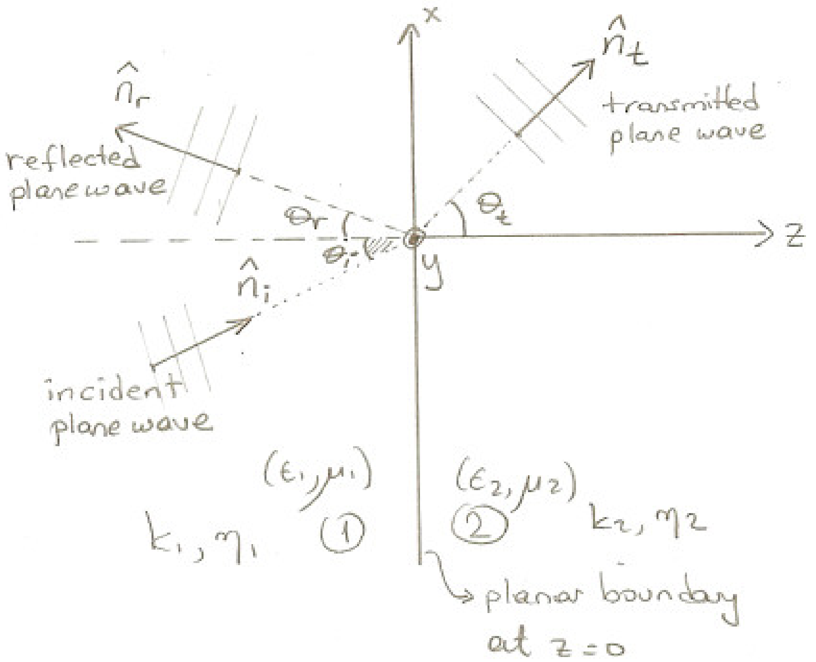
Reflection and Refraction (Transmission) of Plane Waves for OBLIQUE INCIDENCE at a Plane Boundary

Consider the planar boundary at $z=0$ between two lossless dielectrics:



Plane of Incidence (POI) is defined as the plane formed by

- the normal direction to the boundary, and
- the unit vector $\hat{n}_i$ of the incident wave

For the picture shown (x,z) plane is the POI.

θ_i : angle of incidence $\quad 0 \le \theta_i \le \frac{\pi}{2}$

θ_r : " " reflection $\quad 0 \le \theta_r \le \frac{\pi}{2}$ (by definition)

θ_t : " " transmission (refraction) $\quad 0 \le \theta_t \le \frac{\pi}{2}$

in medium 1
$$\begin{cases} \bar{E}^i = \bar{E}_1 \, e^{-j\bar{k}_i \cdot \bar{r}} & \text{with } \bar{k}_i = k_1 \hat{n}_i \quad \text{(for incident p.w)} \\ \bar{E}^r = \bar{E}_2 \, e^{-j\bar{k}_r \cdot \bar{r}} & \text{with } \bar{k}_r = k_1 \hat{n}_r \quad \text{(for reflected p.w)} \end{cases}$$

in medium 2
$$\left\{ \bar{E}^t = \bar{E}_3 \, e^{-j\bar{k}_t \cdot \bar{r}} \quad \text{with } \bar{k}_t = k_2 \hat{n}_t \quad \text{(for transmitted p.w.)} \right.$$

where $k_1 = \omega\sqrt{\epsilon_1 \mu_1}$ and $k_2 = \omega\sqrt{\epsilon_2 \mu_2}$

Apply the B.C. at z=0 that E^{total}_{tang} is continuous across the boundary.

$$\left(\bar{E}^i + \bar{E}^r\right)_{tang}\Big|_{z=0} = \bar{E}^t_{tang}\Big|_{z=0} \quad \text{for all } x,y\ !$$

$$\Rightarrow \left\{ \bar{E}_{1_{tang}} e^{-j\bar{k}_i\cdot\bar{r}} + \bar{E}_{2_{tang}} e^{-j\bar{k}_r\cdot\bar{r}} - \bar{E}_{3_{tang}} e^{-j\bar{k}_t\cdot\bar{r}} \right\}\Big|_{z=0} = 0$$

$$\Rightarrow \left\{ \bar{E}_{1_{tang}} e^{-j(k_{ix}x + k_{iy}y + k_{iz}z)} + \bar{E}_{2_{tang}} e^{-j(k_{rx}x + k_{ry}y + k_{rz}z)} - \bar{E}_{3_{tang}} e^{-j(k_{tx}x + k_{ty}y + k_{tz}z)} \right\}\Big|_{z=0} = 0$$

$$\bar{E}_{1_{tang}} e^{-j(k_{ix}x + k_{iy}y)} + \bar{E}_{2_{tang}} e^{-j(k_{rx}x + k_{ry}y)} - \bar{E}_{3_{tang}} e^{-j(k_{tx}x + k_{ty}y)} = 0 \qquad (*)$$

