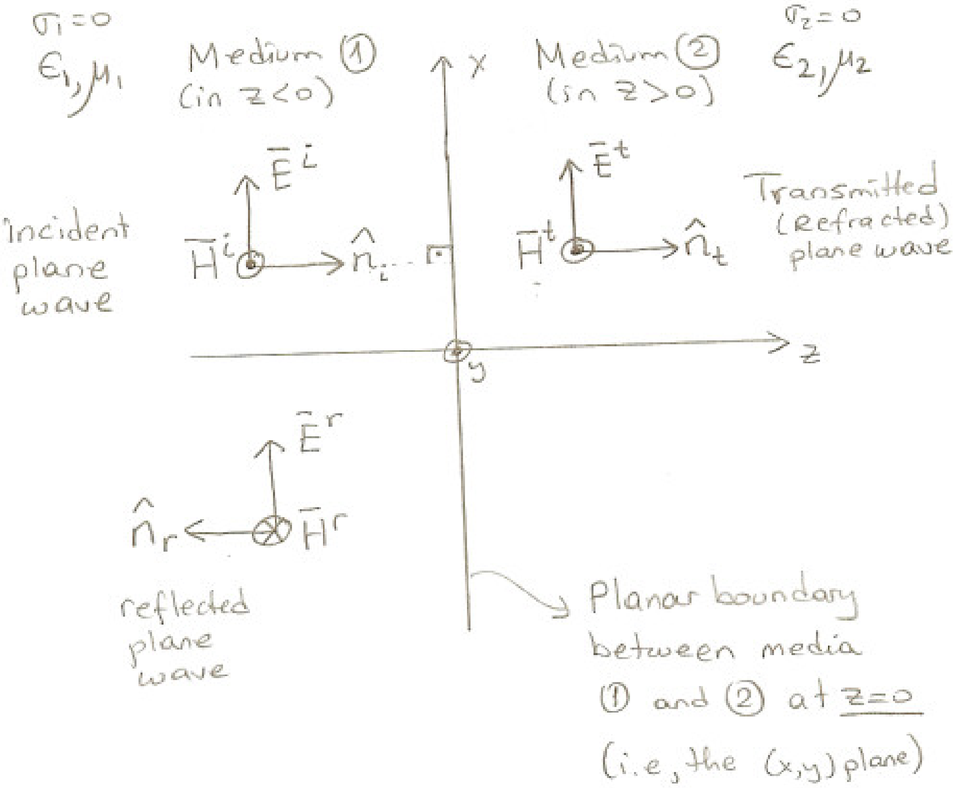
Reflection and Transmission (Refraction) of Plane Waves Under Normal Incidence

We will examine the behavior of <u>plane waves</u> when they hit a <u>planar boundary</u> between two different media under <u>normal incidence</u> (i.e, $\hat{n}_i$: propagation direction of the incident plane wave is perpendicular to the boundary plane).

Case 1: Perfect Dielectric / Perfect Dielectric Boundary



$k_1 = \omega\sqrt{\mu_1\epsilon_1}$, $k_2 = \omega\sqrt{\mu_2\epsilon_2}$ — wave numbers in ① and ②

$\eta_1 = \sqrt{\dfrac{\mu_1}{\epsilon_1}}$, $\eta_2 = \sqrt{\dfrac{\mu_2}{\epsilon_2}}$ — Intrinsic impedances in ① and ②

Note that directions of $\hat{n}_i$, $\hat{n}_r$ and $\hat{n}_t$ are all <u>perpendicular</u> to the boundary plane in this "<u>Normal Incidence</u>" problem.

Using the relation

$$\boxed{\bar{H} = \frac{1}{\eta}\hat{n} \times \bar{E}}$$

($\bar{E}, \bar{H}$: phasor fields, $\hat{n}$: propagation direction)

which is <u>valid for plane waves</u>, we can write the mathematical expressions of $\bar{H}^i$, $\bar{H}^r$ and $\bar{H}^t$ for assumed $\bar{E}^i$, $\bar{E}^r$ and $\bar{E}^t$ phasors.

