EE303 Recitation 10

Q1. A 100 km telephone line has a series resistance of 4 Ω/km, an inductance of 3 mH/km, a leakage conductance of 1 μS/km, and a shunt capacitance of 0.015 μF/km, at an angular frequency $\omega = 5000$ rad/s. The load at the receiving end consists of a 200 Ω resistor and the load current is $I_L = -0.115 + j0.0268$ A.

a. Find the voltage and current as functions of z (the distance from the load), and calculate their values at the midpoint of the line.

b. At the sending end there is a generator operating at 5000 radians per second, in series with a resistance of 300 Ω. Determine the generator voltage.

Solution:

From the given parameters of the transmission line we can calculate the characteristic impedance as

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{4 + j5000 \times 3 \times 10^{-3}}{10^{-6} + j5000 \times 0.015 \times 10^{-6}}} = 10^3 \sqrt{\frac{4 + j15}{1 + j75}}$$

$$= 10^3 \sqrt{\frac{15.52\angle 75^{\circ}}{75\angle 89.2^{\circ}}} = 454.9\angle -7.1^{\circ} \approx 451.5 - j56.1\ \Omega$$

and the propagation factor as

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = 10^{-3}\sqrt{(4 + j15)(1 + j75)}$$
$$= 10^{-3}\sqrt{(15.52\angle 75^{\circ})(75\angle 89.2^{\circ})} = 34.1\angle 82.1^{\circ}$$
$$= (4.66 + j33.8) \times 10^{-3}$$

The voltage and current waves at the load end must satisfy

$$V(z = 0) = [V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}]_{z=0} = V_0^+ + V_0^- = Z_L I_L$$
$$I(z = 0) = \left[\frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}\right]_{z=0} = \frac{V_0^+ - V_0^-}{Z_0} = I_L$$

and we know the values of Z_L and I_L. We can solve for V_0^+ and V_0^- to get

$$V_0^+ + V_0^- = 200(-0.115 + j0.0268)$$
$$V_0^+ - V_0^- = (451.5 - j56.1)(-0.115 + j0.0268)$$
$$V_0^+ = -36.7 + j11.9\ \text{V}; \quad V_0^- = 13.7 - j6.6\ \text{V}$$

a. The voltage and current waves are now easy to write by direct substitution. At the midpoint of the line ($z = -50$ km) we have

$$\gamma z = (4.66 + j33.8) \times 10^{-3} \times (-50) = -0.23 - j1.69$$
$$V(z = -50\ \text{km}) = (-36.7 + j11.9)e^{-(-0.23 - j1.69)} + (13.7 - j6.6)e^{(-0.23 - j1.69)}$$
$$= -15.9 - j57.8\ \text{V}$$
$$I(z = -50\ \text{km}) = \frac{-36.7 + j11.9}{451.5 - j56.1} e^{-(-0.23 - j1.69)} - \frac{13.7 - j6.6}{451.5 - j56.1} e^{(-0.23 - j1.69)}$$
$$= -3.6 - j82.5\ \text{mA}$$

b. At the generator end the line voltage and current can be found as follows:

$$\gamma z = (4.66 + j33.8) \times 10^{-3} \times (-100) = -0.46 - j3.38$$

$$V(z = -100 \text{ km}) = (-36.7 + j11.9)e^{-(-0.46-j3.38)} + (13.7 - j6.6)e^{(-0.46-j3.38)}$$
$$= 53.5 + j1.4 \text{ V}$$
$$I(z = -100 \text{ km}) = \frac{-36.7 + j11.9}{451.5 - j56.1}e^{-(-0.46-j3.38)} - \frac{13.7 - j6.6}{451.5 - j56.1}e^{-(-0.46-j3.38)}$$
$$= 152.1 - j5.1 \text{ mA}$$

which means that the generator side sees an impedance of

$$Z_{in} = \frac{V(z = 100 \text{ km})}{I(z = 100 \text{ km})} = \frac{53.5 + j1.4}{152.1 - j5.1} \approx 350\ \Omega$$

The generator voltage can then be determined from the following equation:

$$V(z = 100 \text{ km}) \approx \frac{350}{350 + 300}V_G$$
$$V_G \approx 53.5 \times \frac{13}{6} \approx 116 \text{ V}.$$

Q2. A transmission line operating at 125 MHz has the characteristics $Z_0 = 40\ \Omega$, $\alpha = 0.173$ dB/m, and $\beta = 0.75$ rad/m. Find the line parameters R, L, G, and C.

