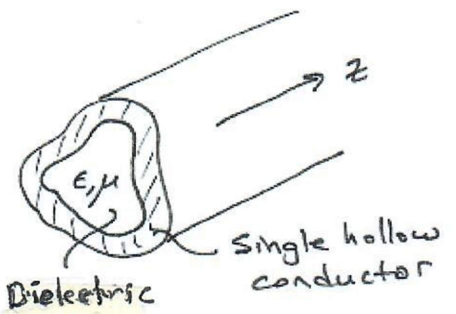
Waveguides

Summary: In analysis, we assumed a waveguide structure extending along the z-axis, having a uniform cross-section and filled by a lossless simple dielectric of parameters ϵ and μ.



(Note that transmission lines are waveguides with two-conductor structure. They can support not only TE and TM waves but also TEM waves.)

In phasor domain, for waves propagating in the $\hat{a}_z$ direction with a propagation constant γ,

$$\bar{E}(x,y,z) = \bar{E}^\circ(x,y)\, e^{-\gamma z} \quad \text{and} \quad \bar{H}(x,y,z) = \bar{H}^\circ(x,y)\, e^{-\gamma z}$$

which satisfy the Helmholtz Eqn:

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0 \quad , \quad \nabla^2 \bar{H} + k^2 \bar{H} = 0 \quad \text{with} \quad k^2 = \omega^2 \mu \epsilon$$

$$\text{where} \quad \nabla^2 = \underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}_{\nabla_t^2} + \frac{\partial^2}{\partial z^2} = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \frac{\partial^2 \bar{E}}{\partial z^2} = \gamma^2 \bar{E}$$

$$\Rightarrow \nabla_t^2 \bar{E} + (\gamma^2 + k^2)\bar{E} = 0 \Rightarrow \boxed{\nabla_t^2 \bar{E}^\circ + (\gamma^2 + k^2)\bar{E}^\circ = 0}$$

(*) Also

$$\boxed{\nabla_t^2 \bar{H}^\circ + (\underbrace{\gamma^2 + k^2}_{h^2})\bar{H}^\circ = 0}$$

In general,

$$\bar{E}^\circ(x,y) = E_x^\circ(x,y)\,\hat{a}_x + E_y^\circ(x,y)\,\hat{a}_y + E_z^\circ(x,y)\,\hat{a}_z$$

$$\bar{H}^\circ(x,y) = H_x^\circ(x,y)\,\hat{a}_x + H_y^\circ(x,y)\,\hat{a}_y + H_z^\circ(x,y)\,\hat{a}_z$$

$E_x^\circ, E_y^\circ, E_z^\circ$, $H_x^\circ, H_y^\circ, H_z^\circ$ and γ need to be determined.

(1-b)

The vectorial partial differential equation (pde)

$$\nabla_t^2 \bar{E}^\circ + (\gamma'^2 + k^2)\bar{E}^\circ = 0$$

can be reduced into three different scalar pde's such that

$$\nabla_t^2 E_x^\circ + (\gamma'^2+k^2)E_x^\circ = 0$$
$$\nabla_t^2 E_y^\circ + (\gamma'^2+k^2)E_y^\circ = 0$$
$$\nabla_t^2 E_z^\circ + (\gamma'^2+k^2)E_z^\circ = 0$$

(Three more scalar pde's can be written for H_x°, H_y° and H_z° similarly)

However, we don't need to solve these six scalar pde's to get $\bar{E}^\circ$ and $\bar{H}^\circ$. Because, various components of $\hat{E}^\circ$ and $\bar{H}^\circ$ are related to each other by Maxwell equations as follows:

Consider, for instance, $\bar{\nabla} \times \hat{E} = -j\omega\mu\bar{H}$ in phasor domain

$$\Rightarrow \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x^\circ e^{-\gamma z} & E_y^\circ e^{-\gamma z} & E_z^\circ e^{-\gamma z} \end{vmatrix} = -j\omega\mu\left(H_x^\circ e^{-\gamma z}\hat{a}_x + H_y^\circ e^{-\gamma z}\hat{a}_y + H_z^\circ e^{-\gamma z}\hat{a}_z\right)$$

Compute the determinant on the LHS, equate corresponding x, y and z components on the LHS and RHS (cancel $e^{-\gamma z}$ terms) to get equations (1), (2) and (3). Note that $\frac{\partial}{\partial z}(e^{-\gamma z})$ is written as $-\gamma' e^{-\gamma z}$.