Note that in this case, both media 1 and 2 are perfect (i.e. lossless) dielectrics. Therefore, we have

$$\left.\begin{aligned} \gamma_1 &= \underbrace{\alpha_1}_{0} + j\underbrace{\beta_1}_{k_1} = jk_1 = j\omega\sqrt{\epsilon_1\mu_1} \\ \eta_1 &= \sqrt{\frac{j\omega\mu_1}{\underset{\text{zero}}{\sigma_1} + j\omega\epsilon_1}} = \sqrt{\frac{\mu_1}{\epsilon_1}} \end{aligned}\right\} \text{in medium } 1$$

and similarly,

$$\left.\begin{aligned} \gamma_2 &= jk_2 = j\omega\sqrt{\epsilon_2\mu_2} \\ \eta_2 &= \sqrt{\frac{\mu_2}{\epsilon_2}} \end{aligned}\right\} \text{in medium } 2$$

Assume that the incident plane wave is linearly polarized along the x-axis and it is sent from medium 1 propagating in $\hat{n}_i = \hat{a}_z$ direction. So, it hits the plane boundary at $z=0$ perpendicularly and has the following phasor fields, $\bar{E}^i$ and $\bar{H}^i$:

$$\bar{E}^i = \hat{a}_x E_1 e^{-\gamma_1 \hat{n}_i \cdot \bar{r}} = \hat{a}_x E_1 e^{-jk_1 \hat{a}_z \cdot \bar{r}} = \boxed{\hat{a}_x E_1 e^{-jk_1 z} = \bar{E}^i}$$

and $$\bar{H}^i = \frac{1}{\eta_1} \hat{n}_i \times \bar{E}^i = \frac{1}{\eta_1} \underbrace{\hat{a}_z \times \hat{a}_x}_{\hat{a}_y} E_1 e^{-jk_1 z} = \boxed{\hat{a}_y \frac{E_1}{\eta_1} e^{-jk_1 z} = \bar{H}^i}$$

where $E_1 = |\bar{E}^i|$ is a known constant. ($\bar{k}_i = k_1 \hat{a}_z$: incident propagation vector)

Also, k_1 and η_1 must be used as the incident p.w. exists (by definition) in medium 1.

(*) When this incident p.w. hits the planar boundary, reflected (back to medium (1)) and transmitted (into medium (2)) plane waves are created with propagation vectors:

$$\bar{k}_r = \underbrace{|\bar{k}_r|}_{k_1}\underbrace{\hat{n}_r}_{-\hat{a}_z} = -\hat{a}_z k_1 \quad (\text{i.e. } \hat{n}_r = -\hat{a}_z)$$ for the reflected p.w. in medium (1)

and

$$\bar{k}_t = \underbrace{|\bar{k}_t|}_{k_2}\underbrace{\hat{n}_t}_{\hat{a}_z} = \hat{a}_z k_2 \quad (\text{i.e. } \hat{n}_t = +\hat{a}_z)$$ for the transmitted p.w. in medium (2).

Then, we have the phasor electric and magnetic fields as:

$$\bar{E}^r = \hat{a}_x E_2 e^{-j\bar{k}_r \cdot \bar{r}} = \hat{a}_x E_2 e^{+jk_1 z}$$

and

$$\bar{E}^t = \hat{a}_x E_3 e^{-j\bar{k}_t \cdot \bar{r}} = \hat{a}_x E_3 e^{-jk_2 z}$$

where E_2 and E_3 are unknown constants.

Also,

$$\bar{H}^r = \frac{1}{\eta_1}\hat{n}_r \times \bar{E}^r = \frac{1}{\eta_1}\underbrace{(-\hat{a}_z)\times\hat{a}_x}_{-\hat{a}_y} E_2 e^{+jk_1 z} = -\hat{a}_y \frac{E_2}{\eta_1} e^{+jk_1 z}$$

and

$$\bar{H}^t = \frac{1}{\eta_2}\hat{n}_t \times \bar{E}^t = \frac{1}{\eta_2}\underbrace{(+\hat{a}_z)\times\hat{a}_x}_{+\hat{a}_y} E_3 e^{-jk_2 z} = \hat{a}_y \frac{E_3}{\eta_2} e^{-jk_2 z}$$

