University of Utah Spring 2024

MATH 2270-002
Midterm 3 Questions

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April 12 2024, 8:35 AM - 9:25 AM

Surname: KEY

First Name:

uNID:

Before turning the page make sure to read and sign the exam policy document, distributed separately.

1. **[90 points]** Consider the following matrix:

$$A = \begin{pmatrix} 7 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix}.$$

20

(a) **[[redacted] points]** Write the characteristic equation of A.

$$S = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow A = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S \end{pmatrix}.$$

$$\mathrm{char}_S(\lambda) = \begin{vmatrix} \lambda - 7 & -2 \\ -2 & \lambda - 4 \end{vmatrix} = (\lambda-7)(\lambda-4) - 4$$

$$= \lambda^2 - 11\lambda + 28 - 4 = \lambda^2 - 11\lambda + 24 = (\lambda-3)(\lambda-8).$$

$$\mathrm{char}_A(\lambda) = \begin{vmatrix} \lambda I - S & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda I - S \end{vmatrix} = \lambda \cdot \mathrm{char}_S(\lambda)^2$$

$$\Rightarrow \boxed{\mathrm{char}_A(\lambda) = \lambda(\lambda-3)^2(\lambda-8)^2}$$

$$\Rightarrow \text{Char eq.:} \ \boxed{\lambda(\lambda-3)^2(\lambda-8)^2 = 0}$$

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(b) **[[redacted]points]** Compute all eigenvalues of A.

eigenvalues of A

= roots of the char. eq.

$\Rightarrow \boxed{0, 3, 8}$

(c) **[15 points]** For each distinct eigenvalue of A find a basis of the associated eigenspace.

$S = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$, $A = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S \end{pmatrix}$. $\mathrm{Spec}(A) = \{8, 3, 0\}$

$\det(S) = 28 - 4 = 24 \neq 0 \Rightarrow S$ inv.

$\boxed{\lambda_1 = 8}$ $\begin{pmatrix} 8-7 & -2 \\ -2 & 8-4 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \Rightarrow w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\rightsquigarrow$ padding: $\boxed{v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}$ basis for espace for 8

$\boxed{\lambda_2 = 3}$ $\begin{pmatrix} 3-7 & -2 \\ -2 & 3-4 \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \Rightarrow w_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\rightsquigarrow$ padding: $\boxed{v_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}}$ basis for espace for 3.

$\boxed{\lambda_3 = 0}$ $AM = 0 \Rightarrow$ espace $= \mathrm{Nul}(A)$

$\Rightarrow \boxed{v_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$ basis for espace for 0

(d) **[10 points]** Compute the algebraic and geometric multiplicities of each distinct eigenvalue of A.

A symmetric $\Rightarrow$ By spec. thm diagonalizable $\Rightarrow$ $\boxed{AM = GM}$

$\boxed{AM(0) = 1;\ AM(3) = 2;\ AM(8) = 2}$

(e) **[10 points]** Is A diagonalizable? If it is, find an invertible matrix P and a diagonal matrix D such that A = PDP^{-1}.

yes by spec. thm.

$$\boxed{D = \begin{pmatrix} \begin{matrix} 8 & 0 \\ 0 & 3 \end{matrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \begin{matrix} 8 & 0 \\ 0 & 3 \end{matrix} \end{pmatrix}}$$

$$P = \begin{pmatrix} | & | & | & | & | \\ v_1 & v_3 & v_5 & v_2 & v_4 \\ | & | & | & | & | \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} \begin{matrix} 2 & 1 \\ 1 & -2 \end{matrix} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \begin{matrix} 2 & 1 \\ 1 & -2 \end{matrix} \end{pmatrix}}$$

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(f) **[[redacted] points]** Is A orthogonally diagonalizable? If it is, find an orthogonal matrix Q and a diagonal matrix E such that $A = QEQ^T$.

yes by spec thm

$$\boxed{E = D = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}}$$

need to normalize the columns of P :

$$Q = \begin{pmatrix} | & | & | & | & | \\ \frac{v_1}{|v_1|} & \frac{v_3}{|v_3|} & \frac{v_5}{|v_5|} & \frac{v_2}{|v_2|} & \frac{v_4}{|v_4|} \\ | & | & | & | & | \end{pmatrix} \qquad q_i = \frac{v_i}{|v_i|}$$

$$|v_1| = |v_2| = |v_3| = |v_4| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$|v_5| = 1$$

$$\Rightarrow \boxed{Q = \begin{pmatrix} 2/\sqrt5 & 1/\sqrt5 & 0 & 0 & 0 \\ 1/\sqrt5 & -2/\sqrt5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2/\sqrt5 & 1/\sqrt5 \\ 0 & 0 & 0 & 1/\sqrt5 & -2/\sqrt5 \end{pmatrix}}$$

