University of Utah Fall 2024

MATH 4800-001 Notes

Instructor: Alp Uzman

Subject to Change; Last Updated: 2024-09-26 11:32:06-06:00

Contents

1 Introduction

These notes serve to document certain technical (and occasionally advanced) topics. Additionally, they contain exercises that can be used for your weekly reports, and possibly your final report. The content is intended to complement the discussions and material covered in class.

Please note that this is a working document; statements, their order, and the enumeration of exercises may be subject to change over time.

Comments, questions and corrections are welcome!

2 Moduli of Uniform Continuity

A modulus of continuity is a gadget one can use to quantify how uniformly continuous a function is[1]. We discussed one family of moduli of uniform continuity; namely Hölder continuity. Recall that for $\theta \in \mathbb{R}_{>0}$, a function $f : X \to Y$ between metric spaces is called **θ-Hölder** if

$$\exists C \in \mathbb{R}_{>0}, \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\theta.$$

A modulus of continuity generalizes the function $t \mapsto Ct^\theta$; note that this is exactly the function applied to the distance between x_1, x_2 in the above defininition. Indeed, we have the following definition:

Definition 1: A function $\omega : [0, \infty] \to [0, \infty]$ is called a **modulus of continuity** if the following properties are satisfied:

(i) $\forall t_1, t_2 \in \mathbb{R}_{>0} : t_1 \leq t_2 \implies \omega(t_1) \leq \omega(t_2)$ (i.e. ω is nondecreasing),

(ii) $\omega(0) = 0$,

(iii) $\lim_{t \to 0} \omega(t) = 0$ (i.e. ω is continuous at 0).

⌟

Exercise 1: For any $C \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}_{>0}$, $\omega(t) = Ct^\theta$ is a modulus of continuity. ⌟

For ω a modulus of continuity, a function $f : X \to Y$ is said to admit **ω as a modulus of continuity** if

[1] Similarly there are other moduli that allow one to quantify some other property, for instance moduli of smoothness.

$$\exists C \in \mathbb{R}_{>0}, \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq \omega(d_X(x_1, x_2)).$$

Exercise 2: Let $f : X \to Y$ be a function between metric space. If f admits a uniform modulus of continuity, then it is uniformly continuous. ⌟

Exercise 3: Let $f : X \to Y$ be a function between metric spaces. Define the function

$$\begin{aligned}\omega_f :&[0, \infty] \to [0, \infty]\\ &t \mapsto \sup\{d_Y(f(x_1), f(x_2)) | x_1, x_2 \in X, d_X(x_1, x_2) \leq t\}.\end{aligned}$$

Then f is uniformly continuous iff ω_f is a modulus of continuity with the following property:

$$\exists \varepsilon \in \mathbb{R}_{>0}, \forall t \in]0, \varepsilon[: \omega_f(t) < \infty.$$

⌟

Lipschitz, Hölder, Dini

There are many important moduli of continuity, in many different contexts. One important example for dynamics (see [Ano69, p.14]) is Dini continuity. A function $f : X \to Y$ is said to be **Dini continuous** if it admits a uniform modulus of continuity ω such that

$$\exists \varepsilon \in \mathbb{R}_{>0} : \int_0^\varepsilon \frac{\omega(t)}{t} dt < \infty.$$

Exercise 4: Every Hölder function is Dini continuous. ⌟

Exercise 5: Find a function that is θ-Hölder for any $\theta \in]0,1[$ but is not Lipschitz. ⌟

Exercise 6: Find a function that is Dini continuous but not Hölder. ⌟

Exercise 7: Find a function that is Dini continuous but not uniformly continuous. ⌟

Exercise 8: Let $f : [0,1] \to \mathbb{R}$ be a function. Then f is θ-Hölder for some $\theta \in \mathbb{R}_{>0}$ iff it is constant. ⌟

Exercise 9: Find a metric space X such that there is a nonconstant θ-Hölder function $f : X \to \mathbb{R}$ with $\theta \in \mathbb{R}_{>0}$. ⌟

Exercise 10: Let X be a metric space such that for any $\theta \in \mathbb{R}_{>0}$, and for any function $f : X \to \mathbb{R}$, f is θ-Hölder iff it is constant. What can be said about X? ⌟

Equivalence(s) of Metrics

Exercise 11: Recall that in class[2] we defined two metrics d_1, d_2 on a set X to be Lipschitz equivalent if

$$\exists C \in \mathbb{R}_{>0}, \forall x, y \in X : \frac{1}{C} d_1(x,y) \leq d_2(x,y) \leq C d_1(x,y).$$

Show that this is equivalent to

$$\exists C_1, C_2 \in \mathbb{R}_{>0}, \forall x, y \in X : C_1 d_1(x,y) \leq d_2(x,y) \leq C_2 d_1(x,y).$$

⌟

Exercise 12: The empty set $X = \emptyset$ has exactly one metric on it.

[2]On 08/20

⌟

Exercise 13: Let X be a finite set. Then any two metrics on X are Lipschitz equivalent. ⌟

Recall that one of the exercises we mentioned in class[3] is that two distances d_1, d_2 on a set X are Lipschitz equivalent iff the identity map $\mathrm{id}_X : (X, d_1) \to (X, d_2)$ is biLipschitz (or a Lipeomorphism)[4]. Generalizing this idea, we may say that, for ω a modulus of continuity, two distances d_1, d_2 on X are ω**-uniformly equivalent** if the identity maps $\mathrm{id}_X : (X, d_1) \to (X, d_2)$ and $\mathrm{id}_X : (X, d_2) \to (X, d_1)$ both admit ω as a uniform modulus of continuity.

Exercise 14: Write explicitly what it means for two metrics to be Hölder equivalent. ⌟

Exercise 15: Hölder/Lipschitz/uniform/topological equivalences of metrics are all equivalence relations. ⌟

[3] On 8/22

[4] A function $f : X \to Y$ is **biLipschitz** if it has an inverse $f^{-1} : Y \to X$ and both f and f^{-1} are Lipschitz.

3 Metric Preserving Function Theory

In class[5] we considered the following example: for X a set, if d is a metric on X, then for any $\lambda \in \mathbb{R}_{>0}$, so is

$$\lambda d : (x_1, x_2) \mapsto \lambda d(x_1, x_2)$$

(of course, for $\lambda = 0$, one gets the indiscrete pseudometric). One would say that the linear function $t \mapsto \lambda t$ is a metric preserving function.

Exercise 16: Call two metrics d_1, d_2 on a set X **proportional** if

$$\exists \lambda \in \mathbb{R}_{>0}, \forall x, y \in X : d_1(x, y) = \lambda d_2(x, y).$$

Proportionality is an equivalence relation on the set of all metrics on X.

How many disproportional metrics are there on a set X with exactly $n \in \mathbb{Z}_{\geq 0}$ elements?

⌟

Another example that is related to Hölder continuity is as follows: if d is a metric on X, then for any $\theta \in]0, 1[$, so is

$$d^\theta : (x_1, x_2) \mapsto d(x_1, x_2)^\theta.$$

Again, the function $t \mapsto t^\theta$ is a metric preserving function[6].

Exercise 17: Let $f : (X, d_X) \to (Y, d_Y)$ be a function between metric spaces and let $\theta \in]0, 1[$. Then $(X, d_X) \to (Y, d_Y)$ is θ-Hölder iff $(X, d_X) \to (Y, d_Y^\theta)$ is Lipschitz.

⌟

More generally, a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called **metric pre-**

[5] On 08/22

[6] Due to its connection to the Koch snowflake fractal and its variants, this function is often called the **snowflake functor** (see e.g. [Gro07, p.406]).

serving[7] (or a **metric transform**) if for any set X and any metric d on X, the pushforward $\phi \circ d$ too is a metric on X:

$$\begin{tikzcd} X \times X \arrow[r, "d"] \arrow[dr, "\phi\circ d"'] & \mathbb{R}_{\geq 0} \arrow[d, "\phi"] \\ & \mathbb{R}_{\geq 0} \end{tikzcd}$$

Exercise 18: Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a metric preserving function. Then ϕ is **subadditive**, that is,

$$\forall t, s \in \mathbb{R}_{\geq 0} : \phi(t+s) \leq \phi(t) + \phi(s).$$

⌟

Exercise 19: Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function with the following properties:

(i) $\forall t \in \mathbb{R}_{\geq 0} : \phi(t) = 0 \Leftrightarrow t = 0$,

(ii) $\forall t_1, t_2 \in \mathbb{R}_{\geq 0} : t_1 < t_2 \implies \phi(t_1) \leq \phi(t_2)$ (i.e. ϕ is nondecreasing),

(iii) $\forall t_1, t_2 \in \mathbb{R}_{\geq 0} : \phi(t_1 + t_2) \leq \phi(t_1)\phi(t_2)$ (i.e. ϕ is subadditive).

Then ϕ is metric preserving.

⌟

Exercise 20: If ϕ and ψ are metric preserving functions, then so is $\psi \circ \phi$. ⌟

Exercise 21: If ϕ and ψ are metric preserving functions, then so is $\max\{\phi, \psi\}$.

⌟

[7] Good references for metric preserving function theory are [Cor99, Dob98].

Exercise 22: If ϕ and ψ are metric preserving functions, then so is $\alpha\phi + \beta\psi$ for any $\alpha, \beta \in \mathbb{R}_{>0}$.

⌟

Exercise 23: For any $\lambda \in \mathbb{R}_{>0}$ and $\theta \in]0, 1]$, $\phi(t) = \lambda t^{\theta}$ is a metric preserving function. ⌟

Exercise 24: Each of the following functions are metric preserving:

(i) $\forall \tau \in \mathbb{R}_{>0} : \phi_{\tau}(t) = \min\{\tau, t\}$

(ii) $\forall \tau \in \mathbb{R}_{>0} : \phi_{\tau}(t) = \begin{cases} \max\{\tau, t\}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$

(iii) $\forall \tau \in \mathbb{R}_{>0} : \phi_{\tau}(t) = \begin{cases} \tau + t, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$

(iv) $\forall \tau \in \mathbb{R}_{>0} : \phi_{\tau}(t) = \dfrac{\log(1 + t)}{\log(1 + \tau)}$

(v) $\phi(t) = \dfrac{t}{1 + t}$

(vi) $\phi(t) = \arctan(t)$

⌟

Note that in order for a function to be metric preserving, the new metric need not be equivalent to the original one.

Exercise 25: Give an example of a metric preserving function that does not preserve the topological equivalence class of some metric it transforms.

⌟

Exercise 26: Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a metric preserving function. Then the following are equivalent:

(i) For any set X and any metric d on X, d and $\phi \circ d$ are topologically equivalent.

(ii) ϕ is continuous.

(iii) ϕ is continuous at 0.

(iv) $\forall \varepsilon \in \mathbb{R}_{>0}, \exists t \in \mathbb{R}_{>0} : \phi(t) < \varepsilon$.

