Time Varying Electromagnetic Fields

Mathematical Review:

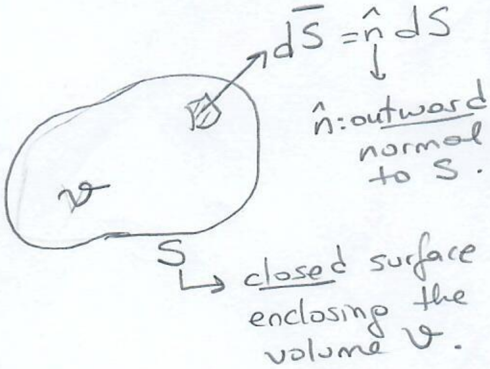
Helmholtz Theorem: A vector field is determined (up to an additive constant) if both its divergence and curl are specified everywhere.

$\bar{\nabla} \times \bar{A}$ represents the vector sources of the field $\bar{A}$.

$\bar{\nabla} \cdot \bar{A}$ " " scalar sources " " " $\bar{A}$.

Divergence Theorem:

$$\int_{v} (\bar{\nabla} \cdot \bar{A})\, dv = \oint_{S} \bar{A} \cdot d\bar{S}$$



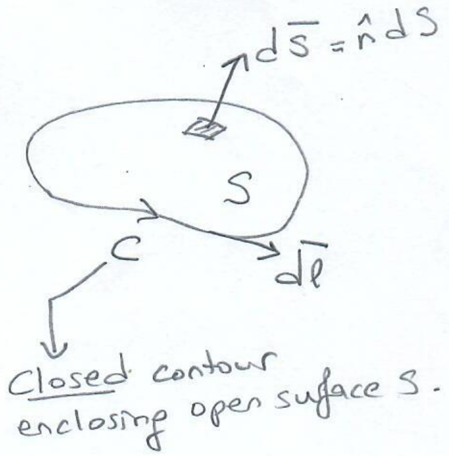
Stoke's Theorem:

$$\int_{S} (\bar{\nabla} \times \bar{A}) \cdot d\bar{S} = \oint_{C} \bar{A} \cdot d\bar{l}$$

$d\bar{l}$: tangent to contour C

$d\bar{S}$: normal to surface S

such that $d\bar{S}$ and $d\bar{l}$ are related by the RHR (Right Hand Rule).



Null Identities:

$$\begin{cases} \bar{\nabla} \times (\bar{\nabla} T) = 0 & T:\ \text{scalar field} \\ \bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) = 0 & \bar{A}:\ \text{vector field} \end{cases}$$

Mathematical Representations of the Static Fields

Differential (Point) Forms — Integral Forms

for Electro-static Fields

$\bar{\nabla} \times \bar{E} = 0$ ← Faraday's Law → $\oint_C \bar{E} \cdot d\bar{l} = 0$

$\bar{\nabla} \cdot \bar{D} = \rho_v$ ← Gauss' Law (Genaralized) → $\oint_S \bar{D} \cdot d\bar{S} = \int_v \rho_v \, dv = Q_{\substack{enclosed \\ free}}$

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for Magneto-static Fields

$\bar{\nabla} \times \bar{H} = \bar{J}$ ← Ampere's Law (Generalized) → $\oint_C \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{S} = I_{\substack{enclosed \\ free}}$

$\bar{\nabla} \cdot \bar{B} = 0$ ← Conservation of magnetic flux → $\oint_S \bar{B} \cdot d\bar{S} = 0$

↓ no magnetic charges exist in nature!

Note that the partial differential equations (pde's) governing the behavior of static electric and magnetic fields are uncoupled $\Longrightarrow$ static electric and magnetic fields can exist independently.

Remember, the "Del operator" can be defined in cartesian coordinate system as

$$\bar{\nabla} = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}$$

and,

$$\nabla^2 = \bar{\nabla} \cdot \bar{\nabla} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right)$$

$$\Rightarrow \boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

is the "Laplacian Operator" in cartesian coordinate system.

