FLOW of ELECTROMAGNETIC POWER

Electromagnetic power is transported between two points in space by electromagnetic waves as follows:

Derivation of POYNTING's Theorem in Time Domain:

Consider a travelling EM wave with fields $\bar{E}(\bar{r},t)$ and $\bar{H}(\bar{r},t)$ which satisfy Maxwell's equations in a linear medium with parameters ϵ and μ.

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \bar{D} = \epsilon \bar{E}$$

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \qquad \bar{B} = \mu \bar{H}$$

Consider the vector identity:

$$\bar{\nabla} \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot \underbrace{(\bar{\nabla} \times \bar{E})}_{-\frac{\partial \bar{B}}{\partial t}} - \bar{E} \cdot \underbrace{(\bar{\nabla} \times \bar{H})}_{\bar{J} + \frac{\partial \bar{D}}{\partial t}}$$

$$= -\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} - \epsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} - \bar{E} \cdot \bar{J}$$

$$\left(\text{Using: } \frac{\partial}{\partial t}(\bar{A} \cdot \bar{B}) = \bar{A} \cdot \frac{\partial \bar{B}}{\partial t} + \bar{B} \cdot \frac{\partial \bar{A}}{\partial t} \quad \text{For } \bar{A} = \bar{B} \Rightarrow \frac{\partial}{\partial t}(\bar{A} \cdot \bar{A}) = 2 \bar{A} \cdot \frac{\partial \bar{A}}{\partial t} \right)$$

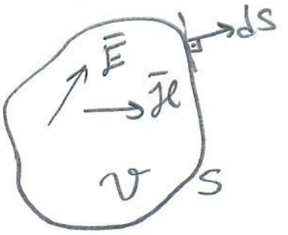
$$\bar{\nabla} \cdot (\bar{E} \times \bar{H}) = -\frac{\mu}{2} \frac{\partial}{\partial t} \underbrace{(\bar{H} \cdot \bar{H})}_{|\bar{H}|^2 = H^2} - \frac{\epsilon}{2} \frac{\partial}{\partial t} \underbrace{(\bar{E} \cdot \bar{E})}_{|\bar{E}|^2 = E^2} - \bar{E} \cdot \bar{J}$$

$$-\bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}}) = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu\mathcal{H}^2 + \frac{1}{2}\epsilon\mathcal{E}^2\right] + \bar{\mathcal{E}}\cdot\bar{\mathcal{J}} \qquad (*)$$

Define $\begin{cases} \omega_M \triangleq \frac{1}{2}\mu\mathcal{H}^2 & \text{: magnetic energy density (joule/m}^3\text{)} \\ \omega_E \triangleq \frac{1}{2}\epsilon\mathcal{E}^2 & \text{: electric energy density (joule/m}^3\text{)} \end{cases}$

$$-\bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}}) = \frac{\partial}{\partial t}(\omega_M + \omega_E) + \bar{\mathcal{E}}\cdot\bar{\mathcal{J}} \qquad (**)$$

Integrate both sides of $(*)$ or $(**)$ over a volume v



$$\underbrace{-\int_v \bar{\nabla}\cdot(\bar{\mathcal{E}}\times\bar{\mathcal{H}})\,dv}_{\oint_S (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{S} \text{ using the Stoke's Thm.}} = \int_v \left[\frac{\partial}{\partial t}(\omega_M + \omega_E) + \bar{\mathcal{E}}\cdot\bar{\mathcal{J}}\right] dv$$

$$\Rightarrow \quad -\oint_S (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{S} = \frac{\partial}{\partial t}\underbrace{\int_v \frac{1}{2}\mu\mathcal{H}^2\,dv}_{\substack{W_m \\ \text{magnetic} \\ \text{energy term}}} + \frac{\partial}{\partial t}\underbrace{\int_v \frac{1}{2}\epsilon\mathcal{E}^2\,dv}_{\substack{W_E \\ \text{electric} \\ \text{energy term}}} + \int_v \bar{\mathcal{E}}\cdot\bar{\mathcal{J}}\,dv$$

$$\boxed{-\oint_S (\bar{\mathcal{E}}\times\bar{\mathcal{H}})\cdot d\bar{S} = \underbrace{\frac{\partial}{\partial t}(W_m + W_E)}_{\substack{\text{rate of change} \\ \text{of total EM energy} \\ \text{stored in volume } v \\ \text{(i.e. a power term)}}} + \int_v \bar{\mathcal{E}}\cdot\bar{\mathcal{J}}\,dv} \quad \text{(Watts)}$$

