

MATH 3210-004 Selected Solutions to PSet Problems

Instructor: Alp Uzman

Subject to Change; Last Updated: 2026-01-22 14:48:51-07:00

In this document you may find solutions (proofs or examples) to selected problems from problem sets. As a general rule of thumb, it is more likely that this document will be updated with the solutions to problems that were graded on accuracy.

Throughout this class your solutions need not match the level of polish, precision and detail of the below solutions. What's most important is grasping the underlying concepts and steadily improving your problem-solving skills as well as your proof-writing and presentation skills.

Most solutions will have follow-up exercises. Just as in the problem sets, these follow-up exercises are stated in a neutral language, and determining whether or not the statement is correct is part of the exercise. A follow-up exercise here may later show up in an exam.

1 PSet 1, Problem 3

Let X and Y be sets. Let R be a relation from X to Y . Then

- (i) The relation $\mathcal{R} = \{ (A, B) \mid B = R(A) \}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is the graph of a function.
- (ii) For any $A, B \subseteq X$, $R(A \cup B) = R(A) \cup R(B)$.
- (iii) For any $A, B \subseteq X$, $R(A \cap B) = R(A) \cap R(B)$.
- (iv) For any $A \subseteq X$, $R(A^c) = R(A)^c$.
- (v) For any $A, B \subseteq X$, $R(A \setminus B) = R(A) \setminus R(B)$.

We'll study each statement separately. The first statement is true:

Claim: The relation $\mathcal{R} = \{ (A, B) \mid B = R(A) \}$ from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is the graph of a function. ┐

Proof: We need to verify that the relation \mathcal{R} from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined by

$$\mathcal{R} = \{(A, B) \mid B = R(A)\} \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$$

is the graph of a function with domain $\mathcal{P}(X)$. For $A \in \mathcal{P}(X)$, by definition $R(A) \in \mathcal{P}(Y)$, so that indeed $\text{dom}(\mathcal{R}) = \mathcal{P}(X)$. Next let $(A_1, B_1), (A_2, B_2) \in \mathcal{R}$. Suppose $A_1 = A_2$. We need to show that $B_1 = B_2$. As $(A_1, B_1), (A_2, B_2) \in \mathcal{R}$, by the definition of \mathcal{R} , we have that

$$B_1 = R(A_1), \quad B_2 = R(A_2).$$

To show the equality of two sets, it suffices to show that either one is a subset of the other one. So let $y \in B_1 = R(A_1)$. Then for some $x \in A_1$, $(x, y) \in R$. But as $A_1 = A_2$, $x \in A_2$ also, and since $(x, y) \in R$,

we have that $y \in R(A_2) = B_2$. Thus $B_1 \subseteq B_2$. Interchanging the roles of B_1 and B_2 in this argument, we immediately obtain also $B_1 \supseteq B_2$, whence $B_1 = B_2$. In other words, \mathcal{R} satisfies the abstract vertical line test, and consequently is the graph of a function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

┘

The second statement is also true:

Claim: For any $A, B \subseteq X$, $R(A \cup B) = R(A) \cup R(B)$.

┘

Proof: Let $A, B \subseteq X$. Then as $A \subseteq A \cup B$, $R(A) \subseteq R(A \cup B)$. Indeed, if $y \in R(A)$, there is an $x \in A$ such that $(x, y) \in R$. But $A \subseteq A \cup B$, so that $x \in A \cup B$, thus $y \in R(A \cup B)$. Similarly $R(B) \subseteq R(A \cup B)$. Then $R(A) \cup R(B) \subseteq R(A \cup B)$.

For the other direction, let $y \in R(A \cup B)$. Then there is an $x \in A \cup B$ such that $(x, y) \in R$. By definition $x \in A$ or $x \in B$, so that $y \in R(A)$ or $y \in R(B)$, that is, $R(A \cup B) \subseteq R(A) \cup R(B)$.

┘

The remaining three statements are false in general. To prove that a given statement is false, it suffices to provide one counterexample to the statement. For instance, the third statement is: "The image of the intersection of two subsets is equal to the intersection of the images". The **logical negation** of this is the statement "The image of the intersection of two subsets may be different than the intersection of the images". Using logical symbols, we may succinctly write:

$$\begin{aligned} & \neg[\forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B)] \\ & \equiv \exists A, B \in \mathcal{P}(X) : R(A \cap B) \neq R(A) \cap R(B). \end{aligned}$$

We'll prove that the given statement is false, that is, the negation

of the given statement if true. As this latter is an existential statement, one example for it is sufficient; the example for the negation is a counterexample for the given statement.

