University of Utah — Spring 2025

MATH 3210-001 Selected Solutions to PSet Problems

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In this document you may find solutions (proofs or examples) to selected problems from problem sets. As a general rule of thumb, it is more likely that this document will be updated with the solutions to problems that were graded on accuracy.

Throughout this class your solutions need not match the level of polish, precision and detail of the below solutions. What's most important is grasping the underlying concepts and steadily improving your problem-solving skills as well as your proof-writing and presentation skills.

Most solutions will have follow-up exercises. Just as in the problem sets, these follow-up exercises are stated in a neutral language, and determining whether or not the statement is correct is part of the exercise. A follow-up exercise here may later show up in an exam.

1 PSet 1, Problem 4

Let X and Y be sets. Then one has:

(i) $\forall R \in \mathcal{P}(X \times Y)$: $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ is a function.

(ii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cup B) = R(A) \cup R(B)$.

(iii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B)$.

(iv) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \setminus B) = R(A) \setminus R(B)$.

Fix a relation R from X to Y.

Claim: The first statement is true, that is, $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ defines a function.

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Proof: We need to verify that the relation $\mathcal{R}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined by

$$\mathcal{R} = \{(A, B) | B = R(A)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$$

is the graph of a function with domain $\mathcal{P}(X)$. For $A \in \mathcal{P}(X)$, by definition $R(A) \in \mathcal{P}(Y)$, so that indeed $\mathrm{dom}(\mathcal{R}) = \mathcal{P}(X)$. Unfolding the definitions, it suffices to verify that

$$\forall A_1, A_2 \subseteq X : A_1 = A_2 \implies R(A_1) = R(A_2).$$

Let A_1 and A_2 be two subsets of X such that $A_1 = A_2$[1]. We claim that $R(A_1) = R(A_2)$. Let $y \in R(A_1)$. Then by definition there is an

[1] This is somewhat awkward, as ultimately we are assuming that we have one set and two different names of it. The point is that while there is only one set here, we have two different labels for it and we want to make sure that the "image under relation" operation nonetheless produces the same outcome. Perhaps a less logically awkward (but equivalent) way of arguing would be to use a proof by contrapositive: suppose $R(A_1) \neq R(A_2)$ and show that $A_1 \neq A_2$.

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$x \in A_1$ such that $(x, y) \in R$. As $x \in A_1 = A_2$, $y \in R(A_2)$ also. Thus $R(A_1) \subseteq R(A_2)$. Interchanging the roles of A_1 and A_2 in this argument gives also $R(A_1) \supseteq R(A_2)$, whence $R(A_1) = R(A_2)$. ⌟

Claim: The second statement is true, that is, for any $A, B \subseteq X$, $R(A \cup B) = R(A) \cup R(B)$.

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Proof: Let $A, B \subseteq X$. Then as $A \subseteq A \cup B$, $R(A) \subseteq R(A \cup B)$. Indeed, if $y \in R(A)$, there is an $x \in A$ such that $(x, y) \in R$. But $A \subseteq A \cup B$, so that $x \in A \cup B$, thus $y \in R(A \cup B)$. Similarly $R(B) \subseteq R(A \cup B)$. Then $R(A) \cup R(B) \subseteq R(A \cup B)$.

For the other direction, let $y \in R(A \cup B)$. Then there is an $x \in A \cup B$ such that $(x, y) \in R$. By definition $x \in A$ or $x \in B$, so that $y \in R(A)$ or $y \in R(B)$, that is, $R(A \cup B) \subseteq R(A) \cup R(B)$.

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Claim: The third statement is false in general. In other words, the image of the intersection of two sets is not necessarily equal to the intersection of the images.

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Proof: Consider $X = [0, 1] = Y$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \to Y$, $x \mapsto 1/2$, that is f is the constant function that is constantly 1/2. Then $A \cap B = \emptyset$, hence $f(A \cap B) = \emptyset$, while $f(A) = f(B) = \{1/2\}$, so that $f(A) \cap f(B) = \{1/2\}$.

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To obtain a correct version of the third statement, either we can weaken the conclusion, or strengthen the hypotheses[2].

Claim: The following version of the third statement with weakened

[2] These are instances of logical perturbations of the original statement given; you do not need to submit such corrected versions, though trying to come up with such corrected versions will help you understand the concepts and prepare you for the exams better.

conclusion is true:

$$\forall A, B \in \mathcal{P}(X) : R\,(A \cap B) \subseteq R(A) \cap R(B).$$

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Proof: Let $A, B \subseteq X$. As $A \cap B \subseteq A$, $R(A \cap B) \subseteq R(A)$. Similarly as $A \cap B \subseteq B$, $R(A \cap B) \subseteq R(B)$. Thus $R(A \cap B) \subseteq R(A) \cap R(B)$.