(1-c)

$$\begin{cases} \dfrac{\partial E_z^\circ}{\partial y} + \gamma' E_y^\circ = -j\omega\mu H_x^\circ & \text{---- (1)} \quad (x\text{-components}) \\ -\gamma' E_x^\circ - \dfrac{\partial E_z^\circ}{\partial x} = -j\omega\mu H_y^\circ & \text{---- (2)} \quad (y\text{-components}) \\ \dfrac{\partial E_y^\circ}{\partial x} - \dfrac{\partial E_x^\circ}{\partial y} = -j\omega\mu H_z^\circ & \text{---- (3)} \quad (z\text{-components}) \end{cases}$$

Now, using $\nabla \times \bar{H} = j\omega\epsilon \bar{E}$ (for $\bar{J}=0$ in source-free lossless dielectric)

obtain three more equations (4), (5) and (6) similarly:

$$\begin{cases} \dfrac{\partial H_z^\circ}{\partial y} + \gamma' H_y^\circ = j\omega\epsilon E_x^\circ & \text{------ (4)} \quad (x\text{-components}) \\ -\gamma' H_x^\circ - \dfrac{\partial H_z^\circ}{\partial x} = j\omega\epsilon E_y^\circ & \text{---- (5)} \quad (y\text{-components}) \\ \dfrac{\partial H_y^\circ}{\partial x} - \dfrac{\partial H_x^\circ}{\partial y} = j\omega\epsilon E_z & \text{---- (6)} \quad (z\text{-components}) \end{cases}$$

Then, use equations (1) through (6) to express

H_x°, H_y°, E_x°, E_y° (transversal components lying on the plane perpendicular to the propagation direction z)

in terms of the axial components H_z° and E_z°.

Therefore, we can solve only H_z° or E_z° from a pde, then find the rest of the components using equations (7) through (10).

* The transversal component unknowns $H_x^\circ, H_y^\circ, E_x^\circ$ and E_y° can be written in terms of axial components E_z°, H_z°

$$\left.\begin{aligned}
&(7)\quad H_x^\circ = \frac{-1}{h^2}\left(\gamma \frac{\partial H_z^\circ}{\partial x} - j\omega\epsilon \frac{\partial E_z^\circ}{\partial y}\right)\\
&(8)\quad H_y^\circ = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^\circ}{\partial y} + j\omega\epsilon \frac{\partial E_z^\circ}{\partial x}\right)\\
&(9)\quad E_x^\circ = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^\circ}{\partial x} + j\omega\mu \frac{\partial H_z^\circ}{\partial y}\right)\\
&(10)\quad E_y^\circ = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^\circ}{\partial y} - j\omega\mu \frac{\partial H_z^\circ}{\partial x}\right)
\end{aligned}\right\} \quad \text{where } \underline{h^2 = \gamma^2 + k^2}$$

<u>TEM WAVES (modes)</u> : $E_z = 0$, $H_z = 0$ by definition

$\Rightarrow$ Eqns. (7–10) can produce non-trivial solutions only if $h=0$.

$h^2 = 0 \Rightarrow \gamma^2 + k^2 = 0$. Let $\gamma = \gamma_{TEM}$ and we know $k^2 = \omega^2\mu\epsilon$

$\Rightarrow \boxed{\gamma_{TEM} = jk = j\omega\sqrt{\mu\epsilon}}$ Also, $v_{TEM} = \frac{\omega}{Im\{\gamma_{TEM}\}} = \frac{\omega}{\omega\sqrt{\mu\epsilon}}$

$$\boxed{v_{TEM} = \frac{1}{\sqrt{\mu\epsilon}}}$$

If TEM waves were present, but they can't be supported within 1-conductor metallic WG structures!

