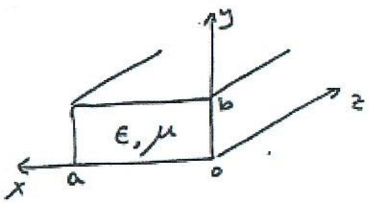
Summary for Rectangular Waveguides



As this is a single-conductor WG, it can support only TM and TE modes. (No TEM modes!)

TM Modes

$H_z = 0$ by definition

$$E_z(x,y,z) = \underbrace{E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}_{E_z^o(x,y)} e^{-\gamma z}$$

} Axial components (Transversal components can be found using the axial components) See eqn. set (12)

for $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$ (zero indices are not allowed)

$\Rightarrow TM_{mn}$ modes : $TM_{11}, TM_{12}, TM_{21}, TM_{13}, \ldots$ ($m=1$, $n=3$) etc, ...

TE Modes

$E_z = 0$ by definition

$$H_z(x,y,z) = \underbrace{H_o \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}_{H_z^o(x,y)} e^{-\gamma z}$$

} Axial components (Transversal components can be obtained later from the axial components using eqn. set (13))

for $m = 0, 1, 2, 3, \ldots$, $n = 0, 1, 2, 3, \ldots$ (except the $m=n=0$ case)

$\Rightarrow TE_{mn}$ modes : TE_{10} ($m=1$, $n=0$), $TE_{01}, TE_{11}, TE_{12}$ ($m=1$, $n=2$), $TE_{21}, TE_{22}, \ldots$ etc., ...

for both TE modes and TM modes :

eigenvalues : $h = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = h_{mn}$

cut-off frequencies: $(f_c)_{mn} = \frac{h_{mn}}{2\pi\sqrt{\mu\epsilon}} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

for a given operation frequency f (i.e., the frequency of the electromagnetic wave conveyed by the WG) and for an arbitrary mode with indices (m,n) :

(i) Mode is propagating if $f > (f_c)_{mn}$

$\Rightarrow \gamma_{mn} = j\beta_{mn}$ (prop. constant is imaginary)

with $\beta_{mn} = \sqrt{k^2 - h_{mn}^2}$

$= \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$

Also, $\lambda_{g_{mn}} = \frac{2\pi}{\beta_{mn}}$, $\lambda_{c_{mn}} = \frac{v}{(f_c)_{mn}} = \frac{2\pi}{h_{mn}}$ (where $v = \frac{1}{\sqrt{\mu\epsilon}}$)

and $v_{g_{mn}} = \frac{\omega}{\beta_{mn}}$

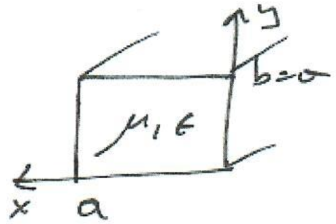
$Z =$ $Z_{TM_{mn}} = \frac{\gamma_{mn}}{j\omega\epsilon}$ OR $Z_{TE_{mn}} = \frac{j\omega\mu}{\gamma_{mn}}$

> NOTE: The subscripts "mn" may be omitted in $\gamma, \beta, \lambda_g, \lambda_c, v_g$ and Z expressions. They are included here for clarity.

(ii) Mode is evanescent (non-propagating) if $f < (f_c)_{mn}$

$\Rightarrow \gamma_{mn} = \alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2}$ (real)

Example 1) Square waveguide where $\boxed{b=a}$



In general,

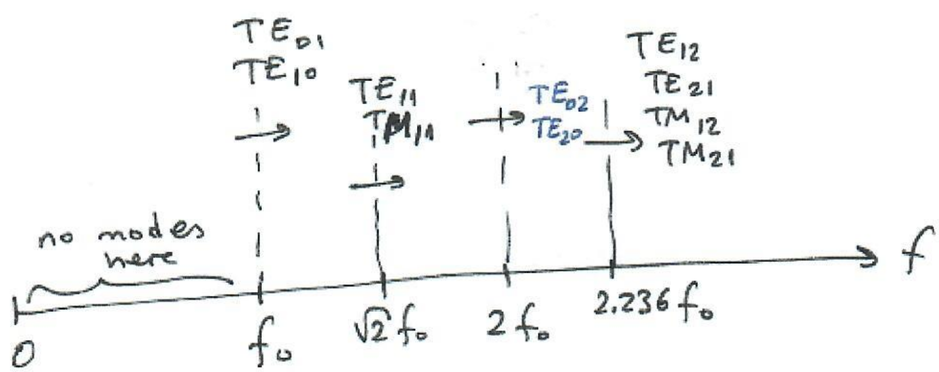
$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Note that $f_{c_{10}} = f_{c_{01}} = \frac{1}{2a\sqrt{\mu\epsilon}} \triangleq f_0$

