University of Utah Spring 2025

MATH 3210-001
Final Exam Questions

Instructor: Alp Uzman

April 25, 2025, 8:00 AM - 10:00 AM

Surname: KEY

First Name:

uNID:

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1. **[5 points]** Let $x_\bullet, y_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be two sequences.

   (a) **[3 points] Prove or disprove the following statement:**

$$\limsup_{n\to\infty} |x_n y_n| \leq \left(\limsup_{n\to\infty} |x_n|\right)\left(\limsup_{n\to\infty} |y_n|\right).$$

Let $N \in \mathbb{Z}_{\geq 0}$. Then $\forall n \in \mathbb{Z}_{\geq N}$:

$$|x_n| \leq \sup_{n\geq N} |x_n|, \quad |y_n| \leq \sup_{n \geq N} |y_n|$$

$$\Rightarrow |x_n y_n| \leq \left(\sup_{n\geq N} |x_n|\right) \cdot \left(\sup_{n\geq N} |y_n|\right)$$

$$\Rightarrow \sup_{n\geq N} |x_n y_n| \leq \left(\sup_{n\geq N} |x_n|\right)\left(\sup_{n\geq N} |y_n|\right)$$

Recall: $\limsup_{n\to\infty} z_n = \inf_{N\geq 0} \sup_{n\geq N} z_n = \lim_{N\to\infty} \sup_{n\geq N} z_n$

Hence we're done by the ~~[illegible]~~ squeeze theorem and limit law that the product of limits is the limit of products. $\square$

KEY

(b) **[2 points] Prove or disprove the following statement:**

$$\limsup_{n\to\infty} x_n y_n \le \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right).$$

Define

$x_\cdot =$	-1	1	-1	1	-1	1	$\dots$
$y_\cdot =$	-1	0	-1	0	-1	0	$\dots$

Then

$(xy)_\cdot =$	1	0	1	0	1	0	$\dots$

$x_n = (-1)^n$, $\quad y_n = \begin{cases} -1, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}$

$\Rightarrow \limsup\limits_{n\to\infty} x_n y_n = 1$ but

$$\left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right) = 1 \cdot 0 = 0.$$

B.

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2. **[50 points]** Let $f : [0, 1] \to \mathbb{R}$ be a function.

   (a) **[40 points] Prove or disprove the following statement:**
   If f is continuous, then there is an $x_* \in [0, 1]$ such that

$$f(x_*) = \int_0^1 f(x)dx.$$

By the Maximum Principle,
$\max f, \min f \in \mathbb{R}$, $\forall x \in [0,1]$:
$\min(f) \leq f(x) \leq \max(f)$

$\Rightarrow \min(f) = \int_0^1 \min(f)\,dx$
$\leq \int_0^1 f(x)dx \leq \int_0^1 \max(f)dx = \max(f)$

$\Rightarrow$ By IVT, $\exists\, x_* \in [0,1]$:

$f(x_*) = \int_I f(x)dx.$

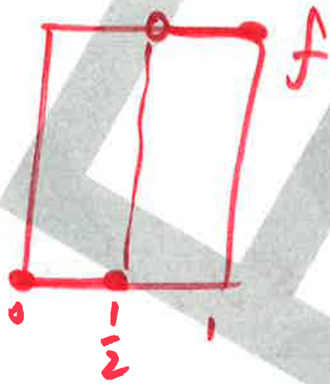
TJ.

(b) **[10 points] Prove or disprove the following statement:** If f is Riemann integrable, then there is an $x_* \in [0,1]$ such that

$$f(x_*) = \int_0^1 f(x)dx.$$

$$f : [0,1] \to \mathbb{R}$$

$$x \longmapsto \begin{cases} 1, & \text{if } x > \frac{1}{2} \\ 0, & \text{else.} \end{cases}$$



$$\Rightarrow \int_0^1 f(x)dx = \frac{1}{2}$$

but $\frac{1}{2} \notin \mathrm{im}(f)$.

