CENG 384 - Signals and Systems for Computer Engineers
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Written Assignment 2

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1. (a)

$$y(t) = \int_{-\infty}^{t} (x(\tau) - 4y(\tau))d\tau$$

.

$$\frac{dy(t)}{dt} + 4y(t) = x(t)$$

is the main equation that will be solved.

(b) Complete solution of differential equation consist of 2 parts:
particular solution $y_p(t)$ and
homogenous solution $y_h(t)$.

This input $(e^{-t} + e^{-2t})u(t)$ can be parsed as $e^{-t}u(t) + e^{-2t}u(t)$.

Particular solution of $e^{-t}u(t)$ for $t > 0$:

$$y_p(t) = Ye^{-t}$$

Place this guess in equation $\frac{dy(t)}{dt} + 4y(t) = x(t)$:

$$-Ye^{-t} + 4Ye^{-t} = e^{-t}$$

$$3Y = 1 \rightarrow y = \frac{1}{3}$$

$$y_p(t) = \frac{1}{3}e^{-t}$$

Particular solution of $e^{-t}u(t)$ when t is less than 0 is zero since $u(t)$ is defined as 0 at that interval.

Particular solution of $e^{-2t}u(t)$ for $t > 0$:

$$y_p(t) = Ye^{-2t}$$

Place this guess in equation $\frac{dy(t)}{dt} + 4y(t) = x(t)$:

$$-2Ye^{-t} + 4Ye^{-t} = e^{-2t}$$

$$2Y = 1 \rightarrow y = \frac{1}{2}$$

$$y_p(t) = \frac{1}{2}e^{-2t}$$

This particular solution is 0 as well since $u(t)$ is 0 when $t < 0$.

Thus, overall particular solution for input $(e^{-t}+e^{-2t})u(t)$ is:

$$y_p(t)=\frac{1}{3}e^{-t}+\frac{1}{2}e^{-2t}$$

Now,homogenous solution should be found.

$$y_h(t)=Ae^{st}$$

Place this guess in equation $\frac{dy(t)}{dt}+4y(t)=0$(right side should be 0 in homogenous solution) :

$$Ase^{st}+4Ae^{st}=0$$

$$Ae^{st}(s+4)=0$$

$$s=-4$$

Then , homogenous solution is:

$$y_h(t)=Ae^{-4t}$$

Total solution,$y_p(t)+y_h(t)$ is:

$$y(t)=y_p(t)+y_h(t)=\frac{1}{3}e^{-t}+\frac{1}{2}e^{-2t}+Ae^{-4t}$$

Since the system is initially rest , $y(0)=0$,

$$y(0)=\frac{1}{3}e^{-0}+\frac{1}{2}e^{-0}+Ae^{-0}$$

$$y(0)=\frac{1}{3}+\frac{1}{2}+A$$

$$A=-\frac{5}{6}$$

Thus,homogenous solution is:

$$y_h(t)=\frac{-5}{6}e^{-4t}$$

and total solution $y(t)$ is:

$$y(t)=y_p(t)+y_h(t)=(\frac{1}{3}e^{-t}+\frac{1}{2}e^{-2t}-\frac{5}{6}e^{-4t})u(t)$$

2. (a)

$We\ know\ that\ x[n]*\delta[n-k]=x[n-k].\ Then,$
$y[n]=x[n]*h[n]$
$=(\delta[n-1]-3\delta[n-2]+\delta[n-3])*(\delta[n+1]+2\delta[n]-3\delta[n-1])$
$=\delta[n]+2\delta[n-1]-3\delta[n-1]-3\delta[n-2]-6\delta[n-2]+\delta[n-2]+9\delta[n-3]+2\delta[n-3]-3\delta[n-4]$
$=\delta[n]-\delta[n-1]-8\delta[n-2]+11\delta[n-3]-3\delta[n-4]$

