University of Utah Spring 2025

MATH 3210-001
Midterm 2 Questions

Instructor: Alp Uzman

March 28, 2025, 9:40 AM - 10:30 AM

Surname: KEY

First Name:

uNID:

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1. **[40 points]** Let $\lambda, \rho \in [-\infty, \infty]$, $\lambda < \rho$, $I =]\lambda, \rho[$, $x_* \in I$, and $f : I \to \mathbb{R}$ be a function.

   (a) **[20 points] Prove or disprove the following statement:** If f is differentiable at x_*, then there is exactly one function $\ell : I \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*. It suffices to solve for A, B.

• f, ℓ are diff. at $x_* \Rightarrow f - \ell$ diff. at $x_* \Rightarrow f - \ell$ cont. at x_*.

$$\Rightarrow \lim_{h \to 0} [f(x_*+h) - \ell(x_*+h)] = f(x_*) - \ell(x_*)$$

• $f(x_*+h) - \ell(x_*+h) = o(h)$

$$\Rightarrow \lim_{h \to 0} [f(x_*+h) - \ell(x_*+h)] = 0$$

$$\Rightarrow \boxed{f(x_*) = \ell(x_*) = Ax_* + B.}$$

• $$0 = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x_*+h) - \ell(x_*+h)}{h}$$

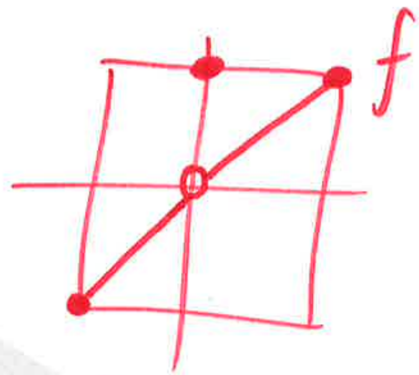
$$= \lim_{\substack{h \to 0 \\ h \neq 0}} \left(\frac{f(x_*+h) - f(x_*)}{h} - A \right)$$

$$= \left(\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x_*+h) - f(x_*)}{h} \right) - A \Rightarrow \boxed{A = f'(x_*)}$$

$$\Rightarrow \boxed{B = f(x_*) - f'(x_*)x_*} \qquad \square$$

(b) **[20 points] Prove or disprove the following statement:** If there is exactly one function $\ell : \mathrm{I} \to \mathbb{R}$ of the form $\ell : x \mapsto Ax + B$ for $A, B \in \mathbb{R}$ that is tangent to f at x_*, then f is differentiable at x_*.

- Put $f : I \to \mathbb{R}$, $I =]-1, 1[$,
$$x \mapsto \begin{cases} x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}.$$



For $|x| < \delta$, $x \neq 0$, $|f(x)| < \delta$

$\Rightarrow$ f is not continuous at $x_* = 0$

$\Rightarrow$ f is not differentiable at x_*.

- Put $\ell : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ $(A = 1, B = 0)$.

$$f(h) - \ell(h) = \begin{cases} 0 & \text{if } h \neq 0 \\ 1, & \text{if } h = 0 \end{cases}$$

$\Rightarrow$ f is tangent to ℓ at $x_* = 0$

- Let $\lambda : \mathbb{R} \to \mathbb{R}$, $x \mapsto Ax + B$ be arbitrary such that f is tangent to λ at x_*.

$$\Rightarrow 0 = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(h) - \lambda(h)}{h} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{h - Ah - B}{h}$$

$$= (1 - A) - B \underbrace{\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{1}{h}}_{= \pm\infty} \Rightarrow \boxed{B = 0, A = 1}$$

$\square$.

