University of Utah Spring 2024

MATH 2270-002 PSet 1 Specification

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Subject to Change; Last Updated: 2024-03-11 13:09:12-06:00

1 Background

This problem set focuses on elementary row operations, echelon forms, vectors and the related concepts of linear combinations and linear spans. While the majority of the problem set aims to develop the basic computational skills, there are also more applications- and geometry-oriented problems.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

For instance problem 5 below is based on Exercise 4 of §1.3 in the book. The book has the answer to Exercise 3 of the same section, which is of the same form as Exercise 4 but is with different vectors. Going forward, and possibly for the rest of this problem set, such correspondences will not be mentioned; it'll be up to you to locate them (if need be)!

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

1. Consider the following three systems of linear equations:

I:

$$\begin{aligned} x_2 + 4x_3 &= -4 \\ x_1 + 3x_2 + 3x_3 &= -2 \\ 3x_1 + 7x_2 + 5x_3 &= 6 \end{aligned}$$

II:

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

III:

$$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

For each of these systems, perform the following tasks:

(a) Solve the system.

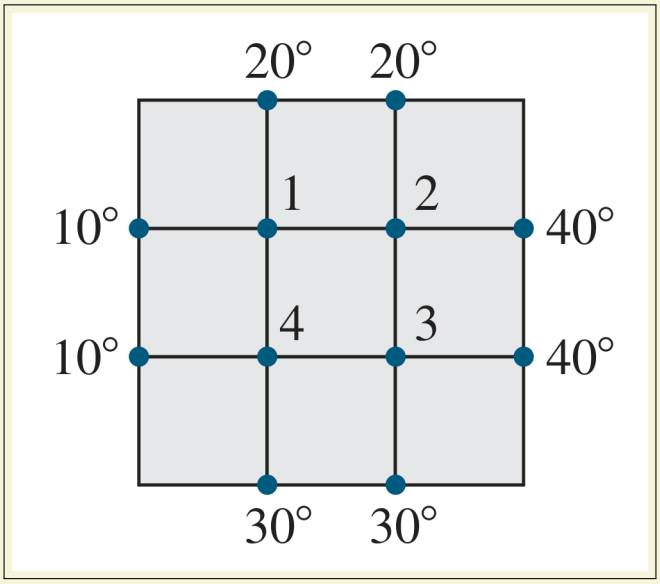
(b) Verify that your solution is correct by plugging the values back into the original system.

2. Suppose a, b, c, and d are constants such that a is not zero and the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d?

$$ax_1 + bx_2 = f$$
$$cx_1 + dx_2 = g$$

3. The heat equation is a partial differential equation that fundamentally relates "spatial acceleration" to temporal velocity. In this regard, time-independent solutions, if they exist, are important in the study of heat diffusion. Time-independent solutions are also called steady-state distributions. Further, to make the heat equation more amenable to computational approaches, one often discretizes the equation via a finite element method. In this problem your job is to apply linear algebra to find the (discretized) steady-state solution in a particular heat transfer setup.

   Consider the thin plate below along whose boundary the temperature distribution is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate.



Let T_1, T_2, T_3 and T_4 denote the temperatures at the four interior nodes of the mesh in the figure. Let's stipulate that the temperature at a node is (approximately) equal to the average of the four nearest nodes—to the left, above, to the right, and below. For instance, the equation for node 1 is:

$$T_1 = \frac{10 + 20 + T_2 + T_4}{4}.$$

(a) Write a system of four linear equations whose solution gives estimates for the temperatures T_1, T_2, T_3, T_4.

(b) Solve the system of equations.

(c) Order T_1, T_2, T_3, T_4. Which of these four temperature is the largest? Could one have guessed this without doing any calculations?

4. List all possible echelon forms of a nonzero 3×5 matrix. Use the symbol ■ for an anonymous nonzero number and the symbol ★ for an anonymous number that is possibly zero.

5. Consider the following two vectors in $\mathbb{R}^2$:

$$u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Display each of the following vectors using arrows on the xy-plane:

(a) u

(b) v

(c) $-v$

(d) $-2v$

(e) $3u + v$

(f) $u + 2v - u$

(g) $3u - v$

(h) $3u - 2v$

(i) $u + v - 2u - 4v + u$

6. Let $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$. Denote the columns of A by a_1, a_2, a_3, and let $W = \text{span}\{a_1, a_2, a_3\}$.

(a) Is b in $\{a_1, a_2, a_3\}$? How many vectors are in $\{a_1, a_2, a_3\}$?

(b) Is b in W? How many vectors are in W?

(c) Is a_1 is in W?

(d) Is $W = \mathbb{R}^3$?

7. Let

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Describe the set of all b for which the equation $Ax = b$ does not have a solution.

8. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{pmatrix}$$

   (a) How many rows of A contain a pivot position?

   (b) Does the equation $Ax = y$ have a solution for each y in $\mathbb{R}^4$?

   (c) Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A above? If the answer is no, give an example of a vector in $\mathbb{R}^4$ that can not be written as such a linear combination.

   (d) Do the columns of A span $\mathbb{R}^4$?

9. Consider the following two vectors in $\mathbb{R}^2$:

$$u = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \qquad v = \begin{pmatrix} -4 \\ 9 \end{pmatrix}.$$

   (a) What is the equation of the line through u parallel to v?

   (b) What is the equation of the line through v parallel to u?

10. Consider the following two vectors in $\mathbb{R}^2$:

$$p = \begin{pmatrix} -6 \\ 3 \end{pmatrix}, \qquad q = \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

What is the equation of the line that passes through both p and q?

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

> Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/y3ZEPXvbbDL5XSQc8.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866300,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on January 19, 2024 at 11:59 PM.

As per the course's syllabus, late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.