CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 1

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1. (a)

$$y[n] = 2x[n] + \frac{3}{4}.y[n-1] - \frac{1}{8}.y[n-2]$$
$$y[n] - \frac{3}{4}.y[n-1] + \frac{1}{8}.y[n-2] = 2x[n]$$

(b) Take Fourier Transform of the difference equation.

$$\frac{1}{8}e^{-2jw}Y(e^{jw}) - \frac{3}{4}e^{-jw}Y(e^{jw}) + Y(e^{jw}) = 2X(e^{jw})$$
$$Y(e^{jw})\ .\ [\frac{1}{8}e^{-2jw} - \frac{3}{4}e^{-jw} + 1] = 2X(e^{jw})$$
$$H(e^{jw}) = \frac{16}{e^{-2jw} - 6e^{-jw} + 8}$$

(c)

$$H(e^{jw}) = \frac{16}{e^{-2jw} - 6e^{-jw} + 8} = \frac{16}{(e^{-jw} - 4)(e^{-jw} - 2)} = \frac{A}{e^{-jw} - 4} + \frac{B}{e^{-jw} - 2}$$
$$A = 8 \quad B = -8$$
$$H(e^{jw}) = \frac{8}{e^{-jw} - 4} - \frac{8}{e^{-jw} - 2}$$
$$= \frac{4}{1 - \frac{1}{2}e^{-jw}} - \frac{2}{1 - \frac{1}{4}e^{-jw}}$$
$$Take\ inverse\ FT$$
$$h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$$

(d)

$$
\begin{aligned}
Fourier\ transform\ of\ x[n] &= (\frac{1}{4})^n u[n]\ is\ \frac{1}{1-\frac{1}{4}e^{-jw}} \\
y[n] = h[n] * x[n] &\longleftrightarrow Y(e^{jw}) = H(e^{jw}).X(e^{jw}) \\
Y(e^{jw}) &= (\frac{8}{e^{-jw}-4} - \frac{8}{e^{-jw}-2}).\frac{1}{1-\frac{1}{4}e^{-jw}} \\
&= (\frac{8}{e^{-jw}-4} - \frac{8}{e^{-jw}-2}).\frac{4}{4-e^{-jw}} \quad let\ z = e^{-jw} \\
&= (\frac{8}{z-4} - \frac{8}{z-2}).\frac{4}{4-z} \\
&= (\frac{8}{z-2} - \frac{8}{z-4}).\frac{4}{z-4} \\
&= \frac{32}{(z-2)(z-4)} - \frac{32}{(z-4)^2} \\
&= \frac{16}{z-4} - \frac{16}{z-2} - \frac{32}{(z-4)^2} \\
&= \frac{-4}{1-\frac{1}{4}e^{-jw}} + \frac{8}{1-\frac{1}{2}e^{-jw}} - \frac{2}{(1-\frac{1}{4}e^{-jw})^2}
\end{aligned}
$$

$Take\ inverse\ Fourier\ transform$

$$y[n] = -4(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n] \ - \ 2(n+1)(\frac{1}{4})^n u[n]$$

2. If two LTI systems are connected parallel, then overall system's impulse response is

$$h[n] = h_1[n] + h_2[n]$$

and frequency response is

$$H(e^{jw}) = H_1(e^{jw}) + H_2(e^{jw})$$

In this question, overall system's frequeny response $H(e^{jw})$ and impulse response of first system $h_1[n]$ are given.In order to find $h_2[n]$,
1-Frequeny response of $h_1[n]$ should be found,
2-Frequeny response of $h_2[n]$ which is $H_2(e^{jw})$ should be found using formula

$$H_2(e^{jw}) \ = \ H(e^{jw}) \ - \ H_1(e^{jw})$$

3-By applying inverse fourier transformation, find $h_2[n]$.

1-Frequency response of

$$h_1[n] = (\frac{1}{3})^n u[n]$$

$$H_1(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}}$$

$$H_1(e^{jw}) = \frac{-3}{e^{-jw}-3}$$

2-Frequeny response of $h_2[n]$,

$$H_2(e^{jw}) \ = \ H(e^{jw}) \ - \ H_1(e^{jw})$$

$$H_2(e^{jw}) \ = \ \frac{5e^{-jw}-12}{e^{-2jw}-7e^{-jw}+12} \ - \ \frac{-3}{e^{-jw}-3}$$

$$H_2(e^{jw}) \ = \ \frac{5e^{-jw}-12}{(e^{-jw}-4)(e^{-jw}-3)} \ - \ \frac{-3}{e^{-jw}-3}$$

$$H_2(e^{jw}) \ = \ \frac{-2}{1 \ - \ \frac{1}{4}e^{-jw}}$$

3-Impulse response of $H_2(e^{jw})$ by using inverse FT,

$$h_2[n] \ = \ -2(\frac{1}{4})^n u[n]$$

3. (a) Input signal x(t) consists of a periodic cosine function and an aperiodic square wave which can be seperated as

$$x(t) \; = \; x_1(t) \; + \; x_2(t)$$

where

$$x_1(t) \; = \frac{sin2\pi t}{\pi t}$$

$$x_2(t) \; = cos3\pi t$$

By taking fourier transformation,

$$F\{x_1(t)\} = X_1(jw) = \begin{cases} 1, & \text{if } |w| < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$X_2(jw) \; = \; \pi[\; \delta(\; \omega \; - \; 3\pi \;) \; + \; \delta(\; \omega \; + \; 3\pi \;)]$$

