POLARIZATION of UNIFORM PLANE WAVES

Definition: Polarization of a u.p.w. describes the time-varying behavior of the $\bar{E}$ vector at a fixed point in space. In general, type of the polarization is specified by the locus (i.e. geometrical place) of the tip of the $\bar{E}(\bar{r},t)$ vector (for $\bar{r}$=constant) against time.

straight line locus $\longrightarrow$ linearly polarized u.p.w.
circular locus $\longrightarrow$ circularly polarized u.p.w.
elliptical locus $\longrightarrow$ elliptically polarized u.p.w

Consider a u.p.w propagating in $\hat{n}=\hat{a}_z$ direction in a simple, lossless, source-free medium with parameters (ϵ,μ):

$$\bar{k}=k\hat{n}=\omega\sqrt{\epsilon\mu}\,\hat{a}_z \quad \Rightarrow \quad \bar{k}\cdot\bar{r}=k\hat{a}_z\cdot(x\hat{a}_x+y\hat{a}_y+z\hat{a}_z)=kz$$

$$\Rightarrow \quad \bar{E}=\bar{E}_0\, e^{-jkz} \quad \text{where} \quad \bar{E}_0=\hat{a}_x \underbrace{E_{0x}+\hat{a}_y E_{0y}}_{\text{complex constants in general}}$$

($\bar{E}$-field phasor)

let $\boxed{\dfrac{E_{0y}}{E_{0x}}=m e^{j\psi}}$

$$\Rightarrow \quad \bar{E}=(E_{0x}\hat{a}_x+E_{0y}\hat{a}_y)\,e^{-jkz}=E_{0x}\left(\hat{a}_x+\frac{E_{0y}}{E_{0x}}\hat{a}_y\right)e^{-jkz}$$

$$\boxed{\bar{E}=E_{0x}\left(\hat{a}_x+m e^{j\psi}\hat{a}_y\right)e^{-jkz}}$$

Assume that E_{0x} is real without any loss of generality.

$$\bar{\mathcal{E}}(z,t) = Re\left\{\bar{E}(z)\, e^{j\omega t}\right\}$$

$$= Re\left\{E_{ox}\left(\hat{a}_x + m e^{j\psi}\hat{a}_y\right) e^{-jkz}\, e^{j\omega t}\right\}$$

$$= Re\left\{\hat{a}_x E_{ox}\, e^{j(\omega t - kz)} + \hat{a}_y E_{ox} m\, e^{j(\omega t - kz + \psi)}\right\}$$

$$\boxed{\bar{\mathcal{E}}(z,t) = \hat{a}_x E_{ox} \cos(\omega t - kz) + \hat{a}_y E_{ox} m \cos(\omega t - kz + \psi)}$$

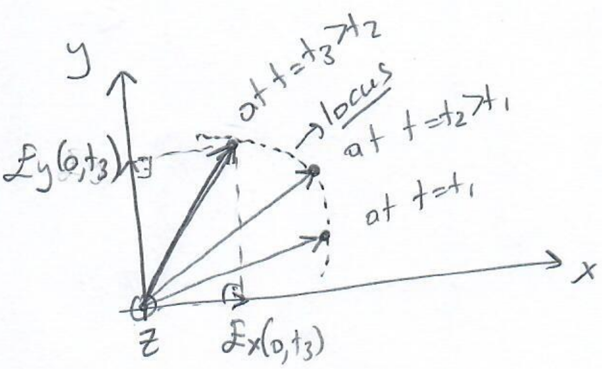
Let's now choose an arbitrary point in space to fix the space dependence. (Here z is the only space variable to consider)

Let, for example, $\boxed{z = 0}$ (The simplest choice!)

$$\Rightarrow \bar{\mathcal{E}}(z=0, t) = \hat{a}_x \underbrace{E_{ox} \cos \omega t}_{\mathcal{E}_x} + \hat{a}_y \underbrace{E_{ox} m \cos(\omega t + \psi)}_{\mathcal{E}_y}$$

Now, examine the locus of the tip of $\bar{\mathcal{E}}(0,t)$ vector as time (t) progresses!