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2. **[40 points] Prove or disprove the following statement:** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and define for any $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, $J(n,x) = \frac{d(f^n)}{dx}(x)$. Then J satisfies the following identity:

$$\forall n, m \in \mathbb{Z}_{\geq 0}, \forall x \in \mathbb{R} : J(n+m, x) = J(n, f^m(x))\, J(m, x).$$

• By the Chain Rule,

$$J(n+m,x) = \frac{d}{dx}(f^{n+m})(x)$$

$$\overset{(*)}{=} \frac{d(f^n)}{dx}(f^m(x)) \cdot \frac{d(f^m)}{dx}(x)$$

$$= J(n, f^m(x)) \cdot J(m,x) \quad \square$$

(*) By def., $f^0 = id$, $f^n = f \circ f^{n-1}$.

• Base step: $\forall m$: $f^{1+m} = f \circ f^m = f^1 \circ f^m$.

• Induction step: Say works for $n=k$, for any m.

Want: works for $n=k+1$, for any m.

$f^{(k+1)+m} = f^{k+(1+m)} = f^k \circ f^{1+m} = f^k \circ (f \circ f^m)$

$= (f^k \circ f) \circ f^m = f^{k+1} \circ f^m \quad \square$

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3. **[20 points]** Let $\lambda, \rho \in \mathbb{R}$, $\lambda < \rho$, $I = [\lambda, \rho]$, $f_\bullet$ be a sequence of real valued Riemann integrable functions on I, and $f_\infty : I \to \mathbb{R}$ be another function.

   (a) **[8 points] Prove or disprove the following statement:** If $f_\bullet \to f_\infty$ uniformly, then f_∞ is Riemann integrable.

• Let $\mathcal{P} \in \mathbb{P}(I)$, $P \in \mathcal{P}$, $x, y \in P$. Then

$$|f_\infty(x) - f_\infty(y)| \le |f_\infty(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_\infty(y)| \le 2 d_{C^0}(f_n, f_\infty) + [\sup(f_n|_P) - \inf(f_n|_P)]$$

$$\Rightarrow \sup(f_\infty|_P) - \inf(f_\infty|_P) \le 2 d_{C^0}(f_n, f_\infty) + \sup(f_n|_P) - \inf(f_n|_P).$$

$$\Rightarrow U(f_\infty; \mathcal{P}) - L(f_\infty; \mathcal{P}) \le 2 d_{C^0}(f_n, f_\infty) \cdot \ell(I) + [U(f_n; \mathcal{P}) - L(f_n; \mathcal{P})] \quad (\star)$$

• Let $\varepsilon > 0$, N be such that $2 d_{C^0}(f_N, f_\infty)\ell(I) < \frac{\varepsilon}{2}$.

$N = N(\varepsilon)$

$\mathcal{P} = \mathcal{P}_{\varepsilon, N} \in \mathbb{P}(I)$ be such that $U(f_N; \mathcal{P}) - L(f_N; \mathcal{P}) < \frac{\varepsilon}{2}$

Then for this $\mathcal{P}$, by $(\star)$,

$U(f_\infty; \mathcal{P}) - L(f_\infty; \mathcal{P}) < \varepsilon$ $\square$.

(b) **[8 points] Prove or disprove the following statement:** If $f_\bullet \to f_\infty$ uniformly and f_∞ is Riemann integrable, then

$$\lim_{n\to\infty}\int_I f_n(x)dx = \int_I f_\infty(x)dx.$$

(by part(a) this is redundant.)

$$\left|\int_I f_n(x)dx - \int_I f_\infty(x)dx\right|$$

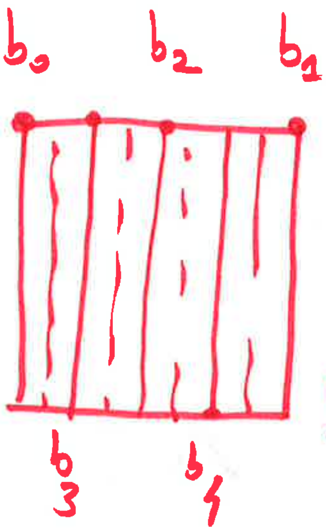
$$= \left|\int_I (f_n(x) - f_\infty(x))dx\right|$$

$$\le \int_I |f_n(x) - f_\infty(x)|dx$$

$$\le d_{C^0}(f_n, f_\infty)\, \ell(I) \xrightarrow{n\to\infty} 0.$$

$\square$

(c) **[2 points] Prove or disprove the following statement:** If $f_\bullet \to f_\infty$ pointwise, then f_∞ is Riemann integrable.



