$$\frac{\partial^2 v^{i}}{\partial z^2} = RG v + (RC+LG)\cdot\frac{\partial v^{i}}{\partial t} + LC\cdot\frac{\partial^2 v^{i}}{\partial t^2}$$ } Telegrapher's Equations

$$\gamma^2 = (R+j\omega L)(G+j\omega C)$$

$$\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0$$

$$V(z) = V^+ \cdot e^{-\gamma z} + V^- \cdot e^{\gamma z}$$

$$\gamma = \alpha + j\cdot\beta$$

$$V(z) = V^+ \cdot e^{-\alpha z} \cdot e^{-j\beta z} + V^- \cdot e^{\alpha z} \cdot e^{j\beta z}$$

* $v = \frac{\omega}{Im(\gamma)} = \frac{\omega}{\beta}$ * $\lambda = \frac{2\pi}{Im(\gamma)} = \frac{2\pi}{\beta}$ $Z_0 \triangleq \frac{1}{Y_0} \triangleq \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ → char empedance

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Lossless

$\gamma = j\omega\sqrt{LC}$ $\Rightarrow$ $\alpha = 0$, $\beta = \gamma$ $Z_0 = \sqrt{\frac{L}{C}}$ $\eta_{upw} = \sqrt{\frac{\mu}{\varepsilon}}$ $v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}}$

Also, $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} = \frac{2\pi}{2\pi f\sqrt{LC}} = \frac{v}{f} \Rightarrow \lambda = \frac{v}{f}$

Low Loss

$$\gamma = \underbrace{\tfrac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)}_{\alpha} + j\underbrace{\omega\sqrt{LC}}_{\beta} = \alpha + j\beta$$

$$\alpha = \tfrac{1}{2}\left(\frac{R}{Z_0} + G\cdot Z_0\right) \qquad \beta = \omega\sqrt{LC} \qquad Z_0 \cong \sqrt{\frac{L}{C}}$$

Reflection Coeff

$V_L = V_L^+ + V_L^-$ $\Gamma_L \triangleq \frac{V_L^-}{V_L^+}$

$$V_s = V(z=0) = V^+ + V^-$$

$$V_L = V(z=\ell) = \underbrace{V^+ \cdot e^{-\gamma\ell}}_{V_L^+} + \underbrace{V^- \cdot e^{+\gamma\ell}}_{V_L^-} = V_L^+ + V_L^-$$

$$I_L = I_L^+ + I_L^- = \frac{V_L^+}{Z_0} - \frac{V_L^-}{Z_0} \qquad \frac{V_L}{I_L} = Z_L$$

$$Z_L = Z_0 \cdot \frac{1+\Gamma_L}{1-\Gamma_L}$$ where Γ_L is complex

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L|\cdot e^{j\theta_L} \Longleftrightarrow \Gamma(d) = \Gamma_L \cdot e^{-2\gamma d}$$ where $\left\{ \begin{array}{l} \gamma = \alpha + j\beta \\ \Gamma_L = |\Gamma_L|\cdot e^{j\theta_L} \end{array} \right.$

$$\Gamma(d) = |\Gamma_L|\cdot e^{-2\alpha d}\cdot e^{j(\theta_L - 2\beta d)}$$

To make complete rotation $2\beta d = 2\pi$ $\quad 2.\frac{2\pi}{\lambda}.d = 2\pi \quad d = \frac{\lambda}{2}$

$|\Gamma(d)| = |\Gamma_L|.e^{-2\alpha d} \longrightarrow$ lossy medium

$\cosh = \frac{e^x + e^{-x}}{2} \quad \sinh \triangleq \frac{e^x - e^{-x}}{2}$

IMPEDANCE

$$\underset{\text{Lossless}}{Z(d)} = Z_0 . \frac{Z_L + jZ_0 . \tan(\beta d)}{Z_0 + jZ_L . \tan(\beta d)}$$

If it is lossey

$$\longrightarrow Z(d) = Z_0 . \frac{Z_L + Z_0 . \tanh(\gamma d)}{Z_0 + Z_L . \tanh(\gamma d)}$$

Important Special Cases

1) Matched Load Case $\quad Z_L = Z_0$

$\Gamma_L = 0 \Rightarrow \Gamma_L = 0 \longrightarrow$ for all d $\quad Z(d) = Z_0 \longrightarrow$ No reflected wave