⌟

4 Metrics on Shift Spaces

The kinds of shift spaces that we'll be interested in the most will be spaces of infinite or bi-infinite sequences taking values in a finite alphabet. Let $m \in \mathbb{Z}_{\geq 1}$ and put $\underline{m} = \{0, 1, ..., m-1\}$. Then by definition the one sided shift space is

$$\Sigma^+ = F(\mathbb{Z}_{\geq 0}; \underline{m}) = \{\omega_\bullet | \forall n \in \mathbb{Z}_{\geq 0} : \omega_n \in \underline{m}\},$$

and similarly the two sided shift space is

$$\Sigma = F(\mathbb{Z}; \underline{m}) = \{\omega_\bullet | \forall n \in \mathbb{Z} : \omega_n \in \underline{m}\}.$$

The reason why these are called shift spaces is due to the fact that both of these spaces carry a natural dynamical system, namely the shift:

$$\sigma : \omega_\bullet \mapsto \omega_{\bullet+1}.$$

There are certain classes of dynamical systems that can be represented via the shift dynamical system; this is the key insight that makes symbolic dynamics so useful.

In dynamics (and metric space) literature there are a few metrics defined on such spaces; here we list some of the most prominent ones and make a comparison. We focus on the two sided space Σ; the discussion for Σ^+ is analogous. Generally speaking, we want distances of Σ that emphasize discrepancies in entries closer to the time-0 entry, modeling the idea that, if one has two real numbers represented in decimal expansion, discrepancies in entries closer to the decimal point matter more when it comes to the distance between the two numbers. This is not to say that we require a metric on shift space to match with the Euclidean distance on the nose.

- Following Bowen (see [Bow08, p.7]), define $N_B : \Sigma \times \Sigma \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$N_B(\omega, \eta) = \max\{n | \forall |k| < n : \omega_k = \eta_k\}.$$

(Here the subscript B is for "Bowen".)

Then for any $\theta \in \mathbb{R}_{>1}$ one can define

$$d_{B;\theta}(\omega, \eta) = \theta^{-N_B(\omega,\eta)}.$$

- Following Ruelle (see [Rue78, p.123]), define $N_R : \Sigma \times \Sigma \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$N_R(\omega, \eta) = \min\{n | \omega_n \neq \eta_n \text{ or } \omega_{-n} \neq \eta_{-n}\}.$$

Then for any $\theta \in \mathbb{R}_{>1}$ one can define

$$d_{R;\theta}(\omega, \eta) = \theta^{-N_R(\omega,\eta)}.$$

- Following Hasselblatt-Katok (see [HK03, pp.214-215]), denote by δ the discrete metric on $\underline{m}$:

$$\delta(i, j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}.$$

Then for any $\theta \in \mathbb{R}_{>1}$ one can define

$$d_{HK;\theta}(\omega, \eta) = \sum_{n \in \mathbb{Z}} \frac{\delta(\omega_n, \eta_n)}{\theta^{|n|}}.$$

- Following Katok-Hasselblatt (see [KH95, p.47]), one may take $\underline{m}$ as a metric subspace of $\mathbb{R}$, and for any $\theta \in \mathbb{R}_{>1}$ one can define

$$d_{KH;\theta}(\omega, \eta) = \sum_{n \in \mathbb{Z}} \frac{|\omega_n - \eta_n|}{\theta^{|n|}}.$$

- A third suggestion from Katok-Hasselblatt (see [KH95, p.55]) that is less common in dynamics is as follows: Taking $\underline{m}$ as a metric subspace of $\mathbb{R}$, for any $\theta \in \mathbb{R}_{>1}$ one can define

$$\widetilde{d}_{KH}(\omega, \eta) = \sum_{n \in \mathbb{Z}} \frac{|\omega_n - \eta_n|}{(n^2 + 1)}.$$

- Generalizing the Katok-Hasselblatt examples, one has the more general family of Fréchet product metrics (see [DD16, p.93]). Let $a_\bullet$ be a sequence of positive numbers such that $\sum a_\bullet = \sum_{n=0}^{\infty} a_n$ is convergent. Then define

$$d_{F;a}(\omega, \eta) = \sum_{n \in \mathbb{Z}} a_{|n|} d(\omega_n, \eta_n).$$

Exercise 27: All of the above are metrics on Σ.

⌟

Let us now compare these metrics. First note that for $\gamma = \dfrac{\log(\alpha)}{\log(\beta)}$, we have $\alpha = \beta^\gamma$. Thus

$$d_{B;\alpha}(\omega, \eta) = (d_{B;\beta}(\omega, \eta))^\gamma,$$

so that for any $\alpha, \beta \in \mathbb{R}_{>1}$, the metrics $d_{B;\alpha}$ and $d_{B;\beta}$ are Hölder equivalent[8]. Similarly for any $\alpha, \beta \in \mathbb{R}_{>1}$, the metrics $d_{R;\alpha}$ and $d_{R;\beta}$ are Hölder equivalent. Next, as $N_R = N_B + 1$, for any $\theta \in \mathbb{R}_{>1}$, the

[8] With Hölder modulus of continuity $t \mapsto t^{\min\{\log(\alpha)/\log(\beta), \log(\beta)/\log(\alpha)\}}$.

metrics $d_{B;\theta}$ and $d_{R;\theta}$ are Lipschitz equivalent. The fact that for any $\theta \in \mathbb{R}_{>1}$, the metrics $d_{HK;\theta}$ and $d_{KH;\theta}$ are equivalent is due to the fact that our alphabet is finite (see Exr. 13).

Let us now compare $d_{B;\theta}$ to $d_{HK;\theta}$. Let $\omega, \eta \in \Sigma$. Abbreviating $N = N_B(\omega, \eta)$ and writing $d_{HK;\theta}(\omega, \eta)$ explicitly, we have

$$\begin{aligned}
&d_{HK;\theta}(\omega, \eta) \\
&= \cdots + \frac{\delta(\omega_{-N}, \eta_{-N})}{\theta^N} \\
&\underbrace{+\frac{\delta(\omega_{-(N-1)}, \eta_{-(N-1)})}{\theta^{N-1}} + \cdots + \frac{\delta(\omega_{-1}, \eta_{-1})}{\theta}}_{=0} \\
&\underbrace{+\delta(\omega_0, \eta_0) + \frac{\delta(\omega_1, \eta_1)}{\theta} + \cdots \frac{\delta(\omega_{N-1}, \eta_{N-1})}{\theta^{N-1}}}_{=0} \\
&+ \frac{\delta(\omega_N, \eta_N)}{\theta^N} + \cdots
\end{aligned}$$

The terms in the middle are zero due to the definition of $N = N_B(\omega, \eta)$, and again due to the definition of N, at least one of

$$\delta(\omega_{-N}, \eta_{-N}), \delta(\omega_N, \eta_N)$$

is equal to 1, so that

$$d_{B;\theta}(\omega, \eta) = \frac{1}{\theta^N} \leq d_{HK;\theta}(\omega, \eta).$$

On the other hand, due again to the vanishing of the middle terms, each summand in the series that defines $d_{HK;\theta}(\omega, \eta)$ has a factor of $\frac{1}{\theta^N}$. Factoring this factor out, we have the tail series dominated by geometric series, so that the total sum of both tails can be bounded by a number $C = C_\theta$ that depends only on θ.

Exercise 28: Compute an explicit expression for $C = C_\theta$. ⌟

But the factor $\frac{1}{\theta^N}$ is exactly $d_{B;\theta}(\omega,\eta)$, so that we have

$$\boxed{d_{B;\theta}(\omega,\eta) \le d_{HK;\theta}(\omega,\eta) \le C_\theta d_{B;\theta}(\omega,\eta)}.$$

Thus $d_{B;\theta}$ is Lipschitz equivalent to $d_{HK;\theta}$.

Next let $\alpha,\beta \in \mathbb{R}_{>1}$ and put $\gamma = \frac{\log(\alpha)}{\log(\beta)}$ as before. By chaining the equivalences already derived, we obtain

$$d_{HK;\alpha}(\omega,\eta) \le C_\alpha d_{B;\alpha}(\omega,\eta) = C_\alpha(d_{B;\beta}(\omega,\eta))^\gamma \le C_\alpha(d_{HK;\beta}(\omega,\eta))^\gamma,$$

so that $d_{HK;\alpha}$ is Hölder equivalent to $d_{HK;\beta}$.

Exercise 29: Verify that $d_{HK;\alpha}$ is Hölder equivalent to $d_{HK;\beta}$ directly, without referring to $d_{B;\alpha}$. ⌟

Exercise 30: Let $\alpha,\beta \in \mathbb{R}_{>1}$. Then

(i) $d_{B;\alpha}$ is not Lipschitz equivalent to $d_{B;\beta}$;

(ii) $d_{HK;\alpha}$ is not Lipschitz equivalent to $d_{HK;\beta}$,

(iii) $\widetilde{d}_{KH}$ is not Hölder equivalent to $d_{B;\alpha}$, but is topologically equivalent to all of the above (including $d_{F;a}$).

⌟

Exercise 31: Let $f : \Sigma \to \mathbb{R}$ be a function. Then f is Hölder relative to one of $d_{B;\theta_1}, d_{R;\theta_2}, d_{HK;\theta_3}, d_{KH;\theta_4}$ iff it is Hölder relative to all (possibly with different exponent or multiplicative constant). ⌟

5 Notation for Sequences and Subsequences

Let X be a set. A **sequence** in X is a function from nonnegative integers to X:

$$x_\bullet : \mathbb{Z}_{\geq 0} \to X.$$

Here the subscript $\bullet$ ("bullet") is the placeholder; it signifies that the input $n \in \mathbb{Z}_{\geq 0}$ is to be plugged into where the bullet is. Thus the value (or position) of the sequence $x_\bullet$ at time n is the point x_n in X. In standard function notation one would and can write $x_n = x(n)$. It is not important if one starts indexing from 0 or 1, except perhaps when X is a familiar space (such as $\mathbb{R}^d$ or $\mathbb{C}^d$) and the value x_n is an explicit function of n. Typically one considers a sequence to be indexed by some infinite subset of $\mathbb{Z}_{\geq 0}$ that is bounded from below; if a sequence is indexed by a subset not bounded below often that is signified by saying e.g. "bi-infinite sequence".

Note that a sequence is fundamentally a dynamical object; its input values being the time parameter. Thus a sequence $x_\bullet$ is different from the set $\{x_n | n \in \mathbb{Z}_{\geq 0}\}$ of its values (or positions); of course this set of values is nothing but the image of the function that is the sequence.

Exercise 32: Let $X = \mathbb{Z}$. How many sequences $x_\bullet$ are there in X such that

$$\{x_n | n \in \mathbb{Z}_{\geq 0}\} = \{0, 1\}?$$

⌟

Let $x_\bullet$ be a sequence in X. A **subsequence** $x_\bullet$ is a sequence $x'_\bullet$ in X such that for some strictly increasing function $\iota : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ one has

$$\forall n \in \mathbb{Z}_{\geq 0} : x'_n = x_{\iota(n)}.$$

Thus using a diagram, one has

$$\begin{tikzcd} & \mathbb{Z}_{\geq 0} \arrow[dr, "x_\bullet"] & \\ \mathbb{Z}_{\geq 0} \arrow[ur, "\iota"] \arrow[rr, "x'_\bullet"] & & X \end{tikzcd}$$

We write $x'_\bullet \leq x_\bullet$ to denote that $x'_\bullet$ is a subsequence of $x_\bullet$[9]. As an example, consider $\iota(n) = n^2$. Thus one starts with a sequence $x_\bullet$, and samples every n^2-th position of it to obtain a subsequence:

$$\begin{aligned} x'_0 &= x_0, \\ x'_1 &= x_1, \\ x'_2 &= x_4, \\ x'_3 &= x_9, \\ x'_4 &= x_{16}, \\ &... \end{aligned}$$

One may consider explicit sampling functions ι as in this example, or one may a subsequence without worrying about an explicit formula for ι.

