EE 303 MT1 Solutions – Fall 2021

Name, surname:

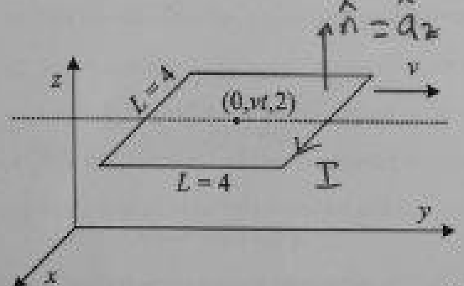
Question 1. (25 points)

The magnetic field in a region of space is given as

$$\bar{B}(\bar{r}) = 3z\hat{a}_x + 5x\hat{a}_y + 2y\hat{a}_z$$

A square loop that lies on the $z = 2$ plane has a total resistance of $R = 5\ \Omega$ and side $L = 4$ m. The loop is moving along the y axis at a velocity of $v = 2$ m/s (its center being at a point ($x = 0, y = vt, z = 2\ m$) at any given time).

a) Determine the current in the loop when the center of the loop is at $y = 3$ m.
b) Indicate the direction of the current on the figure.



$$\Phi(t) = \int_S \bar{B} \cdot d\bar{S} = \int_{vt-2}^{vt+2} \int_{x=-2}^{2} (3z\hat{a}_x + 5x\hat{a}_y + 2y\hat{a}_z) \cdot \hat{a}_z \, dx\, dy$$

$$= 8 \int_{vt-2}^{vt+2} y\, dy = 4 \left[y^2 \right]_{vt-2}^{vt+2}$$

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$$= 4 \left[(vt+2)^2 - (vt-2)^2 \right] = 4 \left[8vt \right] = 32vt = 64t$$

$$\left. \frac{d\Phi}{dt} \right|_{t=0} = 64 \Rightarrow V = 64\ V, \quad I = \frac{64}{5} = 12.8\ A$$

Question 2. (25 points)

Consider a homogeneous space with permittivity ϵ and permeability μ. The **phasor-domain** representation of a **time-harmonic** magnetic vector potential (assuming Lorenz gauge) is given as

$$\bar{A}(\bar{r}) = \hat{a}_y j \exp(-kx),$$

where $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber. Do the following, showing your steps clearly (can be done in any order).

(a) Find the time-domain representation of the magnetic vector potential, i.e., $\bar{A}(\bar{r}, t)$.

(b) Find the phasor-domain representation of the electric scalar potential, i.e., $\Phi(\bar{r})$.

(c) Find the phasor-domain representation of the electric field intensity, i.e., $\bar{E}(\bar{r})$.

(d) Find the phasor-domain representation of the magnetic field intensity, i.e., $\bar{H}(\bar{r})$.

(e) Find the phasor-domain representations of the sources, i.e., the electric current density $\bar{J}_v(\bar{r})$ and the electric charge density $q_v(\bar{r})$.

Caution: Note that this is not a plane wave.

Solution:

(a)

$$\begin{aligned}\bar{A}(\bar{r}, t) &= \text{Re}\{\hat{a}_y j \exp(-kx)\exp(j\omega t)\} = \text{Re}\{\hat{a}_y \exp(-kx)\exp(j\omega t + j\pi/2)\} \\ &= \hat{a}_y \exp(-kx)\cos(\omega t + \pi/2) = \boxed{-\hat{a}_y \exp(-kx)\sin(\omega t)}\end{aligned}$$

(b)

$$\begin{aligned}\nabla \cdot \bar{A}(\bar{r}) &= -j\omega\mu\epsilon\Phi(\bar{r}) \\ &= \frac{\partial}{\partial y}\left[j \exp(-kx)\exp(j\omega t)\right] = 0 \longrightarrow \boxed{\Phi(\bar{r}) = 0}\end{aligned}$$

(c)

$$\begin{aligned}\bar{E}(\bar{r}) &= -\nabla\Phi(\bar{r}) - j\omega\bar{A}(\bar{r}) \\ &= -j\omega\bar{A}(\bar{r}) = -j\omega\hat{a}_y j \exp(-kx) = \boxed{\hat{a}_y \omega \exp(-kx)}\end{aligned}$$

(d)

$$\bar{H}(\bar{r}) = \frac{1}{\mu}\nabla \times \bar{A}(\bar{r}) = \hat{a}_x \times \hat{a}_y \frac{j}{\mu}\frac{\partial}{\partial x}\exp(-kx) = \boxed{-\hat{a}_z \frac{jk}{\mu}\exp(-kx)} = \boxed{-\hat{a}_z \frac{j\omega}{\eta}\exp(-kx)}$$

