University of Utah Spring 2024

# MATH 2270-002 PSet 10 Specification

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## 1 Background

This problem set explores two interconnected areas of linear algebra: the spectral theorem for symmetric matrices and the theory of linear dynamical systems.

The spectral theorem plays a pivotal role in understanding symmetric matrices. It states that every symmetric matrix possesses an eigenbasis consisting of orthonormal vectors. This theorem not only facilitates the diagonalization of symmetric matrices but also underpins the principal axis theorem for quadratic forms and the singular value decomposition of arbitrary matrices. These tools are crucial for decomposing matrices in ways that reveal their intrinsic geometric and algebraic properties.

On the other side of our exploration are linear dynamical systems, which focus on the evolution of systems over time through linear transformations. Despite the prevalence of more complex, nonlinear models in describing natural phenomena, linear systems remain fundamental due to their tractability. In the context of linear dynamics, advancing time corresponds to successive applications of a matrix, or taking its powers, thus making the process of diagonalization and the study of eigenvalues critically important. Diagonalizing a matrix simplifies the computation of its powers, thereby providing a clearer understanding of the system's behavior as time progresses.

## 2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the following quadratic forms:

   **I**: $q(x, y) = 3y^2 + 8xy - 3y^2$

   **II**: $q(x, y) = 4x^2 - 8xy - 2y^2$

**III**: $q(x, y) = -2x^2 - 4xy - 2y^2$

**IV**: $q(x, y, z) = 7x^2 + 5y^2 + 9z^2 - 8xy + 8xz$

For each of the given quadratic forms, perform the following tasks:

(a) Write the symmetric matrix corresponding to the quadratic form.

(b) Classify the quadratic form as positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite.

(c) Find an orthogonal change of coordinates $Q$ such that relative to the new coordinates the quadratic form has no mixed terms.

2. Compute the singular value decomposition of the following matrix:

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}.$$

3. Consider the matrix

$$A = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}.$$

(a) Compute the singular value decomposition of $A$.

(b) Prof. Gil Strang works through this calculation in the video

https://www.youtube.com/watch?v=TX_vooSnhm8&t=710s.

He realizes, however, that his calculation turns out to be incorrect. Find the cause of his calculation error and explain it.

4. **For this problem, the use of a computational tool such as WolframAlpha is allowed and recommended. As an example, the output WolframAlpha produces when the input is the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ can be seen at**

   **https://tinyurl.com/2jh7ut7e.**

   Consider a predator-prey population consisting of the foxes and rabbits living in a certain forest. Initially, there are F(0) foxes and R(0) rabbits; after n months, there are F(n) foxes and R(0) rabbits. We assume that the transition from each month to the next is described by the equations

$$\begin{aligned} F(n+1) &= 0.6\,F(n) + 0.5\,R(n) \\ R(n+1) &= -r\,F(n) + 1.2\,R(n), \end{aligned}$$

   where the predation parameter $r \geq 0$ is the "capture rate" representing the average number of rabbits consumed monthly by each fox.

   (a) If r = 0.16, compute that in the long term the populations of foxes and rabbits stabilize, with 5 foxes for each 4 rabbits.

   (b) If r = 0.175, compute that in the long term the populations of foxes and rabbits both die out.

   (c) If r = 0.135, compute that in the long term the fox and rabbit populations both increase at the rate of 5% per month, maintaining a constant ratio of 10 foxes for each 9 rabbits.

(d) Find all critical values of r such that the behavior of the system changes qualitatively. For the purposes of this problem, the following are among a priori possible qualitatively distinct behaviors:

- mutual population explosion
- mutual extinction
- equilibrium with coexistence
- extinction for one species and population explosion for the other one
- equilibrium with one species

5. One way to generate the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, ...$$

is by considering the following simple population growth model:

- One starts with one pair of juvenile rabbits.
- It takes one month for a pair of juvenile rabbits to become a pair of adult rabbits.
- Any pair of adult rabbits begets a new pair of juvenile rabbits each month.
- Rabbits are immortal.
- Any pair of rabbits always have the appropriate reproductive capacity.
- A rabbit reproduces only with the rabbit that its paired with.

Thus if j represents a juvenile pair and a represents an adult pair,

$$\begin{tikzcd}
n = 0 & n = 1 & n = 2 & n = 3 & n = 4 & \cdots \\
j \arrow[r] & a \arrow[r] \arrow[dr] & a \arrow[r] \arrow[dr] & a \arrow[r] \arrow[dr] & a & \cdots \\
& & j \arrow[dr] & j \arrow[dr] & j & \cdots \\
& & & a \arrow[dr] \arrow[ddr] & a & \cdots \\
& & & & j & \cdots \\
& & & & a & \cdots
\end{tikzcd}$$

Denote by j(n) the number of juvenile pairs of rabbits in month n, by a(n) the number of adult pairs of rabbits in month n, and by $f(n) = j(n) + a(n)$ be the total number of pairs of rabbits in month n.

Perform the following tasks:

(a) Write the stage-matrix A of the population growth model.

(b) Verify that f(n) satisfies the difference equation

$$f(n + 2) = f(n + 1) + f(n).$$

(c) Verify that every third Fibonacci number is even. More precisely for any integer k, f(3k) is an integer divisible by 2.

(d) Compute the eigenvalues and eigenvectors of the stage-matrix A, and diagonalize or orthogonally diagonalize it if possible.

(e) Verify that for any integer n, the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

is an integer by verifying that it is an explicit formula for the nth Fibonacci number.

(f) Verify that Fibonacci's model predicts that there are over 2.5 trillion rabbit pairs after 5 years.

6. **For this problem, the use of a computational tool such as WolframAlpha is allowed and recommended.** In this problem your job is to study a population growth model that is a variant of Fibonacci's model discussed in the previous problem. Start with Fibonacci's model and change only the assumption that rabbits are immortal to:

   - Each rabbit lives exactly 1 year.
   - Any pair of rabbits that are born together die together.

   Perform the following tasks:

   (a) Write the stage-matrix A of the population growth model.

   (b) Compute the exact number of alive pairs of rabbits are there after 3 years.

7. Suppose you have n dollars and can buy a cup of coffee for \$1, a cup of soda for \$2, and a cup of orange juice for \$2. Let C(n) be the number of different ways of spending all your money. Here the order of purchase matters: two cups of the same type are indistinguishable, while two cups of different types are distinguishable Thus for instance C(3) = 5 since either one can buy

   - three cups of coffee, or
   - first one cup of coffee and then one cup of soda, or

- first one cup of soda and then one cup of coffee, or
- first one cup of coffee and then one cup of orange juice, or
- first one cup of orange juice and then one cup of coffee.

Perform the following tasks:

(a) Verify that $C(n)$ satisfies the difference equation

$$C(n) = C(n-1) + 2C(n-2).$$

(b) Compute $C(200)$; this is the number of different ways of spending the budget of \$200 buying cups of coffee, soda, and orange juice, for the purposes of this problem.

**Challenge** Let $\widetilde{C}(n)$ be the number of different ways of spending all your money, where now the order does not matter. Thus now $\widetilde{C}(3) = 3$. Find an explicit formula for $\widetilde{C}(n)$.

# 3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

## 3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

## 3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff

will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

> Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

# 4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/UykrzZrK3AP47nsM8.

You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866309,

  see the Gradescope documentation for instructions.

## 5 When to Submit

This problem set is due on April 19, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.