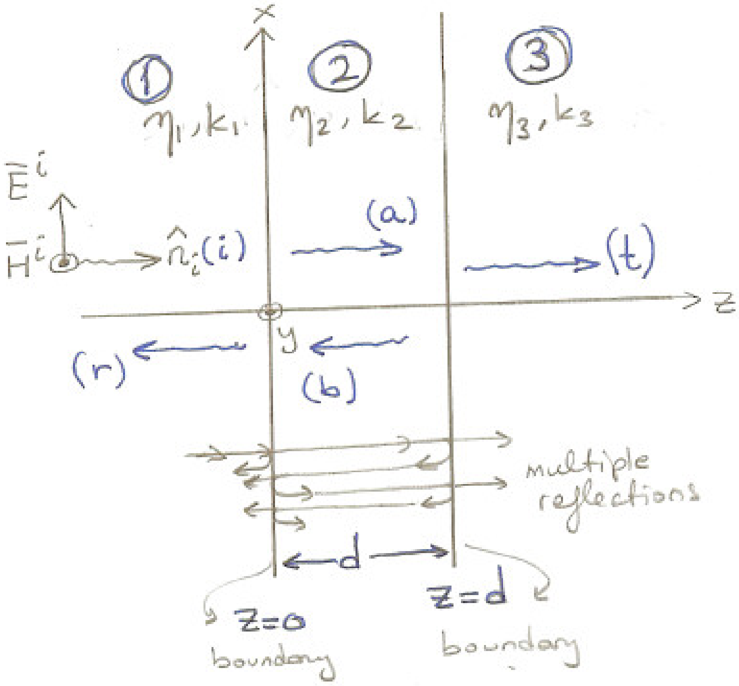
Case 4: Multiple Reflections under normal incidence

Consider the following problem with 3 lossless dielectric media:



At steady state:

(i): Incident u.p.w. in ①

(r): Resultant reflected u.p.w. in ①

(a): Resultant u.p.w propagating in +z direction in ②

(b): Resultant u.p.w propagating in −z direction in ②

(t): Resultant u.p.w transmitted into medium ③.

$$\bar{E}^i = \hat{a}_x E_1 e^{-jk_1 z}, \quad \bar{E}^r = \hat{a}_x E_2 e^{+jk_1 z} \quad \text{in medium ① for } z<0$$

$$\bar{E}^a = \hat{a}_x E_3 e^{-jk_2 z}, \quad \bar{E}^b = \hat{a}_x E_4 e^{+jk_2 z} \quad \text{in medium ② for } 0<z<d$$

$$\bar{E}^t = \hat{a}_x E_5 e^{-jk_3 z} \quad \text{in med. ③ for } z>d$$

Apply the B.C.s for E^{tot}_{tang} and H^{tot}_{tang} at $z=0$ and $z=d$

At $z=0$:

$$E_1 + E_2 = E_3 + E_4 \quad \ldots (\mathrm{I})$$

$$\frac{1}{\eta_1}(E_1 - E_2) = \frac{1}{\eta_2}(E_3 - E_4) \quad \ldots (\mathrm{II})$$

At $z=d$:

$$E_3 e^{-jk_2 d} + E_4 e^{+jk_2 d} = E_5 e^{-jk_3 d} \quad \ldots (\mathrm{III})$$

$$\frac{1}{\eta_2}\left(E_3 e^{-jk_2 d} - E_4 e^{+jk_2 d}\right) = \frac{E_5}{\eta_3} e^{-jk_3 d} \quad \ldots (\mathrm{IV})$$

Equations I, II, III and IV can be simultaneously solved to determine the unknowns E_2, E_3, E_4 and E_5 in terms of the known constant E_1.