2) Impedance Repeater

Let $d = \lambda/2 \quad \beta.d = \frac{2\pi}{\lambda}.d\Big|_{\lambda/2} = \pi = \beta d \qquad \tan(\beta d) \Rightarrow 0$

$$Z_L(d = \lambda/2) = Z_0 . \frac{Z_L + Z_0(0)}{Z_0 + Z_L(0)} = Z_L \qquad d = n.\lambda/2 \quad (n = 1, 2, 3 \ldots)$$

3) Impedance Inverter

Let $d = \lambda/4 \qquad \beta d = \frac{2\pi}{\lambda} . \frac{\lambda}{4} = \frac{\pi}{2}$

$$Z_L(d = \lambda/4) = \frac{Z_L + Z_0 \overbrace{\tan \pi/2}^{\infty}}{Z_0 + Z_L . \tan \pi/2} . Z_0 = \frac{Z_0^2}{Z_L}$$

A lossless TL which is $(2n+1) . \lambda/4 \quad (n = 0, 1, 2, 3, \ldots$

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4) Short Circuited TL $\qquad Z_L = 0 \quad \Gamma_L = -1 \qquad Z(d)\Big|_{Z_L = 0} = Z_0 . \tanh(\gamma d)$

For a lossless TL $\Rightarrow \underset{\text{Lossless}}{Z(d)} = j . Z_0 . \tan(\beta d) = Z_{sc}$

5) Open Circuited

Let $Z_L \longrightarrow \infty \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}\Big|_{Z_L \to \infty} = 1 \qquad \Gamma_L = 1$

$$Z(d)\Big|_{Z_L\to\infty} = \frac{Z_0}{\tanh(\gamma d)} = Z_0 \coth(\gamma d) \longrightarrow \text{lossless}$$

$$Z(d)_{\text{lossless}} = -j\cdot Z_0 \cdot \cot(\beta \cdot d) = Z_{oc}$$

CASE I

Consider a lossless TL of $\ell < \frac{\lambda}{4}$

$\beta\ell < \frac{\lambda}{2}$ $\quad \tan(\beta\ell) > 0$ and $\cot(\beta\ell) > 0$

Short Circuit $\xrightarrow[Z_L=0]{}$ $Z_{in} = j\cdot Z_0 \cdot \tan(\beta\ell) = j\omega \underline{\underline{L}}$

$$\omega L = Z_0 \cdot \tan(\beta\ell)$$

open circuit $\longrightarrow$ $Z_{in} = -j\cdot Z_0 \cdot \cot(\beta\ell) = \frac{1}{j\omega C} = -j\cdot \frac{1}{\underline{\underline{\omega C}}}$

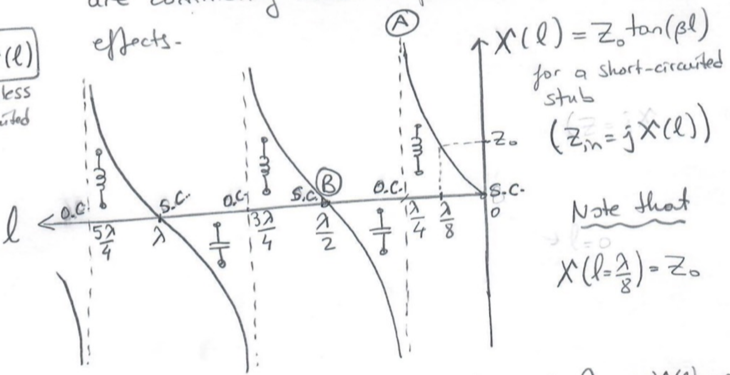
$$\frac{1}{\omega C} = Z_0 \cdot \cot(\beta \cdot \ell)$$

CASE II

$\frac{\lambda}{4} < \ell < \frac{\lambda}{2}$ $\quad \frac{\pi}{2} < \beta\ell < \pi$ $\quad \tan(\beta\ell) < 0$ $\quad \cot(\beta\ell) < 0$

Note: Short-circuited stubs with adjustable length "ℓ" are commonly used to produce capacitor/inductor effects.

$\boxed{Z(\ell) = jX(\ell)}$ for a lossless short-circuited T.L.

