University of Utah — Spring 2024

MATH 2270-002 Handout: LU Type Factorizations

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By definition an LU type factorization of an $m \times n$ matrix A is a factorization of the form

$$A = \Phi_\lambda \Phi_{\lambda-1} \cdots \Phi_1 \widehat{A} \Psi_1 \Psi_2 \cdots \Psi_\rho,$$

where $\widehat{A}$ is a "reduced form" of A and each Φ_i and Ψ_j is an invertible matrix. Typically one also requires that each Φ_i comes from a matrix group G_i, each Ψ_j comes from a matrix group H_j, and no two groups among G_i and H_j repeat. This is a broad, high-level definition: one can interpret it as saying that "A is the same as $\widehat{A}$ up to a change of coordinates", the coordinates being specified by the groups used. In practice in different disciplines certain phrases mean specific factorizations with varying scope.

The main idea is always to package elementary matrices corresponding to elementary operations into groups. The Φ_i's correspond to row operations, whereas Ψ_j's correspond to column operations.

Here is an example. Instead of starting from a matrix A to factorize, I'll instead tell you how one can come up with an informative example: start with a "reduced form", and then pre- and post-process it via row and column operations to hide the reduced form[1]. Consider 3×4 matrices. I'll use

$$\widehat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

One can call this form a "doubly reduced echelon form", since not only are the rows cleared as much as possible, but also the columns are cleared as much as possible[2]. Now let's do row and column operations to get rid of some of the 0's:

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\widehat{A}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{Q}$$

$$= PLD\widehat{A}UQ = \begin{pmatrix} 0 & 2 & 2 & 5 \\ 2 & 0 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{pmatrix}.$$

Here

- D is from the group of 3×3 invertible diagonal matrices,
- L is from the group of 3×3 lower unitriangular matrices,

[1] It's easier to multiply then factorize.

[2] Note that doing "column operations" on B is the same as doing row operations on B^T.

- P is from the group of 3×3 permutation matrices,
- U is from the group of 4×4 upper unitriangular matrices,
- Q is from the group of 4×4 permutatiom matrices.

Further, the matrices P, L, D to the left of $\widehat{A}$ correspond to (inverses) of row operations on A, and the matrices U, Q to the right of $\widehat{A}$ correspond to (inverses) of column operations on A. Moreover, if we wanted instead of using a diagonal to the left of $\widehat{A}$ we could have used a diagonal to the right of $\widehat{A}$ (due to syntax reasons this diagonal would now need to be 4×4)[3].

Say now we don't know the above factorization of A, and we want to come up with an LU type factorization of it. One first needs to determine what type of factorization one wants to end up with. For instance 3×3 invertible lower triangular matrices too is a group (it's a group larger than both the group of lower unitriangulars and diagonals). So the fundamental tradeoff is to use either a small number of large groups, or a large number of small groups.

As a first attempt let's say we want to factorize $A = LU$, where L lower unitriangular and U is a row echelon form of A. In order for this to work, one would need to be able to row reduce A "without going up", that is, one is only allowed to use a row R_i to replace a row R_j with $R_j + cR_i$ for $j > i$. But the top left entry of A is 0, so there is no way to simplify the first column by using the first row only.

Thus we ask for the next best thing; say we want to now factorize $A = PLU$, where P is a permutation, L is lower triangular and U is a row echelon form of A. Sending the row operation matrices to the other side, we have $L^{-1}P^{-1}A = U$. So for this factorization we need to be able to permute the rows of A, and then never permute them

[3] Though in certain contexts it's also of interest to use nonsquare, hence noninvertible, matrices as factors.

again, while row reducing "without going up". But if we saw A for the first time, we don't know what the correct initial permutation is. To figure this out, we can first compute a row echelon form of A without worrying about the factorization; record the pivot entries, and then track them back to the entries of A they came from. Once we know the origins of the pivot entries in the row echelon form of A, we can do an initial permutation to put the rows in the correct order, and then do row reduction "without going up".

Here is how this can be done. First the computation of row echelon form of A:

$$A = \begin{pmatrix} 0 & 2 & 2 & 5 \\ 2 & 0 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{pmatrix} \xrightarrow{\text{scale}} \begin{pmatrix} 0 & 2 & 2 & 5 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}$$

$$\xrightarrow{\text{upper+scale}} \begin{pmatrix} 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{perm}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Mark the pivots of echelon form we obtained:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \boxed{3} \end{pmatrix}$$

Tracking them back to A in the above row reduction. we obtain

$$A = \begin{pmatrix} 0 & 2 & 2 & \boxed{5} \\ 2 & 0 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{pmatrix}$$

Now that we know which row should end up where, we can start the row reduction with a permutation.

$$A = \begin{pmatrix} 0 & 2 & 2 & \boxed{5} \\ 2 & 0 & 2 & 0 \\ 2 & -1 & 1 & -1 \end{pmatrix} \xrightarrow{\text{perm}} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 2 & -1 & 1 & -1 \\ 0 & 2 & 2 & \boxed{5} \end{pmatrix}$$

$$\xrightarrow{\text{lower}} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\text{lower}} \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Thus for $\widehat{A}$ this particular row echelon form of A, we have

$$\text{lower perm } A = \widehat{A}.$$

Isolating A we thus have $A = PL\widehat{A}$; a "permuted LU factorization".

One can of course go further and further column reduce for instance; the idea is the same. To recover the original factorization $A = PLD\widehat{A}UQ$, one needs to order the columns in the beginning and then column reduce "without going left"; since we know a row echelon form of A again one can accomplish this by permuting the third and fourth columns as the first step of column reduction.

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