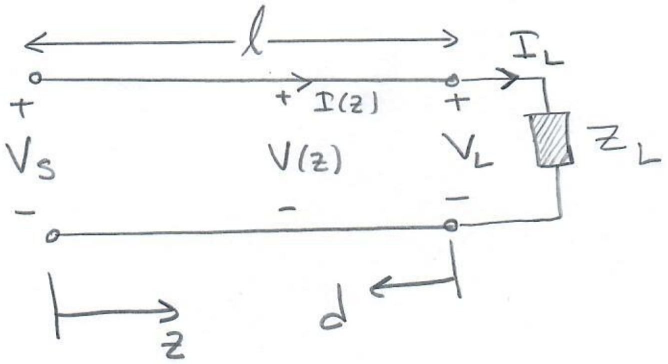
REFLECTION COEFFICIENT in Transmission Lines

Consider a transmission line of length ℓ.



$\boxed{d = \ell - z}$

or

$\boxed{z = \ell - d}$

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$ as found earlier.

$$\boxed{V_s = V(z=0) = V^+ + V^-}$$ at the source end

$$V_L = V(z=\ell) = \underbrace{V^+ e^{-\gamma \ell}}_{\text{call } V_L^+} + \underbrace{V^- e^{+\gamma \ell}}_{\text{call } V_L^-}$$

$$\boxed{V_L = V_L^+ + V_L^-}$$

Similarly,

$$\boxed{I_L = I_L^+ + I_L^- = \frac{V_L^+}{Z_0} - \frac{V_L^-}{Z_0}}$$

$$\boxed{\frac{V_L}{I_L} = Z_L}$$ at the load end.

Now, define the reflection coefficient Γ_L at the load:

$$\boxed{\Gamma_L \triangleq \frac{V_L^-}{V_L^+}} = \frac{\text{Value of backward traveling voltage at load end}}{\text{Value of forward traveling voltage at load end}}$$

Use $Z_L = \dfrac{V_L}{I_L} = \dfrac{V_L^+ + V_L^-}{\frac{1}{Z_0}(V_L^+ - V_L^-)} = Z_0 \dfrac{1 + \frac{V_L^-}{V_L^+}}{1 - \frac{V_L^-}{V_L^+}}$ (divide both num. and denom. by V_L^+)

$$\Rightarrow \boxed{Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}}$$

where Γ_L is complex, in general.

or solve for Γ_L in terms of Z_L:

$$\boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_L}}$$

reflection coefficient at the load

Now, let's obtain the reflection coefficient expression at an arbitrary location over the transmission line. Use the d-axis (measured from the load end) where $d = l - z$.

$$V(d) = V(z)\Big|_{z=l-d} = V^+ e^{-\gamma(l-d)} + V^- e^{\gamma(l-d)}$$

$$V(d) = \underbrace{V^+ e^{-\gamma l}}_{V_L^+} e^{\gamma d} + \underbrace{V^- e^{\gamma l}}_{V_L^-} e^{-\gamma d}$$

(Note that V_L^+ and V_L^- are constants)

$$\boxed{V(d) = \underbrace{V_L^+ e^{\gamma d}}_{\text{forward prop. wave}} + \underbrace{V_L^- e^{-\gamma d}}_{\text{backward prop. wave}}}$$

Similarly,

$$I(d) = I_L^+ e^{\gamma d} + I_L^- e^{-\gamma d}$$

$$= \frac{V_L^+}{Z_0} e^{\gamma d} - \frac{V_L^-}{Z_0} e^{-\gamma d}$$

$$\boxed{I(d) = \frac{1}{Z_0}\left(V_L^+ e^{\gamma d} - V_L^- e^{-\gamma d}\right)}$$

* The <u>reflection coefficient</u> $\Gamma(d)$ at a distance "d" measured from the load is defined as

$$\Gamma(d) = \left.\frac{\text{backward trav. voltage}}{\text{forward trav. voltage}}\right|_{\text{at } d} = \frac{V_L^- e^{-\gamma d}}{V_L^+ e^{+\gamma d}}$$