Claim: The image of the intersection of two subsets under a relation is not necessarily equal to the intersection of the images.

┘

Proof: Consider $X = [0, 1] = Y$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \rightarrow Y$, $x \mapsto 1/2$, that is, f is the **constant function** that is constantly $1/2$. Then $A \cap B = \emptyset$, hence $f(A \cap B) = \emptyset$, while $f(A) = f(B) = \{1/2\}$, so that $f(A) \cap f(B) = \{1/2\}$.

┘

To obtain a correct version of the third statement, either we can weaken the conclusion, or strengthen the hypotheses¹.

Claim: The following version of the third statement with weakened conclusion is true:

$$\forall A, B \in \mathcal{P}(X) : R(A \cap B) \subseteq R(A) \cap R(B).$$

┘

Proof: Let $A, B \subseteq X$. As $A \cap B \subseteq A$, $R(A \cap B) \subseteq R(A)$. Similarly as $A \cap B \subseteq B$, $R(A \cap B) \subseteq R(B)$. Thus $R(A \cap B) \subseteq R(A) \cap R(B)$.

┘

Claim: The following version of the third statement with an additional hypothesis is true:

¹These are instances of logical **perturbations** of the original statement given; you do not need to submit such corrected versions, though trying to come up with such corrected versions will help you understand the concepts and prepare you for the exams better.

$$\begin{aligned} & [\forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2] \\ \iff & [\forall A, B \in \mathcal{P}(X) : R(A \cap B) = R(A) \cap R(B)]. \end{aligned}$$

┘

Proof: (\implies) Suppose R satisfies

$$\forall (x_1, y_1), (x_2, y_2) \in R : y_1 = y_2 \implies x_1 = x_2;$$

one might call this the **injectivity condition** (aka **abstract horizontal line test**) for relations. Let $A, B \subseteq X$, and $y \in R(A) \cap R(B)$. Then there are $a \in A$ and $b \in B$ such that $(a, y), (b, y) \in R$. By the injectivity condition, this forces that $a = b$, and consequently that this element is in $A \cap B$. Thus $y \in R(A \cap B)$, so that $R(A) \cap R(B) \subseteq R(A \cap B)$. Since $R(A) \cap R(B) \supseteq R(A \cap B)$ for any relation R by the previous claim, we have $R(A) \cap R(B) = R(A \cap B)$.

(\impliedby) Suppose R fails the injectivity condition, so that there are $x_1, x_2 \in X$ and $y \in Y$ such that $x_1 \neq x_2$ and $(x_1, y), (x_2, y) \in R$. Define $A = \{x_1\}, B = \{x_2\}$. Then $A \cap B = \emptyset$, so $R(A \cap B) = \emptyset$, but $y \in R(A) \cap R(B)$, that is, $R(A \cap B) \subsetneq R(A) \cap R(B)$.

┘

Claim: The image of the absolute complement of a subset under a relation need not be equal to the absolute complement of the image.

┘

Proof: We may repurpose our previous counterexample to establish this claim also. Put $X = Y = [0, 1]$, $A = [0, 1/4]$, $f : X \rightarrow Y, x \mapsto 1/2$. Then $A^c =]1/4, 1]$, $f(A^c) = \{1/2\}$, but $f(A)^c = \{1/2\}^c$, hence in fact we have that the two subsets are **disjoint**.

┘

Claim: The image of the relative complement of two subsets under a relation need not be equal to the relative complement of the images.

┘

Proof: Recalling that $A \setminus B = A \cap B^c$, the counterexample we gave earlier serves as a counterexample for this claim also. Indeed, put $X = Y = [0, 1]$, $A = [0, 1/4]$, $B = [3/4, 1]$, $f : X \rightarrow Y$, $x \mapsto 1/2$. Then $A \cap B^c = [0, 1/4]$, hence $f(A \setminus B) = \{1/2\}$; on the other hand $f(A) = f(B) = \{1/2\}$, so that $f(A) \setminus f(B) = \emptyset$.

┘

Follow-up Exercise 1 The second and the (correct versions) of the last third statements hold for arbitrary collections $\mathcal{A} \subseteq \mathcal{P}(X)$.

Follow-up Exercise 2 The fourth and fifth statement has correct versions (one involving a "subset of" relation, and one involving the injectivity condition; similar to the correct versions of the third statement).