$$f_{c_{1,1}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{a}\right)^2} = \sqrt{2}\, f_0$$

$$f_{c_{0,2}} = f_{c_{2,0}} = \dots\dots = 2 f_0$$

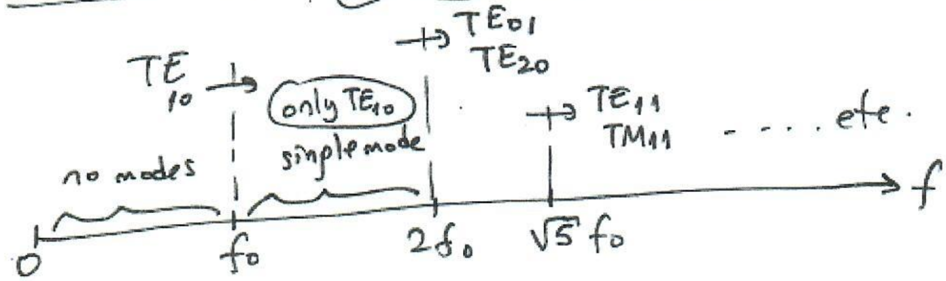
$$f_{c_{1,2}} = f_{c_{2,1}} = \dots\dots = 2.236\, f_0 \quad , \dots\dots \text{etc.}$$



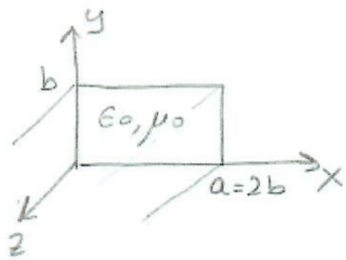
Note that, in square WG single mode operation is not possible.

Example 2) For a $\boxed{a = 2b}$ WG

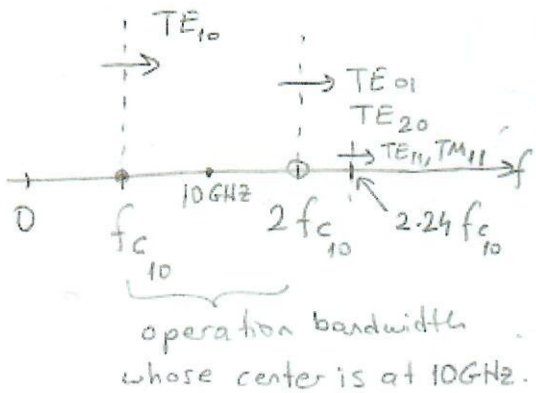
Exercise: write down f_0!



Example: Design an air-filled rectangular WG with $a = 2b$ to transmit single-mode EM waves over a proper bandwidth whose center frequency is 10 GHz.



- We want to allow propagation of a single mode only $\Rightarrow$ It should be the fundamental mode of this WG, which has the smallest cut-off freq.
- Considering all possible TE and TM modes and considering the fact that $a > b \Rightarrow TE_{10}$ mode is that fundamental mode to propagate. ($m=1$, $n=0$)



Remember $f_{c_{mn}} = \dfrac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

(Note that WG is air-filled $\Rightarrow v = \frac{1}{\sqrt{\mu_0\epsilon_0}}$)

$$\Rightarrow f_{c_{mn}} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \text{where} \quad c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 \text{ m/sec.}$$

For the mode TE_{10}, let $m=1$, $n=0$ above to get

$$\boxed{f_{c_{10}} = \frac{c}{2a}}$$

Then, check the cut-off frequency for the next mode(s) to propagate as f increases.

$$f_{c_{01}} = \frac{c}{2b} \quad \text{but } b = \frac{a}{2} \Rightarrow \boxed{f_{c_{01}} = \frac{c}{a} = 2f_{c_{10}}}$$

($m=0$, $n=1$) for TE_{01} mode

$$\text{Also, } f_{c_{20}} = \frac{c}{2}\cdot\frac{2}{a} = \frac{c}{a} \Rightarrow \boxed{f_{c_{20}} = f_{c_{01}} = 2f_{c_{10}}}$$

