Time-Harmonic Electromagnetic Fields

(Sinusoidally varying, monochromatic EM fields)

A time-harmonic electric field that oscillates at a single frequency $\omega = 2\pi f$ (with period $T = \frac{1}{f}$) can be expressed as

$$\mathcal{E}(x,y,z,t) = \hat{a}_x A_x \cos(\omega t + \psi_x) + \hat{a}_y A_y \cos(\omega t + \psi_y) + \hat{a}_z A_z \cos(\omega t + \psi_z).$$

The associated phasor electric field $\bar{E}(x,y,z)$ is:

$$\bar{E}(x,y,z) = \hat{a}_x \underbrace{A_x e^{j\psi_x}}_{R_x + jI_x} + \hat{a}_y \underbrace{A_y e^{j\psi_y}}_{R_y + jI_y} + \hat{a}_z \underbrace{A_z e^{j\psi_z}}_{R_z + jI_z}$$

Define $\left.\begin{array}{l} \bar{R} = \hat{a}_x R_x + \hat{a}_y R_y + \hat{a}_z R_z \\ \bar{I} = \hat{a}_x I_x + \hat{a}_y I_y + \hat{a}_z I_z \end{array}\right\}$ $\bar{E} = \bar{R} + j\bar{I}$

↳ a complex valued field!

The relationship between $\mathcal{E}(\bar{r},t)$ and $\bar{E}(\bar{r})$ can be established as

$$\boxed{\mathcal{E}(\bar{r},t) = \mathrm{Re}\left\{ \bar{E}(\bar{r})\, e^{j\omega t} \right\}}$$

Example:

Let $\bar{\mathcal{E}}(\bar{r},t) = \hat{a}_x \mathcal{E}_x(z,t) = \hat{a}_x \underbrace{3}_{A_x} \cos(\underbrace{10^8}_{\omega} t \underbrace{- 2z}_{\psi_x})$ (where $\omega = 10^8$ rad/sec $= 2\pi f$)

$$\Rightarrow \bar{E}(\bar{r}) = \hat{a}_x A_x e^{j\psi_x} = \hat{a}_x 3\, e^{j(-2z)} = \hat{a}_x 3 \angle{-2z}$$

Let $\underbrace{f(t)}_{\text{Time domain function}} = A\cos(\omega t+\psi) \longleftrightarrow \underbrace{F}_{\text{corresponding phasor}} = A e^{j\psi} = A\angle\psi$

Consider time derivative $\frac{df(t)}{dt}$! What is the phasor corresponding to $\frac{df(t)}{dt}$?

$$\frac{d}{dt}f(t) = \frac{d}{dt}\left(A\cos(\omega t+\psi)\right) = -\omega A \underbrace{\sin(\omega t+\psi)}_{-\cos(\omega t+\psi+\frac{\pi}{2})}$$

$$= \underbrace{\omega A}_{\text{Amplitude}} \cos(\omega t + \underbrace{\psi+\tfrac{\pi}{2}}_{\text{phase}}) \longleftrightarrow \omega A\, e^{j(\psi+\frac{\pi}{2})} = \omega A \underbrace{e^{j\frac{\pi}{2}}}_{j} e^{j\psi}$$

which means

$$\frac{d}{dt}f(t) \longleftrightarrow j\omega \underbrace{Ae^{j\psi}}_{F} = (j\omega)F$$

(<u>Exercise</u>: Show that $\frac{d^2}{dt^2}f(t) \longleftrightarrow (j\omega)^2 F = -\omega^2 F$)

It can be shown in general that

$$\boxed{\frac{d^n}{dt^n}f(t) \longleftrightarrow (j\omega)^n F \qquad \text{where } f(t)\longleftrightarrow F}$$

Based on this general result, i.e. using $\boxed{\frac{d^n}{dt^n} \equiv (j\omega)^n}$

insert $\frac{\partial}{\partial t} \equiv j\omega$ and $\frac{\partial^2}{\partial t^2} \equiv (j\omega)^2 = -\omega^2$

in Maxwell's Equations and in wave equations to express them in "Phasor Domain" (or "Frequency Domain").

General (point) Form of Maxwell's Equations

in time domain		in phasor (frequency) domain
$\bar{\nabla}\times\bar{\mathcal{E}} = -\dfrac{\partial\bar{\mathcal{B}}}{\partial t}$	$\longleftrightarrow$	$\bar{\nabla}\times\bar{E} = -j\omega\bar{B}$
$\bar{\nabla}\times\bar{\mathcal{H}} = \bar{\mathcal{J}} + \dfrac{\partial\bar{\mathcal{D}}}{\partial t}$	$\longleftrightarrow$	$\bar{\nabla}\times\bar{H} = \bar{J} + j\omega\bar{D}$
$\bar{\nabla}\cdot\bar{\mathcal{D}} = \rho_v$	$\longleftrightarrow$	$\bar{\nabla}\cdot\bar{D} = \rho_v$
$\bar{\nabla}\cdot\bar{\mathcal{B}} = 0$	$\longleftrightarrow$	$\bar{\nabla}\cdot\bar{B} = 0$

(where $\bar{E}, \bar{D}, \bar{E}, \bar{H}, \bar{J}$ and ρ_v are all phasor quantities.)

