University of Utah — Spring 2026

MATH 2270-006 Selected Solutions to PSet Problems

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In this document you may find solutions to selected problems from problem sets. As a general rule of thumb, it is more likely that this document will be updated with the solutions to problems that were graded on accuracy.

Please remember that throughout this class your solutions need not match the level of polish and precision of the below solutions. What's most important is grasping the underlying concepts and steadily improving your problem-solving skills as well as your technical presentation skills.

Most solutions will have follow-up exercises. A follow-up exercise here may later show up in an exam.

1 PSet 1, Problem 2

Suppose a, b, c, and d are constants such that the system below is consistent for all possible values of f and g. What can you say about the numbers a, b, c, and d?

$$
\begin{aligned}
ax_1 + bx_2 &= f \\
cx_1 + dx_2 &= g
\end{aligned}
$$

Let's first parse the problem: we are given the following:

- a, b, c, d are fixed but arbitrary numbers.
- f, g are nonfixed arbitrary numbers.
- a, b, c, d are so that no matter what the numerical values of f, g are, the system is consistent.

The problem is asking for a statement about a, b, c, d. While the formulation is somewhat ambiguous, it is safe to assume that the problem is asking for an equality or inequality involving only the numbers a, b, c, d. For instance certainly it can not be the case that

$$a = b = c = d = 0,$$

for otherwise unless $f = g = 0$, the system would fail to be consistent. This is not the most general statement one can obtain however, and to obtain a more general statement one can switch to the augmented matrix notation and do row reduction.

The augmented matrix corresponding to the system is:

$$\left(\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right)$$

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To proceed with row reduction, a case analysis will be useful. Focusing on a, either it is nonzero (the typical case), xor it is zero (the edge case). Let's first handle the typical case, assuming that $a \neq 0$. Then we can leverage division by a and row reduce as follows:

$$\left(\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{1}{a} R_1} \left(\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ c & d & g \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow R_2 - cR_1} \left(\begin{array}{cc|c} 1 & \frac{b}{a} & \frac{f}{a} \\ 0 & d - \frac{bc}{a} & g - \frac{fc}{a} \end{array}\right)$$

Looking at the second row, the only way the system would be inconsistent is if

$$\boxed{d - \frac{bc}{a} = 0} \text{ and simultaneously } \boxed{g - \frac{fc}{a} \neq 0},$$

for fixed a, b, c, d and arbitrary f, g. We can't quite control the inequality, as one can find f, g such that the second inequality holds, e.g. $f = 0$, $g = 1$, which means that in order for the system to be consistent no matter what f and g are, one must have

$$\boxed{d - \frac{bc}{a} \neq 0}$$

which is equivalent to

$$\boxed{ad - bc \neq 0}.$$

Let us now study the edge case of $a = 0$. Then we have

$$\left(\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right) = \left(\begin{array}{cc|c} 0 & b & f \\ c & d & g \end{array}\right)$$

In order for this system to be consistent no matter what the value of f is, it must be the case that $b \neq 0$. Using this, we may proceed with row reduction like so:

$$\left(\begin{array}{cc|c} 0 & b & f \\ c & d & g \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{1}{b}R_1} \left(\begin{array}{cc|c} 0 & 1 & \frac{f}{b} \\ c & d & g \end{array}\right)$$
$$\xrightarrow{R_2 \leftarrow R_2 - dR_1} \left(\begin{array}{cc|c} 0 & 1 & \frac{f}{b} \\ c & 0 & g - \frac{fd}{b} \end{array}\right).$$

Similar to the argument before, we may arrange f, g so as to make $g - \frac{fd}{b}$ nonzero, so in order to guarantee consistency we must have that $c \neq 0$. But then we come to the exact same conclusion as in the typical case:

$$\boxed{ad - bc = -bc \neq 0},$$

as the product of two nonzero numbers is nonzero.

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Follow-up Exercise The problem goes the other way around too: if for some f, g, the system is inconsistent, then $ad - bc = 0$. This is harder to justify, at least using only row reduction.

Follow-up Exercise 2 Same problem, but with 3 equations and 3 unknowns. Using only row reduction the algebra will be quite complicated, but it can be done!

2 PSet 2, Problem 2

Let $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}$. Denote by W the span of the columns of A.

1. Is b in W? How many vectors are in W?
2. Is the first column of A is in W?
3. Is $W = \mathbb{R}^3$?

One can solve this problem either using higher level concepts so as to avoid involved calculations, or by rephrasing what is being asked as certain systems of linear equations. We'll take the high road to showcase the power of higher level concepts.

First two columns of A are linearly independent, since only the first one allows motion with a component in the x_1 direction and only the second one allows motion with a component in the x_2 direction. Next, the third column of A can not be written as a linear combination of the first two columns, since the component in the x_2 direction would require that the coefficient of the second column has to be $-2/3$, which then would force the third component of the third column to be not an integer. Thus the columns of A are linearly independent, and consequently the reduced echelon form of A is the 3×3 identity matrix. This implies that $Ax = y$ has a unique solution for any y in $\mathbb{R}^3$; in other words the span W of the columns of A is the whole $\mathbb{R}^3$, and in particular b is in W. By definition W must also contain any column of A. Finally, spans must be closed under scalar multiplication by definition, and since we have infinitely many scalars, we can conclude that W has infinitely many vectors.

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3 PSet 3, Problem 4

Consider the following matrix:

$$A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$$

Perform the following tasks:

1. Compute the (multiplicative) inverse A^{-1} of A.
2. Verify that $AA^{-1} = I = A^{-1}A$ by multiplying the matrices and comparing the corresponding entries.
3. Verify that $A^{-1}(2, -1)$ is the unique solution to the system

$$\begin{aligned} 8x_1 + 3x_2 &= 2 \\ 5x_1 + 2x_2 &= -1 \end{aligned}$$

To compute A^{-1}, we' use the formula for the inverses of 2×2 matrices. First we compute the determinant of A:

$$\det(A) = 8 \cdot 2 - 3 \cdot 5 = 16 - 15 = 1 \neq 0.$$

Determinant of A is nonzero, hence A is invertible and its inverse is:

$$A^{-1} = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}.$$

If we wanted, we could have also row reduced the augmented matrix $(A|I)$. Especially for 2×2 matrices, the formula is often much more convenient.

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A straightforward matrix-matrix multiplication indeed gives

$$\begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}.$$

Finally, the given system in matrix form looks like:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Multiplying both sides from the left by A^{-1}, we obtain that this system has a unique solution. Further, we may compute this unique solution by:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -18 \end{pmatrix}.$$

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