University of Utah — Fall 2024

Syllabus for MATH 4800-001: Undergraduate Research in Mathematics: Fractal Geometry and Dynamics

Instructor: Alp Uzman

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Contents

1 Staff and Meeting Hours

Instructor: Alp Uzman (uzman@math.utah.edu)

Lectures: Tue Thu 12:25 PM - 1:45 PM @ LCB 323

Office Hours: By appointment

2 Course Description & Prerequisites

MATH 4800 is an introduction to research in mathematics. As outlined in the general catalog, it is worth 3 credits and has listed "Instructor Consent" as the only prerequisite.

The idea of an introduction to research course might seem paradoxical, as conducting mathematical research differs significantly from taking a mathematics course. The fundamental difference lies in the stance and attitude one must adopt. While a standard mathematics course is structured with well-defined tasks and a straightforward path, research is more open-ended and subjective.

MATH 4800 aims to blend these two approaches. While the course includes lectures and weekly graded assignments (courselike aspects), it also incorporates open-ended, student-driven projects, collective assignments, and a final written project and oral presentation instead of a final exam (researchlike aspects). All written submissions must be prepared in Latex.

This Semester's MATH 4800 Each semester, MATH 4800 focuses on a particular topic, as detailed in the MATH 4800 archive. This semester, we will investigate fractals and dynamics. Fractals are objects that appear similar at different scales, and dynamics is the mathematical study of time evolutions of closed systems. Often, a fractal is the "residue" of a dynamical system that has evolved over a long time. For example, the Barnsley fern is the attractor of a specific iterated function system, and the Mandelbrot set is determined by whether the trajectory of a seed runs off to infinity under a certain dynamical system. This deep connection makes studying these two subjects together fruitful. While there are many applications of fractal geometry and dynamics, our starting point will be from a pure mathematics perspective. Exploration of applications will be admissible for assignments.

Prerequisites Interested students should definitely have mastery over the standard calculus series (MATH 1210 and 1220 and 2210 or equivalent) as well as a first course in differential equations and/or linear algebra (MATH 2250 or 2270 or 2280 or equivalent). It is also preferable for students to have mastery over material covered in a single-variable advanced calculus course (MATH 3210 or equivalent) as we will be using many epsilon-delta arguments. Familiarity with topology and measure theory of metric spaces will be beneficial but not necessary, as we will develop what we need in the course.

3 Expectations

As a student, you are expected to:

- attend all lectures,
- complete weekly reports,
- complete a final report,
- give an in-class presentation of your final report.

According to the university regulations, you should expect a workload of approximately 6 to 9 hours per week outside of lecture hours. However, due to the open-ended nature of the course, your actual workload may vary depending on your level of engagement.

4 Learning Objectives

MATH 4800 has two types of learning objectives.

The first type of objectives relates to the course being an introduction to research. These objectives include learning how to:

- study topics that are complementary to the material covered in class,
- study topics in a collective fashion,
- choose a topic to study within a limited amount of time,
- use the document preparation system Latex (through Overleaf),
- present what you have learned, discovered, or invented both in speech and writing in a way that meets contemporary standards of mathematical rigor and presentation.

The second type of objectives pertains to the specific topic of fractal geometry and dynamics. These objectives include learning how to:

- work with metric spaces, in terms of uniformity, topology and measurability,
- work with the Hausdorff distance,
- prove and apply the Banach Contraction Principle,
- think from a dynamical point of view (symmetry = time),
- use iterated function systems to understand complicated shapes with some self-similarity,
- use symbolic dynamics to understand the dynamical nature of attractors of iterated function systems,
- prove and use the Shadowing Theorem for hyperbolic iterated function systems,
- interpret the notion of dimension as a power law,
- derive and use formulas to compute and approximate dimensions of attractors of iterated function systems,
- use probability measures on attractors,
- prove and use Elton Ergodic Theorem for iterated function systems.

