METU Electrical-Electronics Engineering Department

2020-2021 Fall Semester

EE303 Homework #2

Due Date/Time: November 2, 2020 Monday, 1:00 pm

Please upload your solutions to ODTUClass next Monday no later than 1:00 pm.

Problem 1. Show that in a source-free region of space where $\nabla \cdot \mathbf{E} = 0$, the electric and magnetic fields may be found from a magnetic-type vector potential $\mathbf{A}_m$ by means of the equations

$$\mathbf{E} = \nabla \times \mathbf{A}_m$$
$$\mathbf{H} = j\omega\epsilon\mathbf{A}_m - \frac{\nabla\nabla \cdot \mathbf{A}_m}{j\omega\mu}$$

and $\mathbf{A}_m$ is a solution of

$$\nabla^2\mathbf{A}_m + \omega^2\mu\epsilon\mathbf{A}_m = 0$$

The derivation is similar to that for the electric-type vector potential $\mathbf{A}$.

Solution: In a source free-region of space we have $\nabla \cdot \mathbf{E} = 0$, and therefore the electric field can be written as the curl of a vector field, i.e.,

$$\mathbf{E} = \nabla \times \mathbf{A}_m. \tag{1}$$

Using this expression in Ampere's law we get

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} = j\omega\epsilon\nabla \times \mathbf{A}_m \tag{2}$$
$$\nabla \times \mathbf{H} - \nabla \times j\omega\epsilon\mathbf{A}_m = \nabla \times (\mathbf{H} - j\omega\epsilon\mathbf{A}_m) = 0 \tag{3}$$

since the current density $\mathbf{J}$ is also zero in a source-free region of space. Now, $\mathbf{H} - j\omega\epsilon\mathbf{A}_m$ is a curl free field and, we can write it as the gradient of a scalar field, ψ_m

$$\mathbf{H} - j\omega\epsilon\mathbf{A}_m = \nabla\psi_m \tag{4}$$

or

$$\mathbf{H} = j\omega\epsilon\mathbf{A}_m + \nabla\psi_m. \tag{5}$$

Considering Faraday's law, we also get

$$\nabla \times \mathbf{E} = \nabla \times \nabla \times \mathbf{A}_m = -j\omega\mu\mathbf{H} \tag{6}$$

Expanding $\nabla \times \nabla \times \mathbf{A}_m$ to give $\nabla\nabla \cdot \mathbf{A}_m - \nabla^2\mathbf{A}_m$, we get

$$\nabla\nabla \cdot \mathbf{A}_m - \nabla^2\mathbf{A}_m = \omega^2\mu\epsilon\mathbf{A}_m - j\omega\mu\nabla\psi_m \tag{7}$$

The divergence of the magnetic-type vector potential $\mathbf{A}_m$ is not specified yet, so we can choose

$$\nabla \cdot \mathbf{A}_m = -j\omega\mu\psi_m. \tag{8}$$

Using (8) in (7) yields the wave equation for the potential $\mathbf{A}_m$. Solving for ψ_m from (8) and using in (5) yields

$$\mathbf{H} = j\omega\epsilon\mathbf{A}_m - \frac{\nabla\nabla\cdot\mathbf{A}_m}{j\omega\mu} \ .$$

Problem 2. It is almost impossible to obtain solutions to the vector wave equation if the fields are written in terms of their spherical components and if the spherical coordinates are used. Yet for boundary conditions imposed on spherical boundaries, it is equally difficult to utilize rectangular coordinates since the boundary is not a natural one. It turns out, however, that the vector

$$\mathbf{M} = \mathbf{r} \times \nabla\psi$$

where $\mathbf{r}$ is the position vector, satisfies the vector wave equation provided that ψ satisfies the scalar wave equation

$$\nabla^2\psi + \omega^2\mu\epsilon\psi = 0$$

Another solution is

$$\mathbf{N} = \frac{1}{\omega\sqrt{\mu\epsilon}}\nabla\times\mathbf{M}$$

Note that $\mathbf{M}$ and $\mathbf{N}$ may be identified with the $\mathbf{E}$ and $\mathbf{H}$ field, or vice versa. In view of the fact that $\mathbf{M}$ is transverse to spherical surfaces, spherical boundary-value problems may be readily formulated.