Similarly, the wave equations can be transformed into phasor domain similarly:

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$$\nabla^2\bar{\mathcal{E}} - \mu\epsilon\frac{\partial^2\bar{\mathcal{E}}}{\partial t^2} = \mu\frac{\partial\bar{\mathcal{J}}}{\partial t} + \bar{\nabla}\left(\frac{\rho_v}{\epsilon}\right) \longleftrightarrow \nabla^2\bar{E} - \mu\epsilon(j\omega)^2\bar{E} = \mu(j\omega)\bar{J} + \bar{\nabla}\left(\frac{\rho_v}{\epsilon}\right)$$

$$\boxed{\nabla^2\bar{E} + k^2\bar{E} = j\omega\mu\bar{J} + \frac{\bar{\nabla}\rho_v}{\epsilon}}$$

$$\nabla^2\bar{\mathcal{H}} - \mu\epsilon\frac{\partial^2\bar{\mathcal{H}}}{\partial t^2} = -\bar{\nabla}\times\bar{\mathcal{J}} \longleftrightarrow \nabla^2\bar{H} - \mu\epsilon\underbrace{(j\omega)^2}_{-\omega^2}\bar{H} = -\bar{\nabla}\times\bar{J}$$

$$\boxed{\nabla^2\bar{H} + k^2\bar{H} = -\bar{\nabla}\times\bar{J}}$$

in phasor domain

where $k^2 = -\mu\epsilon(j\omega)^2 = (-\mu\epsilon)(-\omega^2) = \omega^2\mu\epsilon$

k: wavenumber $\leftarrow$ $\boxed{k = \omega\sqrt{\mu\epsilon} = \dfrac{\omega}{v}}$ (as $v = \dfrac{1}{\sqrt{\mu\epsilon}}$)

If the medium of concern is a simple, lossless, and source-free medium:

$$\boxed{\nabla^2 \bar{\mathcal{E}} - \mu\epsilon \frac{\partial^2 \bar{\mathcal{E}}}{\partial t^2} = 0} \longleftrightarrow \boxed{\nabla^2 \bar{E} + k^2 \bar{E} = 0}$$

$$\boxed{\nabla^2 \bar{\mathcal{H}} - \mu\epsilon \frac{\partial^2 \bar{\mathcal{H}}}{\partial t^2} = 0} \longleftrightarrow \boxed{\nabla^2 \bar{H} + k^2 \bar{H} = 0}$$

Homogeneous wave equations for $\bar{\mathcal{E}}(\bar{r},t)$ and $\bar{\mathcal{H}}(\bar{r},t)$ in time-domain.

Homogeneous Helmholtz Equations for $\bar{E}(\bar{r})$ and $\bar{H}(\bar{r})$ in phasor domain.

Equations related to scalar and vector potential functions can also be transformed into phasor domain as:

$$\bar{\mathcal{B}} = \bar{\nabla} \times \bar{\mathcal{A}} \longleftrightarrow \boxed{\bar{B} = \bar{\nabla} \times \bar{A}}$$

$$\bar{\mathcal{E}} = -\bar{\nabla} \mathcal{V} - \frac{\partial \bar{\mathcal{A}}}{\partial t} \longleftrightarrow \boxed{\bar{E} = -\bar{\nabla} V - j\omega \bar{A}}$$

$$\nabla^2 \mathcal{V} - \mu\epsilon \frac{\partial^2 \mathcal{V}}{\partial t^2} = -\frac{\rho_v}{\epsilon} \longleftrightarrow \boxed{\nabla^2 V + k^2 V = -\frac{\rho_v}{\epsilon}}$$

$$\nabla^2 \bar{\mathcal{A}} - \mu\epsilon \frac{\partial^2 \bar{\mathcal{A}}}{\partial t^2} = -\mu \bar{\mathcal{J}} \longleftrightarrow \boxed{\nabla^2 \bar{A} + k^2 \bar{A} = -\mu \bar{J}}$$

in time-domain for $\bar{\mathcal{A}}(\bar{r},t)$ and $\mathcal{V}(\bar{r},t)$

in phasor domain for $\bar{A}(\bar{r})$ and $V(\bar{r})$

(where $k^2 = (-\mu\epsilon)(j\omega)^2 = \omega^2\mu\epsilon$)

The solutions for $V(\bar{r})$ and $\bar{A}(\bar{r})$ in phasor domain can be obtained as

$$\boxed{V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_{v'} \rho(\bar{r}') \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \, dv'} \quad \text{(volt)}$$

and

$$\boxed{\bar{A}(\bar{r}) = \frac{\mu}{4\pi} \int_{v'} \bar{J}(\bar{r}') \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \, dv'} \quad (\text{weber}/\text{meter})$$

where the source terms $\rho(\bar{r}')$ and $\bar{J}(\bar{r}')$ are phasor quantities determined in the volume V' that contains sources of the problem.

Exercise: Find time-domain scalar potential function $v(\bar{r},t)$ from the phasor $V(\bar{r})$.

$$v(\bar{r},t) = Re\left\{V(\bar{r})\, e^{j\omega t}\right\} = Re\left\{\frac{e^{j\omega t}}{4\pi\epsilon_0} \int_{v'} \rho(\bar{r}') \frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dv'\right\}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{v'} \rho(\bar{r}') \frac{Re\{e^{j\omega t} e^{-jkR}\}}{R} dv' \quad \left(\text{where } R = |\bar{r}-\bar{r}'|\right)$$

$$= \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho(\bar{r}') \cos(\omega t - kR)}{R} dv'$$

$\omega\sqrt{\mu\epsilon} = \frac{\omega}{v}$

where $\rho(\bar{r}') \cos(\omega t - kR) = \rho(\bar{r}') \cos\left(\omega\left[t - \frac{k}{\omega} R\right]\right)$

$$= \underbrace{\rho(\bar{r}') \cos\left(\omega\left[t - \frac{R}{v}\right]\right)}_{\rho\left(\bar{r}',\, t - \frac{R}{v}\right)}$$

corresponds to term in "Retarded Scalar potential" soln.