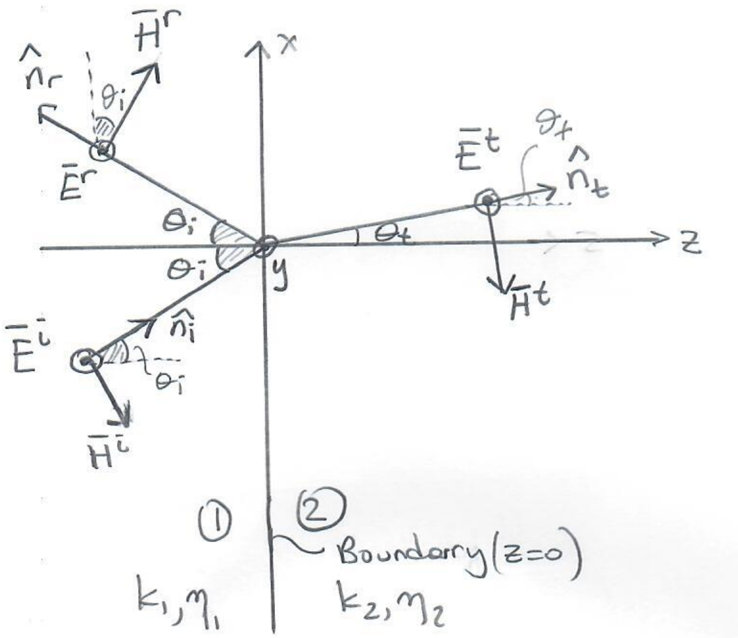
PERPENDICULAR POLARIZATION CASE

($\bar{E}^i$ vector is normal to the POI - plane of incidence)



$\hat{n}_i = \sin\theta_i\, \hat{a}_x + \cos\theta_i\, \hat{a}_z$

$\hat{n}_r = \sin\theta_i\, \hat{a}_x - \cos\theta_i\, \hat{a}_z$

$\hat{n}_t = \sin\theta_t\, \hat{a}_x + \cos\theta_t\, \hat{a}_z$

Plane of Incidence : (x-z) plane

Using

$$\bar{H} = \frac{1}{\eta}\, \hat{n} \times \bar{E}$$

for plane waves, express $\bar{E}$ and $\bar{H}$ phasors for the incident, reflected and transmitted waves.

$$\bar{E}^i = E_1 e^{-jk_1 \hat{n}_i \cdot \bar{r}}\, \hat{a}_y \longrightarrow \bar{H}^i = \frac{1}{\eta_1}\hat{n}_i \times \bar{E}^i = \frac{E_1}{\eta_1}\left(-\cos\theta_i\, \hat{a}_x + \sin\theta_i\, \hat{a}_z\right) e^{-jk_1 \hat{n}_i \cdot \bar{r}}$$

$$\bar{E}^r = E_2 e^{-jk_1 \hat{n}_r \cdot \bar{r}}\, \hat{a}_y \longrightarrow \bar{H}^r = \frac{1}{\eta_1}\hat{n}_r \times \bar{E}^r = \frac{E_2}{\eta_1}\left(\cos\theta_i\, \hat{a}_x + \sin\theta_i\, \hat{a}_z\right) e^{-jk_1 \hat{n}_r \cdot \bar{r}}$$

$$\bar{E}^t = E_3 e^{-jk_2 \hat{n}_t \cdot \bar{r}}\, \hat{a}_y \longrightarrow \bar{H}^t = \frac{1}{\eta_2}\hat{n}_t \times \bar{E}^t = \frac{E_3}{\eta_2}\left(-\cos\theta_t\, \hat{a}_x + \sin\theta_t\, \hat{a}_z\right) e^{-jk_2 \hat{n}_t \cdot \bar{r}}$$

Apply B.C.'s at z=0

1) $\bar{E}^{tot}_{tang①}\Big|_{z=0} = \bar{E}^{tot}_{tang②}\Big|_{z=0} \Rightarrow \boxed{E_1 + E_2 = E_3}$ ⓐ

(Exponential terms cancel out at z=0)