Follow-up Exercise 3 If $f : X \rightarrow Y$ is a function, then the **preimage relation** f^{-1} satisfies the injectivity condition². Consequently, taking preimages **commutes** with unions, intersections and relative complements.

²Recall that f^{-1} really is an abbreviation for $\text{graph}(f)^{-1}$, and f^{-1} is not necessarily a function from Y to X .

2 PSet 1, Problem 6

Let X and Y be two sets. A function from X to Y is **surjunctive** if it is either surjective or non-injective. Then

- (i) X is finite iff any function from X to X is surjunctive.
- (ii) Y has exactly one element iff any function from X to Y is surjunctive.

The first claim is correct without any modifications, while the second statement is not quite correct as given. Still, a minor adjustment suffices to obtain a correct version. We start with the first claim.

For this claim, part of the challenge really is to make intuitive sense of what it means for a set to be finite³. From a set theoretical point of view, this calls for a **rigorous definition**, however we'll keep the notion of finiteness of sets at the naive level⁴. Thus for us, a set X is **finite** if it is either the emptyset or if it can be represented as

$$X = \{x_1, x_2, \dots, x_n\}$$

for some positive integer n . Similarly, a set X is **infinite** if for any positive integer n , there is an element $x_n \in X$ with the property that $x_i \neq x_j$ for $i \neq j$. With these definitions, for instance, it is true that saying that a set is not infinite is equivalent to saying that the set is finite, although proving this equivalence rigorously is surprisingly cumbersome! As we are really focused toward calculus, we'll not investigate this issue any further.

Claim: Let X be a set. Then X is finite iff for any $f \in F(X; X)$, f is

³In *this* sentence, we use the word "intuitive" intuitively and not in a **technical** sense. Throughout this course we'll almost never use "intuition" in a technical sense.

⁴Indeed, it is valid in the grand scheme of things to take the claim we set out to prove as the *definition* of what it means for a set to be finite.

surjunctive.

┘

Proof: Recall that "iff" stands for a necessary and sufficient condition. Thus we'll need to prove two implications.

(\Rightarrow) First suppose $X = \{x_1, x_2, \dots, x_n\}$ is a finite set with exactly n elements and let $f : X \rightarrow X$ be an arbitrary function. We claim that f is surjunctive. That is, either f is surjective, or it is non-injective. To prove this, we may assume f fails to have one of these properties and show that it must have the other property. To that end, suppose f is injective (so that it fails to be non-injective). We need to show that f is surjective.

Suppose for a contradiction that there is a $j \in \{1, 2, \dots, n\}$ such that $x_j \notin \text{im}(f)$. But then $\text{im}(f)$ has at most $n - 1$ distinct elements. On the other hand, by the injectivity of f , for any $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, $f(x_i) \neq f(x_j)$, so that $\text{im}(f)$ has at least n distinct elements, a contradiction⁵.

(\Leftarrow) For the **converse**, we'll argue contrapositively. Suppose X is infinite, so that we have for any positive integer n , an element $x_n \in X$ with the property that $x_i \neq x_j$ for $i \neq j$. We may then define a self-function of X as follows:

$$f : X \rightarrow X, x \mapsto \begin{cases} x_{n+1}, & \text{if } \exists n \in \mathbb{Z}_{\geq 1} : x = x_n \\ x, & \text{else} \end{cases}.$$

Thus f is the function that shifts the index of the specified indexed elements by one, and acts as identity on $X \setminus \{x_n | n \in \mathbb{Z}_{\geq 1}\}$. Then by construction, $x_1 \notin \text{im}(f)$, however f is injective, so that f fails to be surjunctive.

┘

⁵Alternatively, one can refer to the **pigeonhole principle** to derive a contradiction.

The second claim is not correct in general. Indeed, say $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$. Then there are no injections from X to Y , thus any function $X \rightarrow Y$ is automatically surjunctive. The best we can do is to verify one side of the equivalence:

Claim: Let X and Y be two sets. If Y has exactly one element then any function $f : X \rightarrow Y$ is surjunctive.

┘

Proof: If $X = \emptyset$, then there are no functions $X \rightarrow Y$, so we may assume that $X \neq \emptyset$. Put $Y = \{y\}$ and let $f : X \rightarrow Y$ be function. Then f must be surjective, as there is at least one $x \in X$, and $f(x)$ is an element of Y , and consequently $f(x) = y$. Then by definition, f being surjective implies it being surjunctive.

┘

Follow-up Exercise Let Y be a set. Then Y has exactly one element iff for any set X , any function in $F(X; Y)$ is surjunctive.