$\square$

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3. **[15 points]** Let $f \in C^0(\mathbb{R};\mathbb{R})$, $\alpha, \beta \in C^1(\mathbb{R};\mathbb{R})$ and define $F : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \int_{\alpha(x)}^{\beta(x)} f(t)dt.$$

(a) **[10 points] Prove or disprove the following statement:**
For any $x \in \mathbb{R}$, $F'(x) = \beta'(x) f \circ \beta(x) - \alpha'(x) f \circ \alpha(x)$.

Define $G : \mathbb{R} \to \mathbb{R}$, $x \mapsto \int_0^x f(t)\,dt$.
Then by FTC-2, G is differentiable and $G' = f$. By the cocycle property,

$$F(x) = \int_0^{\beta(x)} f(t)dt - \int_0^{\alpha(t)} f(t)\,dt$$

$$= G \circ \beta(x) - G \circ \alpha(x).$$

$\Rightarrow$ By the Chain Rule (and linearity of derivatives)

$$F' = G' \circ \beta \cdot \beta' - G' \circ \alpha \cdot \alpha'$$

$$= f \circ \beta \cdot \beta' - f \circ \alpha \cdot \alpha' \quad \square.$$

(b) **[5 points] Prove or disprove the following statement:** If f is C^p, α is C^q and β is C^r for p, q, r nonnegative integers, then F is C^s, where $s = \min\{p+1, q, r\}$.

Put $G : \mathbb{R} \to \mathbb{R}$, $x \mapsto \int_0^x f(t)\,dt$. Then by FTC-2, $G' = f \in C^p \Rightarrow G \in C^{p+1}$;

$F = G \circ \beta - G \circ \alpha$.

XOR

$\boxed{q = 0 \text{ OR } r = 0}$

$\Downarrow$

$F = \underbrace{G \circ \beta}_{\in C^0} - \underbrace{G \circ \alpha}_{\in C^0} \in C^0$

$s = 0 = \min\{*, 0\}$

(Actually for this case $f \in R$ suffices)

$\boxed{q \geq 1 \text{ AND } r \geq 1}$

$\Downarrow$

$\underbrace{f \circ \beta}_{\in C^{\min\{p, r\}}} \cdot \beta' \in C^{\min\{p, r-1\}}$

and

$f \circ \alpha \cdot \alpha' \in C^{\min\{p, q-1\}}$

$\Rightarrow F' = f \circ \beta \cdot \beta' - f \circ \alpha \cdot \alpha' \in C^{\min\{p, r-1, q-1\}}$

$\Rightarrow F \in C^{\min\{p+1, r, q\}}$. $\square$

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4. **[30 points]** Let $x_\bullet, b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be two sequences.

(a) **[20 points] Prove or disprove the following statement:**
If $b_\bullet$ is bounded and $\sum_{n=0}^{\infty} x_n$ is absolutely convergent, then $\sum_{n=0}^{\infty} x_n b_n$ is absolutely convergent.

Put $B = \sup_{n \in \mathbb{Z}_{\geq 0}} |b_n|$. Then

$$|x_n b_n| \leq |x_n| B$$

$$\Rightarrow x_n b_n = O_{n\to\infty}(x_n).$$

Thus the result follows from the comparison test.

$\square$.

(b) **[10 points] Prove or disprove the following statement:** If $b_\bullet$ is bounded and $\sum_{n=0}^{\infty} x_n$ is convergent, then $\sum_{n=0}^{\infty} x_n b_n$ is convergent.

$\forall n$: $b_n = (-1)^{n+1}$

$x_n = \frac{(-1)^{n+1}}{n+1}$.

Then $\sup_{n \in \mathbb{Z}_{\geq 0}} |b_n| = 1 < \infty$,

$\sum_{n=0}^{\infty} x_n$ convergent by the Alternating series Test

But $\sum_{n=0}^{\infty} x_n b_n = \sum_{n=0}^{\infty} \frac{1}{n+1} = \zeta(1)$

is divergent. $\square$.

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