ELECTROMAGNETIC INDUCTION

Remember, in static fields (i.e $\frac{\partial}{\partial t} \equiv 0$, no time dependence) we have:

$$\bar{\nabla} \times \bar{E} = 0 \qquad \text{and} \qquad \bar{\nabla} \times \bar{H} = \bar{J}$$
$$\bar{\nabla} \cdot \bar{D} = \rho \qquad\qquad\qquad \bar{\nabla} \cdot \bar{B} = 0$$

Equations of electrostatic and magnetostatic fields are separate (decoupled).

In a conducting medium, static electric and magnetic fields may both exist and form an electromagnetostatic field (through $\bar{J} = \sigma \bar{E}$, $\bar{J} \rightarrow \bar{B}$). But these fields can be calculated independently from each other.

In time-varying case, electromagnetic phenomena are described by Maxwell's equations:

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

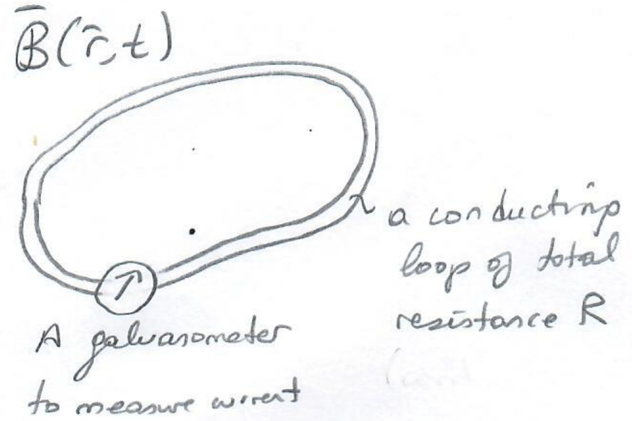
(coupled electric and magnetic fields)

$$\nabla \cdot \bar{D} = \rho \qquad \nabla \cdot \bar{B} = 0$$

(when $\frac{\partial \bar{D}}{\partial t}$ term is neglected, but $\frac{\partial \bar{B}}{\partial t}$ term is retained, the resulting field is a quasi-static electromagnetic field)

$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ is the fundamental postulate for Electromagnetic Induction. It is based on Faraday's experiments and it means that a time-varying magnetic field acts like a source for a time varying electric field.

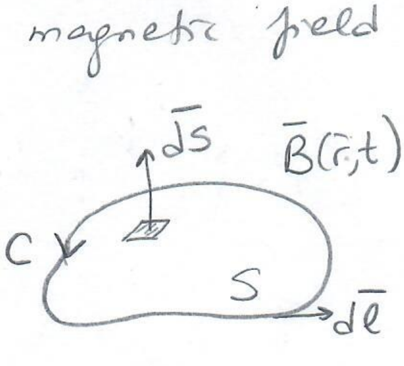
The basic idea and the fundamental results of Faraday's experiments can be summarized as follows:



Consider a conducting loop of total resistance R placed in a time varying magnetic field $\bar{B}(\bar{r},t)$. Note that the circuit does not contain any source of emf (electromotive force).

In this experiment, an induced current is detected by the galvanometer, whose magnitude is proportional to the rate of change of magnetic flux linked by the loop and its direction depends on the case whether the magnetic flux is increasing or decreasing.

Now, consider a closed loop C (can be either a conducting loop or just an imaginary path) in a time-varying magnetic field



$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

integrate both sides on S

$$\int_S \bar{\nabla}\times\bar{E}\cdot d\bar{S} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

use Stoke's Theorem

$$\underbrace{\oint_C \bar{E}\cdot d\bar{l}}_{} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

This LHS is not identically zero in time-varying case.

If we assume a stationary loop (i.e. its shape, location and orientation do not change with time) we can change the order of integration and time-derivation on the LHS:

$$\Rightarrow \quad \underbrace{\oint_C \bar{E}\cdot d\bar{l}}_{\text{Call } v_{ind}} = -\frac{d}{dt}\underbrace{\int_S \bar{B}\cdot d\bar{S}}_{\Phi\text{ : magnetic flux}}$$

$$\Rightarrow \quad \boxed{v_{ind} = -\frac{d\Phi}{dt}}$$

(Volts)

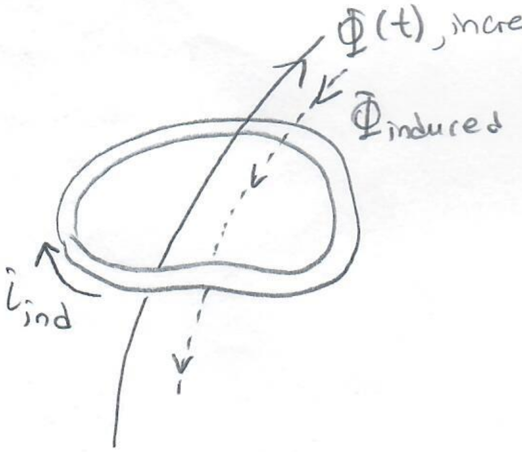
Induced voltage

If C is a conducting path, then a current flows. The direction of this induced current is such that it opposes the change in the primary flux $\Phi(t)$.