Confirm that $\mathbf{M}$ and $\mathbf{N}$ do indeed satisfy the vector wave equations

$$\nabla^2\mathbf{F} + \omega^2\mu\epsilon\mathbf{F} = -\nabla\times\nabla\times\mathbf{F} + \nabla\nabla\cdot\mathbf{F} + \omega^2\mu\epsilon\mathbf{F} = 0$$

where $\mathbf{F}$ may be either $\mathbf{M}$ or $\mathbf{N}$ provided that

$$\nabla^2\psi + \omega^2\mu\epsilon\psi = 0$$

Solution:

We need to evaluate $\nabla\times\nabla\times\mathbf{M}$. Using the vector identity

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

we can evaluate

$$\nabla\times\mathbf{M} = \nabla\times(\mathbf{r}\times\nabla\psi) = \mathbf{r}\nabla\cdot\nabla\psi - \nabla\psi\nabla\cdot\mathbf{r} + (\nabla\psi\cdot\nabla)\mathbf{r} - (\mathbf{r}\cdot\nabla)\nabla\psi \qquad (9)$$

Notice that

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\mathbf{r} &= \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)\mathbf{r} \\ &= A_x\frac{\partial\left(x\hat{\boldsymbol{a}}_x + y\hat{\boldsymbol{a}}_y + z\hat{\boldsymbol{a}}_z\right)}{\partial x} + A_y\frac{\partial\left(x\hat{\boldsymbol{a}}_x + y\hat{\boldsymbol{a}}_y + z\hat{\boldsymbol{a}}_z\right)}{\partial y} \\ &\quad + A_z\frac{\partial\left(x\hat{\boldsymbol{a}}_x + y\hat{\boldsymbol{a}}_y + z\hat{\boldsymbol{a}}_z\right)}{\partial z} = \mathbf{A}\end{aligned}$$

$$\nabla\cdot\mathbf{r} = 3, \quad \nabla\times\mathbf{r} = 0$$

Thus (9) simplifies as

$$\nabla\times\mathbf{M} = \mathbf{r}\nabla^2\psi - 2\nabla\psi - (\mathbf{r}\cdot\nabla)\nabla\psi \qquad (10)$$

Next we evaluate the curl of (10) again and use to vector identity

$$\nabla \times (\psi \mathbf{A}) = \nabla\psi \times \mathbf{A} + \psi \nabla \times \mathbf{A}$$

to obtain

$$\nabla \times \nabla \times \mathbf{M} = \nabla\nabla^2\psi \times \mathbf{r} + \nabla^2\psi\nabla \times \mathbf{r} - 2\nabla \times \nabla\psi - \nabla \times \big((\mathbf{r}\cdot\nabla)\nabla\psi\big)$$

or equivalently,

$$-\nabla \times \nabla \times \mathbf{M} = \mathbf{r} \times \nabla\nabla^2\psi \tag{11}$$

where we have used the fact that

$$\nabla \times \nabla\psi = 0, \tag{12}$$

and

$$\nabla \times (\mathbf{r}\cdot\nabla)\nabla\psi = 0. \tag{13}$$

The identity in (12) is well known. The second identity in (13) is slightly lengthy to prove but it is straightforward to show in Cartesian coordinates. Proof is done later. Next consider $\nabla\nabla\cdot\mathbf{M}$. Using the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} + \mathbf{A}\cdot\nabla\times\mathbf{B}$$

we can write

$$\nabla\nabla\cdot\mathbf{M} = \nabla\nabla\cdot(\mathbf{r}\times\nabla\psi) = \nabla(\nabla\psi\cdot\nabla\times\mathbf{r} + \mathbf{r}\cdot\nabla\times\nabla\psi) = 0. \tag{14}$$

Using (11) and (14) we get

$$\begin{aligned}-\nabla\times\nabla\times\mathbf{M} + \nabla\nabla\cdot\mathbf{M} + \omega^2\mu\epsilon\mathbf{M} &= \mathbf{r}\times\nabla\nabla^2\psi + \omega^2\mu\epsilon(\mathbf{r}\times\nabla\psi)\\ &= \mathbf{r}\times\nabla(\nabla^2\psi + \omega^2\mu\epsilon\psi) = 0\end{aligned}$$

Since it is given that the function ψ satisfies the wave equation.

