Math 230-Spring 2023
Due May 16, 2023 until 23:45 to Moodle.
Homework-3

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Department: CS

Section: 002

Important rules:

- Please use this page as a cover page for your homework. You could attach this page as the first page to your scanned HW or write the above information on the first page.
- No late submitted homework will be accepted!
- The following rules will determine the homework grade:
  - You will get 10 points for each questions you attempted to solve with a reasonable amount of work.
  - Only three randomly chosen questions will be graded in detail with 30 points each. You should show your work to get full credit. Correct answers must have sufficient explanation to get full credit.
  - 10 points for the paper organization. Please keep the order of the questions. If you do not provide any solution for a question, you should write the question number with the note "no solution". There should not be text crossed out. Papers that are messy, disorganized, or unreadable may not be graded.
- There will be a folder in Moodle. You should submit your homework to that folder.
- In the questions, the textbook refers to Probability with Applications in Engineering, Science and Technology, Matthew A. Carlton, Jay L. Devore, 2017, 2nd edition, Springer.

Questions

1. 3.3: 67
   3.4: 77
2. 3.7: 115, 126
3. 3.9: 142, 158.
4. 3.9: 168, 169.
5. Read section 3.8-Simulation of Continuous Random Variables from the textbook and write a short summary. Make sure that you understand all the examples. Then solve the following questions:
   3.8: 137

1 **3.3 : 67**

a) $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$

$$\left.\begin{aligned} u &= \frac{x-\mu}{\sigma} \\ \frac{du}{dx} &= \frac{1}{\sigma} \end{aligned}\right\} \rightarrow = \int_{-\infty}^{\infty} (\sigma u + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \cdot du$$

$$= \sigma \underbrace{\int u \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \cdot du}_{0} + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \cdot du}_{1}$$

$$\boxed{E(x) = \sigma \cdot 0 + \mu \cdot 1 = \mu}$$

b)

$$Var(x) = E((x-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

$$\left.\begin{aligned} u &= \frac{x-\mu}{\sigma} \\ du &= \frac{1}{\sigma}\, dx \end{aligned}\right\} \rightarrow = \int_{-\infty}^{\infty} \sigma^2 u^2 \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2} \cdot du$$

$$= \sigma^2 \int_{-\infty}^{\infty} \underbrace{\underbrace{u^2}_{g(u)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2}}_{f(u)}}_{E(u^2)} \cdot du$$

$f(u) \rightarrow$ pdf of $u \sim N(0,1)$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx$$

$$var(u) = E(u^2) - (E(u))^2 \Rightarrow E(u^2) = Var(u) + (E(u))^2 = 1 + 0^2 = 1$$

$$= \sigma^2 \cdot E(u^2) = \sigma^2 \cdot 1 = \sigma^2 \qquad \boxed{Var(x) = \sigma^2}$$

1 $\quad$ 3.3 : 77

a) $\mu = 24 \quad \sigma = 12 \quad \mu = \alpha\beta \quad \sigma = \beta\sqrt{\alpha} \quad \alpha = \frac{\mu^2}{\sigma^2} = \frac{24^2}{12^2} = 4$

$\beta = \frac{\sigma^2}{\mu} = \frac{12^2}{24} = 6$

$$P(12 < X < 24) = F(24; 4, 6) - F(12; 4, 6)$$
$$= F\left(\tfrac{24}{6}; 4\right) - F\left(\tfrac{12}{6}; 4\right)$$
$$= F(4; 4) - F(2; 4)$$
$$= 0.567 - 0.143 \implies \boxed{P(12 < X < 24) = 0.424}$$

b) $P(x \leq 24) \Rightarrow$ chance of a transistor will last at most 24 weeks

$$P(x \leq 24) = F(24; 4, 6) = F\left(\frac{24}{6}; 4\right) = F(4; 4) \qquad \boxed{P(x \leq 24) = 0.567}$$

c) $\eta_{0.99} \Rightarrow$ 99 th percentile of lifetime distribution

$P(x \leq \eta_{0.99}) = 0.99$ because the lifespan distribution is gamma distribution with $\alpha = 4$, $\beta = 6$

$$P(x \leq \eta_{0.99}) = F(\eta_{0.99}; 4, 6) = F\left(\frac{\eta_{0.99}}{6}; 4\right) = F(10; 4) \Rightarrow \boxed{\eta_{0.99} = 60}$$

d) 0.5 % of all transistors are expected be operational after t weeks.