Note that for the incident, reflected and transmitted uniform planes, the vectors $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are all constant vectors. Therefore, their tangential (to the boundary) components are also constant vectors $\Rightarrow$ $\bar{E}_{1tang}$, $\bar{E}_{2tang}$ and $\bar{E}_{3tang}$ are constant vectors.

As a result, the equation (*) above is satisfied for all possible (x,y) pairs on the z=0 boundary plane if and only if

$$(**) \quad \boxed{k_{ix}x + k_{iy}y = k_{rx}x + k_{ry}y = k_{tx}x + k_{ty}y}$$

(so the exponential terms in (*) can be cancelled at z=0 plane)

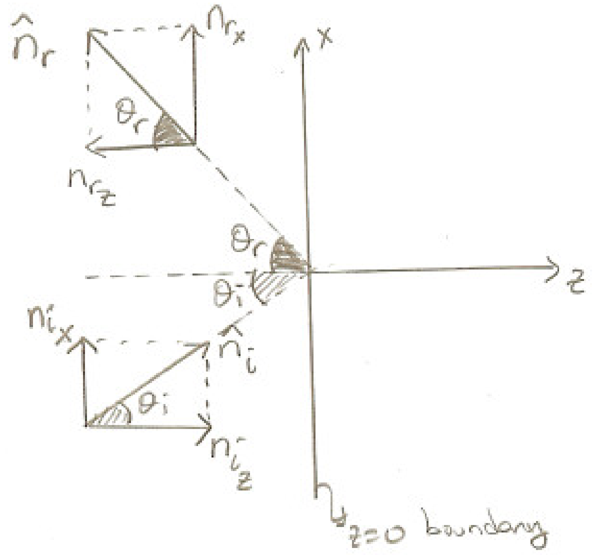
But, we initially assumed that the propagation vector $\bar{k}_i$ of the incident u.p.w lies on the (x-z) plane, so it has no y-component $\Rightarrow \boxed{k_{iy} = 0}$

From $(**)$ and using $k_{i_y}=0$ $\Rightarrow$
$$\begin{cases} \boxed{k_{i_x} = k_{r_x} = k_{t_x}} \quad (***) \\ k_{i_y} = k_{r_y} = k_{t_y} = 0 \end{cases}$$
which means, the propogation vectors $\bar{k}_r$ and $\bar{k}_t$ have no y-components.

Conclusion: Propogation vectors $\bar{k}_i$, $\bar{k}_r$ and $\bar{k}_t$ all lie in the plane of incidence (i.e, the (x-z) plane).

in medium ① $k_1=\omega\sqrt{\epsilon_1\mu_1}$
$$\bar{k}_i = k_1\hat{n}_i = k_1\underbrace{\left(\sin\theta_i\,\hat{a}_x + \cos\theta_i\,\hat{a}_z\right)}_{\hat{n}_i} = \underbrace{k_1\sin\theta_i}_{k_{i_x}}\hat{a}_x + \underbrace{k_1\cos\theta_i}_{k_{i_z}}\hat{a}_z$$
$$\bar{k}_r = k_1\hat{n}_r = \underbrace{k_1\sin\theta_r}_{k_{r_x}}\hat{a}_x - \underbrace{k_1\cos\theta_r}_{k_{r_z}}\hat{a}_z$$

in medium ② $k_2=\omega\sqrt{\mu_2\epsilon_2}$
$$\bar{k}_t = k_2\hat{n}_t = \underbrace{k_2\sin\theta_t}_{k_{t_x}}\hat{a}_x + \underbrace{k_2\cos\theta_t}_{k_{t_z}}\hat{a}_z$$



We just found above in $(***)$
$$k_{i_x} = k_{r_x} = k_{t_x}$$
From $k_{i_x}=k_{r_x} \Rightarrow k_1\sin\theta_i = k_1\sin\theta_r$
$$\Rightarrow \boxed{\theta_i = \theta_r}$$ Snell's Law of Reflection

Conclusion: Being independent from the medium parameters and frequency, the angle of incidence must be equal to the angle of reflection.

