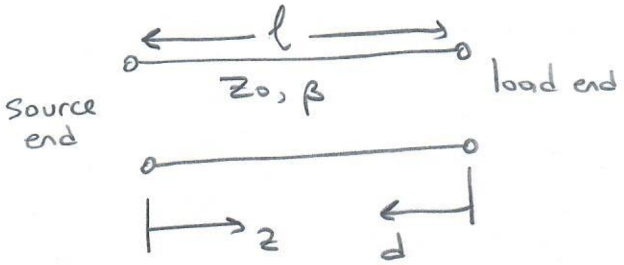
STANDING WAVES in Transmission Lines

Assume lossless T.L. (i.e. $R = G = 0 \Rightarrow \alpha = 0$
$\Rightarrow \gamma = j\beta$)



$(d = \ell - z$, or $z = \ell - d)$

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

: a standing wave created by oppositely traveling waves.

Or, express it in terms of d (distance measured from load end)

$$V(d) = V_L^+ e^{j\beta d} + V_L^- e^{-j\beta d}$$

(Remember; $V_L^+ = V^+ e^{-j\beta \ell}$
$V_L^- = V^- e^{j\beta \ell}$)

$$V(d) = V_L^+ \left(e^{j\beta d} + \frac{V_L^-}{V_L^+} e^{-j\beta d} \right)$$

but $\frac{V_L^-}{V_L^+} \triangleq \Gamma_L$

Load reflection coefficient

$$\boxed{V(d) = V_L^+ \left(e^{j\beta d} + \Gamma_L e^{-j\beta d} \right)}$$

Standing wave Pattern: plot of $|V(d)|$ versus d

(which can be measured by a probe moved along the T.L.)

Case 1: Short Circuit Termination (i.e. $Z_L = 0$)

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L=0} = -1 \implies \boxed{\Gamma_L = -1}$$

$$\Rightarrow V(d) = V_L^+ \underbrace{\left(e^{j\beta d} + (-1)\, e^{-j\beta d}\right)}_{2j\sin(\beta d)} = 2j\, V_L^+ \sin \beta d$$

$$\Rightarrow \boxed{|V(d)| = 2\,|V_L^+|\,|\sin \beta d|}$$

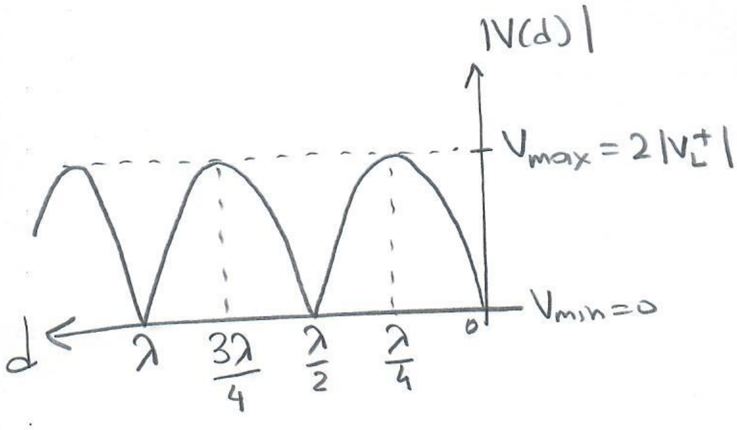
$|V(d)|_{max} = 2\,|V_L^+| = V_{max}$

$|V(d)|_{min} = 0 = V_{min}$

Define VSWR: Voltage Standing Wave Ratio $\triangleq S$

as
$$\boxed{S = \frac{V_{max}}{V_{min}}}$$

For short-circuit termination: $S = \dfrac{V_{max}}{V_{min}} = \dfrac{2|V_L^+|}{0} = \infty$



$$\boxed{S = \infty \quad \text{for } Z_L = 0}$$

Case 2: Open Circuit Termination (i.e. $Z_L = \infty$)

