Wave Equations for $\bar{E}$ and $\bar{H}$ Fields

Assume a simple (i.e. linear + isotropic + homogeneous) medium where ϵ and μ are simple constants.

USE:

Maxwell's Eqns. (M.E.)

$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

$$\bar{\nabla}\cdot\bar{D} = \rho_v$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

Constitutive Relations for a "simple" medium

$$\bar{B} = \mu\bar{H}$$

$$\bar{D} = \epsilon\bar{E}$$

To obtain the wave equation for $\bar{E}$ field, start by computing the curl of both sides of the 1st M.E., i.e. $\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$.

$$\underbrace{\bar{\nabla}\times(\bar{\nabla}\times\bar{E})}_{\bar{\nabla}(\bar{\nabla}\cdot\underbrace{\bar{E}}_{\bar{D}/\epsilon})-\nabla^2\bar{E}} = \bar{\nabla}\times\left(-\frac{\partial \bar{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\bar{\nabla}\times\underbrace{\bar{B}}_{\mu\bar{H}})$$

μ → constant

$$\Rightarrow \bar{\nabla}(\bar{\nabla}\cdot\underbrace{\frac{\bar{D}}{\epsilon}}_{\epsilon \to \text{constant}}) - \nabla^2\bar{E} = -\mu\frac{\partial}{\partial t}(\underbrace{\bar{\nabla}\times\bar{H}}_{\bar{J}+\frac{\partial \bar{D}}{\partial t}})$$

$\bar{D} \to \epsilon\bar{E}$

$$\Rightarrow \bar{\nabla}\left[\frac{1}{\epsilon}(\underbrace{\bar{\nabla}\cdot\bar{D}}_{\rho_v})\right] - \nabla^2\bar{E} = -\mu\frac{\partial \bar{J}}{\partial t} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2}$$

$$\bar{\nabla}\left(\frac{\rho_v}{\epsilon}\right) - \nabla^2\bar{E} = -\mu\frac{\partial \bar{J}}{\partial t} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2}$$

Reorganize as

$$\boxed{\nabla^2\bar{E} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} = \mu\frac{\partial \bar{J}}{\partial t} + \bar{\nabla}\left(\frac{\rho_v}{\epsilon}\right)}$$

Inhomogeneous (right hand side contains source terms) wave eqn. for $\bar{E}$ field!

where $\bar{J}$ and ρ_v are the vector and scalar type sources of the problem.

<u>Exercise:</u> Repeat the same procedure to obtain the wave equation for $\mathcal{H}$ field!

(<u>Hint</u>: Start with the 2nd M.E. $\bar{\nabla}\times\bar{\mathcal{H}} = \bar{J} + \frac{\partial\bar{\mathcal{D}}}{\partial t}$ this time)

Similarly, we can get the wave eqn. for $\bar{\mathcal{H}}$ field as

$$\boxed{\nabla^2\bar{\mathcal{H}} - \epsilon\mu\frac{\partial^2\bar{\mathcal{H}}}{\partial t^2} = -\bar{\nabla}\times\bar{J}}$$

Again, this wave eqn. is inhomogeneous as the right hand side contains the source term $\bar{J}$!

<u>Special Case I:</u> For a [SIMPLE ⊕ LOSSLESS ⊕ SOURCE FREE] medium.

SIMPLE → linear + isotropic + homogeneous

LOSSLESS → zero conductivity $\sigma = 0$

SOURCE FREE → $\bar{J}_{impressed} = 0$, $\rho_v = 0$ (no impressed sources)

In general, $\bar{J}$ in Maxwell's Equations may have the following form:

$$\bar{J} = \underbrace{\bar{J}_{impressed}}_{\substack{\text{forced or impressed current density}\\ \text{which is zero in source-free case.}}} + \underbrace{\bar{J}_{conduction}}_{\substack{\sigma\bar{\mathcal{E}}\text{, which is zero in lossless case}\\ \text{(i.e. when } \sigma=0)}}$$

In this special case, both components of current density are zero. Hence, set $\bar{J} = 0$ in M.E.s and in wave equations.

$$\Rightarrow \boxed{\nabla^2\bar{\mathcal{E}} - \epsilon\mu\frac{\partial^2\bar{\mathcal{E}}}{\partial t^2} = 0} \quad \text{and} \quad \boxed{\nabla^2\bar{\mathcal{H}} - \epsilon\mu\frac{\partial^2\bar{\mathcal{H}}}{\partial t^2} = 0}$$

Consider $\boxed{\nabla^2 \bar{E} - \epsilon\mu \dfrac{\partial^2 \bar{E}}{\partial t^2} = 0}$. for instance

where $\bar{E}(\bar{r}, t) = \hat{a}_x E_x(x,y,z,t) + \hat{a}_y E_y(x,y,z,t) + \hat{a}_z E_z(x,y,z,t)$

in general, in Cartesian coordinate system.