Among alternative, and arguably more common, notations for sequences and subsequences are the following:

$$x_\bullet = \{x_n\} = \{x_n\}_n = \{x_n\}_{n=0}^{\infty} = (x_n) = (x_n)_n = \cdots ,$$

[9]More generally, for two objects a, b of the same type T, $a \leq b$ means that a is a subobject of b, insofar as the "subobject of" relation is clear for objects of type T.

$$x'_\bullet = \{x_{n_k}\} = \{x_{n_k}\}_k = \cdots .$$

While more explicit, there are various issues with these alternative notations. In particular taking multiple subsequences may get unmanageable faster compared to the bullet notation[10]. For instance consider the following syntactic exercise:

Exercise 33: Any subsequence of any subsequence of any subsequence of any subsequence of any subsequence of any subsequence of any subsequence of any subsequence of any subsequence of a sequence $x_\bullet$ is a subsequence of the sequence $x_\bullet$. ⌟

Ultimately though, what notation to use is a matter of taste. For instance one common practice is to say "statement S holds for $x_\bullet$ up to a subsequence", without explicitly worrying about subsequence indices.

As a final remark, the bullet notation is really common for a more general object than a sequence, namely a net, where the "time" parameter is allowed to be more complicated than $\mathbb{Z}_{\geq 0}$. The bullet notation is also common in algebraic topology.

[10] If needed, the specific sampling schemes may stored in the sampling functions ι.

6 Notation for Level Sets of Metrics: Balls, Spheres and all that

We discussed in class[11] that a metric on a set both gives structure to the set (making it a metric space), and is the source of legitimate, continuous functions on it (relative to the metric space structure it creates). We also said that level sets (or more accurately, sublevel sets) of metrics are important subsets of a metric space, as many properties (e.g. continuity and convergence) can be described using unions and intersections of such sets. Here we recall the syntax for such subsets.

Let X be a (nonempty) set, d be a metric. For any point $x \in X$ and any subset $A \subseteq X$, we put

$$d(x, A) = \inf_{a \in A} d(x, a) \in [0, \infty].$$

Exercise 34: Let X be a metric space $A \subseteq X$. Then

$$A = \emptyset \Leftrightarrow \forall x \in X : d(x, A) = \infty.$$

⌟

Exercise 35: Let X be a metric space, and $A, B \subseteq X$. Then the following are equivalent:

(i) $\forall x \in X : d(x, A) = d(x, B)$

(ii) $\overline{A} = \overline{B}$.

⌟

Let further $r \in \mathbb{R}_{>0}$. Then the **open ball** centered at x with radius r is by definition

[11] On 08/27

$$(X, d)[x| < r] = X[x| < r] = [x| < r] = \{y \in X | d(x, y) < r\}.$$

Similarly the **closed ball** centered at x with radius r is defined by

$$(X, d)[x| \leq r] = X[x| \leq r] = [x| \leq r] = \{y \in X | d(x, y) \leq r\}.$$

Instead of using a point as the center, we may use a subset $A \subseteq X$ as "the center", for instance

$$\begin{aligned}(X, d)[A| < r] = X[A| < r] = [A| < r] &= \bigcup_{a \in A} [a| < r] \\ &= \{y \in X | d(y, a) < r \text{ for some } a \in A\}.\end{aligned}$$

The general syntax is as follows:

$$\boxed{(\text{ambient space})[\text{center} \,|\, \text{radius}]}.$$

Based on this syntax, for instance for $a \in A$ one has

$$A[a| < r] = A \cap [a| < r].$$

As can be seen above, if the ambient space, or its metric is clear we merely suppress it for brevity.

Exercise 36: Let A be a subset of a metric space X. Then

(i) $[A| < r] = \{x \in X | d(x, A) < r\}$,

(ii) $[A| \leq r] = \{x \in X | d(x, A) \leq r\}$,

$\lrcorner$

Exercise 37: Let X be a metric space, $A \subseteq X$ be an arbitrary subset and $r \in \mathbb{R}_{>0}$. Then

(i) $[A| < r]$ is open in X.

(ii) $[A| \leq r]$ is closed in X.

$\lrcorner$

The following exercise is useful to get familiar with notation:

Exercise 38: Let $X = \mathbb{R}^2$. We consider X with the standard Euclidean metric (aka ℓ^2 metric) or the Manhattan metric (aka ℓ^1 metric); when we mean the Euclidean metric we suppress it from the notation. Draw the following subsets of X:

(i) $[\{0\}| < 1]$

(ii) $(X, \ell^1)[\{0\}| < 1]$

(iii) $[0| \geq 2]$

(iv) $[0|[1, 2]]$

(v) $(X, \ell^1)[0|]1, 2[]$

(vi) $[[0| < 1]| < 1]$

(vii) $[(X, \ell^1)[0| < 1]| < 1]$

(viii) $(X, \ell^1)[[0| < 1]| < 1]$

(ix) $(X, \ell^1)[(X, \ell^1)[0| < 1]| < 1]$

(x) $[[0| = 1]| < 1/4]$

(xi) $[[0, 1] \times 0| < 1]$

(xii) $([0,1] \times 0)[1/2| < 1]$

(xiii) $[0| < 1][0| < 1]$

(xiv) $[0| < 1][1/2| < 1]$

(xv) $[0| < 1][1/2| < 1][-1/2| < 1]$.

$\lrcorner$

7 Lipschitz Constants

Let X and Y be two metric spaces. For $f : X \to Y$ a function, define the **Lipschitz constant** of f by

$$\mathrm{Lip}(f) = \mathrm{Lip}(f; d_X, d_Y) = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \in [0, \infty].$$

(If X has at most one point, so that the supremum is taken over an empty set, one defines $\mathrm{Lip}(f) = 0$.)

Exercise 39: f is Lipschitz iff $\mathrm{Lip}(f) < \infty$, and f is a contraction iff $\mathrm{Lip}(f) < 1$. ⌟

If f satisfies the Lipschitz property with multiplicative constant C, it satisfies the Lipschitz property with, say, 2C too. Thus the multiplicative constant can be taken as large as possible. On the other hand, shrinking the multiplicative constant, hence making the Lipschitz condition more restrictive, does say something about the function. Indeed, the "smallest" multiplicative constant is exactly the Lipschitz constant Lip(f):

Exercise 40: Let $f : X \to Y$ be a function between metric spaces. Then

$$\mathrm{Lip}(f) = \min\{C \in \mathbb{R}_{\geq 0} | \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)\}.$$

⌟

Linear Algebra

In this section we discuss vector spaces and linear algebra from the point of view of metric spaces. The key insight is that the Lipschitz constant of a function between metric spaces is really a generalization of the notion of the norm of a linear transformation. Let $\mathbb{F}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and let V be a vector space over $\mathbb{F}$. A **norm** on V by definition a function $|\bullet| : V \to \mathbb{R}_{\geq 0}$ such that

- $\forall v \in V : |v| = 0 \Leftrightarrow v = 0,$
- $\forall c \in \mathbb{F}, \forall v \in V : |cv| = |c||v|,$
- $\forall v_1, v_2 \in V : |v_1 + v_2| \leq |v_1| + |v_2|.$

Exercise 41: Let V be $\mathbb{F}^d$, and define for $v = (v_1, v_2, ..., v_d) \in \mathbb{F}^d$,

$$\forall p \in [1, \infty[: |v|_p = \left(\sum_{i=1}^{d} |v_i|^p\right)^{1/p} \quad ; \quad |v|_\infty = \max_{1 \leq i \leq d} |v_i|.$$

Then $\forall p \in [1, \infty]$, $|\bullet|_p$ is a norm on $\mathbb{F}^d$, and the metric induced by $|\bullet|_p$ is the ℓ^p metric on $\mathbb{F}^d$.

⌟

A **normed vector space** is a vector space with a fixed norm. Any normed vector space can be considered to be a metric space:

Exercise 42: Let V be a vector space over $\mathbb{F}$. Then

(i) If $|\bullet|$ is a norm on V, then

$$d : V \times V \to \mathbb{R}_{\geq 0}, (v_1, v_2) \mapsto |v_1 - v_2|$$

is a metric on V, called the **metric induced by the norm** $|\bullet|$. Further, the metric d has the following two properties:

(a) $\forall v_1, v_2, v \in V : d(v_1 + v, v_2 + v) = d(v_1, v_2)$, (aka translation invariance)

(b) $\forall c \in \mathbb{F}, \forall v_1, v_2 \in V : d(cv_1, cv_2) = |c|d(v_1, v_2)$ (aka homogeneity).

(ii) Any translation invariant and homogeneous metric on V is the metric induced by some norm on V.

⌟

Exercise 43: Let V be a vector space. Then any norm on V is Lipschitz with Lipschitz constant 1 relative to the metric induced by it.

⌟

Exercise 44: Any two metrics induced by any two norms on $\mathbb{R}^n$ are Lipschitz equivalent.

⌟

Exercise 45: Let (X, d) be a metric space. If there are two functions

$$\alpha : X \times X \to X, \sigma : \mathbb{F} \times X \to X$$

such that

$$d(\alpha(x, y), \alpha(x, z)) = d(y, z) \text{ and } d(\sigma(c, y), \sigma(c, z)) = |c|d(y, z),$$

is $(X, \alpha, \sigma, d(\bullet, 0))$ necessarily a normed vector space?

⌟

Let now V and W be two vector spaces over $\mathbb{F}$. A function $A : V \to W$ is **linear** if

$$A(v_1 + v_2) = A(v_1) + A(v_2) \text{ and } A(cv) = cA(v).$$

A function $A : V \to W$ is **affine** if there is a linear $B : V \to W$ and a vector $b \in W$ such that for any $v \in V : A(v) = B(v) + b$. Here $B = \mathbb{L}(A)$ is called the **linear part** and $b = \mathbb{T}(A)$ is called the **translation part** of A.

Exercise 46: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there are mn many numbers $a_{ij} \in \mathbb{R}$ such that for any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$:

$$f(x_1, x_2, ..., x_n) = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, ..., \sum_{j=1}^{n} a_{mj}x_j \right).$$

Here we consider both $\mathbb{R}^n$ and $\mathbb{R}^m$ as vector spaces over $\mathbb{R}$.

┘

Exercise 47: Let $| \bullet |_V$ and $| \bullet |_W$ be norms on V and W, respectively, and let $A : V \to W$ be a linear function. Then the following values in $[0, \infty]$ are equal:

(i) $\sup_{\substack{v \in V \\ v \neq 0}} \frac{|Av|_W}{|v|_V}$

(ii) $\sup_{\substack{v \in V \\ |v|_V \leq 1}} |Av|_W$

(iii) $\sup_{\substack{v \in V \\ |v|_V = 1}} |Av|_W$

(iv) $\inf\{C \in \mathbb{R}_{\geq 0} | \forall v \in V : |Av|_W \leq C|v|_V\}$.

