University of Utah | Spring 2025

MATH 3210-001
Midterm 1 Questions

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February 7, 2025, 9:40 AM - 10:30 AM

Surname: KEY

First Name:

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1. **[20 points] Prove or disprove the following statement:** Let $f : \mathbb{R} \to \mathbb{R}$ be a function, $A, B \subseteq \mathbb{R}$ be two subsets. Then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

$(\subseteq)$ $A \cap B \subseteq A, B \Rightarrow f^{-1}(A \cap B) \subseteq f^{-1}(A), f^{-1}(B)$

$\Rightarrow f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ $\square$

$(\supseteq)$ Let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $x \in f^{-1}(A)$ AND $x \in f^{-1}(B) \Rightarrow f(x) \in A$ AND $f(x) \in B$

$\Rightarrow f(x) \in A \cap B \Rightarrow x \in f^{-1}(A \cap B)$

$\Rightarrow f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$. $\square$

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2. **[60 points]** Consider the set S that consists of all positive rational numbers x with the property that $x^2 \leq 2$.

$S = \{x \in \mathbb{Q} \mid 0 < x,\ x^2 \leq 2\}.$

(a) **[20 points] Prove or disprove the following statement:** The supremum of S is $\sqrt{2}$.

$x \in S \Rightarrow x^2 \leq 2 \Rightarrow x \leq \sqrt{2}$

$\Rightarrow \sqrt{2}$ is an upper bound for S

$(1^2 = 1 \leq 2 \Rightarrow 1 \in S \Rightarrow S \neq \emptyset)$

$\Rightarrow -\infty < \sup(S) \leq \sqrt{2}.$

Say $\sup(S) < \sqrt{2}$. ~~Then $0 < \sqrt{2} < \sup(S)$~~

Then by the density of rationals,

$\exists q \in \mathbb{Q}:$ ~~$0 < q < \sqrt{2} < \sup(S) < \sqrt{2}$~~

~~$\Rightarrow q \notin S$~~ $\sup(S) \leq \max\{\sup(S), 0\} < q < \sqrt{2}$

$\Rightarrow 0 < q,\ q^2 < 2$

$\Rightarrow q \in S$, a contradiction.

Hence $\sup(S) = \sqrt{2}$. $\square$

(b) **[20 points] Prove or disprove the following statement:** The infimum of S is $\sqrt{2}$.

$0 < 1,\quad 1^2 = 1 \leq 2$

$\Rightarrow 1 \in S.$

$\Rightarrow \inf(S) \leq 1 < \sqrt{2}$

$\Rightarrow \inf(S) \neq \sqrt{2}.$ $\square$.

(c) **[10 points] Prove or disprove the following statement:**
The set of all upper bounds of S is $[\sqrt{2}, \infty[$.

After part (a), $\sup(S) = \sqrt{2}$.

By definition, $\sup(S)$ is the minimum of all upper bounds for S.

$\Rightarrow \forall u \geq \sup(S), \forall x \in S: x \leq u$

$\Rightarrow [\sqrt{2}, \infty[$ indeed is the set of all upper bounds for S. $\square$.

(d) **[10 points] Prove or disprove the following statement:**
The smallest closed interval containing S is $[0, \sqrt{2}]$.

Claim: $\inf(S) = 0$.

Pf of Claim: By def, $\forall x \in S, 0 < x$
$\rightarrow 0 \leq \inf(S)$. If $0 < \inf(S)$, by density of rationals, $\exists q \in \mathbb{Q}$,
$0 < q < \min\{\inf(S), \sqrt{2}\}$
$\Rightarrow q \in S \Rightarrow q < \inf(S)$, a contradiction $\square$.

We also know from part (a) that $\sup(S) = \sqrt{2}$

$\hookrightarrow$

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(2.(d) continued)

Hence $\forall x \in S : 0 = \inf(S) \leq x \leq \sup(S) = \sqrt{2}$

$\Rightarrow S \subseteq [0, \sqrt{2}]$.