To summarize:

in medium ①
$$\bar{E}^i = \hat{a}_x E_1 e^{-jk_1 z} \longleftrightarrow \bar{H}^i = \hat{a}_y \frac{E_1}{\eta_1} e^{-jk_1 z}$$
$$\bar{E}^r = \hat{a}_x E_2 e^{jk_1 z} \longleftrightarrow \bar{H}^r = -\hat{a}_y \frac{E_2}{\eta_1} e^{jk_1 z}$$

in medium ②
$$\bar{E}^t = \hat{a}_x E_3 e^{-jk_2 z} \longleftrightarrow \bar{H}^t = \hat{a}_y \frac{E_3}{\eta_2} e^{-jk_2 z}$$

Define:
$$\boxed{\Gamma = \frac{E_2}{E_1}} \text{ : reflection coefficient}$$
$$\boxed{T = \frac{E_3}{E_1}} \text{ : transmission coefficient}$$

* To find Γ and T, we need to express E_2 and E_3 in terms of E_1, which can be done by using the Boundary Conditions (BC's) at $z=0$ boundary plane:

B.C. #1: Tangential component of the total $\bar{E}$ field must be continuous at the $z=0$ plane.

$$E^{total}_{tang ①} = E^{total}_{tang ②}$$

Note that, in this case $\bar{E}^i, \bar{E}^r, \bar{E}^t$ have only x-components which are obviously tangential to the boundary.

$$\Rightarrow \left(E^i_{tang} + E^r_{tang}\right)\Big|_{at\ z=0} = \left(E^t_{tang}\right)\Big|_{at\ z=0}$$

$$\left(E_1 e^{-jk_1 z} + E_2 e^{jk_1 z}\right)\Big|_{z=0} = E_3 e^{-jk_2 z}\Big|_{z=0}$$

$$\Rightarrow \boxed{E_1 + E_2 = E_3} \quad (1)$$

B.C. #2 As we have two perfect dielectric media, we know that $\bar{J}_s = 0$ at $z=0$. Then, the tangential component of the total $\bar{H}$ field must be continuous across the boundary $z=0$.

$$H^{total}_{tang_{①}} = H^{total}_{tang_{②}}$$

Note that, in this problem, $\bar{H}$ phasors have only y-components which are obviously tangential to the boundary.

$$\Rightarrow \left(\frac{E_1}{\eta_1} e^{-jk_1 z} - \frac{E_2}{\eta_1} e^{jk_1 z} \right) \Bigg|_{z=0} = \frac{E_3}{\eta_2} e^{-jk_2 z} \Bigg|_{z=0}$$

$$\Rightarrow \boxed{\frac{E_1}{\eta_1} - \frac{E_2}{\eta_1} = \frac{E_3}{\eta_2}} \quad (2)$$

Solving equations (1) and (2) simultaneously, we can get expressions for reflection (Γ) and transmission (τ) coefficients as:

from (1) $\longrightarrow$ $E_1 + E_2 = E_3$ (Divide both sides by E_1)

$$\Rightarrow 1 + \frac{E_2}{E_1} = \frac{E_3}{E_1} \Longrightarrow \boxed{1 + \Gamma = T}$$

Solve for Γ and T

from (2) $\longrightarrow$ (divide both sides by E_1)

$$\frac{1}{\eta_1} - \frac{1}{\eta_1} \cdot \frac{E_2}{E_1} = \frac{1}{\eta_2} \frac{E_3}{E_1} \Rightarrow \boxed{\frac{1}{\eta_1}(1-\Gamma) = \frac{1}{\eta_2} T}$$

$$\frac{1}{\eta_1}(1-\Gamma) = \frac{1}{\eta_2}(1+\Gamma) \Longrightarrow \boxed{\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1}}$$

Then, $T = 1+\Gamma \Longrightarrow \boxed{T = \frac{2\eta_2}{\eta_2+\eta_1}}$

where $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ and $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$ in these lossless media.