┘

The common value in Exr. 47 is defined to be the **operator norm** of A.

Exercise 48: Let V and W be vector spaces and let $| \bullet |_V$ and $| \bullet |_W$ be norms on V and W, respectively. Then

(i) The vector space $\mathcal{L}(V; W)$ of all continuous linear functions from V to W is a vector subspace of the vector space of all linear functions from V to W.

(ii) The operator norm is a norm on $\mathcal{L}(V; W)$.

⌟

Exercise 49: Let $|\bullet|_V$ and $|\bullet|_W$ be norms on V and W, respectively, and let $A : V \to W$ be an affine function. Then one has

$$\mathrm{Lip}(A) = |\mathbb{L}(A)|,$$

where $\mathbb{L}(A)$ is the linear part of A. Furthermore, the following are equivalent:

(i) $|A| < \infty$.

(ii) A is continuous at 0.

(iii) A is continuous at some point in V.

(iv) A is continuous everywhere.

(v) A is Lipschitz.

⌟

Exercise 50: Let V and W be vector spaces. If V is finite dimensional as a vector space, then any affine function $A : V \to W$ is Lipschitz.

⌟

Exercise 51: Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear function and $A^T : \mathbb{R}^d \to \mathbb{R}^d$ be its transpose. For $p \in [1, \infty]$, denote by $\mathrm{Lip}_p(A)$ the Lipschitz constant of A when $\mathbb{R}^d$ is considered with the ℓ^p metric. Then one has:

(i) $\mathrm{Lip}_1(A) = \max_j |a_{\bullet j}|_1 = \max_j \sum_i |a_{ij}| = \mathrm{Lip}_\infty\left(A^T\right)$.

(ii) $\mathrm{Lip}_\infty(A) = \max_i |a_{i\bullet}|_1 = \max_i \sum_j |a_{ij}| = \mathrm{Lip}_1\left(A^T\right)$.

(iii) More generally, $\forall p \in [1,\infty], \mathrm{Lip}_p(A) = \mathrm{Lip}_q(A)$, where q is the Hölder conjugate of p, aka it's the unique value in $[1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

(iv) $\mathrm{Lip}_2(A)$ is the largest singular value of A (aka $\mathrm{Lip}_2(A)$ is the squareroot of the largest eigenvalue of A^TA).

(v) $\mathrm{Lip}_2(A) \leq |A|_F$, where $|A|_F$ is the Frobenius norm of the operator A; by definition $|A|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$ is the ℓ^2 norm of A considered as a vector in $\mathbb{F}^{2d} \cong \mathcal{L}(\mathbb{F}^d; \mathbb{F}^d)$.

⌟

Large-Scale Lipschitz Property

Sometimes the Lipschitz constant Lip(f) is also called the **dilatation** of f (see e.g. [BBI01, p.249]):

$$\mathrm{dilatation}(f) = \mathrm{Lip}(f).$$

Instead of looking at the ratios of distances, one could look at the differences of distances as well; in which case one obtains the **distortion** of f:

$$\mathrm{distortion}(f) = \sup_{x_1, x_2 \in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \in [0,\infty].$$

Exercise 52: Let $f : X \to Y$ be an isometry. Then it has zero distortion.

⌟

Recall that by Exr. 39, a function with finite dilatation must be Lipschitz, hence uniformly continuous. On the other hand a function with finite distortion need not even be continuous:

Exercise 53: Let $f : X \to Y$ be a function between metric spaces. Then[12]

$$\mathrm{distortion}(f) \leq \mathrm{diam}(X) + \mathrm{diam}(Y).$$

In particular if both X and Y have finite diameter (for instance both spaces could be balls in Euclidean space), then any function between them has finite distortion. Consequently finite distortion need not imply continuity.

⌟

One should think of dilatation versus distortion as follows:

- Dilatation (aka Lipschitz constant, aka the multiplicative constant) controls relative changes of distances: if two inputs are close, then the outputs are comparably close.
- Distortion controls absolute changes of distances, or rather, large distortion loosens up changes of distances absolutely: If the distortion of a function is large, no matter how close two inputs are, the distance between the outputs can be as large as the distortion.

One can combine the two constants to obtain what is called the large-scale Lipschitz condition[13]. A function $f : X \to Y$ is called **large-scale Lipschitz** if

[12] The **diameter** of a subset A of a metric space (X, d) is defined by $\mathrm{diam}(A) = \mathrm{diam}(A, d) = \sup_{a_1, a_2 \in A} d(a_1, a_2) \in [0, \infty]$.

[13] See e.g. [Roe03, p.6]. This is also related to quasi-isometries, is important in the context of geometric group theory.

$$\exists A, B \in \mathbb{R}_{>0}, \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B.$$

It is useful to think of the function $\omega(t) = At + B$ as a "modulus of large-scale continuity", even though it is not exactly a modulus of continuity according to Def. 1.

Exercise 54: Give an example of a uniformly continuous function that is large-scale Lipschitz that is not Lipschitz.

⌟

Exercise 55: Define what it means for a function to be large-scale bi-Lipschitz.

⌟

Exercise 56: Define what it means for a function to be large-scale Hölder.

⌟

Exercise 57: Give an example of a uniformly continuous function that is large-scale Hölder that is not Hölder nor large-scale Lipschitz.

⌟

8 Hyperspaces

Let X be a metric space with metric d. Consider the set $\mathcal{H}(X)$ of all nonempty compact subsets of X. A natural metric on $\mathcal{H}(X)$ is the Pompeiu-Hausdorff metric[14] $d_{\mathcal{H}(X)}$. There are multiple definitions of $d_{\mathcal{H}(X)}$:

Exercise 58: Let X be a metric space. Then for any $A, B \in \mathcal{H}(X)$, the following four numbers are equal:

Functional: $\sup_{x\in X} |d(x, A) - d(x, B)|$.

Ball: $\inf\{r \in \mathbb{R}_{>0} | A \subseteq [B| < r] \text{ and } [A| < r] \supseteq B\}$.

Minimax: $\max\{d(A \leftarrow B), d(B \leftarrow A)\}$, where

$$d(E \leftarrow F) = \max_{e\in E} \min_{f\in F} d(e, f).$$

Lipschitz: $\max_{\substack{f:X\to\mathbb{R} \\ \mathrm{Lip}(f)\leq 1}} |\max_{a\in A} f(a) - \max_{b\in B} f(b)|$.

⌟

The common value in Exr.58 is by definition the **Pompeiu-Hausdorff distance** $d_{\mathcal{H}(X)}(A, B)$ between the subsets A and B of X.

Exercise 59: Let X be a metric space, $A, B \in \mathcal{H}(X)$, $r \in \mathbb{R}_{>0}$. Then

(i) $d_{\mathcal{H}(X)}(A, B) < r$ iff $A \subseteq [B| < r]$ and $[A| < r] \supseteq B$.

(ii) $d_{\mathcal{H}(X)}(A, B) \leq r$ iff $A \subseteq [B| \leq r]$ and $[A| \leq r] \supseteq B$.

⌟

Exercise 60: Let X be a metric space. Then $d_{\mathcal{H}(X)}$ is a metric on $\mathcal{H}(X)$.

⌟

[14]Although it's not the only natural metric, nor the implied topology the only natural topology.

Exercise 61: Let X be a metric space. Find conditions that would allow one to identify a metric on $\mathcal{H}(X)$ as the Pompeiu-Hausdorff metric.

⌟

Exercise 62: Let $X = \mathbb{R}^d$, $x, y \in X$, $r, s \in \mathbb{R}_{>0}$. Then

$$d_{\mathcal{H}(X)}([x| \leq r], [y| \leq s]) = |x - y| + |r - s|,$$

Consequently,

$$[\bullet| \leq \bullet] : X \times \mathbb{R}_{>0} \to \mathcal{H}(X)$$

is distance preserving.

⌟

Exercise 63: Let X be a metric space such that every closed ball in X is compact. Find conditions on X such that

$$[\bullet| \leq \bullet] : X \times \mathbb{R}_{>0} \to \mathcal{H}(X)$$

is distance preserving or Lipschitz or uniformly continuous or continuous.

⌟

Exercise 64 (Topological Properties of Logical Operators)**:** Let X be a metric space. Then

(i) The relation $\in$ ("element of") is closed. More specifically,

$$\{(x, A) \in X \times \mathcal{H}(X) | x \in A\}$$

is a closed subset of $X \times \mathcal{H}(X)$.

(ii) The relation $\subseteq$ ("subset of") is closed. More specifically,

$$\{(A,B) \in \mathcal{H}(X) \times \mathcal{H}(X) | A \subseteq B\}$$

is a closed subset of $\mathcal{H}(X) \times \mathcal{H}(X)$.

(iii) $\{(A,B) \in \mathcal{H}(X) \times \mathcal{H}(X) | A \cap B \neq \emptyset\}$ is a closed subset of $\mathcal{H}(X) \times \mathcal{H}(X)$.

(iv) The operation $\bigcup$ ("union") over compact families of compact subsets is Lipschitz. More specifically

$$\bigcup : \mathcal{H}^2(X) \to \mathcal{H}(x), \mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} A$$

is Lipschitz. Here $\mathcal{H}^2(X) = \mathcal{H}(\mathcal{H}(X))$ is the hyperspace of the hyperspace of X.

⌟

Exercise 65 (Functoriality 1)**:** Let $f : X \to Y$ be a function between metric spaces. Then

(i) If f is continuous, then so is $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{H}(Y)$, $A \mapsto f(A)$.

(ii) If f is uniformly continuous with uniform modulus of continuity ω, then so is $\mathcal{H}(f)$ with the same modulus of continuity.

(iii) $\mathrm{Lip}(\mathcal{H}(f)) \leq \mathrm{Lip}(f)$.

⌟

Exercise 66 (Functoriality 2)**:** Let X be a set and d_1, d_2 be two metrics on X. Then the following are equivalent:

(i) d_1 and d_2 are uniformly equivalent.

(ii) $(d_1)_{\mathcal{H}(X)}$ and $(d_2)_{\mathcal{H}(X)}$ are uniformly equivalent.

(iii) $(d_1)_{\mathcal{H}(X)}$ and $(d_2)_{\mathcal{H}(X)}$ are topologically equivalent.

⌟

The Pompeiu-Hausdorff metric is "natural" in the sense that it inherits certain properties of the original metric[15], for instance:

Exercise 67: Let X be a set, and let d be a metric on X. Then d is the discrete metric on X iff $d_{\mathcal{H}(X)}$ is the discrete metric on $\mathcal{H}(X)$.

⌟

Exercise 68 (ℓ^p-Minimax)**:** Let X be a metric space, $A, B \in \mathcal{H}(X)$, $p \in [1, \infty]$ and put

$$d_{p,\mathcal{H}(X)}(A, B) = |(d(A \leftarrow B), d(B \leftarrow A))|_p.$$

Thus for instance $d_{\infty,\mathcal{H}(X)}(A, B) = d_{\mathcal{H}(X)}(A, B)$ and $d_{1,\mathcal{H}(X)}(A, B) = d(A \leftarrow B) + d(B \leftarrow A)$.

(i) $d_{1,\mathcal{H}(X)}$ is a metric on $\mathcal{H}(X)$ that is Lipschitz equivalent to $d_{\mathcal{H}(X)}$.

