General Analysis of PLANE WAVES (propagating along $\hat{n}$)

(in a simple, lossless, source-free medium)

($\hat{n}$: unit vector in an arbitrary direction)

In such a medium, the homogeneous Helmholtz eqn. must be solved to get $\bar{E}$-phasor (or $\bar{H}$-phasor):

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0 \quad (\text{where } k = \omega\sqrt{\epsilon\mu})$$

$$\bar{E} = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z$$

(where $E_x = E_x(x,y,z)$, $E_y = E_y(x,y,z)$, $E_z = E_z(x,y,z)$) in general, in phasor domain.

yields

$$\nabla^2 (\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) + k^2 (\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z) = 0$$

$$\Rightarrow \hat{a}_x \underbrace{(\nabla^2 E_x + k^2 E_x)}_{0} + \hat{a}_y \underbrace{(\nabla^2 E_y + k^2 E_y)}_{0} + \hat{a}_z \underbrace{(\nabla^2 E_z + k^2 E_z)}_{0} = 0$$

zero vector!

$$\Rightarrow \left.\begin{aligned} &\nabla^2 E_x(x,y,z) + k^2 E_x(x,y,z) = 0 \\ &\nabla^2 E_y(x,y,z) + k^2 E_y(x,y,z) = 0 \\ &\nabla^2 E_z(x,y,z) + k^2 E_z(x,y,z) = 0 \end{aligned}\right\}$$

Three scalar Helmholtz Eqn. to be solved (possible in Cartesian coordinates)

Consider the solution of $\nabla^2 E_x + k^2 E_x = 0$.

(solutions for E_y and E_z will be similar)

$$\nabla^2 E_x(x,y,z) + k^2 E_x(x,y,z) = 0$$

Let $E_x(x,y,z) = f(x)\,g(y)\,h(z)$

Use the "separation of variables" technique!

$$\nabla^2 = \overline{\nabla}\cdot\overline{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{(in Cartesian coordinates only!)}$$

$$\Rightarrow \frac{\partial^2 \overbrace{E_x(x,y,z)}^{f(x)g(y)h(z)}}{\partial x^2} + \frac{\partial^2 E_x(x,y,z)}{\partial y^2} + \frac{\partial^2 E_x(x,y,z)}{\partial z^2} + k^2 E_x(x,y,z) = 0$$

$$\Rightarrow g(y)h(z)\frac{d^2 f(x)}{dx^2} + f(x)h(z)\frac{d^2 g(y)}{dy^2} + f(x)g(y)\frac{d^2 h(z)}{dz^2} +$$

$$+ k^2 f(x)g(y)h(z) = 0$$

(Divide both sides by $f(x)g(y)h(z)$)

$$\Rightarrow \boxed{\underbrace{\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}}_{\substack{\text{function of} \\ x\text{-only} \\ -k_x^2}} + \underbrace{\frac{1}{g(y)}\frac{d^2 g(y)}{dy^2}}_{\substack{\text{function of} \\ y\text{-only} \\ -k_y^2}} + \underbrace{\frac{1}{h(z)}\frac{d^2 h(z)}{dz^2}}_{\substack{\text{function of} \\ z\text{-only} \\ -k_z^2}} + \underbrace{k^2}_{\text{a constant}} = 0} \quad (*)$$

a constant $k^2 = \omega^2 \epsilon \mu > 0$

(for a given frequency $\omega = 2\pi f$, ϵ and μ.)

Eqn. (*) can be satisfied only if each of the terms on the left-hand side equals to some constants.

i.e.,

$$\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2} = -k_x^2 \Rightarrow \boxed{f'' + k_x^2 f = 0} \Rightarrow \boxed{f(x): \{ e^{\mp jk_x x} \}}$$

$$\frac{1}{g(y)}\frac{d^2 g(y)}{dy^2} = -k_y^2 \Rightarrow \boxed{g'' + k_y^2 g = 0} \Rightarrow \boxed{g(y): \{ e^{\mp jk_y y} \}}$$

$$\frac{1}{h(z)}\frac{d^2 h(z)}{dz^2} = -k_z^2 \Rightarrow \boxed{h'' + k_z^2 h = 0} \Rightarrow \boxed{h(z): \{ e^{\mp jk_z z} \}}$$

where k_x, k_y and k_z are "Separation Constants" satisfying the "Separation Condition" for the Helmholtz equation

$$\boxed{k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \epsilon}$$

(follows from eqn. (*))