Total fields and Time-Averaged Power Density Vectors in Media 1 and 2

$$\bar{E}_{1} = \bar{E}^i + \bar{E}^r = \hat{a}_x E_1 e^{-jk_1 z} + \hat{a}_x \underbrace{E_2}_{\Gamma E_1} e^{+jk_1 z}$$

(Remember $\Gamma = \frac{E_2}{E_1}$)

Total $\bar{E}$-phasor in medium 1 $\boxed{\bar{E}_{1} = \hat{a}_x E_1 \left(e^{-jk_1 z} + \Gamma e^{+jk_1 z} \right)}$

$$\bar{H}_{1} = \bar{H}^i + \bar{H}^r = \hat{a}_y \frac{E_1}{\eta_1} e^{-jk_1 z} - \hat{a}_y \frac{E_2}{\eta_1} e^{+jk_1 z}$$

(set $E_2 = \Gamma E_1$)

Total $\bar{H}$-phasor in medium 1 $\boxed{\bar{H}_{1} = \hat{a}_y \frac{E_1}{\eta_1} \left(e^{-jk_1 z} - \Gamma e^{+jk_1 z} \right)}$

Then, the Time-Averaged Poynting's vector $\bar{P}_{av_{1}}$ becomes:

$$\bar{P}_{av_{1}} = \frac{1}{2} \mathrm{Re}\left\{ \bar{E}_{1} \times \bar{H}^*_{1} \right\}$$

complex conjugate

$$= \frac{1}{2} \mathrm{Re}\left\{ E_1 \left(e^{-jk_1 z} + \Gamma e^{+jk_1 z} \right) \underbrace{\hat{a}_x \times \hat{a}_y}_{\hat{a}_z} \left[\frac{E_1}{\eta_1} \left(e^{-jk_1 z} - \Gamma e^{jk_1 z} \right) \right]^* \right\}$$

$$\frac{1}{2} \hat{a}_z \mathrm{Re}\left\{ E_1 \left(e^{-jk_1 z} + \Gamma e^{jk_1 z} \right) \frac{E_1^*}{\eta_1^*} \left(e^{jk_1 z} - \Gamma^* e^{-jk_1 z} \right) \right\}$$

Remember that $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ and $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$ are <u>real</u> quantities for these lossless media, i.e., $\eta_1^* = \eta_1$, $\eta_2^* = \eta_2$

$$\Rightarrow \Gamma^* = \frac{\eta_2^* - \eta_1^*}{\eta_2^* + \eta_1^*} = \Gamma \text{ (real)} \Rightarrow \Gamma\Gamma^* = \Gamma^2 \text{ (real)}$$

Hence,

$$\bar{P}_{av_{1}} = \hat{a}_z \frac{1}{2\eta_1} Re\left\{ \underbrace{E_1 E_1^*}_{|E_1|^2} \left(1 + \underbrace{\Gamma e^{j2k_1 z} - \Gamma e^{-j2k_1 z}}_{\Gamma(\underbrace{e^{j2k_1 z} - e^{-j2k_1 z}}_{2j\sin(2k_1 z)})} - \Gamma^2 \right) \right\}$$

$$\bar{P}_{av_{1}} = \hat{a}_z \frac{|E_1|^2}{2\eta_1} Re\left\{ \underbrace{1 - \Gamma^2}_{\text{Real part}} + j \underbrace{2\sin(2k_1 z)}_{\text{Imag. part}} \right\}$$

$$\boxed{\bar{P}_{av_{1}} = \hat{a}_z \underbrace{\frac{1}{2} \frac{|E_1|^2}{\eta_1} (1 - \Gamma^2)}_{\geq 0} \ (W/m^2)}$$ where $|\Gamma| \leq 1$

Time-Avg. Poynting's vector in medium 1 (i.e. for the <u>total</u> field in medium 1)

≥ 0 means <u>positive</u> power flow is towards the boundary (in $+\hat{a}_z$ direction) in medium 1.

<u>Also in medium 2</u> $\bar{P}_{av_{2}} = \frac{1}{2} Re\left\{ \bar{E}^t \times (\bar{H}^t)^* \right\}$ can be computed.

As both media are <u>lossless</u>, we must have

$\bar{P}_{av_{1}} = \bar{P}_{av_{2}}$ in this problem (<u>Prove</u> as an exercise.)

