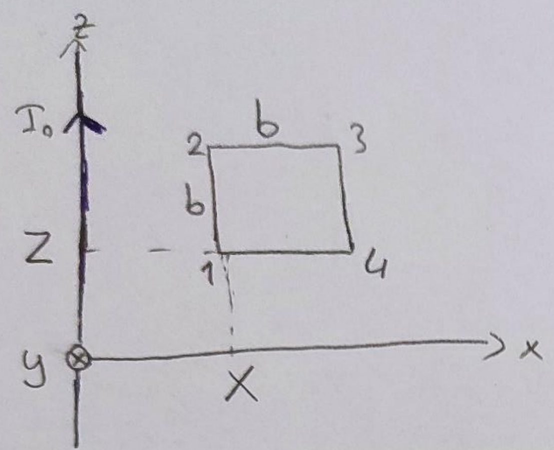
EE303 HW#1 Solution Key

Solution of Problem 1



$\vec{B} = \hat{a}_\phi \dfrac{\mu_o I_o}{2\pi r}$ Tesla

here r: distance from z-axis in $\hat{a}_r$ direction

at $y=0$ plane $\vec{B} = \hat{a}_y \dfrac{\mu_o I_o}{2\pi x}$ T

a) For $\hat{n} = \hat{a}_x$: $V_{ind} = -\dfrac{\partial \phi}{\partial t}$ where $\phi = \int \vec{B} \cdot \vec{ds}$

$\vec{ds} = \hat{a}_y dx\, dz$

$$\phi = \int_{X(t)}^{X(t)+b} \int_{z_o}^{z_o+b} \frac{\mu_o I_o}{2\pi x} dz\, dx = \frac{\mu_o I_o b}{2\pi} \ln\left(\frac{X(t)+b}{X(t)}\right) \quad \text{Wb}$$

$$V_{ind} = -\frac{d\phi}{dt} = -\left(\frac{\mu_o I_o b}{2\pi} \frac{d}{dt}\left(\ln\left(\frac{X(t)+b}{X(t)}\right)\right)\right)$$

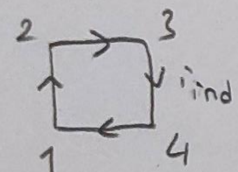
$$\Rightarrow V_{ind} = -\frac{\mu_o I_o b}{2\pi}\left(\frac{-b}{X(t)\,(X(t)+b)}\right)\frac{d(X(t))}{dt}$$

$$X(t) = \underbrace{X(t=0)}_{X_o} + \frac{a_o t^2}{2} \Rightarrow \frac{d(X(t))}{dt} = a_o t$$

Then $$\boxed{V_{ind} = \frac{\mu_o I_o b^2}{2\pi} \frac{a_o t}{\left(x_o + \frac{a_o t^2}{2}\right)\left(x_o + \frac{a_o t^2}{2} + b\right)} \quad \text{Volts}}$$

$$i_{ind} = \frac{V_{ind}}{R} = \frac{b^2 \mu_0 I}{2\pi R} \frac{a_0 t}{\left(x_0 + \frac{a_0 t^2}{2}\right)\left(x_0 + \frac{a_0 t^2}{2} + b\right)} \quad A$$

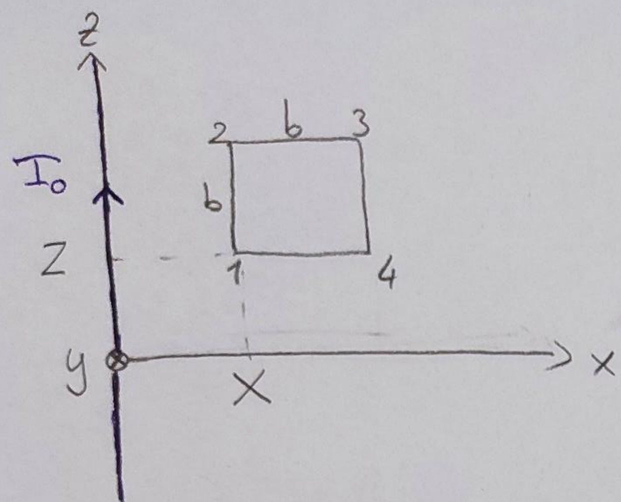
The direction of the induced current can be found by Lenz's Rule. As the square loop moves in $\hat{n} = \hat{a}_x$ direction, Φ (flux) linked by the loop decreases. Therefore, i_{ind} should be in the clockwise direction to support Φ.



b) For $\hat{n} = \hat{a}_z$, there won't be any change in flux as the square loop moves.

Thus, $V_{ind} = \frac{-d\Phi}{dt} = 0$ Volt, $i_{ind} = \frac{V_{ind}}{R} = 0$ A

Alternative Solution of Problem 1



$$V_{ind} = \oint_C \vec{v} \times \vec{B} \cdot \vec{dl}$$

C: The closed contour of the square loop

$C = C_{12} + C_{23} + C_{34} + C_{41}$ then for

C_{12} : $\vec{dl} = \hat{a}_z dz$

C_{23} : $\vec{dl} = \hat{a}_x dx$

C_{34} : $\vec{dl} = \hat{a}_z dz$

C_{41} : $\vec{dl} = \hat{a}_x dx$

a) $\hat{n} = \hat{a}_x \Rightarrow (\vec{v} \times \vec{B}) \cdot \vec{dl} = 0$ for C_{23} and C_{41}

$$(\underset{\downarrow}{\hat{a}_y} \perp \underset{\downarrow}{\hat{a}_x})$$

$$\Rightarrow V_{ind} = \int_{C_{12}} \vec{v} \times \vec{B} \cdot (\hat{a}_z dz) + \int_{C_{34}} \vec{v} \times \vec{B} \cdot (\hat{a}_z dz)$$