Define: $\boxed{\Gamma \triangleq \frac{E_2}{E_1}}$: effective reflection coefficient

(Note that $\Gamma \neq \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ any more)

Special Case: Quarter wave transformer

Let $d = \frac{\lambda_2}{4}$ (The second medium has the thickness of quarter wavelength $\lambda_2/4$ where $\lambda_2 = \frac{2\pi}{k_2}$)

$\Rightarrow$ in equations III and IV $\quad k_2 d = \frac{2\pi}{\lambda_2} d \Big|_{d=\lambda_2/4} = \frac{2\pi}{\lambda_2}\frac{\lambda_2}{4} = \frac{\pi}{2}$

$$\Rightarrow e^{\mp jk_2 d} = e^{\mp j\pi/2} = \mp j$$

Then, equations I – IV can be written as:

(I) $E_1 + E_2 = E_3 + E_4$

(II) $E_1 - E_2 = \frac{\eta_1}{\eta_2}(E_3 - E_4)$

(III) $E_3 - E_4 = j\underbrace{E_5\, e^{-jk_3\lambda_2/4}}_{E_5'} = jE_5'$

(IV) $E_3 + E_4 = j\frac{\eta_2}{\eta_3}E_5'$

Divide eqn. I and II side by side:

$$\frac{E_1+E_2}{E_1-E_2} = \frac{\eta_2}{\eta_1}\frac{E_3+E_4}{E_3-E_4}$$
$E_3+E_4 \rightarrow$ replace using IV
$E_3-E_4 \rightarrow$ replace using III

$$\Downarrow$$

$$\frac{E_1+E_2}{E_1-E_2} = \frac{\eta_2}{\eta_1}\frac{j\frac{\eta_2}{\eta_3}E_5'}{jE_5'} = \frac{\eta_2^2}{\eta_1\eta_3}$$

$$\frac{\cancel{E_1}\left(1+\frac{E_2}{E_1}\right)}{\cancel{E_1}\left(1-\frac{E_2}{E_1}\right)} = \boxed{\frac{1+\Gamma}{1-\Gamma} = \frac{\eta_2^2}{\eta_1\eta_3}} \quad (*)$$

Solve for reflection coefficient Γ in terms of η_1, η_2 and η_3.

To have no reflections in medium (1), i.e. for $E_2 = 0$ or $\Gamma = 0$

Let $\Gamma = 0$ in $(*)$ $\quad \frac{1+0}{1-0} = \frac{\eta_2^2}{\eta_1\eta_3} \Rightarrow \eta_2^2 = \eta_1\eta_3$ or $\boxed{\eta_2 = \sqrt{\eta_1\eta_3}}$

Geometric mean!

$$\boxed{\text{Selecting} \begin{cases} d = \lambda_2/4 \\ \eta_2 = \sqrt{\eta_1\eta_3} \end{cases} \Rightarrow \left.\begin{matrix}\Gamma = 0 \\ (\text{or } E_2 = 0)\end{matrix}\right\} \text{no reflection condition is ensured}}$$

Quarter-Wave Transformer where $\eta_1 \neq \eta_3$, in general.

In other words, a slab of thickness $d = \frac{\lambda_2}{4}$ and intrinsic impedance $\eta_2 = \sqrt{\eta_1 \eta_3}$ acts as an <u>impedance transformer</u> (or as a <u>matching layer</u>) between media ① and ③ to obtain zero reflection in medium ①.

<u>Example:</u> Let, $\mu_1 = \mu_2 = \mu_3 = \mu_0$ (all non-magnetic media)

$$\frac{\lambda_2}{\lambda_1} = \frac{\frac{2\pi}{k_2}}{\frac{2\pi}{k_1}} = \frac{k_1}{k_2} = \frac{\omega\sqrt{\epsilon_1 \mu_0}}{\omega\sqrt{\epsilon_2 \mu_0}} = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$$

Then, $\boxed{d = \frac{\lambda_2}{4} = \frac{\lambda_1}{4}\sqrt{\frac{\epsilon_1}{\epsilon_2}}}$ for the matching layer in ②

