COVER PAGE

MATH 230 - SPRING 2023

Due May 16, 2023 until 23.45 to Moodle.

Homework - 3

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Section : 2

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Question 1: $\boxed{3.3-67}$ suppose $X \sim N(\mu, \sigma)$.

(a) Show via integration that $E(X)=\mu$. [Hint: Make the substitution $u=(x-\mu)/\sigma$, which will create two integrals. For one, use the symmetry of the pdf; for the other, use the fact that the standard normal pdf integrals to 1.]

(b) Show via integration that $Var(X)=\sigma^2$. [Hint: Evaluate the integral for $E[(X-\mu)^2]$ rather than using the variance shortcut formula. Use the same substitution as in part (a).]

Answer

pdf : $f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Standard normal random variable:

$U \sim N(0,1)$

$f(u)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}$

(a) $E(X)=\displaystyle\int_{-\infty}^{\infty} x\, f(x)\, dx$

$=\displaystyle\int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$=\displaystyle\int_{-\infty}^{\infty} (\sigma u+\mu)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2} du$

substitution:

$u=\dfrac{x-\mu}{\sigma}$

$u=\dfrac{1}{\sigma}x-\dfrac{\mu}{\sigma}$

$\dfrac{du}{dx}=\dfrac{1}{\sigma}$

$du=\dfrac{1}{\sigma}dx$

$\sigma u+\mu=x$

$E(x)=\displaystyle\int_{-\infty}^{\infty} x f(x)dx = \underbrace{\sigma\int_{-\infty}^{\infty} u\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du}_{E(u)=0} + \underbrace{\mu\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du}_{1}$

$E(u)=\displaystyle\int_{-\infty}^{\infty} u\, f(u)\, du$

$$\boxed{E(x)=\sigma\cdot 0+\mu\cdot 1=\mu}$$

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Question 1 (3.3-67)

(b) $Var(x) = E((x-\mu)^2)$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \sigma^2 u^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}u^2} du$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

$$u = \frac{x-\mu}{\sigma}$$

$$\sigma u = x - \mu$$

$$du = \frac{1}{\sigma} dx$$

$$= \sigma^2 \underbrace{\int_{-\infty}^{\infty} \underbrace{u^2}_{g(u)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}u^2}}_{f(u)} du}_{E(u^2)}$$

$f(u) \longrightarrow$ pdf of $U \sim N(0,1)$

$$= \sigma^2 \cdot E(u^2)$$

$$= \sigma^2 \cdot 1$$

$$= \sigma^2$$

$$\boxed{Var(x) = \sigma^2}$$

$$Var(u) = E(u^2) - (E(u))^2$$

$$E(u^2) = Var(u) + (E(u))^2$$

$$= 1 + 0^2$$

$$= 1$$

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Question 1 | 3.4 - 77 | Suppose that when a type of transistor is subjected to an accelerated life test, the lifetime X (in weeks) has a gamma distribution with mean 24 weeks and standard deviation 12 weeks.

a) what is the probability that a transistor will last between 12 and 24 weeks?

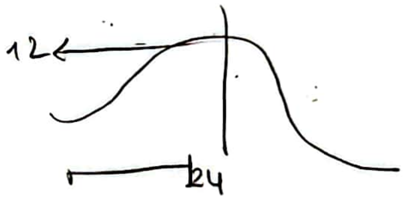
b) what is the probability that a transistor will last at most 24 weeks. Is the median of the lifetime distribution less than 24? Why or why not?

c) what is the 99th percentile of the lifetime distribution?

d) Suppose the test will actually terminated after t weeks. What value of t is such that only .5% of all transistors would still be operating at termination?

Answer: a) $\mu = 24$

$\sigma = 12$



$\mu = \alpha\beta$.