Remember, $E_x(x,y,z) = f(x)\, g(y)\, h(z)$

(choosing solutions with (−) signs) $E_x(x,y,z) = A\, e^{-jk_x x}\, e^{-jk_y y}\, e^{-jk_z z}$

$$\Rightarrow \boxed{E_x(x,y,z) = A\, e^{-j(k_x x + k_y y + k_z z)}} \quad (**)$$

where A is an arbitrary constant.

Let's express this solution for $E_x(x,y,z)$ phasor in a compact way by defining the following vector:

Define $\boxed{\bar{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z}$ Propagation vector

$$|\bar{k}| = \sqrt{\bar{k} \cdot \bar{k}} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

But $k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \epsilon$ (from ✱)

$$\boxed{|\bar{k}| = k = \omega\sqrt{\mu\epsilon}}$$

Express $\bar{k}$ as $\bar{k} = \underbrace{|\bar{k}|}_{k} \hat{n}$ (where $\hat{n} = \frac{\bar{k}}{|\bar{k}|}$)

$\Rightarrow \boxed{\bar{k} = k\hat{n}}$ where $\hat{n}$: unit vector in the direction of propagation vector!

Also remember the definition of "position vector" $\bar{r}$ in Cartesian Coordinate system:

$$\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$
$$\bar{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z$$

$$\boxed{\bar{k} \cdot \bar{r} = k_x x + k_y y + k_z z}$$

which appears in the complex exponent of solution in ✱✱!

$$\Rightarrow \boxed{E_x(x,y,z) = A\, e^{-j(\bar{k} \cdot \bar{r})}} \quad \left(\text{with } \bar{k} = \hat{n}k = \hat{n}\omega\sqrt{\epsilon\mu} = \hat{n}\frac{\omega}{v}\right)$$

in phasor domain.

Solutions for $E_y(x,y,z)$ and $E_z(x,y,z)$ have the same functional form (they satisfy the same p.d.e.)

$$\Rightarrow \boxed{\begin{array}{l} E_y(x,y,z) = B\, e^{-j(\bar{k}\cdot\bar{r})} \\ \hline E_z(x,y,z) = C\, e^{-j(\bar{k}\cdot\bar{r})} \end{array}}$$

where B and C are some arbitrary constants.

Now, combine the components E_x, E_y and E_z to form the vector phasor $\bar{E}$ as

$$\bar{E}(x,y,z) = \hat{a}_x \underbrace{E_x(x,y,z)}_{A\, e^{-j\bar{k}\cdot\bar{r}}} + \hat{a}_y \underbrace{E_y(x,y,z)}_{B\, e^{-j\bar{k}\cdot\bar{r}}} + \hat{a}_z \underbrace{E_z(x,y,z)}_{C\, e^{-j\bar{k}\cdot\bar{r}}}$$

$$\bar{E}(x,y,z) = \underbrace{\left(\hat{a}_x A + \hat{a}_y B + \hat{a}_z C\right)}_{\text{call } \bar{E}_0 \,=\, \text{constant amplitude vector}} e^{-j\bar{k}\cdot\bar{r}}$$

$$\Rightarrow \boxed{\bar{E}(x,y,z) = \bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}}$$

Note that $\bar{E}(x,y,z) = \bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}$ phasor belongs to a uniform plane wave (u.p.w) solution because

constant phase surfaces

phase: $\angle \bar{E} = -\bar{k}\cdot\bar{r} = \text{constant}$

$\Rightarrow S(x,y,z) = \underbrace{(k_x x + k_y y + k_z z)}_{\text{equation of a plane!}} = \text{const.}$

$\Rightarrow$ plane wave (p.w) solution!

