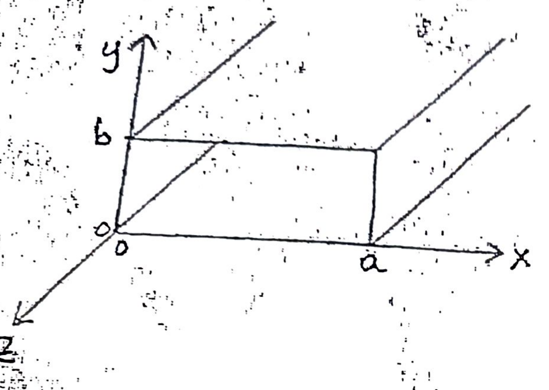
P330

Dec. 21st, 2005

EE 303

Supplementary Problems on Waveguides

Problem: 1 A rectangular waveguide of dimensions a and b $(a > b)$, as shown in Figure ■, is to be operated in a single mode. Determine the smallest ratio of the a/b dimensions that will allow the largest bandwidth of the single-mode operation. State the dominant mode and its largest bandwidth of single-mode operation.



Solution: As the cut-off frequency for the $(mn)^{th}$ mode is given, in general, as

$$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

$$\left.\begin{array}{l} m=0,1,2,\ldots \\ n=0,1,2,\ldots \end{array}\right\} \text{such that } m=n=0 \text{ is excluded.}$$

when $a>b$, TE_{10} is the dominant mode with the cut-off frequency given as $(f_c)_{10} = \dfrac{1}{2a\sqrt{\mu\epsilon}} = \dfrac{v}{2a}$

The mode with the next higher cutoff frequency would be either the TE_{20} or the TE_{01} mode with cutoff frequencies

$$(f_c)_{20} = \frac{1}{a\sqrt{\mu\epsilon}} = 2(f_c)_{10} = \frac{2v}{2a} = \frac{v}{a}$$

and

$$(f_c)_{01} = \frac{1}{2b\sqrt{\mu\epsilon}} = \frac{v}{2b}$$

Obviously, for the single TE_{10} mode operation with the largest possible bandwidth we need to satisfy

$$(f_c)_{10} \le f \le 2(f_c)_{10} = (f_c)_{20} \le (f_c)_{01} \Rightarrow \frac{\cancel{v}}{a} \le \frac{\cancel{v}}{2b}$$

$$\Rightarrow \boxed{\frac{a}{b} \ge 2}$$

... and 25% below the next higher-order TE_{01} mode.

Solution:

$$(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}} = \frac{30 \times 10^9}{2a}$$

Since $f = 10$ GHz must be greater by 25% above the cutoff frequency of the TE_{10} mode, then

$$10 \times 10^9 \geq 1.25\left(\frac{30 \times 10^9}{2a}\right) \Rightarrow a \geq 1.875 \text{ cm} = 0.738 \text{ in.}$$

Since the next higher-order mode is the TE_{01} mode, whose cutoff frequency with a free-space medium in the guide is

$$(f_c)_{01} = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = \frac{30 \times 10^9}{2b}$$

then

$$10 \times 10^9 \leq 0.75\left(\frac{30 \times 10^9}{2b}\right) \Rightarrow b \leq 1.125 \text{ cm} = 0.443 \text{ in.}$$

Problem 3:

A rectangular waveguide with dimensions $a = 2.25$ cm and $b = 1.125$ cm, as shown in Figure 8-1, is operating in the dominant mode.

(a) Assume that the medium inside the guide is free space. Then find the cutoff frequency of the dominant mode.
(b) Assume that the physical dimensions of the guide stay the same (as stated) and that we want to reduce the cutoff frequency of the dominant mode of the guide by a factor of 3. Then find the dielectric constant of the medium that must be used to fill the guide to accomplish this.

Solution:

a. $(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}} = \frac{3\times10^{9}}{2(2.25)} = 6.667\times10^{9}$ Hz.

b. $(f_c')_{10} = \frac{1}{3}(f_c)_{10} = \frac{1}{3(2a\sqrt{\mu_0\varepsilon_0})} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0\varepsilon_r}} \Rightarrow \varepsilon_r = 9$

Problem: 4

A standard X-band (8.2–12.4 GHz) rectangular waveguide with inner dimensions of 0.9 in. (2.286 cm) and 0.4 in. (1.016 cm) is filled with lossless polystyrene ($\varepsilon_r = 2.56$). For the lowest-order mode of the waveguide, determine at 10 GHz the following values. ($a = 2.286$ cm, $b = 1.016$ cm)

(a) Cutoff frequency (in GHz).
(b) Guide wavelength (in cm).
(c) Wave impedance.
(d) Phase velocity (in m/s).
(e) Group velocity (in m/s).

