Example: Let $\bar{E} = 3(j\hat{a}_x + A\hat{a}_y + \sqrt{2}\,\hat{a}_z)\, e^{-j60\pi(x-y)}$ be a <u>phasor</u> $\bar{E}$-field expression for a u.p.w. propagating in a lossless simple medium with parameters (ϵ, μ). Let the wave be oscillating at frequency $f = 1$ GHz.

a) Find $\hat{n}$: unit vector in the direction of propagation.

$$\bar{E} = \underbrace{3(j\hat{a}_x + A\hat{a}_y + \sqrt{2}\,\hat{a}_z)}_{\substack{\bar{E}_0 \text{ (complex valued here)} \\ \text{constant vector}}}\; e^{-j\overbrace{60\pi(x-y)}^{\bar{k}\cdot\bar{r}}}$$

as for a u.p.w, we should have $\bar{E} = \bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}$

(Note that we must have a <u>complex exponential</u> for propagation!)

In general, $\bar{k}\cdot\bar{r} = (k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z)\cdot(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)$

$\Rightarrow \bar{k}\cdot\bar{r} = k_x x + k_y y + k_z z$

Here, $\bar{k}\cdot\bar{r} = 60\pi x - 60\pi y$

Equate coefficients of x, y and z terms

$$\Rightarrow \left.\begin{aligned} k_x &= 60\pi \\ k_y &= -60\pi \\ k_z &= 0 \end{aligned}\right\} \quad \bar{k} = 60\pi\,\hat{a}_x - 60\pi\,\hat{a}_y + 0.\hat{a}_z$$

$$\boxed{\bar{k} = 60\pi\,(\hat{a}_x - \hat{a}_y)\ \text{rad/m.}}$$

propagation vector

We know $\bar{k} = k\hat{n}$ $\Rightarrow$ $k = |\bar{k}| = \sqrt{(60\pi)^2 + (60\pi)^2} = \sqrt{2}\,60\pi$ (rad/m)

and $\hat{n} = \dfrac{\bar{k}}{|\bar{k}|}$

$$\hat{n} = \frac{\cancel{60\pi}(\hat{a}_x - \hat{a}_y)}{\sqrt{2}\,\cancel{60\pi}} \Longrightarrow \boxed{\hat{n} = \frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)}$$

unit vector in the propagation direction – note that $|\hat{n}| = 1$ indeed.

b) Determine the constant A.

We know $\bar{E} \perp \hat{n}$, $\bar{H} \perp \hat{n}$ and $\bar{E} \perp \bar{H}$ for a u.p.w.

Using $\bar{E} \perp \hat{n} \Longrightarrow \bar{E}_0 \cdot \hat{n} = 0$

$$\Longrightarrow 3(j\hat{a}_x + A\hat{a}_y + \sqrt{2}\,\hat{a}_z) \cdot \left(\frac{1}{\sqrt{2}}\hat{a}_x - \frac{1}{\sqrt{2}}\hat{a}_y\right) = 0$$

$$\Longrightarrow \frac{3}{\sqrt{2}}j - \frac{3A}{\sqrt{2}} = 0 \Longrightarrow j - A = 0 \Longrightarrow \boxed{A = j}$$

c) Write down the time-domain expression $\bar{\mathcal{E}}(\bar{r},t)$.

$$\bar{\mathcal{E}}(\bar{r},t) = Re\left\{\bar{E}(\bar{r})\, e^{j\omega t}\right\}$$

$$= Re\left\{3(j\hat{a}_x + j\hat{a}_y + \sqrt{2}\,\hat{a}_z)\, \underbrace{e^{-j60\pi(x-y)}\, e^{j\omega t}}_{e^{j(\omega t - 60\pi(x-y))}}\right\}$$

$$= Re\left\{\left[\overbrace{3\sqrt{2}\,\hat{a}_z}^{\text{real part}} + j\overbrace{3(\hat{a}_x + \hat{a}_y)}^{\text{imag. part}}\right]\left[\overbrace{\cos(\omega t - 60\pi(x-y))}^{\text{real part}} + j\overbrace{\sin(\omega t - 60\pi(x-y))}^{\text{imag. part}}\right]\right.$$

$$\boxed{\bar{\mathcal{E}}(\bar{r},t) = 3\sqrt{2}\cos[\omega t - 60\pi(x-y)]\,\hat{a}_z - 3(\hat{a}_x + \hat{a}_y)\sin[\omega t - 60\pi(x-y)] \quad (V/m)}$$

Note that this expression does not contain any "j" in it after the "Real part" operation!

d) Determine the propagation velocity (phase velocity) v.

We know $k = \omega\sqrt{\epsilon\mu} = \dfrac{\omega}{v}$ $\left(\text{as } v = \dfrac{1}{\sqrt{\epsilon\mu}}\right)$

$$\Rightarrow \underline{\underline{v}} = \frac{\omega}{k} = \frac{2\pi f}{k} = \frac{2\pi \times 1 \times 10^9}{\sqrt{2}\, 60\,\pi} = \underline{\underline{\frac{\sqrt{2}}{6} \times 10^8}} \left(\text{m/sec}\right)$$

$(f = 1 \text{ GHz} = 10^9 \text{ Hz})$

e) Determine the wavelength λ.

We know $\lambda = \dfrac{2\pi}{k}$ or $\lambda = \dfrac{v}{f}$ } use either formula to get λ.

for ex: $\lambda = \dfrac{v}{f} = \dfrac{\sqrt{2}}{6} \times 10^8 \dfrac{1}{10^9}$

$$\boxed{\lambda = \frac{\sqrt{2}}{60} \text{ (m.)}}$$

f) If $\mu_r = 1$ is given, determine ϵ_r.