5 Tentative Schedule

The course's subject matter is divided by weeks within the academic calendar as follows. For your convenience the approximate corresponding chapters in the main textbook are also listed:

Weeks 1-3 (Weeks of 08/19, 08/26, 09/03): Metric Spaces (Ch.2) Distance functions. Convergent and Cauchy sequences. Continuity and uniform continuity. Lipschitz and Hölder continuity. Open, closed, dense, compact, bounded, totally bounded, complete, connected, path-connected, disconnected, totally disconnected subsets of metric spaces. Hyperspaces of subsets of metric spaces.

Weeks 4-5 (Weeks of 09/09, 09/16): Iterated Function Systems (Ch.3) Spaces, Structures, functoriality. Monoids, groups. Endomorphisms, automorphisms. Homomorphisms, actions. Transformation groups of metric spaces. Iterates of active transformations as dynamical systems. Passive transformations as coordinate changes. Equivalences between dynamical systems. Banach Contraction Principle. Iterated function systems (IFS). Deterministic fractals. Attractors of IFSs. Chaos Game. Barnsley collage theorem.

Weeks 6-8 (Weeks of 09/23, 09/30, 10/14): Topological & Symbolic Dynamics (Ch.4) Orbits, invariant subsets, fixed and periodic points. Topological transitivity, weak mixing, and strong mixing. Coding for attractors of IFSs. Cocycles and skew products. Random Coding for attractors of IFSs. Shadowing Theorem for Hyperbolic IFSs. Sensitive dependence on initial conditions. Devaney chaoticity.

Weeks 9-11 (Weeks of 10/21, 10/28, 11/04): Dimension Theory (Ch.5) Topological (aka Lebesgue covering) dimension. Box counting (aka Minkowski–Bouligand) dimension. Hausdorff measures. Hausdorff (aka Hausdorff-Besicovitch) dimension. Moran-Hutchinson formula for dimension of attractors of hyperbolic IFSs.

Weeks 12-14 (Weeks of 11/11, 11/18, 11/25): Invariant Measures (Ch.9) IFSs with probabilities. Borel sigma-algebras of metric spaces. Probability measures on metric spaces. Supports of measures. Carathéodory Theorem. Integrable functions relative to a measure and their integrals. Spaces of probability measures on metric spaces. Markov operators associated to IFSs. Barnsley collage theorem for IFSs with probabilities. Elton Ergodic Theorem for IFSs.

Week 15 (Week of 12/02): Oral Presentations

6 Grades

The final letter grades will be determined according to the following weights and cutoffs. The cutoffs below are not definitive; based on the overall performance they may be adjusted at the end of the semester.

Assignment	Weight
Attendance	5%
Weekly Reports	35%
Final Report	40%
Presentation	20%
Total	100%

Grade	Percent
A	90%
A-	85%
B+	80%
B	75%
B-	70%
C+	65%
C	60%
C-	55%
D+	50%
D	45%
D-	40%
E	else

7 Books & Other Resources

The main textbook for MATH 4800 is the following:

> Barnsley, Michael F.. *Fractals Everywhere, 3e*. Dover Publications. 2012. ISBN: 9780486488707.

Barnsley's book is an introduction to fractal geometry and dynamics through iterated function systems, and is the standard reference for IFSs.

The following textbooks will also be used as secondary sources:

> Petrunin, Anton. *Pure Metric Geometry*. Springer, Cham. 2023. ISBN: 978-3-031-39161-3; 978-3-031-39162-0. Available at https://anton-petrunin.github.io/metric-geometry/pure.html.

Petrunin's book contains basic information on metric spaces, especially the hyperspace of compact nonempty subsets and Hausdorff distance.

> Falconer, Kenneth J.. *Fractal Geometry: Mathematical foundations and applications, 3e*. John Wiley & Sons, Ltd., Chichester. 2014. ISBN: 978-1-119-94239-9.