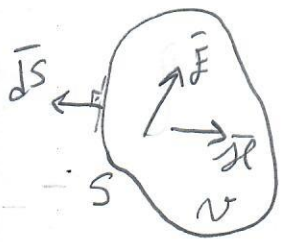
POYNTING'S THEOREM

Starting from Maxwell Equations in time domain, we can derive the following equation:

$$\boxed{-\oint_S (\bar{E}\times\bar{\mathcal{H}})\cdot d\bar{S} = \frac{\partial}{\partial t}\int_v \frac{\mu}{2}|\bar{\mathcal{H}}|^2 dv + \frac{\partial}{\partial t}\int_v \frac{\epsilon}{2}|\bar{E}|^2 dv + \int_v \bar{E}\cdot\bar{J}\,dv} \quad (1)$$

This equation is the mathematical statement of the conservation of electromagnetic power.

(*) Each term in this equation has the units of power (Watt)



$\oint_S (\bar{E}\times\bar{\mathcal{H}})\cdot d\bar{S}$: Electromagnetic power leaving the volume v through its closed surface S, in Watts.

Define: $\bar{\mathcal{P}} = \bar{E}\times\bar{\mathcal{H}}$ (Watts/m^2) : Power flow per unit area in the direction specified by the cross product $\bar{E}\times\bar{\mathcal{H}}$

$\bar{E} \rightarrow$ V/m, $\bar{\mathcal{H}} \rightarrow$ A/m

$\bar{\mathcal{P}} \rightarrow$ Instantaneous Poynting's Vector (Power flow density vector)

(*) Note that $\bar{\mathcal{P}} = \bar{\mathcal{P}}(\bar{r},t)$ is defined as a time-domain vector.

(*) Also note that for a plane wave $\bar{E}\perp\bar{\mathcal{H}}$, $\bar{E}\perp\hat{n}$, $\bar{\mathcal{H}}\perp\hat{n}$

$\Rightarrow \bar{E}\times\bar{\mathcal{H}} = \bar{\mathcal{P}} \,//\, \hat{n} \longrightarrow$ Power flow is in the direction of propagation. ($\bar{\mathcal{P}}\perp\bar{E}$ and $\bar{\mathcal{P}}\perp\bar{\mathcal{H}}$)

In addition to the "Instantaneous Poynting's Vector", we also define the "Complex Poynting's Vector" and the "Time-Averaged Poynting's Vector" as follows:

Let:

$\bar{\mathcal{E}}(\bar{r},t), \bar{\mathcal{H}}(\bar{r},t)$: time-domain monochromatic fields

$\bar{E}(\bar{r}), \bar{H}(\bar{r})$: corresponding phasor fields

$$\bar{\mathcal{P}}(\bar{r},t) \triangleq \bar{\mathcal{E}}(\bar{r},t) \times \bar{\mathcal{H}}(\bar{r},t)$$: Instantaneous Poynting's Vector

complex conjugate

$$\bar{P}(\bar{r}) \triangleq \bar{E}(\bar{r}) \times \bar{H}^*(\bar{r})$$: Complex Poynting's Vector

Then,

$$\boxed{\bar{P}_{av}(\bar{r}) \triangleq \frac{1}{T} \int_{t=0}^{t=T} \bar{\mathcal{P}}(\bar{r},t)\, dt}$$

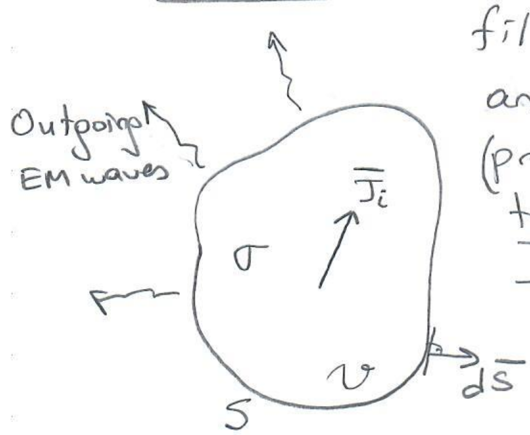
where $T = \frac{1}{f} = \frac{2\pi}{\omega}$ ↓ period of the monochromatic wave