(ii) Is $d_{p,\mathcal{H}(X)}$ a metric for an arbitrary p?[16] Is it equivalent to $d_{\mathcal{H}(X)}$ in some sense, and if yes, what is the regularity of the equivalence?

⌟

Painlevé-Kuratowski Convergence

One can consider convergence for subsets in the sense of Painlevé-Kuratowski also. Convergence relative to the PH-metric turns out to imply PK-convergence, hence it's useful to discuss this weaker type

[15]Although the PH metric does not inherit all properties one might be interested in, hence the interest in other metrics on the hyperspace.

[16]This was mentioned in class on 09/03.

of convergence, especially when one tries to identify points in the PH-limit of a sequence as limits of sequences in the original space.

Let $A_\bullet$ be a sequence of subsets of X. Define the **outer PK-limit** of $A_\bullet$ by

$$\text{PK-limsup}\, A_\bullet = \{x_\infty \in X | \exists A'_\bullet \leq A_\bullet, \exists x'_\bullet \in A'_\bullet : x'_\bullet \to x_\infty\}.$$

Here the expression $x'_\bullet \in A'_\bullet$ is short for

$$\forall n \in \mathbb{Z}_{\geq 1} : x'_n \in A'_n.$$

Similarly one defines the **inner PK-limit** of $A_\bullet$ by

$$\text{PK-liminf}\, A_\bullet = \{x_\infty \in X | A_\bullet, \exists x_\bullet \in A_\bullet : x_\bullet \to x_\infty\}.$$

Any sequence is a subsequence of itself, thus we automatically have

$$\text{PK-liminf}\, A_\bullet \subseteq \text{PK-limsup}\, A_\bullet.$$

The sequence $A_\bullet$ is said to **PK-converge** if

$$\text{PK-liminf}\, A_\bullet = \text{PK-limsup}\, A_\bullet,$$

in which case this common set is defined as the **PK-limit** of $A_\bullet$, and one writes

$$\text{PK-lim}\, A_\bullet = \text{PK-liminf}\, A_\bullet = \text{PK-limsup}\, A_\bullet.$$

Here is a nice result that gives PK-convergence through the Extension Lemma:

Proposition 1: Let X be a metric space. Then any Cauchy sequence

$A_\bullet$ in $\mathcal{H}(X)$ PK-converges.

┘

Before giving the proof, first an exercise that will come in handy:

Exercise 69: Let X be a metric space, $x_\bullet$ be a sequence. If

- $x_\bullet$ is Cauchy and
- $\exists x'_\bullet \leq x_\bullet$, $\exists x_\infty \in X$: $x_\bullet \to x_\infty$,

then $x_\bullet \to x_\infty$.

┘

Proof (of Prop. 1)**:** It suffices to show that the outer PK-limit of $A_\bullet$ is included in the inner PK-limit. Let $x_\infty \in \text{PK-limsup}\, A_\bullet$. Then by definition of the outer PK-limit, $\exists A'_\bullet \leq A_\bullet$, $\exists x'_\bullet \in A'_\bullet$ such that $x'_\bullet \to x_\infty$. $x'_\bullet$ is convergent, hence Cauchy, whence by the Extension Lemma[17] there is a supersequence $x_\bullet \geq x'_\bullet$ that is Cauchy such that $x_\bullet \in A_\bullet$. By Exr. 69, $x_\bullet \to x_\infty$ too; whence by the definition of inner PK-limit we have $x_\infty \in \text{PK-liminf}\, A_\bullet$.

┘

Note that this proposition is different from the "inheritance of completeness" theorem[18]: while the inheritance of completeness theorem requires that X be complete, in the above proposition we don't have this assumption, and the conclusion is convergence, albeit in a sense different than Pompeiu-Hausdorff.

Exercise 70: Let $A_\bullet$ be a sequence of subsets of X. Then the following subsets of X are equal:

(i) $\text{PK-limsup}\, A_\bullet = \{x_\infty \in X | \exists A'_\bullet \leq A_\bullet, \exists x'_\bullet \in A'_\bullet : x'_\bullet \to x_\infty\}$.

[17] [Bar12, p.34, Lem.7.2]
[18] [Bar12, p.35, Thm.7.1]

(ii) $\{x \in X | \forall \varepsilon \in \mathbb{R}_{>0} : x \in [A_n| < \varepsilon] \text{ FIM } n \in \mathbb{Z}_{\geq 1}\}$, where "FIM" is an abbreviation for "for infinitely many".

(iii) $\{x \in X | \liminf_{n\to\infty} d(x, A_n) = 0\}$. Note that the limit inferior inside the set is the limit inferior of a sequence of nonnegative real numbers.

(iv) $\bigcap_{N\in\mathbb{Z}_{\geq 1}} \overline{\bigcup_{n\in\mathbb{Z}_{\geq N}} A_n}$.

⌟

Exercise 71: Let $A_\bullet$ be a sequence of subsets of X. Then the following subsets of X are equal:

(i) $\text{PK-liminf}\, A_\bullet = \{x_\infty \in X | \exists x_\bullet \in A_\bullet : x_\bullet \to x_\infty\}$.

(ii) $\{x \in X | \forall \varepsilon \in \mathbb{R}_{>0} : x \in [A_n| < \varepsilon] \text{ FABFM } n \in \mathbb{Z}_{\geq 1}\}$, where "FABFM" is an abbreviation for "for all but finitely many".

(iii) $\{x \in X | \limsup_{n\to\infty} d(x, A_n) = 0\}$.

(iv) $\bigcap_{\substack{\iota:\mathbb{Z}_{\geq 1}\to\mathbb{Z}_{\geq 1} \\ \text{increasing}}} \overline{\bigcup_{n\in\mathbb{Z}_{\geq 1}} A_{\iota(n)}}$.

(v) $\bigcap_{k\in\mathbb{Z}_{\geq 1}} \bigcup_{N\in\mathbb{Z}_{\geq 1}} \bigcap_{n\in\mathbb{Z}_{\geq N}} [A_n| < 1/k]$.

(vi) $\bigcap_{k\in\mathbb{Z}_{\geq 1}} \bigcup_{N\in\mathbb{Z}_{\geq 1}} \bigcap_{n\in\mathbb{Z}_{\geq N}} [A_n| \leq 1/k]$.

⌟

Exercise 72: Let X be a metric space, and $A_\bullet$ be a sequence of subsets of X. Then both the outer and inner PK-limits of $A_\bullet$ are closed subsets of X.

⌟

Here a comment is in order. After Exr. 59, we have the following factorization for PH-convergence:

$$\begin{aligned}
&\lim_{n\to\infty} d_{\mathcal{H}(X)}(A_n, A_\infty) = 0 \\
&\Leftrightarrow \forall k \in \mathbb{Z}_{\geq 1}, \exists N \in \mathbb{Z}_{\geq 0} : \bigcup_{n\in\mathbb{Z}_{\geq N}} A_n \subseteq [A_\infty| \leq 1/k] \\
&\text{and } \forall k \in \mathbb{Z}_{\geq 1}, \exists N \in \mathbb{Z}_{\geq 0} : A_\infty \subseteq \bigcap_{n\in\mathbb{Z}_{\geq N}} [A_n| \leq 1/k].
\end{aligned}$$

The second statement seems similar to the last characterization of the inner PK-limit in Exr. 71, but it is in fact syntactically *stronger*: indeed, in the statement coming from PH-convergence we need one N that works for any point in A_∞. However, we have the following result for inner PK-limits of Cauchy sequences:

Proposition 2: Let X be a metric space, $A_\bullet$ be a sequence in $\mathcal{H}(X)$. If $A_\bullet$ is Cauchy, then

$$\forall k \in \mathbb{Z}_{\geq 1}, \exists N \in \mathbb{Z}_{\geq 0} : \text{PK-liminf}\, A_\bullet \subseteq \bigcap_{n\in\mathbb{Z}_{\geq N}} [A_n| \leq 1/k].$$

⌟

Proof (of Prop. 2)**:** Let $k \in \mathbb{Z}_{\geq 1}$. Since $A_\bullet$ is Cauchy, there is some $N \in \mathbb{Z}_{\geq 0}$ such that for any $n \in \mathbb{Z}_{\geq N}$ and any $p \in \mathbb{Z}_{\geq 0}$ one has

$$d_{\mathcal{H}(X)}(A_n, A_{n+p}) \leq 1/k.$$

Note that by Exr. 59 this also implies

$$A_{n+p} \subseteq [A_n| \leq 1/k].$$

Fix such an N; this will be the N in the statement we are trying to prove. Let $n \geq N$. We claim that $\text{PK-liminf}\, A_\bullet \subseteq [A_n| \leq 1/k]$. Let

$x \in \text{PK-liminf}\, A_\bullet$. Then by the definition of inner PK-limits, $\exists x_\bullet \in A_\bullet$ such that $x_\bullet \to x$. Then $\exists n' \in \mathbb{Z}_{\geq n}$ such that for any $m \in \mathbb{Z}_{\geq n'}$ one has $d(x_m, x) \leq 1$. But then any such m is larger than N, so that

$$x_m \in A_m \subseteq [A_n| \leq 1/k].$$

Put $x'_m = x_{n'+m}$. Then $x'_\bullet \leq x_\bullet$, $x'_\bullet \to x$ and $x'_\bullet \in [A_n| \leq 1/k]$. Since this latter set is closed by Exr. 37, $x \in [A_n| \leq 1/k]$.

⌟

Corollary 1 (of Prop. 2)**:** Let X be a metric space, $A_\bullet$ be a sequence in $\mathcal{H}(X)$. If $A_\bullet$ is Cauchy, then $\text{PK-liminf}\, A_\bullet$ is totally bounded.

⌟

Proof: Say $\text{PK-liminf}\, A_\bullet$ is not totally bounded. Then there is an $\varepsilon_0 \in \mathbb{R}_{>0}$ and a sequence $x_\bullet$ in $\text{PK-liminf}\, A_\bullet$ such that

$$n \neq m \implies d(x_n, d_m) \geq \varepsilon_0.$$

By Prop. 2, there is an $N \in \mathbb{Z}_{\geq 0}$ such that

$$A_\infty \subseteq [A_N| \leq \varepsilon_0/3].$$

Thus there is a sequence $y_\bullet \in A_N$ such that for any $n \in \mathbb{Z}_{\geq 0}$ one has $d(x_n, y_n) \leq \varepsilon_0/3$. A_N is compact, thus there is a convergent subsequence $y'_\bullet \leq y_\bullet$. Say ι is the associated sampling function, so that $y'_n = y_{\iota(n)}$. Then $x'_n = x_{\iota(n)}$ defines a subsequence of $x_\bullet$, and

$$d(x'_n, x'_{n+p}) \leq d(x'_n, y'_n) + d(y'_n, y'_{n+p}) + d(y'_{n+p}, x'_{n+p}).$$

As $y'_\bullet$ is convergent, the RHS is bounded by ε_0 for large enough n, a contradiction.