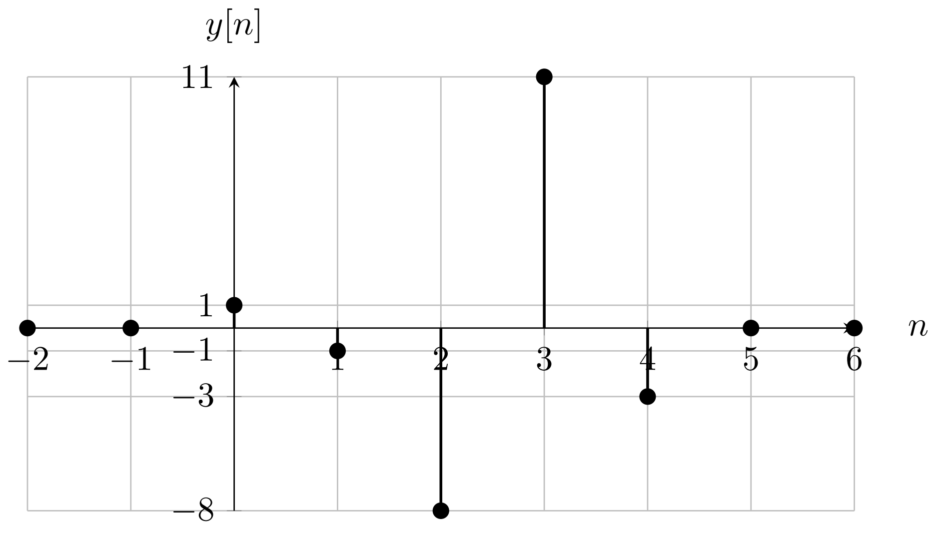

Figure 1: $x[n]*h[n]$

(b)

$$\begin{aligned}
&\frac{dx(t)}{dt} = \delta(t) + \delta(t-1)\\
&y(t) = h(t) * x(t)\\
&= \int_{-\infty}^{\infty} [\delta(t-\tau) + \delta(t-\tau-1)]e^{-2\tau}cos(\tau)u(\tau)d\tau\\
&= \int_{0}^{\infty} \delta(t-\tau)e^{-2\tau}cos(\tau)d\tau + \int_{0}^{\infty} \delta(t-\tau-1)e^{-2\tau}cos(\tau)d\tau\\
&= e^{-2t}cost + e^{-2(t-1)}cos(t-1)
\end{aligned}$$

3. (a)

$$y(t) = x(t) * h(t)$$
$$y(t) = \int_{-\infty}^{\infty} x(\tau) \times h(t-\tau)d\tau$$

.

Input $x(t) = e^{-t}u(t)$ and response $h(t) = e^{-3t}u(t)$.
This integral can be broken up 3 parts which are:
$t < 0$ and $t \geq 0$.

For $t < 0$ part:

$$y(t) = \int_{-\infty}^{0} x(\tau) \times h(t-\tau)d\tau = 0$$

since $x(\tau)$ and $h(t-\tau)$ don't have overlapping area.

For $t \geq 0$ part:

$$x(\tau) = e^{-t}u(t)\ ,\ \ h(t-\tau) = e^{-3(t-\tau)}u(t)$$
$$\begin{aligned}
y(t) &= \int_{0}^{t} x(\tau) \times h(t-\tau)d\tau\\
&= \int_{0}^{t} e^{-\tau} \times e^{-3(t-\tau)}d\tau\\
&= e^{-3t}\int_{0}^{t} e^{2\tau}d\tau\\
&= e^{-3t} \times \frac{e^{2\tau}}{2}|_{0}^{t}\\
&= e^{-3t} \times (\frac{e^{2t}}{2} - \frac{1}{2})\\
&= \frac{e^{-t} + e^{-3t}}{2}
\end{aligned}$$

Thus, convolution of $x(t)$ with $h(t)$:

$$y(t) = x(t) * h(t)$$

equals to

$$y(t) = 0\ ,\ \ t < 0$$
$$y(t) = \frac{e^{-t} + e^{-3t}}{2}u(t)$$

(b) Input $x(t)$ is $u(t-1) - u(t-2)$ is equal to a function that is:

$$1, 1 \leq t \leq 2$$
$$0\ ,\ \ otherwise$$

Then,

$$y(t) = x(t) * h(t)$$
$$y(t) = \int_{-\infty}^{\infty} x(\tau) \times h(t-\tau)d\tau$$

.

