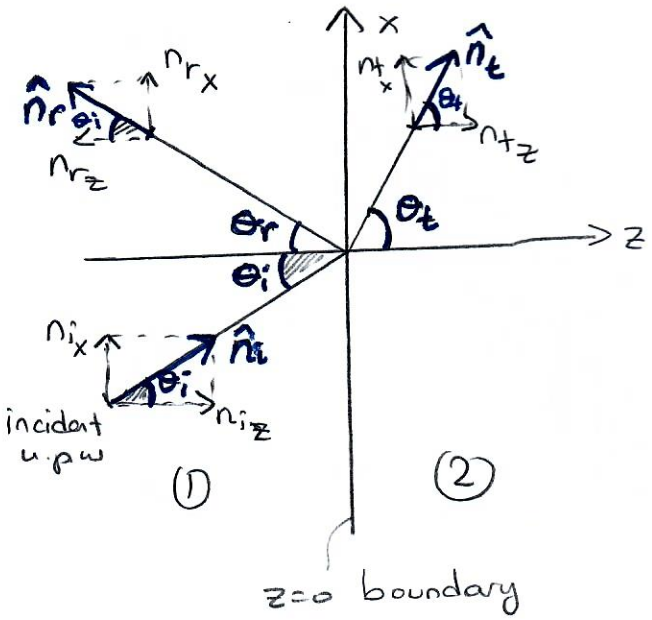
Example: Consider the following "oblique incidence" problem where the planar boundary is at $z=0$, and the medium parameters are given as

$\sigma_1 = 0, \; \mu_1 = \mu_0, \; \epsilon_1 = 6\epsilon_0$

$\sigma_2 = 0, \; \mu_2 = \mu_0, \; \epsilon_2 = 2\epsilon_0$



For a given incident $\bar{E}^i$ phasor of a uniform plane wave (upw), determine:

(a) The angles $\theta_i, \theta_r, \theta_t$.

(b) Reflection and transmission coefficients $\Gamma_{\parallel}, T_{\parallel}, \Gamma_{\perp}, T_{\perp}$.

(c) Reflected $\bar{E}^r$ phasor

(d) Transmitted $\bar{E}^t$ phasor

Solution: The $\bar{E}^i$ phasor of a u.p.w. is given as:

$$\bar{E}^i(x,z) = \left(\frac{\sqrt{3}}{2}\hat{a}_x + \hat{a}_y - \frac{1}{2}\hat{a}_z\right) e^{-jk_1\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)} = \bar{E}_1 e^{-j\bar{k}_i \cdot \bar{r}}$$

(a) $$e^{-j\bar{k}_i\cdot\bar{r}} = e^{-jk_1\hat{n}_i\cdot\bar{r}} = e^{-jk_1(\sin\theta_i\hat{a}_x + \cos\theta_i\hat{a}_z)\cdot(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)}$$

$$= e^{-jk_1(\sin\theta_i x + \cos\theta_i z)} = e^{-jk_1\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$$

$$\Rightarrow \sin\theta_i = \frac{1}{2}, \; \cos\theta_i = \frac{\sqrt{3}}{2} \Longrightarrow \boxed{\theta_i = 30^\circ}$$

Snell's Law of Reflection : $\theta_r = \theta_i \Rightarrow \boxed{\theta_r = 30^\circ}$

Snell's Law of Refraction : $k_1 \sin\theta_i = k_2 \sin\theta_t \Rightarrow \sin\theta_t = \frac{k_1}{k_2}\underbrace{\sin\theta_i}_{1/2}$

$$\Rightarrow \sin\theta_t = \frac{\cancel{\omega}\sqrt{\cancel{\mu_0}6\cancel{\epsilon_0}}}{\cancel{\omega}\sqrt{\cancel{\mu_0}2\cancel{\epsilon_0}}}\;\frac{1}{2} = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta_t = 60^\circ}$$

(b) For parallel polarization:

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

where $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{6\epsilon_0}} = \frac{120\pi}{\sqrt{6}}$ (Ω)

$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{2\epsilon_0}} = \frac{120\pi}{\sqrt{2}}$ (Ω)

$$= \frac{\frac{\cancel{120\pi}}{\sqrt{2}}\frac{1}{\cancel{2}} - \frac{\cancel{120\pi}}{\sqrt{6}}\frac{\sqrt{3}}{\cancel{2}}}{\frac{\cancel{120\pi}}{\sqrt{2}}\frac{1}{\cancel{2}} + \frac{\cancel{120\pi}}{\sqrt{6}}\frac{\sqrt{3}}{\cancel{2}}} = \frac{\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{2}}}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = 0 \Rightarrow \boxed{\Gamma_{\parallel} = 0}$$

As $1 + \Gamma_{\parallel} = T_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right) \Rightarrow 1 + 0 = T_{\parallel}\frac{\cos 60^\circ}{\cos 30^\circ} = T_{\parallel}\frac{\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{1}{\sqrt{3}}$

$$\Rightarrow \boxed{T_{\parallel} = \sqrt{3}}$$

For perpendicular polarization:

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} = \frac{\frac{\cancel{120\pi}}{\sqrt{2}}\frac{\sqrt{3}}{\cancel{2}} - \frac{\cancel{120\pi}}{\sqrt{6}}\frac{1}{\cancel{2}}}{\frac{\cancel{120\pi}}{\sqrt{2}}\frac{\sqrt{3}}{\cancel{2}} + \frac{\cancel{120\pi}}{\sqrt{6}}\frac{1}{\cancel{2}}} = \frac{\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{6}}}{\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{6}}}$$

$$= \frac{\frac{\sqrt{3}\sqrt{3}}{\cancel{\sqrt{6}}} - \frac{1}{\cancel{\sqrt{6}}}}{\frac{\sqrt{3}\sqrt{3}}{\cancel{\sqrt{6}}} + \frac{1}{\cancel{\sqrt{6}}}} = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2} \Rightarrow \boxed{\Gamma_{\perp} = \frac{1}{2}}$$

As $T_{\perp} = 1 + \Gamma_{\perp} = 1 + \frac{1}{2} = \frac{3}{2} \Rightarrow \boxed{T_{\perp} = \frac{3}{2}}$

(c) For the incident u.p.w, $\bar{E}^i = \underbrace{\bar{E}_1}_{\bar{E}_{1\parallel} + \bar{E}_{1\perp}} e^{-j\bar{k}_i \cdot \bar{r}} = (\bar{E}_{1\parallel} + \bar{E}_{1\perp}) e^{-jk_1 \hat{n}_i \cdot \bar{r}}$

$$\bar{E}_1 = \frac{\sqrt{3}}{2}\hat{a}_x + \hat{a}_y - \frac{1}{2}\hat{a}_z = \underbrace{\left(\frac{\sqrt{3}}{2}\hat{a}_x - \frac{1}{2}\hat{a}_z\right)}_{\parallel \text{ components}} + \underbrace{\hat{a}_y}_{\perp \text{ component}}$$

with respect to the plane of incidence (POI) which is the (x-z) plane here!

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Therefore,
$$\begin{cases} \bar{E}_{1_{/\!/}} = \frac{\sqrt{3}}{2}\hat{a}_x - \frac{1}{2}\hat{a}_z = \underbrace{(1)}_{E_{1/\!/}}\underbrace{\left(\frac{\sqrt{3}}{2}\hat{a}_x - \frac{1}{2}\hat{a}_z\right)}_{\hat{u}_{1_{/\!/}} = \text{unit vector}} \\ \bar{E}_{1\perp} = \hat{a}_y = \underbrace{(1)}_{E_{1\perp}}\underbrace{\hat{a}_y}_{\hat{u}_{1\perp} = \text{unitvector}} \end{cases}$$

Then, $\bar{E}^r = \bar{E}_2\, e^{-j\bar{k}_r \cdot \bar{r}} = (\bar{E}_{2_{/\!/}} + \bar{E}_{2_\perp})\, e^{-jk_1\hat{n}_r \cdot \bar{r}}$

As $\Gamma_{/\!/} = 0 \Rightarrow \bar{E}_{2_{/\!/}} = 0 \Rightarrow \bar{E}^r$ has only ($\perp$) component.