$$\bar{\mathcal{E}}(0,t) = \hat{a}_x \mathcal{E}_x(0,t) + \hat{a}_y \mathcal{E}_y(0,t)$$



$$\mathcal{E}_x = E_{ox} \cos \omega t$$

$$\mathcal{E}_y = E_{ox}\, m \underbrace{\cos(\omega t + \psi)}_{\cos\omega t \cos\psi - \sin\omega t \sin\psi}$$

$$\mathcal{E}_y = \underbrace{E_{ox}}\, m \underbrace{\cos\omega t}_{\mathcal{E}_x} \cos\psi - E_{ox} m \sin\omega t \sin\psi$$

Pull out $(E_{ox} \sin \omega t)$ from this eqn.

$$(E_{0x} \sin \omega t)\, m \sin \psi = E_x m \cos \psi - E_y$$

$$\Rightarrow \quad E_{0x} \sin \omega t = \frac{m E_x \cos \psi - E_y}{m \sin \psi}$$

using $\Big(\underbrace{E_{0x} \cos \omega t}_{E_x}\Big)^2 + \Big(\underbrace{E_{0x} \sin \omega t}_{\frac{m E_x \cos \psi - E_y}{m \sin \psi}}\Big)^2 = E_{0x}^2 \underbrace{(\cos^2 \omega t + \sin^2 \omega t)}_{1} = E_{0x}^2$

$$E_x^2 + \frac{(m E_x \cos \psi - E_y)^2}{m^2 \sin^2 \psi} = E_{0x}^2$$

$$\underbrace{E_x^2 m^2 \sin^2 \psi + m^2 E_x^2 \cos^2 \psi}_{E_x^2 m^2 \underbrace{(\sin^2 \psi + \cos^2 \psi)}_{1}} + E_y^2 - 2 m E_x E_y \cos \psi = E_{0x}^2 m^2 \sin^2 \psi$$

$$\boxed{m^2 E_x^2 - 2 m \cos \psi\, E_x E_y + E_y^2 - E_{0x}^2 m^2 \sin^2 \psi = 0} \quad (*)$$

This is a quadratic form in E_x and E_y (components of $\bar{E}(z=0,t)$)

In general, a quadratic form in x and y is given as:

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

which represents an equation of an ellipse if $B^2 - 4AC < 0$

a parabola if $B^2 - 4AC = 0$

a hyperbola if $B^2 - 4AC > 0$

In our case: $(x \leftrightarrow E_x \ , \ y \leftrightarrow E_y)$

$$\left.\begin{array}{l} A = m^2 \\ B = -2m\cos\psi \\ C = 1 \end{array}\right\} \quad \begin{aligned} B^2 - 4AC &= (-2m\cos\psi)^2 - 4(m^2)(1) \\ &= 4m^2\cos^2\psi - 4m^2 \\ &= \underbrace{4m^2}_{>0}\underbrace{(\cos^2\psi - 1)}_{\leq 0} \leq 0 \end{aligned}$$

$\therefore$ $\boxed{B^2 - 4AC \leq 0 \Rightarrow \text{The locus can NOT be a hyperbola}}$

Can we have the "parabola" case?

For that we need to have $B^2 - 4AC = 0$

$$4m^2(\cos^2\psi - 1) = 0$$

$$\Rightarrow \cos^2\psi - 1 = 0 \Rightarrow \cos^2\psi = 1$$

$$\Rightarrow \cos\psi = \mp 1 \Rightarrow \begin{cases} \psi = 0 \\ \text{or } \psi = \pi \end{cases}$$

Remember, $E_x = E_{0x}\cos\omega t$

$$E_y = E_{0x}\, m\cos(\omega t + \psi) \begin{cases} E_{0x}\, m\cos(\omega t + 0) \\ \text{or} \\ E_{0x}\, m\underbrace{\cos(\omega t + \pi)}_{-\cos\omega t} \end{cases}$$

i.e. $\begin{cases} E_y = \pm E_{0x}\, m\cos\omega t \\ E_x = E_{0x}\cos\omega t \end{cases} \Rightarrow \boxed{E_y = \pm m E_x}$

