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MATH 3210-001 Selected Solutions to PSet Problems

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In this document you may find solutions (proofs or examples) to selected problems from problem sets. As a general rule of thumb, it is more likely that this document will be updated with the solutions to problems that were graded on accuracy.

Throughout this class your solutions need not match the level of polish, precision and detail of the below solutions. What's most important is grasping the underlying concepts and steadily improving your problem-solving skills as well as your proof-writing and presentation skills.

Most solutions will have follow-up exercises. Just as in the problem sets, these follow-up exercises are stated in a neutral language, and determining whether or not the statement is correct is part of the exercise. A follow-up exercise here may later show up in an exam.

1 PSet 1, Problem 4

Let X and Y be sets. Then one has:

(i) $\forall R \in \mathcal{P}(X \times Y)$: $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ is a function.

(ii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cup B) = R(A) \cup R(B)$.

(iii) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B)$.

(iv) $\forall R \in \mathcal{P}(X \times Y), \forall A, B \in \mathcal{P}(X) : R(A \setminus B) = R(A) \setminus R(B)$.

Fix a relation R from X to Y.

Claim: The first statement is true, that is, $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto R(A)$ defines a function.

⌟

Proof: We need to verify that the relation $\mathcal{R}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined by

$$\mathcal{R} = \{(A, B) | B = R(A)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$$

is the graph of a function with domain $\mathcal{P}(X)$. For $A \in \mathcal{P}(X)$, by definition $R(A) \in \mathcal{P}(Y)$, so that indeed $\mathrm{dom}(\mathcal{R}) = \mathcal{P}(X)$. Unfolding the definitions, it suffices to verify that

$$\forall A_1, A_2 \subseteq X : A_1 = A_2 \implies R(A_1) = R(A_2).$$

Let A_1 and A_2 be two subsets of X such that $A_1 = A_2$[1]. We claim that $R(A_1) = R(A_2)$. Let $y \in R(A_1)$. Then by definition there is an

[1] This is somewhat awkward, as ultimately we are assuming that we have one set and two different names of it. The point is that while there is only one set here, we have two different labels for it and we want to make sure that the "image under relation" operation nonetheless produces the same outcome. Perhaps a less logically awkward (but equivalent) way of arguing would be to use a proof by contrapositive: suppose $R(A_1) \neq R(A_2)$ and show that $A_1 \neq A_2$.

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$x \in A_1$ such that $(x, y) \in R$. As $x \in A_1 = A_2$, $y \in R(A_2)$ also. Thus $R(A_1) \subseteq R(A_2)$. Interchanging the roles of A_1 and A_2 in this argument gives also $R(A_1) \supseteq R(A_2)$, whence $R(A_1) = R(A_2)$. ⌟

Claim: The second statement is true, that is, for any $A, B \subseteq X$, $R(A \cup B) = R(A) \cup R(B)$.

⌟

Proof: Let $A, B \subseteq X$. Then as $A \subseteq A \cup B$, $R(A) \subseteq R(A \cup B)$. Indeed, if $y \in R(A)$, there is an $x \in A$ such that $(x, y) \in R$. But $A \subseteq A \cup B$, so that $x \in A \cup B$, thus $y \in R(A \cup B)$. Similarly $R(B) \subseteq R(A \cup B)$. Then $R(A) \cup R(B) \subseteq R(A \cup B)$.

For the other direction, let $y \in R(A \cup B)$. Then there is an $x \in A \cup B$ such that $(x, y) \in R$. By definition $x \in A$ or $x \in B$, so that $y \in R(A)$ or $y \in R(B)$, that is, $R(A \cup B) \subseteq R(A) \cup R(B)$.

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Claim: The third statement is false in general. In other words, the image of the intersection of two sets is not necessarily equal to the intersection of the images.

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Proof: Consider $X = [0, 1] = Y$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \to Y$, $x \mapsto 1/2$, that is f is the constant function that is constantly 1/2. Then $A \cap B = \emptyset$, hence $f(A \cap B) = \emptyset$, while $f(A) = f(B) = \{1/2\}$, so that $f(A) \cap f(B) = \{1/2\}$.

⌟

To obtain a correct version of the third statement, either we can weaken the conclusion, or strengthen the hypotheses[2].

Claim: The following version of the third statement with weakened

[2] These are instances of logical perturbations of the original statement given; you do not need to submit such corrected versions, though trying to come up with such corrected versions will help you understand the concepts and prepare you for the exams better.

conclusion is true:

$$\forall A, B \in \mathcal{P}(X) : R(A \cap B) \subseteq R(A) \cap R(B).$$

⌟

Proof: Let $A, B \subseteq X$. As $A \cap B \subseteq A$, $R(A \cap B) \subseteq R(A)$. Similarly as $A \cap B \subseteq B$, $R(A \cap B) \subseteq R(B)$. Thus $R(A \cap B) \subseteq R(A) \cap R(B)$.

⌟

Claim: The following version of the third statement with an additional hypothesis is true:

$$\begin{aligned} & \forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2 \\ \iff \quad & \forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B). \end{aligned}$$

⌟

Proof: ($\Longrightarrow$) Suppose R satisfies

$$\forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2;$$

one might call this the **injectivity condition** (aka **horizontal line test**) for relations. Let $A, B \subseteq X$, and $y \in R(A) \cap R(B)$. Then there are $a \in A$ and $b \in B$ such that $(a, y), (b, y) \in R$. By the injectivity condition, this forces that $a = b$, and consequently that this element is in $A \cap B$. Thus $y \in R(A \cap B)$, so that $R(A) \cap R(B) \subseteq R(A \cap B)$. Since $R(A) \cap R(B) \supseteq R(A \cap B)$ for any relation R by the previous claim, we have $R(A) \cap R(B) = R(A \cap B)$.

($\Longleftarrow$) Suppose R fails the injectivity condition, so that there are $x_1, x_2 \in X$ and $y \in Y$ such that $x_1 \neq x_2$ and $(x_1, y), (x_2, y) \in R$. Define $A = \{x_1\}, B = \{x_2\}$. Then $A \cap B = \emptyset$, so $R(A \cap B)$, but $y \in R(A) \cap R(B)$, that is, $R(A \cap B) \subsetneq R(A) \cap R(B)$.

⌟

Claim: The fourth statement is false in general. In other words, the image of the relative complement of two sets need not be equal to the relative complement of the images.

⌟

Proof: Recalling that $A \setminus B = A \cap B^{\complement}$, the counterexample we gave earlier serves as a counterexample for this claim also. Indeed, put $X = Y = [0,1]$, $A = [0,1/4]$, $B = [3/4,1]$, $f : X \to Y$, $x \mapsto 1/2$. Then $A \cap B^{\complement} = [0,1/4]$, hence $f(A \setminus B) = \{1/2\}$; on the other hand $f(A) = f(B) = \{1/2\}$, so that $f(A) \setminus f(B) = \emptyset$.

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Follow-up Exercise 1 The second and the (correct versions) of the third statements hold for arbitrary collections $\mathcal{A} \subseteq \mathcal{P}(X)$.

Follow-up Exercise 2 The fourth statement has correct versions (one involving a "subset of" relation, and one involving the injectivity condition; similar to the correct versions of the third statement).