$\bar{E}_2 = \bar{E}_{2_\perp} = E_{2\perp}\underbrace{\hat{u}_{2\perp}}_{=\hat{a}_y}$ where $E_{2\perp} = \underbrace{\Gamma_\perp}_{1/2}\underbrace{E_{1\perp}}_{1} \Rightarrow \boxed{E_{2\perp} = \frac{1}{2}}$

$\Rightarrow \boxed{\bar{E}_2 = \frac{1}{2}\hat{a}_y}$ and $\bar{E}^r = \bar{E}_2\, e^{-jk_1\hat{n}_r \cdot \bar{r}}$

$$\hat{n}_r = \underbrace{\sin\theta_r}_{\sin 30^\circ = \frac{1}{2}}\hat{a}_x - \underbrace{\cos\theta_r}_{\cos 30^\circ = \frac{\sqrt{3}}{2}}\hat{a}_z = \frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z$$

$\Rightarrow \boxed{\hat{n}_r \cdot \bar{r} = \frac{1}{2}x - \frac{\sqrt{3}}{2}z} \Rightarrow \boxed{\bar{E}^r = \hat{a}_y \frac{1}{2}\, e^{-jk_1\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)}}$

(d) For the transmitted field, we have

$$\bar{E}^t = \bar{E}_3\, e^{-j\bar{k}_t \cdot \bar{r}} = (\bar{E}_{3_{/\!/}} + \bar{E}_{3\perp})\, e^{-jk_2\hat{n}_t \cdot \bar{r}}$$

where $\hat{n}_t = \underbrace{\sin\theta_t}_{\sin 60^\circ = \frac{\sqrt{3}}{2}}\hat{a}_x + \underbrace{\cos\theta_t}_{\cos 60^\circ = \frac{1}{2}}\hat{a}_z \Rightarrow \boxed{\hat{n}_t = \frac{\sqrt{3}}{2}\hat{a}_x + \frac{1}{2}\hat{a}_z}$

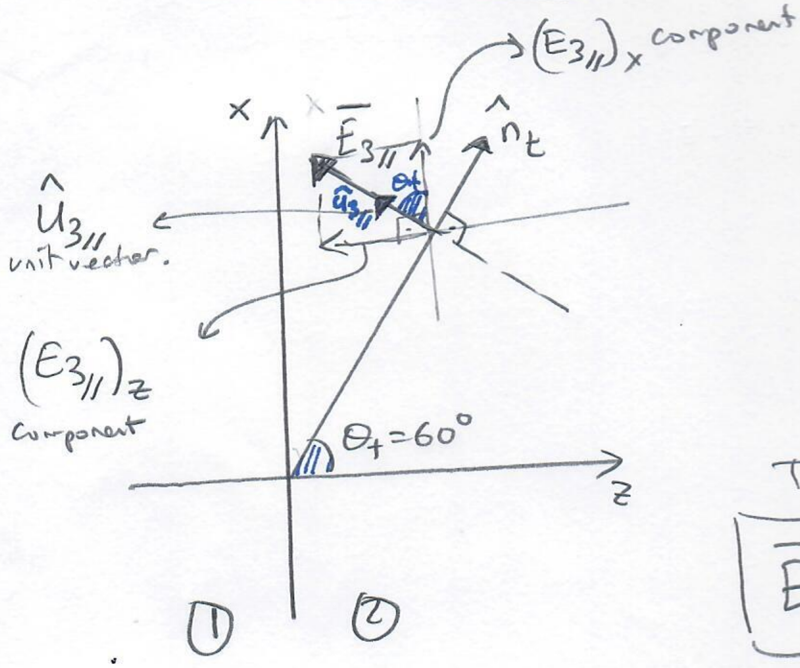
$\Rightarrow \boxed{\hat{n}_t \cdot \bar{r} = \frac{\sqrt{3}}{2}x + \frac{1}{2}z}$

