University of Utah Spring 2025

MATH 3210-001 PSet 5 Specification

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Subject to Change; Last Updated: 2025-02-14 13:28:54-07:00

1 Background

In this problem set, we begin our transition to calculus proper. First part of the problem set focuses on uniform convergence and function spaces in the concrete setting of functions of the unit interval $[0, 1]$, while the second part develops both the strong and weak interpretations of derivatives. Recall that while the strong interpretation of the derivative is that it is the TANGENT LINE (aka it is an approximation of a complicated object by a simpler model), while the weaker interpretation is that it is the SLOPE (aka it is a control on the "rise over run").

2 What to Submit

Submit your detailed solutions to each of the problems below. While the prompts may seem lengthy, the additional text is intended to guide you by providing context and helpful details.

When tackling statements in MATH 3210, follow these four steps (in no particular order):

1. **Assess the truth of the statement:** Determine, either exactly or probabilistically, whether the statement is true.
2. **Build a case:** Come up with examples or begin drafting a proof to identify potential issues or areas where the argument may break down.
3. **Construct a proof:** Provide a formal argument to support your conclusion.
4. **Perturb the statement:** Experiment with logical variations of the statement. Can you prove a stronger/weaker result, or find alternative proofs?

In this problem set, and in later problem sets as well as exams, your work must address the third step (aka constructing a proof). While work regarding the remaining three items likely will help you write a formal proof, understand the material as well as identify connections between different concepts, you do not need to turn in generalizations or multiple proofs of the same statement, nor do you need to provide examples or counterexamples (unless you are specifically asked to). However, demonstrating these steps can contribute to partial credit when appropriate.

Statements in problems will typically be presented in neutral language. For example, a problem may simply state "P" (for P a well-defined statement) rather than "Show that P" or "Prove that P is false". Your task is to determine the truth of the statement and provide a formal proof or refutation.

If a statement is false, it is highly recommended to propose a corrected version of the statement and prove this corrected version. This process, part of "perturbing the statement", will enhance your mathematical error detection and correction skills.

Unless explicitly stated otherwise, all problems require proofs. If you provide an example, you must also prove that the example satisfies the necessary conditions. If you make a calculation, you must prove that your calculation is correct. If an object is claimed to be a well-defined function (and in particular sequence) you need not prove this claim. However if an object is claimed to be a well-defined function and fails to be one really, a proof of this failure is required.

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your work clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

1. Let $\rho \in]0,1[$, $f_\bullet : \mathbb{Z}_{\geq 1} \to F([0,\rho];\mathbb{R})$, $n \mapsto [x \mapsto x^n]$. Then

   (a) $f_\bullet$ converges uniformly to the function that is constantly zero.

   (b) For any $\varepsilon \in]0,1[$, there is a least $N \in \mathbb{Z}_{\geq 1}$ with the following property:

   $$\forall n \in \mathbb{Z}_{\geq N}, \forall x \in [0,\rho] : |f_n(x)| < \varepsilon. \qquad (\star)$$

   (c) For any $\varepsilon \in]0,1[$, the least $N \in \mathbb{Z}_{\geq 1}$ with the property $(\star)$ from the previous part is $\lceil \log(\varepsilon)/\log(\rho) \rceil$, where for r a real number, $\lceil r \rceil$ is the smallest integer not less than r.

2. In this problem your job is to investigate homeomorphisms of the unit interval $I = [0,1]$.

   (a) Any homeomorphism of I is either strictly increasing xor strictly decreasing.

(b) If $f : I \to I$ is a homeomorphism, then either $f(0) = 0$ and $f(1) = 1$, xor $f(0) = 1$ and $f(1) = 0$.

(c) If $f : I \to I$ is a homeomorphism such that for some $p \in \mathbb{Z}_{\geq 1}$, $f^p = \underbrace{f \circ f \circ \cdots \circ f}_{p \text{ times}} = \mathrm{id}$, then either $f = \mathrm{id}$ or $f^2 = f \circ f = \mathrm{id}$.

(d) If $f_* : I \to I$ is a homeomorphism and $\varepsilon \in \mathbb{R}_{>0}$, then there is a piecewise linear homeomorphism $f : I \to I$ such that $d_{C^0}(f, f_*) < \varepsilon$, where a function $f : I \to I$ is **piecewise linear** if its graph consists of finitely many line segments. (More succinctly, any homeomorphism can be C^0-approximated by a PL homeomorphism.)

(e) If $f_\bullet$ is a sequence of homeomorphisms of I and it converges uniformly to a function $f_\infty : I \to I$, then f_∞ is a homeomorphism. (More succinctly, being a homeomorphism is a C^0-closed property.)

(f) If f_* is a homeomorphism of I, then there is an $\varepsilon \in \mathbb{R}_{>0}$ such that for any continuous function $f : I \to I$, if $d_{C^0}(f, f_*) < \varepsilon$, then f is a homeomorphism. (More succinctly, any C^0-perturbation of a homeomorphism is a homeomorphism.)

(g) If $f, g : I \to I$ are two strictly increasing homeomorphisms, then there is a homeomorphism $\Phi : I \to I$ such that $\Phi \circ f = g \circ \Phi$. (More succinctly, any two strictly increasing homeomorphisms are the same up to a C^0 change of variables.)

3. Let X and Y be two metric spaces and $\theta \in \mathbb{R}_{>0}$. For $f : X \to Y$ a function, define the **Lipschitz constant** of f by

$$\mathrm{Lip}(f) = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \in [0, \infty].$$

Then

(a) f is Lipschitz iff $\mathrm{Lip}(f) < \infty$.