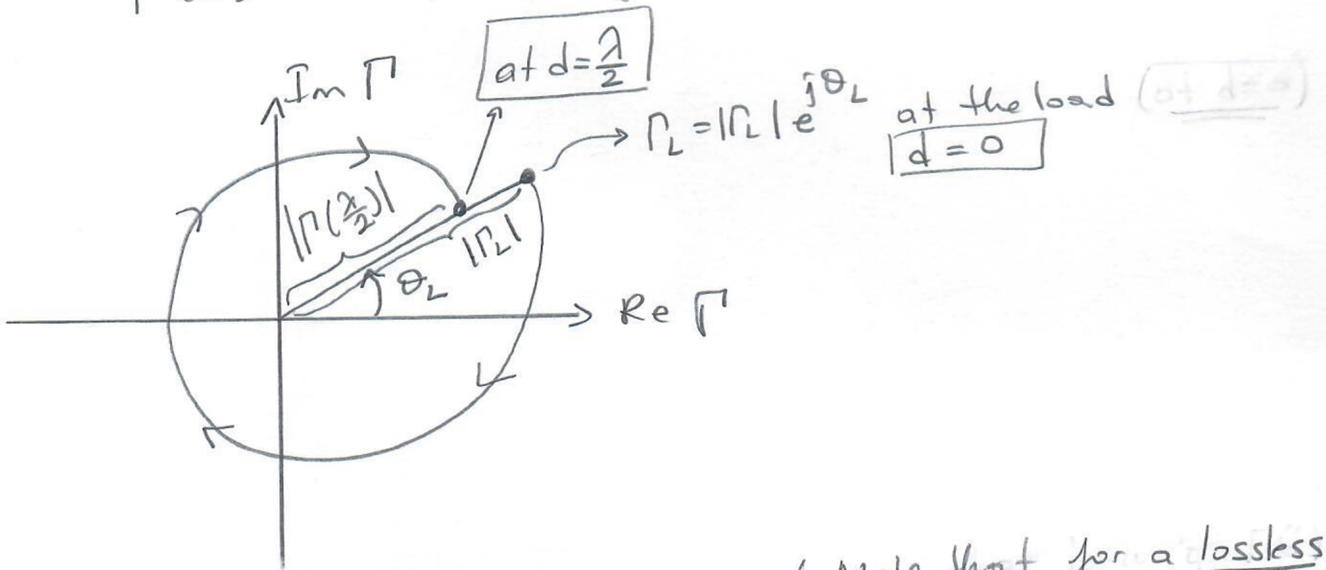
$$\Gamma(d) = \frac{V_L^-}{V_L^+} e^{-2\gamma d} \qquad \text{where } \frac{V_L^-}{V_L^+} = \Gamma_L \quad (\text{ref. coeff at load})$$

$$\Rightarrow \boxed{\Gamma(d) = \Gamma_L e^{-2\gamma d}} \qquad \text{where } \begin{cases} \gamma = \alpha + j\beta \quad (\text{prop. coeff.}) \\ \text{and} \\ \Gamma_L = |\Gamma_L| e^{j\theta_L} \end{cases}$$

Therefore,

$$\boxed{\Gamma(d) = |\Gamma_L|\, e^{-2\alpha d}\, e^{j(\theta_L - 2\beta d)}}$$

$\Gamma(d)$ is complex in general, $\Gamma(d) = \text{Re}\,\Gamma + j\,\text{Im}\,\Gamma$



Γ- plane

(Spiral locus for a lossy T.L. $\alpha \neq 0$)

(Note that for a lossless TL (with $\alpha = 0$), the tip of $\Gamma(d)$ traces a circle)

At the load location (where $d=0$) $\Gamma(d=0) = \Gamma_L = |\Gamma_L| e^{j\theta_L}$

Note that moving towards the source by $d = \frac{\lambda}{2}$ corresponds to making a complete rotation in the Γ-plane, because:

we need $2\beta d = 2\pi$ for a full cycle, where $\boxed{\beta = \frac{2\pi}{\lambda}}$

$$2 \frac{\cancel{2\pi}}{\lambda} d = \cancel{2\pi}$$

$$\Rightarrow \boxed{d = \frac{\lambda}{2}}$$

In a lossy T.L., due to $\alpha \neq 0$,

$|\Gamma(d)| = |\Gamma_L|\, e^{-2\alpha d}$

i.e $|\Gamma(d)| < |\Gamma_L| \Rightarrow$ spiral locus of $\Gamma(d)$ if $\alpha \neq 0$.