$\sigma = \beta\sqrt{\alpha}$

$\alpha = \frac{\mu^2}{\sigma^2} = \frac{24}{12} = 4$

$\beta = \frac{\sigma^2}{\mu} = \frac{12^2}{24} = 6$

$P(12 < X < 24) = F(24; 4, 6) - F(12; 4, 6)$

$= F\left(\frac{24}{6}; 4\right) - F\left(\frac{12}{6}; 4\right)$

$= F(4; 4) - F(2; 4)$

$= 0{,}567 - 0{,}143 = 0.424$

$\boxed{P(12 < x < 24) = 0.424}$

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Question 1 | 3.4-77 | (b) $P(X \le 24)$ is the chance that a transistor will last at most 24 weeks

$P(x \le 24) = F(24; 4, 6)$

$= F\left(\frac{24}{6}; 4\right)$

$= F(4; 4)$

$\boxed{P(X \le 24) = 0.567}$

(c) $\eta_{0.99}$ = 99th percentile of the lifetime distribution

$P(X \le \eta_{0.99}) = 0.99$ Because the lifespan distribution is a gamma distribution with α and β equal to 4 and 6, it is a gamma distribution

$P(x \le \eta_{0.99}) = F(\eta_{0.99}; 4, 6) = F\left(\frac{\eta_{0.99}}{6}; 4\right)$

$\frac{\eta_{0.99}}{6} = 10 \implies \boxed{\eta_{0.99} = 60}$ } The percentile of the lifetime distribution

(d) Only 0.5% of all transistors are expected to be operational after t weeks. This suggests that 99.5% of them have a lifetime of less than that. As a result, t is the 99.5th percentile.

$P(X \le t) = 0.995$

Also,

$P(x \le t) = F(t; 4, 6) = F\left(\frac{t}{6}; 4\right)$

since $\underbrace{\alpha = 4}_{\text{column}}$ and value is 0.995, row = 11

$\frac{t}{6} = 11$ $\boxed{t = 66}$

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Question 2 | 3.7 - 115 | Let X have an exponential distribution with mean 2, so $f_X(x) = \frac{1}{2} e^{-x/2}$, $x > 0$. Find the pdf of $Y = \sqrt{x}$. (Note: Suppose you choose a point in two dimensions randomly, with the horizontal and vertical coordinates chosen independently from the standard normal distribution. Then X has the distribution of the squared distance from the origin and Y has the distribution of the distance from the origin. Y has the Rayleigh distribution.

Answer

$f_x(x) = \frac{1}{2} e^{-x/2}$

$\boxed{f_Y(y) = f_X(h(y)) \cdot (h'(y))}$

$\boxed{Y = \sqrt{x}}$ => monotonic

strictly increasing

$Y = \sqrt{x}$

$y^2 = x$

$h(y) = y^2$

$f_x(h(y)) = \frac{1}{2} e^{-y^2/2}$

$h'(y) = 2y$

$\boxed{f_Y(y) = \frac{1}{2} e^{-y/2} \cdot 2y}$

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Question 2 (3.7 - 126) If a measurement error X is distributed as $N(0,1)$, find the pdf of $|X|$, which is magnitude of the measurement error.

Answer

normally distributed random variable $= x$

mean $= 0$, $\sigma = 1$

pdf:

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , -\infty < x < \infty$$

$$Y = |x|$$

* normal distribution is symmetric with respect to inversion. Taking the absolute value of standard normal variable maps 2 interval $[-\infty, 0)$ and $(0, \infty)$ onto the same interval $[0, \infty]$
* each random variable will be repeated twice

The function is monotonically increasing $x > 0$.

$$f_Y(y) = f_x(h(y)) \cdot (h'(y))$$

For $x > 0$, $h(y) = y \Rightarrow h'(y) = 1$

pdf of y for $x > 0$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \quad y > 0$$

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Question 3 (3.9 - 142) A 12-in. bar clamped at both ends is subjected to an increasing amount of stress until it snaps. Let Y = the distance from the left end at which the break occurs. Suppose Y has pdf

$$f(y) = \begin{cases} \frac{y}{24}\left(1-\frac{y}{12}\right) & 0 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

compute the following:

(a) The cdf of Y, and graph it.