or

$$\boxed{\bar{P}_{av}(\bar{r}) \triangleq \frac{1}{2} Re\{\bar{P}\} = \frac{1}{2} Re\{\bar{E} \times \bar{H}^*\}} \quad (W/m^2)$$

where $\bar{P}_{av}$ is the Time-Averaged Poynting's Vector

Problem: Assume that a region of volume v is filled with a lossy material of conductivity σ and contains an impressed current $\bar{J}_i$ (A/m^2) (produced by external sources such as a transmitting antenna). Apply the Poynting's Theorem to this "Transmitter Case".



Set $\bar{J} = \bar{J}_i + \bar{J}_c = \bar{J}_i + \sigma \bar{E}$ in eqn. (1)

$$-\oint_S \underbrace{(\bar{E} \times \bar{H})}_{\bar{P}} \cdot d\bar{S} = \underbrace{\frac{\partial}{\partial t} \int_v \underbrace{\frac{\mu}{2} H^2}_{w_M} dv}_{W_M:\text{ magnetic energy}} + \underbrace{\frac{\partial}{\partial t} \int_v \underbrace{\frac{\epsilon}{2} E^2}_{w_E} dv}_{W_E:\text{ electric energy}} + \int_v \bar{E} \cdot \underset{\bar{J}_i + \sigma \bar{E}}{\underset{\downarrow}{\bar{J}}}\, dv$$

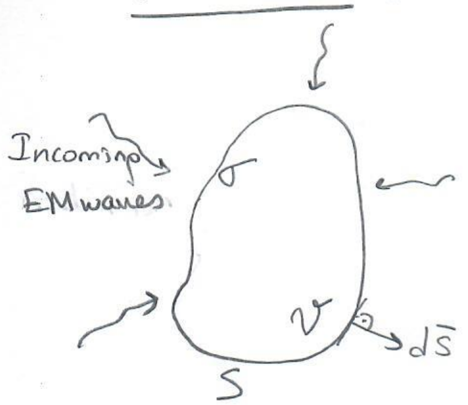
$$-\oint_S \bar{P} \cdot d\bar{S} = \frac{\partial}{\partial t} \underbrace{(W_M + W_E)}_{\substack{W_T \\ \text{Total EM energy}}} + \int_v \bar{E} \cdot \underset{\text{external source}}{\underset{\downarrow}{\bar{J}_i}}\, dv + \int_v \sigma \underbrace{\bar{E} \cdot \bar{E}}_{E^2 \triangleq |\bar{E}|^2} dv$$

Reorganize this equation as

$$\boxed{\underbrace{-\int_v \bar{E} \cdot \bar{J}_i \, dv}_{\substack{\text{Power supplied} \\ \text{by the sources} \\ \text{to volume } v \\ (\text{LHS} > 0)}} = \underbrace{\frac{\partial}{\partial t}(W_T)}_{\substack{\text{Rate of} \\ \text{change of} \\ \text{total EM} \\ \text{energy stored} \\ \text{in volume } v.}} + \underbrace{\oint_S \bar{P} \cdot d\bar{S}}_{\substack{\text{Power} \\ \text{leaving the} \\ \text{volume } v \\ (\text{Radiated} \\ \text{power})}} + \underbrace{\int_v \sigma E^2 dv}_{\substack{\text{Ohmic} \\ \text{Loss due} \\ \text{to } \sigma \neq 0 \\ \text{in the} \\ \text{medium} \\ (\text{Joule} \\ \text{heating})}}}$$

Problem: Assume that a region of volume v is filled with a lossy material of conductivity σ and it is source-free ($\rho_v = 0$, $J_i = 0$).

Apply the Poynting's Theorem to this "Receiver Case".



