Wave Equations for $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ Fields

Assume a simple (i.e. linear + isotropic + homogeneous) medium where ϵ and μ are simple constants.

USE:

Maxwell's Eqns. (M.E.)

$$\begin{cases} \bar\nabla \times \bar{\mathcal{E}} = -\dfrac{\partial \bar{\mathcal{B}}}{\partial t} \\ \bar\nabla \times \bar{\mathcal{H}} = \bar{\mathcal{J}} + \dfrac{\partial \bar{\mathcal{D}}}{\partial t} \\ \bar\nabla \cdot \bar{\mathcal{D}} = \rho_v \\ \bar\nabla \cdot \bar{\mathcal{B}} = 0 \end{cases}$$

Constitutive Relations for a "simple" medium

$$\begin{cases} \bar{\mathcal{B}} = \mu \bar{\mathcal{H}} \\ \bar{\mathcal{D}} = \epsilon \bar{\mathcal{E}} \end{cases}$$

To obtain the wave equation for $\bar{\mathcal{E}}$ field, start by computing the curl of both sides of the 1st M.E., i.e. $\bar\nabla \times \bar{\mathcal{E}} = -\dfrac{\partial \bar{\mathcal{B}}}{\partial t}$.

$$\underbrace{\bar\nabla \times (\bar\nabla \times \bar{\mathcal{E}})}_{\bar\nabla(\underbrace{\bar\nabla \cdot \bar{\mathcal{E}}}_{\bar{\mathcal{D}}/\epsilon}) - \bar\nabla^2 \bar{\mathcal{E}}} = \bar\nabla \cdot \left(-\frac{\partial \bar{\mathcal{B}}}{\partial t}\right) = -\frac{\partial}{\partial t}(\bar\nabla \times \underbrace{\bar{\mathcal{B}}}_{\mu \bar{\mathcal{H}}})$$

constant

$$\Rightarrow \bar\nabla\left(\bar\nabla \cdot \frac{\bar{\mathcal{D}}}{\underbrace{\epsilon}_{\text{constant}}}\right) - \bar\nabla^2 \bar{\mathcal{E}} = -\mu \frac{\partial}{\partial t} \underbrace{(\bar\nabla \times \bar{\mathcal{H}})}_{\bar{\mathcal{J}} + \frac{\partial \bar{\mathcal{D}}}{\partial t}}, \quad \bar{\mathcal{D}} \to \epsilon \bar{\mathcal{E}}$$

$$\Rightarrow \bar\nabla\left[\frac{1}{\epsilon}(\underbrace{\bar\nabla \cdot \bar{\mathcal{D}}}_{\rho_v})\right] - \bar\nabla^2 \bar{\mathcal{E}} = -\mu \frac{\partial \bar{\mathcal{J}}}{\partial t} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2}$$

$$\bar\nabla\left(\frac{\rho_v}{\epsilon}\right) - \bar\nabla^2 \bar{\mathcal{E}} = -\mu \frac{\partial \bar{\mathcal{J}}}{\partial t} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2}$$

Reorganize as

$$\boxed{\nabla^2 \bar{\mathcal{E}} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2} = \mu \frac{\partial \bar{\mathcal{J}}}{\partial t} + \bar\nabla\left(\frac{\rho_v}{\epsilon}\right)}$$

Inhomogeneous (right hand side contains source terms) wave eqn. for $\bar{\mathcal{E}}$ field!

where $\bar{\mathcal{J}}$ and ρ_v are the vector and scalar type sources of the problem.

<u>Exercise:</u> Repeat the same procedure to obtain the wave equation for $\mathcal{H}$ field!

(Hint: Start with the 2nd M.E. $\bar{\nabla} \times \bar{\mathcal{H}} = \bar{J} + \frac{\partial \bar{\mathcal{D}}}{\partial t}$ this time)

Similarly, we can get the wave eqn. for $\mathcal{H}$ field as

$$\boxed{\nabla^2 \bar{\mathcal{H}} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{H}}}{\partial t^2} = -\bar{\nabla} \times \bar{J}}$$

Again, this wave eqn. is inhomogeneous as the right hand side contains the source term $\bar{J}$!

<u>Special Case I:</u> For a [SIMPLE ⊕ LOSSLESS ⊕ SOURCE FREE] medium.