(<u>Lenz Rule</u>)

$$|i_{ind}| = \frac{|v_{ind}|}{R}$$

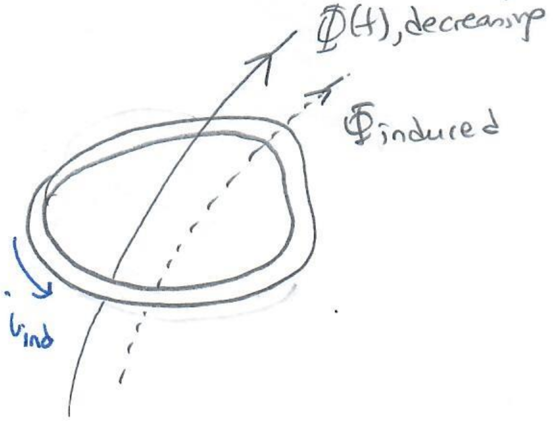
R: total resistance of the loop.



<u>Case 1:</u>

$\Phi(t)\uparrow \Rightarrow \frac{d\Phi(t)}{dt} > 0 \Rightarrow v_{ind} < 0$

$\Phi_{induced}$ is in the opposite direction of $\Phi(t)$ to oppose its increase. The direction of i_{ind} and $\Phi_{induced}$ are related by the RHR (Right hand rule).



<u>Case 2:</u>

$\Phi(t)\downarrow \Rightarrow \frac{d\Phi(t)}{dt} < 0 \Rightarrow v_{ind} > 0$

$\Phi_{induced}$ and $\Phi(t)$ are in the <u>same</u> direction, so that decrease of $\Phi(t)$ is opposed. Again, direction of $i_{induced}$ is found from direction of $\Phi_{induced}$ by the RHR.

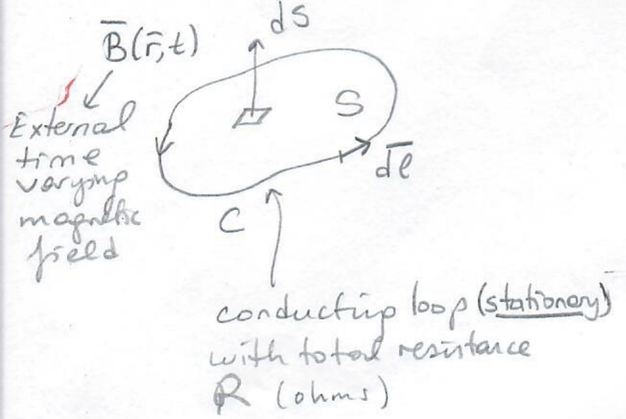
Contin[illegible] ELECTROMAGNETIC INDUCTION

Remember:

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \Rightarrow \underbrace{\oint_C \bar{E}\cdot d\bar{l}}_{v_{ind}} = -\frac{d}{dt}\underbrace{\int_S \bar{B}\cdot d\bar{S}}_{\Phi}$$

v_{ind} ↓ induced voltage

Φ ↓ Magnetic flux (time varying)



$$\Downarrow$$

$$\boxed{v_{ind} = -\frac{d\Phi}{dt}}$$

$$\Rightarrow |v_{ind}| = R\,|i_{ind}| = \left|\frac{d\Phi}{dt}\right|$$

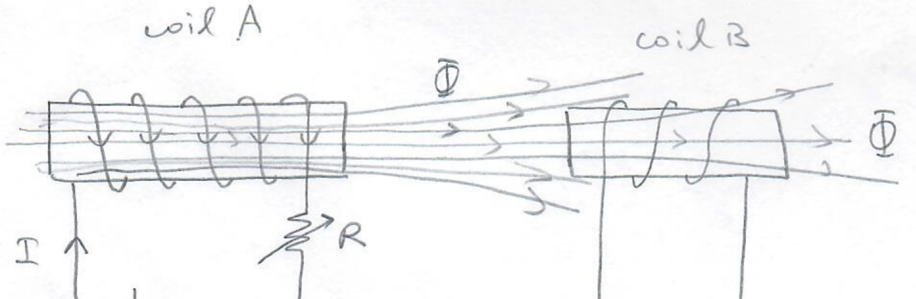
and direction of the induced current i_{ind} is determined from the Lenz Rule, i.e:

If $\Phi(t)\uparrow \Rightarrow \Phi_{induced}$ is in the opposite direction of $\Phi(t)$

" $\Phi(t)\downarrow \Rightarrow$ " " " " same " " "

Then, find the direction of i_{ind} from the direction of $\Phi_{induced}$ using the Right Hand Rule.

Example: Consider two coils A and B shown below:



Answers:
For parts ⓐ and ⓑ: current flows from Q to P

For part ⓒ: current flows from P to Q.