(Also, $\bar{\nabla} S(x,y,z) = \hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z \triangleq \bar{k} = k\hat{n}$ ($\perp$ to "const. phase planes"))

constant amplitude surfaces

$|\bar{E}| = |\bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}|$

$= |\bar{E}_0|\, \underbrace{|e^{-j\bar{k}\cdot\bar{r}}|}_{1}$ (complex exponential)

$|\bar{E}| = |\bar{E}_0| = \text{constant}$

(as $\bar{E}_0 = A\hat{a}_x + B\hat{a}_y + C\hat{a}_z$ const. vector)

$\Rightarrow$ Magnitude of $\bar{E}$ phasor is constant everywhere (independent of x, y, z) including the "constant phase planes". Therefore, the solution is a uniform plane wave (u.p.w.) indeed.

Fact: $\bar{E} \perp \hat{n}$ for this u.p.w. solution in a simple, lossless, source-free medium.

Proof: Start with $\bar{\nabla}\cdot\bar{D} = \underbrace{\rho_v}_{0}$ where $\bar{D} = \epsilon\bar{E}$ (ϵ a constant)

$\Rightarrow \bar{\nabla}\cdot(\epsilon\bar{E}) = \cancel{\epsilon}\,\bar{\nabla}\cdot\bar{E} = 0$ where $\bar{E} = \underbrace{\bar{E}_0}_{\text{a vector}}\, \underbrace{e^{-j\bar{k}\cdot\bar{r}}}_{\text{a scalar}}$

$\Rightarrow \bar{\nabla}\cdot(\bar{E}_0\, e^{-jkr}) = 0$ [Using $\bar{\nabla}\cdot(\underbrace{\bar{A}}_{\text{vector}}\, \underbrace{a}_{\text{scalar function}}) = (\bar{\nabla}\cdot\bar{A})a + \bar{A}\cdot(\bar{\nabla}a)$]

$\Rightarrow \underbrace{\cancel{(\bar{\nabla}\cdot\bar{E}_0)}}_{0}\, e^{-jkr} + \bar{E}_0\cdot\left[\bar{\nabla}(e^{-j\bar{k}\cdot\bar{r}})\right] = 0$

as $\bar{E}_0$ is a constant vector

where $\bar{\nabla}(e^{-j\bar{k}\cdot\bar{r}}) = \bar{\nabla}\left(e^{-j(k_x x + k_y y + k_z z)}\right)$

$$= \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) e^{-jk_x x - jk_y y - jk_z z}$$

$$= -j e^{-j\bar{k}\cdot\bar{r}} \underbrace{(\hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z)}_{\bar{k} \text{ (by definition)}}$$

$$\Rightarrow \boxed{\bar{\nabla}(e^{-j\bar{k}\cdot\bar{r}}) = -j\bar{k}\, e^{-j\bar{k}\cdot\bar{r}}}$$

Therefore,

$$\bar{E}_0 \cdot \bar{\nabla}(e^{-j\bar{k}\cdot\bar{r}}) = \bar{E}_0 \cdot [-j\bar{k}\, \cancel{e^{-j\bar{k}\cdot\bar{r}}}] = 0$$

$$\Rightarrow \boxed{\bar{E}_0 \cdot \bar{k} = 0} \quad \text{or} \quad \boxed{\bar{E}_0 \cdot \hat{n} = 0} \quad (\text{as } \bar{k} = k\hat{n})$$

That means $\boxed{\bar{E}_0 \perp \hat{n}}$ or $\boxed{\bar{E} \perp \hat{n}}$ $(\text{as } \bar{E} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}})$

$\therefore$ $\bar{E}$-vector is perpendicular to the direction of propagation!

Let's now obtain the $\bar{H}$-phasor corresponding to $\bar{E} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$

Using $\bar{\nabla} \times \bar{E} = -j\omega\mu\bar{H}$ in phasor domain ($\bar{B} = \mu\bar{H}$ in this case)

$$\Rightarrow \bar{H} = -\frac{1}{j\omega\mu}\bar{\nabla}\times\bar{E} = -\frac{1}{j\omega\mu}\underbrace{\bar{\nabla}\times[\bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}]}_{\underbrace{\cancel{(\bar{\nabla}\times\bar{E}_0)}}_{0} e^{-j\bar{k}\cdot\bar{r}} - \bar{E}_0 \times \underbrace{\bar{\nabla}(e^{-j\bar{k}\cdot\bar{r}})}_{-j\bar{k}e^{-j\bar{k}\cdot\bar{r}}}}$$