Set $\bar{J} = \underbrace{\bar{J}_i}_{0} + \sigma\bar{E} = \sigma\bar{E}$ in Eqn.(1)

$$-\oint_S (\underbrace{\bar{E}\times\bar{H}}_{\bar{P}})\cdot d\bar{S} = \frac{\partial}{\partial t}(\underbrace{W_M + W_E}_{W_T}) + \int_v \bar{E}\cdot\underbrace{(\sigma\bar{E})}_{\sigma E^2}\,dv$$

$$\boxed{\underbrace{-\oint_S \bar{P}\cdot d\bar{S}}_{\text{Total power carried into the volume } v \text{ by incoming EM waves}} = \underbrace{\frac{\partial}{\partial t}(W_T)}_{\text{Rate of change of total EM energy stored in volume } v} + \underbrace{\int_v \sigma E^2\,dv}_{\text{Ohmic Loss (due to } \sigma\neq 0\text{) in volume } v}}$$

(*) Note that if the medium is lossless ($\sigma = 0$), last term (ohmic loss) is zero. Then all the power carried into the volume by EM waves is stored in the volume.

(*) If $\sigma \neq 0$ in the static case (where $\frac{\partial}{\partial t} \equiv 0$ no time variation) eqn. above reduces to $-\oint_S \bar{P}\cdot d\bar{S} = \int_v \sigma E^2\,dv$ which means that all the power carried into v is dissipated in the volume due to Joule heating mechanism.

(**) Study solved example (7-5) in your textbook (page = 300-301)

Example: Assume a uniform plane wave propagating in a lossy medium for which

$$\gamma = \alpha + j\beta \quad \text{and} \quad \bar{E} = E_o\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_x$$

where E_o is a real constant.

(a) write down the phasor $\bar{H}$.

(b) Find the complex Poynting's vector $\bar{P}$.

(c) Find the time-averaged Poynting's vector $\bar{\mathcal{P}}_{av}$ by two methods.

Solution:

(a) The intrinsic impedance for this lossy medium is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta|\, e^{j\theta} \quad \text{and} \quad \eta = \frac{E_x}{H_y}$$

$$\Rightarrow \boxed{\bar{H} = \frac{E_o}{|\eta|\, e^{j\theta}}\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_y}$$ in the phasor domain.

(b) $\bar{P} = \bar{E} \times \bar{H}^*$ is the complex Poynting's vector where $\bar{E}$ and $\bar{H}$ are phasors.

complex conjugate

$$\bar{P} = (E_o\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_x) \times \left(\frac{E_o}{|\eta|\, e^{j\theta}}\, e^{-\alpha z}\, e^{-j\beta z}\, \hat{a}_y\right)^*$$

$$= (E_o\, e^{-\alpha z}\, \cancel{e^{-j\beta z}}\, \hat{a}_x) \times \left(\frac{E_o}{|\eta|\, e^{-j\theta}}\, e^{-\alpha z}\, \cancel{e^{+j\beta z}}\, \hat{a}_y\right)$$

$$\boxed{\bar{P} = \frac{E_o^2}{|\eta|}\, e^{-2\alpha z}\, e^{j\theta}\, \hat{a}_z \quad \text{watts}/m^2}$$

(c) $$\bar{\mathcal{P}}_{av} = \frac{1}{2} Re\{\bar{P}\} = \frac{1}{2} Re\left\{\frac{E_o^2}{|\eta|}\, e^{-2\alpha z}\, \underbrace{e^{j\theta}}_{\cos\theta + j\sin\theta}\, \hat{a}_z\right\}$$

$$\boxed{\bar{\mathcal{P}}_{av} = \frac{1}{2}\frac{E_o^2}{|\eta|}\, e^{-2\alpha z}\, \cos\theta\, \hat{a}_z \quad W/m^2}$$ is the time-averaged Poynting's vector.

$\bar{P}_{av}$ can also be computed from

$$\bar{P}_{av} = \frac{1}{T}\int_0^T \bar{\mathcal{P}}(\bar{r},t)\,dt \quad \text{where} \quad \underbrace{\bar{\mathcal{P}}(\bar{r},t)}_{\text{instantaneous Poynting's vector}} = \underbrace{\bar{\mathcal{E}}(\bar{r},t) \times \bar{\mathcal{H}}(\bar{r},t)}_{\text{Time-domain quantities}}$$

using
$$\begin{cases} \bar{\mathcal{E}}(\bar{r},t) = Re\{\bar{E}(\bar{r})\,e^{j\omega t}\} = E_0\, e^{-\alpha z}\cos(\omega t - \beta z)\,\hat{a}_x \\ \bar{\mathcal{H}}(\bar{r},t) = Re\{\bar{H}(\bar{r})\,e^{j\omega t}\} = \frac{E_0}{|\eta|}\, e^{-\alpha z}\cos(\omega t - \beta z - \theta)\,\hat{a}_y \end{cases}$$

$$\Rightarrow \bar{\mathcal{P}}(\bar{r},t) = \frac{E_0^2}{|\eta|}\, e^{-2\alpha z} \cdot \underbrace{\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta)}_{\frac{1}{2}[\cos(2\omega t - 2\beta z - \theta) + \cos(\theta)]}\,\hat{a}_z$$

