1.

$$\partial_i \partial_j \left(\frac{1}{r}\right) = -\frac{4\pi}{3}\delta_{ij}\delta(\vec{r}) + \frac{3r_i r_j - r^2\delta_{ij}}{r^5}$$

$$\vec{E} = -\nabla V = -\frac{1}{4\pi\varepsilon}\nabla\left(\vec{p}\cdot\frac{\hat{r}}{r^2}\right)$$

$$= -\frac{1}{4\pi\varepsilon}\left[(\vec{p}\cdot\nabla)\frac{\hat{r}}{r^2} + \left(\frac{\hat{r}}{r^2}\cdot\nabla\right)\vec{p} + \vec{p}\times\left(\nabla\times\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\times(\nabla\times\vec{p})\right]$$

$$= -\frac{1}{4\pi\varepsilon}(\vec{p}\cdot\nabla)\frac{\hat{r}}{r^2}$$

则分量 $E_j = -\frac{1}{4\pi\varepsilon}\left(\sum_i p_i\partial_i\right)\left(-\partial_j\frac{1}{r}\right)$

$$= \frac{1}{4\pi\varepsilon}\sum_i p_i\partial_i\partial_j\left(\frac{1}{r}\right)$$

$$= \frac{1}{4\pi\varepsilon}\sum_i p_i\left[-\frac{4\pi}{3}\delta_{ij}\delta(\vec{r}) + \left(\frac{3r_i r_j - r^2\delta_{ij}}{r^5}\right)\right]$$

$$= -\frac{p_j\delta(\vec{r})}{3\varepsilon} + \frac{1}{4\pi\varepsilon}\frac{3\left(\sum_i p_i r_i\right)r_j - p_j r^2}{r^5} \quad (j=1,2,3)$$

$$\Rightarrow \vec{E} = -\frac{\vec{p}\,\delta(\vec{r})}{3\varepsilon} + \frac{1}{4\pi\varepsilon}\frac{3(\vec{p}\cdot\vec{r})\vec{r} - \vec{p}r^2}{r^5}$$

$$= -\frac{\vec{p}\,\delta(\vec{r})}{3\varepsilon} + \frac{1}{4\pi\varepsilon}\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}$$

2.

$$\begin{cases} +q & \left(\frac{d}{2}, 0, 0\right) \\ -q & \left(-\frac{d}{2}, 0, 0\right) \\ +q & \left(-\frac{d}{2}, 0, l\right) \\ -q & \left(\frac{d}{2}, 0, l\right) \end{cases}$$

$$D_{ij} = \sum_n q_n r_{ni} r_{nj}$$

$$\begin{cases} D_{xx} = D_{yy} = D_{zz} = 0 \\ D_{xy} = D_{yx} = 0 \\ D_{yz} = D_{zy} = 0 \end{cases}$$

$$D_{xz} = D_{zx} = -qdl = -pl$$

$$\Rightarrow \quad \overleftrightarrow{D} = -3pl\left(\vec{e_x}\,\vec{e_z} + \vec{e_z}\,\vec{e_x}\right)$$

$$\varphi = \frac{1}{4\pi\varepsilon}\,\frac{\overleftrightarrow{D} : \vec{e_r}\,\vec{e_r}}{2r^3} = \frac{-3pl\,(\vec{e_x}\cdot\vec{e_r})(\vec{e_z}\cdot\vec{e_r})}{4\pi\varepsilon r^3}$$

$$= \frac{-3pl}{4\pi\varepsilon}\,\frac{\cos\theta\sin\theta\cos\phi}{r^3}$$

3. (a)

$$\begin{cases} +q: (0, l_2, 0) \\ +q: (l_1, 0, 0) \\ -2q: (0, 0, 0) \end{cases} \qquad \begin{cases} Q = \sum_n q_n \\ \vec{p} = \sum_n q_n \vec{r} \\ D_{ij} = \sum_n q_n (3 r_{ni} r_{nj} - r_n^2 \delta_{ij}) \end{cases}$$