Impedance of a Transmission Line, $Z(d)$

At a distance "d" measured from the load end, the impedance of the T.L. is defined as

$$Z(d) \triangleq \frac{V(d)}{I(d)} = \frac{V_L^+ e^{\gamma d} + V_L^- e^{-\gamma d}}{\frac{1}{Z_0}\left(V_L^+ e^{\gamma d} - V_L^- e^{-\gamma d}\right)}$$

(divide both num. and denom. by V_L^+)

$$= Z_0 \frac{e^{\gamma d} + \frac{V_L^-}{V_L^+} e^{-\gamma d}}{e^{\gamma d} - \frac{V_L^-}{V_L^+} e^{-\gamma d}} = Z_0 \frac{e^{\gamma d} + \Gamma_L e^{-\gamma d}}{e^{\gamma d} - \Gamma_L e^{-\gamma d}}$$

(Remember $\frac{V_L^-}{V_L^+} \triangleq \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$)

$$\Rightarrow Z(d) = Z_0 \frac{e^{\gamma d} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma d}}{e^{\gamma d} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma d}}$$

(multiply both num. and denom. by $(Z_L + Z_0)$)

$$= Z_0 \frac{(Z_L + Z_0) e^{\gamma d} + (Z_L - Z_0) e^{-\gamma d}}{(Z_L + Z_0) e^{\gamma d} - (Z_L - Z_0) e^{-\gamma d}}$$

$$= Z_0 \frac{Z_L\left(e^{\gamma d} + e^{-\gamma d}\right) + Z_0\left(e^{\gamma d} - e^{-\gamma d}\right)}{Z_L\left(e^{\gamma d} - e^{-\gamma d}\right) + Z_0\left(e^{\gamma d} + e^{-\gamma d}\right)}$$

(Use $\cosh(x) \triangleq \frac{e^x + e^{-x}}{2}$ and $\sinh(x) \triangleq \frac{e^x - e^{-x}}{2}$)

$$Z(d) = Z_0 \frac{Z_L \cosh(\gamma d) + Z_0 \sinh(\gamma d)}{Z_L \sinh(\gamma d) + Z_0 \cosh(\gamma d)}$$

(divide both num. and denom. by $\cosh(\gamma d)$)

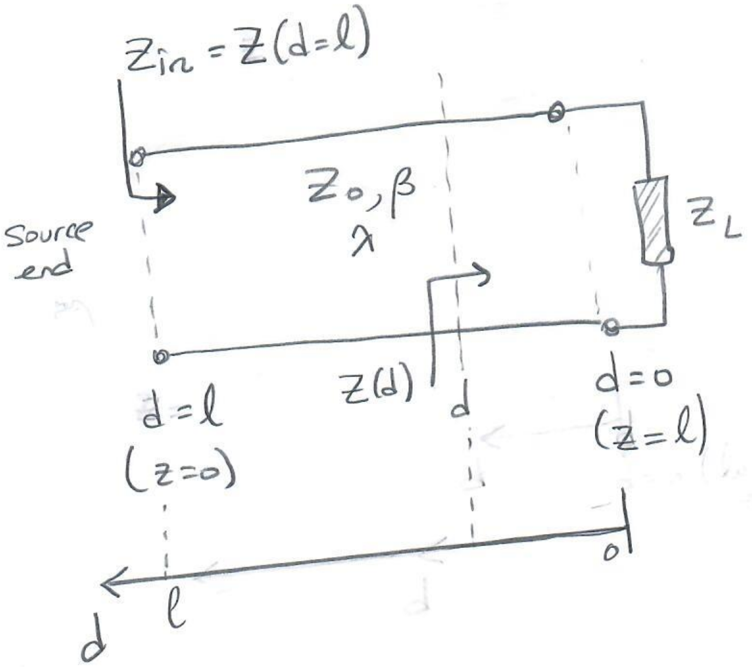
$$\boxed{Z(d) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)}}$$

for a <u>lossy</u> ($\alpha \neq 0$) T.L.

