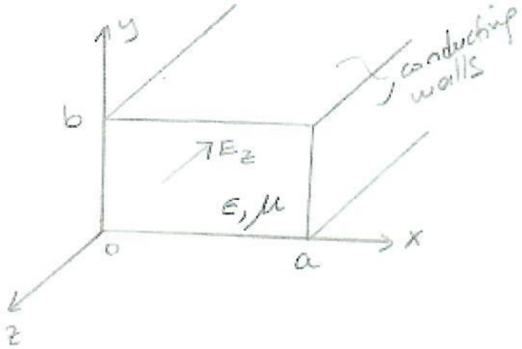
TM Solutions in a Rectangular Waveguide



For TM modes, $\boxed{H_z = 0}$ but $\boxed{E_z \neq 0}$

Solve for $\boxed{E_z(x,y,z) = E_z^\circ(x,y)e^{-\gamma z}}$

first. Then, compute E_x, E_y, H_x, H_y.

Boundary Value Problem (BVP) to be solved is:

BVP

$$\underbrace{\frac{\partial^2 E_z^\circ(x,y)}{\partial x^2} + \frac{\partial^2 E_z^\circ(x,y)}{\partial y^2}}_{\nabla_t^2 E_z^\circ} + h^2 E_z^\circ(x,y) = 0 \quad (*) \qquad \text{where } \boxed{h^2 = \gamma^2 + k^2}$$

such that

$$E_z^\circ(x,y) = 0 \quad \text{at} \quad x = 0, a \text{ and } y = 0, b$$

(as $\bar{E}_{tang}$ is zero at the surface of a perfect conductor)

Use the Method of Separation of Variables:

Let $E_z^\circ(x,y) = X(x)\,Y(y)$ and substitute it in (*)

$$\frac{d^2 X(x)}{dx^2} Y + X(x)\frac{d^2 Y(y)}{dy^2} + h^2 X(x) Y(y) = 0$$

divide both sides by $X(x)Y(y)$ to get

$$\underbrace{\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}}_{-k_y^2} + h^2 = 0 \quad (**)$$

$$\Rightarrow \boxed{h^2 = k_x^2 + k_y^2}$$ Separation Condition (k_x, k_y: Separation constants)

Then, (**) reduces to two ordinary diff. eqns.:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2 \Rightarrow \frac{d^2X(x)}{dx^2} + k_x^2\, X(x) = 0$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2 \Rightarrow \frac{d^2Y(y)}{dy^2} + k_y^2\, Y(y) = 0$$

($s^2+k_x^2=0 \Rightarrow s_{1,2}=\pm j k_x$ etc.)

with solutions

$$\begin{cases} X(x) = A_1 \sin k_x x + A_2 \cos k_x x \\ Y(y) = B_1 \sin k_y y + B_2 \cos k_y y \end{cases}$$

s.t. $E_z^{\circ}(x,y) = X(x)\,Y(y)$

Use B.C.'s next:

$E_z^{\circ} = X(x)\,Y(y) = 0$ at $x=0,a$ $\Rightarrow X(x)=0$ at $x=0,a$

$$X(x=0) = A_1 \cancel{\underbrace{\sin(0)}_{0}} + A_2 \underbrace{\cos(0)}_{1} = 0 \Rightarrow \underline{\underline{A_2=0}} \Rightarrow \underline{X(x)=A_1\sin k_x x}$$

$$X(x=a) = \underbrace{A_1}_{\neq 0} \sin k_x a = 0 \Rightarrow \sin(k_x a) = 0 \Rightarrow k_x a = m\pi$$

i.e, $\boxed{k_x = \frac{m\pi}{a}, \quad m=1,2,3,\dots}$

Similarly,

$E_z^{\circ} = X(x)\,Y(y) = 0$ at $y=0,b$ $\Rightarrow Y(y)=0$ at $y=0,b$

$$Y(y=0) = B_1 \cancel{\underbrace{\sin(0)}_{0}} + B_2 \underbrace{\cos(0)}_{1} = 0 \Rightarrow B_2=0 \Rightarrow \underline{Y(y)=B_1\sin k_y y}$$

$$Y(y=b) = \underbrace{B_1}_{\neq 0} \sin k_y b = 0 \Rightarrow \sin(k_y b) = 0 \Rightarrow k_y b = n\pi$$

i.e, $\boxed{k_y = \frac{n\pi}{b}, \quad n=1,2,3,\dots}$

Soln. to BVP for TM modes

$$\Rightarrow \boxed{\begin{array}{ll} h = \sqrt{k_x^2+k_y^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \ \text{(eigenvalues)} & \text{for } m=1,2,3\dots \\ & \quad\ \ n=1,2,3\dots \\ E_z^{\circ}(x,y) = A_1 B_1 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \ \text{eigenvectors (modes)} & \end{array}}$$