$$\bar{E}_{3\perp} = E_{3\perp}\, \underbrace{\hat{u}_{3\perp}}_{\hat{a}_y \text{ unit vector}} \quad \text{where} \quad E_{3\perp} = \underbrace{T_\perp}_{3/2}\, \underbrace{E_{1\perp}}_{1} = \frac{3}{2}$$

$$\Rightarrow \boxed{\bar{E}_{3\perp} = \frac{3}{2}\hat{a}_y}$$

$$\bar{E}_{3\parallel} = E_{3\parallel}\, \underbrace{\hat{u}_{3\parallel}}_{\text{unit vector in the direction of } \bar{E}_{3\parallel}} \quad \text{where} \quad E_{3\parallel} = \underbrace{T_\parallel}_{\sqrt{3}}\, \underbrace{E_{1\parallel}}_{1} = \sqrt{3}$$

$$\Rightarrow \boxed{\bar{E}_{3\parallel} = \sqrt{3}\,\hat{u}_{3\parallel}}$$



Here $\hat{u}_{3\parallel} = \hat{a}_x \underbrace{\cos\theta_t}_{\cos 60°} - \hat{a}_z \underbrace{\sin\theta_t}_{\sin 60°}$

$$\boxed{\hat{u}_{3\parallel} = \frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z}$$

Then,

$$\boxed{\bar{E}_{3\parallel} = \sqrt{3}\left(\frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z\right)}$$

Then, $\bar{E}^t = (\bar{E}_{3\parallel} + \bar{E}_{3\perp})\, e^{-jk_2\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}z\right)}$

$$\boxed{\bar{E}^t = \left(\frac{\sqrt{3}}{2}\hat{a}_x + \frac{3}{2}\hat{a}_y - \frac{3}{2}\hat{a}_z\right) e^{-jk_2\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}z\right)}}$$

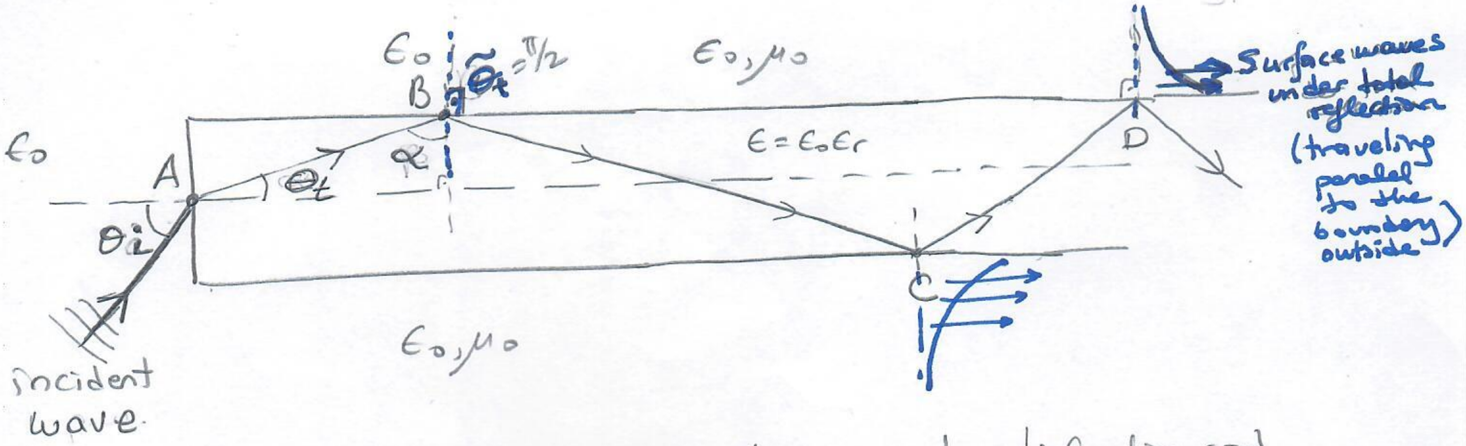
Finally, check if $\left(\bar{E}^i_{tang} + \bar{E}^r_{tang}\right)\Big|_{z=0} \stackrel{?}{=} \bar{E}^t_{tang}\Big|_{z=0}$ (x and y components are tangential wrt z=0 boundary)