SIMPLE → linear + isotropic + homogeneous

LOSSLESS → zero conductivity $\sigma = 0$

SOURCE FREE → $\bar{J}_{impressed} = 0$, $\rho_v = 0$ (no impressed sources)

In general, $\bar{J}$ in Maxwell's Equations may have the following form:

$$\bar{J} = \underbrace{\bar{J}_{impressed}}_{\text{forced or impressed current density which is zero in source-free case.}} + \underbrace{\bar{J}_{conduction}}_{\sigma \mathcal{E}\text{, which is zero in lossless case (i.e. when } \sigma = 0)}$$

In this special case, both components of current density are zero. Hence, set $\bar{J} = 0$ in M.E.s and in wave equations.

$$\Rightarrow \boxed{\nabla^2 \bar{\mathcal{E}} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2} = 0} \quad \text{and} \quad \boxed{\nabla^2 \bar{\mathcal{H}} - \epsilon\mu \frac{\partial^2 \bar{\mathcal{H}}}{\partial t^2} = 0}$$

homogeneous wave equations!

Consider $\boxed{\nabla^2 \bar{E} - \epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$. for instance

where $\bar{E}(\bar{r}, t) = \hat{a}_x E_x(x,y,z,t) + \hat{a}_y E_y(x,y,z,t) + \hat{a}_z E_z(x,y,z,t)$

in general, in Cartesian coordinate system.

For simplicity, assume $\bar{E} = \hat{a}_x E_x(z,t)$

Insert this solution to wave eqn. above:

$$\cancelto{0}{\frac{\partial^2}{\partial x^2}\left[\hat{a}_x E_x(z,t)\right]} + \cancelto{0}{\frac{\partial^2}{\partial y^2}\left[\hat{a}_x E_x(z,t)\right]} + \frac{\partial^2}{\partial z^2}\left[\hat{a}_x E_x(z,t)\right] - \epsilon\mu \frac{\partial^2}{\partial t^2}\left[\hat{a}_x E_x(z,t)\right] = 0$$

$$\Longrightarrow \boxed{\frac{\partial^2 E_x(z,t)}{\partial z^2} - \epsilon\mu \frac{\partial^2 E_x(z,t)}{\partial t^2} = 0} \qquad (a)$$

It can be shown by inspection that the pde (a) has the solutions in the form

$$\left.\boxed{\begin{array}{l} g_1(z - vt) \\ \hline g_2(z + vt) \end{array}}\right\} \text{ where } \boxed{v = \frac{1}{\sqrt{\epsilon\mu}}}$$

and g_1, g_2 are double-differentiable functions with respect to z and t.

Proof: Insert $g_1(z-vt)$, for instance, into pde (a) to see if it is satisfied.

Let $s = z - vt \Rightarrow \frac{\partial s}{\partial z} = 1$ and $\frac{\partial s}{\partial t} = -v = -\frac{1}{\sqrt{\epsilon\mu}}$

$$\frac{\partial g_1}{\partial z} = \frac{\partial g_1}{\partial s}\underbrace{\frac{\partial s}{\partial z}}_{1} = \frac{\partial g_1}{\partial s}$$ (using chain rule in differentiation)

and $$\frac{\partial^2 g_1}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial g_1}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial g_1}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial g_1}{\partial s}\right)\underbrace{\frac{\partial s}{\partial z}}_{1}$$

$$\boxed{\frac{\partial^2 g_1}{\partial z^2} = \frac{\partial^2 g_1}{\partial s^2}} \quad (*)$$

Also,
$$\frac{\partial g_1}{\partial t} = \frac{\partial g_1}{\partial s}\underbrace{\frac{\partial s}{\partial t}}_{-v} = -\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}$$

and
$$\frac{\partial^2 g_1}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}\right) = \frac{\partial}{\partial s}\left[-\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}\right]\underbrace{\frac{\partial s}{\partial t}}_{-\frac{1}{\sqrt{\mu\epsilon}}}$$

$$\boxed{\frac{\partial^2 g_1}{\partial t^2} = \frac{1}{\mu\epsilon}\frac{\partial^2 g_1}{\partial s^2}} \quad (**)$$

Insert (*) and (**) in (a) to see

$$\underbrace{\frac{\partial^2 g_1(z-vt)}{\partial z^2}}_{\frac{\partial^2 g_1}{\partial s^2}} - \cancel{\mu\epsilon}\underbrace{\frac{\partial^2 g_1(z-vt)}{\partial t^2}}_{\frac{1}{\cancel{\mu\epsilon}}\frac{\partial^2 g_1}{\partial s^2}} = 0 \checkmark$$ as claimed!

(Validity of the other solution $g_2(z+vt)$ can be proven similarly.)