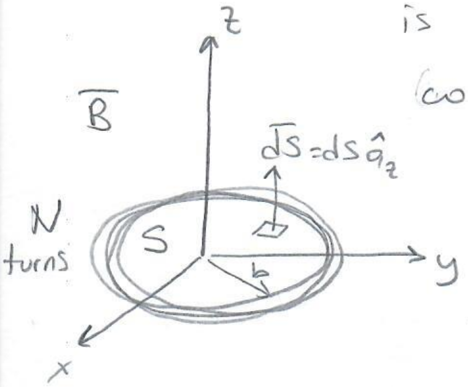
What is the direction of the current flowing through the resistance R_o if

ⓐ coil $\bar{B}$ is brought closer to coil A ⇒ Flux linked to coil B increases

ⓑ resistance R is decreased ⟹ $R\downarrow \Rightarrow I\uparrow \Rightarrow \Phi\uparrow \Rightarrow$ flux linked to coil B increases

ⓒ the switch S is opened ⇒ Φ decreases ⇒ Flux linked to coil B decreases

Example: $\bar{B} = \hat{a}_z B_o \cos\left(\frac{\pi r}{2b}\right) \sin \omega t + \hat{a}_\phi B_1 \sin\left(\frac{\pi r}{2b}\right) \sin \omega t$ is given. Find the induced voltage v_{ind} for the coil placed on the xy-plane as shown in the figure.



$$\Phi_{\substack{\text{simple}\\\text{turn}}} = \int_S \bar{B} \cdot d\bar{S}$$ (flux linked to a simple turn of the circular coil).

where $\bar{B} \cdot d\bar{S} = (B_z \hat{a}_z + B_\phi \hat{a}_\phi) \cdot \underbrace{d\bar{S}}_{\hat{a}_z dS} = B_z \underbrace{dS}_{r dr d\phi}$

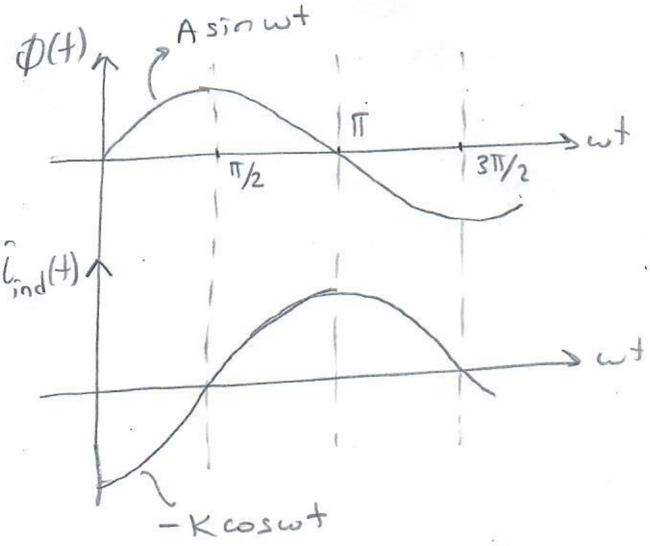
$$\Phi_{\substack{\text{simple}\\\text{turn}}} = \int_{\phi=0}^{2\pi} \int_{r=0}^{b} B_o \cos\left(\frac{\pi r}{2b}\right) \sin \omega t \; r dr d\phi$$

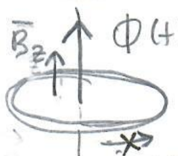
$$= B_o \sin \omega t \, (2\pi) \int_{r=0}^{b} r \cos\left(\frac{\pi r}{2b}\right) dr$$ $\left(\begin{array}{l}\text{Use with } k = \pi/2b \\ \int r\cos(kr)dr = \frac{\cos kr}{k^2} + \frac{r \sin kr}{k}\end{array}\right)$

$$\Rightarrow \Phi_{\substack{\text{simple}\\\text{turn}}} = \underbrace{\frac{8b^2}{\pi}\underbrace{\left(\frac{\pi}{2}-1\right)}_{>0} B_o}_{\text{call A, constant } A>0} \sin \omega t = \underline{\underline{A \sin \omega t}} \quad (\text{weber})$$

$$\Rightarrow v_{ind}\Big|_{\substack{\text{for}\\\text{simple}\\\text{turn}}} = -\frac{d}{dt} \Phi_{\substack{\text{simple}\\\text{turn}}} = -\frac{8b^2}{\pi}\left(\frac{\pi}{2}-1\right) B_o \omega \cos \omega t$$

$$v_{ind_{total}} = N \left(v_{ind}\right)_{\substack{\text{simple}\\\text{turn}}} \Rightarrow \boxed{v_{ind_{total}} = -\frac{8Nb^2}{\pi}\left(\frac{\pi}{2}-1\right) B_o \omega \cos \omega t \quad (\text{Volts})}$$





$\Phi(t) = A \sin \omega t$
$(A>0)$

$i_{ind} = -K \cos \omega t$
$(K>0)$

$$i_{ind} = \frac{v_{ind}}{R}$$

$$\underline{\underline{i_{ind}}} = -\underbrace{\frac{8Nb^2\left(\frac{\pi}{2}-1\right) B_o \omega}{\pi R}}_{\text{call } K>0} \cos \omega t = \underline{\underline{-K \cos \omega t}}$$

for $0 < \omega t < \pi/2$
$\phi(t)$ is increasing ⇒
i_{ind} is in $(-\hat{a}_\phi)$ direction