5

(g) **[[redacted] points]** Compute the orthogonal projection of the vector

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

to the columnspace of A.

$$\mathrm{Col}(A) = \mathrm{Nul}(A^T)^{\perp} = \mathrm{Nul}(A)^{\perp} = E(0;A)^{\perp}$$

$$= \mathrm{Span}(v_5)^{\perp}, \quad v_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{proj}_{v_5}(b) = \frac{(b, v_5)}{|v_5|^2} v_5 = v_5$$

$$\Rightarrow \mathrm{proj}_{Col(A)}(b) = b - \mathrm{proj}_{v_5}(b)$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

(h) **[5 points]** Compute the singular value decomposition of A.

A symmetric $\Rightarrow$ singular values are abs. values of eigenvalues.

$$\Sigma = \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \\ \end{array}$$

$$\Sigma = \left(\begin{array}{cccc|c} 8 & 0 & 0 & 0 & \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 3 & \\ \hline & & 0 & & 0 \end{array}\right)$$

$\sigma_1 = \sigma_2 = 8$

$\sigma_3 = \sigma_4 = 3$

singular values.

$$U = V = \begin{pmatrix} | & | & | & | & | \\ q_1 & q_2 & q_3 & q_4 & q_5 \\ | & | & | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} & 0 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2/\sqrt{5} & 0 & 1/\sqrt{5} & 0 \\ 0 & 1/\sqrt{5} & 0 & -2/\sqrt{5} & 0 \end{pmatrix}$$

$\Rightarrow$ $A = U \Sigma V^T$

or:

$$A = \sum_{i=1}^{4} \sigma_i q_i q_i^T$$

2. **[7 points]** Compute the singular values of the following matrix:

$$A = \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 3 & 4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 15 \\ 15 & 25 \end{pmatrix}.$$

$$0 = \begin{vmatrix} \lambda - 25 & -15 \\ -15 & \lambda - 25 \end{vmatrix} = (\lambda - 25)^2 - 15^2$$

$$= (\lambda - 25 - 15)(\lambda - 25 + 15)$$

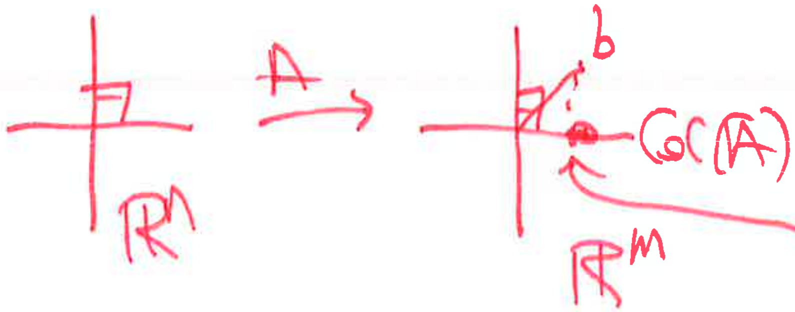
$$= (\lambda - 40)(\lambda - 10)$$

$$\Rightarrow \lambda_1 = 40 > \lambda_2 = 10$$

$$\Rightarrow \boxed{\sigma_1 = 2\sqrt{10} > \sigma_2 = \sqrt{10}}$$

3. **[3 points]** Let A be an m × n matrix and b be a vector in $\mathbb{R}^m$. Assume that the rank of A is n. Verify that the orthogonal projection of b onto the columnspace of A is given by

$$A(A^TA)^{-1}A^Tb.$$



$\hat{b} = proj_{Col(A)}(b)$

$\hat{b} \in Col(A) \Rightarrow$ For some $\hat{x} \in \mathbb{R}^n$: $\boxed{A\hat{x} = \hat{b}}$

$b - \hat{b} \in Col(A)^{\perp} = Nul(A^T) \Rightarrow \boxed{A^Tb = A^T\hat{b} = A^TA\hat{x}}$

$\underset{n \times m}{A^T}\underset{m \times n}{A}$, $rank(A^TA) = rank(A) = n$

$\Rightarrow A^TA$ inv. $\Longrightarrow \hat{x} = (A^TA)^{-1}A^Tb$

$\Rightarrow \boxed{proj_{Col(A)}(b) = \hat{b} = A(A^TA)^{-1}A^Tb}$

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