For a <u>lossless</u> ($\alpha = 0$) TL, $\underline{\underline{\alpha = 0}} \Rightarrow \gamma = j\beta$

$$\Rightarrow \tanh(\gamma d) = \tanh(j\beta d) = j\tan(\beta d)$$

$$\Rightarrow \boxed{Z_{\text{lossless}}(d) = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)}}$$



Z_0: characteristic impedance of the T.L.

β : Phase constant of the T.L.

$\beta = \text{Im}\{\gamma\}$

$\left(\lambda = \frac{2\pi}{\beta}, \; v_p = \frac{\omega}{\beta} \right)$
over the T.L.

IMPORTANT SPECIAL CASES

① Matched Load Case $\boxed{Z_L = Z_0}$

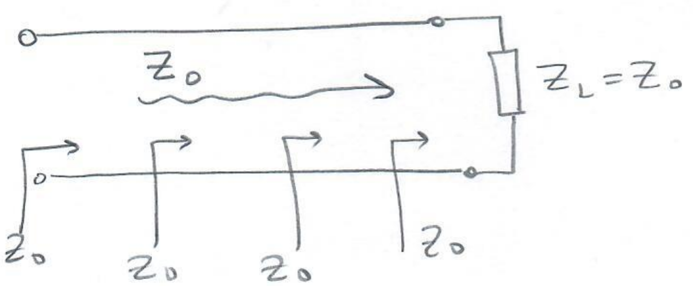
$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}\Bigg|_{Z_L = Z_0} = 0 \Rightarrow \boxed{\Gamma_L = 0} \Rightarrow \boxed{\begin{matrix}\Gamma(d) = 0 \\ Z(d) = Z_0\end{matrix}}$$

for all d

(Remember, $\Gamma(d) = \Gamma_L e^{-\gamma d}\Big|_{\Gamma_L = 0} = 0$ for all d

and $Z(d) = Z_0 \dfrac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)}\Bigg|_{Z_L = Z_0} = Z_0$ for all d)

Matched Line



(There is only the forward traveling wave along the TL)

No reflected wave

② Impedance Repeater

let $d = \frac{\lambda}{2} \Rightarrow \beta d = \frac{2\pi}{\lambda} d\Big|_{d = \frac{\lambda}{2}} = \frac{\cancel{2}\pi}{\cancel{\lambda}}\frac{\cancel{\lambda}}{\cancel{2}} = \pi$

$\beta d = \pi \Rightarrow \tan(\beta d) = \tan(\pi) = 0$

$$\Rightarrow \underset{\text{lossless}}{Z}\left(d = \frac{\lambda}{2}\right) = Z_0 \frac{Z_L + Z_0(0)}{Z_0 + Z_L(0)} = Z_L$$

$$\Rightarrow \boxed{Z_{lossless}\left(d=\frac{\lambda}{2}\right) = Z_L}$$

In fact, the impedance of a lossless T.L. becomes equal to its load impedance Z_L periodically at distances $d = n\frac{\lambda}{2}$ $(n = 1, 2, 3, \ldots)$.

$\Rightarrow$ A lossless T.L. which is $\left(n\frac{\lambda}{2}\right)$ long is called an Impedance Repeater as its input impedance $Z_{in}\left(\ell = n\frac{\lambda}{2}\right)$ is equal to its load impedance Z_L.

(3) Impedance Inverter

Let $d = \frac{\lambda}{4} \rightarrow \beta d = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

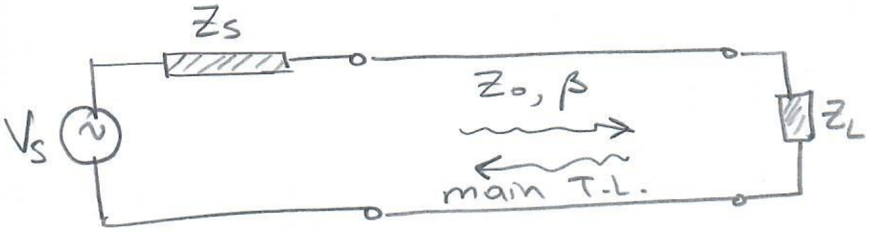
$$\Rightarrow \tan(\beta d) = \tan\left(\frac{\pi}{2}\right) \rightarrow \infty$$

