CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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1. (a) From graph, this signal's period is 4.Therefore,

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

The general formula for spectral coefficients is

$$\frac{1}{N} \sum_{n=<N>} x[n](e^{-jk\omega_0 n})$$

Over one period of summation ([0,3] interval is choosen) is:

$$\frac{1}{4}x[0] + \frac{1}{4}x[1]e^{-jk\omega_0} + \frac{1}{4}x[2]e^{-2jk\omega_0} + \frac{1}{4}x[3]e^{-3jk\omega_0}$$

.

$$0 + \frac{1}{4}e^{-jk\omega_0} + \frac{1}{2}e^{-2jk\omega_0} + \frac{1}{4}e^{-3jk\omega_0}$$

$$0 + \frac{1}{4}(cos(k\pi/2) - jsin(k\pi/2)) + \frac{1}{2}(cos(k\pi) - jsin(k\pi)) + \frac{1}{4}(cos(3k\pi/2) - jsin(3k\pi/2))$$

Thus,

$$a_0 = 0 + 1/4 + 1/2 + 1/4 = 1$$
$$a_1 = 0 - j/4 - 1/2 + j/4 = -1/2$$
$$a_2 = 0 - 1/4 + 1/2 - 1/4 = 0$$
$$a_3 = 0 + j/4 - 1/2 - j/4 = -1/2$$

The magnitude of spectral coefficients:

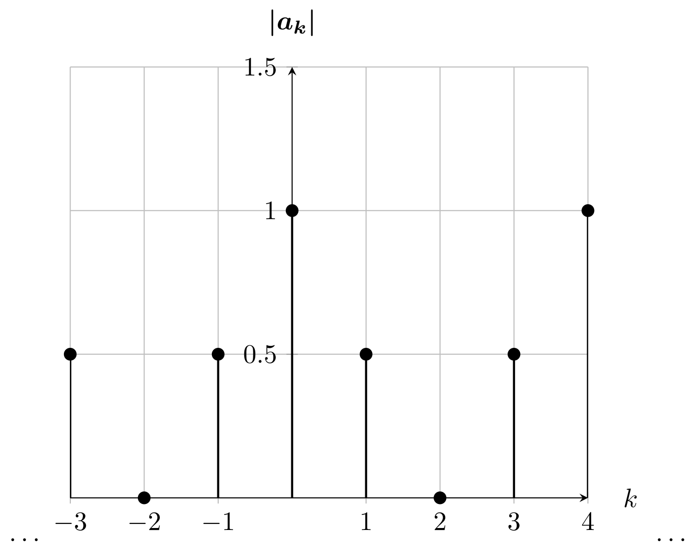

Figure 1: k vs. $|a_k|$.

Phase of the spectral coefficients:

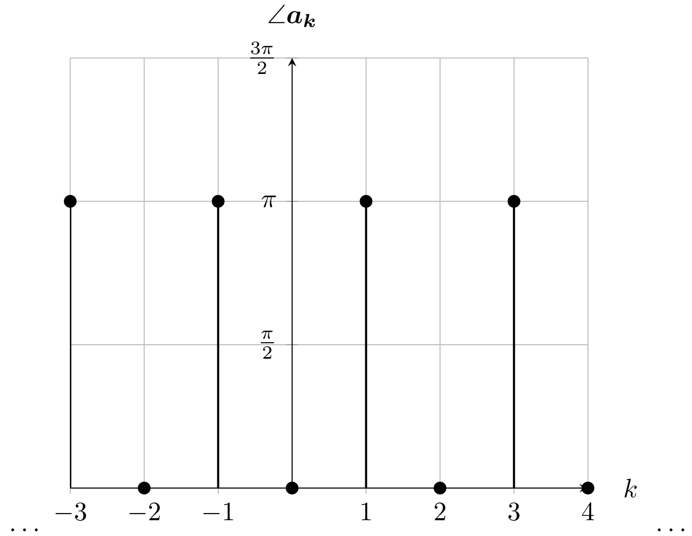

Figure 2: k vs. $\angle a_k$.

