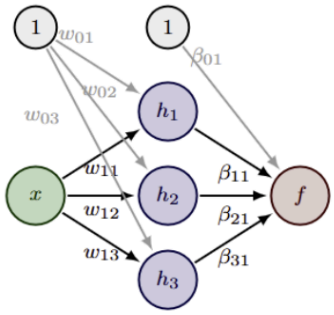
Problem 2 Neural Networks, *(9 points.)*

Consider a small neural network designed to classify a scalar feature x as one of $y \in \{-1, +1\}$. We have three hidden nodes h_1, h_2, h_3 and a single output node f_1.



You are given the weights W of the hidden layer,

$$W = \begin{bmatrix} w_{01} & w_{11} \\ w_{02} & w_{12} \\ w_{03} & w_{13} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ -3 & 3 \end{bmatrix}$$

and the weights B of the output layer,

$$B = \begin{bmatrix} \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 & -1 \end{bmatrix}.$$

(For example, w_{12} is the weight connecting x_1 to h_2; w_{02} is the constant (bias) term for h_2, etc.)

The network uses a ReLU activation function, $a(z) = \max(0, z)$, for the hidden layer, and a logistic sigmoid activation function,

$$\sigma(z) = \frac{1}{1 + \exp(-z)} = \frac{1}{1 + \frac{1}{\exp(z)}} = \frac{\exp(z)}{\exp(z) + 1},$$

for the output layer (the value of which, as usual, corresponds to the model's probability that the class is $+1$). You may find the following values useful:

$\exp(0) = 1$ $\exp(1) = 2.72$ $\exp(2) = 7.39$ $\exp(3) = 20.09$ $\exp(4) = 54.60$ $\exp(5) = 148.40$

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(1) What class is predicted by the model given the input $x_1 = 2$? *(3 points.)*

$x_1 = 2 \Rightarrow r_1 = 0 + 2 = 2 \quad h_1 = 2$

$r_2 = 3 - 4 = -1 \Rightarrow h_2 = 0 \Rightarrow s = -1 + 6 - 0 - 3$

$r_3 = -3 + 6 = 3 \quad h_3 = 3 \quad = 2$

$\sigma(2) = \frac{\exp(2)}{\exp(2)+1} = \frac{7.4}{8.4} = 0.88 > \frac{1}{2}$

$\hat{y} = +1$

(2) What is the model's estimated *probability* $p(y = +1 \mid x_1 = 2)$? *(3 points.)*

0.88

(3) Suppose our input is $x_1 = 0$: what is the probability $p(y = +1 \mid x_1 = 0)$? *(3 points.)*

$x_1 = 0 \Rightarrow r_1 = 0 + 0 = 0 \quad h_1 = 0$

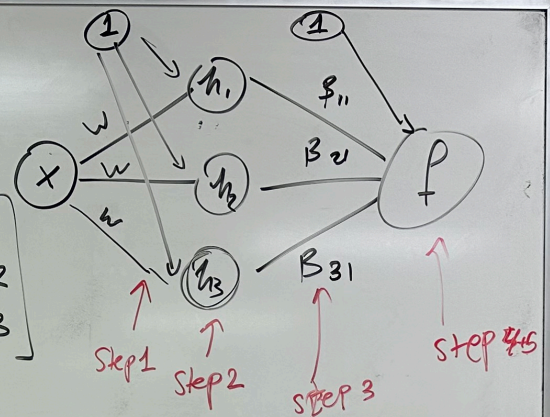
$r_2 = 3 - 0 = 3 \Rightarrow h_2 = 3 \Rightarrow s = -1 + 0 - 3 - 0$

$r_3 = -3 + 0 = -3 \quad h_3 = 0 \quad = -4$

$\sigma(-4) = \frac{1}{1+\exp(4)} = \frac{1}{55.6} \approx .018$

.018

Ex: Consider small N.N. classifies a feature x as one of $y \in \{+1, -1\}$



Given weights of hidden layer

$$W = \begin{bmatrix} w_{01} & w_{11} \\ w_{02} & w_{12} \\ w_{03} & w_{13} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \\ -3 & 3 \end{bmatrix}$$

weights of output

$$B = \begin{bmatrix} \beta_{01} & \beta_{11} & \beta_{21} & \beta_{31} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 & -1 \end{bmatrix}$$

1) What class can predict for $x_1 = 2$

$h(z) = \max(0, z)$

Step 1

$$z_1 = w_{01} \cdot 1 + w_{11} \cdot x = 0 \cdot 1 + 1 \cdot 2 = 2$$
$$z_2 = w_{02} \cdot 1 + w_{12} \cdot x = 3 \cdot 1 + (-2) \cdot 2 = -1$$
$$z_3 = w_{03} \cdot 1 + w_{13} \cdot x = 3$$

move to hidden layers

Step 2

$h(z_1) = 2$

$h(z_2) = 0$

$h(z_3) = 3$

Step 3: Linear score

$$S = \beta_0 \cdot 1 + \beta_1 \cdot h(z_1) + \beta_2 \cdot h(z_2) + \beta_3 (h(z_3))$$
$$= -1 \cdot 1 + 3 \cdot 2 + (-1) \cdot 0 + (-1) \cdot 3 = 2$$

Step 4

$$\sigma(2) = \frac{e^2}{1 + e^2} = 0.88 \geq \frac{1}{2}$$

$\rightarrow$ Predict $\hat{y} = +1$

2) What probability you predict ŷ for $x_1 = 2$,

$\mathbb{P}(\hat{y} = +1 \mid x_1 = 2) = \sigma(2^{\text{Not same 2 (just coincidence)}}) = 0.88$

3) Do same for $x_1 = 0$. ⟶ Predict $\hat{y} = -1$

4) Probability of Model ?

$\sigma(s)$ = Probability of Model getting +1

$\sigma(s) = 0.18$ → Predict $\hat{y} = -1$

→ Probability of predicting +1 is 18%

Formula box (ReLU)

1) Hidden Pre activation $z_j = w_{0j} + w_{1j} x$
2) Hidden layer (ReLU) $h_j \overset{\text{ReLU}}{=} \max(0, z_j)$
3) Linear score $s = \beta_0 + \sum_j \beta_j h_j$
4) Convert to probability (sigmoid) $\sigma(s) = \frac{e^s}{1+e^s}$
5) Decision (Prediction) $\hat{y} = \begin{cases} +1 & \sigma(s) \geq \frac{1}{2} \\ -1 & \sigma(s) < \frac{1}{2} \end{cases}$

Multiple hidden layer
Repeat step 1 & 2
$h_j \longmapsto z_j \longrightarrow$ repeat.