This integral can be broken up 3 parts which are:
$t < 1$, $1 \leq t \leq 2$ and $2 \leq t$.

For $t < 1$ part:

$$y(t) = \int_{-\infty}^{1} x(\tau) \times h(t-\tau) d\tau = 0$$

since $x(\tau)$ and $h(t-\tau)$ don't have overlapping area.

For $1 \leq t \leq 2$ part:

$$\begin{aligned} x(\tau) &= 1, h(t-\tau) = e^{t-\tau} \\ y(t) &= \int_{1}^{t} x(\tau) \times h(t-\tau) d\tau \\ &= \int_{1}^{t} 1 \times e^{t-\tau} d\tau \\ &= e^{t} \int_{1}^{t} e^{-\tau} d\tau \\ &= -e^{t} \times e^{-\tau}|_{1}^{t} \\ &= -e^{t} \times (e^{-t} - e^{-1}) \\ &= -1 + e^{t-1} \end{aligned}$$

For $2 \leq t$ part:

$$\begin{aligned} x(\tau) &= 1 \ , \ h(t-\tau) = e^{t-\tau} \\ y(t) &= \int_{1}^{2} x(\tau) \times h(t-\tau) d\tau \\ &= \int_{1}^{2} 1 \times e^{t-\tau} d\tau \\ &= e^{t} \int_{1}^{2} e^{-\tau} d\tau \\ &= -e^{t} \times e^{-\tau}|_{1}^{2} \\ &= -e^{t} \times (e^{-2} - e^{-1}) \\ &= -e^{t-2} + e^{t-1} \end{aligned}$$

Thus, convolution of $x(t)$ with $h(t)$:

$$y(t) = x(t) * h(t)$$

equals to

$$\begin{aligned} y(t) &= 0 \ , \ t < 1 \\ y(t) &= -1 + e^{t-1}, \ 1 \leq t \leq 2 \\ y(t) &= -e^{t-2} + e^{t-1} \ , \ 2 \leq t \end{aligned}$$

4. (a)

$y[n] - 15y[n-1] + 26y[n-2] = 0.$ *Change each n with $n+2$*
$y[n+2] - 15y[n+1] + 26y[n] = 0$
Let $y[n] = \lambda^n$ then,
$\lambda^{n+2} - 15\lambda^{n+1} + 26\lambda^n = 0$
Divide both sides by λ^n
$\lambda^2 - 15\lambda + 26 = 0$
$\lambda_1 = 13$ *and* $\lambda_2 = 2$
$y[n] = c_1.\lambda_1^n + c_2.\lambda_2^n$
$y[n] = c_1.13^n + c_2.2^n$
$y[0] = c_1 + c_2 = 10$
$y[1] = 13c_1 + 2c_2 = 42$
Then $c_1 = 2$ *and* $c_2 = 8$
$y[n] = 2.13^n + 8.2^n$

(b)

$$
\begin{aligned}
&y[n] - 3y[n-1] + y[n-2] = 0.\ \textit{Change each n with } n+2\\
&y[n+2] - 3y[n+1] + y[n] = 0\\
&\textit{Let } y[n] = \lambda^n \textit{ then,}\\
&\lambda^{n+2} - 3\lambda^{n+1} + \lambda^n = 0\\
&\textit{Divide both sides by } \lambda^n\\
&\lambda^2 - 3\lambda + 1 = 0\\
&\lambda_1 = \frac{3+\sqrt{5}}{2} \textit{ and } \lambda_2 = \frac{3-\sqrt{5}}{2}\\
&y[n] = c_1.(\frac{3+\sqrt{5}}{2})^n + c_2.(\frac{3-\sqrt{5}}{2})^n\\
&y[0] = c_1 + c_2 = 1\\
&y[1] = c_1(\frac{3+\sqrt{5}}{2}) + c_2(\frac{3-\sqrt{5}}{2}) = 2\\
&\textit{Then } c_1 = \frac{\sqrt{5}+1}{2\sqrt{5}} \textit{ and } c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}\\
&y[n] = \frac{\sqrt{5}+1}{2\sqrt{5}}.(\frac{3+\sqrt{5}}{2})^n + \frac{\sqrt{5}-1}{2\sqrt{5}}.(\frac{3-\sqrt{5}}{2})^n
\end{aligned}
$$