(E_y components are tangential to the z=0 boundary surface)

2) $\bar{H}^{tot}_{tang①}\Big|_{z=0} = \bar{H}^{tot}_{tang②}\Big|_{z=0} \Rightarrow \left(H^i_x + H^r_x\right)\Big|_{z=0} = H^t_x\Big|_{z=0}$

$$\Rightarrow \boxed{\frac{1}{\eta_1}(E_2 - E_1)\cos\theta_i = \frac{1}{\eta_2}(-E_3\cos\theta_t)}\ ⓑ$$

Solving equations ⓐ and ⓑ simultaneously for

$\Gamma = \frac{E_2}{E_1}$ and $T = \frac{E_3}{E_1}$, we can get Fresnel formulas:

FRESNEL Formulas for Perpendicular ($\perp$) Polarization

$$\boxed{\begin{aligned} \Gamma_\perp &= \frac{E_2}{E_1} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \\ T_\perp &= \frac{E_3}{E_1} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \end{aligned}}$$

Note that $\boxed{T_\perp = 1 + \Gamma_\perp}$ here!

Special Case 1: Perfect Dielectric / Perfect Conductor Case
(1) (2)

$$\eta_2 = \lim_{\sigma_2 \to \infty} \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = 0$$

Set $\eta_2 = 0$ in Fresnel formulas to get

$\Rightarrow \boxed{\Gamma_\perp = -1}$ and $\boxed{T_\perp = 0}$ for perpendicular polarization.

(i.e. $\boxed{E_2 = -E_1}$ and $\boxed{E_3 = 0}$)

Special Case 2: Consider the case $\mu_1\epsilon_1 > \mu_2\epsilon_2$

(i) Let $\theta_i = \theta_c \Rightarrow \theta_t = \pi/2 \Rightarrow \begin{cases} \sin\theta_t = 1 \\ \cos\theta_t = 0 \end{cases}$

$$\Gamma_\perp = \frac{E_2}{E_1} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos(\pi/2)}{\eta_2 \cos\theta_i + \eta_1 \cancel{\cos(\pi/2)}} = \boxed{1 = \Gamma_\perp} \Rightarrow \boxed{E_2 = E_1}$$

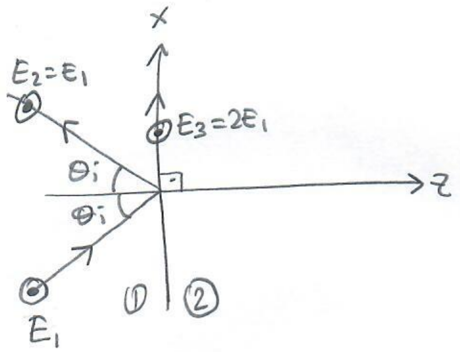
$$T_\perp = \frac{E_3}{E_1} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cancel{\cos(\pi/2)}} = \boxed{2 = T_\perp} \Rightarrow \boxed{E_3 = 2E_1}$$

Then, $\bar{E}^t = E_3 \, e^{-jk_2 \hat{n}_t \cdot \bar{r}} \, \hat{a}_y$

$$= E_3 \, e^{-jk_2(x \overbrace{\sin\theta_t}^{1} + z \overbrace{\cos\theta_t}^{0})} \, \hat{a}_y = \boxed{E_3 \, e^{-jk_2 x} \, \hat{a}_y = \bar{E}^t}$$

$$\bar{H}^t = \frac{E_3}{\eta_2}\left(-\hat{a}_x \underbrace{\cos\theta_t}_{0} + \hat{a}_z \underbrace{\sin\theta_t}_{1}\right) e^{-jk_2 \hat{n}_t \cdot \bar{r}} = \boxed{\frac{E_3}{\eta_2} e^{-jk_2 x} \, \hat{a}_z = \bar{H}^t}$$

with $\boxed{E_3 = 2E_1}$



Power Flow Density in Medium ② becomes:

$$\bar{P}_{av_{②}} = \frac{1}{2} Re\{\bar{E}^t \times \bar{H}^{t*}\}$$

$$= \frac{1}{2} Re\left\{\frac{|E_3|^2}{\eta_2} \hat{a}_y \times \hat{a}_z\right\} = \frac{1}{2}\frac{|E_3|^2}{\eta_2} \hat{a}_x \quad (W/m^2)$$

$$\boxed{\bar{P}_{av_{②}} = \frac{1}{2}\frac{|E_3|^2}{\eta_2} \hat{a}_x}$$

Power flow is in $+x$ direction as expected.