Let $[\lambda, \rho]$, $-\infty < \lambda \leq \rho < \infty$ be such that $[\lambda, \rho] \supseteq S \Rightarrow \lambda$ is a lower bound for S and ρ is an upper bound for S. Then by def,

$\lambda \leq \inf(S)$, $\sup(S) \leq \rho$

$\Leftrightarrow [\inf(S), \sup(S)] = [0, \sqrt{2}] \subseteq [\lambda, \rho]$

$\square$.

3. **[16 points] Prove or disprove the following statement:** Let $x_\bullet$ be a sequence of real numbers that converges to zero. Then for any bounded sequence $b_\bullet$ of real numbers,

$$\lim_{n\to\infty} x_n b_n = 0.$$

Have:

Since $b_\bullet$ is bounded,

$$\exists M \in \mathbb{R}_{>0},\ \forall n \in \mathbb{Z}_{\geq 0}:\ |b_n| \leq M.$$

Since $x_\bullet \to 0$,

$$\forall \eta \in \mathbb{R}_{>0},\ \exists N = N(\eta) \in \mathbb{Z}_{\geq 0},\ \forall n \in \mathbb{Z}_{\geq 0}:$$

"eta" $\quad n \geq N(\eta) \Rightarrow |x_n| < \eta.$

Want: $\forall \varepsilon \in \mathbb{R}_{>0},\ \exists N' = N'(\varepsilon) \in \mathbb{Z}_{\geq 0},\ \forall n \in \mathbb{Z}_{\geq 0}:$

$$n \geq N'(\varepsilon) \Rightarrow |x_n b_n| < \varepsilon.$$

Let $\varepsilon \in \mathbb{R}_{>0}$. Put $\eta = \frac{\varepsilon}{2M} > 0$. Then $\forall n$:

$$n \geq N(\eta) = N(\varepsilon/2M) \Rightarrow |x_n| < \frac{\varepsilon}{2M}$$

$$\Rightarrow |x_n b_n| = |x_n||b_n| \leq \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2} < \varepsilon.$$

Thus for $N'(\varepsilon) = N(\varepsilon/2M)$, we ~~have~~ obtain what we want. $\square$.

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4. **[4 points]** Let $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a sequence of real numbers and x_* be a real number.

   (a) **[1 point] Prove or disprove the following statement:** If $x_\bullet$ converges to x_*, then any subsequence of $x_\bullet$ converges to x_*.

Let $x'_\bullet \leq x_\bullet$. Then $\exists$ strictly increasing j, $\forall n$: $x'_n = x_{j(n)}$. Note: $\boxed{\forall n : n \leq j(n)}$. (*)

Have: $x_\bullet \to x_*$, i.e.,

$\forall \eta \in \mathbb{R}_{>0}, \exists N = N(\eta) \in \mathbb{Z}_{\geq 0}, \forall n \in \mathbb{Z}_{\geq 0}$:

$n \geq N = N(\eta) \Rightarrow |x_n - x_*| < \eta$.

Want: $x'_\bullet \to x_*$, i.e.,

$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N' = N'(\varepsilon) \in \mathbb{Z}_{\geq 0}, \forall n \in \mathbb{Z}_{\geq 0}$:

$n \geq N' = N'(\varepsilon) \Rightarrow |x_{j(n)} - x_*| < \varepsilon$.

Let $\varepsilon \in \mathbb{R}_{>0}$, put $\eta = \varepsilon$. Then $\exists N = N(\varepsilon)$, $\forall n$: $n \geq N(\varepsilon) \Rightarrow |x_n - x_*| < \varepsilon$.

But by (*), if $n \geq N(\varepsilon)$, we also have $j(n) \geq n \geq N(\varepsilon)$, hence defining $N'(\varepsilon) = N(\varepsilon)$ we are done. $\square$.

(b) **[1 point] Prove or disprove the following statement:** If

$$\forall \varepsilon \in \mathbb{R}_{>0}, \forall N \in \mathbb{Z}_{>0}, \exists n \in \mathbb{Z}_{\geq N} : |x_n - x_*| < \varepsilon,$$

then $x_\bullet$ converges to x_*.