We know $v = \dfrac{1}{\sqrt{\epsilon\mu}} = \dfrac{1}{\sqrt{\epsilon_0\epsilon_r\mu_0\mu_r}} = \dfrac{1}{\sqrt{\epsilon_r\mu_r}}\dfrac{1}{\sqrt{\epsilon_0\mu_0}} = \dfrac{c}{\sqrt{\epsilon_r\mu_r}}$

$$\frac{\sqrt{2}}{6} 10^8 = \frac{3 \times 10^8}{\sqrt{\epsilon_r \cdot 1}} \Rightarrow \boxed{\sqrt{\epsilon_r} = \frac{18}{\sqrt{2}}} \text{ or } \boxed{\epsilon_r \simeq 162}$$

g) Find the phasor $\bar{H}(\bar{r})$.

Use either $\bar{H}(\bar{r}) = \dfrac{1}{-j\omega\mu} \bar{\nabla} \times \bar{E}(r)$ (Maxwell's eqn. in phasor domain)

or $\bar{H} = \dfrac{\hat{n} \times \bar{E}}{\eta}$ (as the wave is a u.p.w)

where $\eta = \sqrt{\dfrac{\mu}{\epsilon}} = \sqrt{\dfrac{\mu_0}{\epsilon_r\epsilon_0}} = \dfrac{\eta_0}{\sqrt{\epsilon_r}} = \dfrac{120\pi}{\sqrt{162}} \Rightarrow \boxed{\eta = \dfrac{120\pi}{9\sqrt{2}} \ (\Omega)}$

intrinsic impedance of the medium.

Then, compute $\bar{H}$ phasor.

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$$\bar{H} = \frac{\hat{n}\times\bar{E}}{\eta} = \frac{1}{\eta}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ n_x & n_y & n_z \\ E_x & E_y & E_z \end{vmatrix} = \frac{9\sqrt{2}}{120\pi}\, e^{-j60\pi(x-y)} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 3j & 3j & 3\sqrt{2} \end{vmatrix}$$

Can be computed.

h) Determine the <u>type</u> and <u>sense</u> of polarization.

from part (c):

$$\bar{E}(\bar{r},t) = 3\sqrt{2}\cos[\omega t - 60\pi(x+y)]\,\hat{a}_z - 3(\hat{a}_x+\hat{a}_y)\sin[\omega t - 60\pi(x+y)]$$

let $x=y=z=0$ for example, i.e fix space at origin.

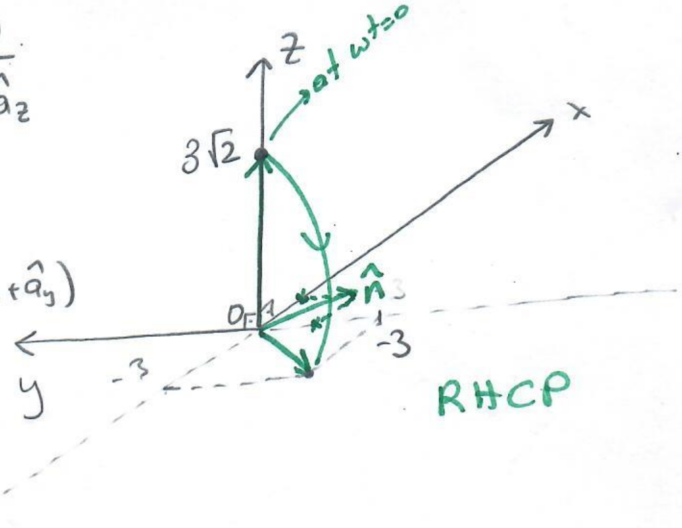
$$\bar{E}(0,t) = 3\sqrt{2}\cos\omega t\;\hat{a}_z - 3(\hat{a}_x+\hat{a}_y)\sin\omega t$$

$$\Rightarrow \left.\begin{aligned} E_x &= E_y = -3\sin\omega t \\ E_z &= 3\sqrt{2}\cos\omega t \end{aligned}\right\} \quad |\bar{E}(t)|^2 = E_x^2 + E_y^2 + E_z^2 = \underbrace{9\sin^2\omega t + 9\sin^2\omega t}_{18\sin^2\omega t} + 18\cos^2\omega t$$

$$\boxed{|\bar{E}(t)| = \sqrt{18} = 3\sqrt{2} \quad \text{for all } t}$$

i.e. the u.p.w. is circularly polarized.

	ωt	E_x	E_y	E_z	$\bar{E}(t)$
0	0	0	0	$3\sqrt{2}$	$\rightarrow 3\sqrt{2}\,\hat{a}_z$
$\frac{T}{8}$	$\frac{\pi}{4}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	3	
$\frac{T}{4}$	$\frac{\pi}{2}$	-3	-3	0	$\rightarrow -3(\hat{a}_x+\hat{a}_y)$



let $t_1 = 0$, $t_2 = \frac{T}{4}$ (Choose $t_2 > t_1$ and both $t_1 < T$, $t_2 < T$)

Check $\bar{E}(t_1)\times\bar{E}(t_2) = 3\sqrt{2}\,\hat{a}_z \times [-3(\hat{a}_x+\hat{a}_y)] = -9\sqrt{2}(\hat{a}_y - \hat{a}_x) = 9\sqrt{2}(\hat{a}_x - \hat{a}_y)$

Remember, $\hat{n} = \frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)$ → in the same direction ⇒ <u>RH</u> CP