Follow-up Exercise 3 If $f : X \to Y$ is a function, then the preimage relation f^{-1} satisfies the injectivity condition[3]. Consequently, taking preimages commutes with unions, intersections and relative complements.

[3] Recall that f^{-1} really is an abbreviation for $\mathrm{graph}(f)^{-1}$, and f^{-1} is not necessarily a function from Y to X.

2 PSet 2, Problem 3

Let $A \subseteq \mathbb{R}$. Then

(i) The set of all upper bounds of A is $[\sup(A), \infty[$.

(ii) The set of all lower bounds of A is $]-\infty, \inf(A)]$.

(iii) The smallest closed interval containing A is

$$[\inf(A), \sup(A)].$$

(iv) The largest open interval contained in A is

$$]\inf(A), \sup(A)[.$$

Let $A \subseteq \mathbb{R}$. First let us set some notation for convenience. Define by $\mathcal{U}(A)$, $\mathcal{L}(A)$ the sets of all upper and lower bounds of A, respectively:

$$\mathcal{U}(A) = \{U \in \mathbb{R} | \forall a \in A : a \leq U\},$$
$$\mathcal{L}(A) = \{L \in \mathbb{R} | \forall a \in A : L \leq a\}.$$

Note that at this stage even though we can define these sets, we make no claim as to whether or not they are nonempty. The problem does not specify whether A is bounded, in which case the supremum and infimum may be $\pm\infty$. With this foresight we further define the sets $\overline{\mathcal{U}}(A)$, $\overline{\mathcal{L}}(A)$ of extended upper and lower bounds[4] of A, respectively:

[4] Almost never is it preferable to take the extended real number ∞ as a possible upper bound; we make these definitions to handle the way this problem is phrased specifically.

$$
\begin{aligned}
\overline{\mathcal{U}}(A) &= \{U \in [-\infty, \infty] | \forall a \in A : a \leq U\}, \\
\overline{\mathcal{L}}(A) &= \{L \in [-\infty, \infty] | \forall a \in A : L \leq a\}.
\end{aligned}
$$

Before continuing with the proofs, let us make the following observation: if U is an upper bound, then any number larger than U too is an upper bound, and similarly one can replace a lower bound by an even lower number. Thus we have:

$$
\begin{aligned}
\mathcal{U}(A) &= \bigcup_{U \in \mathcal{U}(A)} [U, \infty[\ , \qquad & \mathcal{L}(A) &= \bigcup_{L \in \mathcal{L}(A)}]-\infty, L], \\
\overline{\mathcal{U}}(A) &= \bigcup_{U \in \overline{\mathcal{U}}(A)} [U, \infty] \ , \qquad & \overline{\mathcal{L}}(A) &= \bigcup_{L \in \overline{\mathcal{L}}(A)} [-\infty, L]. \qquad (\star)
\end{aligned}
$$

Let us note that separating the cases based on whether or not A is empty will be useful, as many objects mentioned in the problem operate differently (if at all!) for the emptyset.

The $\boxed{A = \emptyset}$ **Case** In this case, depending on the convention we have either that neither the supremum nor the infimum of A is defined, or else by definition we have $\sup(A) = -\infty$, $\inf(A) = \infty$. In the first case, all parts of the problem use undefined terms, hence they are nonsensical and there is nothing to do[5]!

In the second case however we can make sense of at least some of the statements. In this case we have the following vacuously:

[5] Technically this paragraph by itself, that is, noticing that for $A = \emptyset$, relative to a certain convention, the problem does not make sense, is a perfectly valid solution to this problem. Still, not engaging with a problem due to a choice of conventions is not good form; generally it is prefarable to relax conventions to make sense of more.

$$\mathcal{U}(\emptyset) = \mathbb{R} = \mathcal{L}(\emptyset), \qquad \overline{\mathcal{U}}(\emptyset) = [-\infty, \infty] = \overline{\mathcal{L}}(\emptyset),$$

Further, (†) the smallest interval containing $\emptyset$ as well as the largest interval contained in $\emptyset$ both either do not exist or is the emptyset itself (which can be written as for instance $]0, 0[$).

Now onto the four subsets specified in the problem:

(i) $[\sup(\emptyset), \infty[= [-\infty, \infty[$ does not match with $\mathcal{U}(\emptyset)$ nor $\overline{\mathcal{U}}(\emptyset)$, so this part is false for the emptyset.

(ii) Similarly $]-\infty, \inf(\emptyset)] =]-\infty, \infty]$ does not match with $\mathcal{L}(\emptyset)$ nor $\overline{\mathcal{L}}(\emptyset)$, so this part too is false for the emptyset.

(iii) $[\inf(A), \sup(A)] = [\infty, -\infty]$ is not compatible with the interval notation. Even if we were to take it as extended reals with the opposite orientation, this still is incompatible with the observation (†) above, so this part is false for the emptyset.

(iv) Similarly $]\inf(A), \sup(A)[=]\infty, -\infty[$ is either nonsensical or it is incompatible with (†), hence this part too is false for the emptyset.

Thus we see that the case $A = \emptyset$ is a very inconvenient edge case for this problem[6].

The $\boxed{A \neq \emptyset}$ Case Let us now assume that A is nonempty, so that we have

$$-\infty \leq \inf(A) \leq \sup(A) \leq \infty.$$

[6]Typically in mathematics the most trivial objects often end up being edge cases that are unenlightening. For this reason often people don't even consider them (perhaps without stating that they are not) (case in point: this problem is a variant of Exercise 1.4.3 from the textbook). Of course there are also edge cases that end up being very important (especially if there are undetected edge detection errors prior.)

Thus in this case at least all four parts of the problem are syntactic[7].

(i) If A is not bounded from above, then by definition

$$\mathcal{U}(A) = \emptyset, \quad \overline{\mathcal{U}}(A) = \{\infty\}, \quad [\sup(A), \infty[= [\infty, \infty[= \emptyset.$$

Thus the statement is correct (assuming infinite upper bounds are not allowed). If A is bounded from above, then by definition $\sup(A) = \min \mathcal{U}(A) \in \mathcal{U}(A)$, hence by $(\star)$ above,

$$\mathcal{U}(A) = [\min \mathcal{U}(A), \infty[, \quad \overline{\mathcal{U}}(A) = [\min \mathcal{U}(A), \infty],$$
$$[\sup(A), \infty[= [\min \mathcal{U}(A), \infty[.$$

Hence the statement is correct (assuming infinite upper bounds are not allowed).

(ii) Arguing as in the previous part (or alternatively using the above argument with $B = -A$, together with an exercise mentioned in class), the statement is correct (assuming infinite lower bounds are not allowed).

(iii) If A is unbounded from above and below, then $[\inf(A), \sup(A)] = [-\infty, \infty]$ by definition and again by definition it can not be contained in a closed and bounded interval (with finite numbers as endpoints). Thus the third statement is correct for unbounded A. If A bounded both from above and below, then $-\infty \lneq \inf(A) \leq \sup(A) \lneq \infty$, hence $[\inf(A), \sup(A)] \subseteq \mathbb{R}$. By definition $\inf(A)$ is a lower bound for any element in A and similarly $\sup(A)$ is an

[7] Again we could have assumed that the infima and suprema of unbounded sets are not defined, but in this case what we assume in this regard does not make a major difference.