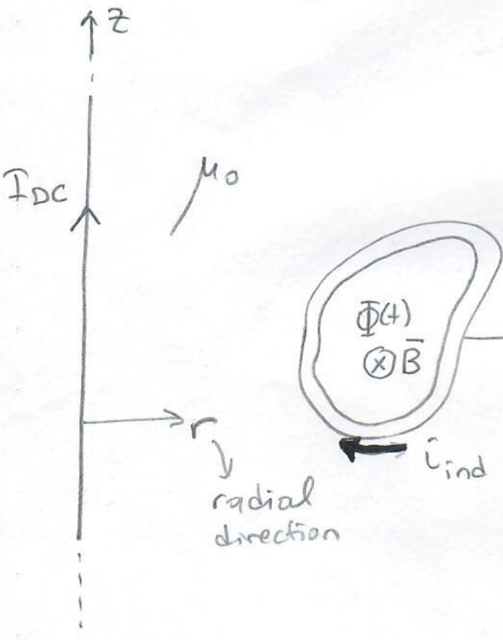
for $\pi/2 < \omega t < 3\pi/2$
$\phi(t)$ is decreasing ⇒
i_{ind} is in $(+\hat{a}_\phi)$ direction.

MOTIONAL EMF

Consider the Faraday's 2nd experiment which can be summarized as follows: There exists a thin long straight wire carrying a steady current I_{DC} that creates a time-invariant magnetic field $\overline{B}$. Consider a conducting loop of arbitrary shape moving with a velocity $\overline{V}$. Although the $\overline{B}$ field does not vary with time, the flux Φ linked by the conducting loop is time-varying because of the movement of the loop. Obviously, this time-varying flux leads to an induced voltage v_{ind} and an induced current i_{ind} such that

$$|i_{ind}| = \frac{|v_{ind}|}{R}$$

where R is total resistance of the conducting loop.



Note that the magnetic field B created by I is:

$$\overline{B} = \frac{\mu_0 I}{2\pi r}\,\hat{a}_\phi$$

The direction of the induced current i_{ind} can be found using the Lenz Rule.

The induced voltage, v_{ind} can be determined in two ways:

(1) Calculate ϕ as a function of time, then differentiate

$$\boxed{v_{ind} = -\frac{d\Phi}{dt}}$$

OR

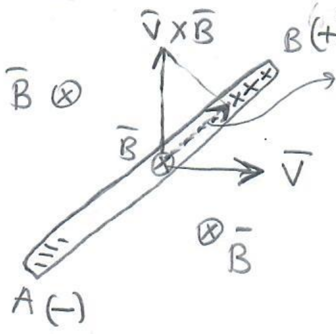
(2) Calculate v_{ind} it as the work done by the force per unit charge

Remember $\overline{F_{mag}} = q\,\overline{V}\times\overline{B}$

$$\Rightarrow \frac{\overline{F_{mag}}}{q} = \overline{V}\times\overline{B} = \text{Force per unit charge}$$

$$\Rightarrow \boxed{v_{ind} = \oint_C (\overline{V}\times\overline{B})\cdot d\overline{l}}$$

To explain the 2^{nd} method consider a conducting bar moving with a velocity $\bar{v}$ in a magnetic field $\bar{B}$.



component of the vector $(\bar{v}\times\bar{B})$ in the main axis of conductor.

As a result of this force vector component, free electrons of the conductor are pushed towards the end A ⇒ end A becomes negatively charged.
⇒ end B " positively " .

⇒ The charges at ends A and B establish an electric force field to oppose the existing magnetic force.

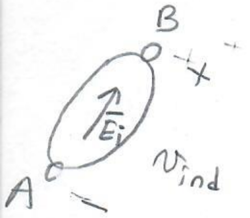
$\bar{v}\times\bar{B}$: impressed electric field $= \bar{E}_i$

$$\Rightarrow \boxed{\text{emf} = |v_{ind}| = \int_{A(-)}^{B(+)} \bar{E}_i \cdot d\bar{l} = \int_{A}^{B} \bar{v}\times\bar{B}\cdot d\bar{l}}$$

$\Big(v_{ind} = \oint_C \bar{v}\times\bar{B}\cdot d\bar{l}$ if a closed conducting loop moves in a given field $\bar{B}$. $\Big)$

Note: Accumulation of charges at the ends of the bar continues for a very short transient period. At the steady state, the induced electric force completely cancels the impressed magnetic force within the conductor.

Note that this moving bar is equivalent to an open-circuited voltage generator whose higher voltage terminal is the end where (+) charges are accumulated.



Important Note: In v_{ind} calculations:

$v_{ind} = -\dfrac{d\Phi}{dt}$ can be used to calculate the motional v_{ind} if there is a closed path.

$v_{ind} = \int_C \bar{v}\times\bar{B}\cdot d\bar{l}$ can be used for open paths as well as closed paths.

Summary:

For a closed path C in a given magnetic field $\bar{B}$, an emf (voltage) is induced along the path if the magnetic flux linked by the path changes with time.