(b) $P(Y \le 4)$, $P(Y > 6)$, and $P(4 \le Y \le 6)$.

(c) $E(Y)$, $E(Y^2)$, and $SD(Y)$.

(d) The probability that the break point occurs more than 2 in from the expected break point.

(e) The expected length of the shorter segment when the break occurs.

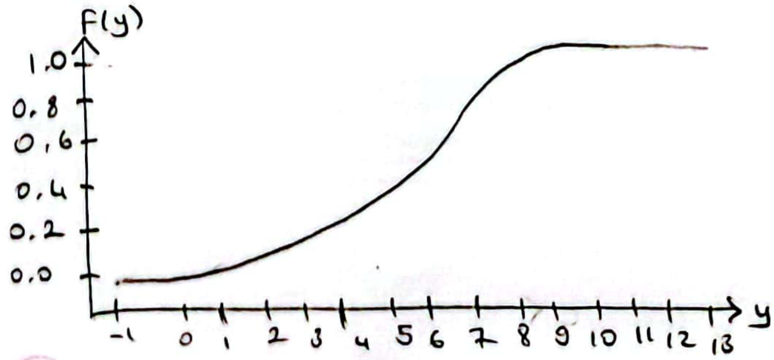
Answer (a) $f(y) = \begin{cases} \frac{1}{24}\left(y-\frac{y^2}{12}\right) & ; 0 \le y \le 12 \\ 0 & ; \text{otherwise} \end{cases}$

$$F(y) = \int_0^y f(y)\,dy = \frac{1}{24}\int_0^y \left(y-\frac{y^2}{12}\right)dy = \frac{1}{24}\left[\frac{y^2}{2}-\frac{y^3}{36}\right]_0^y$$

$$F(y) = \frac{1}{48}\left(y^2-\frac{y^3}{18}\right) \quad \text{cdf of } Y$$

$y \Rightarrow$ x-axis $\quad -1 \le y \le 13$

$F(y) \Rightarrow$ y-axis $\quad 0 \le F(y) \le 1$



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Question 3 | 3.9 - 142 | b

$$P(Y \leq 4) = F(4) = \frac{1}{48}\left[16 - \frac{64}{18}\right] = \underline{\underline{0.254}}$$

$$P(Y > 6) = 1 - P(Y \leq 6) = 1 - F(6) = 1 - \frac{1}{48}\left[36 - \frac{216}{18}\right]$$

$$= 1 - 0.5 = \underline{\underline{0.5}}$$

$$P(4 \leq Y \leq 6) = F(6) - F(4) = 0.5 - 0.259 = \underline{\underline{0.241}}$$

c

$$E(Y) = \int_0^{12} y\, f(y)\, dy = \frac{1}{24}\int_0^{12} \left(y^2 - \frac{y^3}{12}\right) dy = \frac{1}{24}\left[\frac{y^3}{3} - \frac{y^4}{48}\right]_0^{12}$$

$$= \frac{1}{24}\left[\frac{(12)^3}{3} - \frac{(12)^4}{48}\right] = 6 \quad \Big\rangle\; E(Y) = 6$$

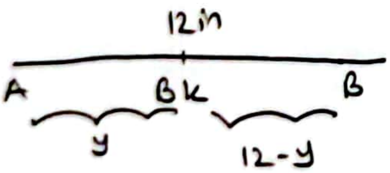
$$E(Y^2) = \int_0^{12} y^2 f(y)\, dy = \frac{1}{12}\int_0^{12}\left(y^3 - \frac{y^4}{12}\right) dy = \frac{1}{24}\left[\frac{y^4}{4} - \frac{y^5}{60}\right]_0^{12}$$

$$= \frac{1}{24}\left[\frac{(12)^4}{4} - \frac{(12)^5}{60}\right] \Rightarrow E(Y^2) = 43.2 \qquad V(Y) = \sigma^2 = E(Y^2) - [E(Y)]^2$$