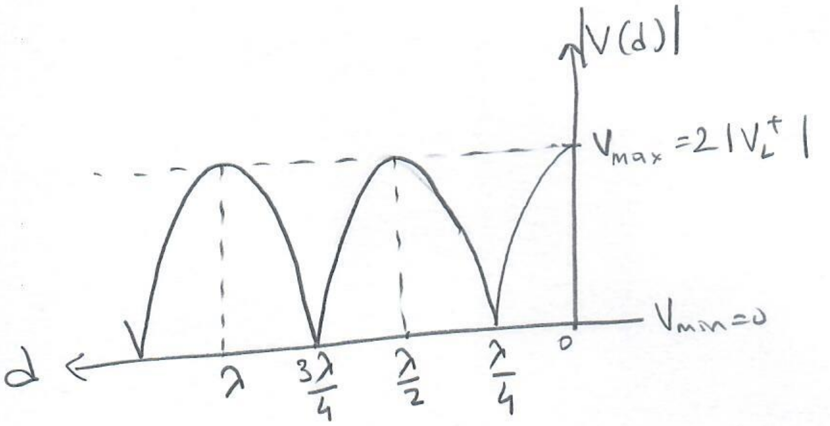
$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L=\infty} = +1 \quad \Rightarrow \quad \boxed{\Gamma_L = +1}$$

$$\Rightarrow \quad V(d) = V_L^+ \big(\underbrace{e^{j\beta d} + (1)\, e^{-j\beta d}}_{2\cos(\beta d)}\big) = 2V_L^+ \cos \beta d$$

$$\Rightarrow \quad \boxed{|V(d)| = 2\,|V_L^+|\,|\cos\beta d|}$$

$\rightarrow |V(d)|_{Max} = V_{Max} = 2|V_L^+|$

$\rightarrow |V(d)|_{min} = V_{min} = 0$



$$S = \frac{V_{max}}{V_{min}} = \frac{2|V_L^+|}{0} = \infty$$

$$\boxed{S = \infty \text{ for } Z_L = \infty}$$

Case 3: Arbitrary load impedance Z_L

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L|\, e^{j\theta_L} \quad \text{where} \quad |\Gamma_L| < 1$$

$$V(d) = V_L^+ e^{j\beta d} + V_L^- e^{-j\beta d}$$

Note that

$|V_L^+| = |V^+ e^{-\gamma \ell}| = |V^+ e^{-\alpha\ell} e^{-j\beta\ell}| = |V^+|\,|e^{-\alpha\ell}| = |V^+| e^{-\alpha\ell}$ for a lossy TL

but if the TL is lossless, i.e $\alpha = 0$, $\Rightarrow$ $|V_L^+| = |V^+|$

Similarly, $|V_L^-| = |V^-|$ } for lossless TL.

V_{max} occurs at those positions along the TL where forward and backward waves are <u>in phase</u>

$$\Rightarrow \boxed{V_{max} = |V^+| + |V^-|}$$

V_{min} occurs at those positions along the TL where forward and backward waves are $180°$ out of phase.

$$\Rightarrow \boxed{V_{min} = |V^+| - |V^-|}$$

$$S = \frac{V_{max}}{V_{min}} = \frac{|V^+| + |V^-|}{|V^+| - |V^-|}$$ (divide num. and denom. by $|V^+|$)

$$S = \frac{1 + \frac{|V^-|}{|V^+|}}{1 - \frac{|V^-|}{|V^+|}}$$

where $\Gamma = \frac{V^-}{V^+} \Rightarrow |\Gamma| = \frac{|V^-|}{|V^+|}$ at a given position d.

$$\Rightarrow \underbrace{S = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

(Note that $\Gamma(d) = \Gamma_L e^{-2\gamma d}$ for a general lossy line with $\gamma = \alpha + j\beta$.)

$$|\Gamma(d)| = |\Gamma_L| \, |e^{-2\alpha d}| \, \underbrace{|e^{-2j\beta d}|}_{1}$$

$|\Gamma(d)| = |\Gamma_L| e^{-2\alpha d}$ for a lossy TL

$$\boxed{|\Gamma(d)| = |\Gamma_L| \text{ if } \alpha = 0 \text{ in a lossless TL}}$$

$$\boxed{S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}}$$

in a lossless TL.