Let $E_0 = A_1 B_1$ (just another constant, note that $A_1 \neq 0$, $B_1 \neq 0$ to get a non-trivial soln.)

$$\Rightarrow \quad E_z(x,y,z) = E_z^{\circ}(x,y)\, e^{-\gamma z}$$

$$\boxed{E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}}$$

linearly independent solutions (modes) for E_z.

$$\text{where} \quad \gamma = \sqrt{h^2 - k^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \epsilon}$$

(as $h^2 = \gamma^2 + k^2$ and $k^2 = \omega^2 \mu \epsilon$)

for $m = 1, 2, \dots \infty$
$n = 1, 2, \dots \infty$

(Note that $m=0$ and $n=0$ cases give trivial soln. as $\sin(0) = 0$.)

Also note that negative signed indices do not produce linearly independent modes as $\sin\frac{(-m)\pi x}{a} = -\sin\frac{(+m)\pi x}{a}$

$\Rightarrow$ Possible TM_{mn} modes are:

$$TM_{\underset{m=1\quad n=1}{11}},\ TM_{12},\ TM_{21},\ TM_{22},\ TM_{\underset{m=1\quad n=3}{1,3}},\ TM_{31},\ TM_{2,3}, \dots$$

Note that

$$\boxed{f_{c_{mn}} = \frac{h_{mn}}{2\pi\sqrt{\mu\epsilon}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

cut-off frequencies of TM_{mn} modes

Propagation of a mode TM_{mn} is possible iff $f > f_{c_{mn}}$

so that

$$\left.\begin{aligned} &\gamma = j\beta \\ &\gamma = \sqrt{h^2 - k^2} = \sqrt{-(k^2 - h^2)} = j\underbrace{\sqrt{k^2 - h^2}}_{\beta} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} &\gamma_{mn} = j\beta_{mn} \\ &\beta_{mn} = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \end{aligned}}$$

As we know β_{mn} for this propagating mode TM_{mn}, we can determine λ_g and v_g at a given frequency $\omega = 2\pi f$ as:

$$\lambda_g = \frac{2\pi}{\beta} \Rightarrow \lambda_{g_{mn}} = \frac{2\pi}{\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

$$v_g = \frac{\omega}{\beta} \Rightarrow v_{g_{mn}} = \frac{\omega}{\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Also,

$$\lambda_{c_{mn}} = \frac{v}{f_{c_{mn}}} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

when $f > f_{c_{mn}} \Rightarrow \lambda < \lambda_{c_{mn}}$, $\gamma_{mn} = j\beta_{mn}$ (TM_{mn} propagates)

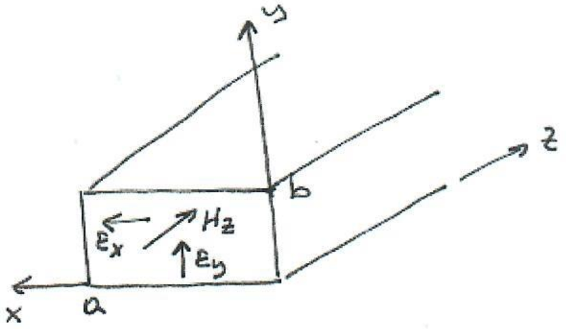
when $f < f_{c_{mn}} \Rightarrow \lambda > \lambda_{c_{mn}}$, $\gamma_{mn} = \alpha_{mn}$ (TM_{mn} is evanescent)

where f is the operation frequency, and "mn" is any combination of the indices m and n chosen from the infinite sets of integers $m = 1, 2, \cdots\cdots$, $n = 1, 2, \cdots\cdots$.

Exercise: Now, you know the solution for E_z. Find H_x, H_y, E_x, E_y using the equation set (12.) given in previous notes.

Continue with Rectangular Waveguides

TE Waves



By definition $E_z = 0$

Solve for $H_z(x,y,z) = H_z^o(x,y)\,e^{-\gamma z}$

$H_z^o(x,y)$ can be obtained by solving the following p.d.e.

$$\underbrace{\frac{\partial^2 H_z^o}{\partial x^2} + \frac{\partial^2 H_z^o}{\partial y^2}}_{\nabla_t^2 H_z^o} + h^2 H_z^o = 0$$

$$\searrow \; h^2 = k^2 + \gamma^2$$

Using Method of Seperation of Variables:

Let $H_z^o(x,y) = X(x)\,Y(y)$, insert it into the p.d.e.