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Claim: The following version of the third statement with an additional hypothesis is true:

$$\begin{aligned} & \forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2 \\ \iff \quad & \forall A, B \in \mathcal{P}(X) : R\,(A \cap B) = R(A) \cap R(B). \end{aligned}$$

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Proof: ($\Longrightarrow$) Suppose R satisfies

$$\forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2;$$

one might call this the **injectivity condition** (aka **horizontal line test**) for relations. Let $A, B \subseteq X$, and $y \in R(A) \cap R(B)$. Then there are $a \in A$ and $b \in B$ such that $(a, y), (b, y) \in R$. By the injectivity condition, this forces that $a = b$, and consequently that this element is in $A \cap B$. Thus $y \in R(A \cap B)$, so that $R(A) \cap R(B) \subseteq R(A \cap B)$. Since $R(A) \cap R(B) \supseteq R(A \cap B)$ for any relation R by the previous claim, we have $R(A) \cap R(B) = R(A \cap B)$.

($\Longleftarrow$) Suppose R fails the injectivity condition, so that there are $x_1, x_2 \in X$ and $y \in Y$ such that $x_1 \neq x_2$ and $(x_1, y), (x_2, y) \in R$. Define $A = \{x_1\}, B = \{x_2\}$. Then $A \cap B = \emptyset$, so $R(A \cap B)$, but $y \in R(A) \cap R(B)$, that is, $R(A \cap B) \subsetneq R(A) \cap R(B)$.

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Claim: The fourth statement is false in general. In other words, the image of the relative complement of two sets need not be equal to the relative complement of the images.

⌟

Proof: Recalling that $A \setminus B = A \cap B^{\complement}$, the counterexample we gave earlier serves as a counterexample for this claim also. Indeed, put $X = Y = [0, 1]$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \to Y$, $x \mapsto 1/2$. Then $A \cap B^{\complement} = [0, 1/4]$, hence $f(A \setminus B) = \{1/2\}$; on the other hand $f(A) = f(B) = \{1/2\}$, so that $f(A) \setminus f(B) = \emptyset$.

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Follow-up Exercise 1 The second and the (correct versions) of the third statements hold for arbitrary collections $\mathcal{A} \subseteq \mathcal{P}(X)$.

Follow-up Exercise 2 The fourth statement has correct versions (one involving a "subset of" relation, and one involving the injectivity condition; similar to the correct versions of the third statement).

Follow-up Exercise 3 If $f : X \to Y$ is a function, then the preimage relation f^{-1} satisfies the injectivity condition[3]. Consequently, taking preimages commutes with unions, intersections and relative complements.

[3] Recall that f^{-1} really is an abbreviation for $\mathrm{graph}(f)^{-1}$, and f^{-1} is not necessarily a function from Y to X.

2 PSet 2, Problem 3

Let $A \subseteq \mathbb{R}$. Then

(i) The set of all upper bounds of A is $[\sup(A), \infty[$.

(ii) The set of all lower bounds of A is $]-\infty, \inf(A)]$.

(iii) The smallest closed interval containing A is

$$[\inf(A), \sup(A)].$$

(iv) The largest open interval contained in A is

$$]\inf(A), \sup(A)[.$$

Let $A \subseteq \mathbb{R}$. First let us set some notation for convenience. Define by $\mathcal{U}(A)$, $\mathcal{L}(A)$ the sets of all upper and lower bounds of A, respectively:

$$\mathcal{U}(A) = \{U \in \mathbb{R} | \forall a \in A : a \leq U\},$$
$$\mathcal{L}(A) = \{L \in \mathbb{R} | \forall a \in A : L \leq a\}.$$

Note that at this stage even though we can define these sets, we make no claim as to whether or not they are nonempty. The problem does not specify whether A is bounded, in which case the supremum and infimum may be $\pm\infty$. With this foresight we further define the sets $\overline{\mathcal{U}}(A)$, $\overline{\mathcal{L}}(A)$ of extended upper and lower bounds[4] of A, respectively:

[4] Almost never is it preferable to take the extended real number ∞ as a possible upper bound; we make these definitions to handle the way this problem is phrased specifically.