(b) i. $y[n]$ graph is the same as $x[n]$ graph at 3 points in a period. At only 1 point where n equals to $(4k\ -1)$ and k is in interval $(-\infty\ ,\ \infty)$, these 2 graphs are different in a period.
For these n values, $x[n]$ is 1 and $y[n]$ is 0.
Therefore, in order to define $y[n]$ in terms of $x[n]$, 1 should be subtracted from x in these n values.

$$\sum_{k=-\infty}^{\infty} \delta[n\ -(4k-1)]\ =\ \sum_{k=-\infty}^{\infty} \delta[n+1\ -4k]$$

is the required term.

Definition of $y[n]$ in terms of $x[n]$ is:

$$y[n]\ =\ x[n]\ -\ \sum_{k=-\infty}^{\infty} \delta[n+1\ -4k]$$

ii. From graph, this signal's period is 4.Therefore,

$$\omega_0\ =\ \frac{2\pi}{4}\ =\ \frac{\pi}{2}$$

The general formula for spectral coefficients is

$$\frac{1}{N}\sum_{n=<N>} x[n](e^{-jk\omega_0 n})$$

Over one period of summation ([0,3] interval is choosen) is:

$$\frac{1}{4}x[0]\ +\ \frac{1}{4}x[1]e^{-jk\omega_0}\ +\ \frac{1}{4}x[2]e^{-2jk\omega_0}\ +\ \frac{1}{4}x[3]e^{-3jk\omega_0}$$

.

$$0\ +\ \frac{1}{4}e^{-jk\omega_0}\ +\ \frac{1}{2}e^{-2jk\omega_0}\ +\ 0$$

$$0\ +\frac{1}{4}(cos(k\pi/2)-jsin(k\pi/2))\ +\frac{1}{2}(cos(k\pi)-jsin(k\pi))\ +0$$

Thus,

$$a_0\ =\ 0\ +1/4\ +\ 1/2\ +\ 0\ =\ 3/4$$
$$a_1\ =\ 0\ -j/4\ -\ 1/2\ +\ 0\ =\ -j/4\ -1/2$$
$$a_2\ =\ 0\ -1/4\ +\ 1/2\ +\ 0\ =\ 1/4$$
$$a_3\ =\ 0\ +j/4\ -\ 1/2\ +\ 0\ =\ j/4\ -1/2$$

The magnitude of spectral coefficients:

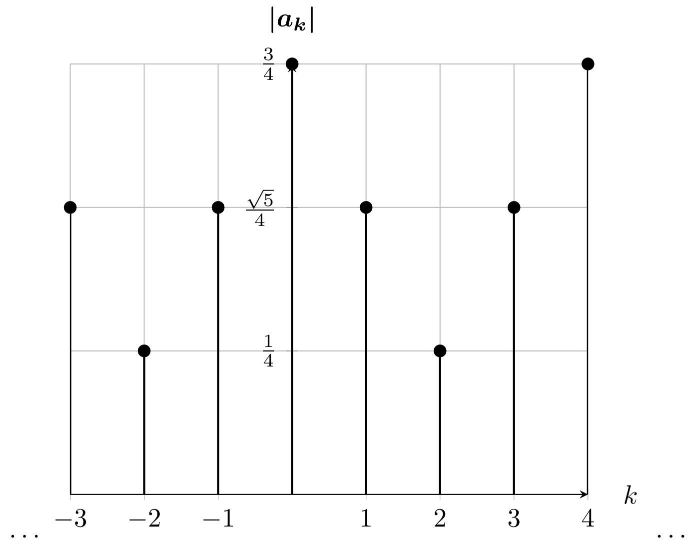

Figure 3: k vs. $|a_k|$.