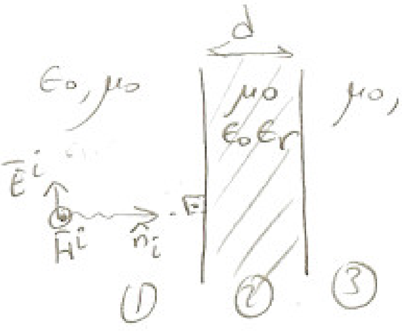
Also, $\eta_2^2 = \eta_1 \eta_3 \Rightarrow \frac{\mu_0}{\epsilon_2} = \sqrt{\frac{\mu_0}{\epsilon_1}\frac{\mu_0}{\epsilon_3}} \Rightarrow \boxed{\epsilon_2 = \sqrt{\epsilon_1 \epsilon_3}}$ in medium ②.

<u>Exercise:</u> Show that selecting $\boxed{d = \frac{\lambda_2}{2} \text{ and } \eta_1 = \eta_3}$ is another way of obtaining $\Gamma = 0$.

<u>Note:</u> It can be shown for this 3-region, normal incidence problem that

$$\Gamma = \frac{E_2}{E_1} \triangleq \frac{\eta_{eq} - \eta_1}{\eta_{eq} + \eta_1} \quad \text{where } \eta_{eq} = \eta_2 \frac{\eta_3 + j\,\eta_2 \tan(\beta_2 d)}{\eta_2 + j\,\eta_3 \tan(\beta_2 d)}$$

Example: A uniform plane wave with free-space wavelength $\lambda_0 = 3\,cm$ is normally incident on a fiberglass slab which has $\epsilon_r = 4.9$, $\mu = \mu_0$; $\sigma = 0$.



a) Find the thickness of fiberglass to have no reflections back in medium ①

b) What percentage of incident power will be reflected back from the fiberglass if the frequency of the incident wave is decreased by 10%, keeping the thickness (d) found in part (a) unchanged?

ⓐ $\eta_1 = \eta_3 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\,(\Omega)$

$\Downarrow$

Use $d = \frac{\lambda_2}{2}$ to provide $\Gamma = 0$.

Find λ_2: $\quad \lambda_2 = \frac{2\pi}{k_2} = \frac{2\pi}{k_0\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{3\,cm.}{\sqrt{4.9}} \cong 1.355\,cm.$

($\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi}{\omega\sqrt{\epsilon_0\mu_0}} = 3\,cm.$ is given in free space)

$\Rightarrow d = \frac{\lambda_2}{2} = \frac{1.355}{2} \Rightarrow \boxed{d \cong 0.678\,cm}$ for no reflections in medium ①

ⓑ $f_{new} = 0.9 f_{old} \Longrightarrow \lambda_{2_{new}} = \frac{\lambda_{2old}}{0.9} = \frac{1.355}{0.9} \cong 1.506\,cm.$ in fiberglass

$\Rightarrow \beta_{2_{new}} d = \frac{2\pi}{\lambda_{2_{new}}} d = \frac{2\pi}{1.506}(0.678) \cong 0.9\pi$ radians $\longrightarrow \sim 162^\circ$

$\Rightarrow$ Using $\eta_1 = \eta_3 = \eta_0 = 120\pi\,(\Omega)$ and $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \frac{\eta_0}{\sqrt{4.9}}\,(\Omega)$ $\Big\} \Rightarrow \eta_{eq} = \eta_2 \frac{\eta_0 + j\eta_2 \tan(\beta_2 d)}{\eta_2 + j\eta_0 \tan(\beta_2 d)} = \dots$

$$\boxed{\eta_{eq} \cong 120\pi\,(0.729 + j\,0.377)\,(\Omega)}$$

$\Rightarrow \boxed{\Gamma = \frac{\eta_{eq} - \eta_0}{\eta_{eq} + \eta_0} \cong 0.263 \angle 113.4^\circ}$ $\quad \Rightarrow \boxed{\frac{P_{av}^{ref}}{P_{av}^{inc}} = |\Gamma|^2 \cong 0.069}$ (6.9 % of the incident power will be reflected back to ①)