Note that the VSWR $\equiv S \geq 1$ always!

$$1 \leq S < \infty$$

happens for a matched load $\boxed{Z_L = Z_0}$

for which

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

$$\Rightarrow S = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 1$$

happens whenever $|\Gamma_L| = 1$ (complete reflection cases)

for $\boxed{Z_L = jX_L}$ purely reactive load

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L = jX_L}$$

$$|\Gamma_L| = \left|\frac{jX_L - Z_0}{jX_L + Z_0}\right| = 1$$

$$\left(\begin{array}{l}\text{where } \theta_L \neq 0 \\ \text{for } \Gamma_L = |\Gamma_L| e^{j\theta_L}\end{array}\right)$$

for $\boxed{Z_L = 0}$ (S.C.)

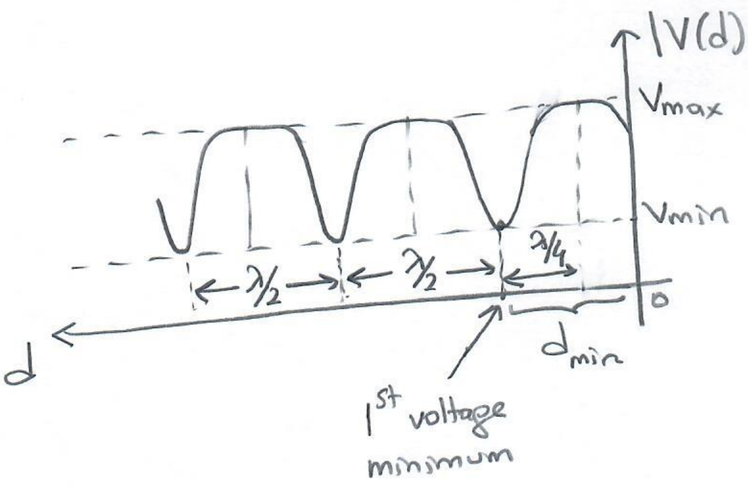
$\Gamma_L = -1$

$|\Gamma_L| = 1$

for $\boxed{Z_L = \infty}$ (O.C.)

$\Gamma_L = 1$

$|\Gamma_L| = 1$



Note that for an arbitrary load impedance Z_L, the voltage standing wave pattern is periodic but not necessarily a sinusoidal plot. Minima are usually sharper than maxima, easier to detect!

d_{min}: the distance of the 1st voltage minimum measured from the load.

Note that in a standing wave pattern:

1) d_{min} depends on the value of load impedance Z_L.

2.a) distance between two successive voltage minima is $\frac{\lambda}{2}$.

2.b) distance between a voltage minimum and the next voltage maximum is $\frac{\lambda}{4}$.

3) At a position where V_{max} occurs, current is minimum (I_{min})
" " " " V_{min} " , " " maximum (I_{max})

$\Rightarrow$ Current and voltage patterns are shifted by $\frac{\lambda}{4}$ with respect to each other.

Because:

$$V = V^+ + V^-$$
$$I = \frac{1}{Z_0}(V^+ - V^-)$$

} when V^- is in phase with V^+ (i.e. at a voltage maximum)

$I^- = -\frac{V^-}{Z_0}$ is 180° out of phase

wrt $I^+ = \frac{V^+}{Z_0}$ (because of (−) sign)

$\Rightarrow$ I_{min} occurs at locations where V_{max} occurs, etc..

4.a) At the positions where (V_{max} and I_{min}) occur, the impedance of TL is purely resistive and becomes maximum

i.e
$$\frac{V_{max}}{I_{min}} \triangleq R_{max} = \frac{|V^+| + |V^-|}{\frac{1}{Z_0}(|V^+| - |V^-|)} = Z_0 \underbrace{\frac{|V^+| + |V^-|}{|V^+| - |V^-|}}_{S} = Z_0 S$$

$$\Rightarrow \boxed{R_{max} = Z_0 S}$$

4.b) Similarly, at a position where (V_{min} and I_{max}) occur, the impedance of the T.L. is purely resistive and becomes minimum.

i.e. $$\frac{V_{min}}{I_{max}} \triangleq R_{min} = \frac{|V^+| - |V^-|}{\frac{1}{Z_0}\left(|V^+| + |V^-|\right)} = \frac{Z_0}{S}$$

$$\boxed{R_{min} = \frac{Z_0}{S}}$$

Example:

Let $Z_L = 50\ \Omega$ and $Z_0 = 75\ \Omega$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50-75}{50+75} = -0.2 \Rightarrow \left(|\Gamma_L| = 0.2\right)$$

$$S = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.2}{1-0.2} = 1.5$$

$R_{max} = Z_0 S = 75 \times 1.5 = 112.5\ \Omega$ (at $\{V_{max}, I_{min}\}$ positions)

$R_{min} = \frac{Z_0}{S} = \frac{75}{1.5} = 50\ \Omega$ (at $\{V_{min}, I_{max}\}$ positions)

Note that $R_{min} = Z_L = 50\ \Omega$ in this example. As the load impedance is repeated that means we are $n\frac{\lambda}{2}$ (n: an integer) away from the load at $\{V_{min}, I_{max}\}$ positions.

Determine the load impedance Z_L

Using the measurements of S and d_{min}

Assume a lossless T.L. with characteristic impedance Z_0:

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}$$ obtained earlier where $\Gamma_L = |\Gamma_L| e^{j\theta_L}$

Need to find $|\Gamma_L|$ and θ_L to determine the load impedance Z_L.

(i) we know $S = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \Rightarrow \boxed{|\Gamma_L| = \frac{S-1}{S+1}}$

where $S \equiv VSWR$ can be measured.

(ii) A voltage minimum occurs when the reflected (backward) wave is 180° out of phase with the incident (forward) wave.

$$\frac{\text{reflected}}{\text{incident}} = \frac{V_L^- e^{-j\beta d}}{V_L^+ e^{j\beta d}} = \underbrace{\frac{V_L^-}{V_L^+}}_{\Gamma_L} e^{-j2\beta d} = \underbrace{\Gamma_L}_{|\Gamma_L| e^{j\theta_L}} e^{-j2\beta d}$$

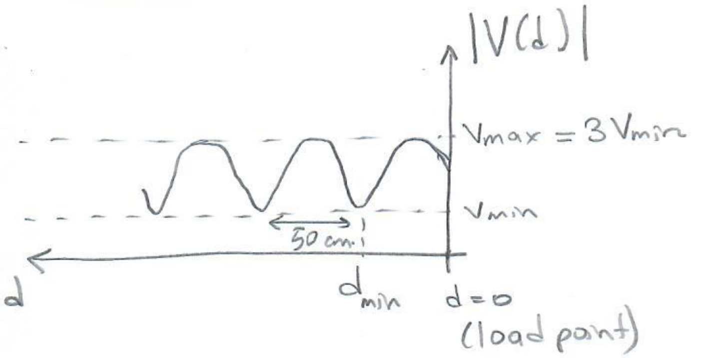
$$= |\Gamma_L| e^{j(\theta_L - 2\beta d)}$$

$\Rightarrow$ At $d = d_{min}$ (at the position of first voltage minimum)

$\theta_L - 2\beta d_{min} = -\pi$ where d_{min} can be measured.

$$\Rightarrow \boxed{\theta_L = 2\beta d_{min} - \pi}$$

Example:



Let $Z_0 = 300 (\Omega)$

$d_{min} = 30$ cm

$\frac{\lambda}{2} = 50$ cm.

$S = \frac{V_{max}}{V_{min}} = 3$

read from the given standing wave pattern.

$\frac{\lambda}{2} = 50$ cm. (distance between two successive minima)

$\Rightarrow \boxed{\lambda = 100 \text{ cm.}} \Rightarrow \beta = \frac{2\pi}{\lambda}$ is known.

$$\theta_L = 2\beta d_{min} - \pi = 2 \, \frac{2\pi}{100} \, 30 - \pi = 0.2\pi \text{ radians}$$

$$\boxed{\theta_L = 0.2\pi \text{ rad} = 36^\circ}$$

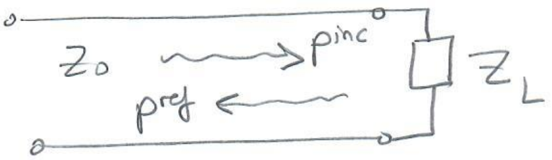
Also, $|\Gamma_L| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = \frac{2}{4} = 0.5 \Rightarrow \boxed{|\Gamma_L| = 0.5}$

$$\Rightarrow \quad Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} = 300 \frac{1 + 0.5 e^{j0.2\pi}}{1 - 0.5 e^{j0.2\pi}} = \ldots$$

$$\boxed{Z_L \cong 649 \angle 38^\circ \cong 510 + j400 \ \Omega}$$

Load impedance with an inductive reactance.