(using $\bar{\nabla}\times(\bar{A}a) = (\bar{\nabla}\times\bar{A})a - \bar{A}\times\bar{\nabla}a$)

Then, we have

$$\bar{H} = -\frac{1}{\cancel{j}\omega\mu}\left[\cancel{+j}\,\bar{E}_0 \times \bar{k}\; e^{-j\bar{k}\cdot\bar{r}}\right]$$

$\hookrightarrow \bar{k} = k\hat{n}$ and $k = \omega\sqrt{\mu\epsilon}$

$$\bar{H} = \frac{-1}{\omega\mu}\underbrace{(\bar{E}_0 \times \hat{n})}_{-\hat{n}\times\bar{E}_0}\,\omega\sqrt{\mu\epsilon}\; e^{-j\bar{k}\cdot\bar{r}} \qquad \left(\text{where } \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} = \sqrt{\frac{\epsilon}{\mu}}\right)$$

$$\bar{H} = \sqrt{\frac{\epsilon}{\mu}}\; \hat{n} \times \underbrace{\bar{E}_0\, e^{-j\bar{k}\cdot\bar{r}}}_{\bar{E}} \qquad \text{and} \qquad \sqrt{\frac{\epsilon}{\mu}} = \frac{1}{\eta}$$

$\rightarrow$ intrinsic impedance of the medium

$$\Rightarrow \boxed{\bar{H} = \frac{1}{\eta}\hat{n} \times \bar{E}} \quad \Rightarrow \quad \boxed{\bar{H} \perp \hat{n}} \text{ and } \boxed{\bar{H} \perp \bar{E}}$$

(because of the cross product)

$$\Downarrow \qquad\qquad \Downarrow$$

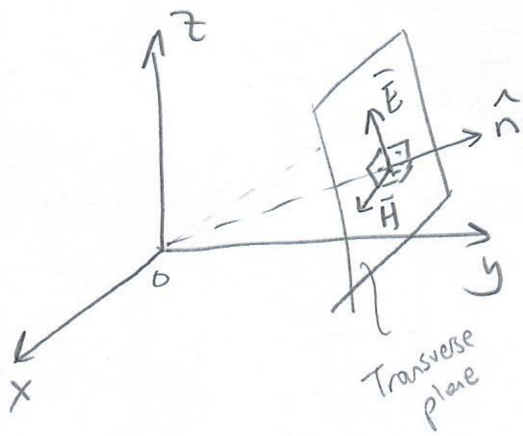
$$\boxed{\bar{H}\cdot\hat{n} = 0} \qquad \boxed{\bar{E}\cdot\bar{H} = 0}$$

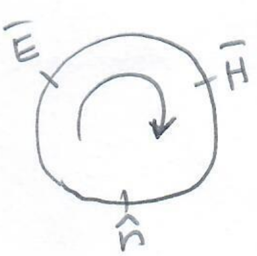
$\left(\text{with } \eta = \sqrt{\frac{\mu}{\epsilon}}\right)$

$\downarrow$

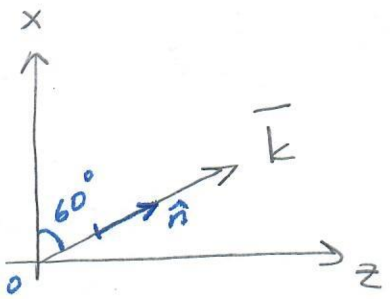
(in ohms)

<u>Conclusion:</u> $\bar{E}$, $\bar{H}$ and $\hat{n}$ vectors are <u>mutually</u> perpendicular to each other obeying the right hand cyclic relation.





Example-1: A u.p.w. (uniform plane wave) is propagating in free space with $f = 3$ GHz and its propagation vector $\bar{k}$ lies in the (x-z) plane making an angle of $60°$ with the x-axis. Express the $\bar{E}$ and $\bar{H}$ phasors for the u.p.w. mathematically.