To show that the wave function $\mathbf{N}$ also satisfies the wave equation, we put it into the wave equation and get

$$\begin{aligned}-\nabla\times\nabla\times\mathbf{N} + \nabla\nabla\cdot\mathbf{N} + \omega^2\mu\epsilon\mathbf{N} &= -\nabla\times\nabla\times\frac{1}{\omega\sqrt{\mu\epsilon}}\nabla\times\mathbf{M} + \nabla\nabla\cdot\frac{1}{\omega\sqrt{\mu\epsilon}}\nabla\times\mathbf{M} + \omega^2\mu\epsilon\frac{1}{\omega\sqrt{\mu\epsilon}}\nabla\times\mathbf{M}\\ &= \frac{1}{\omega\sqrt{\mu\epsilon}}\nabla\times(\nabla\times\nabla\times\mathbf{M} + \omega^2\mu\epsilon\mathbf{M}) = 0\end{aligned}$$

Notice that the second term is middle expression is in the form $\nabla\cdot\nabla\times\mathbf{M}$ and therefore it is identically zero. The final term is zero since we have already shown that $\mathbf{M}$ satisfies the vector wave equation.

Proof of Eq. (13):

$$\begin{aligned}(\mathbf{r}\cdot\nabla)\nabla\psi &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\frac{\partial\psi}{\partial x}\hat{\boldsymbol{a}}_x + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{a}}_y + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{a}}_z\right)\\ &= \left(x\frac{\partial\left(\frac{\partial\psi}{\partial x}\hat{\boldsymbol{a}}_x + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{a}}_y + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{a}}_z\right)}{\partial x} + y\frac{\partial\left(\frac{\partial\psi}{\partial x}\hat{\boldsymbol{a}}_x + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{a}}_y + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{a}}_z\right)}{\partial y} + z\frac{\partial\left(\frac{\partial\psi}{\partial x}\hat{\boldsymbol{a}}_x + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{a}}_y + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{a}}_z\right)}{\partial z}\right)\\ &= \left(x\frac{\partial^2\psi}{\partial x^2} + y\frac{\partial^2\psi}{\partial y\partial x} + z\frac{\partial^2\psi}{\partial z\partial x}\right)\hat{\boldsymbol{a}}_x + \left(x\frac{\partial^2\psi}{\partial x\partial y} + y\frac{\partial^2\psi}{\partial y^2} + z\frac{\partial^2\psi}{\partial z\partial y}\right)\hat{\boldsymbol{a}}_y + \left(x\frac{\partial^2\psi}{\partial x\partial z} + y\frac{\partial^2\psi}{\partial y\partial z} + z\frac{\partial^2\psi}{\partial z^2}\right)\hat{\boldsymbol{a}}_z\end{aligned}$$

Its curl can be evaluated as

$$\nabla \times (\mathbf{r} \cdot \nabla)\nabla\psi = \begin{vmatrix} \hat{\boldsymbol{a}}_x & \hat{\boldsymbol{a}}_y & \hat{\boldsymbol{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(x\dfrac{\partial^2\psi}{\partial x^2} + y\dfrac{\partial^2\psi}{\partial y\partial x} + z\dfrac{\partial^2\psi}{\partial z\partial x}\right) & \left(x\dfrac{\partial^2\psi}{\partial x\partial y} + y\dfrac{\partial^2\psi}{\partial y^2} + z\dfrac{\partial^2\psi}{\partial z\partial y}\right) & \left(x\dfrac{\partial^2\psi}{\partial x\partial z} + y\dfrac{\partial^2\psi}{\partial y\partial z} + z\dfrac{\partial^2\psi}{\partial z^2}\right) \end{vmatrix} = 0$$

Problem 3. (Reading Assignment Problem)

1) Define plane, cylindrical, and spherical waves.
2) Given $\mathbf{E}(\mathbf{r}) = \hat{\boldsymbol{a}}_x E_0 e^{-\alpha y} e^{-j\beta z}$ where E_0, α, and β are positive constants.
   a) Find the corresponding $\mathbf{H}$ phasor,
   b) Is this electromagnetic field a TEM wave?

