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MATH 3210-001 Selected Solutions to PSet Problems

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In this document you may find solutions (proofs or examples) to selected problems from problem sets. As a general rule of thumb, it is more likely that this document will be updated with the solutions to problems that were graded on accuracy.

Throughout this class your solutions need not match the level of polish, precision and detail of the below solutions. What's most important is grasping the underlying concepts and steadily improving your problem-solving skills as well as your proof-writing and presentation skills.

Most solutions will have follow-up exercises. Just as in the problem sets, these follow-up exercises are stated in a neutral language, and determining whether or not the statement is correct is part of the exercise. A follow-up exercise here may later show up in an exam.

1 PSet 1, Problem 4

Let X and Y be sets. Then one has:

(i) $\forall R \in \mathcal{P}(X \times Y)$: $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ is a function.

(ii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cup B) = R(A) \cup R(B)$.

(iii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B)$.

(iv) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \setminus B) = R(A) \setminus R(B)$.

Fix a relation R from X to Y.

Claim: The first statement is true, that is, $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ defines a function.

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Proof: We need to verify that the relation $\mathcal{R}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined by

$$\mathcal{R} = \{(A, B) | B = R(A)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$$

is the graph of a function with domain $\mathcal{P}(X)$. For $A \in \mathcal{P}(X)$, by definition $R(A) \in \mathcal{P}(Y)$, so that indeed $\mathrm{dom}(\mathcal{R}) = \mathcal{P}(X)$. Unfolding the definitions, it suffices to verify that

$$\forall A_1, A_2 \subseteq X : A_1 = A_2 \implies R(A_1) = R(A_2).$$

Let A_1 and A_2 be two subsets of X such that $A_1 = A_2$[1]. We claim that $R(A_1) = R(A_2)$. Let $y \in R(A_1)$. Then by definition there is an

[1] This is somewhat awkward, as ultimately we are assuming that we have one set and two different names of it. The point is that while there is only one set here, we have two different labels for it and we want to make sure that the "image under relation" operation nonetheless produces the same outcome. Perhaps a less logically awkward (but equivalent) way of arguing would be to use a proof by contrapositive: suppose $R(A_1) \neq R(A_2)$ and show that $A_1 \neq A_2$.

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$x \in A_1$ such that $(x, y) \in R$. As $x \in A_1 = A_2$, $y \in R(A_2)$ also. Thus $R(A_1) \subseteq R(A_2)$. Interchanging the roles of A_1 and A_2 in this argument gives also $R(A_1) \supseteq R(A_2)$, whence $R(A_1) = R(A_2)$. ⌟

Claim: The second statement is true, that is, for any $A, B \subseteq X$, $R(A \cup B) = R(A) \cup R(B)$.

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Proof: Let $A, B \subseteq X$. Then as $A \subseteq A \cup B$, $R(A) \subseteq R(A \cup B)$. Indeed, if $y \in R(A)$, there is an $x \in A$ such that $(x, y) \in R$. But $A \subseteq A \cup B$, so that $x \in A \cup B$, thus $y \in R(A \cup B)$. Similarly $R(B) \subseteq R(A \cup B)$. Then $R(A) \cup R(B) \subseteq R(A \cup B)$.

For the other direction, let $y \in R(A \cup B)$. Then there is an $x \in A \cup B$ such that $(x, y) \in R$. By definition $x \in A$ or $x \in B$, so that $y \in R(A)$ or $y \in R(B)$, that is, $R(A \cup B) \subseteq R(A) \cup R(B)$.

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Claim: The third statement is false in general. In other words, the image of the intersection of two sets is not necessarily equal to the intersection of the images.

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Proof: Consider $X = [0, 1] = Y$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \to Y$, $x \mapsto 1/2$, that is f is the constant function that is constantly 1/2. Then $A \cap B = \emptyset$, hence $f(A \cap B) = \emptyset$, while $f(A) = f(B) = \{1/2\}$, so that $f(A) \cap f(B) = \{1/2\}$.

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To obtain a correct version of the third statement, either we can weaken the conclusion, or strengthen the hypotheses[2].

Claim: The following version of the third statement with weakened

[2] These are instances of logical perturbations of the original statement given; you do not need to submit such corrected versions, though trying to come up with such corrected versions will help you understand the concepts and prepare you for the exams better.

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conclusion is true:

$$\forall A, B \in \mathcal{P}(X) : R\,(A \cap B) \subseteq R(A) \cap R(B).$$

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Proof: Let $A, B \subseteq X$. As $A \cap B \subseteq A$, $R(A \cap B) \subseteq R(A)$. Similarly as $A \cap B \subseteq B$, $R(A \cap B) \subseteq R(B)$. Thus $R(A \cap B) \subseteq R(A) \cap R(B)$.

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Claim: The following version of the third statement with an additional hypothesis is true:

$$\begin{aligned} & \forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2 \\ \iff \quad & \forall A, B \in \mathcal{P}(X) : R\,(A \cap B) = R(A) \cap R(B). \end{aligned}$$

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Proof: ($\Longrightarrow$) Suppose R satisfies

$$\forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2;$$

one might call this the **injectivity condition** (aka **horizontal line test**) for relations. Let $A, B \subseteq X$, and $y \in R(A) \cap R(B)$. Then there are $a \in A$ and $b \in B$ such that $(a, y), (b, y) \in R$. By the injectivity condition, this forces that $a = b$, and consequently that this element is in $A \cap B$. Thus $y \in R(A \cap B)$, so that $R(A) \cap R(B) \subseteq R(A \cap B)$. Since $R(A) \cap R(B) \supseteq R(A \cap B)$ for any relation R by the previous claim, we have $R(A) \cap R(B) = R(A \cap B)$.

($\Longleftarrow$) Suppose R fails the injectivity condition, so that there are $x_1, x_2 \in X$ and $y \in Y$ such that $x_1 \neq x_2$ and $(x_1, y), (x_2, y) \in R$. Define $A = \{x_1\}, B = \{x_2\}$. Then $A \cap B = \emptyset$, so $R(A \cap B)$, but $y \in R(A) \cap R(B)$, that is, $R(A \cap B) \subsetneq R(A) \cap R(B)$.

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Claim: The fourth statement is false in general. In other words, the image of the relative complement of two sets need not be equal to the relative complement of the images.

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Proof: Recalling that $A \setminus B = A \cap B^{\complement}$, the counterexample we gave earlier serves as a counterexample for this claim also. Indeed, put $X = Y = [0, 1]$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \to Y$, $x \mapsto 1/2$. Then $A \cap B^{\complement} = [0, 1/4]$, hence $f(A \setminus B) = \{1/2\}$; on the other hand $f(A) = f(B) = \{1/2\}$, so that $f(A) \setminus f(B) = \emptyset$.

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Follow-up Exercise 1 The second and the (correct versions) of the third statements hold for arbitrary collections $\mathcal{A} \subseteq \mathcal{P}(X)$.

Follow-up Exercise 2 The fourth statement has correct versions (one involving a "subset of" relation, and one involving the injectivity condition; similar to the correct versions of the third statement).

Follow-up Exercise 3 If $f : X \to Y$ is a function, then the preimage relation f^{-1} satisfies the injectivity condition[3]. Consequently, taking preimages commutes with unions, intersections and relative complements.

[3] Recall that f^{-1} really is an abbreviation for $\mathrm{graph}(f)^{-1}$, and f^{-1} is not necessarily a function from Y to X.