Phase of spectral coefficients:

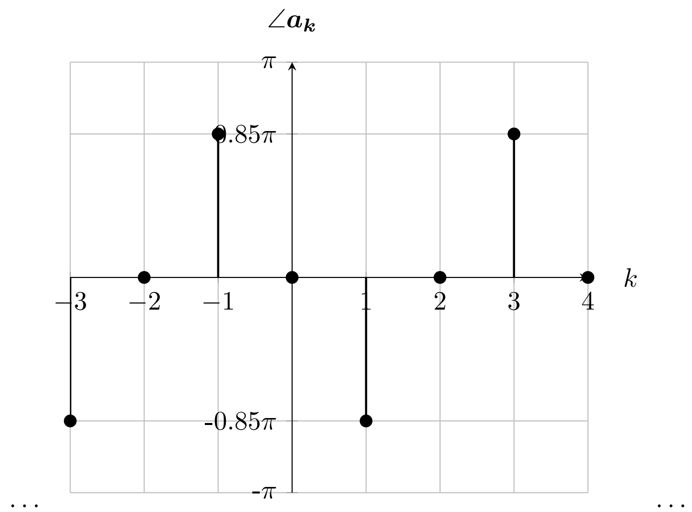

Figure 4: k vs. $\angle a_k$.

2. This discrete signal has following properties:

$\rightarrow$ Its period should be 4.
So

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$\rightarrow$ From (-3) to 4 , there are 2 full periods.Instead of that interval , if [0,3] interval is taken

$$\sum_{n=0}^{3} x[n] = 4$$

$$x[0] + x[1] + x[2] + x[3] = 4$$

$\rightarrow$ Since periodicity is 4 ,

$$a_{-3} = a_1$$

$$a_3 = a_{11}$$

.
$\rightarrow$ One of the a_k is 0.
$\rightarrow$ From e)

$$e^{-j\pi k/2} = cos(-\pi k/2) + jsin(-\pi k/2) = cos(\pi k/2) - jsin(\pi k/2)$$
$$e^{-j\pi 3k/2} = cos(-\pi 3k/2) + jsin(-\pi 3k/2) = cos(\pi 3k/2) - jsin(\pi 3k/2)$$

$\pi/2$ and $3\pi/2$ are related frequency components:

$$sin(\pi 3k/2) \ = \ sin(\pi 3k/2 \ - 2\pi k) \ = \ sin(-\pi k/2) \ = \ -sin(\pi k/2)$$

$$cos(\pi 3k/2) \ = \ cos(\pi 3k/2 \ - 2\pi k) \ = \ cos(-\pi k/2) \ = \ cos(\pi k/2)$$

Thus ,

$$e^{-j\pi k/2}e^{-j\pi 3k/2}$$
$$= \ cos(\pi k/2) - jsin(\pi k/2) \ + \ cos(\pi 3k/2) - jsin(\pi 3k/2)$$
$$= \ cos(\pi k/2) - jsin(\pi k/2) \ + \ cos(\pi k/2) + jsin(\pi 3k/2)$$
$$= \ 2cos(\pi k/2)$$

Hence ,

$$\sum_{k=0}^{3} x[k](e^{-j\pi k/2}e^{-j\pi 3k/2}) \ = \ \sum_{k=0}^{3} x[k](2cos(\pi k/2)) \ = \ 4$$

For $k \ = 1$ and $k \ = 3$, $2cos(\pi k/2) \ = \ 0$.Then

$$2x[0] \ - \ 2x[2] \ = \ 4$$
$$x[0] \ - \ x[2] \ = \ 2$$

These properties are deduced from given conditions.

