University of Utah Spring 2024

MATH 2270-002
Midterm 2 Questions

Instructor: Alp Uzman

March 15 2024, 8:35 AM - 9:25 AM

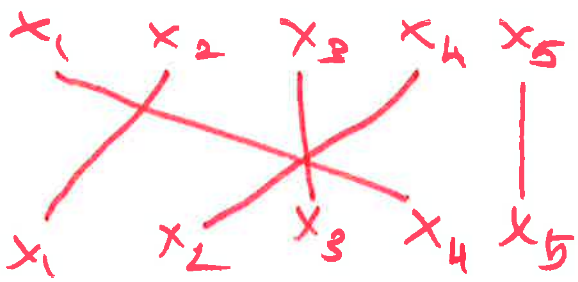
Surname:	Key
First Name:	
uNID:	

Before turning the page make sure to read and sign the exam policy document, distributed separately.

1. **[70 points]** Consider the permutation P defined by

$$P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix}.$$

(a) **[15 points]** Write the permutation P in matrix form.



$$\Rightarrow P_{5\,5} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) **[10 points]** Compute P^T.

$$P^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) **[15 points]** Compute P^{-1} if P is invertible. If P is not invertible, state that P^{-1} does not exist.

P is orthogonal $\Rightarrow P^{-1} = P^T$

(d) **[15 points]** Compute the determinant of P.

$$\det(P) = \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{1}$$

(e) **[15 points]** Compute the determinant of P^T.

$$\det(P^T) = \det(P) = \boxed{1}$$

15

2. **[points]** Let S be the parallelogram determined by the vectors

$$b_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$A = \begin{pmatrix} 3 & -4 \\ -4 & 6 \end{pmatrix}.$$

What is the area of the image of S under the linear transformation A?

$C =$ square determined by $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} -3 & -3 \\ 5 & 8 \end{pmatrix}$. $\Rightarrow S = B(C) \Rightarrow A(S) = AB(C)$,

$\Rightarrow \text{area}(A(S)) = |\det(AB)| \cdot \underbrace{\text{area}(C)}_{=1}$

$= |\det(A)|\,|\det(B)|$

$= 2 \cdot 9 = \boxed{18}$

12

3. **[[redacted] points]** Consider the following two bases of $\mathbb{R}^2$:

$$\beta = \left(\overset{\beta_1}{(7,5)}, \overset{\beta_2}{(-3,-1)}\right), \qquad \gamma = \left(\overset{\gamma_1}{(1,-5)}, \overset{\gamma_2}{(-2,2)}\right).$$

6

(a) **[[redacted] points]** Compute the change-of-basis matrix from β to γ.

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{M}_\beta(v) = \mathcal{M}_\gamma(v)}$$

$$\boxed{\begin{aligned} \beta_1 &= a\gamma_1 + a\gamma_2 \\ \beta_2 &= b\gamma_1 + d\gamma_2 \end{aligned}}$$

$$\boxed{\begin{aligned} 7 &= a(1) + c(-2) \\ 5 &= a(-5) + c\,2 \end{aligned}}$$

$$\boxed{\begin{aligned} -3 &= b(1) + d(-2) \\ -1 &= b(-5) + d\,2 \end{aligned}}$$

$$\Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \frac{1}{-8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \frac{1}{-8} \begin{pmatrix} 24 \\ 40 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}.$$

6

(b) **[[redacted] points]** Compute the change-of-basis matrix from γ to β.

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 8 \\ 16 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

change-of-basis $\gamma \leftarrow \beta$:

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}}$$

change-of-basis $\beta \leftarrow \gamma$:

$$\begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}}$$

4. **[3 point]** Let $\mathcal{M}$ be the vector space of all 2×2 matrices with real entries. What is the dimension of the space of all linear transformations from $\mathcal{M}$ to $\mathcal{M}$ that commute with the transpose operation?

standard basis of $\mathcal{M}$: $\left(\underbrace{\begin{pmatrix}1&0\\0&0\end{pmatrix}}_{\beta_1}, \underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{\beta_2}, \underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{\beta_3}, \underbrace{\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{\beta_4}\right) = \beta$

transpose op: $\mathcal{T}: \mathcal{M} \to \mathcal{M}$, $\mathcal{T}(A) = A^T$.

matrix of transpose relative to std. basis:

$$\mathcal{T}: \begin{array}{l}\beta_1 \to \beta_1\\ \beta_2 \to \beta_3\\ \beta_3 \to \beta_2\\ \beta_4 \to \beta_4\end{array} \Rightarrow \mathcal{T} \rightsquigarrow \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Want: $\mathcal{S}: \mathcal{M} \to \mathcal{M}$ such that $[\mathcal{S}, \mathcal{T}] = 0$.

$\boxed{\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S}}$

$$LHS = \begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}a_{11}&a_{13}&\cancel{a_{12}}&a_{14}\\a_{21}&a_{23}&a_{22}&a_{24}\\\cancel{a_{31}}&\cancel{a_{33}}&a_{32}&\cancel{a_{34}}\\a_{41}&a_{43}&\cancel{a_{42}}&a_{44}\end{pmatrix}$$

$$RHS = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix} = \begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}$$

$$\boxed{\begin{array}{lll}a_{13} = a_{12} & a_{23} = a_{32} & a_{24} = a_{34}\\ a_{21} = a_{31} & a_{22} = a_{33} & a_{43} = a_{42}\end{array}} \Rightarrow 16 - 6 = \boxed{10}$$