A nonzero $\frac{d\Phi}{dt}$ may result from any one of the following cases:

① Path is stationary but $\bar{B}$ is time-varying;
(Transformer emf is induced)

$$v_{ind} = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S}$$

② $\bar{B}$ is time-invariant (static) but the path moves;
Motional emf (generator emf) is induced.

$$v_{ind} = \oint_C \bar{v}\times\bar{B}\cdot d\bar{l}$$

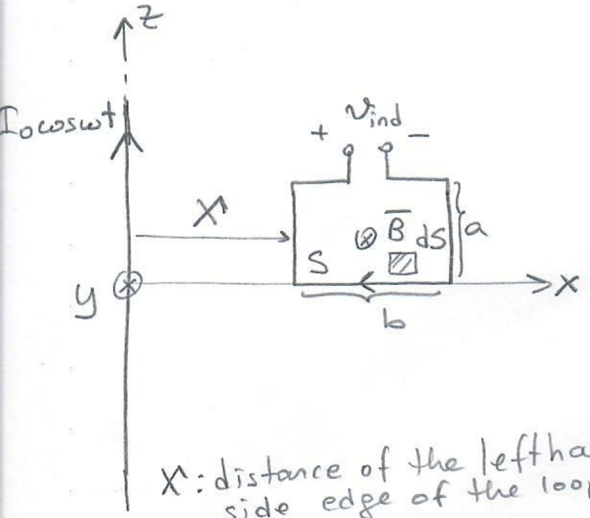
③ The path C moves in a time-varying field $\bar{B}$;
(combination of cases ① and ②)
Both transformer emf and motional emf are induced.

$$v_{ind} = -\frac{d\Phi}{dt}$$

or $$v_{ind} = \underbrace{-\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S}}_{\text{transformer emf}} + \underbrace{\oint_C (\bar{v}\times\bar{B})\cdot d\bar{l}}_{\text{motional emf}}$$

Example: Find v_{ind} in the following cases:

Case I: The ∞-ly long wire along the z-axis carries a time dependent current $I_o \cos wt$. The rectangular conducting loop with sides (a) and (b) is fixed. (i.e. X' is constant)



X': distance of the lefthand side edge of the loop to the wire.

Line current creates a time-varying magnetic field $\bar{B} = \frac{\mu_o I_o \cos wt}{2\pi r} \hat{a}_\phi$

On the loop area, S:

$$\bar{B}\Big|_{on\,S} = \frac{\mu_o I_o \cos wt}{2\pi x} \hat{a}_y$$

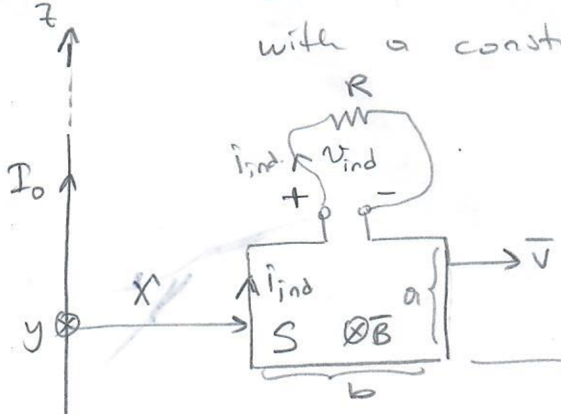
The magnetic flux linked to loop is: $\Phi = \int_S \bar{B} \cdot d\bar{S}$

where $d\bar{S} = dx\,dz\,\hat{a}_y$

$$\Rightarrow \Phi = \int_{z=0}^{a} \int_{x=X'}^{X'+b} \frac{\mu_o I_o \cos wt}{2\pi x}\,dx\,dz = \frac{\mu_o I_o \cos wt}{2\pi}\, a \ln\left(\frac{X'+b}{X'}\right)$$

$$\boxed{v_{ind_I} = -\frac{d\Phi}{dt} = \frac{\mu_o I_o a w}{2\pi} \ln\left(\frac{X'+b}{X'}\right) \sin wt \quad \text{(volts)}}$$ Transformer emf.

Case II: The straight, ∞-ly long wire carries a stationary current I_o. But the rectangular loop moves away from the wire with a constant velocity $\bar{v}$.



Assume $X' = x_o$ at $t=0$

$$\Rightarrow X' = X'(t) = x_o + vt$$

on S: $\bar{B}\Big|_{on\,S} = \frac{\mu_o I_o}{2\pi x} \hat{a}_y$

$$\Phi = \int_S \bar{B} \cdot d\bar{S}$$

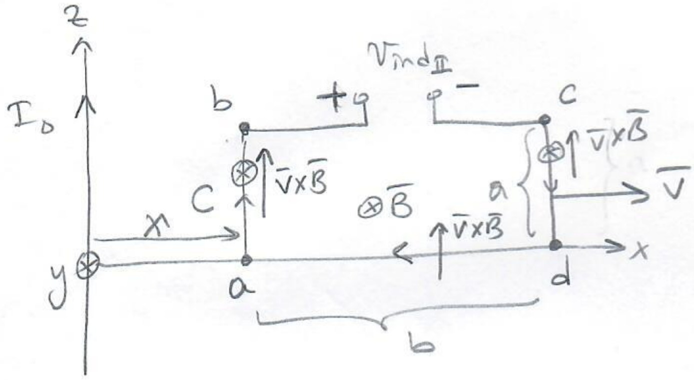
$$= \frac{\mu_o I_o}{2\pi} a \ln\left(\frac{X'+b}{X'}\right)$$