$\Rightarrow$ 总电量：$Q = 0$

电偶极：$\vec{p} = (q l_1 \vec{e_x} + q l_2 \vec{e_y})$

电四极：
$$\overleftrightarrow{D} = (2 q l_1^2 - q l_2^2) \vec{e_x}\vec{e_x} + (2 q l_2^2 - q l_1^2) \vec{e_y}\vec{e_y} + (-q l_1^2 - q l_2^2) \vec{e_z}\vec{e_z}$$

电势一阶近似：
$$\varphi = \frac{1}{4\pi\varepsilon} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\Rightarrow \varphi = \frac{1}{4\pi\varepsilon} \frac{(q l_1 \vec{e_x} + q l_2 \vec{e_y}) \cdot \vec{e_r}}{r^2}$$

$$= \frac{1}{4\pi\varepsilon} \frac{q l_1 \sin\theta\cos\phi + q l_2 \sin\theta\sin\phi}{r^2}$$

$$\Rightarrow \varphi \propto \frac{1}{r^2}$$

(b).

$$\begin{cases} +q: (l_1, l_2, 0) \\ +q: (l_1, 0, 0) \\ -2q: (0, 0, 0) \end{cases}$$

$\Rightarrow$ 总电量： $Q = 0$

电偶极： $\vec{p} = 2ql_1 \vec{e_x} + ql_2 \vec{e_y}$

电四极： $\overleftrightarrow{D} = 3ql_1l_2 (\vec{e_x}\vec{e_y} + \vec{e_y}\vec{e_x})$

$$+ (4ql_1^2 - ql_2^2)\vec{e_x}\vec{e_x}$$

$$+ (-2ql_1^2 + 2ql_2^2)\vec{e_y}\vec{e_y}$$

$$+ (-2ql_1^2 - ql_2^2)\vec{e_z}\vec{e_z}$$

$$\Rightarrow \varphi \simeq \frac{1}{4\pi\varepsilon} \frac{(2ql_1\vec{e_x} + ql_2\vec{e_y})\cdot\vec{e_r}}{r^2}$$

$$= \frac{1}{4\pi\varepsilon} \frac{2ql_1 \sin\theta\cos\phi + ql_2 \sin\theta\sin\phi}{r^2}$$

$$\Rightarrow \varphi \propto \frac{1}{r^2}$$

4.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\begin{cases} x = a r\cos\theta \\ y = b r\sin\theta \\ z = c h \end{cases} , \quad \begin{cases} r \in (0, \sqrt{1-h^2}) \\ \theta \in (0, 2\pi) \\ h \in (-1, 1) \end{cases}$$

二阶矩 $D_{ij} = \int d\tau (\rho r_i r_j)$

$$D_{xy} = \int dx\,dy\,dz \;\rho\, ab r^2 \cos\theta\sin\theta$$

$$= \rho a^3 b^2 c \int r\,dr\,d\theta\,dh \cos\theta\sin\theta$$

$$= \rho a^3 b^2 c \underline{\int_0^{2\pi}\cos\theta\sin\theta\,d\theta} \iint r\,dr\,dh = 0$$

同理: $D_{xy} = D_{yz} = D_{xz} = 0$

而 $D_{xx} = \int \rho a^3 bc\, r^3 dr \cos^2\theta\, d\theta\, dh$

$$= \pi\rho a^3 bc \int_{-1}^{1} dh \int_0^{\sqrt{1-h^2}} r^3 dr$$

$$= \frac{\pi}{4}\rho a^3 bc \left(h - \frac{2}{3}h^3 + \frac{1}{5}h^5\right)\Big|_{-1}^{1} = \frac{1}{5} Q a^2$$