<u>Defn:</u> For a wave propagating in +z direction, the WAVE IMPEDANCE (Z) is defined as

$$Z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

see (eqn(2))

For TEM modes, $Z_{TEM} = \frac{E_x}{H_y} = \frac{E_x^\circ}{H_y^\circ} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{j\omega\mu}{j\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$

$$\boxed{Z_{TEM} = \sqrt{\frac{\mu}{\epsilon}} = \eta}$$

η: Intrinsic imp. of the dielectric filling the waveguide

(same result can be obtained from $Z_{TEM} = -E_y/H_y$ also)

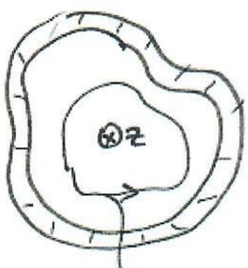
FACT: Single-conductor waveguides can **NOT** support TEM waves.

Assume a TEM wave solution is possible within a one-conductor waveguide extending in z-direction

As $E_z = H_z = 0 \Rightarrow \bar{E}$ and $\bar{H}$ fields must lie on the transverse plane

Also we know from Maxwell Eqns. $\nabla \cdot \bar{B} = 0 \Rightarrow \nabla \cdot \bar{H} = 0$ (as μ is constant)

$\nabla \cdot \bar{H} = 0 \Rightarrow \bar{H}$ field lines must form closed loops in the transverse plane



a closed loop of $\bar{H}$ lines on the (x-y) plane

According to Ampere's Circuital Law:

$$\oint_C \bar{H} \cdot d\bar{l} = I_c + I_d = \text{total axial current through the loop}$$

I_c: Conduction current which is zero as there is no inner conductor to carry the current

I_d: Displacement current which is zero as $I_d = \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$ but $D_z = \epsilon \underbrace{E_z}_{0} = 0$

$$\Rightarrow \oint_C \bar{H} \cdot d\bar{l} = 0 \quad \text{which is a contradiction}$$

$\Rightarrow$ TEM waves can not be supported in single conductor waveguides.

TM Waves: $H_z = 0$ by definition

Obtain $E_z = E_z^{\circ} e^{-\gamma z}$ solving $\nabla_t^2 E_z^{\circ} + h^2 E_z^{\circ} = 0$

with the boundary condition that tangential $\bar{E}$ becomes zero on the perfect conductor walls of the waveguide. Here E_z is the tangential component

$$\Rightarrow \text{Solve} \begin{cases} \nabla_t^2 E_z^{\circ} + h^2 E_z^{\circ} = 0 \\ \text{s.t. } E_z^{\circ} = 0 \text{ on the waveguide walls.} \end{cases}$$

- After you obtain E_z°, you can use eqns. (7-10) to get transversal components. Let $H_z^{\circ} = 0$ in Eq.(7-10)

Eqns. (12)

$$\left.\begin{cases} H_x^{\circ} = -\dfrac{1}{h^2}(-j\omega\epsilon)\dfrac{\partial E_z^{\circ}}{\partial y} \\ H_y^{\circ} = -\dfrac{1}{h^2}(j\omega\epsilon)\dfrac{\partial E_z^{\circ}}{\partial x} \\ E_x^{\circ} = -\dfrac{1}{h^2}\gamma\dfrac{\partial E_z^{\circ}}{\partial x} \\ E_y^{\circ} = -\dfrac{1}{h^2}\gamma\dfrac{\partial E_z^{\circ}}{\partial y} \end{cases}\right\} \Rightarrow Z_{TM} = \frac{E_x^{\circ}}{H_y^{\circ}} = -\frac{E_y^{\circ}}{H_x^{\circ}} \Downarrow \boxed{Z_{TM} = \frac{\gamma}{j\omega\epsilon}}$$

γ: needs to be determined.

TE Waves $E_z = 0$ by definition

Obtain $H_z = H_z^{\circ} e^{-\gamma z}$ by solving $\boxed{\nabla_t^2 H_z^{\circ} + h^2 H_z^{\circ} = 0}$ subject to proper B.C.'s.

