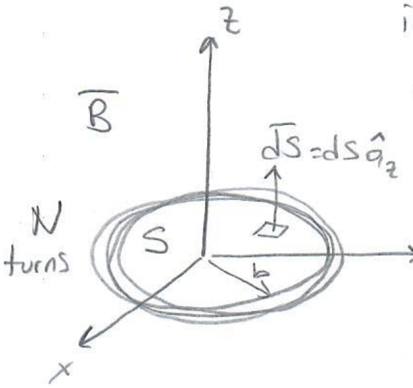
Example: $\bar{B} = \hat{a}_z B_0 \cos\left(\frac{\pi r}{2b}\right)\sin\omega t + \hat{a}_\phi B_1 \sin\left(\frac{\pi r}{2b}\right)\sin\omega t$

is given. Find the induced voltage v_{ind} for the coil placed on the xy-plane as shown in the figure.



$$\Phi_{\substack{\text{single}\\\text{turn}}} = \int_S \bar{B}\cdot d\bar{S}$$ (flux linked to a single turn of the circular coil).

where $\bar{B}\cdot d\bar{S} = (B_z\hat{a}_z + B_\phi\hat{a}_\phi)\cdot \underbrace{d\bar{S}}_{\hat{a}_z dS} = B_z \underbrace{dS}_{r\,dr\,d\phi}$

$$\Phi_{\substack{\text{single}\\\text{turn}}} = \int_{\phi=0}^{2\pi}\int_{r=0}^{b} B_0\cos\left(\frac{\pi r}{2b}\right)\sin\omega t\; r\,dr\,d\phi$$

$$= B_0 \sin\omega t\,(2\pi)\int_{r=0}^{b} r\cos\left(\frac{\pi r}{2b}\right)dr$$

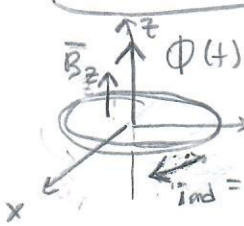
(Use with $k = \pi/2b$: $\int r\cos(kr)\,dr = \frac{\cos kr}{k^2} + \frac{r\sin kr}{k}$)

$$\Rightarrow \Phi_{\substack{\text{single}\\\text{turn}}} = \frac{8b^2}{\pi}\underbrace{\left(\frac{\pi}{2}-1\right)}_{>0}B_0 \sin\omega t = \underline{A\sin\omega t} \quad \text{(weber)}$$

call A, constant, $A>0$

$$\Rightarrow v_{ind}\Big|_{\substack{\text{for}\\\text{single}\\\text{turn}}} = -\frac{d}{dt}\Phi_{\substack{\text{single}\\\text{turn}}} = -\frac{8b^2}{\pi}\left(\frac{\pi}{2}-1\right)B_0\,\omega\cos\omega t$$

$$v_{ind_{total}} = N\left(v_{ind}\right)_{\substack{\text{single}\\\text{turn}}} \Rightarrow \boxed{v_{ind_{total}} = -\frac{8Nb^2}{\pi}\left(\frac{\pi}{2}-1\right)B_0\,\omega\cos\omega t \quad \text{(Volts)}}$$



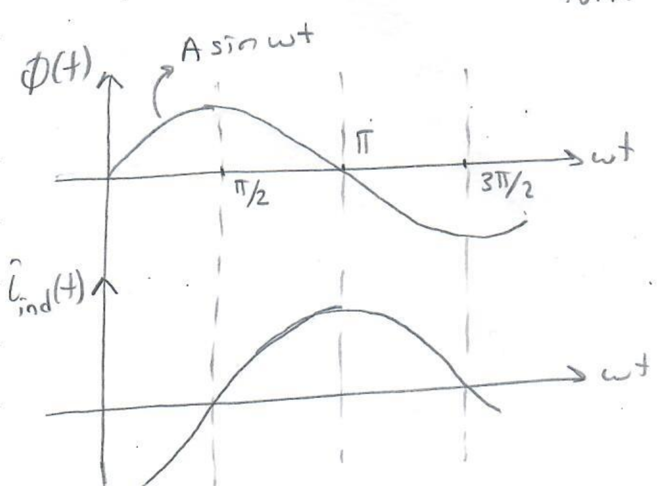
$\phi(t) = A\sin\omega t$ $(A>0)$

$i_{ind} = \frac{v_{ind}}{R}$

$i_{ind} = -K\cos\omega t$ (direction is in $-\hat{a}_\phi$) for $0<\omega t<\pi/2$)

$$\underline{i_{ind}} = -\frac{8Nb^2\left(\frac{\pi}{2}-1\right)B_0\,\omega}{\pi R}\cos\omega t = \underline{\underline{-K\cos\omega t}}$$

call $K>0$



for $0<\omega t<\pi/2$
$\phi(t)$ is increasing $\Rightarrow$

for $\pi/2<\omega t<3\pi/2$
$\phi(t)$ is decreasing $\Rightarrow$
$(\ldots\hat{a}_\phi)$ direction.

Summary:

For a closed path C in a given magnetic field $\bar{B}$, an emf (voltage) is induced along the path if the magnetic flux linked by the path changes with time.

A <u>nonzero</u> $\frac{d\Phi}{dt}$ may result from any one of the following cases:

1. Path is stationary but $\bar{B}$ is time-varying; (Transformer emf is induced)

$$v_{ind} = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S}$$

2. $\bar{B}$ is time-invariant (static) but the path moves; Motional emf (generator emf) is induced.

$$v_{ind} = \oint_C \bar{v}\times\bar{B}\cdot d\bar{l}$$

3. The path C moves in a time-varying field $\bar{B}$; (combination of cases ① and ②)

Both transformer emf and motional emf are induced.

$$v_{ind} = -\frac{d\Phi}{dt}$$

or $$v_{ind} = \underbrace{-\int_S \frac{d\bar{B}}{dt}\cdot d\bar{S}}_{\text{transformer emf}} + \underbrace{\oint_C (\bar{v}\times\bar{B})\cdot d\bar{l}}_{\text{motional emf}}$$