(Note: as $t\uparrow \Rightarrow X'\uparrow \Rightarrow B\Big|_{on\,S}$ and $\Phi' \downarrow$
$\Rightarrow \Phi_{ind}$ is in $\hat{a}_y$ direction to support original Φ $\Rightarrow$ Right hand rule wrt ϕ_{ind} direction gives i_{ind} direction and the polarity of v_{ind} as shown)

$$v_{ind} = -\frac{d\Phi}{dt} = -\frac{\mu_0 I_0 a}{2\pi} \underbrace{\frac{d}{dt}\left[\ln\left(\frac{x+b}{x}\right)\right]}_{\underbrace{\frac{d}{dx}\left[\ln\left(\frac{x+b}{x}\right)\right]}_{-\frac{b}{x(x+b)}}\underbrace{\left(\frac{dx}{dt}\right)}_{V}}$$ (use chain rule)

$$\Rightarrow \boxed{v_{ind_{II}} = \frac{\mu_0 I_0 a b V}{2\pi} \frac{1}{(x_0+Vt)(x_0+b+Vt)} \quad \text{(Volts)}}$$ Motional emf.

An alternative solution for $v_{ind_{II}}$ using $\oint_C \bar{V}\times\bar{B}\cdot d\bar{l}$



$$v_{ind_{II}} = \oint_C \bar{V}\times\bar{B}\cdot d\bar{l}$$

where $\left\{\begin{array}{l}\bar{B} = \frac{\mu_0 I_0}{2\pi x}\hat{a}_y \\ \bar{V} = V\hat{a}_x\end{array}\right.$

$$\hookrightarrow \bar{V}\times\bar{B} = \frac{\mu_0 I_0 V}{2\pi x}\hat{a}_z$$

The closed contour C around the rectangular loop can be decomposed in four parts as:

$$C = \underset{d\bar{l}=dz\hat{a}_z}{C_{ab}} + \underset{d\bar{l}=dx\hat{a}_x}{C_{bc}} + \underset{d\bar{l}=dz\hat{a}_z}{C_{cd}} + \underset{d\bar{l}=dx\hat{a}_x}{C_{da}}$$

$(\bar{V}\times\bar{B})\cdot d\bar{l} = 0$ as $\hat{a}_z \perp \hat{a}_x$

No contribution comes from sides C_{bc} and C_{da} to the closed contour integral.

$$\Rightarrow v_{ind_{II}} = \underset{(C_{ab})}{\int_{z=0}^{a}} \frac{\mu_0 I_0 V}{2\pi x}\underbrace{\hat{a}_z\cdot\hat{a}_z}_{1}dz + \underset{(C_{cd})}{\int_{z=a}^{0}} \frac{\mu_0 I_0 V}{2\pi(x+b)}\underbrace{\hat{a}_z\cdot\hat{a}_z}_{1}dz = \cdots\cdots$$

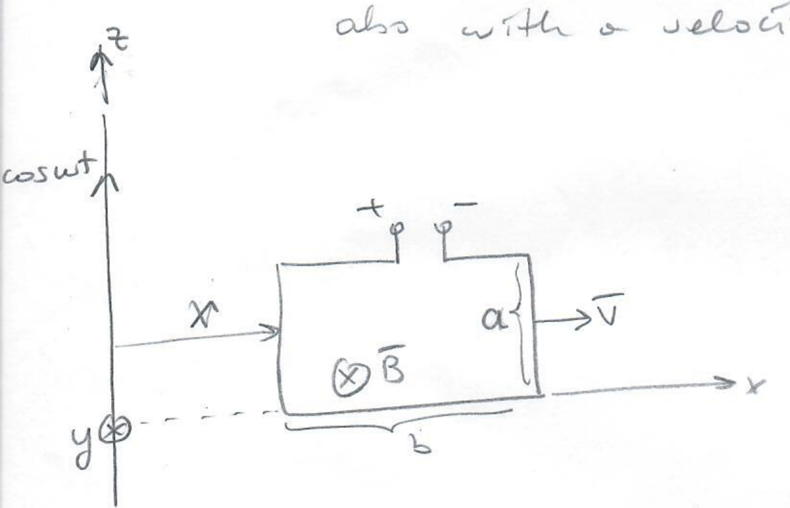
$$v_{ind_{II}} = \cdots = \frac{\mu_0 I_0 V a b}{2\pi}\frac{1}{x(x+b)} = \frac{\mu_0 I_0 V a b}{2\pi}\frac{1}{(x_0+Vt)(x_0+b+Vt)}$$ Volts.

as found earlier.

Note that $|\bar{v}\times\bar{B}|\Big|_{\text{at } x=x^{\wedge}} > |\bar{v}\times\bar{B}|\Big|_{\text{at } x=x^{\wedge}+b}$ as $B \propto \frac{1}{x}$

so the direction of $(\bar{v}\times\bar{B})$ vector at $x=x^{\wedge}$ side points the terminal which is positively charged. (Negatively charged free electrons of the conductor atoms are pushed to the opposite direction of $\bar{v}\times\bar{B}$ vector component along the loop side closer to the ∞-ly long current filament)

Case III: The ∞-ly long wire carries a time-varying current $I_o \cos\omega t$ and the rectangular loop also with a velocity $\bar{v}$ away from the wire.