Eqns. (13)

$$\left.\begin{cases} H_x^{\circ} = -\dfrac{1}{h^2}\gamma\dfrac{\partial H_z^{\circ}}{\partial x} \\ H_y^{\circ} = -\dfrac{1}{h^2}\gamma\dfrac{\partial H_z^{\circ}}{\partial y} \\ E_x^{\circ} = -\dfrac{1}{h^2}(j\omega\mu)\dfrac{\partial H_z^{\circ}}{\partial y} \\ E_y^{\circ} = -\dfrac{1}{h^2}(-j\omega\mu)\dfrac{\partial H_z^{\circ}}{\partial x} \end{cases}\right\}$$

found by using eqns. (7-10) with $E_z^{\circ} = 0$

$$Z_{TE} = \frac{E_x^{\circ}}{H_y^{\circ}} = -\frac{E_y^{\circ}}{H_x^{\circ}} \Rightarrow \boxed{Z_{TE} = \frac{j\omega\mu}{\gamma}}$$

γ: needs to be determined.

Cut-off Phenomena in Waveguides

- For both TE and TM modes, we must solve the diff. eqn. (with proper B.C.'s)

$$\nabla_t^2 \phi + h^2 \phi = 0 \Rightarrow \boxed{\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} + h^2 \phi(x,y) = 0} \quad (11)$$

where $h^2 = \gamma^2 + k^2$
and $\phi(x,y)$ is either E_z° or H_z°

This Boundary Value Problem (BVP) has non-trivial solutions only for some discrete values of "h".

Definition: The discrete values of h for which non-trivial solutions to Eqn. (11) exists are called Eigenvalues or Characteristic Values of the BVP.

Each eigenvalue corresponds to an eigenvector which is in fact a particular TM or TE mode in the waveguide.

Consider $h^2 \triangleq \gamma^2 + k^2 \Rightarrow \gamma = \sqrt{h^2 - k^2}$
$\gamma = \sqrt{h^2 - \omega^2 \mu \epsilon}$

Case 1: If at a given frequency, $\omega^2 \mu \epsilon < h^2 \Rightarrow h^2 - \omega^2 \mu \epsilon > 0$

$\Rightarrow \gamma$ is a real number $\Rightarrow \boxed{\gamma = \alpha}$: real

$\Rightarrow$ The solution has a term: $e^{-\gamma z} = e^{-\alpha z}$: represents attenuation (Evanescent wave)

$\therefore$ For $\omega^2 \mu \epsilon < h^2 \Rightarrow$ wave does not propagate, just attenuates. It is called EVANESCENT solution.

Case 2: If $\omega^2\mu\epsilon > h^2 \Rightarrow h^2 - \omega^2\mu\epsilon < 0$

$\Rightarrow \gamma'$ is a purely imaginary number

$$\gamma' = \sqrt{\underbrace{h^2-\omega^2\mu\epsilon}_{<0}} = \sqrt{-(\omega^2\mu\epsilon - h^2)} = j\sqrt{\underbrace{\omega^2\mu\epsilon-h^2}_{\text{positive}}} = \boxed{j\beta = \gamma'}$$

$\Rightarrow$ The solution has a term : $e^{-\gamma z} = e^{-j\beta z}$: represents a PROPAGATING wave.

Cut-off Condition refers to $\boxed{\gamma' = 0}$ case

$\gamma' = \sqrt{h^2-\omega^2\mu\epsilon}$ in general, at cut-off let $\omega = \omega_c$

$$\gamma' = 0 \Rightarrow \left.\begin{array}{l} h^2 = \omega_c^2\mu\epsilon \\ h = \omega_c\sqrt{\mu\epsilon} \end{array}\right\} \Rightarrow \boxed{h = \frac{\omega_c}{v}} \text{ where } \boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$$