For simplicity, assume $\bar{E} = \hat{a}_x E_x(z,t)$

Insert this solution to wave eqn. above:

$$\cancel{\frac{\partial^2}{\partial x^2}\left[\hat{a}_x E_x(z,t)\right]}^{0} + \cancel{\frac{\partial^2}{\partial y^2}\left[\hat{a}_x E_x(z,t)\right]}^{0} + \frac{\partial^2}{\partial z^2}\left[\hat{a}_x E_x(z,t)\right] - \epsilon\mu \frac{\partial^2}{\partial t^2}\left[\hat{a}_x E_x(z,t)\right] = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 E_x(z,t)}{\partial z^2} - \epsilon\mu \frac{\partial^2 E_x(z,t)}{\partial t^2} = 0} \qquad (a)$$

It can be shown by inspection that the pde (a) has the solutions in the form

$\left.\boxed{\begin{array}{l} g_1(z - vt) \\ \hline g_2(z + vt) \end{array}}\right\}$ where $\boxed{v = \dfrac{1}{\sqrt{\epsilon\mu}}}$ and g_1, g_2 are double-differentiable functions with respect to z and t.

Proof: Insert $g_1(z-vt)$, for instance, into pde (a) to see if it is satisfied.

Let $s = z - vt \Rightarrow \dfrac{\partial s}{\partial z} = 1$ and $\dfrac{\partial s}{\partial t} = -v = -\dfrac{1}{\sqrt{\epsilon\mu}}$

$$\frac{\partial g_1}{\partial z} = \frac{\partial g_1}{\partial s}\underbrace{\frac{\partial s}{\partial z}}_{1} = \frac{\partial g_1}{\partial s} \quad \text{(using chain rule in differentiation)}$$

and $$\frac{\partial^2 g_1}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial g_1}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial g_1}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial g_1}{\partial s}\right)\underbrace{\frac{\partial s}{\partial z}}_{1}$$

$$\boxed{\frac{\partial^2 g_1}{\partial z^2} = \frac{\partial^2 g_1}{\partial s^2}} \quad (*)$$

Also,
$$\frac{\partial g_1}{\partial t} = \frac{\partial g_1}{\partial s}\underbrace{\frac{\partial s}{\partial t}}_{-v} = -\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}$$

and
$$\frac{\partial^2 g_1}{\partial t^2} = \frac{\partial}{\partial t}\left(-\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}\right) = \frac{\partial}{\partial s}\left[-\frac{1}{\sqrt{\epsilon\mu}}\frac{\partial g_1}{\partial s}\right]\underbrace{\frac{\partial s}{\partial t}}_{-\frac{1}{\sqrt{\mu\epsilon}}}$$

$$\boxed{\frac{\partial^2 g_1}{\partial t^2} = \frac{1}{\mu\epsilon}\frac{\partial^2 g_1}{\partial s^2}} \quad (**)$$

Insert $(*)$ and $(**)$ in (a) to see

$$\underbrace{\frac{\partial^2 g_1(z-vt)}{\partial z^2}}_{\frac{\partial^2 g_1}{\partial s^2}} - \cancel{\mu\epsilon}\underbrace{\frac{\partial^2 g_1(z-vt)}{\partial t^2}}_{\frac{1}{\cancel{\mu\epsilon}}\frac{\partial^2 g_1}{\partial s^2}} = 0 \quad \checkmark \text{ as claimed!}$$

(Validity of the other solution $g_2(z+vt)$ can be proven similarly.)