$$\sigma = \sqrt{43.2 - 36}$$

$$\boxed{\sigma = SD(Y) = \underline{\underline{2.683}}}$$

d



$$E(Y) = 6$$

$$P(Y > 8) = \int_8^{12} f(y)\, dy = \frac{1}{24}\int_8^{12}\left(y - \frac{y^2}{12}\right) dy$$

$$= \frac{1}{24}\left[\frac{y^2}{2} - \frac{y^3}{36}\right]_8^{12} = \frac{1}{24}\left[\frac{(12)^2}{2} - \frac{(12)^3}{36} - \frac{(8)^2}{2} + \frac{(8)^3}{36}\right]$$

$$P(Y > 8) = \underline{\underline{0.259}}$$

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Question 3 $\boxed{3.9-142}$ e)

Let $x = \min(y, 12-y)$ $0 \le y \le 12$ when $y \le 6$, $x = y$ and when $y \geq 6$, $x = 12-y$

$$x = \begin{cases} y \; ; \; y \le 6 \\ 12-y \; ; \; y \geq 6 \end{cases} \qquad E(x) = \int_0^6 y f(y)\,dy + \int_0^{12} y f(y)\,dy$$

$$E(x) = \frac{1}{24} \int_0^6 \left(y^2 - \frac{y^3}{24}\right)dy + \int_6^{12} (12-y) \cdot \frac{y}{24}\left(1-\frac{y}{12}\right)dy$$

$$= \frac{1}{24}\left[\frac{y^3}{3} - \frac{y^4}{48}\right]_0^6 + \frac{1}{24.12}\int_6^{12} (12-y)^2 y\,dy$$

$$= \frac{1}{24}\left[\frac{6^3}{3} - \frac{6^4}{48}\right] + \frac{1}{288}\int_6^{12} y\,(144 - 24y + y^2)\,dy$$

$$= \frac{15}{8} + \frac{1}{288}\left[144\frac{y^2}{2} - 24\frac{y^3}{3} + \frac{y^4}{4}\right]_6^{12}$$

$$= \frac{15}{8} + \frac{1}{288}\left[12\,(12)^2 - 8(12)^3 + \frac{(12)^4}{4} - 12\,(6)^2 + 8\,(6)^3 - \frac{(6)^4}{4}\right]$$

$$= \frac{15}{8} + \frac{15}{8} = \frac{30}{8} = 3.75$$

$$\boxed{E(x) = 3.75}$$

Question 3 $\boxed{3.9-158}$ The article "Statistical Behavior Modeling for Driver-Adaptive Precrash Systems" (IEEE trans. on Intelligent Transp. Systems, 2013 : 1-9) Proposed the following distribution for modeling the behavior of what the author called "the criticality level of a situation' x.

$$f(x; \lambda_1, \lambda_2, p) = \begin{cases} p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Question 3 | 3.9-158 | This is often called the hyperexponential or mixed exponential distribution.

a) what is the cdf $F(x; \lambda_1, \lambda_2, p)$?

b) If $p = 0.5$, $\lambda_1 = 40$, $\lambda_2 = 200$ (values of the λs suggested in the cited article), calculate $P(X > 0.01)$.

c) If X has $f(x; \lambda_1, \lambda_2, p)$ as its pdf, what is $E(X)$?

d) Using the fact that $E(X^2) = 2/\lambda^2$ when X has an exponential distribution with parameter λ, compute $E(X^2)$ when X has pdf $f(x; \lambda_1, \lambda_2, p)$. Then compute $Var(X)$.

e) The coefficient of variation of a random variable (or distribution) is $CV = \sigma/\mu$. What is the CV for an exponential rv? What can you say about the value of CV when X has an hyperexponential distribution?

f) What is the CV for an Erlang distribution with parameters λ and n as defined in Section 3-4? [Note: In applied work, the sample CV is used to decide which of the three distributions might be appropriate.]

g) For the parameter values given in (b), calculate the probability that X is within one standard deviation of its mean value. Does this probability depend upon the values of the λs (it does not depend on λ when X has an exponential distribution)?