to obtain

$$\left.\begin{aligned} X(x) &= A_1 \sin k_x x + A_2 \cos k_x x \\ Y(y) &= B_1 \sin k_y y + B_2 \cos k_y y \end{aligned}\right\} \text{Unknowns can be determined using B.C.'s.}$$

Boundary Condition to be applied here are:

· $E_y = 0$ at $x = 0, a$

$E_y = 0 \Rightarrow E_y^o = 0$

where $E_y^o = \frac{j\omega\mu}{h^2}\frac{\partial H_z^o}{\partial x}$

$\Rightarrow \frac{\partial H_z^o}{\partial x} = 0$ at $x = 0, a$

$\Downarrow$

$\frac{d X(x)}{dx} = 0$ at $x = 0, a$

$\Rightarrow k_x(A_1 \cos k_x x - A_2 \sin k_x x)\Big|_{x=0} = 0$

$\Rightarrow \boxed{A_1 = 0} \Rightarrow X(x) = A_2 \cos k_x x$

$-k_x A_2 \sin k_x x\Big|_{x=a} = 0 \Rightarrow \boxed{k_x = \frac{m\pi}{a}, \; m = 0, 1, 2, \dots}$

$E_x = 0$ at $y = 0, b$

$E_x = 0 \Rightarrow E_x^o = 0$

where $E_x^o = -\frac{j\omega\mu}{h^2}\frac{\partial H_z^o}{\partial y}$

$\Downarrow$

$\frac{\partial H_z^o}{\partial y} = 0$ at $y = 0, b$

$\Downarrow$

$\frac{d Y(y)}{dy} = 0$ at $y = 0, b$

Similarly, $\Rightarrow \boxed{B_1 = 0}$

and $\boxed{k_y = \frac{n\pi}{b} \quad n = 0, 1, 2, \dots}$

Therefore,
$$\left.\begin{aligned} X(x) &= A_2 \cos\left(\frac{m\pi}{a}x\right) \\ Y(y) &= B_2 \cos\left(\frac{n\pi}{b}y\right) \end{aligned}\right\} \Rightarrow H_z^o(x,y) = \underbrace{A_2B_2}_{\text{let, } H_o} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \boxed{H_z^o(x,y) = H_o \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad m,n: 0,1,2,\cdots}$$

$$\text{with} \quad h^2 = k_x^2 + k_y^2 \Rightarrow \boxed{h_{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

Eigenvalues for TE modes

<u>Note</u>: For TE modes, either n or m can be zero, but they can't be zero at the same time

i.e. $m, n: 0, 1, 2, \cdots$

but $m = n = 0$ is <u>not</u> included.

we may have TE_{10}, TE_{01}, TE_{11}, $TE_{1,2}$, $TE_{3,1}$, ... etc.

<u>Definition</u>: <u>FUNDAMENTAL MODE</u> is the mode having the <u>lowest cut-off frequency</u>. It is also called as the <u>dominant mode</u>.

<u>Note that</u> for a rectangular waveguide with $a > b$ TE_{10} mode is the dominant mode.

Because, for both TE and TM modes h is defined as
$$h_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \Rightarrow (f_c)_{mn} = \frac{h_{mn}}{2\pi\sqrt{\mu\epsilon}} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

For $a > b$ case $TE_{\underset{m=1}{1}\,\underset{n=0}{0}}$ gives the lowest possible cut-off frequency $f_{c_{TE10}}$.

$$\boxed{\underbrace{(f_c)_{TE_{10}}}_{\text{cut-off freq.}} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{v}{2a}}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$

($m = 1, n = 0 \Rightarrow$ no y-variation)

$$\boxed{h_{10} = \frac{\pi}{a}} \qquad (\lambda_c)_{TE_{10}} = \frac{2\pi}{h} = \frac{2\pi}{\pi/a} = 2a \Rightarrow \boxed{\lambda_{c_{TE_{10}}} = 2a}$$

Eigenvalue for $m=1, n=0$.

cut-off wavelength

Field expressions for TE_{10} mode

$$H_z^0 = H_0 \cos\frac{\pi x}{a} \Longrightarrow \boxed{H_z = H_0 \cos\frac{\pi x}{a}\, e^{-\gamma z}}$$

(independent of y)

$$H_x^0 = -\frac{1}{h^2}\gamma \frac{\partial H_z^0}{\partial x} = -\frac{1}{h^2}\gamma H_0\left(-\frac{\pi}{a}\right)\sin\frac{\pi x}{a} = \frac{\gamma}{\pi/a} H_0 \sin\frac{\pi x}{a}$$