free space $\Rightarrow$ (ϵ_0, μ_0), $\sigma = 0$, source-free case

u.p.w $\Rightarrow$ $\boxed{\bar{E} = \bar{E}_0 \, e^{-j\bar{k}\cdot\bar{r}}}$

where $\bar{E}_0$ is a constant vector (may be complex-valued)

and $\bar{k} = k\hat{n}$: propagation vector

$$k = |\bar{k}| = \omega\sqrt{\epsilon_0\mu_0} = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi \times 3\times10^9}{3\times10^8} = 20\pi \ (\text{rad/m})$$

$$\hat{n} = \frac{\bar{k}}{|\bar{k}|} = \frac{\bar{k}}{k}$$

$\hat{n}$: unit vector in the dir. of propagation

Here $\bar{k}$ and hence $\hat{n}$ lies on the (x,z) plane

$$\Rightarrow \hat{n} = \underbrace{n_x}_{\cos 60°}\hat{a}_x + \underbrace{n_z}_{\sin 60° = \cos 30°}\hat{a}_z$$

$$\Rightarrow \boxed{\hat{n} = \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z}$$

Check that $|\hat{n}| = 1$ ✓

$$\Rightarrow \boxed{\bar{k} = k\hat{n} = 20\pi\left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right) \ \text{rad/m}}$$

$$\Rightarrow \bar{k}\cdot\bar{r} = 20\pi\left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right)\cdot\left(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z\right)$$

$$\Rightarrow \boxed{\bar{k}\cdot\bar{r} = 10\pi\left(x + \sqrt{3}\,z\right) \ (\text{radian})}$$

So, we know $e^{-j\bar{k}\cdot\bar{r}}$ term now.

Next, we should try to determine $\overline{E}_0$ constant vector:

$$\overline{E}_0 = E_{0x}\hat{a}_x + E_{0y}\hat{a}_y + E_{0z}\hat{a}_z \quad \text{in general.}$$

For a u.p.w, we know that $\overline{E} \perp \overline{k}$

As $\overline{k} = k\hat{n} \Rightarrow \overline{E} \perp \hat{n} \Rightarrow \overline{E} \cdot \hat{n} = 0$

$$\Rightarrow \left(\overline{E}_0 e^{-j\overline{k}\cdot\overline{r}}\right) \cdot \hat{n} = 0 \quad \Rightarrow \boxed{\overline{E}_0 \cdot \hat{n} = 0} \text{ must hold.}$$

$$\Rightarrow \left(E_{0x}\hat{a}_x + E_{0y}\hat{a}_y + E_{0z}\hat{a}_z\right) \cdot \left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right) = 0$$

$$\Rightarrow \frac{1}{2}E_{0x} + \frac{\sqrt{3}}{2}E_{0z} = 0 \quad \Rightarrow \boxed{E_{0x} = -\sqrt{3}\,E_{0z}}$$

must be satisfied

$\boxed{E_{0y} \text{ can be chosen arbitrarily.}}$

Let $\left.\begin{matrix} E_{0z} = A \\ E_{0y} = B \end{matrix}\right\} \Rightarrow \boxed{\overline{E}_0 = -\sqrt{3}\,A\,\hat{a}_x + B\,\hat{a}_y + A\,\hat{a}_z}$

(Constants A and B could be determined if additional information was provided.)

$$\Rightarrow \boxed{\overline{E} = \overline{E}_0\, e^{-j\overline{k}\cdot\overline{r}} = \left(-\sqrt{3}A\,\hat{a}_x + B\,\hat{a}_y + A\,\hat{a}_z\right)e^{-j10\pi(x+\sqrt{3}z)}} \quad (V/m)$$

As we consider a u.p.w., $\overline{H}$ phasor can be found by using $\boxed{\overline{H} = \frac{1}{\eta}\,\hat{n}\times\overline{E}}$ where $\boxed{\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \text{ (ohms)}}$ in free space.

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$$\bar{H} = \frac{1}{120\pi} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ n_x & n_y & n_z \\ E_x & E_y & E_z \end{vmatrix} = \frac{e^{-j10\pi(x+\sqrt{3}z)}}{120\pi} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\sqrt{3}A & B & A \end{vmatrix}$$

$$\Rightarrow \boxed{\bar{H} = \frac{1}{120\pi}\left(-\frac{\sqrt{3}}{2}B\,\hat{a}_x - 2A\,\hat{a}_y + \frac{B}{2}\,\hat{a}_z\right) e^{-j10\pi(x+\sqrt{3}z)} \quad (A/m)}$$

Note: $\bar{H}$ may also be fund from the Maxwell's Law: $\bar{\nabla} \times \bar{E} = -j\omega\mu_0\bar{H}$ (Faraday's Law in phasor domain)

Do this as an exercise!

