University of Utah Spring 2026

MATH 3210-004 PSet 3 Specification

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Subject to Change; Last Updated: 2026-01-23 11:42:48-07:00

1 Background

This problem set focuses on sequences and their convergence properties. While one can talk about the convergence of a sequence in the abstract setting of a metric space (or even a topological space), the sequences considered in this problem set consist exclusively of real numbers. While not necessary, it is a good exercise to investigate, as part of logical perturbations, whether a given statement about sequences of real numbers remains valid for sequences taking values in a more general space X—or even whether the statement (has a logical perturbation that) makes sense in such a setting!

The final problem too is about sequences: Indeed, starting with a sequence $x_\bullet$ and transitioning to the sequence of partial sums $s_\bullet : n \mapsto \sum_{k=1}^{n} x_k$ can be thought of as an instance of a discrete analog of integration. Based on this, you'll investigate elements of calculus with sequences.

2 Generative AI and Computer Algebra Systems Regulations

This section applies only if you choose to use either a generative AI tool (e.g. chatbots) or a computer algebra system (CAS) while working on this problem set. If you do not use such tools, you may skip this section.

The use of these tools while completing this problem set is permitted, provided it is done responsibly and in a manner that supports your learning. Note however that in exams you will not be allowed to use any AI or CAS.

2.1 Disclosure

If you use such tools, you must disclose this fact in the designated section of the form you will complete as part of your submission. You must also share your reflections on which parts of the problem required your own mathematical judgment, insight, or decision-making, and which parts could reasonably be delegated to a computational or generative tool.

2.2 Guidelines for Responsible Use

If you use a generative AI tool or chatbot, you must adhere to the following guidelines:

- During the chat you may share with the chatbot parts or all of this specification document, as well as parts of the textbook or other sources.

- Directly asking the tool for complete problem solutions is prohibited.
- Appropriate uses include asking for conceptual explanations, checking intermediate steps, clarifying definitions or theorems, or exploring alternative solution approaches, provided that the final work submitted is your own.
- Regardless of the tools used, you are expected to understand, justify, and be able to reproduce all submitted work independently. The use of AI or CAS does not reduce or replace this expectation.

3 What to Submit

Submit your detailed solutions to each of the problems below. While the prompts may seem lengthy, the additional text is intended to guide you by providing context and helpful details.

When tackling statements in MATH 3210, follow these four steps (in no particular order):

1. **Assess the truth of the statement:** Determine, either exactly or probabilistically, whether the statement is true.
2. **Build a case:** Come up with examples or begin drafting a proof to identify potential issues or areas where the argument may break down.
3. **Construct a proof:** Provide a formal argument to support your conclusion.
4. **Perturb the statement:** Experiment with logical variations of the statement. Can you prove a stronger/weaker result, or find

alternative proofs?

In this problem set, and in later problem sets as well as exams, your work must address the third step (aka constructing a proof). While work regarding the remaining three items likely will help you write a formal proof, understand the material as well as identify connections between different concepts, you do not need to turn in generalizations or multiple proofs of the same statement, nor do you need to provide examples or counterexamples (unless you are specifically asked to). However, demonstrating these steps can contribute to partial credit when appropriate.

Statements in problems will typically be presented in neutral language. For example, a problem may simply state "P" (for P a well-defined statement) rather than "Show that P" or "Prove that P is false". Your task is to determine the truth of the statement and provide a formal proof or refutation.

If a statement is false, it is highly recommended to propose a corrected version of the statement and prove this corrected version. This process, part of "perturbing the statement", will enhance your mathematical error detection and correction skills.

Unless explicitly stated otherwise, all problems require proofs. If you provide an example, you must also prove that the example satisfies the necessary conditions.

Some problems may be (variants of) theorems or exercises from the textbook. Going forward, and possibly for the rest of this problem set, such correspondences will not be mentioned; it'll be up to you to locate them (if need be)!

Make sure that each solution is properly enumerated and organized. Start the solution to each problem on a new page, and consider using headings or subheadings to structure your work clearly. This will not only aid in your thought process but also ensure that no part of your solution is overlooked during grading.

1. Represent visually each of the following statements. Your visual representations may vary from impressionistic to metaphysical. Ultimately, your visualizations should be so that they are useful to you personally as intuition pumps. **There is nothing to prove in this problem!**

   (a) $x_\bullet$ is a sequence converging to x_∞, in short:

   $$\lim_{n\to\infty} x_n = x_\infty.$$

   (b) $x_\bullet$ is a sequence that fails to converge to $x_\dagger$ (in short, $\lim_{n\to\infty} x_n \neq x_\dagger$).

   (c) The limit of the sequence of sums is the sum of the limits (in short, $\lim_{n\to\infty}(x_n + y_n) = (\lim_{n\to\infty} x_n) + (\lim_{n\to\infty} y_n)$).

   (d) The limit of the sequence of products is the product of the limits (in short, $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$).

   (e) The limit of the sequence of maximums is the maximum of the limits (in short, $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\}$).

