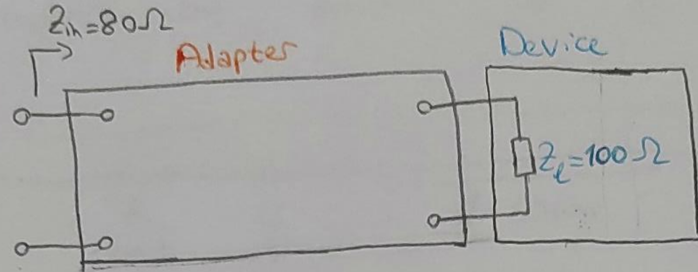
EE303 HW#9 SOLUTIONS

1)



$f = 1$ GHz and we have

- 50 cm lossless coaxial cable with $C_c = 25$ pF/m and $L_c = 62.5$ nH/m
- $40\ \Omega$ resistor, 5.3 pF capacitor and 12.7 nH inductor

• For lossless transmission line,

$$\boxed{Z_o = \sqrt{\frac{L_c}{C_c}} = \sqrt{\frac{62.5 \times 10^{-9}}{25 \times 10^{-12}}} = 50 \text{ Ohm}}$$

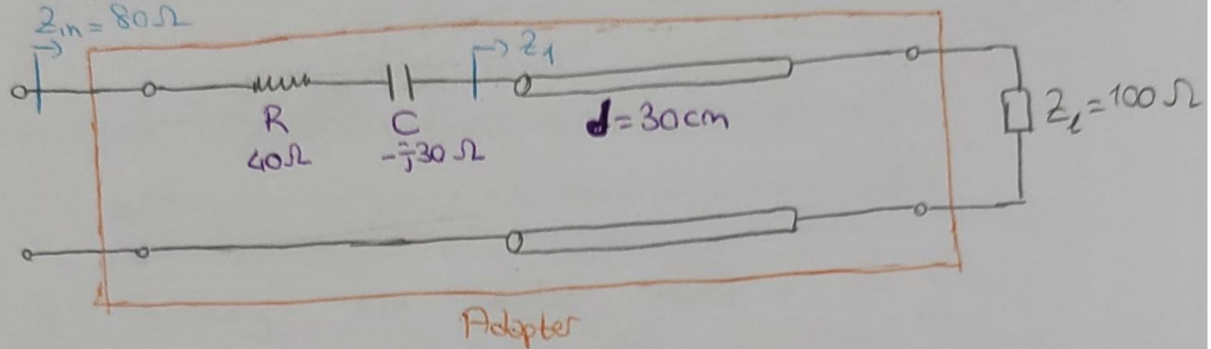
$$\boxed{\beta = \omega\sqrt{L_c C_c} = 2\pi f \sqrt{L_c C_c} = 2\pi\, 10^9 \sqrt{62.5 \times 10^{-9} \times 25 \times 10^{-12}} = 7.854 \text{ rad/m}}$$

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{7.854} = 0.8 \text{ m}}$$

$$\boxed{Z_c = \frac{1}{j\omega C} = \frac{1}{j 2\pi \times 10^9 \times 5.3 \times 10^{-12}} = -j30 \text{ Ohm}}$$

$$\boxed{Z_L = j\omega L = j\, 2\pi \times 10^9 \times 12.7 \times 10^{-9} = j80 \text{ Ohm}}$$

• There can be multiple solutions to this design problem and all correct solutions will be acceptable. One solution is given here as an example:



$$Z_1 = Z_0 \left(\frac{Z_\ell + jZ_0 \tan(\beta d)}{Z_0 + jZ_\ell \tan(\beta d)} \right)$$