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$$\overline{\mathcal{U}}(A) = \{U \in [-\infty, \infty] | \forall a \in A : a \leq U\},$$
$$\overline{\mathcal{L}}(A) = \{L \in [-\infty, \infty] | \forall a \in A : L \leq a\}.$$

Before continuing with the proofs, let us make the following observation: if U is an upper bound, then any number larger than U too is an upper bound, and similarly one can replace a lower bound by an even lower number. Thus we have:

$$\mathcal{U}(A) = \bigcup_{U \in \mathcal{U}(A)} [U, \infty[\ , \qquad \mathcal{L}(A) = \bigcup_{L \in \mathcal{L}(A)}]-\infty, L],$$
$$\overline{\mathcal{U}}(A) = \bigcup_{U \in \overline{\mathcal{U}}(A)} [U, \infty]\ , \qquad \overline{\mathcal{L}}(A) = \bigcup_{L \in \overline{\mathcal{L}}(A)} [-\infty, L]. \qquad (\star)$$

Let us note that separating the cases based on whether or not A is empty will be useful, as many objects mentioned in the problem operate differently (if at all!) for the emptyset.

The $\boxed{A = \emptyset}$ **Case** In this case, depending on the convention we have either that neither the supremum nor the infimum of A is defined, or else by definition we have $\sup(A) = -\infty$, $\inf(A) = \infty$. In the first case, all parts of the problem use undefined terms, hence they are nonsensical and there is nothing to do[5]!

In the second case however we can make sense of at least some of the statements. In this case we have the following vacuously:

[5] Technically this paragraph by itself, that is, noticing that for $A = \emptyset$, relative to a certain convention, the problem does not make sense, is a perfectly valid solution to this problem. Still, not engaging with a problem due to a choice of conventions is not good form; generally it is prefarable to relax conventions to make sense of more.

$$\mathcal{U}(\emptyset) = \mathbb{R} = \mathcal{L}(\emptyset), \qquad \overline{\mathcal{U}}(\emptyset) = [-\infty, \infty] = \overline{\mathcal{L}}(\emptyset),$$

Further, (†) the smallest interval containing $\emptyset$ as well as the largest interval contained in $\emptyset$ both either do not exist or is the emptyset itself (which can be written as for instance $]0, 0[$).

Now onto the four subsets specified in the problem:

(i) $[\sup(\emptyset), \infty[= [-\infty, \infty[$ does not match with $\mathcal{U}(\emptyset)$ nor $\overline{\mathcal{U}}(\emptyset)$, so this part is false for the emptyset.

(ii) Similarly $]-\infty, \inf(\emptyset)] =]-\infty, \infty]$ does not match with $\mathcal{L}(\emptyset)$ nor $\overline{\mathcal{L}}(\emptyset)$, so this part too is false for the emptyset.

(iii) $[\inf(A), \sup(A)] = [\infty, -\infty]$ is not compatible with the interval notation. Even if we were to take it as extended reals with the opposite orientation, this still is incompatible with the observation (†) above, so this part is false for the emptyset.

(iv) Similarly $]\inf(A), \sup(A)[=]\infty, -\infty[$ is either nonsensical or it is incompatible with (†), hence this part too is false for the emptyset.

Thus we see that the case $A = \emptyset$ is a very inconvenient edge case for this problem[6].

The $\boxed{A \neq \emptyset}$ Case Let us now assume that A is nonempty, so that we have

$$-\infty \leq \inf(A) \leq \sup(A) \leq \infty.$$

[6]Typically in mathematics the most trivial objects often end up being edge cases that are unenlightening. For this reason often people don't even consider them (perhaps without stating that they are not) (case in point: this problem is a variant of Exercise 1.4.3 from the textbook). Of course there are also edge cases that end up being very important (especially if there are undetected edge detection errors prior.)

Thus in this case at least all four parts of the problem are syntactic[7].

(i) If A is not bounded from above, then by definition

$$\mathcal{U}(A) = \emptyset, \quad \overline{\mathcal{U}}(A) = \{\infty\}, \quad [\sup(A), \infty[= [\infty, \infty[= \emptyset.$$

Thus the statement is correct (assuming infinite upper bounds are not allowed). If A is bounded from above, then by definition $\sup(A) = \min \mathcal{U}(A) \in \mathcal{U}(A)$, hence by $(\star)$ above,

$$\mathcal{U}(A) = [\min \mathcal{U}(A), \infty[, \quad \overline{\mathcal{U}}(A) = [\min \mathcal{U}(A), \infty],$$
$$[\sup(A), \infty[= [\min \mathcal{U}(A), \infty[.$$

Hence the statement is correct (assuming infinite upper bounds are not allowed).

(ii) Arguing as in the previous part (or alternatively using the above argument with $B = -A$, together with an exercise mentioned in class), the statement is correct (assuming infinite lower bounds are not allowed).

