University of Utah Spring 2024

MATH 2270-002 PSet 5 Specification

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Subject to Change; Last Updated: 2024-03-11 13:10:52-06:00

1 Background

This problem set explores the determinant, a key concept in linear algebra that bridges algebraic computations and geometric interpretations. The set is divided into two main focuses: calculating determinants and understanding their properties. These properties are not only theoretically interesting but also practically valuable, as they can greatly simplify the computation of determinants.

At the heart of the determinant's algebraic properties is its multiplicative nature:

$$\det(AB) = \det(A)\det(B).$$

This property allows for the application of elementary row operations to simplify a square matrix before calculating its determinant, with careful attention to how these operations affect the determinant's value.

Geometrically, the determinant quantifies the rate at which a square matrix distorts lengths in one dimension, areas in two dimensions, and volumes in three (or more) dimensions. This property is crucial for volume computations and reveals the determinant's role in understanding the spatial effects of linear transformations.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Compute the following determinants.

   (a) $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 7 & 6 & 8 & 4 \\ 0 & 0 & 0 & 6 \\ 8 & 7 & 9 & 3 \\ 0 & 4 & 0 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 6 & 0 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

2. Suppose $a, b, c, d, e, f, g, h, i$ are nine real numbers such that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute each of the following determinants.

(a) $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix}$

(b) $\begin{vmatrix} a & b & c \\ 8d & 8e & 8f \\ g & h & i \end{vmatrix}$

(c) $\begin{vmatrix} -g & -h & -i \\ a & b & c \\ d & e & f \end{vmatrix}$

(d) $\begin{vmatrix} b & e & h \\ a & d & g \\ c & f & i \end{vmatrix}$

3. Verify the formula

$$\det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{pmatrix} = (b-a)(c-a)(c-b).$$

by computing the determinant on the LHS and expanding the product on the RHS.

4. Consider a basis

$$\beta = (\beta_1, \beta_2, ..., \beta_n) \in \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n\text{ times}}$$

and define B to be the $n \times n$ matrix whose ith column is β_i. The basis β is called

- right-handed if $\det(B) > 0$.
- left-handed if $\det(B) < 0$.

Which of the following ordered lists of vectors form right-handed bases of $\mathbb{R}^3$?

(a)

$$\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right)$$

(b)

$$\left(\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right)$$

(c)

$$\left(\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right)$$

(d)

$$\left(\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right)$$

5. Consider the following four permutations:

I:

$$P \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ u \end{pmatrix},$$

II:

$$P \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \\ a \\ b \end{pmatrix},$$

III:

$$P \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix},$$

IV:

$$P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix}.$$

For each of the four given permutations, perform the following tasks:

(a) Write the permutation P in matrix form.

(b) Compute the determinant of P.

(c) Compute P^{-1}.

6. For each shape S in $\mathbb{R}^2$ or $\mathbb{R}^3$ listed below, compute the area of S if S is in $\mathbb{R}^2$ and compute the volume of S if S is in $\mathbb{R}^3$.

   (a) S is the parallelogram whose corners are at $(0, -2)$, $(5, -2)$, $(-3, 1)$, $(2, 1)$.

   (b) S is the image of the parallelogram determined by the vectors
   $$b_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$
   under the linear transformation
   $$A = \begin{pmatrix} 3 & -4 \\ -4 & 6 \end{pmatrix}.$$

   (c) S is the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -6)$, $(1, 3, 5)$, and $(6, 7, 0)$.

7. Consider the following family of diagonal matrices:
$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where the parameters a, b, c are positive real numbers. Let S be the unit ball in $\mathbb{R}^3$ centered at the origin; formally

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

(a) Verify that the image of B under D is

$$D(B) = \{(x, y, z) \in \mathbb{R}^3 \mid (x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1\}.$$

(b) Sketch the unit ball B and its image D(B).

(c) Compute the ratio of the volume of D(B) to the volume of B.

8. Let Q be a 3×3 orthogonal matrix; thus by definition all that is known about Q is that it satisfies the following equations:

$$Q^T Q = I = QQ^T.$$

(a) Verify that the determinant of Q is either 1 or –1.