The general formula for a_k's is

$$\frac{1}{N}\sum_{n=<N>} x[n](e^{-jk\omega_0 n})$$

In this case , period is 4 and over one period of summation ([0,3] interval is choosen) is:

$$\frac{1}{4}x[0] \ + \ \frac{1}{4}x[1]e^{-jk\omega_0} \ + \ \frac{1}{4}x[2]e^{-2jk\omega_0} \ + \ \frac{1}{4}x[3]e^{-3jk\omega_0}$$

.
Since $\omega_0 = \pi/2$,

$$a_0 \ = \ \frac{1}{4}x[0] \ + \ \frac{1}{4}x[1] \ + \ \frac{1}{4}x[2] \ + \ \frac{1}{4}x[3]$$

.

$$a_0 \ = \frac{1}{4}(\ x[0] \ + \ x[1] \ + \ x[2] \ + \ x[3] \ )$$

This sum in the paranthesis is known from part(b) and it is 4.Therefore,

$$a_0 \ = \ 1$$

It is given that a_1 and a_3 are conjugate complex numbers.
Let a_1 be $x \ + \ jy$.
Then a_3 will be a_1 be $x \ - \ jy$.
$|a_1 \ - \ a_3| \ = \ |2yj| \ = 1 \rightarrow$ at least imaginary parts of these coefficients are exist.
Thus , a_0 ,a_1 and a_3 cannot be 0.However, it is given that one coeeficient is 0.That means

$$a_2 \ = \ 0$$

By the way ,

$$a_2 \ = \frac{1}{4}(\ x[0] \ - \ x[1] \ + \ x[2] \ - \ x[3] \ )$$

From general formula of a_k's , a_1 can be derived that

$$a_1 \ = \frac{1}{4}x[0] \ + \ \frac{1}{4}x[1]e^{-j\omega_0} \ + \ \frac{1}{4}x[2]e^{-2j\omega_0} \ + \ \frac{1}{4}x[3]e^{-3j\omega_0}$$

.

$$= \ \frac{1}{4}x[0] \ + \ \frac{1}{4}x[1](cos(\pi/2) - jsin(\pi/2)) \ + \ \frac{1}{4}x[2](cos(\pi) - jsin(\pi)) \ + \ \frac{1}{4}x[3](cos(3\pi/2) - jsin(3\pi/2))$$

$$= \frac{1}{4}x[0] \ - \ \frac{1}{4}jx[1] \ - \ \frac{1}{4}x[2] \ + \ \frac{1}{4}jx[3]$$

Similarly , a_3 is:

$$a_3 \ = \frac{1}{4}x[0] \ + \ \frac{1}{4}x[1]e^{-3j\omega_0} \ + \ \frac{1}{4}x[2]e^{-6j\omega_0} \ + \ \frac{1}{4}x[3]e^{-9j\omega_0}$$

.

$$= \frac{1}{4}x[0] \ + \ \frac{1}{4}x[1](cos(3\pi/2) - jsin(3\pi/2)) \ + \ \frac{1}{4}x[2](cos(3\pi) - jsin(3\pi)) \ + \ \frac{1}{4}x[3](cos(9\pi/2) - jsin(9\pi/2))$$

$$= \frac{1}{4}x[0] \ + \ \frac{1}{4}jx[1] \ - \ \frac{1}{4}x[2] \ - \ \frac{1}{4}jx[3]$$

From part b and previous calculations, it is known that

$$|a_1 \ - \ a_3| \ = \ 1$$

Put the values of a_1 and a_3 ,

$$|\frac{1}{4}(2j(x[3] \ - \ x[1]))| = \ 1$$

$$|\frac{1}{2}j(x[3] \ - \ x[1])| = \ 1$$

$$|j(x[3] \ - \ x[1]) \ | = \ 2$$

Thus,

$$|x[3] \ - \ x[1]| \ = \ 2$$

In this point both $(x[3] \ - \ x[1])$ or $(x[1] \ - \ x[3])$ can be 2. Let me choose $(x[1] \ - \ x[3]) = 2$.
Recall that