> Falconer, Kenneth J.. *Techniques in fractal geometry*. John Wiley & Sons, Ltd., Chichester. 1997. ISBN: 0-471-95724-0.

The above two books by Falconer are standard references in fractal geometry, the second being the sequel to the first. They are more advanced than the main textbook; we will use the first book especially in the context of dimension theory.

> Pesin, Yakov and Climenhaga, Vaugn. *Lectures on fractal geometry and dynamical systems*. American Mathematical Society, Providence, RI. 2009. ISBN: 978-0-8218-4889-0. Rough draft available at https://www.math.uh.edu/~climenha/fractals.html.

The book by Pesin and Climenhaga grew from a PSU MASS 2008 course, similar in spirit to this semester's MATH 4800. The emphasis is more on dynamics and it presumably requires more familiarity with analysis than the main textbook.

> Schroeder, Manfred. *Fractals, chaos, power laws: Minutes from an infinite paradise*. W. H. Freeman and Company, New York. 1991. ISBN: 0-7167-2136-8.

Schroeder's book is more along the lines of popular science and is meant to serve as inspiration. It includes many references to works in mathematics, physics and beyond.

The following notes for a PSU MASS 2012 course too will be useful for general information regarding dynamics:

Rodríguez Hertz, Federico and Vinhage, Kurt. *An Introduction to Geometric Topology and Dynamical Systems*. 2012. Available at https://www.math.utah.edu/~vinhage/teaching/archive/4800-fa23/useminar_bookdraft.pdf.

Further resources will be communicated if needed.

Scope Generally speaking, the scope of the course is limited to have "citation distance one" to one of the above resources. More specifically, the problem or project you are studying for a weekly or final report should be about a topic covered in a written work (article or book) that either cites one of the above references, or one of the above references should cite it.

8 Lectures

Attendance at all lectures is expected. Attendance at "random" lectures will be taken and will have an effect on the final grades. The number of lectures when attendance will be taken will also be "random".

Lecture Recordings Video recordings of lectures, excluding those reserved for oral presentations, will be uploaded weekly to the following YouTube playlist:

https://www.youtube.com/playlist?list=PL40ydqvvyXfMdJMim52HXAOamgWJOajUM.

This is an experimental feature of the course. Recordings may become unavailable at any time and without notice; do not rely solely on them.

9 Weekly Reports

Students are required to submit a weekly report each week, totaling 15 weekly reports. Each weekly report is due by 11:59 PM on the Sunday ending the week. Late submissions will not be accepted unless you contact the course staff with a valid excuse before the deadline. A certain number of weekly reports with lowest scores may be dropped at the end of the semester.

Content The content of a weekly report will depend on the student's personal preferences, as long as it is within the scope of the class and within reasonable limits[1]. The length of a weekly report should be between 2 to 5 pages not counting the acknowledgements and references. A weekly report may include:

- detailed and rigorous solutions of problems from one of the references,
- detailed and rigorous solutions of exercises suggested in class,
- discussion and study of generalizations of theorems or formulas included in one of the references or discussed in class,
- completions of gaps in proofs of theorems discussed in class,
- study of the preliminaries of a certain article,
- ...

Weekly projects will be collective: Collaboration with other students is allowed and encouraged. However, in case of content overlap in the reports of two different students, the earlier submission will be

[1] For instance, one could arguably write all weekly reports on properties and theorems about metric spaces. While this is within the scope of the course, it is not reasonable.

assessed, and the later submission will receive an automatic zero for the overlapping content.

Latex and Overleaf All weekly reports must be prepared using Latex. A convenient interface for preparing documents in Latex is Overleaf. As familiarity with Latex and Overleaf is not assumed, Weekly Report 0 is reserved for Latex practice.

Grading Criteria Weekly reports will be graded on accuracy, validity and presentation/documentation.