(b) For any function $f : X \to Y$, its Lipschitz constant $\mathrm{Lip}(f)$ is equal to the smallest $C \in [0, \infty]$ with the following property:

$$\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$$

(c) If $f : X \to Y$ is a function with a functional inverse $f^{-1} : Y \to X$, then

$$\frac{1}{\mathrm{Lip}(f^{-1})} = \inf_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}.$$

4. Let $\lambda, \rho \in [-\infty, \infty]$, $\lambda < \rho$, $I =]\lambda, \rho[$, $x_* \in I$ and $f : I \to \mathbb{R}$. Then

(a) If $f(x_* + h) = O_{h \to 0}(h^2)$, then $f(x_* + h) = o_{h \to 0}(h)$.

(b) If $f(x_* + h) = o_{h \to 0}(h)$, then $f(x_* + h) = O_{h \to 0}(h^2)$.

(c) If α and β are positive real numbers such that $\alpha \geq \beta$, then $f(x_* + h) = O_{h \to 0}(h^\alpha)$ implies $\dfrac{f(x_* + h)}{(x_* + h)^\beta} = O_{h \to 0}(h^{\alpha - \beta})$.

5. Let $\lambda, \rho \in [-\infty, \infty]$, $\lambda < \rho$, $I =]\lambda, \rho[$, and $x_* \in I$. Two functions $f, g : I \to \mathbb{R}$ are said to be **tangent** at x_* if

$$f(x_* + h) = g(x_* + h) + o_{h \to 0}(h).$$

Then

(a) "Being tangent at x_*" is an equivalence relation on $F(I; \mathbb{R})$.

(b) If $f, g : I \to \mathbb{R}$ are two functions tangent at x_*, then $f(x_*) = g(x_*)$.

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(c) If $f : I \to \mathbb{R}$ is a function, then there is at most one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*.

(d) If $f : I \to \mathbb{R}$ is a function, then f is differentiable at x_* iff there is exactly one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*.

(e) If $f : I \to \mathbb{R}$ is differentiable at x_*, then for any function $g : I \to \mathbb{R}$ tangent to f at x_* and differentiable at x_*, $f'(x_*) = g'(x_*)$.

6. For any $x_* \in \mathbb{R}$, $M \in \mathbb{R}_{>0}$, and any function $f : \mathbb{R} \to \mathbb{R}$, define the **zoom** of f centered at x_* with magnification factor M to be the function

$$\mathcal{Z}_{x_*,M}(f) : \mathbb{R} \to \mathbb{R}, \quad x \mapsto M\left(f\left(x_* + \frac{x}{M}\right) - f(x_*)\right).$$

Then for any $x_* \in \mathbb{R}$ and any function $f : \mathbb{R} \to \mathbb{R}$:

(a) If f is differentiable at x_*, then as $M \to \infty$, $\mathcal{Z}_{x_*,M}(f)$ converges pointwise to the function whose graph is the line through the origin with slope $f'(x_*)$.

(b) If f is differentiable at x_*, then as $M \to \infty$, $\mathcal{Z}_{x_*,M}(f)$ converges uniformly to the function whose graph is the line through the origin with slope $f'(x_*)$.

(c) The graph of f is a line iff

$$\forall M \in \mathbb{R}_{>0} : \mathcal{Z}_{x_*,M}(f) = f.$$

(d) If f is such that $\mathcal{Z}_{x_*,M}(f)$ converges pointwise as $M \to \infty$, then f is differentiable at x_*.

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3 Generative AI and Computer Algebra Systems Regulations

This section applies if you decide to use either a generative AI tool or a computer algebra system for this problem set. If not, you may skip this section.

3.1 Providing Logs

If you use such tools, you are required to provide logs of your interactions. Here are some ways to submit them:

- If the tool generates a URL for the interaction (e.g. ChatGPT), list such URLs in the appropriate section of the form you will be filling as part of your submission.
- For tools without direct URL generation, use an appropriate external service to archive the session. An example of such a tool that might work is the Wayback Machine, see the documentation for the "Save Page Now" feature.
- If the tool allows PDF export of the interaction (e.g., Microsoft's Copilot), attach these PDFs to your Gradescope submission.

It is your responsibility to ensure that an archiving method is available for the tool you choose to use. If none of the archiving methods works, then that service is prohibited. If you use a service under the assumption that archiving is available, but it turns out not to be, you must report this in the submission form. Future assignments will be monitored accordingly.

3.2 Chat Guidelines; Prompt Engineering

For chatbot use, follow these guidelines:

- During the chat you may copy and paste parts of this specification document, as well as parts of the textbook or other sources.
- Directly asking the tool for complete problem solutions is prohibited.
- You are required to start any chat with a prompt that ideally would structure the chatbot's responses to you. This practice, known as prompt engineering, is your responsibility, and the staff will evaluate the reasonableness of your attempts. When in doubt, check with the course staff to ensure your guardrails prompt is appropriate.

  Here are some guides regarding prompt engineering:

  - https://platform.openai.com/docs/guides/prompt-engineering
  - https://developers.google.com/machine-learning/resources/prompt-eng
  - https://www.ibm.com/topics/prompt-engineering
  - https://aws.amazon.com/what-is/prompt-engineering/

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/cp6ULdW9TApBneF5A.

You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/934441/assignments/5503413,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on February 21, 2025 at 11:59 PM.

As per the course's syllabus, late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.