The solution for $v_{ind_{III}}$ can be found from the superposition of v_{ind_I} and $v_{ind_{II}}$ as:

$$v_{ind_{III}} = v_{ind_I}\Big|_{x^{\wedge}\to x_o+vt} + v_{ind_{II}}\Big|_{I_o \to I_o\cos\omega t}$$

$x^{\wedge} = x_o + vt$
$\downarrow$
initial position

$$\Rightarrow \boxed{v_{ind_{III}} = \frac{\mu_o I_o a\omega}{2\pi}\ln\left(\frac{x_o+b+vt}{x_o+vt}\right)\sin\omega t + \frac{\mu_o I_o\cos\omega t\, abv}{2\pi(x_o+vt)(x_o+b+vt)}} \text{ (volts)}$$

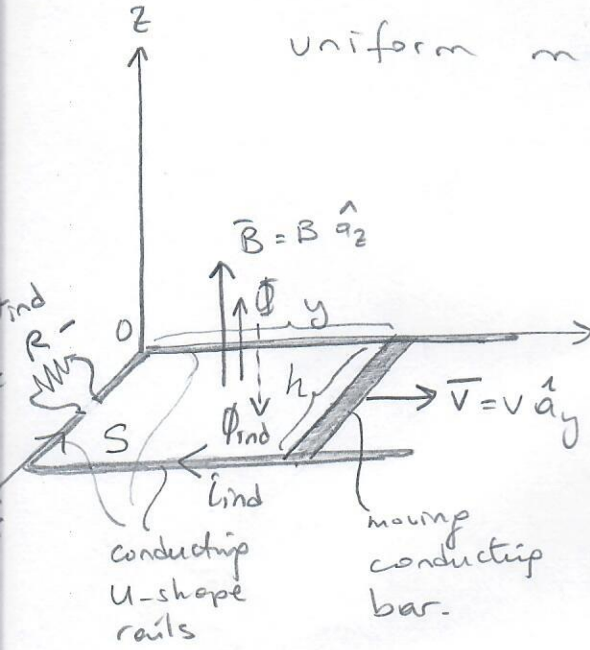
(transformer emf + motional emf)

Exercise: Obtain the same result (without superposing previous results) directly from

$$\Phi = \int_S \bar{B}\cdot d\bar{S} = \int_0^a \int_{x_o+vt}^{x_o+b+vt} \frac{\mu_o I_o\cos\omega t}{2\pi x}\,dx\,dz$$

Then, $v_{ind} = \frac{-d\Phi(t)}{dt} = - - - - -.$

Example: Consider the U-shape conducting rails with a moving conductor bar as shown in the figure. Find v_{ind} and i_{ind}, in the presence of a uniform magnetic field $\bar{B} = B_0 \hat{a}_z$ (where B_0: constant)



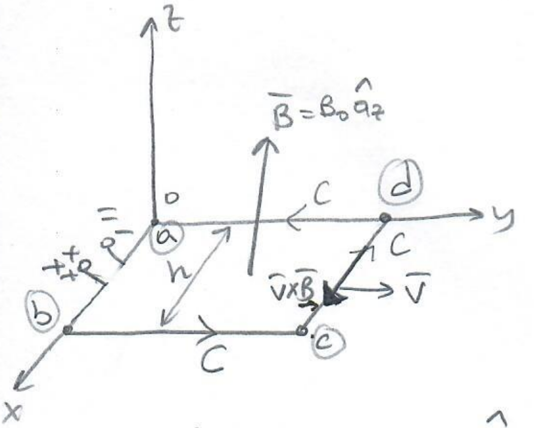
Using $v_{ind} = -\frac{d\phi}{dt}$ expression:

$$\Phi = \int_S \bar{B} \cdot d\bar{S} = B_0 S = B_0 h y$$

$$v_{ind} = -\frac{d\Phi}{dt} = -B_0 h \underbrace{\frac{dy}{dt}}_{V} = -B_0 h v \quad \text{Volts}$$

Note that as $\bar{v}$ pulls the moving bar in $+\hat{a}_y$ direction, the loop area S increases.
$\Rightarrow$ Flux Φ increases $\Rightarrow$ Φ_{ind}, the induced flux becomes in the opposite direction ($-\hat{a}_z$ dir.)
Then, from the RHR, i_{ind} is found to flow in the clockwise direction.