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upper bound for any element in A, thus $A \subseteq [\inf(A), \sup(A)]$. If $[\lambda, \rho] \subseteq \mathbb{R}$ were a closed interval containing A, then λ would be a lower bound and ρ would be an upper bound for A; hence by definition $\lambda \leq \inf(A)$ and $\sup(A) \leq \rho$. That is $[\inf(A), \sup(A)] \subseteq [\lambda, \rho]$. In other words, $[\inf(A), \sup(A)]$ indeed is the smallest closed interval containing A.

(iv) This part breaks down even for bounded subsets A. For instance consider $A =]-1, 0[\cup]0, 1[$. Then $\inf(A) = -1$ and $\sup(A) = 1$, but certainly $]\inf(A), \sup(A)[=]-1, 1[$ is not a subset of A, as $0 \notin A$.

Let us summarize the statement we proved, without mentioning the emptyset, as it is rather unlikely that the statement for the emptyset will be of any use in the future:

Claim: Let $A \subseteq \mathbb{R}$. If $A \neq \emptyset$, then

(i) $\mathcal{U}(A) = [\sup(A), \infty[\subseteq \mathbb{R}$,

(ii) $\mathcal{L}(A) =]-\infty, \inf(A)] \subseteq \mathbb{R}$,

(iii) The smallest closed interval containing A is $[\inf(A, \sup(A)] \subseteq [-\infty, \infty]$

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We make a final remark to contextualize this problem: We had discussed the "clamp" heuristic for supremum and infimum in class, and the problem really is a formalization of this heuristic: increasing L in $]-\infty, L]$ corresponds to tightening the lower clamp, decreasing U in $[U, \infty[$ corresponds to tighthening the upper clamp, and the numbers L and U that correspond to the tighthest possible configuration are the infimum and supremum, respectively. The heuristic works for clamping from without. While there are physical devices that clamp from within, they are different in principle than the abstract notions on infimum and supremum, whence the fourth part is incorrect.

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Follow-up Exercise 1 Let $A \subseteq \mathbb{R}$. Then A is a singleton if and only if $\inf(A) = \sup(A)$.

Follow-up Exercise 2 There are conditions one can impose on a nonempty and bounded $A \subseteq \mathbb{R}$ that would guarantee that the largest open interval contained in A is $]\inf(A), \sup(A)[$.

Follow-up Exercise 3 Let $A \subseteq \mathbb{R}$ be nonempty and bounded. Then there is a property P (distinct from the property of "being equal to $]\inf(A), \sup(A)[$") that any given interval may or may not have such that $]\inf(A), \sup(A)[$ is the largest open interval with property P.

3 PSet 2, Problem 4

Let $A, B \subseteq \mathbb{R}$. Then

(i) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

(ii) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$.

Again we separate cases: either at least one of A and B is empty, xor neither A nor B is empty.

The $\boxed{A = \emptyset \textbf{ or } B = \emptyset}$ **Case** WLOG, we may assume $A \neq \emptyset = B$. Then $A \cup B = A$ and

$$\sup(A \cup B) = \sup(A) > -\infty = \sup(\emptyset) = \sup(B),$$
$$\inf(A \cup B) = \inf(A) < \infty = \inf(\emptyset) = \inf(B).$$

Since the maximum of an extended real number R and $-\infty$ is R by definition, the first part follows. Similarly since the minimum of an extended real number R and ∞ is R by definition the second part is also true.

The $\boxed{A \neq \emptyset \neq B}$ **Case** For the first part, WLOG we may assume that $\sup(A) \geq \sup(B)$. Thus it suffices to show that $\sup(A) = \sup(A \cup B)$. First note that as $A \subseteq A \cup B$, by the monotonicity of supremum we have

$$-\infty \lneq \sup(B) \leq \sup(A) \leq \sup(A \cup B) \leq \infty.$$

Either $A \cup B$ is unbounded from above, xor it is bounded from above. If $A \cup B$ is unbounded from above, then at least one of A or

B would have to be unbounded from above. If A is unbounded from above, then $\sup(A) = \infty = \sup(A \cup B)$ concludes the proof. If B is unbounded from above, then $\infty = \sup(B) \le \sup(A) \le \sup(A \cup B) \le \infty$, hence again $\sup(A) = \sup(A \cup B)$ (and consequently A is unbounded from above also).

Now suppose $A \cup B$ is bounded from above; then the same is true for both A and B, and we have

$$-\infty \lneq \sup(B) \le \sup(A) \le \sup(A \cup B) \lneq \infty.$$

Say $\sup(A) \lneq \sup(A \cup B)$. Then $\sup(A)$ is not an upper bound for $A \cup B$, so that there is an $x \in A \cup B$ such that

$$\sup(A) < x.$$

Since x exceeds the least upper bound for A, $x \notin A$, thus it must be the case that $x \in B$. But then

$$x \le \sup(B) \le \sup(A) < x;$$

a contradiction.

One can argue analogously for the statement involving infima. Alternatively one can argue as follows[8]:

$$\begin{aligned} \inf(A \cup B) &= -\sup(-(A \cup B)) \\ &= -\sup((-A) \cup (-B)) \\ &= -\max\{\sup(-A), \sup(-B)\} \\ &= -\max\{-\inf(A), -\inf(B)\} \\ &= \min\{\inf(A), \inf(B)\} \end{aligned}$$

[8] You should check these steps if they are not clear!

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We have thus proved that indeed the statements in the problem are correct without any modifications:

Claim: $\forall A, B \subseteq \mathbb{R}$:

(i) $\sup(A \cup B) = \max\{\sup(A)\,,\ \sup(B)\}$

(ii) $\inf(A \cup B) = \min\{\inf(A)\,,\ \inf(B)\}$.

⌟

Follow-up Exercise 1 $\forall \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$:

(i) $\sup \left(\bigcup_{A \in \mathcal{A}} A\right) = \sup_{A \in \mathcal{A}} \sup(A)$,

(ii) $\inf \left(\bigcup_{A \in \mathcal{A}} A\right) = \inf_{A \in \mathcal{A}} \inf(A)$.

Follow-up Exercise 2 $\forall \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$:

(i) $\sup \left(\bigcap_{A \in \mathcal{A}} A\right) = \inf_{A \in \mathcal{A}} \sup(A)$,

(ii) $\inf \left(\bigcap_{A \in \mathcal{A}} A\right) = \sup_{A \in \mathcal{A}} \inf(A)$.