(iii) If A is unbounded from above and below, then $[\inf(A), \sup(A)] = [-\infty, \infty]$ by definition and again by definition it can not be contained in a closed and bounded interval (with finite numbers as endpoints). Thus the third statement is correct for unbounded A. If A bounded both from above and below, then $-\infty \lneq \inf(A) \leq \sup(A) \lneq \infty$, hence $[\inf(A), \sup(A)] \subseteq \mathbb{R}$. By definition inf(A) is a lower bound for any element in A and similarly sup(A) is an

[7] Again we could have assumed that the infima and suprema of unbounded sets are not defined, but in this case what we assume in this regard does not make a major difference.

upper bound for any element in A, thus $A \subseteq [\inf(A), \sup(A)]$. If $[\lambda, \rho] \subseteq \mathbb{R}$ were a closed interval containing A, then λ would be a lower bound and ρ would be an upper bound for A; hence by definition $\lambda \leq \inf(A)$ and $\sup(A) \leq \rho$. That is $[\inf(A), \sup(A)] \subseteq [\lambda, \rho]$. In other words, $[\inf(A), \sup(A)]$ indeed is the smallest closed interval containing A.

(iv) This part breaks down even for bounded subsets A. For instance consider $A =]-1, 0[\cup]0, 1[$. Then $\inf(A) = -1$ and $\sup(A) = 1$, but certainly $]\inf(A), \sup(A)[=]-1, 1[$ is not a subset of A, as $0 \notin A$.

Let us summarize the statement we proved, without mentioning the emptyset, as it is rather unlikely that the statement for the emptyset will be of any use in the future:

Claim: Let $A \subseteq \mathbb{R}$. If $A \neq \emptyset$, then

(i) $\mathcal{U}(A) = [\sup(A), \infty[\subseteq \mathbb{R}$,

(ii) $\mathcal{L}(A) =]-\infty, \inf(A)] \subseteq \mathbb{R}$,

(iii) The smallest closed interval containing A is $[\inf(A, \sup(A)] \subseteq [-\infty, \infty]$

⌟

We make a final remark to contextualize this problem: We had discussed the "clamp" heuristic for supremum and infimum in class, and the problem really is a formalization of this heuristic: increasing L in $]-\infty, L]$ corresponds to tightening the lower clamp, decreasing U in $[U, \infty[$ corresponds to tighthening the upper clamp, and the numbers L and U that correspond to the tighthest possible configuration are the infimum and supremum, respectively. The heuristic works for clamping from without. While there are physical devices that clamp from within, they are different in principle than the abstract notions on infimum and supremum, whence the fourth part is incorrect.

Follow-up Exercise 1 Let $A \subseteq \mathbb{R}$. Then A is a singleton if and only if $\inf(A) = \sup(A)$.

Follow-up Exercise 2 There are conditions one can impose on a nonempty and bounded $A \subseteq \mathbb{R}$ that would guarantee that the largest open interval contained in A is $]\inf(A), \sup(A)[$.

Follow-up Exercise 3 Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Then there is a property P (distinct from the property of "being equal to $]\inf(A), \sup(A)[$") that any given interval may or may not have such that $]\inf(A), \sup(A)[$ is the largest open interval with property P.

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3 PSet 2, Problem 4

Let $A, B \subseteq \mathbb{R}$. Then

(i) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

(ii) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$.

Again we separate cases: either at least one of A and B is empty, xor neither A nor B is empty.

The $\boxed{A = \emptyset \textbf{ or } B = \emptyset}$ **Case** WLOG, we may assume $A \neq \emptyset = B$. Then $A \cup B = A$ and

$$\sup(A \cup B) = \sup(A) > -\infty = \sup(\emptyset) = \sup(B),$$
$$\inf(A \cup B) = \inf(A) < \infty = \inf(\emptyset) = \inf(B).$$

Since the maximum of an extended real number R and $-\infty$ is R by definition, the first part follows. Similarly since the minimum of an extended real number R and ∞ is R by definition the second part is also true.

The $\boxed{A \neq \emptyset \neq B}$ **Case** For the first part, WLOG we may assume that $\sup(A) \geq \sup(B)$. Thus it suffices to show that $\sup(A) = \sup(A \cup B)$. First note that as $A \subseteq A \cup B$, by the monotonicity of supremum we have

$$-\infty \lneq \sup(B) \leq \sup(A) \leq \sup(A \cup B) \leq \infty.$$

Either $A \cup B$ is unbounded from above, xor it is bounded from above. If $A \cup B$ is unbounded from above, then at least one of A or

B would have to be unbounded from above. If A is unbounded from above, then $\sup(A) = \infty = \sup(A \cup B)$ concludes the proof. If B is unbounded from above, then $\infty = \sup(B) \leq \sup(A) \leq \sup(A \cup B) \leq \infty$, hence again $\sup(A) = \sup(A \cup B)$ (and consequently A is unbounded from above also).