(using $\cos(A+B)$ and $\cos(A-B)$ expansions)

$$\Rightarrow \bar{P}_{av}(\bar{r}) = \frac{1}{T}\int_0^T \frac{E_0^2}{2|\eta|}\, e^{-2\alpha z}\left[\cos(2\omega t - 2\beta z - \theta) + \cos\theta\right]\hat{a}_z\, dt$$

$$= \frac{E_0^2}{2|\eta| T}\, e^{-2\alpha z}\,\hat{a}_z \left[\underbrace{\int_0^T \cos(2\omega t - 2\beta z - \theta)\, dt}_{0} + \underbrace{\int_0^T \cos\theta\, dt}_{\cos\theta \int_0^T dt = T\cos\theta}\right]$$

$$\Rightarrow \boxed{\bar{P}_{av}(\bar{r}) = \frac{1}{2}\frac{E_0^2}{|\eta|}\, e^{-2\alpha z}\cos\theta\;\hat{a}_z \quad W/m^2}$$ as found earlier.

<u>Special Case</u>: In a <u>lossless</u> medium, $\gamma = j\beta$ as $\alpha = 0$ and $\beta = k$. Also, $\eta = \sqrt{\frac{\mu}{\epsilon}} = |\eta|$ with $\theta = 0$

$$\Rightarrow \bar{P}_{av\text{-}lossless}(\bar{r}) = \frac{1}{2}\frac{E_0^2}{\eta}\,\hat{a}_z \quad W/m^2$$

Let $\bar{P}_{av} = 2\ W/m^2$ in free space where $\eta = \eta_0 \cong 377\ \Omega = 120\pi\ (\Omega)$

$\Rightarrow E_0 = \sqrt{2\eta_0 P_{av}} = \sqrt{2 \times 377 \times 2} \cong 38.8\ V/m.$ and $H_0 = \frac{E_0}{\eta} = \frac{38.8}{377} \cong 0.103\ A/m$

Example: Remember the problem which was largely solved earlier:

$$\bar{E}(\bar{r}) = 3\left(j\hat{a}_x + A\hat{a}_y + \sqrt{2}\,\hat{a}_z\right) e^{-j60\pi(x-y)} = \bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}$$

is the phasor E-field expression for a uniform plane wave propagating in a simple, lossless medium ($\sigma=0$, ϵ, μ) with $f = 1$ GHz.

a) Find $\hat{n}$! $\Longrightarrow$ $\bar{k}\cdot\bar{r} = 60\pi(x-y) \Rightarrow \bar{k} = 60\pi(\hat{a}_x - \hat{a}_y)$ rad/m

$$\Rightarrow \boxed{\hat{n} = \frac{\bar{k}}{|k|} = \frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)}$$

b) Determine A! $\Rightarrow$ Using $\bar{E} \perp \hat{n}$ for a u.p.w

$$\Rightarrow \bar{E}_0 \cdot \hat{n} = 0 \Rightarrow \boxed{A = j}$$

c) Find $\bar{\mathcal{E}}(\bar{r},t)$! $\quad \bar{\mathcal{E}}(\bar{r},t) = Re\{\bar{E}(\bar{r})e^{j\omega t}\}$

$$\Rightarrow \bar{\mathcal{E}}(\bar{r},t) = 3\sqrt{2}\cos(\omega t - 60\pi(x-y))\hat{a}_z - 3(\hat{a}_x + \hat{a}_y)\sin(\omega t - 60\pi(x-y)) \ (V/m)$$

d) Phase velocity $v = ?$ $\quad v = \frac{\omega}{k} = \frac{2\pi f}{k} = \frac{\sqrt{2}}{6} \times 10^8$ (m/s)

e) Wavelength $\lambda = ?$ $\quad \lambda = \frac{2\pi}{k} = \frac{v}{f} = \frac{\sqrt{2}}{60}$ m.