4 PSet 3, Problem 3

Let $x_\bullet$ be a sequence of real numbers that converges to zero. Then for any bounded sequence $b_\bullet$ of real numbers,

$$\lim_{n\to\infty} x_n b_n = 0.$$

This claim is indeed true without any modifications:

Claim: $\forall x_\bullet, b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$: if $x_\bullet \to 0$ and $\sup_m b_m < \infty$, then $x_\bullet b_\bullet \to 0$. ⌟

Proof: Instead of starting with an $\epsilon > 0$, it's often useful to start from the end, in other words with the expression we are trying to bound. For any $n \in \mathbb{Z}_{\geq 0}$, one has

$$|x_n b_n| = |x_n|\,|b_n| \leq |x_n| \underbrace{\sup_{m\in\mathbb{Z}_{\geq 0}} |b_m|}_{<\infty} \qquad (\star)$$

Put $M = \sup_{m\in\mathbb{Z}_{\geq 0}} |b_m| \in \mathbb{R}_{\geq 0}$. We know that this is a finite nonnegative number, as the sequence $b_\bullet$ (and consequently the sequence $|b_\bullet|$) is bounded. Thus to bound $|x_n b_n|$ from above by ε, we can bound $|x_n|$ by ε/M. Indeed, let $\varepsilon \in \mathbb{R}_{>0}$. Then as $x_\bullet \to 0$, there is an $N \in \mathbb{Z}_{\geq 0}$ such that for any $n \in \mathbb{Z}_{\geq N}$, we have

$$|x_n| = |x_n - 0| < \frac{\varepsilon}{M}.$$

Here we used the number $\frac{\varepsilon}{M}$ as the "epsilon" in the definition of $x_\bullet$ converging to 0. For any $n \in \mathbb{Z}_{\geq N}$, $(\star)$ also holds; hence combining the two we have that for any $n \in \mathbb{Z}_{\geq N}$:

$$|x_n b_n - 0| = |x_n b_n| < \frac{\varepsilon}{M} M = \varepsilon.$$

$\lrcorner$

5 PSet 4, Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\forall x, y \in \mathbb{R} : f(y) - f(x + y) + f(x) = 0$. Then the following are equivalent:

(i) f is linear.

(ii) f is continuous at x_* for some $x_* \in \mathbb{R}$.

(iii) f is continuous at x_* for any $x_* \in \mathbb{R}$.

The statement is true without any modifications. To see this we first will look into the consequences of the functional equation in the hypothesis of the problem; we package these consequences into the following lemma:

Lemma: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\forall x, y \in \mathbb{R} : f(y) - f(x + y) + f(x) = 0$. Then

(i) $f(0) = 0$.

(ii) $\forall x \in \mathbb{R} : f(-x) = -f(x)$.

(iii) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} : f(nx) = nf(x)$.

(iv) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} : n \neq 0 \implies f(x) = \frac{f(nx)}{n}$.

(v) $\forall x \in \mathbb{R}, \forall q \in \mathbb{Q} : f(qx) = qf(x)$.

⌟

Proof: • The first part follows from

$$f(0) = f(0 + 0) = f(0) + f(0) = 2f(0).$$

• The second part follows from the first part by

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x).$$

• The third part follows by induction. Indeed by the second part it suffices to verify the equation for $n \in \mathbb{Z}_{\geq 1}$. Indeed, for the base step we have $f(2x) = f(x + x) = 2f(x)$ and for the induction step we have $f(nx) = f((n-1)x + x) = f((n-1)x) + f(x)$.

• The fourth part follows immediately from the third part.

• Finally for the fifth part, say $q = \frac{m}{n}$ with m and n integers. Then

$$f(qx) = f\left(\frac{mx}{n}\right) = \frac{f(mx)}{n} = \frac{mf(x)}{n} = qf(x),$$

where the second equality follows from the fourth part and the third equality follows from the third part.

⌟

In the above claim the most important consequence is the last part; namely it says that any additive group homomorphism of the reals must be linear over the rationals.

Claim: Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\forall x, y \in \mathbb{R} : f(y) - f(x + y) + f(x) = 0$. Then the following are equivalent:

(i) f is linear.

(ii) f is continuous at x_* for any $x_* \in \mathbb{R}$.

(iii) f is continuous at x_* for some $x_* \in \mathbb{R}$.

⌟

Proof: The implications (i) $\implies$ (ii) $\implies$ (iii) are straightforward; indeed if f is linear, so that $f(x) = xf(1)$, then by the limit law that a constant multiple of a convergent sequence is convergent, f is continuous. Moreover if f is continuous at any point then in particular it is continuous at some point (say e.g. at 0).

Thus it suffices to show that (iii) $\implies$ (i). Let $x_* \in \mathbb{R}$ be such that f is continuous at x_*. Furthermore let $x \in \mathbb{R}$ be arbitrary. We claim that $f(x) = xf(1)$.

By the density of rationals axiom, there is a sequence $q_\bullet$ of rational numbers such that $q_\bullet \to x$ [9]. Then by limit laws

$$\lim_{n\to\infty} q_n - x + x_* = x_*.$$

As f is continuous at x_*, it must be the case that

$$\lim_{n\to\infty} f(q_n - x + x_*) = f(x_*).$$

Moreover by the last part of the lemma and the hypothesis we have $f(q_\bullet - x + x_*) = q_\bullet f(1) - f(x) + f(x_*)$, so that

$$\lim_{n\to\infty} q_n f(1) - f(x) + f(x_*) = f(x_*).$$

Again by the limit laws we thus have

$$f(x) = \lim_{n\to\infty} q_n f(1) = xf(1).$$

⌟

[9] More explicitly one can construct a $q_\bullet$ as follows: $x - 1 < x$, hence by the density of rationals there is a $q_1 \in]x-1, x[$. Then again by the density of rationals there is a $q_2 \in]q_1, x[$. By the principle of mathematical induction we thus obtain a strictly increasing sequence of rational numbers that converges to x.

6 PSet 5, Problem 5

Let $\lambda, \rho \in [-\infty, \infty]$, $\lambda < \rho$, $I =]\lambda, \rho[$, and $x_* \in I$. Two functions $f, g : I \to \mathbb{R}$ are said to be **tangent** at x_* if $f(x_* + h) = g(x_* + h) + o_{h \to 0}(h)$. Then

(i) "Being tangent at x_*" is an equivalence relation on $F(I; \mathbb{R})$.

(ii) If $f, g : I \to \mathbb{R}$ are two functions tangent at x_*, then $f(x_*) = g(x_*)$.

(iii) If $f : I \to \mathbb{R}$ is a function, then there is at most one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*.

(iv) If $f : I \to \mathbb{R}$ is a function, then f is differentiable at x_* iff there is exactly one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*.

(v) If $f : I \to \mathbb{R}$ is differentiable at x_*, then for any function $g : I \to \mathbb{R}$ tangent to f at x_* and differentiable at x_*, $f'(x_*) = g'(x_*)$.

The first part is true without any modifications:

Claim: "Being tangent at x_*" is an equivalence relation on $F(I; \mathbb{R})$.

⌟

Proof: Let us abbreviate "being tangent at x_*" by $\sim_{x_*}$, so that the relation in question formally is

$$\mathcal{R}_{x_*} = \{(f, g) \in F(I; \mathbb{R}) \times F(I; \mathbb{R}) | f \sim_{x_*} g\}.$$

We need to verify three properties; namely reflexivity, symmetry and transitivity.

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For reflexivity, let $f \in F(I;\mathbb{R})$. Then as $0 = o(h)$[10], $f(x_* + h) = f(x_* + h) + o(h)$, so that $f \sim_{x_*} f$.