Here, $Z_0 = 50$ Ohm, $Z_\ell = 100$ Ohm, $\beta = 7.854$ rad/m

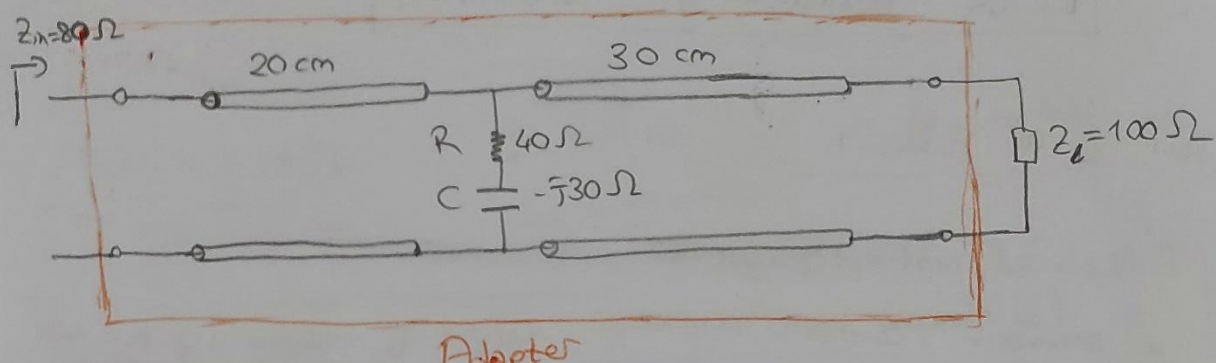
and $d = 30$ cm $= 0.3$ m

Then $Z_1 = 50 \left(\frac{100 + j50 \tan(7.854 \times 0.3)}{50 + j100 \tan(7.854 \times 0.3)} \right) = 40 + j30\ \Omega$

Since $Z_{in} = R + Z_c + Z_1 = 40 + (-j30) + (40 + j30) = 80$ Ohm

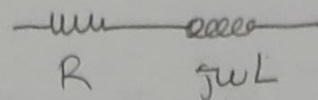
the design is successful.

an alternative design could be the one shown below:



By following the similar procedure, we can confirm that $Z_{in}=80\Omega$ when we use this design.

(2) Load :



R = 10 Ohm

$j\omega L = j 2\pi f L$ $\quad f = 10^9$ Hz $\quad \Rightarrow j\omega L = j30\ \Omega$

$L = 4.77 \times 10^{-9}$ H

$\boxed{Z_\ell = R + j\omega L = 10 + j30 \text{ Ohm}}$ (Load impedance)

Z_0 is given as 20 Ohm

- $\Gamma_\ell = \dfrac{Z_\ell - Z_0}{Z_\ell + Z_0} = \dfrac{10 + j30 - 20}{10 + j30 + 20} = \dfrac{-10 + j30}{30 + j30} = 0.333 + j0.666$

$$VSWR = \frac{1 + |\Gamma_\ell|}{1 - |\Gamma_\ell|} = \frac{1 + |0.333 + j0.666|}{1 - |0.333 + j0.666|} = \boxed{6.85}$$

- $V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$

$\gamma = j\beta$ (for lossless transmission line)

$\Rightarrow V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$

When the distance is measured from the load:

$$V(z') = V(z)\Big|_{z = l - z'} = V^+ e^{-j\beta l} e^{j\beta z'} + V^- e^{j\beta l} e^{-j\beta z'}$$

z': the distance from the load, l: the length of transmission line

$$V(z') = V^+ e^{-j\beta l} e^{j\beta z'} + V^- e^{j\beta l} e^{-j\beta z'} = V_L^+ e^{j\beta z'} + V_L^- e^{-j\beta z'}$$

$(V_L^+ = V^+ e^{-j\beta l}$ and $V_L^- = V^- e^{j\beta l})$

$V_L^+ = 1V$ (given in the question)

$\Gamma_l = \dfrac{V_L^-}{V_L^+} \Rightarrow V_L^- = V_L^+ \Gamma_l = = (0.333 + j0.666)$ Volt

$$\Rightarrow \boxed{V(z') = e^{j\beta z'} \left(1 + (0.333 + j0.666)\, e^{-j2\beta z'}\right) \text{ Volt}}$$

$$\bullet\ |V(z')| = \underbrace{|e^{j\beta z'}|}_{1}\ |1+\Gamma_\ell e^{-j2\beta z'}|\ V$$

$\Rightarrow |V(z')|$ becomes maximum when $|1+\Gamma_\ell e^{-j2\beta z'}|$ is maximum.