5. (a) This is a continuous LTI system so when an input $x(t) = \delta(t)$ (impulse),then the output $y(t)$ is $h(t)$ (impulse response).

A general representations of these systems are:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t)$$

Solution of these systems are consist of 2 parts:
particular solution $y_p(t)$ and
homogenous solution $y_h(t)$.

Particular solution of this system which is $y_p(t)$ is corresponding to inputs when $t > 0$.However , when $x(t) = \delta(t)$ is given as input,$x(t) = \delta(t)$ is 0 for $t > 0$ and $y_p(t)$ become ineffective.
Thus, finding an impulse response of this LTI system is simply finding a homogenous solution of this system.

Solution of homogenous part:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 0$$

In that point , a guess about solution should be done.

$$y_h(t) = Ae^{st}$$

Then

$$
\begin{gathered}
y'' + 6y' + 8y = 0\\
s^2 + 6s + 8 = 0\\
(s+4)(s+2) = 0\\
s_1 = -4 \textit{ and } s_2 = -2
\end{gathered}
$$

Placing these s values in the guess and that will construct the general solution for system:

$$A_1 e^{-4t} + A_2 e^{-2t}$$

Constants A_1 and A_2 should be found. Since system is initally rest:

$$A_1 + A_2 = 0$$
$$-2A_1 - 4A_2 = 2$$

Then

$$A_1 = 1 \; and \; A_2 = -1$$

Thus,impulse response of this system is

$$h(t) = (e^{-2t} - e^{-4t})u(t)$$

(b) i) An LTI system is causal if and only if $h(t) = 0$ for $t < 0$
$h(t) = (e^{-2t} - e^{-4t})u(t)$
Since $u(t) = 0$ for $t < 0$, $h(t)$ also equals to 0. Therefore, the given LTI system is causal.

ii) In order a LTI system to be memoryless $h(t) = 0$ for $t \neq 0$, so basically $h(t) = k.\delta(t)$
for $t < 0$, $h(t) = 0$ because $u(t) = 0 \; for \; t < 0$
for $t > 0$, $h(t) = e^{-2t} - e^{-4t} \neq 0$.
Therefore, given LTI system is not memoryless.

iii) A continuous-time LTI system is stable if and only if $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

$$\int_{-\infty}^{\infty} |(e^{-2t} - e^{-4t})u(t)|dt$$
$$= \int_{0}^{\infty} (e^{-2t} - e^{-4t})dt$$
$$= (\frac{e^{-2t}}{-2} - \frac{e^{-4t}}{-4}) \;|_0^\infty$$
$$= 0 - (\frac{-1}{4}) = \frac{1}{4} < \infty.$$

Therefore, given LTI system is stable.

iv) An LTI system is invertible if there is a $h^{-1}(t)$ such that $y(t) = x(t) * h(t) * h^{-1}(t) = x(t)$ which means $h(t) * h^{-1}(t) = \delta(t)$

We need to find such $h^{-1}(t)$ that $(e^{-2t} - e^{-4t})u(t) * h^{-1}(t) = \delta(t)$

$\int_{-\infty}^{\infty}(e^{-2t} - e^{-4t}).u(t).h^{-1}(t)dt$

$= \int_0^\infty(e^{-2t} - e^{-4t}).h^{-1}(t)dt$

However, there is no $h^{-1}(t)$ that makes the value of this integral $\delta(t)$, so the given LTI system is not invertible.