For symmetry, let $f, g \in F(I;\mathbb{R})$. Then

$$\begin{aligned}
f \sim_{x_*} g &\Leftrightarrow f(x_* + h) = g(x_* + h) + o(h) \\
&\Leftrightarrow \lim_{h \to 0} \frac{f(x_* + h) - g(x_* + h)}{h} = 0 \\
&\Leftrightarrow \lim_{h \to 0} \frac{g(x_* + h) - f(x_* + h)}{h} = 0 \\
&\Leftrightarrow g(x_* + h) = f(x_* + h) + o(h) \Leftrightarrow g \sim_{x_*} f.
\end{aligned}$$

Finally for transitivity, let $f, g, k \in F(I;\mathbb{R})$ be such that

$$f \sim_{x_*} g \sim_{x_*} k.$$

We need to show that $f \sim_{x_*} k$. This is indeed true since

$$\begin{aligned}
f(x_* + h) &= g(x_* + h) + o(h) \\
&= k(x_* + h) + o(h) + o(h) \\
&= k(x_* + h) + o(h).
\end{aligned}$$

where in the last equation we used the fact that the sum of two functions that decay faster than h does as $h \to 0$, too decays faster than h does as $h \to 0$.

⌟

The second part is not quite correct as given:

Claim: There are functions $f, g : I \to \mathbb{R}$ tangent at x_*, such that

[10] Throughout we abbreviate the more explicit $o_{h\to0}(h)$ by $o(h)$ as it is clear from context that the asymptotics are as $h \to 0$.

$f(x_*) \neq g(x_*)$.

┘

Proof: Consider for instance $I =]-1, 1[$, and $f, g : I \to \mathbb{R}$ defined by

$$f : x \mapsto x, \qquad g : x \mapsto \begin{cases} x, & \text{if } x \neq 0 \\ 1, & \text{if } x=0 \end{cases} .$$

Finally put $x_* = 0$. Then for any $h \neq 0$, $f(x_* + h) = g(x_* + h)$, hence a fortiori, $f(x_* + h) = g(x_* + h) + o(h)$, so that f and g are tangent at x_*, but these two functions don't take the same value at x_*.

┘

Note that in the counterexample we gave above one of the functions is discontinuous. Indeed, if we assume continuity at x_*, the statement in the second part is true:

Claim: For any $f, g \in F(I; \mathbb{R})$, if f and g are both continuous at x_*, and $f \sim_{x_*} g$ then $f(x_*) = g(x_*)$.

┘

Proof: Let $f, g : I \to \mathbb{R}$ be two functions continuous at x_*. Note that we can describe the continuity of these two functions by

$$f(x_* + h) = f(x_*) + o(1), \qquad g(x_* + h) = g(x_*) + o(1),$$

as $o(1) = o_{h \to 0}(1)$ represents a function that decays to zero as $h \to 0$ (albeit possibly not faster than h itself does). Suppose further that f is tangent to g at x_*. We'll use the epsilon of room principle to show that $f(x_*) = g(x_*)$, that is, we'll show that $|f(x_*) - g(x_*)|$ is bounded by an expression that is arbitrarily small:

$$\begin{aligned}|f(x_*) - g(x_*)| &\leq |f(x_*) - f(x_* + h)| \\ &\quad + |f(x_* + h) - g(x_* + h)| \\ &\quad + |g(x_* + h) - g(x_*)| \\ &\leq o(1) + o(h) + o(1) \\ &= o(1).\end{aligned}$$

⌟

On the other hand, without the continuity assumption, the sufficiency breaks down:

Claim: There is a function $f : I \to \mathbb{R}$ such that there is a unique affine function $\ell : I \to \mathbb{R}$ tangent to f at x_*, however f is not differentiable at x_*.

⌟

Proof: We can build on the previous counterexample we gave. Consider $I =]-1, 1[$, $x_* = 0$ and $f, g : I \to \mathbb{R}$ defined by

$$f : x \mapsto x, \qquad g : x \mapsto \begin{cases} x, & \text{if } x \neq 0 \\ 1, & \text{if } x{=}0 \end{cases}.$$

Then f is an affine function and is tangent to g at x_* as discussed in the previous counterexample. Any other affine function ℓ tangent to g at x_* must also be tangent to f at x_* by the first claim, and we'll show in the next claim that $\ell = f$. Consequently there is a unique affine function tangent to g at x_*, however g is not continuous at x_*, let alone differentiability.

⌟

The third statement is true without modification.

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Claim: If $f : I \to \mathbb{R}$ is a function, then there is at most one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*.

⌟

Proof: Let $f, \ell_1, \ell_2 : I \to \mathbb{R}$ be three functions such that

$$\ell_1 \sim_{x_*} f \sim_{x_*} \ell_2.$$

Further assume that there are numbers $A_1, B_1, A_2, B_2 \in \mathbb{R}$ such that

$$\ell_1 : x \mapsto A_1x + B_1, \qquad \ell_2 : x \mapsto A_2x + B_2.$$

As being tangent at x_* is an equivalence relation, we have that ℓ_1 is tangent to ℓ_2 at x_*. We claim that $A_1 = A_2$ and $B_1 = B_2$. To show these equalities, note that

$$\begin{aligned} \ell_1(x_* + h) &= \ell_2(x_* + h) + o(h) \\ &\Leftrightarrow (A_1 - A_2)(x_* + h) + (B_1 - B_2) = o(h) \\ &\Leftrightarrow [(A_1 - A_2)x_* + (B_1 - B_2)] + (A_1 - A_2)h = o(h). \end{aligned}$$

The only way this can happen is if $A_1 - A_2 = 0$, as the $(A_1 - A_2)h$ term can not decay faster than h does otherwise[11]. This further implies that $(B_1 - B_2) = 0$ for otherwise the last expression would not be decaying to zero[12].

⌟

Note that the previous claim does not say anything about whether or not an tangent affine function exists for a given arbitrary function

[11] Indeed, its decay matches exactly the decay of h.

[12] Alternatively we could have used the previous claim together with the observation that both ℓ_1 and ℓ_2 are polynomials and hence continuous functions.

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f; indeed the existence of a tangent affine function is equivalent to differentiability, but if we additionally assume continuity:

Claim: If $f : I \to \mathbb{R}$ is a function, then f is differentiable at x_* iff f is continuous at x_* and there is exactly one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ such that $f \sim_{x_*} \ell$.

⌟

Proof: ($\Rightarrow$) If f is differentiable at x_*, then it is continuous at x_* (as discussed in class). Further, putting $A = f'(x_*)$ and $B = f(x_*)$ defines an affine function tangent to f at x_*, and by the previous claim this is the only affine function tangent to f at x_*.

($\Leftarrow$) If f is continuous at x_* and $\ell : x \mapsto Ax + B$ is the (an) affine function tangent to f at x_*, then as ℓ is a polynomial it is continuous, and so by a previous claim $f(x_*) = Ax_* + B$. Then

$$\begin{aligned} f(x_* + h) &= \ell(x_* + h) + o(h) \\ &= (Ax_* + B) + Ah + o(h) \\ &= f(x_*) + Ah + o(h); \end{aligned}$$

This is equivalent to the statement that f is differentiable at x_*, and $f'(x_*) = A$.