$$\Rightarrow Z_{lossless}\left(d=\frac{\lambda}{4}\right) = \lim_{\tan(\beta d)\to\infty} Z_0 \frac{Z_L + Z_0\tan(\beta d)}{Z_0 + Z_L\tan(\beta d)} = \frac{Z_0^2}{Z_L}$$

$$\boxed{Z_{lossless}\left(d=\frac{\lambda}{4}\right) = \frac{Z_0^2}{Z_L}}$$

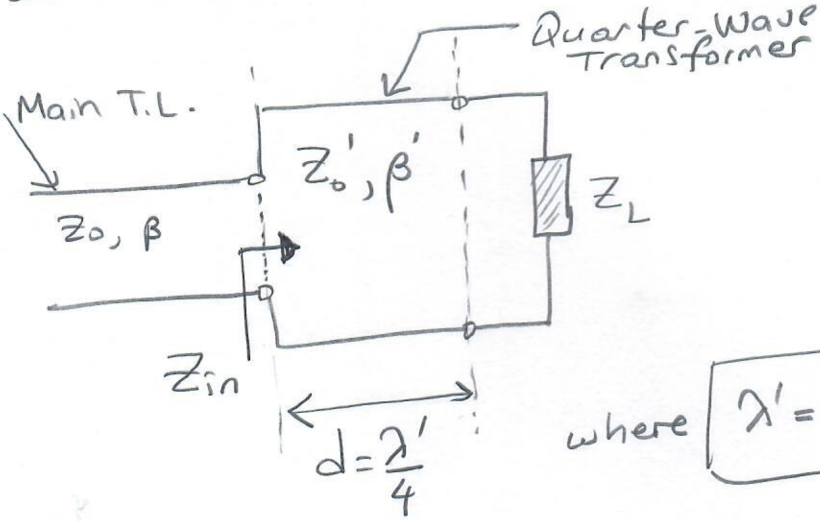
A lossless T.L. which is $(2n+1)\frac{\lambda}{4}$ $(n = 0, 1, 2, \ldots)$ long is called Impedance Inverter as its input impedance is inversely proportional to its load impedance Z_L.

Example: Quarter-Wave Transformer



As long as $Z_L \neq Z_o$, there will be reflections from the load. These reflections should be minimized to deliver maximum possible power to the load.

As a solution, a quarter-wave length TL can be connected between the "main T.L." and the load Z_L.



where $\boxed{\lambda' = \frac{2\pi}{\beta'}}$ ($\lambda \neq \lambda'$ in general)

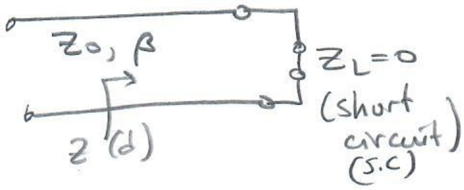
To eliminate reflected waves on the main T.L., we must provide $\boxed{Z_{in} = Z_o}$

where $Z_{in} = Z(d=\frac{\lambda'}{4}) = \frac{(Z_o')^2}{Z_L}$

$\Big\} \; \frac{(Z_o')^2}{Z_L} = Z_o$ is required

$$\Rightarrow \boxed{\text{Choosing} \begin{cases} d = \lambda'/4 \\ Z_o' = \sqrt{Z_o Z_L} \end{cases} \text{eliminates reflections on the main line}}$$

(4) Short-Circuited TL



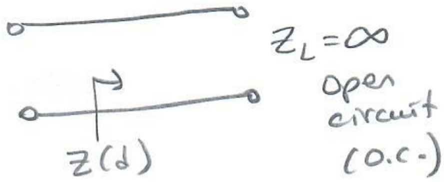
let $Z_L = 0$

$$\Rightarrow \Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L=0} = -1 \quad \Rightarrow \boxed{\Gamma_L = -1}$$

$$\Rightarrow Z(d)\Big|_{Z_L=0} = Z_0 \frac{\cancel{Z_L} + Z_0 \tanh(\gamma d)}{Z_0 + \cancel{Z_L \tanh(\gamma d)}} = Z_0 \tanh(\gamma d)$$