Using $k_{ix} = k_{tx} \Rightarrow \boxed{k_1 \sin\theta_i = k_2 \sin\theta_t}$ Snell's Law of Refraction

insert $k_1 = \omega\sqrt{\mu_1\epsilon_1}$ and $k_2 = \omega\sqrt{\mu_2\epsilon_2}$ above,

$$\cancel{\omega}\sqrt{\mu_1\epsilon_1}\,\sin\theta_i = \cancel{\omega}\sqrt{\mu_2\epsilon_2}\,\sin\theta_t$$

$$\boxed{\sqrt{\mu_1\epsilon_1}\,\sin\theta_i = \sqrt{\mu_2\epsilon_2}\,\sin\theta_t}$$

or $\boxed{\dfrac{\sin\theta_t}{\sin\theta_i} = \sqrt{\dfrac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}}$ This ratio is independent of frequency.

For these lossless media, we can define:

$$v_1 = \frac{1}{\sqrt{\mu_1\epsilon_1}} \text{ and } v_2 = \frac{1}{\sqrt{\mu_2\epsilon_2}} \Rightarrow \boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{v_2}{v_1}}$$

Index of refraction $= n \triangleq \dfrac{c}{v}$ in general where $c = 3\times10^8$ m/s, speed of light in vacuum $\left(c = \frac{1}{\sqrt{\epsilon_0\mu_0}}\right)$

$$\left.\begin{aligned} n_1 &\triangleq \frac{c}{v_1} = \sqrt{\mu_{r_1}\epsilon_{r_1}} \\ n_2 &\triangleq \frac{c}{v_2} = \sqrt{\mu_{r_2}\epsilon_{r_2}} \end{aligned}\right\} \Rightarrow \frac{\sin\theta_t}{\sin\theta_i} = \frac{v_2}{v_1} = \frac{c/n_2}{c/n_1} = \frac{n_1}{n_2} \Rightarrow \boxed{n_1\sin\theta_i = n_2\sin\theta_t}$$

Consider two possible cases:

Case 1

$\mu_1 \epsilon_1 < \mu_2 \epsilon_2$

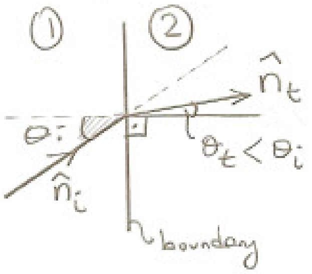
(second medium is denser)

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} < 1$$

$\Rightarrow \sin\theta_t < \sin\theta_i$

$\Rightarrow \boxed{\theta_t < \theta_i}$

The transmitted wave will be bent closer to the normal of the boundary.



Case 2

$\mu_1 \epsilon_1 > \mu_2 \epsilon_2$

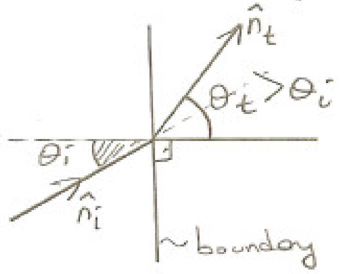
(First medium is denser)

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} > 1$$

$\Rightarrow \sin\theta_t > \sin\theta_i$

$\Rightarrow \boxed{\theta_t > \theta_i}$

The transmitted wave will be bent away from the normal of the boundary.



Question: How much can we increase θ_i (incidence angle) before θ_t becomes $\frac{\pi}{2}$ in case 2 where $\theta_t > \theta_i$?