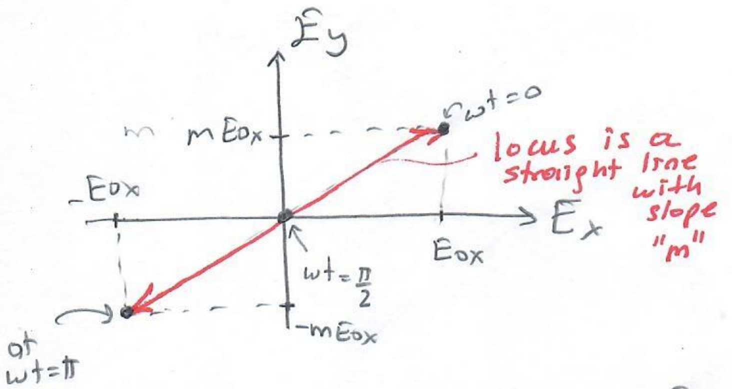
Equation of a straight line in (E_x, E_y) plane

$\Downarrow$

$\left(\bar{E}(z=0,t) = \hat{a}_x E_x + \hat{a}_y E_y\right) \Rightarrow$ Locus of $\bar{E}(0,t)$ is NOT a parabola but a straight line!

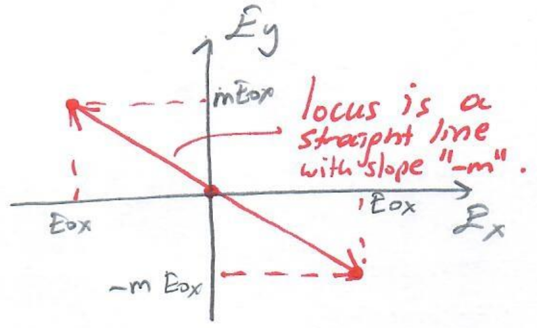
For $\psi = 0$ case

$$\mathcal{E}_y = m\,\mathcal{E}_x$$



For $\psi = \pi$ case

$$\mathcal{E}_y = -m\,\mathcal{E}_x$$



ωt	t	$\mathcal{E}_x = E_{ox}\cos\omega t$	$\mathcal{E}_y = m E_{ox}\cos\omega t$
0	0	E_{ox}	$m E_{ox}$
$\frac{\pi}{2}$	$\frac{T}{4}$	0	0
π	$\frac{T}{2}$	$-E_{ox}$	$-m E_{ox}$

LINEAR POLARIZATION

Conclusion: The quadratic form in (*) can NOT represent a hyperbola or a parabola. It can only represent an ellipse. As special cases of ELLIPTICAL POLARIZATION, we can also have LINEAR and CIRCULAR polarizations for u.p.waves.

Note that if $\mathcal{E}_x$ and $\mathcal{E}_y$ are in-phase (i.e. $\psi = 0$), or out-of-phase (i.e. $\psi = \pi$), or if one of them is zero (i.e. $\mathcal{E}_x = 0$ or $\mathcal{E}_y = 0$), then the resultant u.p.w. is LINEARLY POLARIZED.

CIRCULAR Polarization happens when $\begin{cases} m = 1 & \text{(same magnitude)} \\ \psi = \mp \frac{\pi}{2} & \text{(phase quadrature)} \end{cases}$

Consider the case $\boxed{m=1 \text{ and } \psi = \frac{\pi}{2}}$

Remember: $\bar{E} = E_{ox}(\hat{a}_x + m e^{j\psi} \hat{a}_y) e^{-jkz} = E_{ox}(\hat{a}_x + \underbrace{e^{j\pi/2}}_{j} \hat{a}_y) e^{-jkz}$

$\Rightarrow \boxed{\bar{E} = E_{ox}(\hat{a}_x + j\hat{a}_y) e^{-jkz}}$ for $\bar{E}$-phasor.

for $\bar{\mathcal{E}}(0,t)$ at $z=0$:

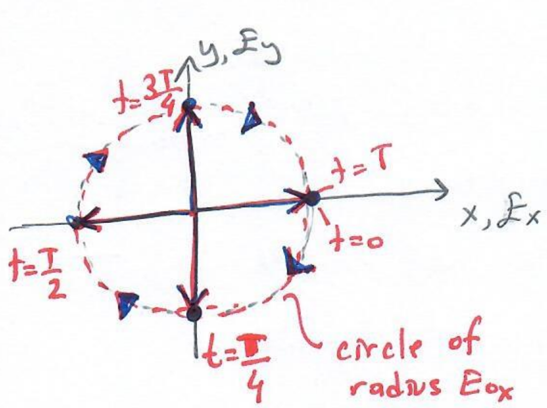
$$\begin{cases} \mathcal{E}_x = E_{ox} \cos \omega t \\ \mathcal{E}_y = E_{ox} \underbrace{\cos(\omega t + \frac{\pi}{2})}_{-\sin \omega t} = -E_{ox} \sin \omega t \end{cases}$$

$\mathcal{E}_x^2 + \mathcal{E}_y^2 = E_{ox}^2 \cos^2 \omega t + E_{ox}^2 \sin^2 \omega t$

$\boxed{\mathcal{E}_x^2 + \mathcal{E}_y^2 = E_{ox}^2}$

Equation of a circle with radius Eox.

ωt	t	$\mathcal{E}_x = E_{ox} \cos \omega t$	$\mathcal{E}_y = -E_{ox} \sin \omega t$	$\bar{\mathcal{E}}(z=0,t)$
0	0	E_{ox}	0	$E_{ox} \hat{a}_x$
$\frac{\pi}{2}$	$\frac{T}{4}$	0	$-E_{ox}$	$-E_{ox} \hat{a}_y$
π	$T/2$	$-E_{ox}$	0	$-E_{ox} \hat{a}_x$
$\frac{3\pi}{2}$	$3T/4$	0	E_{ox}	$E_{ox} \hat{a}_y$
2π	T	E_{ox}	0	$E_{ox} \hat{a}_x$



Tip of the $\bar{\mathcal{E}}(z=0,t)$ vector moves around the circle of radius Eox in clock-wise direction with an angular speed of $\omega = 2\pi f$ (radian/sec).

(LEFT-HAND CIRCULAR POLARIZATION

Definition: when the fingers of right hand follow the direction of rotation of $\bar{E}(\bar{r}=\text{const}, t)$ vector while the thumb points the direction of propagation ($\hat{n}$ or $\bar{k}=k\hat{n}$ vector direction), the polarization of the u.p.w. is called the RIGHT-HAND CIRCULAR polarization (RHCP). Otherwise, it is called LEFT-HAND CIRCULAR polarization (LHCP).

Exercise: For $\boxed{m=1, \ \psi=-\frac{\pi}{2}}$ show that the associated $\bar{E}$-phasor $\boxed{\bar{E} = E_{0x}\,(\hat{a}_x - j\,\hat{a}_y)\,e^{-jkz}}$ belongs to a Right-Hand circularly polarized (RHCP) u.p.w.

Note that a u.p.w is elliptically polarized (EP) when it is not linearly polarized (LP) or circularly polarized (CP). Definitions of right-hand and left-hand elliptical polarizations (RHEP and LHEP) are similar to the definition given for RHCP and LHCP.

Also note that directions of $\bar{E}$ and $\bar{H}$ vectors are not independent of each other. Therefore, it is enough to examine the $\bar{E}$ vector to determine the polarization (type and sense) of a u.p.w. It is redundant to examine $\bar{H}$ vector also.

Example: Show that a linearly polarized wave can be decomposed into a RHCP wave and a LHCP wave of equal amplitude.

Let $\hat{n} = \hat{a}_z \Rightarrow \bar{E}(z) = E_0 e^{-jkz} \hat{a}_x$ is the phasor for a u.p.w.