(e)

$$\nabla^2 \bar{A}(\bar{r}) + k^2 \bar{A}(\bar{r}) = -\mu \bar{J}_v(\bar{r}) \longrightarrow \bar{J}_v(\bar{r}) = -\frac{1}{\mu}\left[\nabla^2 \bar{A}(\bar{r}) + k^2 \bar{A}(\bar{r})\right]$$

$$\bar{J}_v(\bar{r}) = -\frac{1}{\mu}\left[\hat{a}_y j k^2 \exp(-kx) + k^2 \hat{a}_y j \exp(-kx)\right]$$

$$= \boxed{-\hat{a}_y \frac{2jk^2}{\mu} \exp(-kx)} = \boxed{-\hat{a}_y 2j\omega^2 \epsilon \exp(-kx)}$$

$$\nabla \cdot \bar{J}_v(\bar{r}) = -j\omega q_v(\bar{r}) \longrightarrow q_v(\bar{r}) = -2j\omega^2 \epsilon \frac{\partial}{\partial y}\left[\exp(-kx)\right] = \boxed{0}$$

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Question 3. (25 points)

Consider a uniform plane wave ($\bar{E} = \bar{E}_0 e^{-j\bar{k}.\bar{r}}$) propagating in a non-magnetic material (μ_r=1) at a frequency of 31.25 MHz with a wavelength of $\frac{24}{5}$ m. Point P_1: x=3, y=2, z=3 and point P_2: x=0, y= -2, z=6 are located on a constant phase surface. The phasor electric field intensity is measured on this constant phase surface as: $\bar{E} = \hat{a}_z 4e^{j\pi/3}$ V/m.

a) Find the wave number.
b) Find the unit vector in the direction of propagation. Note that both $\hat{a}_n$ and -$\hat{a}_n$ are valid solutions. You may choose either of these two solutions.
c) Find complex vector $\bar{E}_0$.
d) Write the electric field intensity in time domain in terms of x, y, z and t.
e) Find the relative permittivity (ε_r) of the medium.
f) Find the magnetic field intensity in phasor domain.

Note that $\bar{r}$ is the position vector.

a) $k = \frac{2\pi}{\lambda} = \frac{2\pi\, 5}{24} = \frac{5\pi}{12}$ rad/m

b) The vector from P_1 to P_2 lies on constant phase surface

$\bar{r}_{12} = (3-0)\,\hat{a}_x + (2+2)\,\hat{a}_y + (3-6)\,\hat{a}_z$

$= 3\hat{a}_x + 4\hat{a}_y - 3\hat{a}_z$

$\bar{E}$ field vector also lies on constant phase surface. The unit vector in the direction of propagation should be perpendicul to this constant phase surface. Then it can be found from the cross product of $\bar{r}_{12}$ and $\bar{E}$ vectors

$(3\hat{a}_x + 4\hat{a}_y - 3\hat{a}_z) \times (\hat{a}_z) = -3\hat{a}_y + 4\hat{a}_x$

$\hat{a}_n = \frac{4\hat{a}_x - 3\hat{a}_y}{5}$

Note that $\bar{E} \times \bar{r}_{12}$ is also a valid solution

b) Alternative solution

$\bar{k} = k_x \hat{a}_x + k_y \hat{a}_y$ since $\hat{a}_n \cdot \bar{E} = 0$ $k_z = 0$

Let $\bar{E}_0 = E_0 e^{j\psi_0} \hat{a}_z$ where E_0 is real

$\bar{E} = E_0 e^{-j(k_x x + k_y y - \psi_0)} \hat{a}_z$

By equating phases at P_1 and P_2

$3k_x + 2k_y - \psi_0 = -2k_y - \psi_0$

$3k_x + 4k_y = 0 \qquad k_x = -\frac{4}{3} k_y$

$k_x^2 + k_y^2 = k^2 = \frac{25\pi^2}{144} = \frac{16}{9} k_y^2 + k_y^2 = \frac{25\pi^2}{144}$

$k_y = \mp \frac{\pi}{4}$ choose $k_y = -\frac{\pi}{4} \Rightarrow k_x = \frac{\pi}{3}$

$\hat{a}_n = \dfrac{\frac{\pi}{3}\hat{a}_x - \frac{\pi}{4}\hat{a}_y}{\frac{5\pi}{12}} = \dfrac{4\hat{a}_x - 3\hat{a}_y}{5}$

You may choose $k_y = \frac{\pi}{4} \Rightarrow \hat{a}_n =$

c) $\bar{E}\Big|_{\text{at } P_1} = E_0 e^{-j\left(\frac{\pi}{3}x - \frac{\pi}{4}y - \psi_0\right)} \hat{a}_z = E_0 e^{-j\left(\pi - \frac{\pi}{2} - \psi_0\right)} \hat{a}_z$

$= 4 e^{j\pi/3} \hat{a}_z$

$E_0 = 4 \qquad -\frac{\pi}{2} + \psi_0 = \frac{\pi}{3} \Rightarrow \psi_0 = \frac{5\pi}{6}$

$\bar{E}_0 = 4 e^{j5\pi/6} \hat{a}_z$ V/m

d) $\bar{\mathcal{E}}(\bar{r}, t) = 4 \cos\left(62.5 \times 10^6 \pi t - \frac{\pi}{3} x + \frac{\pi}{4} y + \frac{5\pi}{6}\right) \hat{a}$ V/m

e) $v_p = \frac{\omega}{k} = \frac{62.5 \times 10^6 \cancel{\pi}}{5\cancel{\pi}/12} = 150 \times 10^6 = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3 \times 10^8}{\sqrt{\varepsilon_r}}$