⌟

Claim: If $f : I \to \mathbb{R}$ is differentiable at x_*, then for any function $g : I \to \mathbb{R}$ tangent to f at x_* and differentiable at x_*, $f'(x_*) = g'(x_*)$.

⌟

Proof: If f and g are differentiable and tangent at x_*, then by a previous claim, the unique affine function tangent to f at x_* is

$$\ell : x \mapsto Ax + B$$

for $A = f'(x_*)$ and $B = f'(x_*)x_* + f(x_*)$. But this affine function ℓ also is the unique affine function tangent to g at x_*, so that again we have $f'(x_*) = A = g'(x_*)$[13].

⌟

[13]We also have $f(x_*) = g(x_*)$ from a previous claim as differentiability implies continuity.

Follow-up Exercise Put for $f, g \in F(I;\mathbb{R})$,

$$f \sim'_{x_*} g \Leftrightarrow \limsup_{h\to 0} \frac{f(x_*+h)-g(x_*+h)}{h},$$

where for a real valued function ϕ,

$$\limsup_{h\to 0} \phi(h) = \inf_{\delta\in\mathbb{R}_{>0}} \sup_{h\in]0,\delta[} \phi(h).$$

Then

(i) $\sim'_{x_*}$ too is an equivalence relation on $F(I;\mathbb{R})$.

(ii) If $f, g \in F(I;\mathbb{R})$, then $f \sim'_{x_*} g$ iff $f \sim_{x_*} g$.

Follow-up Exercise 2 Let $\beta : I \to \mathbb{R}_{>0}$ be a positive function. Then

(i) The following defines an equivalence relation on $F(I;\mathbb{R})$:

$$f \sim_{x_*,\beta} g \Leftrightarrow f(x_*+h) = g(x_*+h) + o_{h\to 0}(\beta(h)).$$

(ii) Interpret $\sim_{x_*,\beta}$ geometrically for $\beta : h \mapsto h^2$.

Follow-up Exercise 3 The relation "being tangent up to order β at a point" makes sense for functions between metric spaces.

7 PSet 6, Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function define

$$J : \mathbb{Z}_{\geq 0} \times \mathbb{R} \to \mathbb{R}, (n, x) \mapsto \frac{d(f^n)}{dx}(x).$$

Then J satisfies the following identity:

$$\forall n, m \in \mathbb{Z}_{\geq 0}, \forall x \in \mathbb{R} : J(n + m, x) = J(n, f^m(x))\, J(m, x).$$

The statement given is true without any modifications:

Claim: For any differentiable $f : \mathbb{R} \to \mathbb{R}$,

$$\forall n, m \in \mathbb{Z}_{\geq 0}, \forall x \in \mathbb{R} : J(n + m, x) = J(n, f^m(x))\, J(m, x).$$

⌟

Proof: This is a matter of applying the chain rule. Let n and m be nonnegative integers, and $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} J(n + m, x) &= \frac{d(f^{n+m})}{dx}(x) \\ &= \frac{d(f^n \circ f^m)}{dx}(x) \\ &= \frac{d(f^n)}{dx}(f^m(x))\frac{d(f^m)}{dx}(x) \\ &= J(n, f^m(x))J(m, x), \end{aligned}$$

where in the second equality we used the fact that $f^{n+m} = f^n \circ f^m$[14], and in the third equality we used the chain rule.

⌟

[14] In case a more formal account is needed, this can be proved using induction and the associativity of function composition.

8 PSet 7, Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then

(i) If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist and are finite, then $\lim_{x\to\infty} f'(x) = 0$.

(ii) If $\lim_{x\to\infty} f'(x) = 0$, then $\lim_{x\to\infty} f(x)$ exists and is finite.

The first part is correct without any modification:

Claim: For any differentiable $f : \mathbb{R} \to \mathbb{R}$, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist and are finite, then $\lim_{x\to\infty} f'(x) = 0$.

┘

Proof: By the Mean Value Theorem, for any $x \in \mathbb{R}$, there is a $y_x \in]x, x+1[$ such that

$$f(x+1) - f(x) = f'(y_x). \qquad (\dagger)$$

Say $\lim_{x\to\infty} f(x) = A$, and let $x_\bullet$ be a strictly increasing sequence of real numbers such that $x_\bullet \to \infty$. Then $x_\bullet + 1 \to \infty$ and $y_{x_\bullet} \to \infty$. Thus plugging in the sequence $x_\bullet$ in $(\dagger)$ and taking the limit as $n \to \infty$, by hypothesis we have

$$\begin{aligned} 0 &= \lim_{x\to\infty} f(x) - \lim_{x\to\infty} f(x) \\ &= \lim_{n\to\infty} f(x_n + 1) - \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty} (f(x_n + 1) - f(x_n)) \\ &= \lim_{n\to\infty} f'(y_{x_n}) \\ &= \lim_{x\to\infty} f'(x). \end{aligned}$$

Here in the third equality we used a limit law and in the fourth equality we used (†).

⌟

The second part is incorrect; indeed there are functions that increase less and less but still does not converge to a finite limit:

Claim: There are differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} f'(x) = 0$ but, $\lim_{x\to\infty} f(x) = \infty$

⌟

Proof: First consider $g(x) = \log(x)$. While this function is not defined everywhere on $\mathbb{R}$, it has the desired asymptotic properties; indeed its derivative is $1/x$, so that $\lim_{x\to\infty} g'(x) = 0$, but $\lim_{x\to\infty} g(x) = \infty$.

To get a true example we can modify g a bit. For instance instead of g, consider

$$f(x) = \begin{cases} \log(x) & \text{, if } x \geq 1 \\ x - 1 & \text{, if } x < 1 \end{cases}.$$

Then it is immediate that everywhere except at $x = 1$ $f(x)$ is differentiable. For continuity at $x = 1$, we note that the two piecewise formulas for f match at $x = 1$ and their common value is 0, so that f is continuous everywhere. For differentiability at $x = 1$, we note that the right limit and left limit at $x = 1$ of the ratio that defines derivatives are both 1, hence f is differentiable at $x = 1$ also, so that f is differentiable everywhere. Certainly f has the same asymptotic properties as $x \to \infty$ as g.

⌟

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9 PSet 8, Problem 5

Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$, $f_\bullet : \mathbb{Z}_{\geq 0} \to R(I;\mathbb{R})$, $f_\infty \in F(I;\mathbb{R})$. If $f_\bullet \to f_\infty$ uniformly, then

(i) f_∞ is Riemann integrable.

(ii) $\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f_\infty(x)dx$.

Both of the claims are true without modification:

Claim: The uniform limit of a sequence of Riemann integrable functions is Riemann integrable.

⌟

Proof: Let $f_\bullet : I \to \mathbb{R}$ be a sequence of Riemann integrable functions converging uniformly to the function $f_\bullet : I \to \mathbb{R}$. We claim that $f_\bullet$ is Riemann integrable. Riemann integrability is defined only for bounded functions; so first we prove that $f_\bullet$ is bounded[15]. By uniform convergence, choosing $\varepsilon = 1$, there is an N such that

$$d_{C^0}(f_N, f_\infty) \leq 1.$$

But then for any $x \in I$, by the triangular inequality we have

$$\begin{aligned} |f_\infty(x)| &\leq |f_\infty(x) - f_N(x)| + |f_N(x)| \\ &\leq d_{C^0}(f_N, f_\infty) + \sup(|f_N|) \\ &\leq 1 + \sup(|f_N|). \end{aligned}$$

Consequently $\sup(|f_\infty|) \leq 1 + \sup(|f_N|) < \infty$, so that f_∞ is bounded.