Using $v_{ind} = \oint_C \bar{v} \times \bar{B} \cdot d\bar{l}$ expression:



$$C = \underbrace{C_{ab} + C_{bc} + C_{da}}_{\text{fixed edges}} + C_{cd}$$

fixed edges
$\Rightarrow \bar{v} = 0$
$\Rightarrow \bar{v} \times \bar{B} = 0$
$\Rightarrow$ no contribution

C_{cd} only contributing edge moving with velocity $\bar{v} = v \hat{a}_y$

$$\Rightarrow \oint_C \bar{v} \times \bar{B} \cdot d\bar{l} = \int_{C_{cd}} \bar{v} \times \bar{B} \cdot d\bar{l} = v_{ind}$$

$$\Rightarrow v_{ind} = \int_{x=h}^{0} (v B_0 \hat{a}_x) \cdot (\hat{a}_x dx)$$

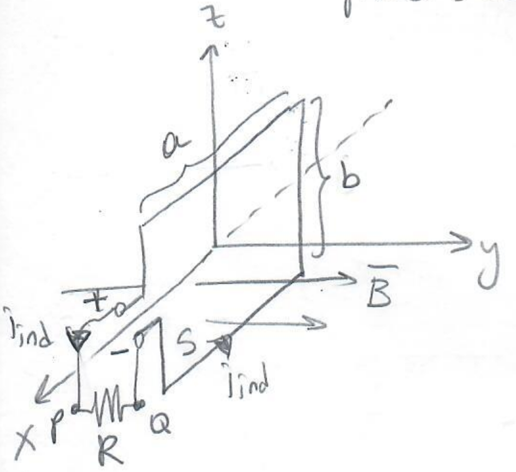
$$\bar{v} = v \hat{a}_y, \quad \bar{B} = B_0 \hat{a}_z \Bigg\} \underbrace{\bar{v} \times \bar{B} = v B_0 \hat{a}_x}$$

This force vector direction determines the (+) terminal
(electrons are pushed to the opposite direction of $\bar{v} \times \bar{B}$)

$$\boxed{V_{ind} = -v B_0 h \text{ (volts)}}$$

same as found earlier.

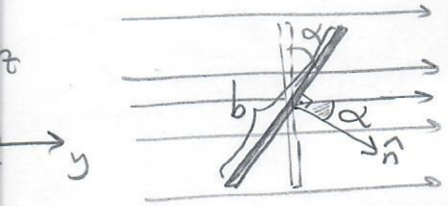
Example: Consider a rotating rectangular conducting loop (with sides a and b) in a uniform, static magnetic field. Compute v_{ind}.



Let $\overline{B} = B_0 \hat{a}_y$ and the loop rotates around the x-axis with an angular velocity ω rad/sec.

The magnetic flux linked to the rotating loop is:

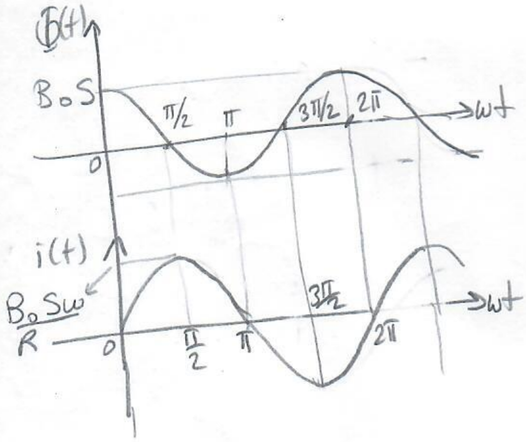
$$\Phi = \int_S \overline{B} \cdot d\overline{S} = \int_S B_0 \underbrace{\hat{a}_y \cdot \hat{n}}_{\cos\alpha} dS = B_0 \cos\alpha \underbrace{\int dS}_{S}$$



where $\boxed{\alpha = \omega t}$ (radians)

$$\Rightarrow \boxed{\Phi(t) = B_0 S \cos(\omega t)} \text{ (weber)}$$

$$\Rightarrow v_{ind} = -\frac{d\Phi}{dt} = -\left(-\omega B_0 S \sin \omega t\right)$$



$$\boxed{v_{ind} = BS\omega \sin \omega t \text{ (volts)}}$$

$$\boxed{i_{ind} = \frac{v_{ind}}{R} = \frac{BS\omega \sin\omega t}{R} \text{ (Amp)}}$$

where R is the resistance connected between the terminals.

For $0 < \omega t < \pi \Rightarrow \Phi \downarrow \Rightarrow \Phi_{ind}$ must be in the same direction with Φ (in $\hat{a}_y$ dir.) ⇒ using the RH Rule i_{ind} flows from P to Q over R.

For $\pi < \omega t < 2\pi \Rightarrow \Phi \uparrow \Rightarrow \Phi_{ind}$ is the opposite dir. of $-\Phi$ (in $-\hat{a}_y$ dir.) ⇒ i_{ind} flows from Q to P over resistance R.

(Exercise: Show that $v_{ind} = -BS\omega \cos 2\omega t$ for $\overline{B} = B \sin \omega t \, \hat{a}_y$)

Soln: $\Phi = \int_S \overline{B} \cdot d\overline{S} = \int B_0 \sin(\omega t) \underbrace{\hat{a}_y \cdot \hat{n}}_{\cos\alpha} dS$ ↳ $\alpha = \omega t$ $= B_0 \underbrace{\cos(\omega t) \sin(\omega t)}_{\frac{1}{2}\sin(2\omega t)} \underbrace{\int dS}_{S} = \frac{B_0 S}{2} \sin 2\omega t$

$$\boxed{v_{ind} = -\frac{d\Phi}{dt} = -B_0 S\omega \cos(2\omega t)}$$