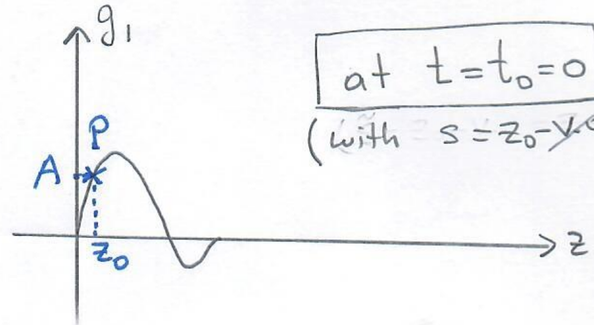
<u>Some Examples to Wave Solutions</u>

$K e^{\alpha(z \mp vt)}$, $K \cos\{\alpha(z \mp vt)\}$, $K(z \mp vt)^3$, etc...

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, K and α are some arbitrary constants.

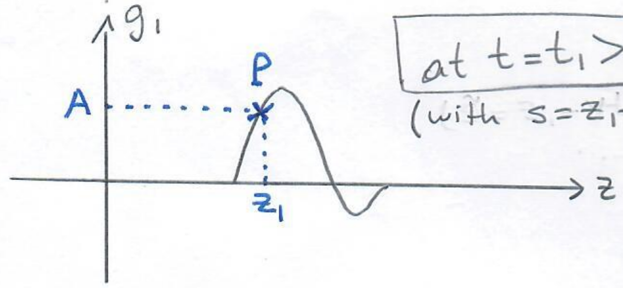
Behavior of Wave Solutions: Traveling Waves

Let's examine the behavior of $g_1(z-vt)$ types of solutions. Call the argument of the function as $\boxed{s=z-vt}$. Plot an arbitrary g_1 function versus distance (z) at different time (t) instants as follows:



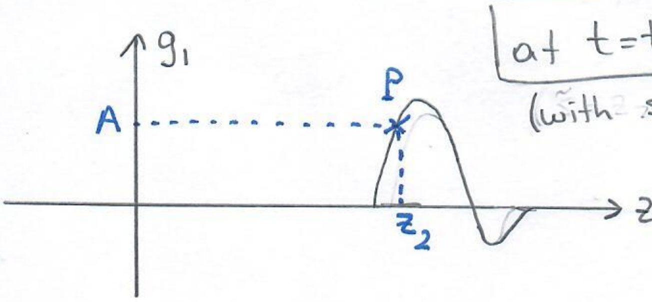
$\boxed{\text{at } t=t_0=0}$

(with $s=z_0-\cancel{V.0}=z_0$)



$\boxed{\text{at } t=t_1>t_0}$

(with $s=z_1-Vt_1$)



$\boxed{\text{at } t=t_2>t_1>0}$

(with $s=z_2-vt_2$)

Choose an arbitrary point P on the curve. Coordinates of this particular point P are

(z_0, A) at $t=t_0=0$

(z_1, A) at $t=t_1$

(z_2, A) at $t=t_2$

⋮

Note that the value of the argument must remain the same as we keep tracking the same point P of the curve. In other words,

for P $\boxed{s=\tilde{s}=z_0-V\underbrace{(0)}_{t_0=0}=z_1-v_1 t=z_2-v_2t_2=\cdots}$

Because:

(*) As long as the value of argument (s) remains constant at $s=\tilde{s}$ (for point P) at different combinations of distance (z) and time (t) variables, we keep reading the same functional value $g_1(\tilde{s})=A$.

More generally, $\vec{\tilde{s}} = z - vt$ for point P where $\tilde{s}$ = constant

Compute total derivatives of both sides

$$\underbrace{d(\tilde{s})}_{0} = d(z-vt) = dz + d(vt) \quad \text{where } v = \frac{1}{\sqrt{\mu\epsilon}} \text{ is a constant}$$

as $\tilde{s}$ is a constant

$$\Rightarrow \quad 0 = dz - v\,dt$$

$$\Rightarrow \quad \boxed{\frac{dz}{dt} = v = \frac{1}{\sqrt{\mu\epsilon}}}$$

Here, $\frac{dz}{dt}$ is a velocity term by definition.

(*) Therefore, $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the "velocity of propagation" of point P along the z-axis.

(*) The same discussion can be repeated for all other points of the wave form $g_1(z-vt) \Rightarrow$ The complete wave solution g_1 moves in $(+z)$ direction with velocity $\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$ as time increases preserving its shape.

(*) Note that we need to use the condition

$$\boxed{s = z \mp vt = \text{constant}}$$

to explain traveling wave behavior.

For $g_1(z-vt)$ (Forward traveling wave)

$$\boxed{z - vt = \text{constant}} \Rightarrow \boxed{\frac{dz}{dt} = v}$$

As time (t) increases, distance (z) must also increase to keep the value of $(z-vt)$ constant.

$\Rightarrow g_1(z-vt)$ travels in $(+z)$ direction as time passes.