($m=2$, $n=0$) for TE_{20} mode

Also check the modes TE_{11} and TM_{11} for which

$$f_{c_{11}} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{(a/2)^2}} = \frac{c}{2}\sqrt{\frac{1+4}{a^2}} = \frac{c}{a}\underbrace{\frac{\sqrt{5}}{2}}_{\approx 1.118}$$

$$f_{c_{11}} \approx 1.118\,\frac{c}{a} > f_{c_{01}} = f_{c_{20}} \qquad \left(f_{c_{11}} \approx 2.24\, f_{c_{10}}\right)$$

$\Longrightarrow$ The single-mode propagation BW should be chosen as

$$f_{c_{10}} \leq f < f_{c_{20}} = f_{c_{01}} = 2 f_{c_{10}}$$

such that $f_{center} = 10\text{ GHz} = \dfrac{f_{c_{10}} + (2 f_{c_{10}})}{2} = \dfrac{3}{2} f_{c_{10}}$

$\Downarrow$

$f_{c_{10}} = \frac{20}{3}$ GHz

$\Downarrow$

$$\boxed{BW = \left[\tfrac{20}{3}, \tfrac{40}{3}\right] \text{ GHz}}$$

$$\Longrightarrow 10\times10^9 \text{ Hz} = \frac{3}{2}\,\frac{c}{2a} = \frac{3}{2}\,\frac{3\times10^8}{2a}$$

$$\Longrightarrow a = 0.0225\text{ m} \Longrightarrow \boxed{\begin{aligned} a &= 2.25\text{ cm.} \\ \text{and} \quad b = \frac{a}{2} &= 1.125\text{ cm.}\end{aligned}}$$

Example 3) For the air-filled WG designed in the previous problem, determine the maximum allowable time-averaged power that can be transmitted without causing breakdown inside the WG.

(Hint: Dielectric strength, i.e. $E_{breakdown} = 2\times10^6$ V/m for air.

Using a safety factor of 10, try to keep the maximum E-field anywhere in the waveguide less than $\dfrac{E_{breakdown}}{10}$)

Solution: The propagating mode in the previous example is the TE_{10} fundamental mode for which

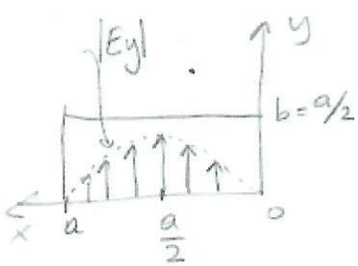
$E_z = 0$, $E_x = 0$ and $E_y = E_0 \sin \frac{\pi x}{a} e^{-\gamma z}$

in phasor domain where $\gamma = j\beta_{10}$.

In time domain, $\mathcal{E}_y(\bar{r},t) = Re\{E_y(\bar{r})\, e^{j\omega t}\}$

$\Rightarrow \mathcal{E}_y(x,z,t) = E_0 \sin\frac{\pi x}{a} \cos(\omega t - \beta z)$ (V/m.)

whose maximum magnitude occurs at $\frac{\pi x}{a} = \frac{\pi}{2} \Rightarrow x = \frac{a}{2}$.



(Note that $E_y = 0$ at $x = 0, a$)

$\Rightarrow Max\{|E_y|\} = E_0 \leq \frac{E_{breakdown}}{10} = 2\times10^5$ (V/m.)

$\Rightarrow$ Max. allowable $E_0 = 2\times10^5$ (V/m)

$P_{av} = \frac{E_0^2\, a\, b}{4\, Z_{TE_{10}}}$ where $Z_{TE_{10}} = \frac{\omega\mu_0}{\beta_{10}} = \frac{\omega\mu_0}{\sqrt{\omega^2\mu_0\epsilon_0 - (\frac{\pi}{a})^2}}$

$\left(\beta_{10} = \sqrt{\omega^2\mu_0\epsilon_0 - (\frac{\pi}{a})^2}\right)$

If the EM wave will be transmitted at $f = 10$ GHz basically, then

$\beta_{10} \simeq 49.69\pi$ rad/m.

$Z_{TE_{10}} \cong 505.8\ \Omega$ (Note that $Z_{TE_{10}}$ is real)

$P_{av} \approx 5$ kW.

$\downarrow$

(Max. avg. power transmitted at 10 GHz without causing breakdown inside the WG)

Also, at $f = 10$ GHz,

$\lambda_g = \frac{2\pi}{\beta} \cong 4.02$ cm.

$\lambda = \frac{v}{f} = \frac{c}{f} = 3$ cm.

$\lambda_c = \frac{v}{f_{c_{10}}} = \frac{c}{f_{c_{10}}} = 4.5$ cm. (Note that $\lambda_c = 2a$ in this case also)

(Check if $\frac{1}{\lambda^2} \overset{?}{=} \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}$)