Instead of $E_0 \hat{a}_x$, write $\underbrace{\frac{E_0}{2}\hat{a}_x + \frac{E_0}{2}\hat{a}_x}_{E_0 \hat{a}_x} + \underbrace{j\frac{E_0}{2}\hat{a}_y - j\frac{E_0}{2}\hat{a}_y}_{0}$

$$\Rightarrow \bar{E}(z) = \left(\frac{E_0}{2}\hat{a}_x + \frac{E_0}{2}\hat{a}_x + j\frac{E_0}{2}\hat{a}_y - j\frac{E_0}{2}\hat{a}_y\right) e^{-jkz}$$

$$= \underbrace{\frac{E_0}{2}\left(\hat{a}_x + j\hat{a}_y\right) e^{-jkz}}_{\text{a LHCP u.p.w}} + \underbrace{\frac{E_0}{2}\left(\hat{a}_x - j\hat{a}_y\right) e^{-jkz}}_{\text{a RHCP u.p.w}}$$

having the same amplitude!

Example: For $\hat{n} = \hat{a}_z$, $m = 1$, $\psi = \frac{\pi}{4}$ determine the polarization of the u.p.w with phasor $\bar{E}(z)$.

$$\bar{E}(z) = E_{0x}\left(\hat{a}_x + m e^{j\psi} \hat{a}_y\right) e^{-jkz}$$

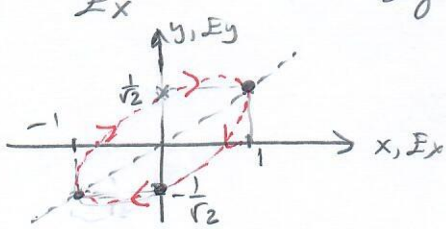
$$= E_{0x}\left(\hat{a}_x + \underbrace{e^{j\pi/4}}_{\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}} \hat{a}_y\right) e^{-jkz}$$

$$\bar{E}(z) = E_{0x}\left(\hat{a}_x + \frac{1}{\sqrt{2}}(1+j)\hat{a}_y\right) e^{-jkz}$$ in phasor domain

$$\bar{\mathcal{E}}(z=0,t) = Re\left\{\bar{E}(z) e^{j\omega t}\right\}\Big|_{z=0} = \underbrace{E_{0x}\cos\omega t\, \hat{a}_x}_{\mathcal{E}_x} + \underbrace{E_{0x}\cos\left(\omega t + \frac{\pi}{4}\right)\hat{a}_y}_{\mathcal{E}_y}$$

(Let $E_{0x} = 1$ for simplicity)

ωt	t	$\mathcal{E}_x$	$\mathcal{E}_y$
0	0	1	$\frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4}) \cong 0.707$
$\frac{\pi}{2}$	$\frac{T}{4}$	0	$-\frac{1}{\sqrt{2}} = \cos(3\pi/4)$
π	$T/2$	-1	$-\frac{1}{\sqrt{2}} = \cos(5\pi/4)$



$\boxed{\text{LHEP}}$

Exercise: Determine the type and sense of polarization for the following phasors.

a) $\bar{E} = (j\hat{a}_x + \hat{a}_y)\, e^{-jkz}$ (Ans: RHCP)

b) $\bar{E} = [(1+j)\,\hat{a}_y + (1-j)\,\hat{a}_z]\, e^{-jkx}$ (Ans: RHCP)

c) $\bar{E} = [(2+j)\hat{a}_x + (3-j)\,\hat{a}_z]\, e^{-jky}$ (Ans: LHEP)

d) $\bar{E} = [j\hat{a}_x + j\,2\,\hat{a}_y]\, e^{-jkz}$ (Ans: LP)

Example: The $\bar{E}$-phasor of a u.p.w. is given as

$$\bar{E}(z) = [2\,\hat{a}_x + (1+j)\,\hat{a}_y]\, e^{-jkz} \quad (V/m)$$

(α) Find time-domain $\mathcal{E}(z,t)$ field for a given frequency $\omega = 2\pi f$.