For $g_2(z+vt)$ (Backward traveling wave)

$$\boxed{z + vt = \text{constant}} \Rightarrow \boxed{\frac{dz}{dt} = -v}$$

As t increases, z must decrease (because of the $(+)$ sign in between) to keep the value of $(z+vt)$ constant.

$\Rightarrow g_2(z+vt)$ travels in $(-z)$ direction as time passes.

(*) Check the unit of $\frac{1}{\sqrt{\epsilon\mu}} = v$!

$$\mu \longleftrightarrow \frac{\text{Henry}}{\text{meter}} \equiv \frac{(\text{weber}/\text{Amp})}{\text{meter}}$$

$$\epsilon \longleftrightarrow \frac{\text{Farad}}{\text{meter}} \equiv \frac{(\text{Coulomb}/\text{Volt})}{\text{meter}}$$

$$\mu\epsilon \longleftrightarrow \frac{\overbrace{(\text{Weber})}^{(\cancel{\text{Volt}}).(\text{sec})}\ \overbrace{(\text{Coulomb})}^{(\cancel{\text{Amp}})(\text{sec})}}{m^2\ (\cancel{\text{Amp}})\ (\cancel{\text{Volt}})} = \frac{(\text{sec})^2}{m^2}$$

$$\boxed{\frac{1}{\sqrt{\mu\epsilon}} \longleftrightarrow \frac{1}{\sqrt{\frac{\text{sec}^2}{m^2}}} = \left(\frac{m}{\text{sec}}\right) : \text{unit of velocity indeed!}}$$

(*) In vacuum, $v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3 \times 10^8$ m/sec : velocity of light in vacuum.

In other natural media: $\mu = \mu_0 \mu_r$, $\epsilon = \epsilon_0 \epsilon_r$

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}\ \sqrt{\mu_r \epsilon_r}} = \frac{c}{\sqrt{\mu_r \epsilon_r}} < c$$

(as $\mu_r \geqslant 1$, $\epsilon_r \geqslant 1$)

(*) The argument $(z \pm vt)$ of traveling waves can be expressed in a different but equivalent way:

$$g(z - vt) = g\Big(\underbrace{-v}_{\text{a constant}}\big(t - \frac{z}{v}\big)\Big) \Rightarrow f\left(t - \frac{z}{v}\right)$$

$g(z - vt)$, $f\left(t - \frac{z}{v}\right)$ } forward waves traveling in $(+z)$ direction as time progresses	$g(z + vt)$, $f\left(t + \frac{z}{v}\right)$ } backward waves traveling in $(-z)$ direction as time progresses

Special Case II : For a [SIMPLE ⊕ LOSSY ⊕ SOURCE FREE] medium.

LOSSY → $\sigma \neq 0$; SOURCE FREE: $\bar{J}_{impressed} = 0$, $\rho_v = 0$

As being different from the special case I, now we have $\sigma \neq 0$ in the lossy medium.

$$\sigma \neq 0 \Rightarrow \bar{J} = \underbrace{\cancel{\bar{J}_{impressed}}}_{\substack{0 \text{ as the} \\ \text{medium is} \\ \text{source-free}}} + \underbrace{\bar{J}_{conduction}}_{\sigma \bar{E} \neq 0} \Rightarrow \boxed{\bar{J} = \sigma \bar{E}}$$

must be inserted in general expression!

For instance, $\bar{\nabla} \times \bar{H} = \bar{J} + \dfrac{\partial \bar{D}}{\partial t} = \sigma \bar{E} + \dfrac{\partial \bar{D}}{\partial t}$ in this case.

Using the general inhomogeneous wave equations with $\begin{cases} \rho_v = 0 \\ \bar{J} = \sigma \bar{E} \end{cases}$

$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = \mu \frac{\partial \bar{J}}{\partial t} + \bar{\nabla}\Big(\overset{0}{\frac{\rho_v}{\epsilon}}\Big)$$

$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = \mu \frac{\partial}{\partial t}(\sigma \bar{E}) = \mu\sigma \frac{\partial \bar{E}}{\partial t}$$

or,

$$\boxed{\nabla^2 \bar{E} - \mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

$$\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = -\bar{\nabla} \times \bar{J}$$

$$\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = -\bar{\nabla} \times (\sigma \bar{E}) = -\sigma \underbrace{\bar{\nabla} \times \bar{E}}_{-\frac{\partial \bar{B}}{\partial t}} = \sigma\mu \frac{\partial \bar{H}}{\partial t}$$

$\bar{B} \rightarrow \mu \bar{H}$

$$\boxed{\nabla^2 \bar{H} - \mu\sigma \frac{\partial \bar{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = 0}$$

Partial differential equations describing wave propagation in a simple, source-free but lossy medium.