Answer | a) $F(x; \lambda_1, \lambda_2, p) = \int_0^x f(y; \lambda_1, \lambda_2, p)\, dy$

$$= \int_0^x \left[p\lambda_1 e^{-\lambda_1 y} + (1-p)\lambda_2 e^{-\lambda_2 y}\right] dy = p\lambda_1 \left(\frac{e^{-\lambda_1 y}}{-\lambda_1}\right)\Bigg|_0^x + (1-p)\lambda_2 \left(\frac{e^{-\lambda_2 y}}{-\lambda_2}\right)\Bigg|_0^x$$

$$= -p\left(e^{-\lambda_1 x} - 1\right) + (1-p)\left[\frac{e^{-\lambda_2 x} - 1}{-1}\right] = p\left(1 - e^{-\lambda_1 x}\right) + (1-p)\left(1 - e^{-\lambda_2 x}\right)$$

$$F(x; \lambda_1, \lambda_2, p) = \begin{cases} 0 \; ; & x < 0 \\ p\left(1 - e^{-\lambda_1 x}\right) + (1-p)\left(1 - e^{-\lambda_2 x}\right)\Big] \; ; & x \geq 0 \end{cases}$$

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Question 3 | 3.9 - 158

(b) $P = 0.5, \ \lambda_1 = 40 \qquad \lambda_2 = 200$

$$P(x > 0.01) = (1-P)(x \le 0.01) = 1 - F(0.01) = 1 - \left[0.5\left(1 - e^{-40(0.01)}\right) + 0.5\left(1 - e^{-200(0.01)}\right)\right]$$

$$P(x > 0.01) = \underline{\underline{0.4028}}$$

(c) let x_1 and x_2 be expo rv with parameters λ_1 and λ_2

$$\boxed{x = p x_1 + (1-p) x_2} \qquad E(x) = p\,E(x_1) + (1-p)\,E(x_2)$$

$$E(x_1) = \frac{1}{\lambda_1} \quad \text{and} \quad E(x_2) = \frac{1}{\lambda_2}$$

$$\boxed{E(x) = \frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}}$$

(d) $E(x^2) = \dfrac{2}{\lambda_2}$

By definition, $E(x^2) = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 \left[p\lambda_1 e^{\lambda_1 x} + (1-p)\lambda_2 e^{\lambda_2 x}\right] dx$

$$= p\,\lambda_1 \int_0^\infty x^2 e^{-\lambda_1 x} dx + (1-p)\lambda_2 \int_0^\infty x^2 e^{-\lambda_2 x} dx$$

$$E(x^2) = P\,E(x_1^2) + (1-p)E(x_2^2)$$

$$E(x^2) = \frac{p^2}{\lambda_1^2} + \frac{(1-p)^2}{\lambda_2^2} = \frac{2p}{\lambda_1^2} + \frac{2(1-p)}{\lambda_2^2}$$

$$\boxed{V(x) = E(x^2) - \left[E(x)\right]^2 = \frac{2p}{\lambda_1^2} + \frac{2(1-p)}{\lambda_2^2} - \left(\frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}\right)^2}$$

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Question 3 (3.9 - 158)