This suggests that 99.5 % of them have a lifetime of less. So t is 99.5 th percentile

$$P(x \leq t) = 0.995$$

$$P(x \leq t) = F(t; 4, 6) = F\left(\frac{t}{6}; 4\right)$$

Since $\alpha = 4$ & value is 0,995 row = 11 $\qquad \frac{t}{6} = 11 \Rightarrow \boxed{t = 66}$

② $\underline{3.7 : 115}$

$f_x(x) = \frac{1}{2} e^{-x/2}$ $\quad f_y(y) = f_x\,(h(y)) \cdot (h'(y))$

$y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow h(y) = y^2 \Rightarrow f_x(h(y)) = \frac{1}{2} e^{-y^2/2} \Rightarrow h'(y) = 2y$

$$\Rightarrow \boxed{F_y(y) = \frac{1}{2} e^{-y/2} \cdot 2y}$$

② $\underline{3.7 : 126}$

mean $= 0$, $\sigma = 1$ $\quad$ Pdf $\Rightarrow f_x\,|x| = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$

$y = |x| \Rightarrow$ monotonically increasing $x > 0$

pdf of y for $x>0$

$$f_y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2} \quad y>0$$

3

3.9 : 142

a)

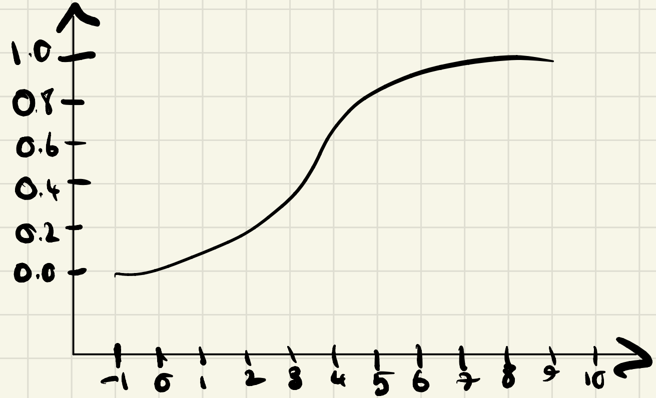
$$f(y) \begin{cases} \frac{1}{24}\left(\frac{y-y^2}{12}\right) & ; 0 \le y \le 12 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f(y) = \int_0^y f(y)\,dy = \frac{1}{24}\int_0^y \left(\frac{y-y^2}{12}\right) dy = \frac{1}{24}\left[\frac{y^2}{2} - \frac{y^3}{36}\right]_0^y$$

$$\boxed{F(y) = \frac{1}{48}\left(y^2 - \frac{y^3}{18}\right)}$$

$y \Rightarrow -1 \leq y \leq 13$

$F(y) \Rightarrow 0 \leq F(y) \leq 1$



b) $P(y \leq 4) = F(4) = \frac{1}{48}\left[16 - \frac{64}{18}\right] = 0.254$

$P(y > 6) = 1 - P(y \leq 6) = 1 - F(6) = 1 - \frac{1}{48}\left[36 - \frac{216}{18}\right] = 1 - 0.5 = \boxed{0.5}$

$P(4 \leq y \leq 6) = F(6) - F(4) = 0.5 - 0.259 = \boxed{0.241}$

c) $E(y) = \int_0^{12} y\, f(y)\, dy = \frac{1}{24} \int_0^{12} \left(y^2 - \frac{y^3}{12}\right) dy = \frac{1}{24} \left[\frac{y^3}{3} - \frac{y^4}{48}\right]_0^{12}$

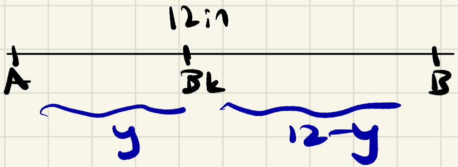
$= \frac{1}{24} \left[\frac{12^3}{3} - \frac{12^4}{48}\right] \Rightarrow \boxed{E(y) = 6}$

$E(y^2) = \int_0^{12} y^2 \cdot f(y)\, dy = \frac{1}{12} \int_0^{12} \left(y^3 - \frac{y^4}{12}\right) dy = \frac{1}{24} \left[\frac{y^4}{4} - \frac{y^5}{60}\right]_0^{12}$

$= \frac{1}{24} \left[\frac{12^4}{4} - \frac{12^5}{60}\right] \Rightarrow \boxed{E(y^2) = 43.2}$

$V(y) = \sigma^2 = E(y^2) - [E(y)]^2$

$\sigma = \sqrt{43.2 - 36} \Rightarrow \boxed{\sigma = 2.683}$

d) 