⌟

Note that while the outer and inner limits of sequences of sets are well-defined subsets of X, they may fail to be elements of the hyperspace $\mathcal{H}(X)$, even when $A_\bullet$ is a sequence in $\mathcal{H}(X)$. However we have the following:

Exercise 73 (Monotone Convergence)**:** Let X be a metric space, $A_\bullet$ be a sequence of subsets of X. Then

(i) If $A_\bullet$ is increasing, that is, $\forall n : A_n \subseteq A_{n+1}$, then it PK-converges and

$$\text{PK-lim}\, A_\bullet = \overline{\bigcup_{n \in \mathbb{Z}_{\geq 1}} A_n}.$$

(ii) If $A_\bullet$ is decreasing, that is, $\forall n : A_n \supseteq A_{n+1}$, then it PK-converges and

$$\text{PK-lim}\, A_\bullet = \bigcap_{n \in \mathbb{Z}_{\geq 1}} \overline{A_n}.$$

┘

Exercise 74: Give an example of a sequence of subsets of a metric space X that PK-converges to some subset, but the PK-limit subset is not in $\mathcal{H}(X)$.

┘

Exercise 75: Let X be a metric space, and let $A_\bullet$ be a sequence in $\mathcal{H}(X)$ that PK-converges to some subset A_∞ of X. If $A_\infty \in \mathcal{H}(X)$, then does it follow that A_∞ is also the PH-limit of $A_\bullet$?

┘

We shall not discuss further details of PK-convergence, see [RW98] or [Bee93] for further details.

Exercise 76: Let X be a metric space, $A_\bullet$ be a sequence of subsets of X, and $A_\infty \subseteq X$ be a closed subset. Then[19]

$$\begin{aligned} &\text{PK-lim}\, A_\bullet = A_\infty \\ \iff &\forall x \in X : \lim_{n\to\infty} |d(x, A_n) - d(x, A_\infty)| = 0. \end{aligned}$$

⌟

[19]Note that one side of this equivalence merely states that the associated Lipschitz functions of subsets converge pointwise to the associated Lipschitz function of the candidate limit set. Indeed this is yet another, weaker, way of talking about convergence of sets; in this setting this is referred to as convergence in the sense of Wijsman (see also [Wij64, Wij66], [Bee94]).

9 Baire Category Theory

In this section we give a brief outline of Baire Category Theory, which classifies subsets of a metric space to belong to the "first category" or "second category"[20]. "First category" (aka meager) subsets are negligible subsets from the point of view of topology, and "second category" (aka nonmeager) subsets are not topologically negligible.

Let X be a metric space. X is said to be **Baire** if for any countable family $F_\bullet$ of closed subsets of X, one has the following property:

$$\bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathrm{int}(F_n) = \emptyset \implies \mathrm{int}\left(\bigcup_{n \in \mathbb{Z}_{\geq 0}} F_n\right) = \emptyset.$$

In words, X is Baire if union of any countable family of closed subsets with empty interiors has empty interior. Recall that the interior of a subset is the largest open set contained in it; thus a subset has empty interior if the only open subset it contains is the empty set.

Exercise 77: Let X be a metric space, $A \subseteq X$. Then A has empty interior if and only if its complement $X \setminus A$ is dense.

⌟

Exercise 78: Let X be a metric space. Then X is Baire iff for any countable family $U_\bullet$ of open dense subsets, $\bigcap_{n \in \mathbb{Z}_{\geq 0}} U_n$ is dense.

⌟

Before going into further details, here is the cornerstone of the Baire Category Theory:

Theorem 1 (Baire Category)**:** Any complete metric space is a Baire space.

⌟

[20] This is a different sense of "category" than the sense we discussed in class.

Proof: Let $F_\bullet$ be a sequence of closed subsets of X such that $\bigcup_{n\in\mathbb{Z}_{\geq 1}} \mathrm{int}(F_n) = \emptyset$. We claim that the union $\bigcup_{n\in\mathbb{Z}_{\geq 1}} F_n$ too has empty interior. Let U be a nonempty open subset of X. We need to show that

$$U \setminus \left(\bigcup_{n\in\mathbb{Z}_{\geq 1}} F_n \right) \neq \emptyset.$$

First consider F_1. We know by hypothesis that F_1 is closed and has empty interior. Thus $U \setminus F_1$ is a nonempty open subset. Thus

$$\exists x_1 \in U \setminus F_1, \exists r_1 \in \mathbb{R}_{>0} : [x_1| \leq r_1] \subseteq U \setminus F_1.$$

Next, $[x_1| < r_1]$ is a nonempty open subset, and since F_2 is closed and has empty interior, $[x_1| \leq r_1] \setminus F_2$ is a nonempty open subset, so that

$$\begin{aligned} &\exists x_2 \in [x_1| < r_1] \setminus F_2, \exists r_2 \in \mathbb{R}_{>0} : \\ &r_2 < \min\{r_1, 1/2\} \text{ and } [x_2| \leq r_2] \subseteq [x_1| < r_1] \setminus F_2. \end{aligned}$$

Inductively arguing, one obtains a sequence of closed balls:

$$\forall k \in \mathbb{Z}_{\geq 1} : r_k < \min\{r_{k-1}, 1/k\} \text{ and } [x_k| \leq r_k] \subseteq [x_{k-1}| < r_{k-1}] \setminus F_k.$$

Note that

$$d(x_k, x_{k+q}) \leq r_k < 1/k,$$

whence $x_\bullet$ is Cauchy, so that by the completeness of X, there is a unique point $\lim_{n\to\infty} x_n = x_\infty$ such that

$$\{x_\infty\} = \bigcap_{n\in\mathbb{Z}_{\geq 1}} [x_k| \leq r_k] \subseteq U \setminus \left(\bigcup_{n\in\mathbb{Z}_{\geq 1}} F_n \right).$$

⌟

Exercise 79: Give an example of a metric space that is Baire but not complete.

⌟

Exercise 80: Let X be a Baire metric space, $A \subseteq X$. Then

(i) If A is open in X, then A is Baire.

(ii) If A is G-delta, then A is Baire.

⌟

Recall that arbitrary unions of open subsets is open and dually arbitrary intersections of closed subsets are closed. However, the opposite operations may not preserve these classes of subsets, even for countable families of subsets. For instance the intersection of countably many open sets need not be open:

Exercise 81: Let $X = \mathbb{R}$.

(i) Find a countable family of open subsets of X whose intersection is not open.

(ii) Find a countable family of closed subsets of X whose union is not closed.

⌟

Still, knowing that a subset can be written as the union of countably many closed subsets does say something about the subset[21]. Because of this we introduce the following hierarchy of subsets:

[21] Indeed, that it is Borel measurable, for instance.

Definition 2: Let X be a metric space and A $\subseteq$ X. Then

(i) A is **F-sigma** (F_σ for short) if it can be written as the union of countably many closed subsets of X.

(ii) A is **G-delta** (G_δ for short) if it can be written as the intersection of countably many open subsets of X.

(iii) A is **F-sigma-delta** ($F_{\sigma\delta}$ for short) if it can be written as the intersection of countably many F-sigma subsets of X.

(iv) A is **G-delta-sigma** ($F_{\sigma\delta}$ for short) if it can be written as the union of countably many G-delta subsets of X.

⌟

Exercise 82: Give the definition of a $F_{\sigma\delta\sigma\delta\sigma\delta\sigma\delta}$ subset of a metric space.

⌟

Definition 3: Let X be a metric space and A $\subseteq$ X. Then

(i) A is **meager** (or **first category**) if it is contained in the union of countably many closed subsets with empty interior.

(ii) A is **nonmeager** (or **second category**) if it is not meager.

(iii) A is **residual** (or **comeager**) if its complement X \ A is meager.

⌟

Whether or not a metric space is Baire is a topological property, in the following sense:

Corollary 2: If (X, d) is a complete metric space, then for any metric d' on X that is topologically equivalent to d, (X, d') is Baire.

⌟

Proof: Let d' be a metric on X that is topologically equivalent to d. Recall that this means by definition that $\mathrm{id}_X : (X, d) \to (X, d')$ is a homeomorphism. Consequently, a subset F of X is closed relative to d iff it is closed relative to d', and similarly a subset U of X is open relative to d iff it is open relative to d'. In particular, the notion of interiors is preserved if one changes metrics. The defining property for a metric space is stated only with closed subsets and interiors, consequently if the statement holds for d, it must hold for d' also.

⌟

Exercise 83: Let $X = \mathbb{R}$. Find two topologically equivalent metrics d_1, d_2 on X such that (X, d_1) is complete but (X, d_2) is not.

⌟

Exercise 84: Consider $X = \mathbb{Q}$ to be the set of rational numbers. Then there is no metric on X topologically equivalent to the Euclidean metric relative to which X is complete.

⌟

10 Notation for Dynamically Defined Sets

In this section we discuss certain dynamically defined sets. Many of these definitions have analogs for more general monoid or group actions (or even equivalence relations), but to keep technicalities and complications to a minimum we focus on the case of dynamics with stroboscopic time, that is to say, iterates of one transformation. Let X be a set and $f : X \to X$ be a function. Recall that when considering f as a dynamical system, one considers really the monoid action

$$f^{\bullet} : \mathbb{Z}_{\geq 0} \to F(X; X), n \mapsto f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ many}}$$

in case f is not a bijection and the group action

$$f^{\bullet} : \mathbb{Z} \to \mathrm{Bij}(X), n \mapsto f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ many}}$$

in case f is a bijection. For brevity let's let T stand for $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}$, with the understanding that when f is not invertible only "future" makes sense and when f is invertible both "future" and "past" make sense. Sometimes these two cases simultaneously are called a **cascade** (see e.g. [Ano69, p. 1]). Thus X models "space", T models "time", and $f^{\bullet}$ models the "time evolution". Many of the dynamically defined sets are subsets of the "space" object, although there are also dynamically defined sets that are subsets of the "time" object.

A subset $A \subseteq X$ is called **invariant** if $f^{-1}(A) = A$. Recall that with the notation f^{-1} we don't mean that f is a bijection necessarily; rather the expression on the LHS is the preimage of the set A under f. Saying that A is an invariant subset under f means that any point $a \in A$ returns to A in one iteration ($f(a) \in A$), and simultaneously, any point $a' \in A$ must have been in A in the previous timestep ($\exists a'' \in A : f(a'') = a'$).

There are other notions of invariance for subsets, with names such as "forward subinvariant" etc., but the definition we give here is the most generally applicable and robust one.

For $x \in X$, the **orbit** (or **trajectory**) of x under f is

$$\mathcal{O}_x(f) = \{f^n(x) | n \in T\}.$$

In this case we also call x an **initial condition** (or **seed**). If we are interested in a certain time segment $T' \subseteq T$ of an orbit we write

$$\mathcal{O}_x(f; T') = \{f^n(x) | n \in T'\}.$$

Exercise 85: Let X be a set and $f : X \to X$ be a function. Define a relation on X as follows:

$$x \sim_f y \Leftrightarrow \mathcal{O}_x(f) \cap \mathcal{O}_y(f) \neq \emptyset.$$

Then $\sim_f$ is an equivalence relation. The associated partition $\mathcal{O}(f)$ of X is called the **orbit partition** of f.

