University of Utah Spring 2024

MATH 2270-002 PSet 7 Specification

Instructor: Alp Uzman

Subject to Change; Last Updated: 2024-03-20 21:57:50-06:00

1 Background

This problem set focuses on the concept of linear coordinates and elementary digital signal processing (DSP). Linear coordinates, established by bases, facilitate a practical interpretation of:

- (potentially) abstract vectors as concrete column matrices of appropriate sizes, and
- (potentially) abstract linear transformations as concrete matrices of suitable sizes.

Linear coordinates are specific isomorphisms and by applying them in series one can construct isomorphisms between seemingly different finite dimensional vector spaces, insofar as the vector spaces in question have the same dimension.

In the realm of digital signal processing, signals form a vector space where the true utility of linear algebra emerges through the shift operator ρ. This operator models time progression, advancing each signal by one time unit. Within DSP, a "filter" is essentially a (possibly nonlinear, possibly nondeterministic) transformation. Notably, LTI (Linear Time-Invariant) filters—linear transformations that commute with the shift operator—are among the most significant types of filters, impacting how signals are processed and analyzed.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the following four vectors in $\mathbb{R}^3$:

$$(1,-8,6), \quad (2,-3,0), \quad (2,-5,7), \quad (3,9,-4)$$

(a) List all distinct bases of $\mathbb{R}^3$ that can be constructed using some of the above vectors.

(b) For each basis in your list, write the matrix representation of the remaining vector(s) relative to that basis.

2. Let $\mathcal{P}_2$ be the space of polynomial functions of degree at most 2.

(a) Verify that

$$\beta = (\, 1 - t^2 \,,\, t - t^2 \,,\, 2 - 2t + t^2 \,).$$

is a basis of $\mathcal{P}_2$.

(b) Compute the matrix representation of the vector

$$p(t) = 3 + t - 6t^2$$

relative to the basis β.

(c) Construct another basis γ of $\mathcal{P}_2$ such that no vector in β is in γ.

(d) Compute the matrix representation of the vector

$$p(t) = 3 + t - 6t^2$$

relative to the basis γ.

(e) Compute the change-of-basis matrix from β to γ.

(f) Compute the change-of-basis matrix from γ to β.

(g) Compute the matrix representation of differentiation $D : \mathcal{P}_2 \to \mathcal{P}_2$ with β as both the input basis and the output basis.

(h) Compute the matrix representation of twice-differentiation $\mathrm{D}^2 : \mathcal{P}_2 \to \mathcal{P}_2$ with β as the input basis and γ as the output basis.

3. Let V be a vector space and let $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$ be two bases of V. Suppose the vectors in β and γ are related as follows:

$$\beta_1 = -\gamma_1 + 4\gamma_2, \qquad \beta_2 = 5\gamma_1 - 3\gamma_2.$$

(a) Compute the dimension of V.

(b) Let $v = 5\beta_1 + 3\beta_2$.

i. Compute the matrix representation of v relative to β.

ii. Compute the matrix representation of v relative to γ.

(c) Let $v = 5\beta_1 + 3\gamma_2$.

i. Compute the matrix representation of v relative to β.

ii. Compute the matrix representation of v relative to γ.

(d) Compute the change-of-coordinates matrix from β to γ.

(e) Compute the change-of-coordinates matrix from γ to β.

(f) Construct an isomorphism from V to the space $\mathbb{R}^2$ of ordered pairs real numbers.

(g) Construct an isomorphism from V to that subspace of the function space $\mathcal{F}$ that is spanned by sin(t) and cos(t).

(h) Construct an isomorphism from V to a subspace of the space of polynomial functions $\mathcal{P}$.

(i) Construct an isomorphism from V to a subspace of the space of 2×3 matrices.

(j) Construct an isomorphism from V to a subspace of the space $\mathcal{L}(\mathcal{F};\mathcal{F})$ of the linear transformations from function space to itself.

4. Let $\mathcal{S} = \mathcal{F}(\mathbb{Z};\mathbb{R})$ be the space of signals and consider the filter $A : \mathcal{S} \to \mathcal{S}$ defined by:

$$A(x)(n) = \frac{x(n-2) + x(n-1) + x(n)}{3}.$$

(a) Write A as a linear combination of powers of the shift operator ρ.

(b) Verify that A is an LTI filter.

(c) Find a nonzero signal in the kernel ker(A) of A.

(d) Find a basis for ker(A).

(e) Compute the dimension of ker(A).

(f) Find a nonzero signal in the image of A.

(g) Find a two dimensional subspace of the image im(A) of A.

(h) Compute the dimension of im(A).

(i) Is A invertible? If it is, compute A^{-1}. If A is not invertible, find a signal that is not in the image of A. It is possible that there is no such signal! [1]

Challenge Let $\mathcal{V} \subseteq \mathcal{S}$ be a finite dimensional subspace of the space of signals. Find a function $f = f_{\mathcal{V}}$ such that for any positive integer n

[1] Note that while a linear transformation from a finite dimensional vector space to itself is one-to-one if and only if onto if and only if invertible, for infinite dimensional vector spaces (such as the signal space $\mathcal{S}$), a transformation may be onto but still not invertible!

$$1 \leq \frac{\dim(A^n(\mathcal{V}))}{f(n)} \leq C$$

for some number C that is independent of n, where A is the above moving average filter.

Challenge Let $\mathcal{V} \subseteq \mathcal{S}$ be a finite dimensional subspace of the space of signals. Find two functions $l = l_{\mathcal{V}}$ and $u = u_{\mathcal{V}}$ such that for any positive integer n, no signal in $A^n(\mathcal{V})$ has an entry smaller than $l(n)$ and larger than $u(n)$. Here again A is the above moving average filter.

Challenge

(a) Give an example of a linear filter that is not time-invariant.

(b) Give an example of a time-invariant filter that is not linear.

(c) Describe the space of all LTI filters.

Challenge

(a) Give an example of an affine time-invariant (ATI) filter that is not linear.

(b) Describe the space of all ATI filters.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.

- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

  > Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

  Here are some guides regarding prompt engineering:

  – https://platform.openai.com/docs/guides/prompt-engineering
  – https://developers.google.com/machine-learning/resources/prompt-eng
  – https://www.ibm.com/topics/prompt-engineering
  – https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/ZxVJFzjmF8MouaBx8.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866306,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on March 22, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.