$$\left(\frac{\sqrt{3}}{2}\hat{a}_x + \hat{a}_y\right) e^{-jk_1\frac{x}{2}} + \frac{1}{2}\hat{a}_y\, e^{-jk_1 x/2} \stackrel{?}{=} \left(\frac{\sqrt{3}}{2}\hat{a}_x + \frac{3}{2}\hat{a}_y\right) e^{-jk_2\frac{\sqrt{3}}{2}x}$$

As $k_1 \sin\theta_i = k_2 \sin\theta_t \Rightarrow \frac{k_1}{2} = k_2\frac{\sqrt{3}}{2} \Rightarrow$ exponentials are cancelled and equality is shown!

BC is satisfied

Example: Consider a dielectric rod with $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_0$, $\sigma = 0$.
Assume the outside medium is air. (optical fiber problem)



at point A : wave enters from air to dielectric rod

Apply Snell's Law $\quad \sqrt{\epsilon_0 \mu_0}\,\sin\theta_i = \sqrt{\epsilon_0 \epsilon_r \mu_0}\,\sin\theta_t$

$$\Rightarrow \boxed{\sin\theta_t = \frac{1}{\sqrt{\epsilon_r}}\sin\theta_i} \quad (1)$$

at points B, C, D, etc., "total reflection" must occur to keep the wave escaping from the rod into air!

So, we need $\quad \alpha \geq \theta_c \quad$ where $\quad \boxed{\sin\theta_c = \frac{1}{\sqrt{\epsilon_r}}} \quad (2)$
$(\sin\alpha \geq \sin\theta_c)$

(Apply Snell's Law for $\alpha = \theta_c$, $\tilde{\theta}_t = \frac{\pi}{2}$ $\Rightarrow \sqrt{\mu_0 \epsilon_0 \epsilon_r}\,\sin\theta_c = \sqrt{\mu_0 \epsilon_0}\,\underbrace{\sin\frac{\pi}{2}}_{1}$)

But, $\alpha = \frac{\pi}{2} - \theta_t \Rightarrow \sin\alpha = \sin\left(\frac{\pi}{2} - \theta_t\right) = \cos\theta_t$

For "total reflection", we need $\quad \boxed{\sin\alpha = \cos\theta_t \geq \sin\theta_c = \frac{1}{\sqrt{\epsilon_r}}}$

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} \overset{\text{use (1)}}{=} \sqrt{1 - \left(\frac{1}{\sqrt{\epsilon_r}}\sin\theta_i\right)^2} \geq \frac{1}{\sqrt{\epsilon_r}}$$

$$\Rightarrow 1 - \left(\frac{1}{\sqrt{\epsilon_r}}\sin\theta_i\right)^2 \geq \left(\frac{1}{\sqrt{\epsilon_r}}\right)^2 \Rightarrow \epsilon_r - \sin^2\theta_i \geq 1$$

$$\Rightarrow \boxed{\epsilon_r \geq 1 + \sin^2\theta_i}$$

is needed to guide EM wave within the rod!

or

maximum acceptance angle $\leftarrow \boxed{\theta_{i_{max}} = \sin^{-1}\left(\sqrt{\epsilon_r - 1}\right)} \quad \Leftarrow$ or