University of Utah Spring 2024

MATH 2270-002 PSet 6 Specification

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Subject to Change; Last Updated: 2024-03-11 13:11:17-06:00

1 Background

This problem set is designed to expand your understanding of vector spaces and linear transformations, with a particular emphasis on practical applications of linear algebra rather than purely abstract concepts. For example, we will explore partial fraction decompositions within the realm of rational functions—an essential technique for integration. Additionally, we'll delve into affine graph transformations of single-variable real-valued functions and examine linear transformations of matrices.

2 What to Submit

Submit your detailed solutions to each of the problems below. Though they may seem long, the additional text is meant to guide you by providing further context.

When documenting your solutions, be thorough. Your goal is not just to find the answer, but to create a clear, logical pathway to it that you or anyone else could follow in the future. It is likely that the textbook has the answers to some problems that are similar to some problems in this problem set; without further notice you may refer to these answers and reverse engineer them.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your response clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

Some of the problems have "challenge" parts; you may attempt these parts if you feel comfortable with the rest of the problems. Challenge questions will not be graded, however you are more than welcome to reach out to staff to discuss them!

1. Let V be a vector space and u, v, w be three vectors.

   (a) Verify that linear independence of $\{u, v\}$ is equivalent to the linear independence of $\{u + v, u - v\}$. More explicitly,

      i. Verify that if u and v are linearly independent, then so are $u + v$ and $u - v$.

      ii. Verify that if $u + v$ and $u - v$ are linearly independent, then so are u and v.

   (b) Verify that linear independence of $\{u, v, w\}$ is equivalent to the linear independence of $\{u + v, v + w, w + u\}$.

2. For each of the following sets of functions, determine whether the set is linearly independent or linearly dependent.

   (a) $f(x) = e^x \sin x, g(x) = e^x \cos x$

   (b) $f(x) = 2 \cos x + 3 \sin x, g(x) = 3 \cos x - 2 \sin x$

   (c) $f(x) = 1, g(x) = x, h(x) = x^2$

(d) $f(x) = e^x, g(x) = e^{2x}, h(x) = e^{3x}$

3. Find the constants denoted by capital letters in each of the following partial fraction decompositions.

(a)
$$\frac{x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

(b)
$$\frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

(c)
$$\frac{2x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

(d)
$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

4. Let $\mathcal{F} = \mathcal{F}(\mathbb{R};\mathbb{R})$ be the vector space of all single variable, real valued functions. Consider the following family of transforms:

$$\mathcal{G} = \mathcal{G}_{a,b,c,d} : \mathcal{F} \to \mathcal{F}, \qquad \mathcal{G}(f)(t) = c\,f(at+b)+d,$$

where the parameters a, b, c, d are real numbers.

(a) Verify that for any a, b, c, d, $\mathcal{G}_{a,b,c,d}$ is an affine transformation.

(b) Verify that if $d = 0$, then $\mathcal{G}$ is a linear transformation.

(c) Verify that if $\mathcal{G}_{a,b,c,d}$ is a linear transformation, then $d = 0$.

(d) For which values of a, b, c, d is $\mathcal{G}_{a,b,c,d}$ an invertible linear transformation of $\mathcal{F}$?

Challenge Fix three functions $\alpha, \beta, \gamma \in \mathcal{F}$. Verify that $\mathcal{G}_{\alpha,\beta,\gamma}$ defined by

$$\mathcal{G}_{\alpha,\beta,\gamma}(f)(t) = \beta(t)\; f(\,\alpha(t)\,) + \gamma(t)$$

is an affine transformation.

Challenge Let A : $\mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation of the plane. For f : $\mathbb{R} \to \mathbb{R}$ an arbitrary function, one says that the graph transform of f by A exists if there is a function g : $\mathbb{R} \to \mathbb{R}$ such that the graph of g is the image of the graph of f under A; more concisely

$$A(\,\mathrm{graph}(f)\,) = \mathrm{graph}(g).$$

(a) Verify that for the affine transformation

$$A_{a,b,c,d}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix}$$

one has

$$A_{a,b,c,d}(\,\mathrm{graph}(f)\,) = \mathrm{graph}(\,\mathcal{G}_{a,b,c,d}(f)\,),$$

where $\mathcal{G}_{a,b,c,d}(f)(t) = c\,f(at + b) + d$.

(b) Find conditions on A such that the set of functions whose graph transform by A exists contains a two dimensional linear subspace of functions.