Consider the sequence $b_\bullet$ made up of integer multiples of powers of $\frac{1}{2}$:

$$b_\bullet = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \dots$$

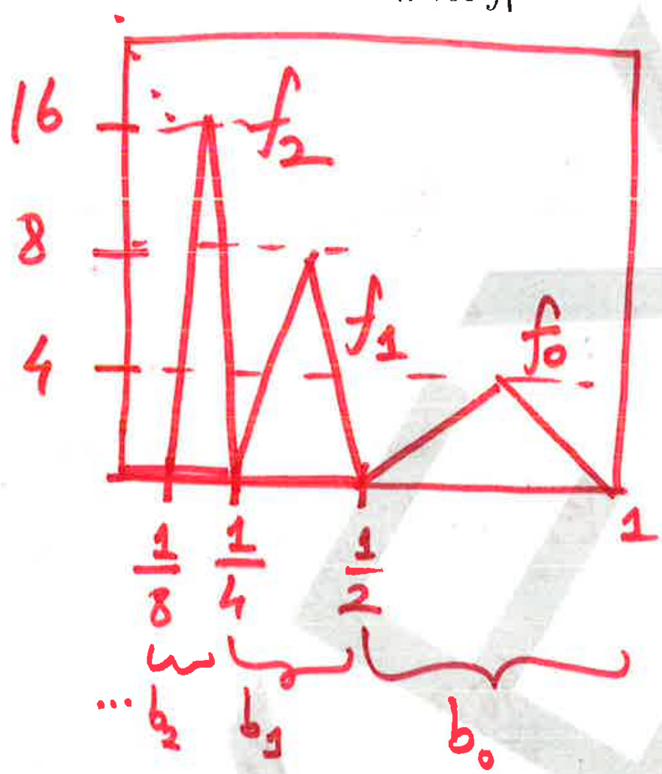
Define $\forall n$: $f_n : [0,1] \to \mathbb{R},\ x \mapsto \begin{Bmatrix} 1, & \text{if } x = b_i,\ i \le n \\ 0, & \text{else.} \end{Bmatrix}$

$f_\infty : [0,1] \to \mathbb{R},\ x \mapsto \begin{Bmatrix} 1, & \text{if } x = b_n\ \exists n \\ 0, & \text{else.} \end{Bmatrix}$

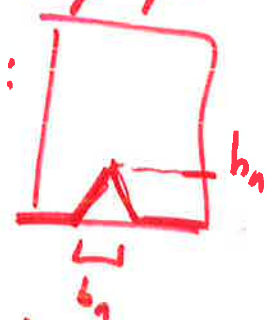
Then $\forall n: f_n \in \mathcal{R}([0,1];\mathbb{R})$, $f_\bullet \to f_\infty$ pointwise, but $f_\infty \notin \mathcal{R}([0,1];\mathbb{R})$.

(d) **[2 points] Prove or disprove the following statement:** If $f_\bullet \to f_\infty$ pointwise and f_∞ is Riemann integrable, then

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I f_\infty(x)dx.$$



Let f_n have graph of the form:



$\Rightarrow \int f_n = \frac{1}{2} b_n \cdot h_n$

$b_n = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$

$h_n = 2^{n+2}$

$\Rightarrow \int f_n = 1, \forall n.$ f_n is continuous hence Riemann integrable.

$f_\bullet \to f_\infty \equiv 0$ pointwise,

$\int f_\infty = 0.$ $\square$

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