$$\bar{\mathcal{E}}(z,t) = Re\{\bar{E}(z)\, e^{j\omega t}\} = Re\{[2\,\hat{a}_x + (1+j)\,\hat{a}_y]\, \underbrace{e^{-jkz}\, e^{j\omega t}}\}$$

$$= Re\{[(2\hat{a}_x + \hat{a}_y) + j\hat{a}_y]\,[\cos(\omega t - kz) + j\sin(\omega t - kz)]\}$$

$$= (2\hat{a}_x + \hat{a}_y)\cos(\omega t - kz) - \hat{a}_y \sin(\omega t - kz)$$

$$\boxed{\bar{\mathcal{E}}(z,t) = 2\cos(\omega t - kz)\,\hat{a}_x + [\cos(\omega t - kz) - \sin(\omega t - kz)]\,\hat{a}_y}$$

or, using $\cos(\omega t - kz) - \sin(\omega t - kz) = \sqrt{2}\,\cos(\omega t - kz + \frac{\pi}{4})$

$$\boxed{\bar{\mathcal{E}}(z,t) = 2\cos(\omega t - kz)\,\hat{a}_x + \sqrt{2}\,\cos(\omega t - kz + \tfrac{\pi}{4})\,\hat{a}_y}$$

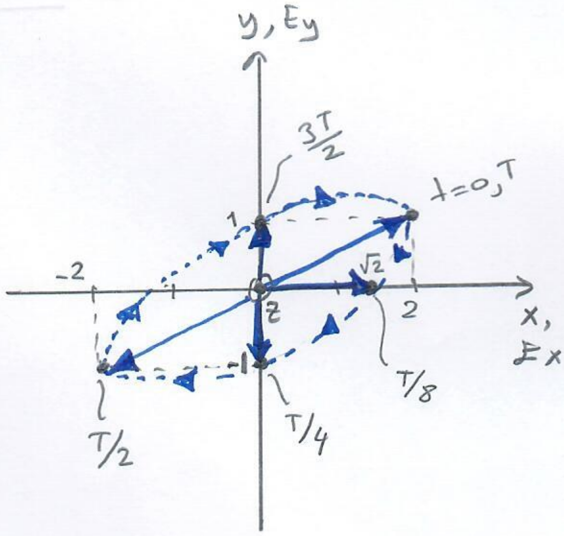
(b) Find the value of $\bar{E}(z=0,t)$ at $t=0, \frac{T}{8}, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$

where $T = \frac{2\pi}{\omega} = \frac{1}{f}$ is the period of u.p.w in time.

Sketch these vectors!

$$\bar{E}(z=0,t) = \underbrace{2\cos\omega t}_{E_x}\ \hat{a}_x + \underbrace{\sqrt{2}\cos(\omega t + \tfrac{\pi}{4})}_{E_y}\ \hat{a}_y$$

t	ωt	E_x	E_y	$\bar{E}(0,t)$
0	0	2	1	$2\hat{a}_x + \hat{a}_y$
$T/8$	$\pi/4$	$\sqrt{2}$	0	$\sqrt{2}\,\hat{a}_x$
$T/4$	$\pi/2$	0	-1	$-\hat{a}_y$
$T/2$	π	-2	-1	$-2\hat{a}_x - \hat{a}_y$
$3T/4$	$3\pi/2$	0	1	$\hat{a}_y$
T	2π	2	1	$2\hat{a}_x + \hat{a}_y$



(c) What is the type and sense of polarization?

LHEP (using the sense rule)

$\left(m \neq 1 \text{ as } |2\hat{a}_x| \neq |(1+j)\hat{a}_y| \right.$
$\left. \text{Also, } \angle E_x = \angle 2 = 0,\ \angle E_y = \angle 1+j = \frac{\pi}{4} \right)$

$$\psi \neq 0, \pi, \mp\frac{\pi}{2}$$

(So, linear or circular polarizations are NOT possible. It can be only an elliptical polarization as revealed by the sketch also)

Note: To find the sense of polarization, you may evaluate $\bar{E}(t_1)$ and $\bar{E}(t_2)$ at $t_2 > t_1$ (for $t_2 - t_1 < \frac{T}{2}$ = half period). Then, check the direction of $\boxed{\bar{E}(t_1) \times \bar{E}(t_2)}$. If the result is in the same direction as $\hat{n}$ (dir. of propagation), sense of polarization is "Right-Hand". Apply this rule to the example above.