5. Consider the following functions:

I:

$$f(t) = \begin{cases} 1/2 & \text{, if } 0 \leq t \leq 2 \\ -1 & \text{, if } -1 \leq t < 0 \\ 0 & \text{, else} \end{cases}$$

II:

$$f(t) = e^{2t}$$

III:

$$f(t) = \cos(3t) + |t|$$

IV:

$$f(t) = \begin{cases} e^{2t} + \cos(3t) + 2|t| + 1/2 & \text{, if } 0 \leq t \leq 2 \\ e^{2t} + \cos(3t) + 2|t| - 1 & \text{, if } -1 \leq t < 0 \\ e^{2t} + \cos(3t) + 2|t| & \text{, else} \end{cases}$$

Compute the even and odd parts of each of the given functions.

6. Consider the family of functions

$$f_\alpha(t) = \sin(t + \alpha),$$

where the parameter α is a real number.

(a) Find all α values such that f_α is an even function.

(b) Find all α values such that f_α is an odd function.

(c) Let S be the linear span of all f_α's for arbitrary α; more precisely:

$$S = \text{Span}\{f_\alpha \mid \alpha \in \mathbb{R}\}.$$

i. Verify that S is a linear subspace of the vector space of all single variable real valued functions.

ii. Find a basis for S.

iii. Is S finite dimensional? If S is finite dimensional, compute its dimension.

7. Let $\mathcal{M} = \mathcal{M}(2 \times 2; \mathbb{R})$ be the vector space of all 2×2 matrices with real entries. Define $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{S}(A) = \frac{A + A^T}{2}.$$

(a) Verify that $\mathcal{S}$ is a linear transformation.

(b) Let B be a symmetric matrix; that is

$$B^T = B.$$

Find a matrix A such that $\mathcal{S}(A) = B$.

(c) Verify that the image of $\mathcal{S}$ is the subspace of symmetric matrices.

(d) Verify that $\mathcal{S}$ is idempotent, that is, $\mathcal{S}^2 = \mathcal{S}$.

(e) Describe the kernel of $\mathcal{S}$.

8. Let $\mathcal{M} = \mathcal{M}(2\times2; \mathbb{R})$ be the vector space of all 2×2 matrices with real entries. The trace $\operatorname{tr}(A)$ of a matrix $A \in \mathcal{M}$ is by definition the sum of the diagonal entries of A. More precisely,

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

(a) Verify that $\operatorname{tr} : \mathcal{M} \to \mathbb{R}$ is a linear transformation.

(b) Verify that if $C = [A, B]$ is the commutator of two matrices, then C is in the kernel of tr.

(c) Verify that if A and B are two matrices such that for some invertible matrix P, $A = PBP^{-1}$, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

(d) Let $\tau : \mathcal{M} \to \mathbb{R}$ be an arbitrary linear transformation. Find a matrix $A = A_\tau$ such that for any $X \in \mathcal{M}$,

$$\tau(X) = \mathrm{tr}(AX).$$

Challenge Consider the map

$$\Phi : \mathcal{M} \to \mathcal{L}(\mathcal{M}; \mathbb{R}), \quad \Phi(A)(X) = \mathrm{tr}(AX).$$

Here $\mathcal{L}(\mathcal{M}; \mathbb{R})$ is the vector space of all linear transformations from $\mathcal{M}$ to $\mathbb{R}$. Verify that this is an invertible linear transformation.

Challenge Verify that if C is in the kernel of tr, then there are matrices A, B such that $C = [A, B]$. Thus the kernel of trace is precisely commutators.

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.

- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. Here is an example of such a "guardrails" prompt that worked reasonably well in this context for ChatGPT as of October 2023:

  > Hello. I am working on a linear algebra problem as part of a university class. My instructor has permitted the use of ChatGPT, but only under specific guidelines to encourage independent critical thinking. Please assist me by asking probing questions, encouraging reflection, and providing general insights about the concepts involved. Do not offer direct hints, strategies, solutions, or step-by-step guidance. I seek to understand the underlying principles and want to develop my own approach to the problem. Your role is to facilitate my learning process without directly leading me to the answer. Thank you!

  Here are some guides regarding prompt engineering:

  – https://platform.openai.com/docs/guides/prompt-engineering
  – https://developers.google.com/machine-learning/resources/prompt-eng
  – https://www.ibm.com/topics/prompt-engineering
  – https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

https://forms.gle/o7k56VcsRXDyxD5s9.

You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/694951/assignments/3866305,

  see the Gradescope documentation for instructions.

5 When to Submit

~~This problem set is due on March 1, 2024 at 11:59 PM.~~
This problem set is due on March 3, 2024 at 11:59 PM.

Late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.