f) If $\mu_r = 1$, find ϵ_r! $\quad v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{\sqrt{\epsilon_r\mu_r}} = \frac{3\times10^8}{\sqrt{\epsilon_r}} = \frac{\sqrt{2}}{6}\times10^8$ m/s

$$\Rightarrow \sqrt{\epsilon_r} = \frac{18}{\sqrt{2}} = 9\sqrt{2} \Rightarrow \boxed{\epsilon_r \cong 162}$$

g) Find phasor $\bar{H}(r)$.

Use either $\bar{H}(\bar{r}) = \frac{1}{-j\omega\mu}\bar{\nabla}\times\bar{E}(\bar{r})$ (Maxwell's Eqn. – always useful)

or $\quad \bar{H}(\bar{r}) = \frac{1}{\eta}\hat{n}\times\bar{E}(r)$ where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ to get the answer.

$$\boxed{\eta = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{120\pi}{9\sqrt{2}} \ (\Omega)}$$

(h) Determine the type and sense of polarization for this u.p.w.

$$\bar{E}(\bar{r},t)\Big|_{x=y=z=0} = 3\sqrt{2}\cos\omega t\,\hat{a}_z - 3(\hat{a}_x+\hat{a}_y)\sin\omega t = \bar{E}(\bar{r}=0,t)$$

(at the origin, for example)

$$\Rightarrow \left.\begin{array}{l} E_x = -3\sin\omega t \\ E_y = -3\sin\omega t \\ E_z = 3\sqrt{2}\cos\omega t \end{array}\right\} |\bar{E}(0,t)| = \sqrt{E_x^2+E_y^2+E_z^2}$$

$$= \sqrt{\underbrace{9\sin^2\omega t + 9\sin^2\omega t}_{18\sin^2\omega t} + 18\cos^2\omega t}$$

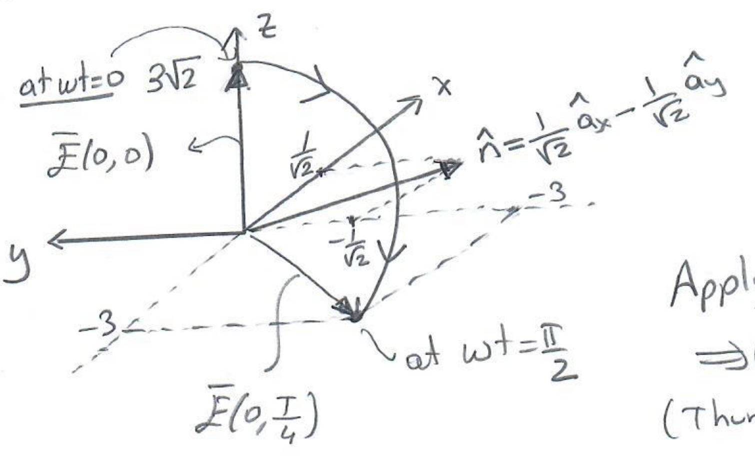
$$= \sqrt{18(\underbrace{\sin^2\omega t+\cos^2\omega t}_{1})} = \sqrt{18} = 3\sqrt{2}$$

As $|\bar{E}(\bar{r}=0,t)| = 3\sqrt{2}$ (constant vector length) for all time t, it is clear that we have a <u>circular polarization</u>.

To determine the <u>sense</u> of polarization, sketch the $\bar{E}(0,t)$ vector for a few time instants:

$\omega = 2\pi f = 2\pi/T$

t	ωt	$E_x=-3\sin\omega t$	$E_y=-3\sin\omega t$	$E_z=3\sqrt{2}\cos\omega t$
0	0	0	0	$3\sqrt{2}$
$T/8$	$\pi/4$	$-3\frac{1}{\sqrt{2}}$	$-3\frac{1}{\sqrt{2}}$	$3\sqrt{2}\frac{1}{\sqrt{2}}=3$
$T/4$	$\pi/2$	-3	-3	0



Apply Right Hand Rule
$\Rightarrow$ (RHCP) <u>Right Hand Circular Pol.</u>
(Thumb showes $\hat{n}$ direction)

i) Find the time-average Poynting's vector $\bar{P}_{av}$!