$\vec{B} = \hat{a}_y \dfrac{\mu_o I_o}{2\pi x}$ Tesla at $y=0$ plane

$\vec{v} = \hat{a}_x a_o t$

$$\Rightarrow V_{ind} = \int_{z_0}^{z_0+b} (\hat{a}_x a_0 t) \times \left(\hat{a}_y \frac{\mu_0 I_0}{2\pi X(t)}\right) \cdot (\hat{a}_z dz) + \int_{z_0+b}^{z_0} (\hat{a}_x a_0 t) \times \left(\hat{a}_y \frac{\mu_0 I_0}{2\pi (X(t)+b)}\right) \cdot (\hat{a}_z dz)$$

$$\Rightarrow V_{ind} = \frac{\mu_0 I_0 a_0 t b^2}{2\pi X(t) (X(t)+b)} \quad \text{Volts}$$

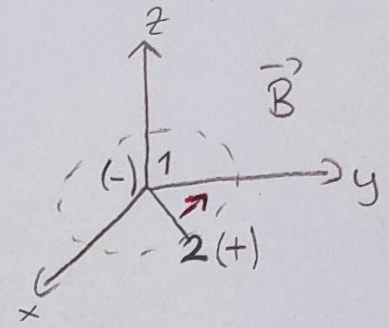
$$X(t) = X(t=0) + \frac{a_0 t^2}{2} = x_0 + \frac{a_0 t^2}{2}$$

$$\Rightarrow \boxed{V_{ind} = \frac{\mu_0 I_0 a_0 t b^2}{2\pi} \frac{1}{\left(x_0 + \frac{a_0 t^2}{2}\right)\left(x_0 + \frac{a_0 t^2}{2} + b\right)} \quad \text{Volts}}$$

↳ The same as the result of 1^{st} solution

Solution of Problem 2

a) $\vec{B} = B_o \hat{a}_z$ Tesla



$V_{ind} = \int \vec{v} \times \vec{B} \cdot \vec{dl}$

$\vec{v} = |\vec{v}|\, \hat{a}_\phi$

$\vec{dl} = \hat{a}_r\, dr$

N revolutions in 1 min (60 sec)

$\Rightarrow \frac{N}{60}$ revolutions in 1 sec

At each revolution, the rotational distance taken $= 2\pi r$ (where r is the distance of an arbitrary point on the conducting rod to the origin.)

$\Rightarrow$ With $\frac{N}{60}$ revolutions/sec, $\underbrace{\text{the distance taken in 1 sec} = 2\pi r \left(\frac{N}{60}\right) \text{m}}_{\text{the definition of velocity in m/s}}$

$$\Rightarrow \boxed{\vec{v} = \frac{2\pi r N}{60}\, \hat{a}_\phi \ \text{m/s}}$$

$$V_{21} = \int_0^L \left(\hat{a}_\phi \frac{2\pi r N}{60}\right) \times \left(\hat{a}_z B_o\right) \cdot \left(\hat{a}_r\, dr\right)$$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_r$ and $\hat{a}_r \cdot \hat{a}_r = 1$

$$\Rightarrow V_{21} = \frac{2\pi N B_o}{60} \int_0^L r\, dr = \frac{2\pi N B_o}{60} \left(\frac{r^2}{2} \Big|_0^L \right)$$

$$\boxed{V_{21} = \frac{\pi N B_o L^2}{60} \ \text{Volts}}$$

b) $V_{ind} = \int \vec{v} \times \vec{B} \cdot \vec{dl}$ However, this time $\vec{v} \times \vec{B} = 0$

$(\hat{a}_\phi \times \hat{a}_\phi = 0)$

$$\Rightarrow \boxed{V_{21} = 0 \ \text{V}}$$

Solution of Problem 3.

a) Maxwell's Equations:
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{D} = \rho_v$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

Constitutive relations:
$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \varepsilon\vec{E}$$

Assuming a simple medium, ε and μ are constants.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$\downarrow$

Taking the curl of $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H}$$

Since $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\cdot\vec{H} = 0 \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\nabla^2\vec{H}$

Then, $-\nabla^2\vec{H} = \vec{\nabla}\times\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = \vec{\nabla}\times\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{D})$

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{D}) = \frac{\partial}{\partial t}(\varepsilon\,\vec{\nabla}\times\vec{E}) = \varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = \varepsilon\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{D}) = -\varepsilon\frac{\partial^2 \vec{B}}{\partial t^2} = -\varepsilon\mu\frac{\partial^2 \vec{H}}{\partial t^2} \quad (\text{using } \vec{B} = \mu\vec{H})$$

$$\Rightarrow -\nabla^2 \vec{H} = \vec{\nabla} \times \vec{J} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2}$$ which can be written also

as $$\boxed{\nabla^2 \vec{H} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = -\vec{\nabla} \times \vec{J}}$$

b) $\vec{\nabla} \cdot \vec{B} = 0$

using the null identity $\vec{\nabla} \cdot (\vec{\nabla} \times \underset{\text{a vector field}}{\vec{V}}) = 0$

$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$, so that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ is satisfied

$\vec{A}$: Vector potential function

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t}\right) \Rightarrow \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

Using the null identity $\vec{\nabla} \times (\vec{\nabla} \underset{\text{a scalar field}}{S}) = 0$,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$$ Φ : Scalar potential function

$$\boxed{\nabla \Phi = -\left(E + \frac{\partial \vec{A}}{\partial t}\right)}$$

c) $\vec{B} = \vec{\nabla} \times \vec{A}$ (the curl of vector potential function)

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

(For static problems, $\frac{d}{\partial t} = 0$, $\vec{E} = -\vec{\nabla}\phi$)