(ii) Let $\theta_i > \theta_c$ $\Rightarrow \theta_t$ is a complex angle

$$\left.\begin{array}{l} \sin\theta_t = a > 1 \\ \cos\theta_t = -jb \end{array}\right\} \text{ where } \left\{\begin{array}{l} a = \sqrt{\dfrac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} \sin\theta_i \\ b = \sqrt{\dfrac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2} \sin^2\theta_i - 1} \end{array}\right.$$

Then,

$$\Gamma_\perp = \left.\frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}\right|_{\cos\theta_t = -jb} = \boxed{\frac{A + jB}{A - jB} = 1 \, e^{j\phi_\perp} = \Gamma_\perp}$$

i.e., $\Gamma_\perp$ = complex, $|\Gamma_\perp| = 1$

$$T_\perp = \left.\frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}\right|_{\cos\theta_t = -jb} = \boxed{\frac{C}{A - jB} = D \, e^{j\psi}}$$

$T_\perp$ = complex

Exercise:

Show that
$$\begin{cases} \bar{E}^t = E_3\, e^{-\alpha_z z}\, e^{-j\beta_x x}\, \hat{a}_y \\ \bar{H}^t = \frac{E_3}{\eta_2}\, e^{-\alpha_z z}\, e^{-j\beta_x x}\,(\hat{a}_x\, jb + \hat{a}_z\, a) \\ \bar{P}_{av_{(2)}} = \frac{1}{2}\frac{|E_3|^2}{\eta_2}\, e^{-2\alpha_z z}\, a\, \hat{a}_x \end{cases}$$

(α_z and β_x are as defined earlier)

Special Case 3: Normal Incidence at Perpendicular Polarization

Normal incidence $\Rightarrow \theta_i = 0 \quad \Longrightarrow \quad \theta_r = \theta_i = 0$

$$\sqrt{\mu_1\epsilon_1}\,\underbrace{\sin\theta_i}_{0} = \sqrt{\mu_2\epsilon_2}\,\sin\theta_t \Rightarrow \theta_t = 0$$

$\therefore \boxed{\theta_i = \theta_r = \theta_t = 0}$ from Snell's Laws.

$$\Rightarrow \cos\theta_i = \cos\theta_t = 1$$

$$\Rightarrow \Gamma_\perp = \frac{E_2}{E_1} = \left.\frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}\right|_{\substack{\cos\theta_i = 1 \\ \cos\theta_t = 1}} \Rightarrow \boxed{\Gamma_\perp = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

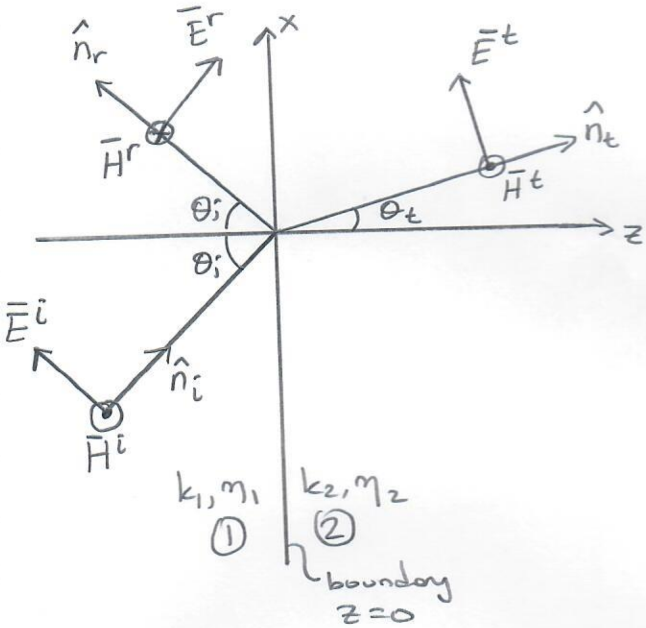
as obtained earlier.