For a lossless T.L. ($\alpha = 0$, $\gamma = j\beta$) $\Rightarrow$ $\boxed{Z_{lossless}(d) = j Z_0 \tan(\beta d) = Z_{s.c.}}$

(5) Open-Circuited TL



Let $Z_L \to \infty$

$$\Rightarrow \Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L \to \infty} = +1 \quad \Rightarrow \boxed{\Gamma_L = +1}$$

$$\Rightarrow Z(d)\Big|_{Z_L \to \infty} = \frac{Z_0}{\tanh(\gamma d)} = Z_0 \operatorname{cotanh}(\gamma d)$$

For a lossless T.L. $\Rightarrow$ $\boxed{Z_{lossless}(d) = -j Z_0 \operatorname{cotan}(\beta d) = Z_{oc}}$

Use of short-circuited / open-circuited Transmission Lines to design circuit elements (capacitors/inductors)

As $f \uparrow \Rightarrow \lambda \downarrow \Rightarrow$ Dimensions of conventional lumped circuit elements like capacitors and inductors become radiating.

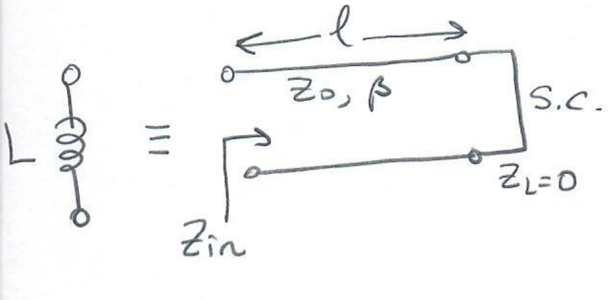
$\Rightarrow$ Simple rules of circuit theory can not explain their performance any more.

$\Rightarrow$ As an alternative, short-circuited or open-circuited TLs (stubs) can be used as capacitors and inductors at UHF and higher frequencies.

<u>Case I</u> : Consider a lossless T.L. of $\boxed{\underbrace{\text{length } \ell < \frac{\lambda}{4}}}$

for which $\beta \ell = \frac{2\pi}{\lambda} \ell \Big|_{\ell < \frac{\lambda}{4}} < \frac{\pi}{2}$ (in the first quadrant)

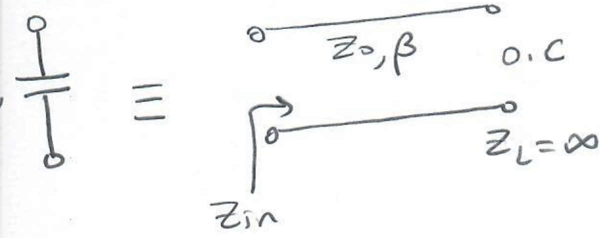
if $\beta \ell < \frac{\pi}{2} \Rightarrow \tan(\beta \ell) > 0$ and $\cot(\beta \ell) > 0$



$Z_{in} = j Z_o \underbrace{\tan \beta \ell}_{>0} \equiv j\omega L$

$\Rightarrow \boxed{\omega L = Z_o \tan(\beta \ell)}$

using a short-circuited stub!



$Z_{in} = -j Z_o \underbrace{\cot(\beta \ell)}_{>0} \equiv \frac{1}{j\omega C} = -j \frac{1}{\omega C}$

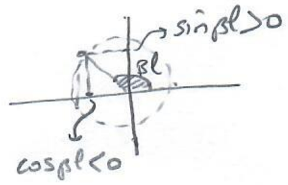
(minus sign)

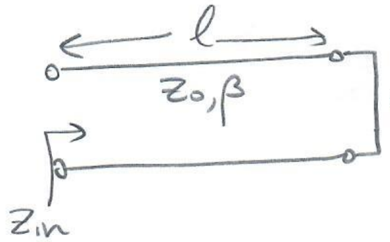
$\Rightarrow \boxed{\frac{1}{\omega C} = Z_o \cot(\beta \ell)}$

using an open-circuited stub

Case II : Now consider a lossless T.L. of length l

$$\boxed{\frac{\lambda}{4} < l < \frac{\lambda}{2}} \Rightarrow \begin{cases} \beta l = \frac{2\pi}{\lambda} l \Rightarrow \boxed{\frac{\pi}{2} < \beta l < \pi \text{ (in the 2nd quadrant)}} \\ \tan \beta l < 0 \quad \text{and} \quad \cot(\beta l) < 0 \end{cases}$$