$$\left(V = \frac{4\pi}{3} abc ,\quad \rho V = Q \right)$$

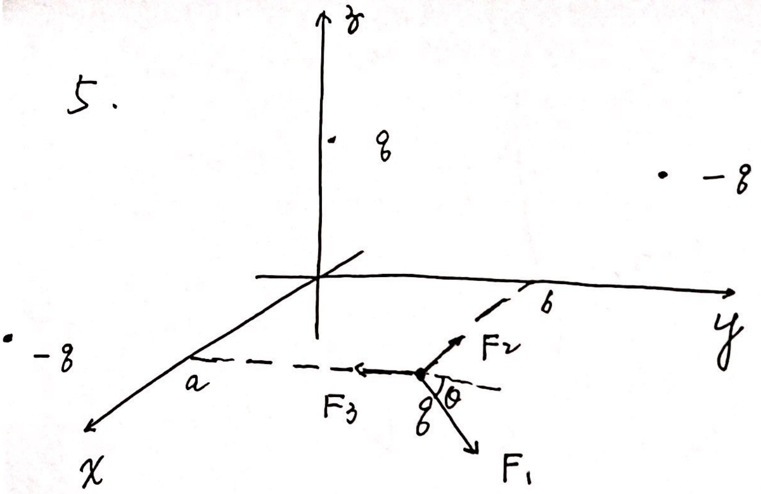
$$\Rightarrow \quad \overset{\rightrightarrows}{D} = \begin{pmatrix} \frac{1}{5}Qa^2 & 0 & 0 \\ 0 & \frac{1}{5}Qb^2 & 0 \\ 0 & 0 & \frac{1}{5}Qc^2 \end{pmatrix}$$

在本题中，取 $a = b \Leftarrow a$ ， $c \Leftarrow b$

则：

$$\overset{\rightrightarrows}{D} = \begin{pmatrix} \frac{Q}{5}(a^2-b^2) & 0 & 0 \\ 0 & \frac{Q}{5}(a^2-b^2) & 0 \\ 0 & 0 & \frac{2Q}{5}(b^2-a^2) \end{pmatrix}$$

5.



$$\begin{cases} F_1 = \dfrac{q^2}{4\pi\varepsilon}\dfrac{1}{(2a)^2+(2b^2)} = \dfrac{q^2}{16\pi\varepsilon}\dfrac{1}{a^2+b^2} \\ F_2 = \dfrac{q^2}{4\pi\varepsilon}\dfrac{1}{(2a)^2} = \dfrac{q^2}{16\pi\varepsilon}\dfrac{1}{a^2} \\ F_3 = \dfrac{q^2}{4\pi\varepsilon}\dfrac{1}{(2b)^2} = \dfrac{q^2}{16\pi\varepsilon}\dfrac{1}{b^2} \end{cases}$$