Maxwell's Equations

I) Divergence type pde's

(i) $\boxed{\bar{\nabla}\cdot\bar{\mathcal{D}} = \rho_v}$ and (ii) $\boxed{\bar{\nabla}\cdot\bar{\mathcal{B}} = 0}$

are still valid in time-varying case, where

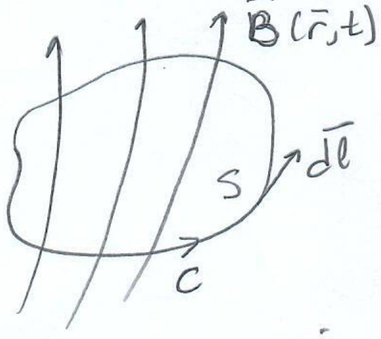
$$\left.\begin{aligned} \bar{\mathcal{D}} &= \bar{\mathcal{D}}(\bar{r},t) \\ \bar{\mathcal{B}} &= \bar{\mathcal{B}}(\bar{r},t) \\ \rho_v &= \rho_v(\bar{r},t) \end{aligned}\right\}$$ functions of space variables and time!

(Notation: Use "italic" letters to denote time-varying quantities!)

II) Curl type pde's need to be modified in time-varying case.

(i) Remember the Faraday's Law:

$$\underbrace{v_{ind}}_{\oint_C \bar{\mathcal{E}}\cdot d\bar{l}} = -\frac{d\Phi}{dt} \quad \text{where} \quad \Phi = \Phi(t) = \int_S \bar{\mathcal{B}}(\bar{r},t)\cdot d\bar{S}$$



$\bar{\mathcal{B}}(\bar{r},t)$: Time-varying magnetic flux density vector

$\Phi(t)$: Time-varying magnetic flux linked by the path C.

$$\underbrace{\oint_C \bar{E}\cdot d\bar{l}}_{\substack{\text{by Stoke's Thm.}\\ \int_S (\bar{\nabla}\times\bar{E})\cdot d\bar{S}}} = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

If the contour C and the surface S are fixed in time!

$$\Rightarrow \int_S (\bar{\nabla}\times\bar{E})\cdot d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

$$\Rightarrow \int_S \left(\bar{\nabla}\times\bar{E} + \frac{\partial \bar{B}}{\partial t}\right)\cdot d\bar{S} = 0$$

This equation holds for an arbitrary surface S!

$$\Rightarrow \bar{\nabla}\times\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$$

$$\Rightarrow \boxed{\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$

Faraday's Law (experimentally established!)

Note that $\frac{\partial}{\partial t} \equiv 0$ in static problems as static fields have no time variation. Therefore, in static case, $\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$ reduces to $\bar{\nabla}\times\bar{E} = 0$ as expected.

(ii) Do we need any modification in the other curl type equation $\bar{\nabla} \times \bar{\mathcal{H}} = \bar{J}$ in time-varying problems?

Maxwell analytically modified this equation using the "Conservation of Charge Principle" as follows:

We know $\boxed{\bar{\nabla} \cdot \bar{J} = -\dfrac{\partial \rho_v}{\partial t}}$ Continuity Eqn. (mathematical expression of conservation of charge principle)

$\bar{J} = \bar{J}(\bar{r}, t)$: Volume current density

$\rho_v = \rho_v(\bar{r}, t)$: Volume charge density

Now, compute the divergence of both sides of

$\bar{\nabla} \times \bar{\mathcal{H}} = \bar{J}$

$$\underbrace{\bar{\nabla} \cdot (\bar{\nabla} \times \bar{\mathcal{H}})}_{0} = \bar{\nabla} \cdot \bar{J} \Rightarrow \bar{\nabla} \cdot \bar{J} = 0$$

Contradiction in time-varying case!

by the null identity

$\Rightarrow$ Needs modification!