$$T_\perp = \frac{E_2}{E_1} = \left.\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}\right|_{\substack{\cos\theta_i = 1 \\ \cos\theta_t = 1}} \Rightarrow \boxed{T_\perp = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

$$\boxed{T_\perp = 1 + \Gamma_\perp}$$

PARALLEL POLARIZATION CASE

($\bar{E}^i$ vector lies in the plane of incidence)



In this case (i.e., if both propagation vector and $\bar{E}$ phasors lie on the plane of incidence (x-z) plane), the $\bar{H}$ phasors must be perpendicular to the P.O.I.

Let

$$\bar{H}^i = \hat{a}_y H_1 e^{-j\bar{k}_i \cdot \bar{r}}$$

$$\bar{H}^r = -\hat{a}_y H_2 e^{-j\bar{k}_r \cdot \bar{r}}$$

$$\bar{H}^t = \hat{a}_y H_3 e^{-j\bar{k}_t \cdot \bar{r}}$$

Using $\bar{E} = \eta \bar{H} \times \hat{n}$

$$\bar{E}^i = \underbrace{\eta_1 H_1}_{E_1} (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)\, e^{-j\bar{k}_i \cdot \bar{r}}$$

$$\bar{E}^r = \underbrace{\eta_1 H_2}_{E_2} (\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i)\, e^{-j\bar{k}_r \cdot \bar{r}}$$

$$\bar{E}^t = \underbrace{\eta_2 H_3}_{E_3} (\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t)\, e^{-j\bar{k}_t \cdot \bar{r}}$$

Apply B.C. s at $z=0$ plane

B.C. 1 (continuity of $\bar{H}_{\substack{total \\ tang}}$) $\Rightarrow H_1 - H_2 = H_3 \Rightarrow \boxed{\frac{1}{\eta_1}(E_1 - E_2) = \frac{1}{\eta_2} E_3}$

B.C. 2 (continuity of $\bar{E}_{\substack{total \\ tang}}$) $\Rightarrow \boxed{(E_1 + E_2)\cos\theta_i = E_3 \cos\theta_t}$

(i.e. use the x-components)

Solving these last two equations simultaneously,

FRESNEL Formulas for Parallel (//) Polarization

$$\boxed{\Gamma_{//} = \frac{E_2}{E_1} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad T_{//} = \frac{E_3}{E_1} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}}$$

Note that

$$\boxed{1 + \Gamma_{//} = T_{//}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)}$$

in (//) case.

Special Case 1: Let medium (2) be a perfect conductor

i.e., $\eta_2 = 0$

$\Rightarrow \boxed{\Gamma_{//} = -1}$ and $\boxed{T_{//} = 0}$ as in the case of ($\perp$) polarization

Special Case 2: For $\epsilon_1\mu_1 > \epsilon_2\mu_2$ and $\theta_i > \theta_c$ case,

it can be shown that both $\Gamma_{//}$ and $T_{//}$ are complex.

In particular, $\Gamma_{//} = e^{j\phi_{//}}$ i.e $|\Gamma_{//}| = 1$ and $\phi_{//} \neq \phi_{\perp}$, in general.

Special Case 3: Normal incidence case, i.e, $\theta_i = 0$

From Snell's Laws $\Rightarrow$ $\theta_i = \theta_r = \theta_t = 0$ $\Rightarrow$ $\begin{cases} \cos\theta_i = 1 \\ \cos\theta_t = 1 \end{cases}$

From Fresnel formulas with $\cos\theta_i = \cos\theta_t = 1$ $\Rightarrow$ $\boxed{\Gamma_{//} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$ and $\boxed{T_{//} = \frac{2\eta_2}{\eta_2 + \eta_1}}$

as in the ($\perp$) polarization case.

Conclusion: For normal incidence case, use of ($\perp$) or (//) polarizations does not make any difference, same expressions for Γ and T apply in both polarizations.