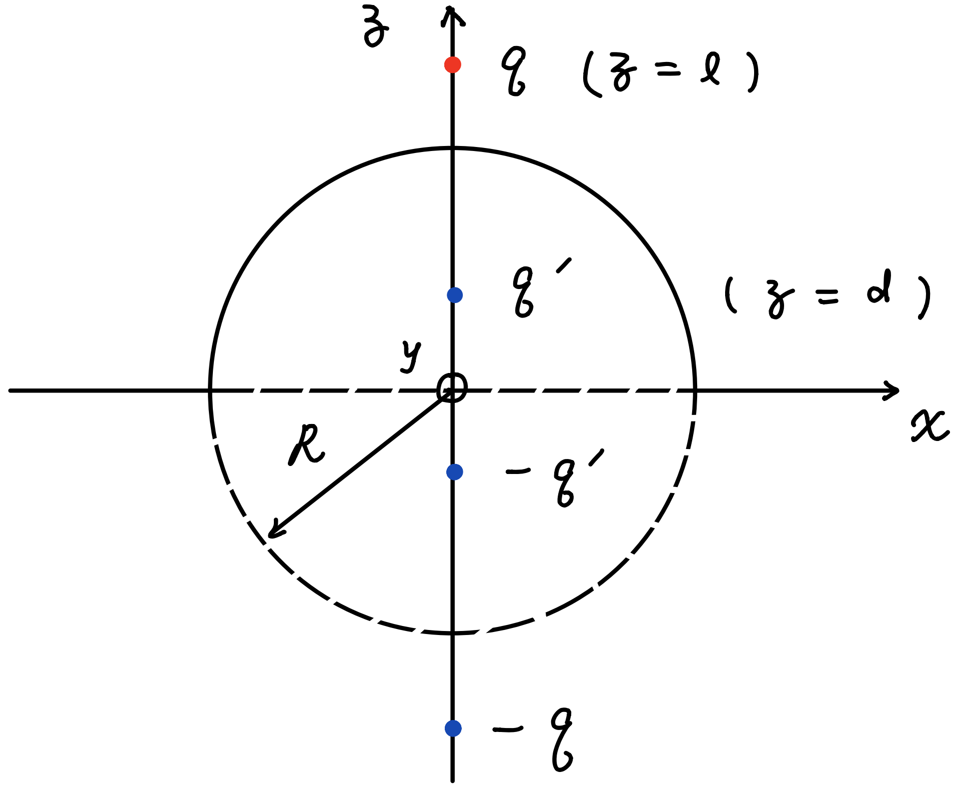
$$\vec{F} = \frac{q^2}{16\pi\varepsilon}\left(\frac{1}{a^2+b^2}\cdot\frac{a}{\sqrt{a^2+b^2}} - \frac{1}{a^2}\right)\vec{e_x} + \frac{q^2}{16\pi\varepsilon}\left(\frac{1}{a^2+b^2}\cdot\frac{b}{\sqrt{a^2+b^2}} - \frac{1}{b^2}\right)\vec{e_y}$$

此构型总能量： $E = \frac{1}{4}\sum_{i=1}^{4}\frac{1}{2}q_i U_i$

$$\begin{cases} U_+ = -\dfrac{q}{4\pi\varepsilon}\dfrac{1}{2a} - \dfrac{q}{4\pi\varepsilon}\dfrac{1}{2b} + \dfrac{q}{4\pi\varepsilon}\dfrac{1}{2\sqrt{a^2+b^2}} \\ U_- = -U_+ \end{cases}$$

$$\Rightarrow \quad W = -\frac{q^2}{16\pi\varepsilon}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{\sqrt{a^2+b^2}}\right)$$