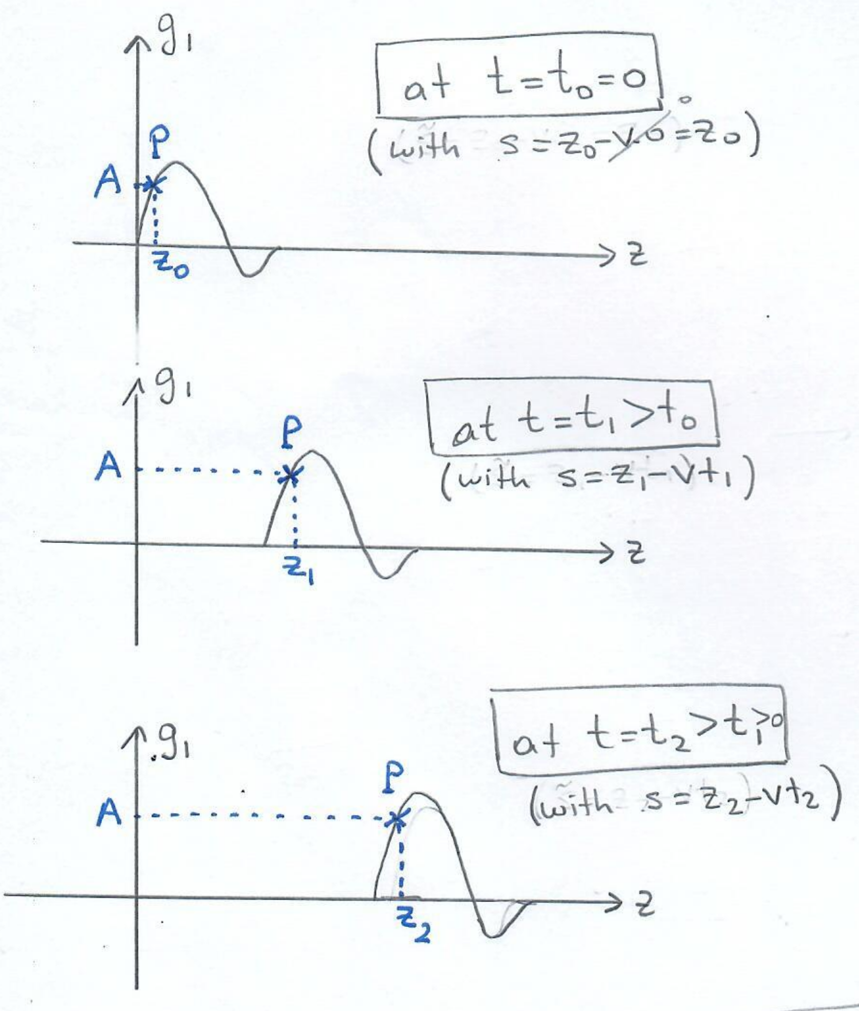
<u>Some Examples to Wave Solutions</u>

$$K e^{\alpha(z \mp vt)}, \quad K\cos\{\alpha(z \mp vt)\}, \quad K(z \mp vt)^3, \text{ etc...}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, K and α are some arbitrary constants.

Behavior of Wave Solutions: Traveling Waves

Let's examine the behavior of $g_1(z-vt)$ types of solutions. Call the argument of the function as $\boxed{s=z-vt}$. Plot an arbitrary g_1 function versus distance (z) at different time (t) instants as follows:



Choose an arbitrary point P on the curve. Coordinates of this particular point P are

(z_0, A) at $t=t_0=0$

(z_1, A) at $t=t_1$

(z_2, A) at $t=t_2$

$\vdots$

Note that the value of the argument must remain the same as we keep tracking the same point P of the curve. In other words,

for P $\boxed{s=\tilde{s}=z_0-v\underbrace{(0)}_{t_0=0}=z_1-v_1t=z_2-v_2t_2=\cdots}$

Because:

As long as the value of argument (s) remains constant at $s=\tilde{s}$ (for point P) at different combinations of distance (z) and time (t) variables, we keep reading the same functional value $g_1(\tilde{s})=A$.

More generally, $\tilde{s} = z - vt$ for point P where $\tilde{s} = \text{constant}$

Compute total derivatives of both sides

$$\underbrace{d(\tilde{s})}_{0} = \underbrace{d(z - vt)}_{dz - d(vt)} \quad \text{where } v = \frac{1}{\sqrt{\mu\epsilon}} \text{ is a constant}$$

as $\tilde{s}$ is a constant

$$\Rightarrow \quad 0 = dz - v\,dt$$

$$\Rightarrow \quad \boxed{\frac{dz}{dt} = v = \frac{1}{\sqrt{\mu\epsilon}}}$$

Here, $\frac{dz}{dt}$ is a velocity term by definition.

* Therefore, $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the "velocity of propagation" of point P along the z-axis.

* The same discussion can be repeated for all other points of the wave form $g_1(z-vt) \Rightarrow$ The complete wave solution g_1 moves in $(+z)$ direction with velocity $\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$ as time increases preserving its shape.

* Note that we need to use the condition

$$\boxed{S = z \mp vt = \text{constant}}$$

to explain traveling wave behavior.

For $g_1(z-vt)$ (Forward traveling wave)

$$\boxed{z - vt = \text{constant}} \Rightarrow \boxed{\frac{dz}{dt} = v}$$

As time (t) increases, distance (z) must also increase to keep the value of $(z-vt)$ constant.