Solution:

a. $(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon}} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = \frac{v_0}{2a\sqrt{\varepsilon_r}} = \frac{30\times10^{9}}{2(2.286)\sqrt{2.56}} = 4.101$ GHz

b. $\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda_0/\sqrt{\varepsilon_r}}{\sqrt{1-\left(\frac{4.101}{10}\right)^2}} = \frac{30\times10^{9}/(10\times10^{9}\sqrt{2.56})}{\sqrt{1-(0.4101)^2}} = \frac{1.875}{0.912} = 2.056$ cm

c. $Z_H = \frac{\eta}{\sqrt{1-(f_c/f)^2}} = \frac{377/\sqrt{2.56}}{0.912} = 258.36$ Ohms

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Department of Electrical and Electronic Engineering

EE 303Midterm 2

2004/2005

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- 110 MINUTES
- 5 QUESTIONS OF EQUAL POINTS
- CLOSED BOOK
- SIMPLE CALCULATORS ALLOWED

Please Fill in Your

LASTNAME	
NAME	
SECTION	
STUDENT ID:	

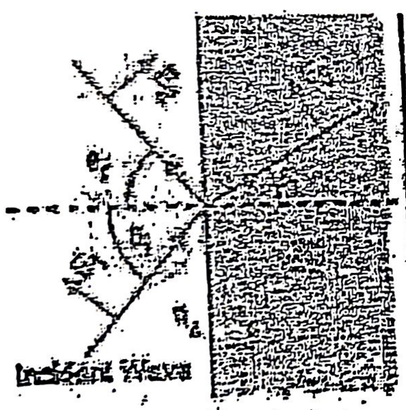
Question	Grade
Q1	
Q2	
Q3	
Q4	
Q5	
TOTAL	

For air and free space:

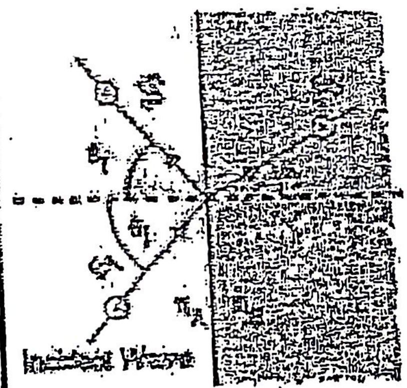
$\mu_0 = 4\pi 10^{-7}$ H/m

$\varepsilon_0 = \dfrac{10^{-9}}{36\pi}$ F/m

Fresnel Formulas.



E // Plane of incidence Case



E ⊥ Plane of incidence Case

$$\frac{E_2}{E_1} = \Gamma_{//} = \frac{\eta_B \cos\theta_t - \eta_A \cos\theta_i}{\eta_B \cos\theta_t + \eta_A \cos\theta_i} \qquad \frac{E_2}{E_1} = \Gamma_{\perp} = \frac{\eta_B \cos\theta_i - \eta_A \cos\theta_t}{\eta_B \cos\theta_i + \eta_A \cos\theta_t}$$

Problem: 5

An empty X-band (8.2–12.4 GHz) rectangular waveguide, with dimensions of 2.286 $= a$ cm by 1.016 cm, is to be connected to an X-band waveguide of the same dimensions but filled with lossless polystyrene ($\varepsilon_r = 2.56$). To avoid reflections, an X-band waveguide (of the same dimensions) quarter-wavelength long section is inserted between the two. Assume dominant-mode propagation and that matching is to be made at 10 GHz.

(a) Determine the wave impedance of the quarter-wavelength section waveguide.
(b) Determine the dielectric constant of the lossless medium that must be used to fill the quarter-wavelength section waveguide.
(c) Determine the length (in cm) of the quarter-wavelength section waveguide.