(b) Give an example of a 3×3 orthogonal matrix Q such that

- Q is not the identity matrix, and
- det(Q) = 1.

(c) Give an example of a 3×3 orthogonal matrix Q such that

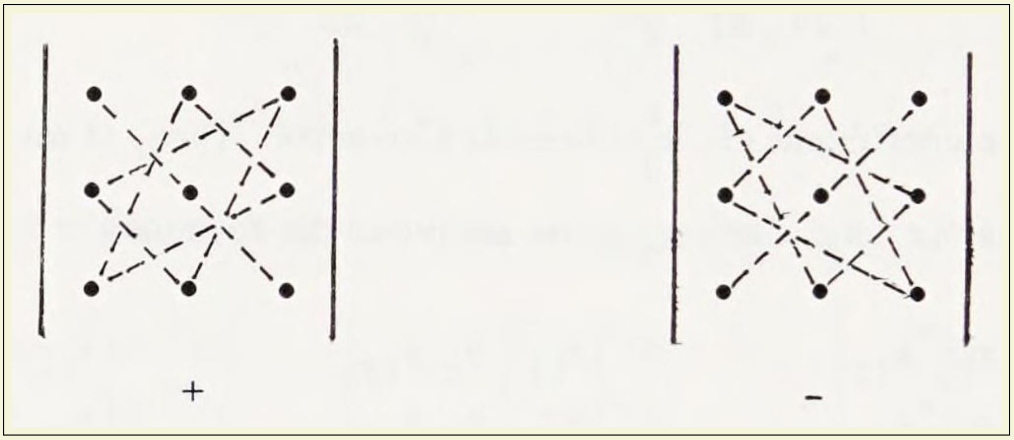
- Q is not a permutation matrix, and
- det(Q) = –1.

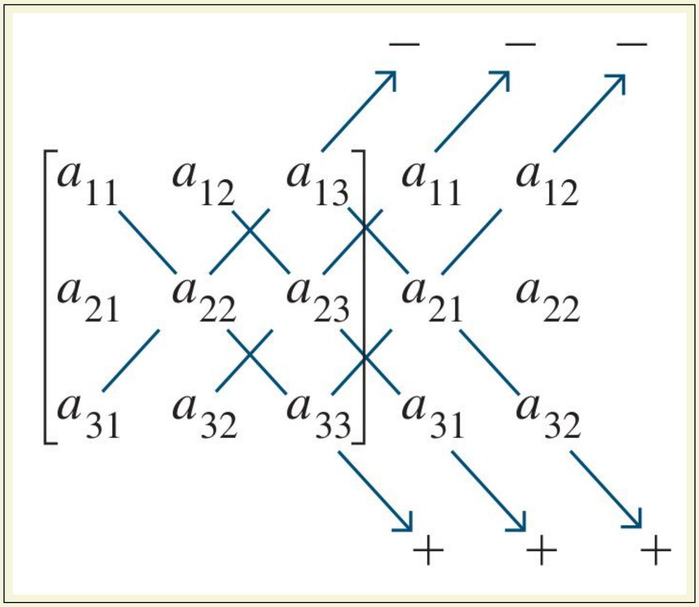
(d) Let C be the cube in $\mathbb{R}^3$ determined by the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Verify that the image of C under Q is a cube with each edge of length 1.

Challenge Generalize the above problems to $n \times n$ orthogonal matrices and C an arbitrary hypercube in $\mathbb{R}^n$.

Challenge For A a square matrix, $\det(A) = \det\left(A^T\right)$. Demonstrate this for A a 2×2 matrix via a geometric argument.

Challenge There are mnemonic devices for the computation of the determinants of 3×3 matrices, often referred to as the Rule of Sarrus. Here are two versions of it:





Construct a mnemonic device for computing determinants of 4×4 matrices.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

  > Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

Here are some guides regarding prompt engineering:

- https://platform.openai.com/docs/guides/prompt-engineering
- https://developers.google.com/machine-learning/resources/prompt-eng
- https://www.ibm.com/topics/prompt-engineering
- https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/MmvXy8fzoAZ6EhhLA.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866304/,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on February 23, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.