Further we have that f is a bijection if and only if

$$x \sim_f y \Leftrightarrow \mathcal{O}_x(f) = \mathcal{O}_y(f).$$

┘

For two subsets $A, B \subseteq X$, the **hitting time** from A to B is by definition the following subset of the time object:

$$\mathrm{Hit}^f(B \leftarrow A) = \{n \in T | f^n(A) \cap B \neq \emptyset\}.$$

Thus $n \in \mathrm{Hit}^f(B \leftarrow A)$ if there is some point $a \in A$ whose trajectory hits B in exactly n timesteps. If A or B is a singleton, we often drop the set braces, so for instance $\mathrm{Hit}^f(B \leftarrow x) = \mathrm{Hit}^f(B \leftarrow \{x\})$.

We say that x is a **fixed point** of f if $f(x) = x$, and we denote by

Fix(f) the set of all fixed points of f.

Exercise 86: Let X be a set, $f : X \to X$ be a function, and $x \in X$. Then the following are equivalent:

(i) $x \in \mathrm{Fix}(f)$

(ii) $\mathcal{O}_x(f) = \{x\}$

(iii) $1 \in \mathrm{Hit}^f(x \leftarrow x)$

(iv) $\mathrm{Hit}^f(x \leftarrow x) = \mathrm{T}$

⌟

x is a **periodic point** of f if $\exists p \in \mathbb{Z}_{\geq 1} : f^p(x) = x$. The set of all periodic points of f is denoted by Per(f). If $x \in \mathrm{Per}(f)$, then there is a minimum $p \in \mathbb{Z}_{\geq 1}$ such that $f^p(x) = x$; that minimum time is called the **minimal period** (or **prime period**) of x under p. The set of all periodic points of f of minimal period p is denoted by $\mathrm{Per}_p(f)$.

Exercise 87: Let X be a set and $f : X \to X$ be a function. Then

$$\mathrm{Per}(f) = \bigcup_{n \in \mathbb{Z}_{\geq 1}} \mathrm{Fix}(f^n) = \biguplus_{p \in \mathbb{Z}_{\geq 1}} \mathrm{Per}_p(f).$$

Here the latter union is a union of disjoint sets.

⌟

x is an **eventually periodic point** of f if $\mathcal{O}_x(f) \cap \mathrm{Per}(f) \neq \emptyset$.

Exercise 88: Let X be a set and $f : X \to X$ be a function. Then any periodic point is an eventually periodic point. If f is a bijection, any eventually periodic point is a periodic point.

⌟

Fixed and periodic points of a dynamical system signify the simplest notions of recurrence. It is however more reasonable to expect

https://alpuzman.github.io/
uzman@math.utah.edu

not recurrence on the nose but approximate recurrence, for this we need to assume that the space object X has some structure that allows one to talk about approximations. Let us now assume that X is a metric space. Note that we still don't assume f to be continuous[22].

For $x \in X$, the **omega limit set** of x under f is defined by

$$\omega_x(f) = \bigcap_{N \in \mathbb{Z}_{\geq 0}} \overline{\{f^n(x) | n \in \mathbb{Z}_{\geq N}\}}.$$

Exercise 89: Let X be a metric space, $f : X \to X$ be a function, $x, y \in X$. Then the following are equivalent:

(i) $y \in \omega_x(f)$.

(ii) $\forall \varepsilon \in \mathbb{R}_{>0}, \forall N \in \mathbb{Z}_{\geq 0}, \exists n \in \mathbb{Z}_{\geq N} : d(f^n(x), y) < \varepsilon$.

(iii) $\exists n_\bullet \uparrow \infty : \lim_{k \to \infty} d(f^{n_k}(x), y) = 0$.

(iv) $\forall \varepsilon \in \mathbb{R}_{>0} : \# \mathrm{Hit}^f([y| < \varepsilon] \leftarrow x) = \infty$.

⌟

If f is a bijection, then the **alpha limit set** of x under f is the omega limit set of x under f^{-1}:

$$\alpha_x(f) = \omega_x\left(f^{-1}\right).$$

Exercise 90: Let X be a metric space, $f : X \to X$ be a function, $x \in X$. Then the omega limit set $\omega_x(f)$ of x is closed. If f is a bijection, the alpha limit set $\alpha_x(f)$ of x is also closed.

⌟

x is a **recurrent point** of f if $x \in \omega_x(f)$; Rec(f) denotes the set of all recurrent points of f.

[22]While it is natural to assume from the get go that the dynamics is continuous from the point of view of topological dynamics, for general dynamical purposes one is often interested in dynamics much rougher than continuous.

Exercise 91: Let X be a metric space, $f : X \to X$ be a function. Then $\mathrm{Per}(f) \subseteq \mathrm{Rec}(f)$.

⌟

x is a **nonwandering point** of f if $\forall \varepsilon \in \mathbb{R}_{>0} : \mathrm{Hit}^f([x| < \varepsilon] \leftarrow [x| < \varepsilon]) \neq \emptyset$; NW(f) denotes the set of all nonwandering points of f. If x is not nonwandering, it's called **wandering**.

Exercise 92: Let X be a metric space, $f : X \to X$ be a function, $x \in X$. Then the following are equivalent:

(i) $x \in \mathrm{NW}(f)$.

(ii) $\forall \varepsilon \in \mathbb{R}_{>0} : \# \mathrm{Hit}^f([x| < \varepsilon] \leftarrow [x| < \varepsilon]) = \infty$.

(iii) $\forall \varepsilon \in \mathbb{R}_{>0}, \exists n \in \mathbb{Z}_{\geq 1} : f^{-n}([x| < \varepsilon]) \cap [x| < \varepsilon] \neq \emptyset$.

⌟

Exercise 93: Let X be a metric space, $f : X \to X$ be a function, $x \in X$. Then the following are equivalent:

(i) $x \in X \setminus \mathrm{NW}(f)$.

(ii) $\exists \varepsilon \in \mathbb{R}_{>0} : \mathrm{Hit}^f([x| < \varepsilon] \leftarrow [x| < \varepsilon]) = \emptyset$.

(iii) $\exists \varepsilon \in \mathbb{R}_{>0} : \# \mathrm{Hit}^f([x| < \varepsilon] \leftarrow [x| < \varepsilon]) < \infty$.

(iv) $\exists \varepsilon \in \mathbb{R}_{>0}, \forall n \in \mathbb{Z}_{\geq 0}, \forall p \in \mathbb{Z}_{\geq 1} : f^{-(n+p)}([x| < \varepsilon]) \cap f^{-n}([x| < \varepsilon]) = \emptyset$.

⌟

Exercise 94: Let X be a metric space, $f : X \to X$ be a function. Then the set NW(f) of nonwandering points of f is closed.

⌟

Exercise 95: Let X be a metric space, $f : X \to X$ be a function. Then

(i) $\overline{\bigcup_{x \in X} \omega_x(f)} \subseteq NW(f)$.

(ii) $\overline{Rec(f)} \subseteq NW(f)$.

(iii) If f is a bijection, one also has $\overline{\bigcup_{x \in X} \alpha_x(f)} \subseteq NW(f)$.

⌟

Next we discuss the notion of chain recurrence, which is a form of recurrence that is weaker than being nonwandering. Let $x, y \in X$. Then for $\varepsilon \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{\geq 1}$, an ε**-pseudo-orbit of length** n from x to y under f is by definition a function $x_\bullet : \{0, 1, ..., n\} \to X$ such that

$$\begin{aligned} x &= x_0, \\ d(f(x_0), x_1) &< \varepsilon, \\ d(f(x_1), x_2) &< \varepsilon, \\ &\vdots \\ d(f(x_{n-1}), x_n) &< \varepsilon, \\ x_n &= y \end{aligned}$$

Exercise 96: Let X be a metric space, $f : X \to X$ be a function, $x \in X$. Then for any $p \in \mathbb{Z}_{\geq 0}$ and for any $\varepsilon \in \mathbb{R}_{>0}$, $(x, f(x), f^2(x), ..., f^p(x))$ is an ε-pseudo-orbit from x to $f^p(x)$.

⌟

x is called a **chain recurrent point** of f if $\forall \varepsilon \in \mathbb{R}_{>0}$, there is an ε-pseudo-orbit from x to itself under f. cRec(f) denotes the set of chain recurrent points of f.

Exercise 97: Let X be a metric space, $f : X \to X$ be a function. Then $NW(f) \subseteq cRec(f)$.

⌟

Exercise 98: Let X be a metric space, $f : X \to X$ be a function. If f is continuous, then cRec(f) is closed.

┘

Exercise 99: For each of the following conditions, give an example of a homeomorphism $f : X \to X$ of a compact metric space X satisfying the condition.

(i) $\mathrm{Fix}(f) \neq \mathrm{Per}(f)$.

(ii) $\mathrm{Per}(f) \neq \mathrm{Rec}(f)$.

(iii) $\mathrm{Per}(f) \neq \overline{\mathrm{Per}(f)}$.

(iv) $\mathrm{Rec}(f) \neq \overline{\mathrm{Rec}(f)}$.

(v) $\overline{\mathrm{Per}(f)} \neq \mathrm{NW}(f)$.

(vi) $\overline{\left(\bigcup_{x\in X} \alpha_x(f)\right) \cup \left(\bigcup_{x\in X} \omega_x(f)\right)} \neq \mathrm{NW}(f)$.

(vii) $\overline{\mathrm{Rec}(f)} \neq \mathrm{NW}(f)$.

(viii) $\mathrm{NW}(f) \neq \mathrm{cRec}(f)$.

┘

The **global stable set** of x under f is

$$\mathcal{S}_x(f) = \{y \in X | \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\}.$$

For $\varepsilon \in \mathbb{R}_{>0}$, the **local stable set** of x of size ε under f is

$$\mathcal{S}_{x,\varepsilon}(f) = \bigcap_{n\in\mathbb{Z}_{\geq 0}} \{y \in X | d(f^n(y), f^n(x)) \leq \varepsilon.\}$$

Exercise 100: Let X be a metric space, $f : X \to X$ be a function. Then

$$\forall x \in X : x \in \mathcal{S}_x(f) \cap \left(\bigcap_{\varepsilon \in \mathbb{R}_{>0}} \mathcal{S}_{x,\varepsilon}(f) \right).$$

┘

Exercise 101: Let X be a metric space, $f : X \to X$ be a function, $x, y \in X$. Then

$$\mathcal{S}_x(f) \cap \mathcal{S}_y(f) \neq \emptyset \Leftrightarrow \mathcal{S}_x(f) = \mathcal{S}_y(f).$$

┘

After Exr. 100 and Exr. 101, we have that the global stable sets of f partition X, in other words, X can be written as a union of global stable sets of f. The partition $\mathcal{S}(f)$ of X into the global stable sets of f is called the **stable partition** of f.

Exercise 102: Let X be a metric space, $f : X \to X$ be a function, $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$. If f is continuous, then $\mathcal{S}_{x,\varepsilon}(f)$ is closed.

