The Poincaré Duality Theorem and its Applications

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What is a Vector Space

A **vector space** is a set V, with addition and multiplication such that the following holds for all $u, v, w \in V$ and $a, b \in F$: Commutative, Associative, Additive identity, Additive inverse, Multiplicative identity, and the Distribution laws

A **linear map** from V to W is a function $T : V \to W$ with the following properties for all $u, v \in V$ and $\lambda \in F$:

$$T(u + v) = Tu + Tv, \quad T(\lambda v) = \lambda T(v)$$

An **isomorphism** is an invertible linear map

$\mathcal{L}(\mathbf{V}, \mathbf{W})$ is the set of all linear maps from V to W.

A **linear functional** on V is a linear map from V to F, that is element of $\mathcal{L}(V, F)$.

A **dual space** of V, denoted by V^*, is the vector space of all linear functionals on V.

The Five Lemma

A sequence of maps $d_0, d_1, \ldots d_n$, is an **exact sequence** if

$$Im(d_{k-1}) = Ker(d_k)$$

A **short exact sequence** is of the form:

$$0 \to A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{0}$$

A **long exact sequence** is of $f_0, f_1, \ldots f_n$, has the from

$$\ldots \xrightarrow{f_0} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \to \ldots$$

The five lemma Given a commutative diagram of Abelian groups and group homomorphisms as in Figure **??** below, in which the rows are exact sequence, if the maps α, β, δ, and ε are isomorphism, then γ is also an isomorphism.

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\varepsilon"'] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "f_1"'] & B' \arrow[r, "f_2"'] & C' \arrow[r, "f_3"'] & D' \arrow[r, "f_4"'] & E' \arrow[r] & \cdots
\end{tikzcd}$$

Figure 1:A commutative diagram to show the Five Lemma

Smooth Manifold

A **diffeomorphism** is a map $f : X \to Y$ such that f is a homeomorphism, and both f and f^{-1} are smooth(differentiable).

A **smooth manifold of dimension m** is a subset $M \subset \mathbb{R}^n$ such that for each $x \in M$, x has a neighborhood $W \cap M$ that is diffeomorphic to an open subset U of the euclidean space $\mathbb{R}^m$.

A basis $(b_1, \ldots, b_n)$ determines some **orientation** as basis $(b'_1, \ldots, b'_n)$ if: $b'_i = \sum_j a_{i,j} b_i, \quad det(a_{i,j}) > 0$.

A **oriented smooth manifold** consists of a manifold M and a choice of orientation for each tangent TM_x.

A **good cover** is an open cover $U = \{U_\alpha\}$ of a manifold M of dimension m. M where all nonempty finite intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^m$.

A **finite good cover** is a good cover U of M which is finite. Equivalently M is **of finite type**.

Outline

In this talk I will explain the duality between the deRham cohomology of a manifold M and the compactly supported cohomology on the same space. This phenomenon is entitled "Poincaré duality" and it describes a general occurrence in differential topology, a duality between spaces of closed, exact differentiable forms on a manifold and their compactly supported counterparts. In order to define and prove this duality I will start with the simple definition of the dual space of a vector space, with the definition of a positive definite inner product on a vector space, then define the concept of a manifold. I will continue with the definition of differential forms on a differentiable manifold and their corresponding spaces necessary to this analysis. I will then introduce the concepts of a good cover of a manifold, manifolds of finite type, and orientation, all necessary concepts towards the goal of defining and proving Poincaré duality. I will finish with the proof of the Poincaré duality in the case of M orientable and admits a finite good cover, with examples.

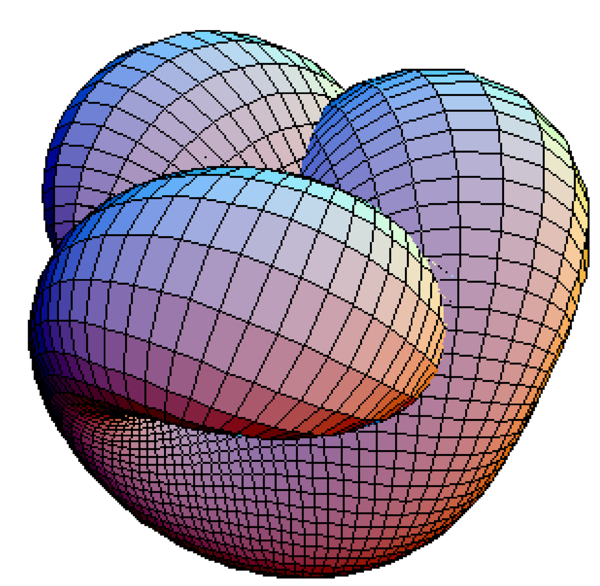

Figure 2:A smooth manifold

Poincaré duality for deRham cohomology

Lemma 5.6 *The two Mayer-Vietoris sequences ... and ..., may be paired togther to form a sign-commutative diagram*

Theorem. *For an oriented manifold M there is a paring*

$$\int : H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R},$$

given by the integral of the wedge product of two forms. Then the Poincaré duality asserts that this paring is nondegenerate whenever M is orientable and has a finite good cover; equivalently

$$H^q(M) \simeq (H_c^{n-q}(M))^*$$

Proof idea. The proof is a proof by induction as follows:

- Let M be a manifold $M = \bigcup_{k=1}^{l} U_k$.
- Induction basis: By lemma 5.6 we have $U_1 \cup U_2$.
- Induction Hypothesis: Assume $(U_1 \cup \cdots \cup U_k)$.
- Induction Step: $(U_1 \cup \ldots U_k) \cup U_{k+1}$
- We have

$$\begin{aligned} H^*(U_1 \cup \cdots \cup U_k) \cup H^*(U_{k+1}) &\implies H^{*+1}((U_1 \ldots U_k) \cap U_{k+1}) \\ &\implies H^{*+1}((U_1 \ldots U_k) \cap U_{k+1}) \end{aligned}$$

- Then by the induction step and the Five Lemma: we get

$$H^{*+1}(U_1 \ldots U_{k+1})$$

Other forms of the Poincaré duality

The theorem can be extended to any orientable manifold by the Mayer-Vietoris theorem, as follows:

Theorem. *If M is an orientable manifold of dimension n, whose cohomology is not necessarily finite dimension, then*

$$H^q(M) \simeq (H_c^{n-q}(M))^*$$

for any integer q.

Proof idea. *The finitness assumption on the good cover is not necessary, then by closer of analysis of topology of a manifold can be extended by the Mayer-Vietoris theorema.*

Remark. *One should note that the the reverse implication that te following is not always true:*

$$H_c^q(M) \simeq (H^{n-q}(M))^*$$

The Euclidian space $\mathbb{R}^n$

Example. *By the Five Lemma if Poincaré duality holds for U, V, and $U \cap V$, then it holds for $U \cup V$. By induction on the cardinality of a good cover. Considering M diffeomorphic to $\mathbb{R}^n$, and from the Poincaré lemmas*

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0 \\ 0 & \text{elsewhere} \end{cases}, H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{elsewhere} \end{cases}$$

The Poincaré duality follows.

The Sphere space $\mathbb{S}^n$

Let S^n are the point $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, such that

$$x_1^1 + \cdots + x_{n+1}^2 = 1$$

Example. *Let $\mathbb{S}^n = U \cup V$ where $U \cap V$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$. Then, through the Mayer-Vietoris sequence,*

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimensions } 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Which can be written as:

$$\begin{aligned} H^0(\mathbb{S}^n) &= \mathbb{R} \\ H^n(\mathbb{S}^n) &= \mathbb{R} \\ H^k(\mathbb{S}^n) &= 0, \quad k \neq 0, n \end{aligned}$$

Hence we have by the Poincaré dual we know $H^q(s^n) \simeq (H^{n-q}(S^n))^*$. For $q = 0$ we have $H^n(S^n) = \mathbb{R}$. For $q = n$, $H^0(S^n) \simeq \mathbb{R}$. And since $\mathbb{R} = \mathbb{R}^*$, we obtain, $H_c^n(S^n) = \mathbb{R}$.

Poincaré duals of a point in $\mathbb{R}^n$

Since $H^n(\mathbb{R}^n) = 0$, the closed Poincaré dual is μ_p is trivial, and can be represented by any closed n–form on $\mathbb{R}^n$, but the compact Poincaré dual is the nontrivial class in $H_c^n(\mathbb{R}^n)$ represented by a bump from with total integral 1.

Möbius strip

Counter example. *One may suspect that for cohomology with we compact support would have: $H_c^*(E) \simeq H_c^{*-n}(M)$. However this is not generally true; the open Möbius strip which is a vector bundle over S^1, is a counter example. The compact cohomology of the Möbius strip is identically 0; but S^1 does not match that, hence the Poincaré duality will not hold.*

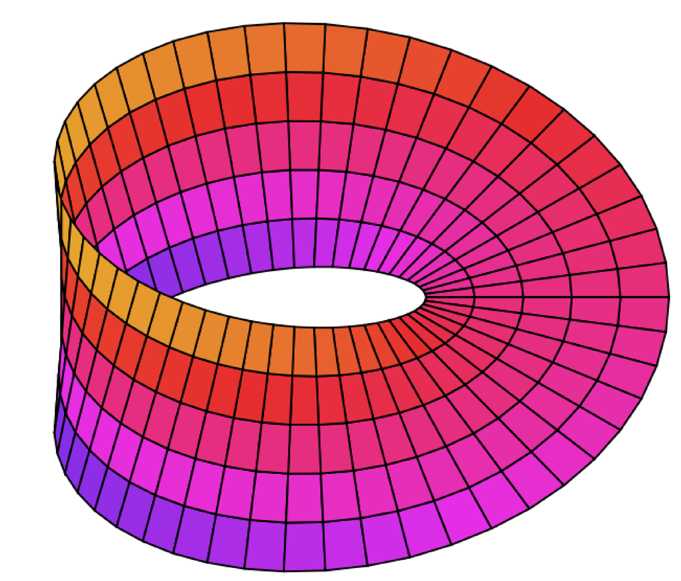

Figure 3:Möbius strip

Example. *But if E and M are finite orientable manifolds, and thus the equation would hold using the Poincaré duality (P.D):*

$$\begin{aligned} H_c^*(E) &\simeq (H^{m+n-*}(E))^* && \text{By applying the P.D thoerm on } E \\ &\simeq (H^{m+n-*}(M))^* && \text{By deRham cohomology homopoy} \\ &\simeq H_c^{*-n}(M) && \text{By P.D on } M \end{aligned}$$

Conclusion

Poincaré duality describes a general occurrence in differential topology, a duality between spaces of closed, exact differentiable forms on a manifold and their compactly supported counterparts.

$$H^q(M) \simeq (H_c^{(n-q)}(M))$$

The duality between the deRham cohomology of a manifold M and the compactly supported cohomology on the same space.

References

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[3] J. W. Milnor, *Topology From the Differentiable Viewpoint*