Let $\bar{G}$ be a non-zero vector field such that

$$\boxed{\bar{\nabla} \times \bar{\mathcal{H}} = \bar{J} + \bar{G}}$$

$$\underbrace{\bar{\nabla} \cdot (\bar{\nabla} \times \bar{\mathcal{H}})}_{0} = \bar{\nabla} \cdot (\bar{J} + \bar{G}) = \bar{\nabla} \cdot \bar{J} + \bar{\nabla} \cdot \bar{G} = 0$$

$$\Rightarrow \bar{\nabla} \cdot \bar{G} = - \underbrace{\bar{\nabla} \cdot \bar{J}}_{= \frac{\partial \rho_v}{\partial t}} \quad \text{(from Continuity Eqn.)}$$

$$\Rightarrow \bar{\nabla} \cdot \bar{G} = \frac{\partial \rho_v}{\partial t} \quad \text{where} \quad \rho_v = \bar{\nabla} \cdot \bar{D}$$

$$\Rightarrow \bar{\nabla} \cdot \bar{G} = \frac{\partial}{\partial t}(\bar{\nabla} \cdot \bar{D}) = \bar{\nabla} \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\Rightarrow \text{Choose } \boxed{\bar{G} = \frac{\partial \bar{D}}{\partial t}} \quad \left(\text{or, you may choose } \bar{G} = \frac{\partial \bar{D}}{\partial t} + \text{any vector with zero divergence} \right)$$

$$\Rightarrow \boxed{\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \quad \text{valid in time-varying problems!}$$

<u>Note</u> again that in static problems fields are time independent $\Rightarrow \frac{\partial \bar{D}}{\partial t} = 0 \Rightarrow \bar{\nabla} \times \bar{H} = \bar{J}$ as expected.

Displacement Current ($\bar{J}_d$)

$\frac{\partial \bar{D}}{\partial t}$: $\frac{\text{Coul}/m^2}{\text{sec}} = \text{Amp}/m^2$ has units of volume current density!

$$\Rightarrow \boxed{\bar{J}_d = \frac{\partial \bar{D}}{\partial t}} \quad (\text{Amp}/m^2)$$

Displacement current density due to time-varying displacement vector $\bar{D}(\bar{r}, t)$.

$$\Rightarrow \boxed{\bar{\nabla} \times \bar{\mathcal{H}} = \underbrace{\bar{J}}_{\text{conduction and/or convection current density}} + \underbrace{\bar{J}_d}_{\text{displacement current density}}}$$

where $\bar{J}_d = \dfrac{\partial \bar{\mathcal{D}}}{\partial t}$

Integral form of this equation can be obtained by integrating both sides of the above equation over a surface S (enclosed by a contour C)

$$\underbrace{\int_S (\bar{\nabla} \times \bar{\mathcal{H}}) \cdot d\bar{S}}_{\oint_C \bar{\mathcal{H}} \cdot d\bar{l}} = \underbrace{\int_S \bar{J} \cdot d\bar{S}}_{I} + \underbrace{\int_S \frac{\partial \bar{\mathcal{D}}}{\partial t} \cdot d\bar{S}}_{I_d}$$

I: conduction and/or convection current through surface S

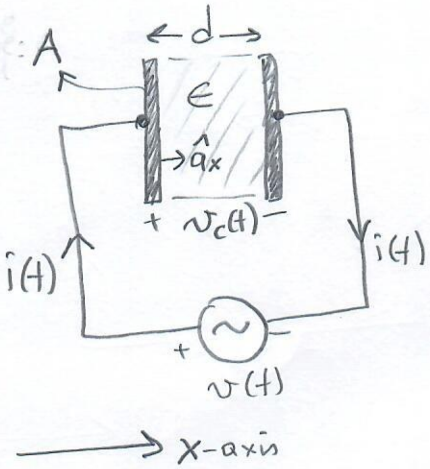
I_d: displacement current through surface S

$$\boxed{\oint_C \bar{\mathcal{H}} \cdot d\bar{l} = I + I_d}$$

Remember that

- $\bar{J} = \sigma \bar{E}$ — conduction current density (in metalic conductors); σ: conductivity
- $\bar{J} = \rho_v \bar{v}$ — convection current density due to moving charges; ρ_v: density of moving charges; $\bar{v}$: average velocity of moving charges

Example: (about the displacement current $i_d(t)$)



Consider a parallel plate capacitor filled by a lossless ($\sigma=0$) simple dielectric of permittivity ϵ. Neglect fringe fields, hence $\boxed{\bar{E} = \hat{a}_x \frac{v_c(t)}{d}}$ (a uniform electric field) exists within the capacitor. as $\boxed{v_c(t) = v(t)}$ as shown in the figure.