Now suppose $A \cup B$ is bounded from above; then the same is true for both A and B, and we have

$$-\infty \lneq \sup(B) \leq \sup(A) \leq \sup(A \cup B) \lneq \infty.$$

Say $\sup(A) \lneq \sup(A \cup B)$. Then $\sup(A)$ is not an upper bound for $A \cup B$, so that there is an $x \in A \cup B$ such that

$$\sup(A) < x.$$

Since x exceeds the least upper bound for A, $x \notin A$, thus it must be the case that $x \in B$. But then

$$x \leq \sup(B) \leq \sup(A) < x;$$

a contradiction.

One can argue analogously for the statement involving infima. Alternatively one can argue as follows[8]:

$$\begin{aligned}
\inf(A \cup B) &= -\sup(-(A \cup B)) \\
&= -\sup((-A) \cup (-B)) \\
&= -\max\{\sup(-A), \sup(-B)\} \\
&= -\max\{-\inf(A), -\inf(B)\} \\
&= \min\{\inf(A), \inf(B)\}
\end{aligned}$$

[8] You should check these steps if they are not clear!

We have thus proved that indeed the statements in the problem are correct without any modifications:

Claim: $\forall A, B \subseteq \mathbb{R}$:

(i) $\sup(A \cup B) = \max\{\sup(A)\,,\,\sup(B)\}$

(ii) $\inf(A \cup B) = \min\{\inf(A)\,,\,\inf(B)\}$.

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Follow-up Exercise 1 $\forall \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$:

(i) $\sup\left(\bigcup_{A\in\mathcal{A}} A\right) = \sup_{A\in\mathcal{A}} \sup(A)$,

(ii) $\inf\left(\bigcup_{A\in\mathcal{A}} A\right) = \inf_{A\in\mathcal{A}} \inf(A)$.

Follow-up Exercise 2 $\forall \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$:

(i) $\sup\left(\bigcap_{A\in\mathcal{A}} A\right) = \inf_{A\in\mathcal{A}} \sup(A)$,

(ii) $\inf\left(\bigcap_{A\in\mathcal{A}} A\right) = \sup_{A\in\mathcal{A}} \inf(A)$.

4 PSet 3, Problem 3

Let $x_\bullet$ be a sequence of real numbers that converges to zero. Then for any bounded sequence $b_\bullet$ of real numbers,

$$\lim_{n\to\infty} x_n b_n = 0.$$

This claim is indeed true without any modifications:

Claim: $\forall x_\bullet, b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$: if $x_\bullet \to 0$ and $\sup_m b_m < \infty$, then $x_\bullet b_\bullet \to 0$. ⌟

Proof: Instead of starting with an $\epsilon > 0$, it's often useful to start from the end, in other words with the expression we are trying to bound. For any $n \in \mathbb{Z}_{\geq 0}$, one has

$$|x_n b_n| = |x_n|\,|b_n| \leq |x_n| \underbrace{\sup_{m\in\mathbb{Z}_{\geq 0}} |b_m|}_{<\infty} \qquad (\star)$$

Put $M = \sup_{m\in\mathbb{Z}_{\geq 0}} |b_m| \in \mathbb{R}_{\geq 0}$. We know that this is a finite nonnegative number, as the sequence $b_\bullet$ (and consequently the sequence $|b_\bullet|$) is bounded. Thus to bound $|x_n b_n|$ from above by ε, we can bound $|x_n|$ by ε/M. Indeed, let $\varepsilon \in \mathbb{R}_{>0}$. Then as $x_\bullet \to 0$, there is an $N \in \mathbb{Z}_{\geq 0}$ such that for any $n \in \mathbb{Z}_{\geq N}$, we have

$$|x_n| = |x_n - 0| < \frac{\varepsilon}{M}.$$

Here we used the number $\frac{\varepsilon}{M}$ as the "epsilon" in the definition of $x_\bullet$ converging to 0. For any $n \in \mathbb{Z}_{\geq N}$, $(\star)$ also holds; hence combining the two we have that for any $n \in \mathbb{Z}_{\geq N}$:

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$$|x_n b_n - 0| = |x_n b_n| < \frac{\varepsilon}{M} M = \varepsilon.$$

⌟