[15] More generally, the uniform limit of a sequence of bounded functions is bounded. Further, a direct proof of Riemann integrability of f_∞ simultaneously implies that f_∞ must be bounded.

Next we prove that f_∞ is Riemann integrable. Let $\mathcal{P} \in \mathrm{IP}(I)$, $P \in \mathcal{P}$ and $x, y \in P$. Then for n arbitrary we have

$$\begin{aligned} f_\infty(x) - f_\infty(y) &\le |f_\infty(x) - f_\infty(y)| \\ &\le |f_\infty(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_\infty(y)| \\ &\le 2d_{C^0}(f_n, f_\infty) + |f_n(x) - f_n(y)| \\ &\le 2d_{C^0}(f_n, f_\infty) + \sup(f_n|_P) - \inf(f_n|_P). \end{aligned}$$

Thus we have

$$\sup(f_\infty|_P) - \inf(f_\infty|_P) \le 2d_{C^0}(f_n, f_\infty) + \sup(f_n|_P) - \inf(f_n|_P).$$

(Note that this bound also implies that f_∞ must be bounded, as there are only finitely many elements P in $\mathcal{P}$.)

Taking the lower and upper Darboux sums we obtain, for any $\mathcal{P}$ and any n:

$$\begin{aligned} U(f_\infty; \mathcal{P}) - L(f_\infty; \mathcal{P}) &= \sum_{P \in \mathcal{P}} \sup(f_\infty|_P)\ell(P) - \sum_{P \in \mathcal{P}} \inf(f_\infty|_P)\ell(P) \\ &= \sum_{P \in \mathcal{P}} [\sup(f_\infty|_P) - \inf(f_\infty|_P)]\ell(P) \\ &\le \sum_{P \in \mathcal{P}} [2d_{C^0}(f_n, f_\infty) + \sup(f_n|_P) - \inf(f_n|_P)]\ell(P) \\ &= 2d_{C^0}(f_n, f_\infty)\ell(I) + U(f_n; \mathcal{P}) - L(f_n; \mathcal{P}). \end{aligned}$$

Let $\varepsilon \in \mathbb{R}_{>0}$. Then there is an $N = N(\varepsilon)$ such that

$$2d_{C^0}(f_n, f_\infty)\ell(I) < \varepsilon/2.$$

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f_N is Riemann integrable, thus by one of the characterizations of Riemann integrability, there is a partition $\mathcal{P} = \mathcal{P}(\varepsilon, N) \in \mathrm{IP}(I)$ such that

$$U(f_N; \mathcal{P}) - L(f_N; \mathcal{P}) < \varepsilon/2.$$

Thus for this particular partition $\mathcal{P}$, we have

$$\begin{aligned} U(f_\infty; \mathcal{P}) - L(f_\infty; \mathcal{P}) &= 2d_{C^0}(f_n, f_\infty)\ell(I) + U(f_n; \mathcal{P}) - L(f_n; \mathcal{P}) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

┘

Now that we have guaranteed the Riemann integrability of the uniform limit of a sequence of Riemann integrable functions, we move on to the convergence of the Riemann integrals.

Claim: If $f_\bullet \to f_\infty$ uniformly on I and $f_\bullet$ is a sequence of Riemann integrable functions, then

(i) $\int_I |f_\bullet(x) - f_\infty(x)| dx \to 0$.

(ii) $\int_I f_\bullet(x) dx \to \int_I f_\infty(x) dx$.

┘

Proof: We first note that it suffices to verify the first statement, as for any n,

$$\left| \int_I f_n(x) dx - \int_I f_\infty(x) dx \right| \leq \int_I \left| f_n(x) - f_\infty(x) \right| dx.$$

Note that for any $x \in I$, $\left| f_n(x) - f_\infty(x) \right| \leq d_{C^0}(f_n, f_\infty)$, hence by the monotonicity of Riemann integrals, we have

$$0 \leq \int_I \left| f_n(x) - f_\infty(x) \right| dx \leq d_{C^0}(f_n, f_\infty)\ell(I).$$

Taking limits as $n \to \infty$, we are done.

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Follow-Up Exercise Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$, $f_\bullet : \mathbb{Z}_{\geq 0} \to R(I;\mathbb{R})$, $f_\infty \in F(I;\mathbb{R})$. If $f_\bullet \to f_\infty$ pointwise, then f_∞ is Riemann integrable.

Follow-Up Exercise 2 Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$, $f_\bullet : \mathbb{Z}_{\geq 0} \to R(I;\mathbb{R})$, $f_\infty \in F(I;\mathbb{R})$. If $f_\bullet \to f_\infty$ pointwise and f_∞ is Riemann integrable, then

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f_\infty(x)dx.$$

Follow-Up Exercise 3 Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$, $f_\bullet : \mathbb{Z}_{\geq 0} \to R(I;\mathbb{R})$, $f_\infty \in F(I;\mathbb{R})$. If $f_\bullet \to f_\infty$ pointwise, f_∞ is Riemann integrable, and $\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f_\infty(x)dx$, then

$$\lim_{n\to\infty} \int_I \left| f_n(x) - f_\infty(x) \right| dx = 0.$$

10 PSet 9, Problem 1

Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$. Then for any $f \in C^0(I; \mathbb{R})$, there is an $x_* \in]\lambda, \rho[$ such that

$$f(x_*) = \frac{1}{\ell(I)} \int_I f(x)dx.$$

The statement is correct without any modifications:

Claim: If $f : I \to \mathbb{R}$ is a continuous function defined on a closed and bounded interval $I = [\lambda, \rho]$ with positive length, then there is an $x_* \in]\lambda, \rho[$ such that

$$f(x_*) = \frac{1}{\ell(I)} \int_I f(x)dx.$$

⌟

We give two different proofs of this claim; one using the Fundamental Theorem of Calculus and the other using the Intermediate Value Theorem:

Proof (Using FTC)**:** Consider the function $F : I \to \mathbb{R}$, $x \mapsto \int_\lambda^x f(t)dt$. Then by the Second Fundamental Theorem of Calculus, as f is continuous, F is (continuously) differentiable with $F' = f$. Applying the Mean Value Theorem to F on I, we obtain that for some $x_* \in]\lambda, \rho[$,

$$F'(x_*) = \frac{F(\rho) - F(\lambda)}{\rho - \lambda}.$$

But then

$$f(x_*) = F'(x_*) = \frac{F(\rho) - F(\lambda)}{\rho - \lambda} = \frac{1}{\ell(I)} \int_\lambda^\rho f(t)dt,$$

where in the last equality we used the First Fundamental Theorem Calculus.

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⌟

For the proof using IVT we first establish a lemma:

Lemma: Let $g : I \to \mathbb{R}$ be a Riemann integrable function defined on a closed and bounded interval $I = [\lambda, \rho]$ with positive length. If

- for any $x \in I$, $g(x) \gneq 0$ and
- there is some point μ where g is continuous[16],

then $\int_I g(x)dx \gneq 0$.

