University of Utah Spring 2024

MATH 2270-002 PSet 8 Specification

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Subject to Change; Last Updated: 2024-03-29 16:12:04-06:00

1 Background

This problem set explores spectral theory, which studies the eigenequation

$$Av = \lambda v,$$

for a square matrix A or more generally a linear transformation from a vector space to itself. Here, v is a nonzero eigenvector, and λ is its corresponding eigenvalue, indicating that matrix multiplication by A can be simplified to scalar multiplication by λ, but only in the direction of v.

When A has enough eigenvectors to constitute a basis for the vector space it operates on, it can be thought of as a diagonal matrix. Diagonalization simplifies many operations, as all off-diagonal entries of a diagonal matrix are zero; hence it is crucial to be able to recognize the matrices that are diagonalizable, and diagonalize them accordingly.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

   Perform the following tasks:

   (a) Compute the eigenvalues of A.

   (b) Sketch the spectrum of A as a subset of the complex plane.

(c) For each eigenvalue of A, compute a basis for the associated eigenspace.

(d) Compute the algebraic and geometric multiplicities of each distinct eigenvalue of A.

(e) Determine if A is diagonalizable. If A is diagonalizable, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

2. Consider the family of matrices

$$A_\alpha = \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix},$$

where α is a parameter. Perform the following tasks. It is likely that your final answers to each part will depend on the parameter α.

(a) Compute the eigenvalues of A_α.

(b) Sketch how the spectrum of A_α changes as one varies the parameter α.

(c) For each eigenvalue of A_α, compute a basis for the associated eigenspace.

(d) Sketch each eigenspace of A_α.

(e) Compute the algebraic and geometric multiplicities of each distinct eigenvalue of A_α.

(f) For which values of α is A_α diagonalizable?

(g) For any parameter α such that A_α is diagonalizable, diagonalize A_α.

3. Consider the following family of matrices:

$$S_{p,q} = \begin{pmatrix} p & 1-q \\ 1-p & q \end{pmatrix}.$$

parameterized by two numbers $0 \le p \le 1$ and $0 \le q \le 1$.

Perform the following tasks. It is likely that your final answers to each part will depend on the parameters p, q.

   (a) Compute the eigenvalues of $S_{p,q}$.
   (b) For each eigenvalue of $S_{p,q}$, compute a basis for the associated eigenspace.
   (c) Sketch each eigenspace of $S_{p,q}$.
   (d) Compute the algebraic and geometric multiplicities of each distinct eigenvalue of $S_{p,q}$.
   (e) For which values of p and q is $S_{p,q}$ diagonalizable?
   (f) For any parameters p, q such that $S_{p,q}$ is diagonalizable, diagonalize $S_{p,q}$.

4. (a) Verify that for any permutation matrix P, the vector of ones each of whose entries is 1 is an eigenvector associated to the eigenvalue 1.
   (b) Let A be an arbitrary square matrix. The ith row sum of A is by definition the sum of all the entries in the ith row of A. Verify that if each row sum of A is equal to the number r, then r is an eigenvalue of A.
   (c) Similarly, for A again an arbitrary square matrix, the jth column sum of A is by definition the sum of all the entries in the jth column of A. Verify that if each column sum of A is equal to the number c, then c is an eigenvalue of A.

Challenge Verify that any eigenvalue of any permutation matrix has modulus 1. More precisely, if P is a permutation matrix and if $\lambda = a + ib$ is an eigenvalue of P, then $a^2 + b^2 = 1$.

Challenge

(a) Verify that taking the ith row sum is a linear transformation on any space of matrices. Similarly verify that taking the jth column sum is a linear transformation on any space of matrices.

(b) Verify that the following is an isomorphism of vector spaces:

$$\Phi : \mathcal{M}(n \times n; \mathbb{R}) \to \mathcal{L}(\mathcal{M}(n \times n; \mathbb{R}); \mathbb{R}), \quad \Phi(A)(X) = \mathrm{tr}(AX).$$

(c) Find the $n \times n$ matrix that corresponds to taking the ith row sum under the isomorphism Φ. Similarly find the $n \times n$ matrix that corresponds to taking the jth column sum under the isomorphism Φ.

5. Consider the following diagonalizable matrix:

$$A = \begin{pmatrix} 15 & -36 \\ 6 & -15 \end{pmatrix}.$$

Perform the following tasks:

(a) Diagonalize A by finding a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

(b) Compute A^{500}. You may use the symbol ▲ to denote the integer 3^{500}.

6. In each part of this problem, A is a square matrix with real entries. Your job is to determine, based on the spectral information provided, the type of A among the following three types:

   I: definitely diagonalizable, or

   II: definitely nondiagonalizable, or

   III: possibly diagonalizable and possibly nondiagonalizable.

   Further, if A is of type III, give two examples for A with the matching spectral data such that the first example is diagonalizable and the second example is not diagonalizable.

   (a) A is 5×5 with exactly two distinct eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional.

   (b) A is 3×3 with exactly two distinct eigenvalues. Each eigenspace is one-dimensional.

   (c) A is 7×7 with exactly three distinct eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional.

   (d) A is 4×4 with at least two distinct eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional.

7. Let $\mathcal{S}$ be the space of signals and consider the moving average filter $A : \mathcal{S} \to \mathcal{S}$ defined by:

$$A(x)(n) = \frac{x(n-2) + x(n-1) + x(n)}{3}.$$

(a) Find all eigenvalues of A.

(b) Sketch the spectrum of A.

(c) For each eigenvalue of A, find a basis for the associated eigenspace.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

> Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/wyhGCBjkBtvKLWB26.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866307,

  see the Gradescope documentation for instructions.

5 When to Submit

~~This problem set is due on March 29, 2024 at 11:59 PM.~~
This problem set is due on March 31, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.