$|1+(0.333+j0.666)e^{-j2\beta z'}|$ is max. for $e^{-j2\beta z'} = \dfrac{\Gamma_\ell^*}{|\Gamma_\ell|}$

$$\Rightarrow e^{-j2\beta z'_{max}} = \frac{0.333-j0.666}{|0.333+j0.666|}$$

$$\Rightarrow 2\beta z'_{max} = 1.071 \text{ radian} \Rightarrow \boxed{z'_{max} = \frac{1.071}{4\pi}\lambda = 0.0852\lambda}$$

We know that transmission lines are periodic with $\frac{\lambda}{2}$, the set of points, where $|V(z')|$ is maximum, are $z' = 0.0852\lambda, 0.5852\lambda, 1.0852\lambda \ldots$

1^{st} maximum point

$\hookrightarrow 2^{nd}$

$\hookrightarrow 3^{rd}$

$|V(z')|$ is minimum when $\left|1+(0.333+j0.666)e^{-j2\beta z'}\right|$

is minimum

$\left|1+(0.333+j0.666)e^{-j2\beta z'}\right|$ is min. for $e^{-j2\beta z'} = \dfrac{-\Gamma_\ell^*}{|\Gamma_\ell|}$

$$\Rightarrow e^{-j2\beta z'_{min}} = \frac{-(0.333-j0.666)}{|0.333+j0.666|}$$

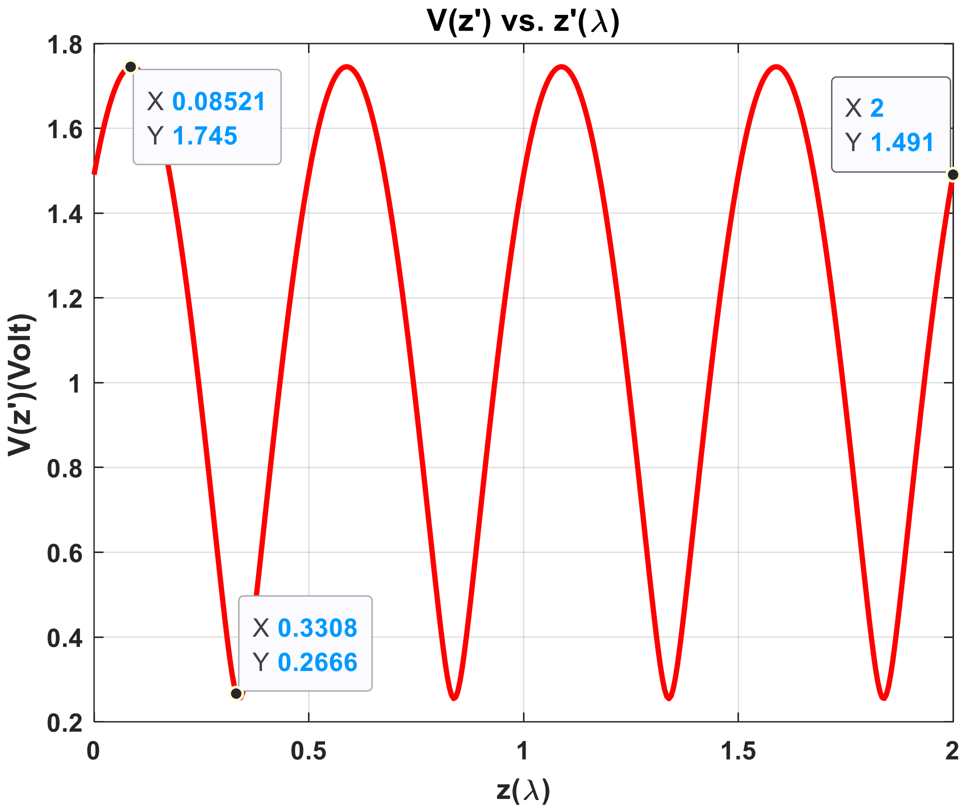
$$\Rightarrow 2\beta z'_{min} = 4.2487 \text{ radian} \quad \Rightarrow \boxed{z'_{min} = \frac{4.2487}{4\pi}\lambda = 0.3381\lambda}$$