┘

If f is a bijection, the **global unstable set** of x under f is the global stable set of x under f^{-1}, and similarly for the **local unstable set** of x of size ε:

$$\mathcal{U}_x(f) = \mathcal{S}_x(f^{-1}), \quad \mathcal{U}_{x,\varepsilon}(f) = \mathcal{S}_{x,\varepsilon}(f^{-1}).$$

11 Banach Contraction Principle

In this section we discuss the Banach Contraction Principle; which guarantees that contractions on (nonempty) complete metric spaces have unique fixed points:

Theorem 2 (Banach Contraction Principle)**:** Let X be a complete metric space and $f : X \to X$ be a function. If $\mathrm{Lip}(f) < 1$, then there is a unique $x_* \in X$ such that

$$f(x_*) = x_*.$$

⌟

Proof: Let us make the abbreviation $L = \mathrm{Lip}(f)$. We first show the "uniqueness" part, that is $\#\,\mathrm{Fix}(f) \leq 1$. Let $x_*, x_\dagger \in \mathrm{Fix}(f)$ be such that $x_* \neq x_\dagger$. Then

$$0 < d(x_*, x_\dagger) = d(f(x_*), f(x_\dagger)) \leq L d(x_*, x_\dagger) < d(x_*, x_\dagger).$$

No positive number is strictly greater than itself, hence this is a contraction; thus $x_* = x_\dagger$.

Next we show the "existence" part, that is $\#\,\mathrm{Fix}(f) \geq 1$. Let $x \in X$ and consider the sequence $x_\bullet$ defined by $x_n = f^n(x)$; thus $x_\bullet$ is the sequence of iterates of x under f. We claim that $x_\bullet$ is Cauchy. Indeed,

$$\begin{aligned} d(f^n(x), f^{n+p}(x)) &\leq \sum_{i=0}^{p-1} d(f^{n+i}(x), f^{n+i+1}(x)) \\ &\leq \sum_{i=0}^{p-1} L^{n+i} d(x, f(x)) \\ &= L^n \frac{1-L^p}{1-L} d(x, f(x)) \\ &< \frac{L^n}{1-L} d(x, f(x)). \end{aligned}$$

By completeness, $x_\bullet$ must converge to a point, say x_∞. Finally one has

$$\begin{aligned} d(x_\infty, f(x_\infty)) &\leq d(x_\infty, f^{n+1}(x)) + d(f^{n+1}(x), f(x_\infty)) \\ &\leq d(x_\infty, f^{n+1}(x)) + Ld(f^n(x), x_\infty) \end{aligned}$$

Note that defining $x'_n = f^{n+1}(x)$ gives a subsequence $x'_\bullet \leq x_\bullet$, hence $x'_\bullet \to x_\infty$, whence one has $x_\infty = f(x_\infty)$.

┘

With minor modifications to the argument in the proof of the contraction principle, one obtains certain estimates that are useful in practice:

Exercise 103: Let X be a complete metric space, $f : X \to X$ be a contraction with x_* the unique fixed point. Then one has the following estimates:

(i) $\forall x, y \in X, \forall n \in \mathbb{Z}_{\geq 0} : d(f^n(x), f^n(y)) \leq \mathrm{Lip}(f)^n d(x, y)$.

(ii) $\forall x \in X, \forall n \in \mathbb{Z}_{\geq 0} : d(f^n(x), x_*) \leq \dfrac{\mathrm{Lip}(f)^n}{1 - \mathrm{Lip}(f)} d(x, f(x))$. (The case n = 0 is called the **collage estimate**, due to its use in the context of the Barnsley Collage Theorem for B-IFS's.)

(iii) $\forall x \in X, \forall n \in \mathbb{Z}_{\geq 0} : d(f^{n+1}(x), x_*) \leq \frac{\mathrm{Lip}(f)}{1 - \mathrm{Lip}(f)} d(f^n(x), f^{n+1}(x))$.

(iv) $\forall x \in X, \forall n \in \mathbb{Z}_{\geq 0} : d(f^{n+1}(x), x_*) \leq \mathrm{Lip}(f) d(f^n(x), x_*)$.

⌟

Typically BCP is applied to complicated contractions acting on complicated spaces, and as it is rather general it is very useful in many fields. On the other hand from a dynamical point of view the conclusion is that a contraction does not have interesting dynamics. The next few exercises make this observation more rigorous:

Exercise 104: A function $f : X \to X$ on a metric space X is called **contractive** if

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \implies d(f(x_1), f(x_2)) < d(x_1, x_2),$$

(i) All contractions are contractive and all contractive functions are Lipschitz.

(ii) Give an example of a contractive function that is not a contraction.

(iii) Give an example of a Lipschitz function that is not contractive.

(iv) Any contractive function has at most one fixed point.

(v) Any contractive function on a compact metric space has a unique fixed point.

(vi) Give an example of a contractive function on a complete metric space that has no fixed points.

⌟

Exercise 105: Let X be a metric space and $f : X \to X$ be a function. If f is contractive, then

$$\mathrm{Fix}(f) = \mathrm{Per}(f) = \bigcup_{x \in X} \omega_x(f) = \mathrm{Rec}(f) = \mathrm{NW}(f) = \mathrm{cRec}(f).$$

In particular, if X is complete and f is a contraction with x_* the unique fixed point, then all of these dynamically defined sets are equal to the singleton $\{x_*\}$.

⌟

Exercise 106: Let X be a complete metric space, $f : X \to X$ be a contraction with unique fixed point x_*. Then for any $x \in X$ one has

$$\mathcal{S}_x(f) = \mathcal{S}_{x_*}(f) = X.$$

⌟

We mentioned above that BCP is often applied to a complicated contraction acting on a complicated space. Often even establishing the contraction property globally turns out to be hard and even unnecessary; as there are "local" versions of BCP:

Exercise 107: Let X be a complete metric space, $x_0 \in X$, $r \in \mathbb{R}_{>0}$, $f : [x_0| \leq r] \to X$ be a function. If f is a contraction and

$$\mathrm{Lip}(f) \leq 1 - \frac{d(x_0, f(x_0))}{r},$$

then f maps $[x_0| \leq r]$ into itself and in it it has a unique fixed point.

⌟

Exercise 108: Let X be a complete metric space, $x_0 \in X$, $r \in \mathbb{R}_{>0}$, $f : [x_0| < r] \to X$ be a function. If f is a contraction and

$$\mathrm{Lip}(f) < 1 - \frac{d(x_0, f(x_0))}{r},$$

then f maps $[x_0| < r]$ into itself and in it it has a unique fixed point.

⌟

Dependence on a Parameter

Let X and S be two sets. A **family** of functions of X is by definition a function

$$f_\bullet : S \to F(X; X).$$

Thus for each $s \in S$, we have that $f_s : X \to X$ is a dynamical system on its own. One can think of a family of functions of X as a single, bivariate function also, by way of unCurrying:

$$f : S \times X \to X, f(s, x) = f_s(x).$$

BCP associates to each contraction on a complete metric space a unique fixed point. The following proposition shows that in fact this association is a continuous function.

Proposition 3: Let X be a complete metric space and let $f_\bullet : S \to F(X; X)$ be a family of functions parameterized by a metric space S. If

(i) $\sup_{s \in S} \mathrm{Lip}(f_s) < 1$, and

(ii) $\forall x \in X : f_\bullet(x) : S \to X, s \mapsto f_s(x)$ is continuous,

then there is a unique continuous function $\phi : S \to X$ such that

$$\forall s \in S : \mathrm{Fix}(f_s) = \{\phi(s)\}.$$

⌟

Proof: Uniqueness of the function ϕ is immediate by the BCP. Thus it suffices to show that $\phi : S \to X$ is indeed continuous. Fix $s_0 \in S$; we'll show that ϕ is continuous at s_0.

Let $s \in S$. Applying the collage estimate from Exr. 103 to the contraction f_s, we have

$$\begin{aligned} d_X(\phi(s), \phi(s_0)) &\leq \frac{1}{1 - \mathrm{Lip}(f_s)} d_X(\phi(s_0), f_s(\phi(s_0))) \\ &= \frac{1}{1 - \mathrm{Lip}(f_s)} d_X(f_{s_0}(\phi(s_0)), f_s(\phi(s_0))). \quad (\dagger) \end{aligned}$$

Let $\varepsilon \in \mathbb{R}_{>0}$. By hypothesis, $f_\bullet(\phi(s_0)) : S \to X$, $s \mapsto f_s(\phi(s_0))$ is continuous. Thus there is a $\delta \in \mathbb{R}_{>0}$ such that if $s \in S$ is with $d_S(s, s_0) < \delta$,

$$d_X(f_{s_0}(\phi(s_0)), f_s(\phi(s_0))) \leq (1 - L)\varepsilon,$$

where $L = \sup_{s \in S} \mathrm{Lip}(f_s)$. Then by ($\dagger$) one has

$$d_X(\phi(s), \phi(s_0)) < \frac{1 - L}{1 - \mathrm{Lip}(f_s)} \varepsilon \leq \varepsilon.$$

⌟

Exercise 109: Let X be a complete metric space and $s \in]0, 1[$. Consider the following set of functions:

$$\Lambda_s(X) = \{f \in F(X; X) | \mathrm{Lip}(f) \leq s\}.$$

Endow $\Lambda_s(X)$ with the metric

$$d_{C^0}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Then the function $\Phi : \Lambda_s(X) \to X$ defined by $\mathrm{Fix}(f) = \{\Phi(f)\}$ is continuous.

$\lrcorner$

References

[Ano69] D. V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder, American Mathematical Society, Providence, R.I., 1969. MR 0242194 4, 47

[Bar12] M.F. Barnsley, *Fractals everywhere*, third ed., Dover Books on Mathematics, Dover Publications, 2012. 36

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR 1835418 28

[Bee93] Gerald Beer, *Topologies on closed and closed convex sets*, Mathematics and its Applications, vol. 268, Kluwer Academic Publishers Group, Dordrecht, 1993. MR 1269778 40

[Bee94] ———, *Wijsman convergence: a survey*, Set-Valued Anal. **2** (1994), no. 1-2, 77–94, Set convergence in nonlinear analysis and optimization. MR 1285822 41

[Bow08] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, revised ed., Lecture Notes in Mathematics, vol. 470, Springer-Verlag, Berlin, 2008, With a preface by David Ruelle, Edited by Jean-René Chazottes. MR 2423393 12

[Cor99] Paul Corazza, *Introduction to metric-preserving functions*, Amer. Math. Monthly **106** (1999), no. 4, 309–323. MR 1682385 8

[DD16] Michel Marie Deza and Elena Deza, *Encyclopedia of distances*, fourth ed., Springer, Berlin, 2016. MR 3559482 13

[Dob98] Jozef Doboš, *Metric preserving functions*, Štroffek, Košice, 1998. 8

[Gro07] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, english ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR 2307192 7

[HK03] Boris Hasselblatt and Anatole Katok, *A first course in dynamics*, Cambridge University Press, New York, 2003, With a panorama of recent developments. MR 1995704 12

[KH95] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza. MR 1326374 13

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003. MR 2007488 29

[Rue78] David Ruelle, *Thermodynamic formalism*, Encyclopedia of Mathematics and its Applications, vol. 5, Addison-Wesley Publishing Co., Reading, Mass., 1978, The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota. MR 511655 12

[RW98] R. Tyrrell Rockafellar and Roger J.-B. Wets, *Variational analysis*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 317, Springer-Verlag, Berlin, 1998. MR 1491362 40

[Wij64] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions*, Bull. Amer. Math. Soc. **70** (1964), 186–188. MR 157278 41

[Wij66] ———, *Convergence of sequences of convex sets, cones and functions. II*, Trans. Amer. Math. Soc. **123** (1966), 32–45. MR 196599 41