Example 3. If the given electromagnetic wave is a superposition of two or more uniform plane waves, you may proceed as follows: for example, let

$$\bar{E} = \hat{a}_x E_{01} e^{+jkz} + \hat{a}_y E_{02} e^{-jkx}$$

where $k = 2\pi f\sqrt{\epsilon\mu}$, E_{01} and E_{02} are some real constants

$$\bar{E}(\bar{r}) = \underbrace{\hat{a}_x E_{01} e^{jkz}}_{\bar{E}_1} + \underbrace{\hat{a}_y E_{02} e^{-jkx}}_{\bar{E}_2} = \bar{E}_1 + \bar{E}_2$$

first u.p.w. propagating in $\hat{n}_1 = -\hat{a}_z$ direction

Second u.p.w propagating in $\hat{n}_2 = +\hat{a}_x$ direction

$\bar{k}_1 \cdot \bar{r} = -kz$

$\Rightarrow \boxed{\bar{k}_1 = -k\hat{a}_z}$

$\bar{k}_2 \cdot \bar{r} = kx$

$\Rightarrow \boxed{\bar{k}_2 = k\hat{a}_x}$

(by fitting the exponential parts above to $e^{-j\bar{k}\cdot\bar{r}}$ form)

$$\bar{\mathcal{E}}(\bar{r},t) = \text{Real}\left\{\bar{E}(\bar{r})e^{j\omega t}\right\}$$

$$= \text{Re}\left\{\hat{a}_x E_{01} e^{j(\omega t+kz)} + \hat{a}_y E_{02} e^{j(\omega t-kx)}\right\}$$

$$\boxed{\bar{\mathcal{E}}(\bar{r},t) = \hat{a}_x E_{01}\cos(\omega t+kz) + \hat{a}_y E_{02}\cos(\omega t-kx)}$$

to keep $\omega t + kz$ = constant
as t increases $\Rightarrow$ z must decrease
$\Rightarrow$ propagation in $-\hat{a}_z$ dir.

$\omega t - kx$ = constant
as t increases $\Rightarrow$ x must also increase
$\Rightarrow$ propagation in $+\hat{a}_x$ dir.

To find $\bar{H}(\bar{r})$ corresponding to the given $\bar{E}(\bar{r})$,

(i) You may use $\bar{\nabla} \times \bar{E} = -j\omega\mu \bar{H}$ (Maxwell Eqn. in phasor domain)

$$\Rightarrow \bar{H} = \frac{1}{-j\omega\mu} \bar{\nabla} \times \left[\hat{a}_x E_{01} e^{jkz} + \hat{a}_y E_{02} e^{-jkx}\right] \quad (*)$$

(ii) Or, you may apply the $\bar{H} = \frac{1}{\eta} \hat{n} \times \bar{E}$ rule to each individual u.p.w separately. Namely, compute

$$\bar{H}_1 = \frac{1}{\eta} \hat{n}_1 \times \bar{E}_1$$

$$\bar{H}_1 = \frac{1}{\eta} (-\hat{a}_z) \times \hat{a}_x E_{01} e^{jkz}$$

$$\bar{H}_1 = -\hat{a}_y \frac{E_{01}}{\eta} e^{jkz}$$

$$\bar{H}_2 = \frac{1}{\eta} \hat{n}_2 \times \bar{E}_2$$

$$\bar{H}_2 = \frac{1}{\eta} \hat{a}_x \times \hat{a}_y E_{02} e^{-jkx}$$

$$\bar{H}_2 = \hat{a}_z \frac{E_{02}}{\eta} e^{-jkx}$$

Then superpose $\bar{H}_1$ and $\bar{H}_2$

$$\bar{H} = \bar{H}_1 + \bar{H}_2$$

$$\boxed{\bar{H} = -\hat{a}_y \frac{E_{01}}{\eta} e^{jkz} + \hat{a}_z \frac{E_{02}}{\eta} e^{-jkx}}$$

Check that this result can be obtained from (*) above after performing the curl operation and using $k = \omega\sqrt{\epsilon\mu}$.