Note that time-averaged Poynting's vectors for the reflected and incident plane waves can be computed individually:

$$\bar{P}^{i}_{av} = \frac{1}{2} Re\left\{ \bar{E}^{i} \times (\bar{H}^{i})^{*} \right\} = \underbrace{\frac{1}{2}\frac{|E_1|^2}{\eta_1}}_{P^{i}_{av}} \hat{a}_z = P^{i}_{av}\, \hat{a}_z$$

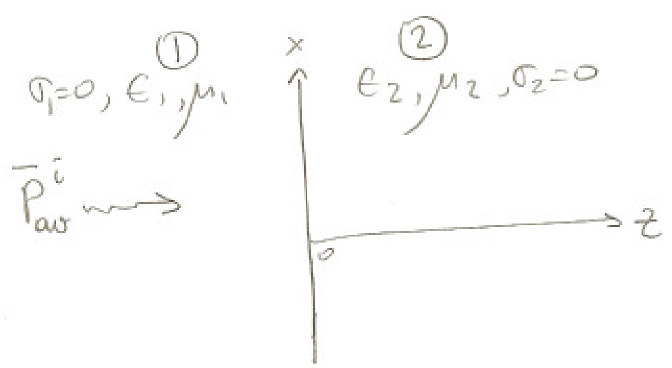
and

$$\bar{P}^{r}_{av} = \frac{1}{2} Re\left\{ \bar{E}^{r} \times (\bar{H}^{r})^{*} \right\} = \underbrace{\frac{1}{2}\frac{\Gamma^2 |E_1|^2}{\eta_1}}_{P^{r}_{av}} (-\hat{a}_z) = P^{r}_{av}(-\hat{a}_z)$$

Note that $\boxed{P^{r}_{av} = \Gamma^2 P^{i}_{av}}$

Also, $\bar{P}^{i}_{av} + \bar{P}^{r}_{av} = \bar{P}_{av_{1}}$ (This relation is valid in normal incidence problem only!)

Example: For the following perfect dielectric / perfect dielectric problem (boundary at $z=0$), the incident power flow density $\bar{P}^{i}_{av}$ is given to be $\bar{P}^{i}_{av} = 150\,\hat{a}_z$ (W/m^2). Also the reflection coefficient is given to be $\Gamma = 0.7$. Find $\bar{P}^{r}_{av}$ and $\bar{P}^{t}_{av}$!

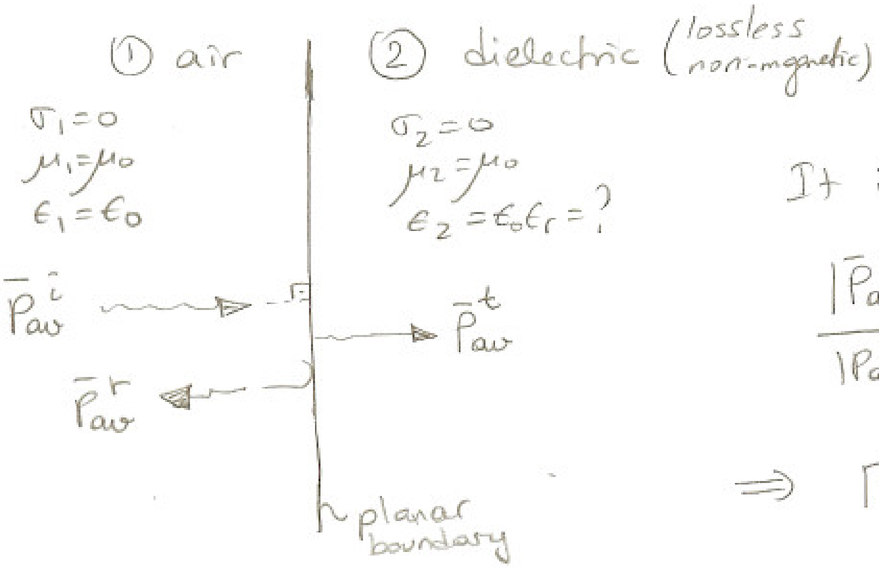


(Normal incidence and lossless problem)

$$\bar{P}^{r}_{av} = (-\hat{a}_z)\,\Gamma^2 P^{i}_{av} = (-\hat{a}_z)\,(0.7)^2 \times 150 = (-\hat{a}_z)\,73.5 \ (W/m^2)$$

$$\underbrace{\bar{P}_{av_{2}}}_{=\bar{P}^{t}_{av}} = \bar{P}_{av_{1}} \Rightarrow \bar{P}^{t}_{av} = \bar{P}^{i}_{av} + \bar{P}^{r}_{av} = 150\,(\hat{a}_z) + 73.5\,(-\hat{a}_z) = 76.5\,\hat{a}_z \ (W/m^2)$$