Again, due to periodicity with $\frac{\lambda}{2}$, the set of point where $|V(z')|$ is minimum:

$$z'_{min} = \underbrace{0.3381\lambda}_{1^{st} \text{ minimum point}},\ 0.8381\lambda,\ 1.3381\lambda, \ldots$$

$z'_{min,first} = 0.3381\lambda$, $z'_{max,first} = 0.0852\lambda$

$z'_{min\,first} - z'_{max,first} = 0.25\lambda$ as expected ✓



• $V(z') = e^{j\beta z'} \left(1 + \Gamma_\ell e^{-j2\beta z'}\right)$ V in phasor domain

$$V(z',t) = Re\left\{e^{j\omega t}\, V(z')\right\}$$

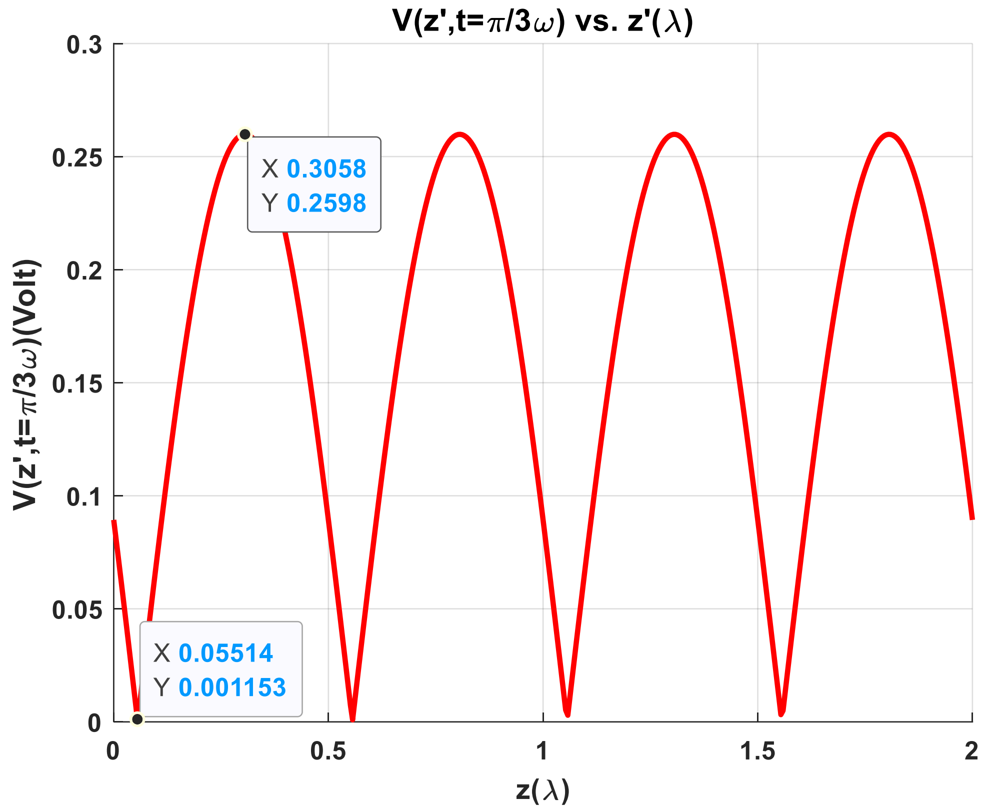
$$\Rightarrow V(z',t) = Re\left\{e^{j(\omega t + \beta z')} + \Gamma_\ell e^{j(\omega t - \beta z')}\right\}$$