⌟

Proof: Since g is continuous at μ, for $\varepsilon \in]0, g(\mu)[$, there is a closed subinterval $J \subseteq I$ containing μ of positive length such that and for any $x \in J$, $|g(x) - g(\mu)| < \varepsilon$. Thus

$$\forall x \in J : 0 \lneq g(\mu) - \varepsilon \lneq g(x).$$

The restriction of g to the subinterval J too is Riemann integrable, and by the monotonicity and interval additivity of Riemann integrals we have

[16] This assumption actually is redundant; indeed, the Lebesgue-Vitali Theorem states that a bounded function defined on a closed and bounded interval is Riemann integrable iff it is continuous off a "negligible" set. Consequently any Riemann integrable function must be continuous at least at some point. The reason we state the lemma with this additional redundant assumption is that the Lebesgue-Vitali Theorem uses concepts we haven't covered in MATH 3210. These concepts are related to the more robust theory of integration, compared to Riemann integration, often attributed to Lebesgue.

$$\begin{aligned}
0 &\lneq \ell(J)(g(\mu) - \varepsilon) \\
&= \int_J (g(\mu) - \varepsilon)dx \\
&\leq \int_J g(x)dx \\
&= \int_{[\lambda,\min(J)]} 0dx + \int_J g(x)dx + \int_{[\max(J),\rho]} 0dx \\
&\leq \int_{[\lambda,\min(J)]} g(x)dx + \int_J g(x)dx + \int_{[\max(J),\rho]} g(x)dx \\
&= \int_I g(x)dx.
\end{aligned}$$

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Proof (Using IVT)**:** For any $x \in I$, we have

$$\min(f) \leq f(x) \leq \max(f).$$

Considering the minimum and maximum of f as constant functions and integrating, we obtain

$$\min(f)\ell(I) = \int_I \min(f)dx \leq \int_I f(x)dx \leq \int_I \max(f)dx = \max(f)\ell(I).$$

Dividing the terms by the length of I, we obtain

$$\min(f) \leq \frac{1}{\ell(I)} \int_I f(x)dx \leq \max(f),$$

so that the average of f is in the image of f[17]. Thus by the Intermediate Value Theorem, there is an $x_\dagger \in [\lambda, \rho]$ such that

[17] Recall that the image of a closed and bounded interval under a continuous function is again a closed and bounded interval (of possibly zero length).

$$f(x_\dagger) = \frac{1}{\ell(I)} \int_I f(x)dx.$$

If $x_\dagger \in]\lambda, \rho[$, we are done as we can choose $x_* = x_\dagger$. Otherwise, suppose $x_\dagger \in \{\lambda, \rho\}$. If f is a constant function, then again we are done (as any point can be taken as x_*). Thus suppose f is not constant. We claim that for some $y_\dagger, z_\dagger \in]\lambda, \rho[$,

$$f(y_\dagger) \leq \frac{1}{\ell(I)} \int_I f(x)dx \leq f(z_\dagger).$$

Indeed, if for any $y \in]\lambda, \rho[$, $f(y) \gneq \frac{1}{\ell(I)} \int_I f(x)dx$, then considering the lower bound as a constant function and integrating would give

$$\int_I f(y)dy \gneq \int_I \left(\frac{1}{\ell(I)} \int_I f(x)dx \right) dy = \left(\frac{1}{\ell(I)} \int_I f(x)dx \right) \ell(I) = \int_I f(x)dx,$$

a contradiction. Here in the strict inequality we used the above lemma with $g = f - c$, where c is the mean of f.

A similar argument gives the existence of a $z_\dagger$ as claimed. If $f(y_\dagger) = \frac{1}{\ell(I)} \int_I f(x)dx$ or $\frac{1}{\ell(I)} \int_I f(x)dx = f(z_\dagger)$, we are done. Otherwise again applying the Intermediate Value Theorem to the restriction of f to the closed and bounded interval with endpoints $y_\dagger$ and $z_\dagger$ guarantees the existence of an x_* as claimed.

$\lrcorner$

11 PSet 10, Problem 5

Let $b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a bounded sequence. Then for any sequence $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $\sum_n x_n$ is absolutely convergent iff $\sum_n x_n b_n$ is absolutely convergent.

Only one part of this statement works. We give two proofs; one using Cauchy sequences and one using the Comparison Test:

Claim: Let $b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a bounded sequence. Then for any sequence $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that $\sum_n x_n$ is absolutely convergent $\sum_n x_n b_n$ is absolutely convergent.

⌟

Proof (Using Cauchy Sequences)**:** Recall that convergence of a formal series of real numbers is really the convergence of the sequence of partial sums. Recall also that convergence of a sequence of real numbers is equivalent to the Cauchy property. If each term of $b_\bullet$ is zero, then there is nothing to prove. So suppose otherwise, and put $B = \sup_{n \in \mathbb{Z}_{\geq 0}} |b_n| \in \mathbb{R}_{>0}$.

Let $\varepsilon \in \mathbb{R}_{>0}$. Then by the Cauchy property of the sequence $n \mapsto \sum_{k=0}^{n} |x_k|$, we have that there is an $N \in \mathbb{Z}_{\geq 0}$ such that for any $n \in \mathbb{Z}_{\geq N}$ and for any $p \in \mathbb{Z}_{\geq 0}$,

$$\sum_{k=n}^{n+p} |x_k| \leq \frac{\varepsilon}{B}.$$

Then again for $n \geq N$ and any $p \geq 0$, we have

$$\sum_{k=n}^{n+p} |x_k b_k| \leq \sum_{k=n}^{n+p} |x_k| B < \frac{\varepsilon}{B} B = \varepsilon,$$

so that the sequence $n \mapsto \sum_{k=0}^{n} |x_k b_k|$ too is Cauchy, hence convergent; so that $\sum_n |x_n b_n|$ is convergent, that is to say $\sum_n x_n b_n$ is absolutely convergent.

⌟

Proof (Using Comparison Test)**:** Put $B = \sup_n |b_n|$. Then we have

$$|x_n b_n| \leq B|x_n| = O_{n\to\infty}(|x_n|),$$

hence by the Comparison Test we are done.

⌟

Claim: There is a bounded sequence $b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that for some sequence $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $\sum_n x_n$ is divergent but $\sum_n x_n b_n$ is absolutely convergent.

⌟

Proof: For instance consider $b_\bullet$ and $x_\bullet$ defined by

$$\forall n \in \mathbb{Z}_{\geq 0} : \qquad b_n = 0, \qquad x_n = \frac{1}{n+1}.$$

Then $\sum_n x_n = \sum_n \frac{1}{n+1}$ is divergent as this is the harmonic series as we discussed in class; whereas $\sum_n |b_n x_n| = \sum_n 0 = 0$.

⌟

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Follow-up Exercise: Let $b_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a bounded sequence such that $\inf_{n \in \mathbb{Z}_{\geq 0}} |b_n| > 0$. Then for any sequence $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $\sum_n x_n$ is absolutely convergent iff $\sum_n x_n b_n$ is absolutely convergent.