(e) $c.v = \frac{\sigma}{\mu} = \frac{1/\lambda}{1/\lambda} = 1$ c.v for exp. dist $= 1$

use (c) and (d) to find cv of hyper exponential dist

$$\frac{\sigma}{\mu} = \frac{\sqrt{\frac{2p}{\lambda_1^2} + \frac{2(1-p)}{\lambda_2^2} - \left(\frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}\right)^2}}{\frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}} = \sqrt{\frac{\frac{2p}{\lambda_1^2} + \frac{2(1-p)}{\lambda_2^2} - \left(\frac{p\lambda_2 + \lambda_1(1-p)}{\lambda_1\lambda_2}\right)^2}{\left(\frac{p\lambda_2 + \lambda_1(1-p)}{\lambda_1\lambda_2}\right)^2}}$$

$$= \sqrt{\frac{\frac{2p\lambda_2^2 + 2(1-p)\lambda_1^2}{\cancel{\lambda_1^2\lambda_2^2}}}{\frac{p\lambda_2 + \lambda_1(1-p)}{\cancel{\lambda^2\lambda_2^2}}} - 1} = \sqrt{2\frac{p\lambda_2^2 + (1-p)\lambda_1^2}{(p\lambda_2 + \lambda_1(1-p))^2} - 1}$$

$$\frac{\sigma}{\mu} \geq \sqrt{2(1)-1} \Rightarrow \frac{\sigma}{\mu} \geq 1$$

(f) for Erlang distb $cv. = \frac{\sigma}{\mu} = \frac{\sqrt{n}/\lambda}{n/\lambda}$ $c.v = \frac{1}{\sqrt{n}}$

(g) no solution.

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Question 4 (3.9 - 168) Let X have the pdf $f(x) = 1/[\pi(1+x^2)]$ for $-\infty < x < \infty$ (a central Cauchy distribution), and show that $Y = 1/X$ has the same distribution. [Hint: Consider $P(|Y| \le y)$, the cdf of $|Y|$, then obtain its pdf and show it is identical to the pdf of $|X|$.]

Answer $x = 0$ $(-\infty, 0), (0, \infty)$

$y =) (0, -\infty)$

$$f_Y(y) = f_X[h(y)][h'(y)]$$

$$f[h(x)] = \frac{1}{\pi}\left(\frac{y^2}{1+y^2}\right) \quad -\infty < y < 0$$

$$h'(y) = \frac{-1}{y^2}$$

$$\boxed{f_Y(y) = \left(\frac{1}{\pi}\right)\left(\frac{\cancel{y^2}}{1+y^2}\right)\left(\frac{1}{\cancel{y^2}}\right) = \left(\frac{1}{\pi}\cdot\frac{1}{y^2+1}\right)}$$

Question 4 (3.9 - 169) Let X have a Weibull distribution with shape parameter α and scale parameter β. Show that the transformed variable $Y = \ln(X)$ has an extreme value distribution as defined in Section 3-6, with $\theta_1 = \ln(\beta)$ and $\theta_2 = 1/\alpha$.

Answer Weibull CDF

$$F(x) = 1 - e^{-(x/\beta)^\alpha} \quad x \ge 0$$

Extreme value CDF

$$F(z) = 1 - e^{(z-\theta_1)/\theta_2} \quad -\infty < z < \infty$$

Show $Y = \ln(x) \sim$ Extreme Value $(\theta_1 - \ln(\beta), \theta_2 = 1/\alpha)$

$$F(y) = P(Y \le y) = P(\ln(x) \le y) = P(x \le e^y) = 1 - e^{-(e^y/\beta)^\alpha}$$

$$= 1 - e^{-y^\alpha/\beta^\alpha}$$

$$F(z) = 1 - e^{-[(z-\ln(\beta))/1/\alpha]} = 1 - e^{-e^{\alpha z - \alpha\ln(\beta)}} = 1 - e^{-e^{\alpha z}/e^{\alpha\ln(\beta)}}$$

$$= 1 - e^{-e^{\alpha z}/[e^{\ln(\beta)}]^\alpha} = 1 - e^{-\alpha z/\beta^\alpha}$$

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$$\boxed{Y \sim \text{Ext. val. Dist. with } \theta_1 = \ln(\beta),\ \theta_2 = 1/\alpha}$$

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Question 5 Summary of 3.8 - Simulation of Continuous Random Variables from the textbook

The Inverse CDF Method:

Theorem: Consider any continuous distribution with pdf f and cdf F. Let $U \sim$ Unif $[0,1)$, and define a random variable X by

$X = F^{-1}(U)$ Then the pdf of X is f.