$$Re\left\{e^{j(\omega t + \beta z')}\right\} = \cos(\omega t + \beta z')$$

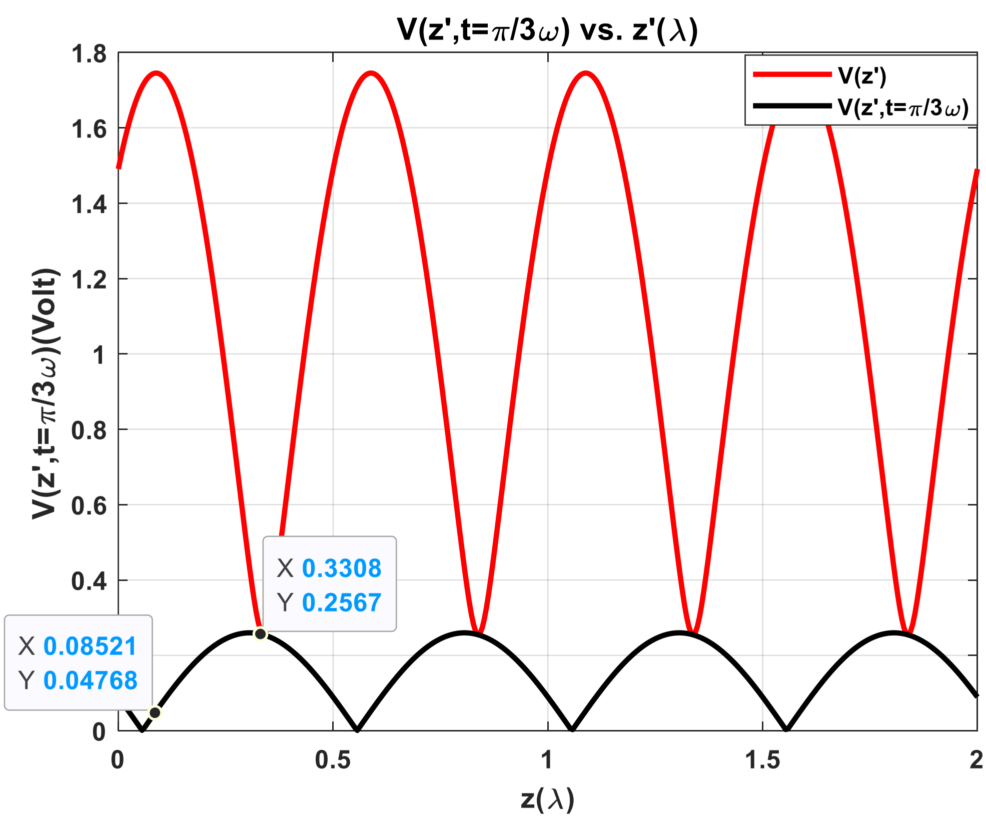
$$Re\left\{(0.333 + j0.666)e^{j(\omega t - \beta z')}\right\} = 0.333\cos(\omega t - \beta z') - 0.666\sin(\omega t - \beta z')$$

$$\Rightarrow \boxed{V(z',t) = \cos(\omega t + \beta z') + 0.333\cos(\omega t - \beta z') - 0.666\sin(\omega t - \beta z')}$$

Volt



V(z',t=π/3ω) vs. z'(λ)
V(z',t=π/3ω)(Volt)
z(λ)
X 0.3058
Y 0.2598
X 0.05514
Y 0.001153
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2



V(z',t=π/3ω) vs. z'(λ)
V(z')
V(z',t=π/3ω)
V(z',t=π/3ω)(Volt)
z(λ)
X 0.3308
Y 0.2567
X 0.08521
Y 0.04768
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0
0
0.5
1
1.5
2

By looking at the plot, we can say that

$$\left|V\left(z' = z'_{max,first}, t=\frac{\pi}{3\omega}\right)\right| < \left|V\left(z'=z'_{min,first}, t=\frac{\pi}{3\omega}\right)\right|$$

$\Rightarrow$ The wave is propagating, not a standing wave, so the magnitudes at min & max points change as time passes.