The capacitor current $\boxed{i_c(t) = i(t)}$ (flowing through metal wires) can be written as

$$i(t) = C\frac{dv(t)}{dt} \quad \text{where} \quad C = \epsilon\frac{A}{d} \text{ is the capacitance.}$$

$$\Rightarrow \boxed{i(t) = \epsilon\frac{A}{d}\frac{dv(t)}{dt}} \quad (*)$$

As the capacitor is filled by a lossless material, no conduction current can flow through it. But continuation of current at the input and output nodes of capacitor (as required by KCL) can be explained by the "Displacement Current" $i_d(t)$ as follows:

$$i_d(t) = \int_S \bar{J}_d \cdot d\bar{S} \quad \text{where} \quad d\bar{S} = \hat{a}_x\, dS \quad \text{and} \quad \bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon \underbrace{\bar{E}}_{\hat{a}_x \frac{v(t)}{d}})$$

$$\Rightarrow i_d(t) = \int_S \frac{\partial}{\partial t}\left(\epsilon\frac{v(t)}{d}\hat{a}_x\right)\cdot \hat{a}_x\, dS$$

$$= \frac{\epsilon}{d}\frac{dv(t)}{dt}\underbrace{\int_S dS}_{A} \quad \Rightarrow \boxed{i_d(t) = \epsilon\frac{A}{d}\frac{dv(t)}{dt}} \quad (**)$$

It is seen from (*) and (**) that

$\boxed{i(t) = i_d(t)}$ as expected by KCL (Kirchoff's Current Law)

Summary of MAXWELL'S Equations

Differential (point) Forms

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\bar{\nabla}\cdot\bar{D} = \rho_v$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

Integral Forms

$$\oint_C \bar{E}\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S} = -\frac{d\Phi}{dt}$$

$$\oint_C \bar{H}\cdot d\bar{l} = \underbrace{\int_S \bar{J}\cdot d\bar{S}}_{I_{free}} + \underbrace{\int_S \frac{\partial \bar{D}}{\partial t}\cdot d\bar{S}}_{I_d}$$

$$\oint_S \bar{D}\cdot d\bar{S} = \int_v \rho_v \, dv = Q_{\substack{enclosed \\ free}}$$

$$\oint_S \bar{B}\cdot d\bar{S} = 0$$

$$\boxed{\bar{\nabla}\cdot\bar{J} = -\frac{\partial \rho_v}{\partial t}}$$

Continuity equation (dependent on maxwell's equations)

Constitutive Relations

valid in general $\begin{cases} \bar{D} = \epsilon_0 \bar{E} + \bar{P} \\ \bar{B} = \mu_0 (\bar{H} + \bar{M}) \end{cases}$

$\left.\begin{array}{l} \bar{D} = \epsilon \bar{E} \\ \bar{B} = \mu \bar{H} \end{array}\right\}$ valid only in linear media

$\bar{J} = \sigma \bar{E}$ and/or $\bar{J} = \rho_v \vec{V}$

Notes: The set of maxwell's equations is composed of 4 linearly independent pde's. If the free sources $\bar{J}(\bar{r},t)$ and $\rho_v(\bar{r},t)$ are known, then the fields $\bar{E}(\bar{r},t)$, $\bar{D}(\bar{r},t)$, $\bar{H}(\bar{r},t)$ and $\bar{B}(\bar{r},t)$ can be solved using maxwell's equations and constitutive relations.