$\Rightarrow \sqrt{\varepsilon_r} = 2 \qquad \boxed{\varepsilon_r = 4}$

f) $\eta = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{120\pi}{2} = 60\pi \ \Omega$

$$\bar{H} = \frac{1}{\eta} \hat{a}_n \times \bar{E} = \frac{1}{60\pi}\left(\frac{4\hat{a}_x - 3\hat{a}_y}{5}\right) \times \hat{a}_z \, 4 \, e^{j5\pi/6} e^{-j\bar{k}\cdot\bar{r}}$$

$$= \frac{1}{75\pi}\left(-4\hat{a}_y - 3\hat{a}_x\right) e^{j5\pi/6} e^{-j\left(\frac{\pi}{3}x - \frac{\pi}{4}y\right)} \ A/m$$

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Question 4. (25 points)

a) The phasor electric field intensity of a uniform plane wave is given as

$$\bar{E}_1 = (4\hat{a}_x - 10j\hat{a}_y + 3\hat{a}_z)e^{-j10\pi(3x-4z)} \quad \text{V/m}.$$

Find the type and sense of polarization of $\bar{E}_1$.

b) Another uniform plane wave, $\bar{E}_2$, of the same frequency propagates at the same direction. Its phasor expression is given by,

$$\bar{E}_2 = (8\hat{a}_x + A\hat{a}_y + B\hat{a}_z)e^{-j10\pi(3x-4z)} \quad \text{V/m}.$$

Given that total electric field of the medium is represented by $\bar{E}_3 = \bar{E}_1 + \bar{E}_2$ and $\bar{E}_3$ is a right hand circularly polarized plane wave, find the constants A and B.

Note that in both parts, you ***must verify your answers with sketches/explanations.**. Answers consisting of a single word will not be fully credited.*

$\bar{k}\cdot\bar{r} = 10\pi(3x-4z) \Rightarrow \bar{k} = 10\pi(3\hat{a}_x - 4\hat{a}_z)$

$\hat{a}_n = \frac{\bar{k}}{|\bar{k}|} = \frac{3}{5}\hat{a}_x - \frac{4}{5}\hat{a}_z$

a) at $(x,y,z) = (0,0,0)$

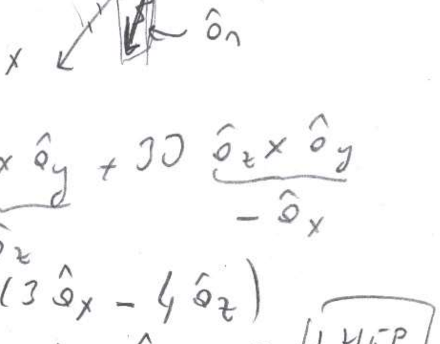
$\bar{\mathcal{E}}_1(t) = (4\hat{a}_x + 3\hat{a}_z)\cos\omega t + 10\hat{a}_y \underbrace{\cos(\omega t - \frac{\pi}{2})}_{\sin\omega t}$

$|\bar{E}_1|^2 = E_x^2 + E_y^2 + E_z^2 = (16+9)\cos^2\omega t + 100\sin^2\omega t$

$|\bar{\mathcal{E}}_1(t)|$ changes with time

$t=0 \quad \bar{\mathcal{E}}_1 = 4\hat{a}_x + 3\hat{a}_z$

$\omega t = \frac{\pi}{2} \quad \bar{\mathcal{E}}_1 = 10\hat{a}_y$



$\boxed{\text{LHEP}}$

or $\bar{\mathcal{E}}_1\big|_{t=0} \times \bar{\mathcal{E}}_1\big|_{\omega t=\frac{\pi}{2}} = 40\underbrace{\hat{a}_x \times \hat{a}_y}_{\hat{a}_z} + 30\underbrace{\hat{a}_z \times \hat{a}_y}_{-\hat{a}_x}$

$= -10(3\hat{a}_x - 4\hat{a}_z)$

opposite to $\hat{a}_n \Rightarrow \boxed{\text{LHEP}}$

b) $\bar{E}_2$ is a UPW $\Rightarrow$ $\bar{E}_2 \cdot \hat{a}_n = 0$

$8 \times 3 - B \times 4 = 0 \Rightarrow B = 6$

$\bar{E}_3 = \bar{E}_1 + \bar{E}_2 = (12\,\hat{a}_x + (A - 10j)\,\hat{a}_y + 9\,\hat{a}_z)\, e^{-j10\pi(3x-4z)}$

To have circular polarization, the two orthogonal components of phasor $\bar{E}_3$ must have equal magnitudes and a 90° phase difference

i.e. $|12\,\hat{a}_x + 9\,\hat{a}_z| = |A - 10j| = \sqrt{144 + 81} = 15$

$A = 25j$ would give us a RHCP wave