Submission Details For each weekly report, you are required to submit the TEX file containing your work on Canvas, together with a rendered PDF and any supporting files (such as a BIB or PNG file). Detailed submission instructions will be provided later as specification documents.

10 Final Report and Presentation

Instead of a final exam, there will be a final report and an in-class oral presentation based on the final report. Here are the relevant deadlines:

- **Declaration of topic and primary source(s):** by 11:59 PM on October 27 (Sunday of Week 9)
- **First draft for final report:** by 11:59 PM on November 17 (Sunday of Week 12)
- **First draft and plan for presentation:** by 11:59 PM on November 24 (Sunday of Week 13)

- **Presentation:** in class on December 3 or December 5 (the two classes in Week 15)
- **Final report:** by 11:59 PM on December 8 (Sunday of Week 15)

Broadly speaking, the topic should be according to the scope of the class. The complexity and depth of the project depends on personal preference (within reasonable limits) and could vary from an exposition or expansion of a certain chapter in one of the books to an exposition of a research article, to generalization of a simpler problem, and even discussion or solution of an original problem. The final report must be prepared using Latex and should be between 5 to 10 pages, not counting the acknowledgements and references. The final report and presentation will be collective, in the sense that while collaboration with other students is allowed and encouraged, only one student will be assessed per report.

The presentation may be a chalk talk or digital as a Beamer presentation. The length of the presentation will be such that the total time for the two lectures in Week 15 is equidistributed among students.

11 Exams

There will be no midterm exams nor a final exam.

12 Accessibility

The University of Utah seeks to provide equal access to its programs, services, and activities for people with disabilities. If you will need accommodations in this class, reasonable prior notice needs to be given to the Center for Disability & Access, SSB Basement Level 65, (801) 581-5020. CDA will work with you and the staff to make arrangements for accommodations. All written information in this course can be made available in an alternative format with prior notification to the Center for Disability & Access.

13 Inclusivity and Safety

Title IX makes it clear that violence and harassment based on sex and gender (which includes sexual orientation and gender identity/expression) is a civil rights offense subject to the same kinds of accountability and the same kinds of support applied to offenses against other protected categories such as race, national origin, color, religion, age, status as a person with a disability, veteran's status or genetic information.

If you or someone you know has been harassed or assaulted, you are encouraged to report it to:

- **Title IX Coordinator** in the Office of Equal Opportunity and Affirmative Action, PARK 135, (801) 581-8365, or
- **Office of the Dean of Students**, UNION 270, (801) 581-7066.

For support and confidential consultation, contact the Center for Student Wellness, SSB 328, (801) 581-7776. To report to the police, contact the Department of Public Safety, (801) 585-2677(COPS). For further safety resources, see SafeU.

14 Academic Honesty

You are expected to adhere to University of Utah policies regarding academic honesty. This means that ultimately all work you submit must be your own, created without unauthorized assistance, and all external resources, including generative tools and computer algebra systems, must be properly cited and documented within your submissions. Any student who engages in academic dishonesty or who violates the professional and ethical standards may be subject to academic sanctions as per the University of Utah's Student Code.

Generative AI and Computer Algebra Systems In this course, you may use generative tools like ChatGPT as well as computer algebra systems such as MATLAB under specific guidelines. These tools are permitted to guide your understanding of topics. The use of generative tools in a submission for Latex purposes is allowed, provided that it is acknowledged in the submission. If applicable, you must acknowledge your use of such tools in your submissions. You are responsible for ensuring the validity, accuracy, and relevance of your submissions, whether or not you use these tools. While encouraged, the use of such technology is not required for success in this class.

Regret Clause If you commit an unreasonable act but bring it to the staff's attention within 48 hours of the relevant submission, sanctions may be limited to that submission only, rather than leading to further disciplinary action. This clause will not be applied in the case of repeated violations.

15 Acknowledgements

The staff used ChatGPT in retouching this syllabus for clarity and readability.