Example: It is measured that 30% of the power carried by a plane wave (hits the air/unknown dielectric planar boundary as shown in the figure below) is reflected back. If the dielectric material is known to be non-magnetic (i.e $\mu_2=\mu_0$) find its relative permittivity ϵ_r!



It is given that

$$\frac{|\bar{P}_{av}^{r}|}{|\bar{P}_{av}^{i}|} = \Gamma^2 = 0.3$$

$$\Rightarrow \Gamma = \mp\sqrt{0.3} \quad \left(\text{sign must be decided!}\right)$$

We know that

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}}-\sqrt{\frac{\mu_0}{\epsilon_0}}}{\sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}}+\sqrt{\frac{\mu_0}{\epsilon_0}}}$$

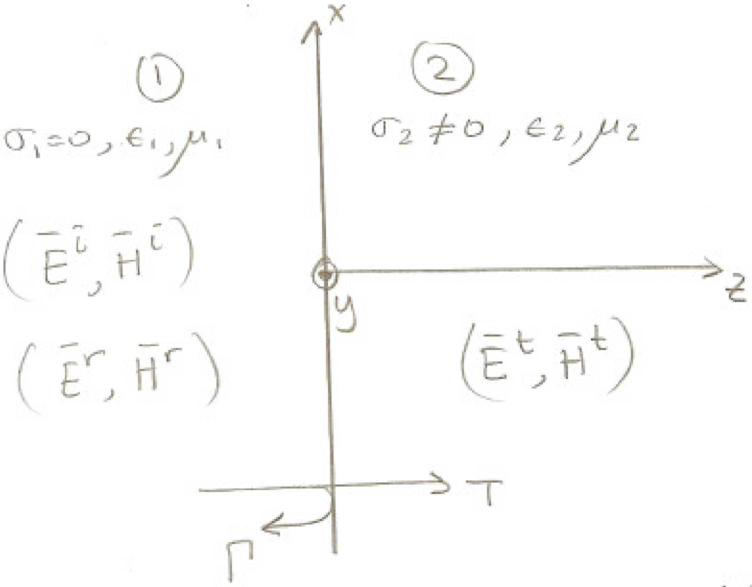
$$\Gamma = \frac{\frac{1}{\sqrt{\epsilon_r}}-1}{\frac{1}{\sqrt{\epsilon_r}}+1} = \frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}}$$

As $\epsilon_r > 1 \Rightarrow \sqrt{\epsilon_r} > 1 \Rightarrow \boxed{\Gamma < 0} \Rightarrow \boxed{\Gamma = -\sqrt{0.3}}$

$(\Gamma \cong -0.548)$

$$\Rightarrow \frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} = -\sqrt{0.3} \Rightarrow \sqrt{\epsilon_r} = \frac{1+\sqrt{0.3}}{1-\sqrt{0.3}} \cong 3.422 \Rightarrow \boxed{\epsilon_r \cong 11.7}$$

Case 2: Perfect Dielectric / Lossy Dielectric Boundary



Formulas for Γ and T are still valid as

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$T = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_1}$$

where $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$ (Medium ① is lossless)

but $\boxed{\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma + j\omega\epsilon}}}$ (Medium ② is lossy)

$\Rightarrow$ Therefore, in this case, Γ and T will be complex.

Also, the expressions of transmitted fields in in medium ② must be modified by using the complex-valued propagation constant γ_2 where

$$\boxed{\gamma_2 = \alpha_2 + j\beta_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)}}$$

such that

$$\bar{E}^t = \hat{a}_x E_3 e^{-\gamma_2 z} \quad \text{and} \quad \bar{H}^t = \hat{a}_y \frac{E_3}{\eta_2} e^{-\gamma_2 z}.$$