Solution:

In a distance of 1 m, the voltage is scaled by $-\alpha = -0.173$ dB or by a factor of $10^{-0.173/20} = 0.98$. This ratio must be $e^{-\alpha_{,Np}}$, thus the attenuation constant is $\alpha = 0.02$ Np/m. Since Z_0 is real, the line is distortionless, i.e.,

$$R + j\omega L = k(G + j\omega C)$$

for some real k. Equating the real and imaginary parts we get

$$k = \frac{R}{G} = \frac{j\omega L}{j\omega C}$$

If we substitute $R = LG/C$ into the expression for Z_0, we get

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{LG/C + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{G + j\omega C}{G + j\omega C}} = \sqrt{\frac{L}{C}} \ \Rightarrow\ \sqrt{L} = 40\sqrt{C}$$

Now, if we substitute $R = LG/C$ into the expression for the propagation factor, we get

$$\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(LG/C + j\omega L)(G + j\omega C)}$$
$$= \sqrt{L/C(G + j\omega C)(G + j\omega C)} = \sqrt{\frac{L}{C}}(G + j\omega C)$$

Which yields

$$\alpha = G\sqrt{\frac{L}{C}};\quad \beta = \omega\sqrt{LC}$$

From the equation for β we can find

$$\beta = 2\pi f\sqrt{LC} = 2\pi f\sqrt{L}\sqrt{C} = 2\pi f Z_0 C \ \Rightarrow$$
$$C = \frac{\beta}{2\pi f Z_0} = \frac{0.75}{2 \times 125 \times 10^6 \times 40 \times \pi} = 23.9 \text{ pF/m}$$
$$L = Z_0^2 C = 38.2 \text{ nH/m}$$

From $\alpha = G\sqrt{L/C}$ and $RC = LG$, we find

$$G = 0.5\ \mathrm{mS/m} \quad \text{and} \quad R = \alpha Z_0 = 0.8\ \Omega/\mathrm{m}.$$

Q3. Consider a lossless transmission line having distributed parameters $L = 245\ \mathrm{nH/m}$ and $C = 200\ \mathrm{pF/m}$. The line is terminated with a resistor $R_L = 100\ \Omega$. The operating frequency is $f = 1\ \mathrm{GHz}$.

a. Determine the characteristic impedance and phase velocity of the line.

b. Determine the input impedance seen looking into the input terminals of the line at $1\ \mathrm{GHz}$, if the length of the line is $35.7\ \mathrm{mm}$.

c. Determine the VSWR of the load.

Solution:

a. For a lossless transmission line the characteristic impedance is given by

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{245 \times 10^{-9}}{200 \times 10^{-12}}} = 35\ \Omega$$

$$\beta = \mathrm{Im}(\gamma) = \omega\sqrt{LC} = 2\pi 10^9 \sqrt{245 \times 10^{-9} \times 200 \times 10^{-12}} = 44\ \mathrm{rad/m}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{245 \times 10^{-9} \times 200 \times 10^{-12}}} = 1.429 \times 10^8\ \mathrm{m/s}$$

b. The input impedance is given by

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

Using the numerical values $Z_L = 100\ \Omega$, $Z_0 = 35\ \Omega$, and $\beta l = 44 \times 0.0357 = \pi/2$. The value of $\tan(\beta l)$ goes to infinity and in the limit we have

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{35^2}{100} = 12.25\ \Omega$$

c. The voltage standing wave ratio (VSWR) is the value of the maximum magnitude of the voltage wave to its minimum value, which is

$$s = \frac{|V_0^+ + V_0^-|}{|V_0^+ - V_0^-|} = \frac{1 + \left|\frac{V_0^-}{V_0^+}\right|}{1 - \left|\frac{V_0^-}{V_0^+}\right|}$$

where $|V_0^-/V_0^+|$ is the reflection coefficient. At the load end we have

$$V_0^+ + V_0^- = R_L I_L$$
$$V_0^+ - V_0^- = Z_0 I_L$$

which gives

$$\frac{V_0^-}{V_0^+} = \Gamma = \frac{R_L - Z_0}{R_L + Z_0} = \frac{100 - 35}{100 + 35} = 0.481$$

and

$$s = \frac{20}{7} = 2.857$$