MATLAB scripts for plots:

```
clc
clear all
close all

omega=2*pi*1e9; % rad/s
R=10; % Ohm
Z_L=1i*omega*4.77e-9; % Ohm
Z_load=10+30*i; % Ohm
Z0=20; % Ohm
ref_coef_l=(Z_load-Z0)/(Z_load+Z0);

ind_z=1;

% z' is changing from 0 to 2*lambda, so beta*z' is changing from 0 to 4*pi
for beta_z=linspace(0,4*pi,400)



mag_V(ind_z)=abs(1+ref_coef_l*exp(-2i*beta_z)); % magnitude of V(z') Volt

ind_z=ind_z+1;
end

figure
plot(linspace(0,2,400),mag_V,'r','Linewidth',2)
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontWeight','bold','FontSize',10);
grid on
xlabel('z(\lambda)')
ylabel('V(z'')(Volt)','Fontweight','bold','Fontsize',12)
title('V(z'') vs. z''(\lambda)','Fontweight','bold','Fontsize',12)
```

```
clear all
% close all



omega=2*pi*1e9; % rad/s
R=10; % Ohm
Z_L=1i*omega*4.77e-9; % Ohm
Z_load=10+30*i; % Ohm
Z0=20; % Ohm
ref_coef_l=(Z_load-Z0)/(Z_load+Z0);

ind_z=1;
omega_t=0;

% z' is changing from 0 to 2*lambda, so beta*z' is changing from 0 to 4*pi
for beta_z=linspace(0,4*pi,400)



mag_V(ind_z)=abs((exp(1i*omega_t)*(exp(1i*beta_z)+ref_coef_l*exp(-1i*beta_z)))); % magnitude of V(z') Volt

ind_z=ind_z+1;
end

%hold on
figure
plot(linspace(0,2,400),mag_V,'k','Linewidth',2)
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontWeight','bold','FontSize',10);
grid on
xlabel('z(\lambda)')
ylabel('V(z'',t=\pi/3\omega)(Volt)','Fontweight','bold','Fontsize',12)
title('V(z'',t=\pi/3\omega) vs. z''(\lambda)','Fontweight','bold','Fontsize',12)

% legend('V(z'')','V(z'',t=\pi/3\omega)')
```

(3) A rectangular waveguide with $a = 50$ cm
$b = 30$ cm

$f_{op} = 1.5$ GHz (operating frequency)

First, we need to find cut-off frequencies for possible modes:

Note: TEM wave cannot propagate in rectangular waveguide, so at least one of m and n should be greater than 0.

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

- For $m=1$, $n=0$ $\quad f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\pi}{a}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}$

$\mu = \mu_0$ $\varepsilon = \varepsilon_0$ for vacuum $\Rightarrow$ $f_c = 3\times10^8$ Hz $< f_{op}$

Thus, TE_{10} mode can propagate because their cut-off frequencies are below our operating frequency.

- For $m=0$, $n=1$ $\quad f_c = \frac{1}{2b\sqrt{\mu\varepsilon}} = 5\times10^8$ Hz $< f_{op}$

$\Rightarrow$ TE_{01} mode can also propagate.

- For $m=1$, $n=1$ $\quad f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = 5.83\times10^8$ Hz $< f_{op}$

$\Rightarrow$ TE_{11} and TM_{11} can propagate.

* Remember that for TM waves the fundamental mode is TM_{11}, so m and n cannot be zero for TM.

- For $m=2,\ n=0$ $\quad f_c = \dfrac{1}{a\sqrt{\mu\varepsilon}} = 6\times10^8\ Hz < f_{op}$

$\Rightarrow TE_{20}$ can propagate.

- For $m=0,\ n=2$ $\quad f_c = \dfrac{1}{b\sqrt{\mu\varepsilon}} = 1\,GHz < f_{op}$

$\Rightarrow TE_{02}$ can propagate

- For $m=2,\ n=1$ $\quad f_c = \dfrac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\dfrac{2\pi}{a}\right)^2+\left(\dfrac{\pi}{b}\right)^2} = 7.81\times10^8\ Hz < f_{op}$

$\Rightarrow TE_{21}$ and TM_{21} can propagate

- For $m=1,\ n=2$ $\quad f_c = \dfrac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\dfrac{\pi}{a}\right)^2+\left(\dfrac{2\pi}{b}\right)^2} = 1.044\ GHz < f_{op}$

$\Rightarrow TE_{12}$ and TM_{12} can propagate.