Power Transfer by a Transmission Line



P_L : Power delivered to the load.

$$P_{av} = \frac{1}{2} \mathrm{Re}\{V I^*\}$$

$$P^{inc} = \frac{1}{2} \frac{|V^+|^2}{Z_0} \quad \text{(assuming } Z_0 \text{ is real)}$$

$$P^{ref} = |\Gamma_L|^2 P^{inc}$$

$$\boxed{P_L = P^{inc} - P^{ref} = (1 - |\Gamma_L|^2) P^{inc}}$$

or $\boxed{\dfrac{P_L}{P^{inc}} = 1 - |\Gamma_L|^2}$

Note that $P_L = 0$ for a load
- $Z_L = \infty$ (open cct)
- $Z_L = 0$ (short cct)
- $Z_L = jX_L$ (purely reactive)

for which $|\Gamma_L| = 1$.

Determination of Attenuation Constant (α)

(i) If R, L, G, C and $\omega = 2\pi f$ are known, then

$$\alpha = Re\{\gamma\} = Re\left\{\sqrt{(R+j\omega L)(G+j\omega C)}\right\}$$

(ii) α can also be determined based on power relations:

Consider for example, a forward traveling wave along the T.L. with

$$\left.\begin{aligned} V(z) &= V^+ e^{-\alpha z} e^{-j\beta z} \\ I(z) &= \frac{V^+}{Z_0} e^{-\alpha z} e^{-j\beta z} \end{aligned}\right\} \quad P_{av}(z) = \frac{1}{2} Re\{V(z) I^*(z)\}$$

Average power propagating along the T.L.

$$P_{av}(z) = \frac{1}{2} Re\left\{V^+ e^{-\alpha z} \cancel{e^{-j\beta z}} \frac{(V^+)^*}{Z_0^*} e^{-\alpha z} \cancel{e^{+j\beta z}}\right\}$$

$$= \frac{1}{2} Re\left\{|V^+|^2 e^{-2\alpha z} \frac{1}{\underbrace{Z_0^*}} \frac{Z_0}{\underbrace{Z_0}}\right\}$$

$$= \frac{1}{2} Re\left\{|V^+|^2 e^{-2\alpha z} \frac{Z_0}{\underbrace{Z_0^* Z_0}_{|Z_0|^2}}\right\} \quad \begin{pmatrix}\text{divide and} \\ \text{multiply by} \\ Z_0\end{pmatrix}$$

$$= \frac{1}{2} \frac{|V^+|^2}{|Z_0|^2} e^{-2\alpha z} Re\{Z_0\}$$

$$\boxed{P_{av}(z) = \underbrace{\frac{1}{2} \frac{|V^+|^2}{|Z_0|^2} Re\{Z_0\}}_{P(0) \,=\, \text{power at } z=0} e^{-2\alpha z}}$$

$$\Rightarrow \boxed{P_{av}(z) = P(0)\, e^{-2\alpha z}}$$

$$\Rightarrow \frac{d P_{av}(z)}{dz} = -2\alpha \underbrace{P(0)\, e^{-2\alpha z}}_{P_{av}(z)}$$

$$\Rightarrow \boxed{\frac{d P_{av}(z)}{dz} = -2\alpha P_{av}}$$

Let $P_{loss}(z) = -\frac{d P_{av}(z)}{dz}$ Watts/m : Time-average power loss per unit length on the T.L.

$$-P_{loss}(z) = -2\alpha P_{av}(z)$$

$$\Rightarrow \boxed{\alpha = \frac{P_{loss}(z)}{2 P_{av}(z)}}$$

where $P_{loss}(z)$ can be obtained from:

$$\boxed{P_{loss}(z) = \frac{1}{2}\left\{ R\,|I(z)|^2 + G\,|V(z)|^2 \right\}}$$