3) a)

Definition 1: An electromagnetic wave is called a PLANE WAVE if its "constant phase surfaces" are PLANES.

Definition 2: A plane wave is called a UNIFORM PLANE WAVE (u.p.w.) if the "constant magnitude surfaces" are the same as the "constant phase planes".

Example:

Let $\bar{E} \cong \hat{a}_\theta E_0(\theta) \frac{e^{-jkR}}{R}$ in spherical coordinates (R, θ, ϕ) (for a small dipole at far field)

phase of $\bar{E}$ = $\angle\bar{E} = -kR$ (k: propagation constant)

const. phase surfaces: $-kR = \text{const} \Rightarrow \boxed{R = \text{const.}}$ equation for a family of spherical surfaces

$\Downarrow$

Given $\bar{E}$-phasor belongs to a SPHERICAL WAVE!

Example:

Let $\bar{E} \cong \hat{a}_z E_0(\beta) \frac{e^{-j\beta r}}{\sqrt{r}}$ in cylindrical coordinates (r, ϕ, z) (far field generated by an ∞'ly long line current)

phase of $\bar{E}$ = $\angle\bar{E} = -\beta r$ (β = propagation constant)

const. phase surfaces: $-\beta r = \text{const.} \Rightarrow \boxed{r = \text{constant}}$ equation for a family of cylindrical surfaces.

$\Downarrow$

Given $\bar{E}$-phasor belongs to a CYLINDRICAL WAVE!

3) b)

Exercise:

Given $\bar{E}(\bar{r}) = \hat{a}_x \underbrace{E_0 e^{-\alpha y} e^{-j\beta z}}_{E_x}$ (V/m) (E_0, α, β are all real positive constants.)

a) Find the corresponding $\bar{H}$ phasor.

b) Is this given electromagnetic field a TEM wave?

(TEM Wave: Transverse Electromagnetic wave)

Solution:

a) Given $\bar{E}$ phasor belongs to a non-uniform plane wave as constant phase sufaces are $-\beta z = \text{const} \Rightarrow z = \text{Constant}$ planes

and constant magnitude sufaces are $E_0 e^{-\alpha y} = \text{const.} \Rightarrow y = \text{const.}$ planes

Any Electromagnetic wave should satisfy the Maxwell equations $\Rightarrow \bar{\nabla} \times \bar{E} = -j\omega\mu\bar{H}$ should be satisfied (assuming a linear medium)

$$\Rightarrow \bar{H} = \frac{1}{-j\omega\mu} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \frac{j}{\omega\mu}\left[-\hat{a}_y\left(0 - \frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(-\frac{\partial E_x}{\partial y}\right)\right]$$

$$\bar{H} = \frac{j}{\omega\mu}\left[\hat{a}_y \underbrace{\frac{\partial}{\partial z}\left(E_0 e^{-\alpha y} e^{-j\beta z}\right)}_{-j\beta E_0 e^{-\alpha y} e^{-j\beta z}} - \hat{a}_z \underbrace{\frac{\partial}{\partial y}\left(E_0 e^{-\alpha y} e^{-j\beta z}\right)}_{-\alpha E_0 e^{-\alpha y} e^{-j\beta z}}\right]$$

$$\boxed{\bar{H} = \frac{\beta}{\omega\mu} E_0 e^{-\alpha y} e^{-j\beta z} \hat{a}_y + \alpha E_0 e^{-\alpha y} e^{-j\beta z} \hat{a}_z}$$

(Note that using $\bar{H} = \frac{1}{\eta} \hat{n} \times \bar{E}$ gives <u>incorrect</u> results here as the wave is <u>not</u> a u.p.w.)

b) As the $\bar{H}$ field has a component in the propagation direction ($\hat{n} = \hat{a}_z$), this electromagnetic wave does not belong to TEM waves.