- For $m=2,\ n=2$ $\quad f_c = \dfrac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\dfrac{2\pi}{a}\right)^2+\left(\dfrac{2\pi}{b}\right)^2} = 1.166\ GHz < f_{op}$

TE_{22} and TM_{22} can propagate

- For $m=3,\ n=0$ $\quad f_c = \dfrac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\dfrac{3\pi}{a}\right)^2} = 9\,GHz < f_{op}$

TE_{30} can propagate

- For $m=3$, $n=1$ $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{3\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2} = 1.029\ GHz$

  $f_c < f_{op}$ => TE_{31} and TM_{31} can propagate

- For $m=3$, $n=2$ $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{3\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2} = 1.345\ GHz < f_{op}$

  => TE_{32} and TM_{32} can propagate

- For $m=0$, $n=3$, $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{3\pi}{b}\right)^2} = 1.5\ GHz$

  So TE_{03} cannot propagate.

- For $m=4$, $n=0$, $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{4\pi}{a}\right)^2} = 1.2\ GHz < f_{op}$

  TE_{40} mode can propagate.

- For $m=4$, $n=1$, $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{4\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2} = 1.3\ GHz < f_{op}$

  => TE_{41} and TM_{41} can propagate.

- For $m=4$, $n=2$ $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{4\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2} = 1.562\ GHz$

  => TE_{42} and TM_{42} cannot propagate.

Propagating modes: TE_{01}, TE_{10}, TE_{11}, TM_{11}, TE_{02}, TE_{20}, TE_{12}, TE_{21}, TM_{12}, TM_{21}, TE_{22}, TM_{22}, TE_{03}, TE_{31}, TM_{31}, TE_{32}, TM_{32}, TE_{40}, TE_{41}, TM_{41}

Phase constant $\beta_{mn} = \sqrt{\omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$

wavelength $\lambda_{mn} = \dfrac{2\pi}{\beta_{mn}}$

phase velocity $v_{mn} = \dfrac{\omega}{\beta_{mn}}$

MATLAB script to find the modes, phase constants, wavelengths, and phase velocities:

```
clc
clear all

f=1.5e9; % Hz
omega=2*pi*f; % rad/s
a=0.5; % m
b=0.3; % m
for m=0:1:5
    for n=0:1:5

f_c=3e8*(sqrt(((m*pi/a)^2)+((n*pi/b)^2)))/(2*pi); % Hz cut-off frequency
if f>f_c

disp(['when m is ',num2str(m),' and n is ',num2str(n)]);

beta=sqrt((omega^2)/((3e8)^2)-((m*pi/a)^2)-((n*pi/b)^2));
lambda=2*pi/beta;
v=omega/beta;
disp(['phase constant is ',num2str(beta),' rad/m',' wavelength is ',num2str(lambda),' m',' and phase velocity is ',num2str(v),' m/s'])
end
    end
end
```

Mode	m,n	β_{mn} (rad/m)	λ_{mn} (m)	v_{mn} (m/s)
TE_{10}	m=1 n=0	30.78	0.2041	3.06×10^8
TE_{01}	m=0 n=1	29.62	0.2121	3.18×10^8
TE_{11} TM_{11}	m=1 n=1	28.95	0.2171	3.26×10^8
TE_{20}	m=2 n=0	28.79	0.2182	3.27×10^8
TE_{02}	m=0 n=2	23.42	0.2683	4.025×10^8
TE_{21} TM_{21}	m=2 n=1	26.82	0.2343	3.51×10^8
TE_{12} TM_{12}	m=1 n=2	22.56	0.2785	4.18×10^8
TE_{22} TM_{22}	m=2 n=2	19.76	0.3180	4.77×10^8
TE_{30}	m=3 n=0	25.13	0.2500	3.75×10^8
TE_{31} TM_{31}	m=3 n=1	22.85	0.2750	4.13×10^8
TE_{32} TM_{32}	m=3 n=2	13.89	0.4523	6.78×10^8
TE_{40}	m=4 n=0	18.85	0.333	5×10^8
TE_{41} TM_{41}	m=4 n=1	15.67	0.401	6.01×10^8

Throughout the solution I took $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s

but if you take it as 2.99792×10^8 m/s

TE_{03} mode can propagate, too.