$$a_0 \ = \frac{1}{4}(\ x[0] \ + \ x[1] \ + \ x[2] \ + \ x[3] \ ) \ = \ 1$$

$$a_2 \ = \frac{1}{4}(\ x[0] \ - \ x[1] \ + \ x[2] \ - \ x[3] \ ) \ = \ 0$$

If inside the paranthesis of a_0 and a_2 are :

$$x[0] \ + \ x[1] \ + \ x[2] \ + \ x[3] \ = \ 4$$

$$x[0] \ - \ x[1] \ + \ x[2] \ - \ x[3] \ = \ 0$$

If these values are added:

$$2x[0] + 2x[2] \ = \ 4 \rightarrow x[0] \ + x[2] \ = \ 2$$

From e and previous calculations,

$$x_0 \ - x_2 \ = \ 2$$

Thus ,

$$x[0] \ = 2$$

$$x[2] \ = 0$$

Placing this values into equation $(x[0] \ + \ x[1] \ + \ x[2] \ + \ x[3] \ = \ 4)$,

$$2 \ + \ x[1] \ + \ 0 \ + \ x[3] \ = \ 4$$

$$x[1] \ + \ x[3] \ = \ 2$$

From previous calculations,

$$x[1] \ - \ x[3] \ = \ 2$$

Thus,

$$x[1] \ = 2$$

$$x[3] \ = 0$$

Whole values are:

$$x[0] \ = 2 \ , \ x[1] \ = 2 \ , \ x[2] \ = 0 \ , \ x[3] \ = 0$$

Graph of $x[n]$ is:

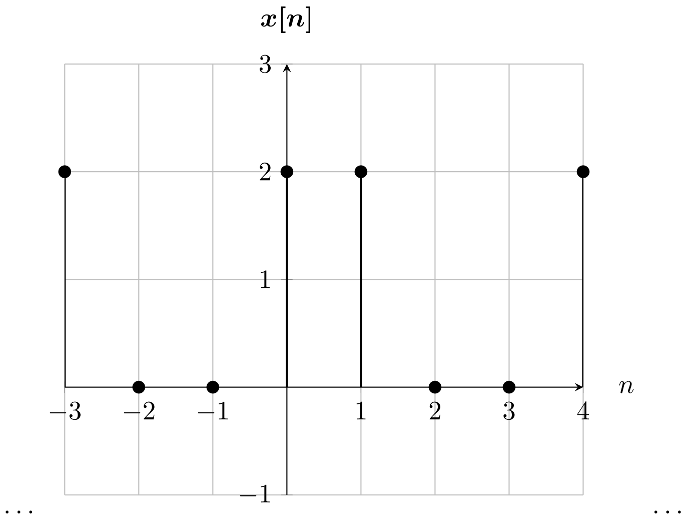

Figure 5: n vs. $x[n]$.

3.

$$
\begin{aligned}
&Normally\ y(t) \ = \ h(t) * x(t)\\
&In\ the\ question\ it\ is\ said\ that\ x(t) \ = \ h(t) * [x(t) + r(t)]\\
&When\ we\ apply\ Fourier\ Transform\ to\ this,\ we\ get\\
&X(jw) = H(jw).[X(jw) + R(jw)]\\
&X(jw) = H(jw).X(jw) + H(jw).R(jw)\\
&R(jw) = 0,\ as\ r(t)\ is\ composed\ of\ only\ high\ frequency\ components.\ Therefore,\\
&X(jw) = H(jw).X(jw)\\
&Hence,\ H(jw) = 1\ for\ |w| \leq K2\pi/T\\
&Now\ we\ need\ to\ find\ h(t)\ by\ applying\ inverse\ Fourier\ Transform\ to\ H(jw)\\
&h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)e^{jwt}dw
\end{aligned}
$$

$$
\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-K2\pi/T}^{K2\pi/T} e^{jwt}dw\\
&= \frac{1}{2\pi}.\frac{e^jwt}{jt}|_{-K2\pi/T}^{K2\pi/T}\\
&= \frac{1}{2\pi}(\frac{e^{jK2\pi t/T}}{jt} - \frac{e^{-jK2\pi t/T}}{jt})\\
&= \frac{1}{\pi t}(\frac{e^{jK2\pi t/T} - e^{-jK2\pi t/T}}{2j})\\
So\ h(t) &= \frac{1}{\pi t}.sin(\frac{K2\pi t}{T})
\end{aligned}
$$