BREWSTER ANGLE (θ_B)

It is the value of incidence angle for which $\Gamma = \frac{E_2}{E_1} = 0$ (i.e. reflected wave is zero).

Case 1: Perpendicular Polarization

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

at $\theta_i = \theta_{B_\perp} \Rightarrow \Gamma_\perp = 0$

$$\Rightarrow \boxed{\eta_2 \cos\theta_{B_\perp} - \eta_1 \cos\theta_t = 0}$$

using

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$$

$$\sin\theta_t = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}} \sin\theta_i \Big|_{\theta_i = \theta_{B_\perp}} \quad \text{(Snell's Law)}$$

$$\cos\theta_t = \sqrt{1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} \sin^2\theta_{B_\perp}}$$

gives

$$\boxed{\sin^2\theta_{B_\perp} = \frac{1 - \left(\mu_1\epsilon_2 / \mu_2\epsilon_1\right)}{1 - \left(\mu_1/\mu_2\right)^2}}$$

For $\mu_1 = \mu_2$ case, $\sin\theta_{B_\perp} \to \infty$

$\therefore$ $\theta_{B_\perp}$ does <u>not</u> exist

For non-magnetic media with $\mu_1 = \mu_2 = \mu_0$, for example, a Brewster angle $\theta_{B_\perp}$ can <u>not</u> be found to make reflections zero!

Case 2: Parallel Polarization

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

at $\theta_i = \theta_{B_\parallel} \Rightarrow \Gamma_\parallel = 0$

$$\boxed{\eta_2 \cos\theta_t - \eta_1 \cos\theta_{B_\parallel} = 0}$$

Using Snell's Law of refraction, it can be shown that

$$\boxed{\sin^2\theta_{B_\parallel} = \frac{1 - \left(\mu_2\epsilon_1 / \mu_1\epsilon_2\right)}{1 - \left(\epsilon_1/\epsilon_2\right)^2}}$$

$\therefore$ $\theta_{B_\parallel}$ exists unless $\epsilon_1 = \epsilon_2$

For $\mu_1 = \mu_2$ case,

$$\boxed{\theta_{B_\parallel} = \sin^{-1}\left(\frac{1}{\sqrt{1 + \frac{\epsilon_1}{\epsilon_2}}}\right) = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}}$$

Note: When the incident u.p.w has both perpendicular and parallel polarization components, setting the incidence angle as $\theta_i = \theta_{B//}$ (assume $\epsilon_1 \neq \epsilon_2$), only the ($\perp$) component will be reflected back as the reflection due to the (//) component is nullified. Then, the Brewster angle is also called the "Polarizing Angle" due to this polarization selectivity.

Note: Consider the (//) polarization case:
Let $\mu_1 = \mu_2 = \mu$ and $\theta_i = \theta_{B//}$

Snell's Law: $k_1 \sin\theta_{B//} = k_2 \sin\theta_t$

Brewster condition: $\eta_1 \cos\theta_{B//} = \eta_2 \cos\theta_t$

insert expressions for k_1, k_2, η_1, η_2 simplify:

$$\Rightarrow \sqrt{\epsilon_1}\,\sin\theta_{B//} = \sqrt{\epsilon_2}\,\sin\theta_t$$

$$\frac{1}{\sqrt{\epsilon_1}}\cos\theta_{B//} = \frac{1}{\sqrt{\epsilon_2}}\cos\theta_t$$

multiply side by side:

$\times$

$$\sin\theta_{B//}\cos\theta_{B//} = \sin\theta_t \cos\theta_t$$

$$\Rightarrow \sin 2\theta_{B//} = \sin 2\theta_t \quad \not\Rightarrow \quad \theta_{B//} = \theta_t$$

as this would violate the Snell's Law with $\epsilon_1 \neq \epsilon_2$.

Instead, $\boxed{2\theta_{B//} = \pi - 2\theta_t}$

or $\boxed{\theta_{B//} = \frac{\pi}{2} - \theta_t}$ is valid when $\mu_1 = \mu_2$ and $\theta_i = \theta_{B//}$