Thus, fourier transform of $x(t)$,

$$X(jw) \; = \; X_1(jw) \; + \; X_2(jw)$$

And the graph of $X(jw)$,

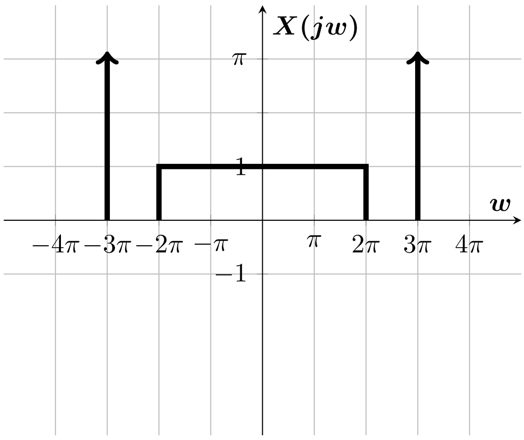

Figure 1: w vs. $X(jw)$.

(b) From part a), Nyquist frequency is 3π.
Since

$$w_s = 2 \times w_m$$

$$w_s = 6\pi$$

Nyquist rate is 6π.
Sampling rate is

$$T \; = \; \frac{2\pi}{6\pi} \; = \; \frac{1}{3}$$

(c) By using formula,

$$X_p(jw) \; = \; \frac{1}{T} \sum_{k= \; -\infty}^{\infty} X(j(\omega \; - \; k\omega_s))$$

$$X_p(jw) \; = \; 3 \sum_{k= \; -\infty}^{\infty} X(j(\omega \; - \; 6\pi k))$$

(Graph of this question is below, top of the next page)

4. (a)

$$N = \frac{2\pi}{w_s} = 2$$

$$X_p(jw) = \frac{1}{T} \sum_{\forall k} X(j(w - kw_s))$$

$$X_d(e^{jw}) = X_p(j\frac{w}{T})$$

$$X_d(e^{jw}) = \begin{cases} \frac{2}{\pi}w & if \; |w| \leq \frac{\pi}{2} \\ 0 & otherwise \end{cases} \quad , X_d(e^{jw}) = X_d(e^{j(w+N)})$$

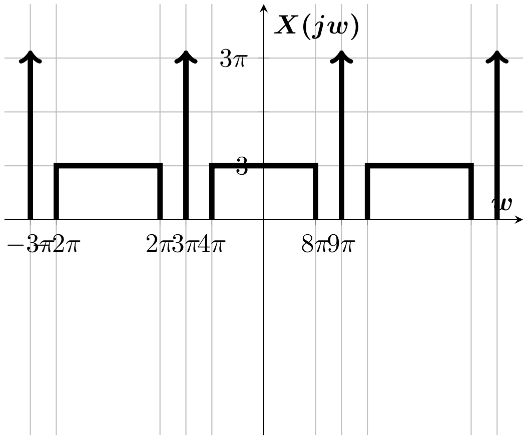

Figure 2: w vs. $X(j(w-6\pi k))$.

(b) From discrete time Fourier transform table, we know that

$$e^{jw_0n} \longleftrightarrow 2\pi \sum_{\forall k} \delta(w - w_0 - 2\pi k)$$
$$h[n] = cos\pi n = \frac{1}{2}(e^{j\pi n} + e^{-j\pi n})$$
$$Then,\ H(e^{jw}) = \frac{1}{2}(2\pi \sum_{\forall k} \delta(w - \pi - 2\pi k) + 2\pi \sum_{\forall k} \delta(w + \pi - 2\pi k))$$
$$= \pi(\sum_{\forall k} \delta(w - \pi - 2\pi k) + \delta(w + \pi - 2\pi k))$$

(c)

$$y_d[n] = x_d[n].h[n] \longleftrightarrow Y_d(e^{jw}) = \frac{1}{2\pi} X_d(e^{jw}) * H(e^{jw})$$
$$Convolution\ over\ 1\ period\ (-\pi\ to\ \pi)$$
$$Y_d(e^{jw}) = \frac{1}{2\pi}.\pi(\sum_{\forall k} \delta(w - \pi) + \delta(w + \pi)) * X_d(e^{jw})$$
$$Shift\ X_d(e^{jw})\ to\ the\ left\ and\ right\ by\ \pi.\ Hence,$$
$$Y_d(e^{jw}) = \begin{cases} \frac{1}{\pi}w & , \frac{\pi}{2} \leq |w| \leq \frac{3\pi}{2} \\ 0 & otherwise \end{cases} \quad , Y_d(e^{jw}) = X_d(e^{j(w+2\pi)})$$