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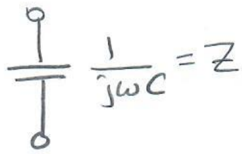

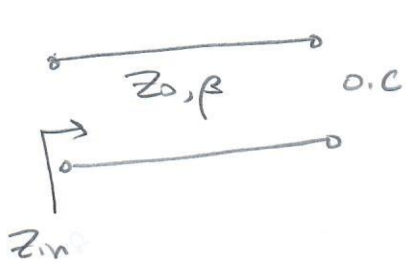

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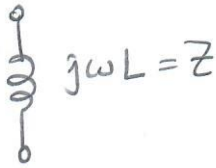
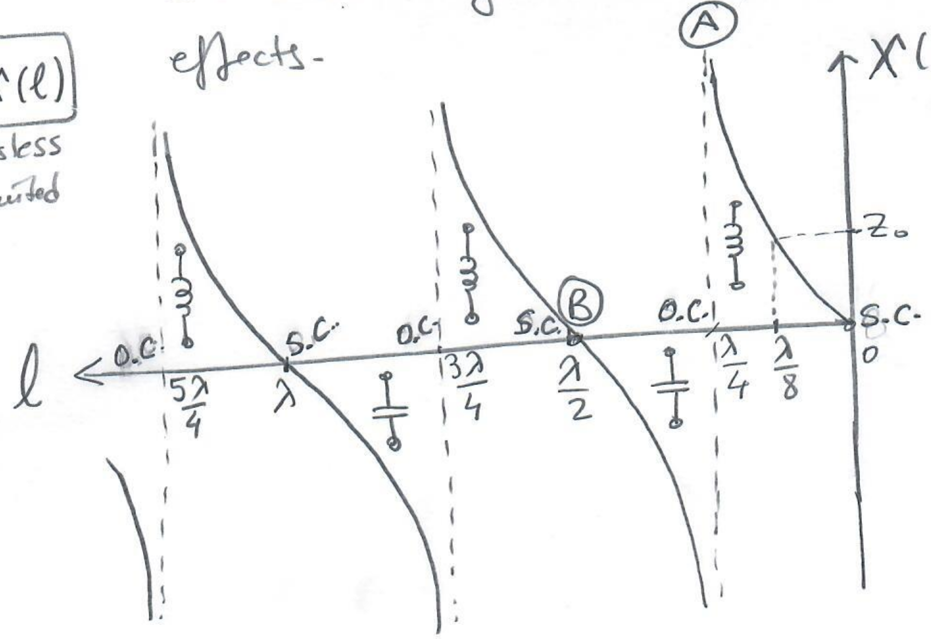


Note : Short-circuited stubs with adjustable length "l" are commonly used to produce capacitor/inductor effects.

$\boxed{Z(l) = jX(l)}$ for a lossless short-circuited T.L.

$X(l) = Z_0 \tan(\beta l)$ for a short-circuited stub $\left(Z_{in} = jX(l)\right)$

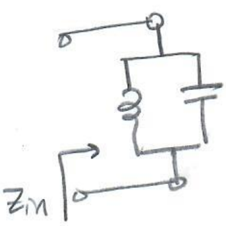


Note that

$$X\left(l = \frac{\lambda}{8}\right) = Z_0$$

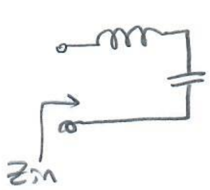
at (A) $l \approx \frac{\lambda}{4} \Rightarrow X(l) \to \infty$

Short-circuited stub behaves like a parallel resonant circuit

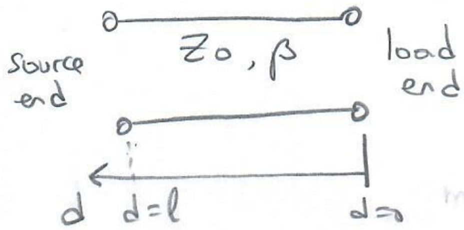


at (B) $l \approx \frac{\lambda}{2} \Rightarrow X(l) \to 0$

Short-circuited stub behaves like a series resonant circuit.