$\omega_c = 2\pi f_c$ — ω_c: Cut-off angular freq. (in rad/sec); f_c: cut-off frequency (in Hz)

$$\boxed{f_c = \frac{\omega_c}{2\pi} = \frac{h}{2\pi\sqrt{\mu\epsilon}} = \frac{hv}{2\pi}}$$

$$\gamma' = \sqrt{h^2-\omega^2\mu\epsilon} = \sqrt{\omega_c^2\mu\epsilon - \omega^2\mu\epsilon} = \sqrt{\omega^2\mu\epsilon}\sqrt{\frac{\omega_c^2}{\omega^2}-1} = k\sqrt{\left(\frac{f_c}{f}\right)^2-1}$$

or

$$\gamma' = \sqrt{h^2-\omega^2\mu\epsilon} = \sqrt{\omega_c^2\mu\epsilon - \omega^2\mu\epsilon} = \underbrace{\sqrt{\omega_c^2\mu\epsilon}}_{h}\sqrt{1-\frac{\omega^2}{\omega_c^2}} = h\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$

$$\gamma' = \begin{cases} \alpha = k\sqrt{\left(\frac{f_c}{f}\right)^2-1} & \text{for } f<f_c \rightarrow \text{EVANESCENT MODE} \\ 0 & \text{for } f=f_c \rightarrow \text{CUT-OFF} \\ j\beta = j\underbrace{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}}_{\beta} & \text{for } f>f_c \rightarrow \text{PROPAGATING MODE} \end{cases}$$

For a propagating mode

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{(2\pi/k)}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

where λ_g : guided wavelength $(2\pi/\beta)$

λ : wavelength of a plane wave with freq. f in an unbounded lossless medium with (ϵ, μ)

$$\left(\lambda = 2\pi/k = \frac{2\pi}{2\pi f\sqrt{\mu\epsilon}} = \frac{1}{f\sqrt{\mu\epsilon}} = \frac{v}{f}\right)$$

$\lambda_g > \lambda$ for a propagating wave as $f > f_c$

Also define λ_c : cut-off wavelength (value of λ at $f = f_c$)

$$\boxed{\lambda_c = \frac{v}{f_c}} \quad \text{or} \quad \lambda_c = \frac{v\,2\pi}{f_c 2\pi} = \frac{2\pi v}{\omega_c} = \frac{2\pi}{\omega_c/v}$$

$\Rightarrow$ remember $h = \frac{\omega_c}{v}$

$$\Rightarrow \boxed{\lambda_c = \frac{v}{f_c} = \frac{2\pi}{h}}$$

(h is the eigenvalue)

- Note that λ_g, λ and λ_c are related by

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$$

Phase velocity of a propagating wave $= v_g = \frac{\omega}{\beta}$

$$v_g = \frac{\omega}{\beta} = \frac{\omega}{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \boxed{\frac{v}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = v_g}$$

As $f > f_c \Rightarrow v_g > v$ for a propagating wave.

Note that $v_g = v_g(f)$ is a function of frequency.

$\Rightarrow$ If propagating signal has different frequency components, each will propagate at a different speed.

i.e. The single-conductor waveguides are <u>DISPERSIVE</u>

Wave Impedances

	TM	TE
	$Z_{TM} = \dfrac{\gamma}{j\omega\epsilon}$	$Z_{TE} = \dfrac{j\omega\mu}{\gamma}$
For $f > f_c$ ($\gamma = j\beta$) Propagating Modes	$Z_{TM} = \dfrac{j\beta}{j\omega\epsilon} = \dfrac{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{\omega\epsilon}$ (where $k = \omega\sqrt{\mu\epsilon}$) $Z_{TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$: real $= Z_{TM}(f)$ $< \eta = \sqrt{\frac{\mu}{\epsilon}}$	$Z_{TE} = \dfrac{j\omega\mu}{j\beta} = \dfrac{\omega\mu}{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$ $Z_{TE} = \dfrac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$: real $= Z_{TE}(f)$ $> \eta$
For $f < f_c$ ($\gamma = \alpha$) Evanescent Modes	$Z_{TM} = \dfrac{\alpha}{j\omega\epsilon} = j\underbrace{\left(-\dfrac{\alpha}{\omega\epsilon}\right)}_{<0}$	$Z_{TE} = \dfrac{j\omega\mu}{\alpha} = j\underbrace{\left(\dfrac{\omega\mu}{\alpha}\right)}_{>0}$

<u>Note</u>: for $f < f_c$, Z_{TM} and Z_{TE} are both purely reactive indicating that no power flow is associated with evanescent waves.