It is desired to simulate n values from a distribution with pdf $f(x)$. Let $F(x)$ be the corresponding cdf. Repeat n times:

1. Use a random-number generator (RNG) to produce a value, u, from $[0,1)$.
2. Assign $x = F^{-1}(u)$.

Then resulting values $x_1, \ldots, x_n$ form a simulation of a random variable with the original pdf, $f(x)$.

Accept-Reject Method:

It is desired to simulate n values from a distribution with pdf $f(x)$. Let $g(x)$ be some other pdf such that the ratio f/g is bounded, i.e., there exists a constant c such that $f(x)/g(x) \le c$ for all x. (The constant c is sometimes called the majorization constant.) proceed as follows:

1. Generate a variate, y, from the distribution g. This value y is called a candidate.
2. Generate a standard uniform variate, u.
3. If $u \cdot c \cdot g(y) \le f(y)$, then assign $x = y$ (i.e., "accept" the candidate). Otherwise, discard ("reject") y and return to step 1.

These steps are repeated until n candidate values have been accepted. The resulting accepted values $x_1, \ldots, x_n$ form a simulation of a random variable with the original pdf, $f(x)$.

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Question 5 | (3.8 - 137) The half-normal distribution has the following pdf:

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}} \cdot e^{-x^2/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is distribution of $|Z|$, where $Z \sim N(0,1)$; equivalently, it's the pdf that arises by "folding" the standard normal distribution in half along its line of symmetry. Consider simulating values from this distribution using the accept-reject method with a candidate distribution $g(x) = e^{-x}$ for $x \geq 0$ (i.e., an exponential pdf with $\lambda = 1$).

a) Find the inverse cdf corresponding to $g(x)$. (This will allow us to simulate values from the candidate distribution.)

b) Find the smallest majorization constant c so that $f(x)/g(x) \leq c$ for all $x \geq 0$. [Hint: Use calculus to determine where the ratio $f(x)/g(x)$ is maximized.]

c) On the average, how many candidate values will be required to generate 10.000 "accepted" values?

d) Write a program to construct 10,000 values from a half-normal distribution.

Answer |

a)
$$f(x) = \begin{cases} \frac{2}{\pi} e^{-x^2/2} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases} \qquad g(x) = e^{-x},\ x \geq 0$$

$$G(x) = 1 - e^{-x}, \quad x \geq 0$$

$$u = G(x) \Rightarrow u = 1 - e^{-x} \Rightarrow x = -\ln(1-u) = G^{-1}(x)$$

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16/03/2023

Question 5 $\boxed{3.8-137}$

(b) $\dfrac{f(x)}{g(x)} = \sqrt{\dfrac{2}{\pi}}\, e^{-\frac{x^2}{2}+x}$ $\qquad \dfrac{d}{dx} - \dfrac{x^2}{2} + x = 0$

$\Rightarrow -x+1=0 \Rightarrow x=1$

$c = \sqrt{\dfrac{2}{\pi}}\, e^{\frac{1}{2}} = 1.3165$

(c) $n \cdot c = (10000)(1.3155) = 13155$

(d) $u \cdot c \cdot g(y) \le f(y) \Rightarrow u \le \dfrac{f(y)}{g(y)} \cdot \dfrac{1}{c} \Rightarrow u \le e^{-(y-1)^2/2}$ (*)

1. draw from $g(x) = y$
2. draw from u
3. accept $x = y$ if (*) holds

```
x <- NULL
i <- 0
while (i < 10000) {
    y <- -1 * log(1-runif(1))
    u <- runif(1)
    if (u <= (1/4) * y^2 * (3-y)) {
        i <- i+1
        x[i] <- y
    }
}
```

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CamScanner ile tarandı