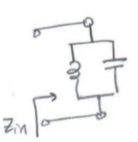


Note that

$X(\ell = \frac{\lambda}{8}) = Z_0$

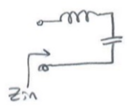
at (A) $\ell \approx \frac{\lambda}{4} \Rightarrow X(\ell) \to \infty$

Short-circuited stub behaves like a parallel resonant circuit



at (B) $\ell \approx \frac{\lambda}{2} \Rightarrow X(\ell) \to 0$

Short-circuited stub behaves like a series resonant circuit.



$$Z_{sc} = j\cdot Z_0 \cdot \tan(\beta d)$$

$$Z_{oc} = -j\cdot Z_0 \cdot \cot(\beta d)$$

$$Z_0 = \sqrt{Z_{sc}\cdot Z_{oc}}$$

STANDING WAVES

$z = \ell - d$

Assume TL is lossless $\quad V(z) = V_+ \cdot e^{-j\beta z} + V_- \cdot e^{j\beta z}$ $\quad V_L^+ = V^+ \cdot e^{-j\beta\ell}$

$$V(d) = V_L^+ \cdot e^{j\beta d} + V_L^- \cdot e^{-j\beta \cdot d}$$

$$V(d) = V_L^+ \Big(e^{j\beta d} + \underbrace{\frac{V_L^-}{V_L^+}}_{\Gamma_L} \cdot e^{-j\beta d}\Big) \qquad \text{but} \quad \frac{V_L^-}{V_L^+} \triangleq \Gamma_L$$

CASE 1

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L=0} = -1 \Rightarrow \Gamma_L = -1$$

$$V(d) = V_L^+ \Big(\underbrace{e^{j\beta d} + (-1) \cdot e^{-j\beta d}}_{2j \cdot \sin(\beta d)} \Big) = 2j \cdot V_L^+ \cdot \sin \beta d$$

$$|V(d)| = 2|V_L^+| \cdot |\sin \beta d| \xrightarrow{=1} |V(d)|_{max} = 2|V_L^+| = V_{max}$$

$$|\sin\beta d| \xrightarrow{=0} |V(d)|_{min} = 2|V_L^-| = V_{min} = 0$$

Define VSWR

$$S = \frac{V_{max}}{V_{min}} \Rightarrow \frac{2|V^+|}{0} = \infty \Rightarrow \text{For short-circuit termination}$$

$S = \infty \quad Z_L = 0$

CASE 2 : Open Circuit

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 1 \qquad \Gamma_L = 1$$

$$V(d) = V_L^+ \Big(\underbrace{e^{j\beta d} + \overset{\Gamma_L}{(1)} \cdot e^{-j\beta d}}_{2 \cdot \cos(\beta d)} \Big) = 2 V_L^+ \cdot \cos \beta d$$

$$|V(d)| = 2|V_L^+| \, |\cos \beta d| \rightarrow |V(d)|_{max} = 2|V_L^+|$$

$$\rightarrow |V(d)|_{min} = V_{min} = 0$$

$S = \infty \quad Z_L = \infty$

CASE 3 Arbitrary Load Impedance Z_L

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| \cdot e^{j\theta_L} \qquad |\Gamma_L| < 1$$

$$|V_L^+| = |V^+ \cdot e^{-\gamma \ell}| = |V^+ \cdot e^{-\alpha \ell} \cdot \underbrace{e^{-j\beta \ell}}| = |V_+| \cdot e^{-\alpha \ell} \longrightarrow \text{lossy}$$

$$\downarrow$$

$$|\cos(\beta \ell) + j \cdot \sin(\beta \ell)| = 1$$

for lossless $\longrightarrow$ $|V_+| = |V_L^+|$

$|V_-| = |V_L^-|$

$V_{max} = |V^+| + |V^-|$ → In phase

$V_{min} = |V^+| - |V^-|$ → out of phase

$|\Gamma| = \dfrac{|V^-|}{|V_+|}$

$S = \dfrac{V_{max}}{V_{min}}$

$|\Gamma(d)| = |\Gamma_L| \, |e^{-2\alpha d}|$ → lossy TL if $\alpha = 0$ → $|\Gamma(d)| = |\Gamma_L|$

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

Note that VSWR ≥ 1 always

$1 \leq S < \infty$

Happens for a matched load

$Z_L = Z_0$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 1$$

happens whenever $|\Gamma_L| = 1$

for $Z_L = jX_L$ purely reactive load

$$|\Gamma_L| = \left| \frac{jX_L - Z_0}{jX_L + Z_0} \right| = 1$$

↓

$\theta_L \neq 0$

$\Gamma_L = |\Gamma_L| \cdot e^{j\theta_L}$

for $Z_L = 0$ s.c.