6. 使用如下镜像法



镜像方程：

$$\begin{cases} d = \dfrac{R^2}{l} \\ q' = -\dfrac{qR}{l} \end{cases}$$

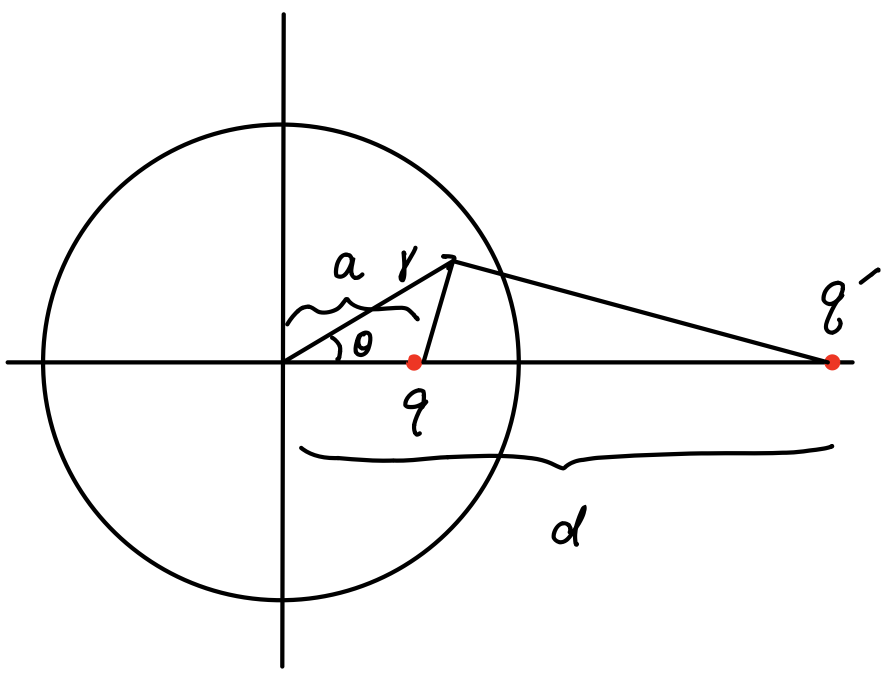
可知：

$$V(x,y,z) = \frac{q}{4\pi\varepsilon}\left[\frac{1}{\sqrt{x^2+y^2+(z-l)^2}}\right.$$

$$\left. + \frac{-\frac{R}{l}}{\sqrt{x^2+y^2+\left(z-\frac{R^2}{l}\right)^2}} + \frac{\frac{R}{l}}{\sqrt{x^2+y^2+\left(z+\frac{R^2}{l}\right)^2}}\right]$$

$$+ \frac{-1}{\sqrt{x^2+y^2+(z+l)^2}} \Big]$$

7. 使用如下镜像法



镜像方程：

$$\begin{cases} d = \dfrac{R^2}{a} \\ q' = -\dfrac{d}{R} q = -\dfrac{R}{a} q \end{cases}$$

内部电势：

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon} \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}}$$

$$- \frac{1}{4\pi\varepsilon} \frac{\frac{R}{a} q}{\sqrt{r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta}}$$

内表面电荷是 $-q$，分布：

$$\sigma_{in} = \varepsilon \vec{E} \cdot \vec{n} \Big|_{r=R} = \varepsilon \nabla\varphi \cdot \hat{r} \Big|_{r=R} = \varepsilon \frac{\partial \varphi}{\partial r} \Big|_{r=R}$$

$$\frac{\partial \varphi}{\partial r} = \frac{q}{4\pi\varepsilon} \frac{a\cos\theta - r}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}}$$

$$- \frac{\frac{Rq}{a}}{4\pi\varepsilon} \frac{\frac{R^2}{a}\cos\theta - r}{\left(r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta\right)^{3/2}}$$

$$\Rightarrow \sigma_{in}(\theta) = \frac{q}{4\pi} \frac{a\cos\theta - R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} - \frac{\frac{Rq}{a}}{4\pi} \frac{\frac{R^2}{a}\cos\theta - R}{\left(R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\theta\right)^{3/2}}$$

外表面是均匀分布，共 $q+Q$

$$\Rightarrow \sigma_{out}(\theta) = \frac{q+Q}{4\pi R^2} = \sigma_{out}$$

$$\sigma(\theta) = \sigma_{in}(\theta) + \sigma_{out}$$

$$= \frac{q+Q}{4\pi R^2} + \frac{q}{4\pi} \frac{a\cos\theta - R}{(R^2 + a^2 - 2Ra\cos\theta)^{3/2}} - \frac{\frac{Rq}{a}}{4\pi} \frac{\frac{R^2}{a}\cos\theta - R}{\left(R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\theta\right)^{3/2}}$$

8 真空区域的电势解是：

$$\varphi(r,\theta)=\sum_{l=0}^{+\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

边界条件：

$$\varphi(\infty,\theta)=0$$

$$\varphi(R,\theta)=\varphi_0\left(1+2\cos\theta+\cos^2\theta\right)$$

$$=\varphi_0\left[\frac{2}{3}P_2(\cos\theta)+2P_1(\cos\theta)+\frac{4}{3}P_0(\cos\theta)\right]$$

得：

$$\varphi(r,\theta)=\varphi_0\left[\frac{2R^3}{3r^3}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)+\frac{2R^2}{r^2}\cos\theta+\frac{4}{3}\frac{R}{r}\right]$$