θ_c : Critical Angle = Value of incidence angle θ_i for which $\theta_t = \frac{\pi}{2}$.

$$\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} = \left.\frac{\sin\theta_t}{\sin\theta_i}\right|_{\theta_i=\theta_c} = \frac{\sin\frac{\pi}{2}}{\sin\theta_c} = \frac{1}{\sin\theta_c} \Longrightarrow \boxed{\sin\theta_c = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}}$$

or $\theta_c = \sin^{-1}\left(\sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}\right)$

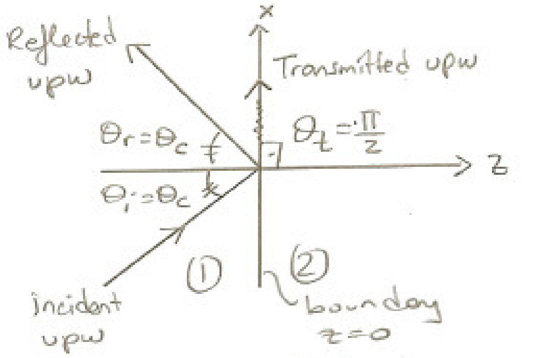
For $\theta_i = \theta_c$ when $\mu_1\epsilon_1 > \mu_2\epsilon_2$ (Case 2)

$$\Rightarrow \theta_t = \frac{\pi}{2} \Rightarrow \hat{n}_t = \hat{a}_x \Rightarrow \bar{k}_t = k_2\hat{n}_t = k_2\hat{a}_x$$

$$\Rightarrow \bar{k}_t \cdot \bar{r} = k_2\hat{a}_x \cdot (x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) = k_2 x$$

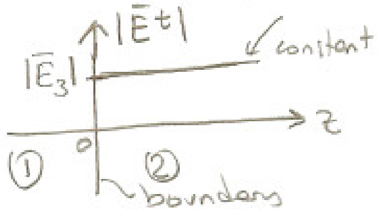
$$\Rightarrow \bar{E}^t = \bar{E}_3 e^{-j\bar{k}_t\cdot\bar{r}} = \underbrace{\bar{E}_3}_{\text{constant vector}} e^{-jk_2x}$$

(a uniform plane wave propagating in (+x) direction) in medium (2) for $z>0$



Note that $|\bar{E}^t| = |\bar{E}_3|$ for all $z>0$

There exists a u.p.w in medium (2) propagating in $\hat{n}_t = \hat{a}_x$ direction with constant $\bar{E}$-phasor magnitude everywhere in $z>0$ region.



Question: What happens (in case 2) if $\theta_i > \theta_c$?

$$\theta_i > \theta_c \Rightarrow \sin\theta_i > \sin\theta_c$$

As $\sin\theta_c = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}} \Rightarrow \sin\theta_i > \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}$

From Snell's Law of refraction $\Rightarrow \sin\theta_t = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} \sin\theta_i$

$$\sin\theta_t > \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}$$

$$\boxed{\sin\theta_t > 1}$$

Conclusion: For $\left\{\begin{matrix}\mu_1\epsilon_1 > \mu_2\epsilon_2 \\ \theta_i > \theta_c\end{matrix}\right\} \Rightarrow \boxed{\sin\theta_t > 1}$ which is NOT possible for a real valued angle

$$\Rightarrow \boxed{\theta_t \text{ must be a complex angle}}$$

Consider the trigonometric identity

$\cos^2\theta_t + \sin^2\theta_t = 1$ which is valid for complex angles also!

$$\cos\theta_t = \mp\sqrt{\underbrace{1-\sin^2\theta_t}_{<0 \text{ as } \sin\theta_t > 1}} = \mp\sqrt{-(\underbrace{\sin^2\theta_t - 1}_{>0})}$$

$$\cos\theta_t = \mp\underbrace{\sqrt{-1}}_{j}\sqrt{\sin^2\theta_t - 1} = \mp j\sqrt{\sin^2\theta_t - 1}$$

use Snell's Law

$\sqrt{\mu_1\epsilon_1}\sin\theta_i = \sqrt{\mu_2\epsilon_2}\sin\theta_t$

$$\boxed{\cos\theta_t = \mp j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}}$$

Conclusion: $\cos\theta_t$ is a purely imaginary number

$\Rightarrow$ θ_t is indeed a complex quantity if $\theta_i > \theta_c$.