   (f) If two sequences squeeze a third one, and if all three sequences are convergent, then the limit of the squeezed sequence is the squeezed between the limits (in short, if $x_n \leq y_n \leq z_n$ for all but finitely many n and $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$, $\lim_{n\to\infty} z_n$ all exist, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n$).

   (g) $x_\bullet$ does not converge to x'_∞, but a subsequence of it does (in short, $\lim_{n\to\infty} x_n \neq x'_\infty$ but for some strictly increasing j, $\lim_{n\to\infty} x_{j(n)} \neq x'_\infty$).

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2. This problem is about limit laws for sequences. Let $x_\bullet, y_\bullet$ be two convergent sequences of real numbers with limits $x_\infty, y_\infty \in \mathbb{R}$ respectively:

$$\lim_{n\to\infty} x_n = x_\infty, \qquad \lim_{n\to\infty} y_n = y_\infty.$$

Then

   (a) $\lim_{n\to\infty}(x_n + y_n) = x_\infty + y_\infty$.
   (b) For any real number α, $\lim_{n\to\infty}(\alpha x_n) = \alpha x_\infty$.
   (c) $\lim_{n\to\infty}(x_n y_n) = x_\infty y_\infty$.
   (d) $\lim_{n\to\infty} \max\{x_n, y_n\} = \max\{x_\infty, y_\infty\}$.

3. Let $x_\bullet$ be a sequence of real numbers that converges to zero. Then for any bounded sequence $b_\bullet$ of real numbers,

$$\lim_{n\to\infty} x_n b_n = 0.$$

4. Let $x_\bullet$ be a sequence of positive real numbers and R be a real number. Then

   (a) $\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n} = R$ if and only if $\lim_{n\to\infty} x_n^{1/n} = R$.
   (b) There is an $\alpha \in \mathbb{R}$ such that for any $n \in \mathbb{Z}_{\geq 0}$, $x_n = n^\alpha$ iff $\lim_{n\to\infty} x_n^{1/n} = 1$.

5. This problem is about discrete analogs of Fundamental Theorems of Calculus. Fundamental Theorems of Calculus come before calculus, and indeed, one can establish these discrete versions without any calculus proper!

Let $X = \mathbb{R}$, and put $\mathfrak{S}(X) = F(\mathbb{Z}; X)$. We'll suppress the bullets in the standard bullet notation for sequences for clarity in notation when appropriate. Define the following functions:

$$\mathfrak{D} : \mathfrak{S}(X) \to \mathfrak{S}(X), x_\bullet \mapsto [n \mapsto x_n - x_{n-1}]$$

$$\forall n, m \in \mathbb{Z} : \mathfrak{I}_n^m : \mathfrak{S}(X) \to X, x_\bullet \mapsto \begin{cases} \sum_{k=n+1}^{m} x_k, & \text{if } n \lneq m \\ 0, & \text{if } n{=}m \\ -\sum_{k=m+1}^{n} x_k, & \text{if } n \gneq m \end{cases}$$

One can consider $\mathfrak{D}$ as a discrete analog of "differentiation", and $\mathfrak{I}_n^m$ as a discrete analog of "(signed) integration (from n to m)".

(Here, "$\mathfrak{S}$" is "S" for "signal", "$\mathfrak{D}$" is "D" for "derivative" and "$\mathfrak{I}$" is "I" for "integral"; all in Fraktur font.)

(a) For any $x \in \mathfrak{S}(X)$, x is constant iff $\mathfrak{D}(x) = 0$.

(b) $\mathfrak{D}$ is a bijection.

(c) For any $x \in \mathfrak{S}(X)$ and any $m, n, p \in \mathbb{Z}$,

$$\mathfrak{I}_m^n(x) + \mathfrak{I}_n^p(x) + \mathfrak{I}_p^m(x) = 0.$$

(d) For any $x \in \mathfrak{S}(X)$ and any $n, m \in \mathbb{Z}$,

$$\mathfrak{I}_n^m(\mathfrak{D}(x)) = x_m - x_n.$$

(e) For any $x \in \mathfrak{S}(X)$ and any $p \in \mathbb{Z}$,

$$\mathfrak{D}(\mathfrak{I}_p^\bullet(x)) = x.$$

4 How to Submit

- **Step 1 of 2:** Submit the form at the following URL:

  https://forms.gle/YE6TfTWFA8BWGJqLA.

  You will receive a zero for this assignment if you skip this step, even if you submit your work on Gradescope on time.

- **Step 2 of 2:** Submit your work on Gradescope at the following URL:

  https://www.gradescope.com/courses/1212058/assignments/7390769,

  see the Gradescope documentation for instructions.

5 When to Submit

This problem set is due on January 30, 2026 at 11:59 PM.

As per the course's syllabus, late submissions, up to 24 hours after this deadline, will be accepted with a 10% penalty. Submissions more than 24 hours late will not be accepted unless you contact the course staff with a valid excuse before the 24-hour extension expires.