \xrightarrow{E_2} \begin{pmatrix} 1 & 3 & 8 & -1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{pmatrix}$$

$$\downarrow E_3$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

reduced echelon form.

E_1: $\boxed{\begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}}$

E_2: $\boxed{R_2 \leftarrow -\frac{1}{6} R_2}$

E_3: $\boxed{\begin{array}{l} R_1 \leftarrow R_1 - 3R_2 \\ R_3 \leftarrow R_3 - R_2 \end{array}}$

(e) **[10 points]** Find all solutions of the system.

$$\boxed{\begin{array}{l} x_1 = x_3 - 2x_4 \\ x_2 = -3x_3 + x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{array}}$$

(f) **[10 points]** What is the reduced echelon form of the coefficient matrix A?

$$(A|b) \sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ Reduced echelon form of A:

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(g) **[5 points]** What is the rank and nullity of the coefficient matrix A?

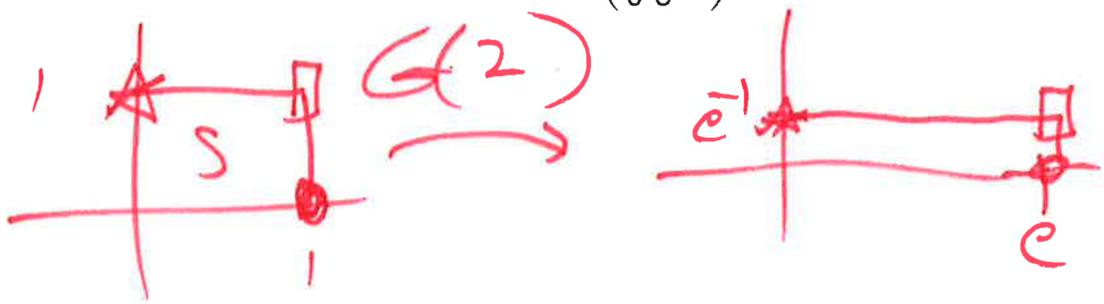
$\operatorname{rank}(A) = 2$

$\operatorname{nullity}(A) = 2$

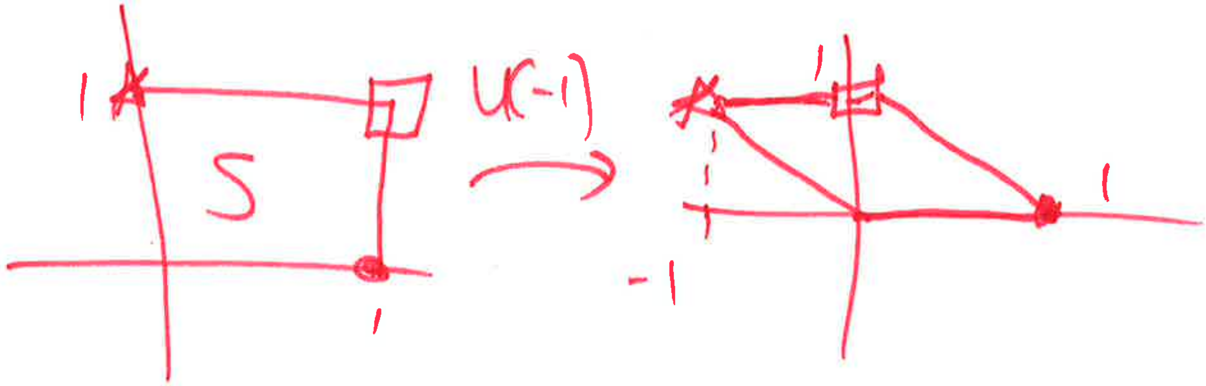
2. **[32 points]** Consider the following two families of linear transformations of the plane:

$$G(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad U(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

(a) **[10 points]** Consider the unit square S on the plane with corners at $(0,0), (1,0), (1,1), (0,1)$. Sketch the image of S under the transformation $G(2) = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$.



(b) **[10 points]** Consider the unit square S on the plane with corners at $(0,0), (1,0), (1,1), (0,1)$. Sketch the image of S under the transformation $U(-1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.



(c) **[4 points]** Compute $G(t)^{-1}$.

$$G(t)^{-1} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}^{-1} = \frac{1}{\boxed{1}} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$$

$$e^{t/2} e^{-t/2} - 0 = 1$$

(d) **[4 points]** Compute $U(s)^{-1}$.

$$U(s)^{-1} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{\boxed{1}} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$$

$$1 \cdot 1 - 0 \cdot s = 1$$

(e) **[4 points]** Compute $G(t)\,U(s)\,G(t)^{-1}$.

$$G(t)\,U(s)\,G(t)^{-1} = G(t) \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$$

$$= G(t) \begin{pmatrix} e^{-t/2} & s e^{t/2} \\ 0 & e^{t/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} e^{-t/2} & s e^{t/2} \\ 0 & e^{t/2} \end{pmatrix} = \begin{pmatrix} 1 & s e^{t} \\ 0 & 1 \end{pmatrix}$$

$$= U(s e^{t}).$$

3. **[2 points]** Verify that the interchange ($R_i \leftrightarrow R_j$) of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types (replacement $R_i \leftarrow aR_i + bR_j$ and scaling $R_i \leftarrow cR_i$).

$$\begin{tikzcd}
\begin{pmatrix} R_i \\ R_j \end{pmatrix} \arrow[rr, "E_1"] \arrow[d, "E_2"'] & & \begin{pmatrix} R_j \\ R_i \end{pmatrix} \\
\begin{pmatrix} R_i \\ R_j+R_i \end{pmatrix} \arrow[r, "E_3"] & \begin{pmatrix} -R_j \\ R_j+R_i \end{pmatrix} \arrow[r, "E_4"] & \begin{pmatrix} -R_j \\ R_i \end{pmatrix} \arrow[u, "E_5"']
\end{tikzcd}$$

$E_1 : \boxed{R_i \leftrightarrow R_j}$ interchange

$E_2 : \boxed{R_j \leftarrow R_j + R_i}$ replace

$E_3 : \boxed{R_i \leftarrow R_i - R_j}$ replace

$E_4 : \boxed{R_j \leftarrow R_j + R_i}$ replace

$E_5 : \boxed{R_i \leftarrow -R_i}$ scale

<u>Check:</u>

$E_1 \overset{?}{=} E_5 E_4 E_3 E_2$

$$\underbrace{\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}}$$

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_1$$

✓.

4. **[1 point]** How many distinct 5×9 reduced echelon forms are there?

$5\times 9 \Rightarrow \mathbb{R}^9 \to \mathbb{R}^5$. rank + nullity = 9, rank ≤ 5

0	9
1	8
2	7
3	6
4	5
5	4

- rank = # of pivot columns.
- any choice of pivot columns determines a unique reduced echelon form.

$$\Rightarrow \binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\binom{9}{4}+\binom{9}{5}$$

$$= 1 + 9 + \frac{9\cdot 8}{2} + \frac{9\cdot 8\cdot 7}{3\cdot 2} + 2\cdot\frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2}$$

$$= 10 + 36 + 3\cdot 4\cdot 7 + 9\cdot 4\cdot 7$$

$$= 46 + 84 + 252$$

$$= 130 + 252 = \boxed{382}$$

$$\begin{array}{r} 12 \\ \times 7 \\ \hline 84 \end{array} \qquad \begin{array}{r} 36 \\ \times 7 \\ \hline 252 \end{array}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{"n choose k"}}$$