$h = \frac{\pi}{a}$

from Eqn. set ⑬

$$\Longrightarrow \boxed{H_x = \frac{\gamma}{\pi/a} H_0 \sin\frac{\pi x}{a}\, e^{-\gamma z}}$$

$$H_y^0 = -\frac{1}{h^2}\gamma \cancel{\frac{\partial H_z^0}{\partial y}}^{\,0} = 0 \Longrightarrow \boxed{H_y = 0}$$

$$E_x^0 = -\frac{1}{h^2}(j\omega\mu)\cancel{\frac{\partial H_z^0}{\partial y}}^{\,0} = 0 \Longrightarrow \boxed{E_x = 0}$$

$$E_y^0 = \frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial x} = \frac{j\omega\mu}{(\pi/a)^2} H_0\left(-\frac{\pi}{a}\right)\sin\frac{\pi x}{a} = -\frac{j\omega\mu}{\pi/a} H_0 \sin\frac{\pi x}{a}$$

$$\Longrightarrow \boxed{E_y = \underbrace{\left(-\frac{j\omega\mu}{\pi/a} H_0\right)}_{\text{may call } E_0} \sin\frac{\pi x}{a}\, e^{-\gamma z}},$$

we also know by definition $\boxed{E_z = 0}$

$\Rightarrow \therefore$ For TE_{10} mode, H_z, H_x and E_y are the only non-zero field components.

Also, in the expressions above,

$$\gamma = \gamma_{1,0} = \sqrt{h_{1,0}^2 - k^2} \Rightarrow \boxed{\gamma_{1,0} = \sqrt{\left(\frac{\pi}{a}\right)^2 - \omega^2\mu\epsilon}}$$

For $\omega > \omega_c$ $(f > f_c)$ $\Rightarrow$ $\gamma_{1,0} = j\beta_{1,0} = j\underbrace{\sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{a}\right)^2}}_{\beta_{1,0}}$

$$\boxed{\beta_{1,0} = \sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{a}\right)^2}}$$ for a propagating TE_{10} mode.

Also, $$\boxed{Z_{TE_{10}} = -\frac{E_y^{\circ}}{H_x^{\circ}} = \frac{j\omega\mu}{\gamma_{1,0}}}$$

(λ_g and v_g can also be computed using $\beta_{1,0}$)

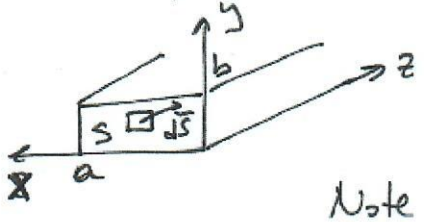
Power Transmission In waveguides (Rectangular WGs)

$$\overline{P}_{av} = \frac{1}{2} Re\{\bar{E} \times \bar{H}^*\} \Rightarrow P_{av} = \frac{1}{2} Re \int_S \bar{E} \times \bar{H}^* \cdot d\bar{S}$$

$\overline{P}_{av}$ ↓ Time-avg. power density

P_{av} ↳ Time-average power

where $d\bar{s} = \underbrace{dx\,dy}_{dS}\,\hat{a}_z$



$$\bar{E} \times \bar{H}^* = \hat{a}_x (E_y H_z^* - E_z H_y^*) + \hat{a}_y (E_z H_x^* - E_x H_z^*) + \hat{a}_z (E_x H_y^* - E_y H_x^*)$$

Note that $E_z = 0$ for TE modes
$H_z = 0$ for TM modes

$$\Rightarrow P_{av} = \frac{1}{2} Re \left\{ \int_S (E_x H_y^* - E_y H_x^*) dS \right\}$$

where $E_x = Z H_y$, $E_y = -Z H_x$ with Z (wave impedance) is either Z_{TM} or Z_{TE}.

$$\Rightarrow P_{av} = \frac{1}{2} Re\{ Z\} \underbrace{\int_{y=0}^{b} \int_{x=0}^{a} (|H_x|^2 + |H_y|^2)\, dx\, dy}_{\text{positive, real}}$$

$$\Rightarrow P_{av} = \frac{1}{2} Re\left\{\frac{1}{Z}\right\} \underbrace{\int_0^b \int_0^a (|E_x|^2 + |E_y|^2)\, dx\, dy}_{\text{pos., real}}$$

Note that

for $f > f_c$ mode propagates, Z is real

for $f < f_c$ mode is evanescent, Z is purely imaginary

$\Rightarrow P_{av} = 0$ for $f < f_c$

as expected.

Exercise: Show that for the $TE_{1,0}$ mode, we can get

$$\boxed{P_{av} = \frac{|E_0|^2 ab}{4 Z_{TE_{10}}}}$$

where

$$\boxed{Z_{TE_{1,0}} = \frac{\omega\mu}{\beta_{1,0}} = \frac{\omega\mu}{\sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{a}\right)^2}}}$$