$$\bar{P}_{av} = \frac{1}{2} Re\{\bar{E} \times \bar{H}^*\} = \frac{1}{2} Re\left\{\bar{E}_0 \, e^{-j\bar{k}\cdot\bar{r}} \times \left(\frac{1}{\underbrace{\eta}_{\text{real here.}}} \hat{n} \times \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}\right)^*\right\}$$

$$= \frac{1}{2} Re\left\{\bar{E}_0 \, \cancel{e^{-j\bar{k}\cdot\bar{r}}} \times \left(\frac{1}{\eta} \hat{n} \times \bar{E}_0^* \, \cancel{e^{+j\bar{k}\cdot\bar{r}}}\right)\right\}$$

$$= \frac{1}{2\eta} Re\left\{\bar{E}_0 \times (\hat{n} \times \bar{E}_0^*)\right\}$$

Using $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$$\Rightarrow \quad \bar{E}_0 \times (\hat{n} \times \bar{E}_0^*) = \underbrace{(\bar{E}_0 \cdot \bar{E}_0^*)}_{|\bar{E}_0|^2} \hat{n} - \underbrace{(\bar{E}_0 \cdot \hat{n})}_{0 \text{ for plane waves}} \bar{E}_0^*$$

$$\Rightarrow \bar{P}_{av} = \frac{1}{2\eta} Re\left\{\underbrace{\hat{n}\,|\bar{E}_0|^2}_{\text{real}}\right\}$$

where $\hat{n} = \frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)$ is real !

$$\bar{E}_0 = 3\left(j\hat{a}_x + j\hat{a}_y + \sqrt{2}\,\hat{a}_z\right)$$

$$\Rightarrow |\bar{E}_0|^2 = \bar{E}_0 \cdot \bar{E}_0^* = 3(j\hat{a}_x + j\hat{a}_y + \sqrt{2}\,\hat{a}_z) \cdot 3(-j\hat{a}_x - j\hat{a}_y + \sqrt{2}\,\hat{a}_z)$$

$$= 9(1 + 1 + 2) = 36$$

$$\Rightarrow \boxed{\bar{P}_{av} = \frac{1}{2\eta} |\bar{E}_0|^2 \, \hat{n}}$$

in this lossless simple medium

Numerically,

$$\bar{P}_{av} = \frac{1}{2} \frac{36}{\frac{120\pi}{9\cancel{\sqrt{2}}}} \frac{1}{\cancel{\sqrt{2}}} (\hat{a}_x - \hat{a}_y)$$

$$\boxed{\bar{P}_{av} \cong 0.43\,(\hat{a}_x - \hat{a}_y) \quad (W/m^2)}$$

Example:

Due to various effects, human exposure to electromagnetic radiation is considered to be harmful above a certain time-average value of the Poynting's vector. This treshold value depends on frequency, and differs greatly among different countries in the world. A commonly accepted limit for power flux density is $\frac{f}{2000}$ (mW/cm^2) where f is the frequency of radiation in MHz. For example, at $f = 1\,GHz$, this treshold value is $\frac{1000\ MHz}{2000} = 0.5$ (mW/cm^2).

Assume that the following $|\bar{E}|$ field values are measured around certain GSM base stations. Assuming plane waves at the measurement points, so using $\boxed{\bar{P}_{av_{lossless}} = \frac{1}{2}\frac{|E_0|^2}{\eta_0}\hat{n}}$ with $\eta_0 \cong 377\ \Omega$ for air, determine for each measurement point whether the safety limits are exceeded or not.

GSM Base Station No.	$\lvert\bar{E}\rvert$ (V/m)	f (MHz)	Treshold $\frac{f}{2000}$ (mW/cm^2)	P_{av} - Computed mW/cm^2	Is $P_{av_{comp}} \leq$ Treshold
1	5	900	0.45	$\frac{1}{2}\frac{(5)^2}{377} \times \frac{10^3}{10^4} \cong 0.0033$	✓
2	10	900	0.45	~0.0133	✓
3	44	900	0.45	~0.256	✓
4	21	1800	0.90	~0.058	✓
5	50	1800	0.90	~0.332	✓

Although all of the P_{av} power flux density measurements are below the safety limits, measurement made for the base station #3 gives the relatively worse result.