<u>Problem:</u> Can we obtain characteristic impedance Z_0 using the short-circuit and open-circuit impedance measurements?



Consider a lossless T.L. Measure (for an arbitrary d)

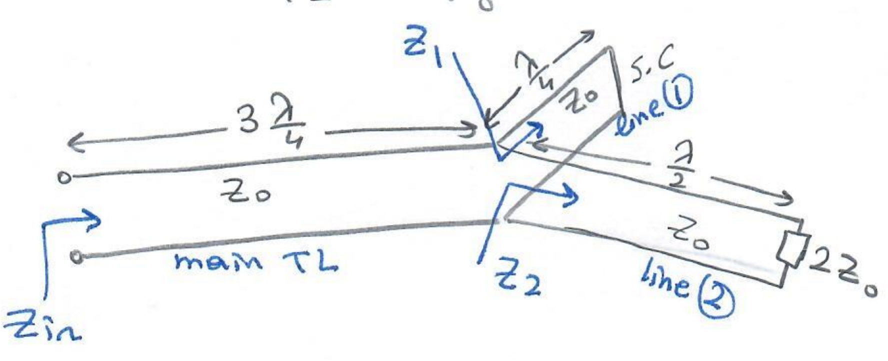
$$\left.\begin{aligned} Z_{sc} &= jZ_0 \tan(\beta d) \\ Z_{oc} &= -jZ_0 \cot(\beta d) \end{aligned}\right\} \text{multiply them!}$$

$$Z_{sc} Z_{oc} = (jZ_0 \tan \beta d)(-jZ_0 \cot(\beta d)) = Z_0^2$$

$$\Rightarrow \boxed{Z_0 = \sqrt{Z_{sc} Z_{oc}}}$$

<u>Geometric mean</u> of the open circuited impedance and short circuited impedance gives the characteristic imp.

<u>Example:</u> Calculate the input impedance Z_{in} for the following TL configuration.



Z_1: input imp. of TL1

Z_2: input imp. of TL2

Z': load impedance seen by main TL

<u>TL ①</u>: $\frac{\lambda}{4}$ long → Impedance inverter, load $Z_{L_1} = 0$ (s.c.) $\Rightarrow Z_1 = \left.\frac{Z_0^2}{Z_{L_1}}\right|_{Z_{L_1}=0} = \infty$ (o.c.)

<u>TL ②</u>: $\frac{\lambda}{2}$ long → Impedance repeater, load $Z_{L_2} = 2Z_0 \Rightarrow Z_2 = Z_{L_2} = 2Z_0$

$$Z' = Z_1 \,//\, Z_2 = (\infty) \,//\, 2Z_0 = 2Z_0 \Rightarrow \boxed{Z' = 2Z_0}$$

TL① and TL② are connected <u>in parallel</u> and act like an equivalent load for the main transmission line.

Main TL: $\frac{3\lambda}{4}$ long $\rightarrow$ Impedance Inverter $\Rightarrow Z_{in} = \frac{Z_0^2}{Z'}$

$$Z_{in} = \frac{Z_0^2}{2Z_0} = \frac{Z_0}{2} \Rightarrow \boxed{Z_{in} = \frac{Z_0}{2}}$$

Note that in this example, we are given three TLs of the same type having the same characteristic impedance Z_0 and the same wavelength λ. In general, however, we could have different Z_0 (Z_{01}, Z_{02}, Z_{03}) and different β ($\beta_1, \beta_2, \beta_3$) leading to different wavelength values λ ($\lambda_1, \lambda_2, \lambda_3$).

Example: Put an O.C. instead of a S.C. at the end of TL 1. Then find Z_{in} in the problem above.

Answer: $Z_1 = \left.\frac{Z_0^2}{Z_{L_1}}\right|_{Z_{L_1}\to\infty} = 0$ (S.C.) $\Rightarrow Z' = 0 // Z_2 = 0$ (S.C.)

$$\Rightarrow Z_{in} = \left.\frac{Z_0^2}{Z'}\right|_{Z'=0} = \infty \text{ (O.C.)}$$