Consider $x_\bullet : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $n \mapsto (-1)^n$,

and put $x_* = 1$.

Let $\varepsilon \in \mathbb{R}_{>0}$, $N \in \mathbb{Z}_{\geq 0}$. Then for $n = j(N) = 2N \geq N$, and

$$|x_n - x_*| = |x_{2N} - 1| = 0 < \varepsilon.$$

Thus $x_\bullet$ and x_* satisfy the hypothesis.

However $x_\bullet$ does not converge to x_* (proved in class). $\square$.

(c) **[1 point] Prove or disprove the following statement:** If

$n = n(\varepsilon, N)$

$$\forall \varepsilon \in \mathbb{R}_{>0}, \forall N \in \mathbb{Z}_{\geq 0}, \exists n \in \mathbb{Z}_{\geq N} : |x_n - x_*| < \varepsilon,$$

then some subsequence of $x_\bullet$ converges to x_*.

$\varepsilon = 1, N = 1 \Rightarrow \exists\, n_1 : n_1 \geq 1, |x_{n_1} - x_*| < 1.$

$\varepsilon = \frac{1}{2}, N = n_1 + 1 \Rightarrow \exists\, n_2 > n_1, |x_{n_2} - x_*| < \frac{1}{2}.$

$\varepsilon = \frac{1}{3}, N = n_2 + 1 \Rightarrow \exists\, n_3 > n_2 : |x_{n_3} - x_*| < \frac{1}{3}.$

$\vdots$

$\forall k \in \mathbb{Z}_{\geq 1}$, for $\varepsilon = \frac{1}{k}, N = n_{k-1} + 1, \exists\, n_k > n_{k-1}$:

(*) $|x_{n_k} - x_*| < \frac{1}{k}$ by the Principle of Mathematical Induction.

$1 \leq n_1 \lneq n_2 \lneq \cdots \lneq n_k \lneq \cdots$

$\Rightarrow x_\bullet^!$ defined by $k \mapsto x_{n_k}$ is a subsequence of $x_\bullet$.

Let $\eta \in \mathbb{R}_{>0}$. Then $\exists K \in \mathbb{Z}_{\geq 1} : 0 < \frac{1}{K} < \eta$ by the Archimedean Principle. $\forall k : k \geq K$

$\Rightarrow 0 < \frac{1}{k} \leq \frac{1}{K} < \eta \overset{(*)}{\Rightarrow} |x_k^! - x_*| < \frac{1}{k} < \eta \Rightarrow x_\bullet^! \to x_*$ ∎.

(Or you can use the squeeze theorem for sequences to conclude that $x_\bullet^! \to x_*$)

(d) **[1 point] Prove or disprove the following statement:** If some subsequence of $x_\bullet$ converges to x_*, then

$$\forall \varepsilon \in \mathbb{R}_{>0}, \forall N \in \mathbb{Z}_{\geq 0}, \exists n \in \mathbb{Z}_{\geq N} : |x_n - x_*| < \varepsilon.$$

Let $x'_\bullet \leq x_\bullet$ be such that $x'_\bullet \to x_*$.

Then $\exists$ strictly increasing j, $\forall n$:

$x'_n = x_{j(n)}$.

Let $\varepsilon \in \mathbb{R}_{>0}$, $N \in \mathbb{Z}_{\geq 0}$. Since $x'_\bullet \to x_*$,

$\exists N' = N'(\varepsilon) \in \mathbb{Z}_{\geq 0}$, $\forall n \in \mathbb{Z}_{\geq 0}$:

$n \geq N' = N'(\varepsilon) \Rightarrow |x_{j(n)} - x_*| < \varepsilon$. (*).

Want: $\boxed{\exists n = n(\varepsilon, N) : n \geq N \text{ and } |x_n - x_*| < \varepsilon.}$

Put $n(\varepsilon, N) = j(\max\{N, N'\})$.

Then $n(\varepsilon, N) \geq N, N'$ (as j is strictly increasing)

and by (*), $|x_{n(\varepsilon,N)} - x_*| < \varepsilon$ $\square$.

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