4. (a)

$$
\begin{aligned}
&First\ of\ all,\ when\ we\ give\ x(t) = e^{jwt}\ to\ a\ system,\ we\ get\ y(t) = H(jw).e^{jwt}\\
&Secondly,\ the\ differential\ equation\ that\ represents\ this\ system\ is\ y''(t) = x(t) - 6y(t) - 5y'(t) + 4x'(t)\\
&Now\ put\ the\ corresponding\ values\ we\ found\ in\ the\ first\ part\ to\ this\ equation\\
&H(jw).(jw)^2.e^{jwt} = e^{jwt} - 6H(jw).e^{jwt} - 5H(jw).jw.e^{jwt} + 4jw.e^{jwt}\\
&We\ need\ to\ find\ H(jw),\ so\ put\ the\ parts\ containing\ H(jw)\ to\ the\ left\ side\ of\ the\ equality.\\
&H(jw).(jw)^2.e^{jwt} + 6H(jw).e^{jwt} + 5H(jw).jw.e^{jwt} = e^{jwt} + 4jw.e^{jwt}\\
&H(jw)e^{jwt}.[(jw)^2 + 5jw + 6] = e^{jwt}.[4jw + 1]\\
&So,\ frequency\ response\ is\ H(jw) = \frac{4jw+1}{(jw)^2 + 5jw + 6} = \frac{4jw+1}{(3+jw)(2+jw)}
\end{aligned}
$$

(b)

In order to find the impulse response of this system, we need to apply inverse Fourier Transform to the freque

$$H(jw) = \frac{4jw+1}{(3+jw)(2+jw)} = \frac{A}{3+jw} + \frac{B}{2+jw}$$

$$2A + Ajw + 3B + Bjw = 4jw + 1$$

$$2A + 3B = 1$$

$$A + B = 4$$

Then, $A = 11$ *and* $B = -7$

$$H(jw) = 11.\frac{1}{3+jw} - 7.\frac{1}{2+jw}$$

After applying inverse Fourier Transform to $H(jw)$*, we get*

$$\begin{aligned} h(t) &= 11e^{-3t}u(t) - 7e^{-2t}u(t) \quad (e^{-|a|t}u(t) \longleftrightarrow \frac{1}{|a|+jw}) \\ &= (11e^{-3t} - 7e^{-2t})u(t) \end{aligned}$$

(c)

Since $Y(jw) = H(jw).X(jw)$*, we first need to find the Fourier Transform of* $x(t)$*.*
Then, we will calculate $Y(jw)$ *and apply inverse Fourier Transform to it.*
Apply F.T

$$x(t) = \frac{1}{4}e^{\frac{-t}{4}}u(t) \longleftrightarrow X(jw) = \frac{1}{4}.\frac{1}{\frac{1}{4}+jw} = \frac{1}{1+4jw} \quad (e^{-|a|t}u(t) \longleftrightarrow \frac{1}{|a|+jw})$$

$$\begin{aligned} Y(jw) &= H(jw).X(jw) \\ &= \frac{4jw+1}{(3+jw)(2+jw)}.\frac{1}{1+4jw} \\ &= \frac{1}{(3+jw)(2+jw)} \\ &= \frac{1}{2+jw} - \frac{1}{3+jw} \end{aligned}$$

Apply inverse F.T

$$\begin{aligned} y(t) &= e^{-2t}u(t) - e^{-3t}u(t) \\ &= (e^{-2t} - e^{-3t})u(t) \end{aligned}$$