$E(y)=6$ $\qquad P(y>8)=\int_8^{12} f(y)\,dy = \frac{1}{24}\int_8^{12}\left(y-\frac{y^2}{12}\right)dy$

$$= \frac{1}{24}\left[\frac{y^2}{2}-\frac{y^3}{36}\right]_8^{12} = \frac{1}{24}\left[\frac{12^2}{2}-\frac{12^3}{36}-\frac{8^2}{2}+\frac{8^3}{36}\right]$$

$$\boxed{P(y>8)=0.259}$$

e)

$$x=\begin{cases} y \,; & y\le 6 \\ 12-y \,; & y\ge 6 \end{cases} \qquad E(x)=\int_0^6 y\,f(y)\,dy+\int_0^{12} y\,f(y)\,dy$$

$$E(x)=\frac{1}{24}\int_0^1\left(y^2-\frac{y^3}{24}\right)dy+\int_6^{12}(12-y)\cdot\frac{y}{24}\left(1-\frac{y}{12}\right)dy$$

$$=\frac{1}{24}\left[\frac{y^3}{3}-\frac{y^4}{48}\right]_0^6+\frac{1}{24\cdot 12}\int_6^{12}(12-y)^2\,y\,dy$$

$144-24y+y^2$

$$=\frac{1}{24}\left[\frac{6^3}{3}-\frac{6^4}{48}\right]+\frac{1}{288}\left[144\,\frac{y^2}{2}-24\,\frac{y^3}{3}+\frac{y^4}{4}\right]_6^{12}$$

$$= \frac{15}{8} + \frac{1}{288}\left[12\cdot 12^2 - 8\cdot 12^3 + \frac{12^4}{4} - 12\cdot 6^2 + 8\cdot 6^3 - \frac{6^4}{4}\right]$$

$$= \frac{15}{8} + \frac{15}{8} \Rightarrow \boxed{E(x) = 3.75}$$

3

3.9: 158

a) $$F(x; \lambda_1, \lambda_2, P) = \int_0^x f(y; \lambda_1, \lambda_2, P)\, dy$$

$$= \int_0^x p\lambda_1 e^{-\lambda_1 y} + (1-p)\lambda_2 e^{-\lambda_2 y}]\, dy = p\lambda_1\left(\frac{e^{-\lambda_1 y}}{-\lambda_1}\right) + (1-p)\lambda_2\left(\frac{e^{-\lambda_2 y}}{-\lambda_2}\right)_0^x$$

$$= p(1-e^{-\lambda_1 x}) + (1-p)(1-e^{-\lambda_2 x})$$

$$F(x; \lambda_1, \lambda_2, P) = \begin{cases} 0; & x < 0 \\ p(1-e^{-\lambda_1 x}) + (1-p)(1-e^{-\lambda_2 x}) \end{cases}$$

b) $p = 0.5, \ \lambda_1 = 40 \quad \lambda_2 = 200$

$$P(x > 0.01) = (1-p)\,(x \leq 0.01) = 1 - F(0.01) \Rightarrow \boxed{P(x > 0.01) = 0.4028}$$

c) $x_1, x_2 \Rightarrow$ expo with parameters $\lambda_1 \ \lambda_2$

$$x = p x_1 + (1-p) x_2 \qquad E(x) = p\,\underbrace{E(x_1)}_{1/\lambda_1} + (1-p)\,\underbrace{E(x_2)}_{1/\lambda_2}$$

$$\boxed{E(x) = \frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}}$$

d e f g $\Rightarrow$ <u>No Solution</u>

4

3.9: 168

$x = 0 \qquad (-\infty, 0], \ (0, \infty)$

$y \Rightarrow (0, -\infty)$

$f_y(y) = f_x[h(y)]\,[h'(y)]$

$f[h(x)] = \frac{1}{\pi}\left(\frac{y^2}{1+y^2}\right) \quad -\infty < y < 0$

$h'(y) = \frac{-1}{y^2}$

$$\boxed{f_y(y) = \left(\frac{1}{\pi}\right)\left(\frac{y^2}{1+y^2}\right)\left(\frac{1}{y^2}\right) = \left(\frac{1}{\pi} \cdot \frac{1}{y^2+1}\right)}$$

4

3.9: 168

Weibull CDF

$$F(x) = 1 - e^{-(x/B)^{\alpha}} \quad x \geq 0$$

Extreme value CDF

$$F(x) = 1 - e^{(x-\theta_1)/\theta_2}$$

$$-\infty < z < \infty$$

$y = \ln x \sim$ Extreme value $(\theta_1 - \ln B,\ \theta_2 = 1/\alpha)$

$$F(y) = P(x \leq y) = P(\ln x \leq y) = P(x \leq e^y) = 1 - e^{-(e^y/B)^{\alpha}}$$

$$= 1 - e^{-y^{\alpha}/B^{\alpha}}$$

$$F(z) = 1 - e^{-[(z - \ln B)/(1/\alpha)]} = 1 - e^{-e^{\alpha z - \alpha \ln B}}$$

$$= 1 - e^{-\alpha z/B^{\alpha}}$$

5 No Solution