Hence, for the transmitted wave in medium (2):

(Remember $\bar{k}_t = k_2\hat{n}_t$ and $\hat{n}_t = \sin\theta_t\hat{a}_x + \cos\theta_t\hat{a}_z$)

$$\bar{E}^t = \bar{E}_3\, e^{-j\bar{k}_t\cdot\bar{r}}$$

$$= \bar{E}_3\, e^{-j(k_2\sin\theta_t x + k_2\cos\theta_t z)}$$

$$= \bar{E}_3\, e^{-jk_2\sin\theta_t x}\, e^{-jk_2\cos\theta_t z}$$

(insert $\cos\theta_t$ expression)

$$= \bar{E}_3\, e^{-jk_2(\sin\theta_t)x}\, e^{-jk_2\left(-j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}\right)z}$$

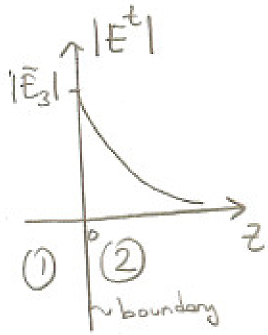
$$= \bar{E}_3\, \underbrace{e^{-jk_2\sin\theta_t x}}_{\text{Complex exponential}}\, \underbrace{e^{-k_2\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}\, z}}_{\text{Real exponential}}$$

(combine with $\bar{E}_3$: constant amplitude vector)

Let $\boxed{\begin{aligned} &\alpha_z \triangleq k_2 \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2} \sin^2\theta_i - 1} \\ &\beta_x \triangleq k_2 \sin\theta_t \end{aligned}}$ $\Rightarrow$ $\boxed{\bar{E}^t = \bar{E}_3 \, \underbrace{e^{-\alpha_z z}}_{\text{Attenuation in z-direction}} \, \underbrace{e^{-j\beta_x x}}_{\text{Propagation in x-direction}}}$

$$\bar{E}^t = \bar{E}_3 \, e^{-\alpha_z z} \, e^{-j\beta_x x}$$

Amplitude of transmitted wave $\quad |\bar{E}^t| = |\bar{E}_3| \underbrace{|e^{-\alpha_z z}|}_{e^{-\alpha_z z}} \underbrace{|e^{-j\beta_x x}|}_{1} = \underbrace{|\bar{E}_3|}_{\text{constant vector}} e^{-\alpha_z z}$



$|\bar{E}^t|$ is kept constant if $z = \text{constant}$ $\Rightarrow$ $\boxed{\text{"Constant Amplitude" surfaces are } z = \text{Constant planes.}}$

Phase of transmitted wave $\quad \angle \bar{E}^t = -\beta_x x \quad$ is kept constant if $x = \text{constant}$

$\Rightarrow$ $\boxed{\text{"Constant Phase" surfaces are } x = \text{Constant planes}}$

Conclusion: In medium ②, we still have "plane wave" type transmitted electromagnetic field but it is now a NON-UNIFORM PLANE WAVE because the constant amplitude surfaces (z=const. planes) are not the same as constant phase surfaces (x=const. planes).

NOTES:

* The non-uniform plane wave in medium 2 can also be called "<u>Surface Wave</u>" if the attenuation parameter $\alpha_z = k_2 \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2} \sin^2\theta_i - 1}$ is large enough. Then, the attenuation of the transmitted field in +z direction is sufficiently rapid, power associated with transmitted wave is concentrated in a region close to the boundary.

* The non-uniform plane wave solution is medium 2 is also called "<u>Slow Wave</u>" as: (remember $k_2 = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v}$)

For $\theta_i \leq \theta_c$, the phase velocity of the u.p.w in med. 2 $= v = \frac{\omega}{k_2}$

For $\theta_i > \theta_c$, the phase velocity of the non-uniform p.w in medium 2 $= v_{SW} = \frac{\omega}{\beta_x} = \frac{\omega}{k_2 \sin\theta_t}$

As $\sin\theta_t > 1$ (for $\theta_i > \theta_c$) $\Rightarrow v_{SW} = \frac{\omega}{k_2 \sin\theta_t} < \frac{\omega}{k_2} = v$

i.e., $\boxed{v_{SW} < v}$ (wave propagation is slower)

* When $\theta_i \geq \theta_c$, it is said that we have "Total Reflection" (TR) case where there is <u>no</u> wave propagation <u>into</u> the 2nd medium. Instead, propagation direction is parallel to the boundary.

* Remember that we have examined a "<u>lossless</u>" case. The parameter α_z in $\bar{E}^t = \bar{E}_3 e^{-\alpha_z z} e^{-j\beta_x x}$ expression has nothing to do with the α parameter of lossy medium problems. The decay $e^{-\alpha_z z}$ is <u>not</u> due to medium loss ($\sigma_2 = 0$ is assumed at the beginning.)