... $(+z)$ direction

For $g_2(z+vt)$ (Backward traveling wave)

$$\boxed{z + vt = \text{constant}} \Rightarrow \boxed{\frac{dz}{dt} = -v}$$

As t increases, z must decrease (because of the $(+)$ sign in between) to keep the value of $(z+vt)$ constant.

$\Rightarrow g_2(z+vt)$ travels in $(-z)$

(*) Check the unit of $\frac{1}{\sqrt{\epsilon\mu}} = v$!

$$\mu \leftrightarrow \frac{\text{Henry}}{\text{meter}} \equiv \frac{(\text{weber/Amp})}{\text{meter}}$$

$$\epsilon \leftrightarrow \frac{\text{Farad}}{\text{meter}} \equiv \frac{(\text{Coulomb/Volt})}{\text{meter}}$$

$$\mu\epsilon \leftrightarrow \frac{\overbrace{(\text{Weber})}^{\cancel{(\text{Volt})}(\text{sec})}\overbrace{(\text{Coulomb})}^{\cancel{(\text{Amp})}(\text{sec})}}{m^2\,\cancel{(\text{Amp})}\,\cancel{(\text{Volt})}} = \frac{(\text{sec})^2}{m^2}$$

$$\boxed{\frac{1}{\sqrt{\mu\epsilon}} \leftrightarrow \frac{1}{\sqrt{\frac{\text{sec}^2}{m^2}}} = \left(\frac{m}{\text{sec}}\right) : \text{unit of velocity indeed!}}$$

(*) In vacuum, $v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\times10^8$ m/sec : velocity of light in vacuum.

In other natural media: $\mu = \mu_0\mu_r$, $\epsilon = \epsilon_0\epsilon_r$

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0}\,\sqrt{\mu_r\epsilon_r}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} < c$$

$(\text{as } \mu_r \geqslant 1, \ \epsilon_r \geqslant 1)$

(*) The argument $(z \pm vt)$ of traveling waves can be expressed in a different but equivalent way:

$$g(z-vt) = g\Big(\underset{\text{a constant}}{-v}\big(t-\tfrac{z}{v}\big)\Big) \equiv f\left(t-\frac{z}{v}\right)$$

$g(z-vt)$, $f(t-\frac{z}{v})$ } forward waves traveling in $(+z)$ direction as time progresses	$g(z+vt)$, $f(t+\frac{z}{v})$ } backward waves traveling in $(-z)$ direction as time progresses

Special Case II : For a [SIMPLE ⊕ LOSSY ⊕ SOURCE FREE] medium.

$\underbrace{\text{LOSSY}}_{\sigma \neq 0}$ $\underbrace{\text{SOURCE FREE}}_{\bar{J}_{impressed}=0,\ \rho_v=0}$

As being different from the special case I, now we have $\sigma \neq 0$ in the lossy medium.

$$\sigma \neq 0 \Rightarrow \bar{J} = \underbrace{\cancel{\bar{J}_{impressed}}}_{\substack{0 \text{ as the} \\ \text{medium is} \\ \text{source-free}}} + \underbrace{\bar{J}_{conduction}}_{\sigma\bar{E}\neq 0} \Rightarrow \boxed{\bar{J} = \sigma \bar{E}}$$

must be inserted in general expressions!

For instance, $\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} = \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$ in this case.

Using the general inhomogeneous wave equations with $\begin{cases} \rho_v = 0 \\ \bar{J} = \sigma \bar{E} \end{cases}$

$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = \mu \frac{\partial \bar{J}}{\partial t} + \bar{\nabla}\Big(\overset{0}{\frac{\rho_v}{\epsilon}}\Big)$$

$$\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = \mu \frac{\partial}{\partial t}(\sigma \bar{E})$$

$$= \mu\sigma \frac{\partial \bar{E}}{\partial t}$$

or,

$$\boxed{\nabla^2 \bar{E} - \mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

$$\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = -\bar{\nabla} \times \bar{J}$$

$$\nabla^2 \bar{H} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = -\bar{\nabla} \times (\sigma \bar{E})$$

$$= -\sigma \underbrace{\bar{\nabla} \times \bar{E}}_{-\frac{\partial \bar{B}}{\partial t}}, \quad \bar{B} \to \mu \bar{H}$$

$$= \sigma\mu \frac{\partial \bar{H}}{\partial t}$$

$$\boxed{\nabla^2 \bar{H} - \mu\sigma \frac{\partial \bar{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{H}}{\partial t^2} = 0}$$

Partial differential equations describing ... propagation in a simple, source-free but