$\Gamma_L = -1$

$|\Gamma_L| = 1$

for $Z_L = \infty$ o.c.

$\Gamma_L = 1$

$|\Gamma_L| = 1$

Standing wave pattern

1) d_{min} depends on the value of Z_L

2 a) two succesive voltage min $\frac{\lambda}{2}$

2 b) min – max $\lambda/4$

3) (V_{max}, I_{min}) (V_{min}, I_{max}) → Current and Voltage patterns are shifted by $\frac{\lambda}{4}$ with respect to each other.

4a) Impedance should be purely resistive and becomes maximum.

$$\frac{V_{max}}{I_{min}} \triangleq R_{max} = \frac{|V^+| + |V^-|}{\frac{1}{Z_0}(|V_+| - |V_-|)} = Z_0 \cdot S = R_{max}$$

S

Impedance should be purely resistive and becomes minimum

$$\frac{V_{min}}{I_{max}} \triangleq R_{max} = \frac{|V^+| - |V^-|}{1/Z_0 \, (|V_+| + |V_-|)} = \frac{Z_0}{S}$$

$R_{min} \cdot S = Z_0$

i) $z_L = z_0 \cdot \dfrac{1+\Gamma_L}{1-\Gamma_L} \Rightarrow$ i) $S = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|} \Rightarrow S = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|}$

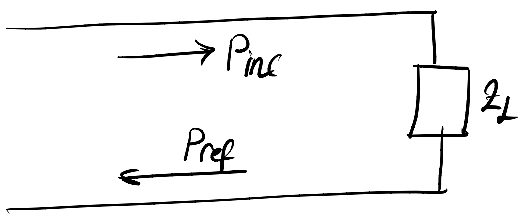
$$\frac{\text{reflected}}{\text{incident}} = \frac{V_L^- \cdot e^{-j\beta d}}{V_L^+ \cdot e^{j\beta d}} = \Gamma_L \cdot e^{-j2\beta d} = |\Gamma_L| \cdot e^{j\theta_L} \cdot e^{-j2\beta d}$$

$= |\Gamma_L| \cdot e^{j(\theta_L - 2\beta d)}$ θ_L : phase angle of reflection coefficient

At $d = d_{min} \Rightarrow$ position of V_{min}

$\theta_L - 2\beta d_{min} = -\pi$ $\theta_L = 2\beta \cdot d_{min} - \pi$

Power Transfer



P_L : Power delivered to the load

$P_{av} = \dfrac{1}{2} Re\{V \cdot I^*\}$

$P_{ref} = |\Gamma_L|^2 \cdot P_{inc}$ $P_{inc} = \dfrac{1}{2} \dfrac{|V^+|^2}{Z_0}$ → real

$P_L = P_{inc} - P_{Ref} = (1 - |\Gamma_L|^2) \cdot P_{inc}$ $\dfrac{P_L}{P_{inc}} = 1 - |\Gamma_L|^2$ → $\Gamma_L = 1 \rightarrow$ o.c, s.c, jX_c

Determination of Attenuation Constant

$$\alpha = Re\{\gamma\} = Re\{\sqrt{(R+j\omega L)(G+j\omega C)}\}$$

$$P_{av}(z) = \underbrace{\tfrac{1}{2} \cdot \frac{|V_+|^2}{|z_0|^2} \cdot Re\{z_0\}}_{P(0)} \cdot e^{-2\alpha z}$$

$P_{av}(z) = P(0) \cdot e^{-2\alpha z}$

Derivative

$$\frac{dP_{av}(z)}{dz} = -2\alpha \cdot \underbrace{P(0) \cdot e^{-2\alpha z}}_{P_{av}(z)}$$

Let $P_{loss}(z) = -\dfrac{dP_{av}(z)}{dz}$ watts/m $\Longrightarrow$ $-P_{Loss}(z) = -2\alpha \cdot P_{av}(z)$

$$\alpha = \frac{P_{loss}(z)}{2 \cdot P_{av}(z)}$$

$$P_{Loss}(z) = \frac{1}{2}\left\{ R \cdot |I(z)|^2 + G \cdot |V(z)|^2 \right\}$$