University of Utah Spring 2025

MATH 3210-001 PSet 3 Specification

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Subject to Change; Last Updated: 2025-01-31 10:51:00-07:00

1 Background

This problem set focuses on sequences and their convergence properties. While one can talk about the convergence of a sequence in the abstract setting of a metric space (or even a topological space), the sequences considered in this problem set consist exclusively of real numbers. While not necessary, it is a good exercise to investigate, as part of logical perturbations, whether a given statement about sequences of real numbers remains valid for sequences taking values in a more general space X—or even whether the statement (has a logical perturbation that) makes sense in such a setting!

2 What to Submit

Submit your detailed solutions to each of the problems below. While the prompts may seem lengthy, the additional text is intended to guide you by providing context and helpful details.

When tackling statements in MATH 3210, follow these four steps (in no particular order):

1. **Assess the truth of the statement:** Determine, either exactly or probabilistically, whether the statement is true.

2. **Build a case:** Come up with examples or begin drafting a proof to identify potential issues or areas where the argument may break down.

3. **Construct a proof:** Provide a formal argument to support your conclusion.

4. **Perturb the statement:** Experiment with logical variations of the statement. Can you prove a stronger/weaker result, or find alternative proofs?

In this problem set, and in later problem sets as well as exams, your work must address the third step (aka constructing a proof). While work regarding the remaining three items likely will help you write a formal proof, understand the material as well as identify connections between different concepts, you do not need to turn in generalizations or multiple proofs of the same statement, nor do you need to provide examples or counterexamples (unless you are specifically asked to). However, demonstrating these steps can contribute to partial credit when appropriate.

Statements in problems will typically be presented in neutral language. For example, a problem may simply state "P" (for P a well-defined statement) rather than "Show that P" or "Prove that P is false". Your task is to determine the truth of the statement and provide a formal proof or refutation.

If a statement is false, it is highly recommended to propose a corrected version of the statement and prove this corrected version. This process, part of "perturbing the statement", will enhance your mathematical error detection and correction skills.

Unless explicitly stated otherwise, all problems require proofs. If you provide an example, you must also prove that the example satisfies the necessary conditions. If you make a calculation, you must prove that your calculation is correct. **In this problem set and future assignments, if an object is claimed to be a well-defined function (and in particular sequence) you need not prove this claim. However if an object is claimed to be a well-defined function and fails to be one really, a proof of this failure is required.**

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your work clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

1. This problem is about limit laws for sequences. Let $x_\bullet, y_\bullet$ be two convergent sequences of real numbers with limits $x_\infty, y_\infty \in \mathbb{R}$ respectively:

$$\lim_{n\to\infty} x_n = x_\infty, \qquad \lim_{n\to\infty} y_n = y_\infty.$$

Then

(a) $\lim_{n\to\infty}(x_n + y_n) = x_\infty + y_\infty$.

(b) $\lim_{n\to\infty}(x_n - y_n) = x_\infty - y_\infty$.

(c) For any real number α, $\lim_{n\to\infty}(\alpha x_n) = \alpha x_\infty$.

(d) $\lim_{n\to\infty}(x_n y_n) = x_\infty y_\infty$.

(e) If $\forall n : y_n \neq 0$ and $y_\infty \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x_\infty}{y_\infty}$.

(f) $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{x_\infty, y_\infty\}$.

(g) $\lim_{n\to\infty} \min\{x_n, y_n\} = \min\{x_\infty, y_\infty\}$.

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2. Represent visually each of the following statements. Your visual representations may vary from impressionistic to metaphysical. Ultimately, your visualizations should be so that they are useful to you personally as intuition pumps. **There is nothing to prove in this problem!**

   (a) $x_\bullet$ is a sequence converging to x_∞ (aka $\lim_{n\to\infty} x_n = x_\infty$).

   (b) $x_\bullet$ is a sequence that fails to converge to $x_\dagger$ (aka $\lim_{n\to\infty} x_n \neq x_\dagger$).

   (c) $x_\bullet$ does not converge to x'_∞, but a subsequence of it does (aka $\lim_{n\to\infty} x_n \neq x'_\infty$ but for some strictly increasing j, $\lim_{n\to\infty} x_{j(n)} \neq x'_\infty$).

   (d) The limit of the sequence of sums is the sum of the limits (aka $\lim_{n\to\infty}(x_n + y_n) = (\lim_{n\to\infty} x_n) + (\lim_{n\to\infty} y_n)$).

   (e) The limit of the sequence of products is the product of the limits (aka $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$).

   (f) The limit of the sequence of maximums is the maximum of the limits (aka $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\}$).

3. Let $x_\bullet$ be a sequence of real numbers that converges to zero. Then for any bounded sequence $b_\bullet$ of real numbers,

$$\lim_{n\to\infty} x_n b_n = 0.$$

4. Let $x_\bullet$ be a sequence of positive real numbers. Then $x_\bullet$ converges to zero if and only if $\lim_{n\to\infty} \frac{x_n}{1+x_n} = 0$.

5. Let $x_\bullet$ be a sequence of positive real numbers and R be a real number. Then $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = R$ if and only if $\lim_{n\to\infty} x_n^{1/n} = R$.

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6. $\lim_{n\to\infty} n^{1/n} = 1$.

7. Let $x_\bullet$ be a sequence of real numbers and $x_\infty \in \mathbb{R}$. Then the following are equivalent:

   (a) $\lim_{n\to\infty} x_n = x_\infty$. (In words, the sequence $x_\bullet$ converges to x_∞.)

   (b) $\lim_{n\to\infty} x_n^+ = x_\infty^+$ and $\lim_{n\to\infty} x_n^- = x_\infty^-$, where for $r \in \mathbb{R}$, $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$ are the positive and negative parts of r, respectively.

   (c) $\lim_{n\to\infty} x_{2n} = x_\infty$ and $\lim_{n\to\infty} x_{2n+1} = x_\infty$.

   (d) $\forall x'_\bullet \leq x_\bullet, \exists x''_\bullet \leq x'_\bullet : \lim_{n\to\infty} x''_n = x_\infty$. (In words, any subsequence of $x_\bullet$ has a subsequence that converges to x_∞.)

   (e) $\forall x'_\bullet \leq x_\bullet, \exists x''_\bullet \leq x'_\bullet, \exists x''_\infty \in \mathbb{R} : \lim_{n\to\infty} x''_n = x''_\infty$. (In words, any subsequence of $x_\bullet$ has a convergent subsequence.)

   (f) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} x_k = x_\infty$. (In words, the sequence of time averages converges to x_∞.)

3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the

staff will evaluate the reasonableness of your attempts. When in doubt, check with the course staff to ensure your guardrails prompt is appropriate.

Here are some guides regarding prompt engineering:

– https://platform.openai.com/docs/guides/prompt-engineering
– https://developers.google.com/machine-learning/resources/prompt-eng
– https://www.ibm.com/topics/prompt-engineering
– https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/XuCAvjPfGxP9virn8.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/934441/assignments/5503408,

  see the Gradescope documentation for instructions.

5 When to Submit

~~This problem set is due on January 31, 2025 at 11:59 PM.~~
This problem set is due on February 2, 2025 at 11:59 PM.

As per the course's syllabus, late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.

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