Solution:

a. For the empty $(f_c)_{10} = 6.56 \times 10^9 \Rightarrow Z_{w0} = \dfrac{\eta_0}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-\left(\frac{6.56}{10}\right)^2}} = \dfrac{377}{0.7547} = 499.5$

For the filled waveguide, from Problem 8.2

$$(f_c)_{10} = 4.101 \times 10^9 \Rightarrow Z_{w2} = 258.36$$

Therefore for the intermediate waveguide ($\lambda/4$ section)

$$Z_{w1} = \sqrt{Z_{w0} Z_{w2}} = \sqrt{499.5(258.36)} = 359.235 \text{ Ohms}$$

b. $Z_{w1} = 359.235 = \dfrac{377/\sqrt{\epsilon_{r1}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \dfrac{377/\sqrt{\epsilon_{r1}}}{\sqrt{1-\left(\frac{f_{c0}/\sqrt{\epsilon_{r1}}}{f}\right)^2}} = \dfrac{377/\sqrt{\epsilon_r}}{\sqrt{1-\epsilon_r\left(\frac{6.56}{10}\right)^2}} = \dfrac{377/\sqrt{\epsilon_{r1}}}{\sqrt{1-0.43\epsilon_{r1}}}$

$\epsilon_{r1} = 1.531$

c. $l = \dfrac{\lambda_{g1}}{4} = \dfrac{\lambda_0/\sqrt{\epsilon_{r1}}}{4\sqrt{1-\left(\frac{f_{c1}}{f}\right)^2}} = \dfrac{30\times10^9/(10\times10^9\sqrt{1.531})}{4\sqrt{1-\left(\frac{f_{c0}/\sqrt{\epsilon_{r1}}}{f}\right)^2}} = \dfrac{3/\sqrt{1.531}}{4\sqrt{1-\left(\frac{6.56}{\sqrt{1.531}(10)}\right)^2}} = 0.7149 \text{ cm}$

Problem: 6

The dominant TE_{10} mode, transmitting an average power P_{av}, propagates through a rectangular waveguide of sides a and b. Find the largest amplitude of the electric field intensity in the waveguide.

Answer.

$$E_{max} = 4\mu f_c \sqrt{\sqrt{\frac{\varepsilon}{\mu}} \frac{a P_{av}}{b\sqrt{1-f_c^2/f^2}}}$$

Problem: 7

In a rectangular waveguide, with sides a and b, two signals are simultaneously emitted. The frequency spectrum of one is in the immediate vicinity of frequency f_1, that of the other in the vicinity of f_2. Both propagate as TE_{10} waves. Find the time interval Δt between the arrival of the two signals at the receiving point, which is at a distance L from the sending point, if the dielectric is air.

Answer. For $f_1 = 10$ GHz, $f_2 = 12$ GHz, $a = 2$ cm, $b = 1$ cm, and $L = 10$ m, $\Delta t = 7.69 \times 10^{-9}$ s.

Q1

Determine the types and senses of polarization for the following uniform plane waves. Verify your answers (by sketching and/or explaining).

a) $\bar{E}(\bar{r},t)=\hat{a}_x\cos(\omega t-kz+\frac{\pi}{6})-\hat{a}_y\sin(\omega t-kz-\frac{\pi}{3})$

b) $\bar{E}(\bar{r})=(\hat{a}_x+j3\hat{a}_y)e^{-jkz}$

c) $\bar{E}(\bar{r})=(\hat{a}_x e^{-j\pi/4}+\hat{a}_z e^{j\pi/4})e^{-jky}$

a) $\sin(\omega t-kz-\frac{\pi}{3}) = \sin(\omega t-kz+\frac{\pi}{6}-\frac{\pi}{2}) = -\cos(\omega t-kz+\frac{\pi}{6})$

$\bar{E}(\bar{r},t)=(\hat{a}_x+\hat{a}_y)\cos(\omega t-kz+\pi/6)$

no phase difference between orthogonal components

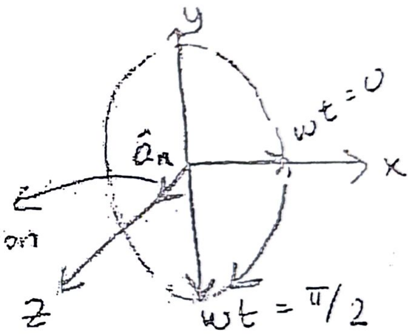
$\Rightarrow$ Linear polarization

b) $\bar{E}(\bar{r},t)=\hat{a}_x\cos(\omega t-kz)-3\hat{a}_y\sin(\omega t-kz)$

$z=0$ at $\omega t=0$ $\bar{E}=\hat{a}_x$

at $\omega t=\pi/2$ $\bar{E}=-3\hat{a}_y$

direction of propagation



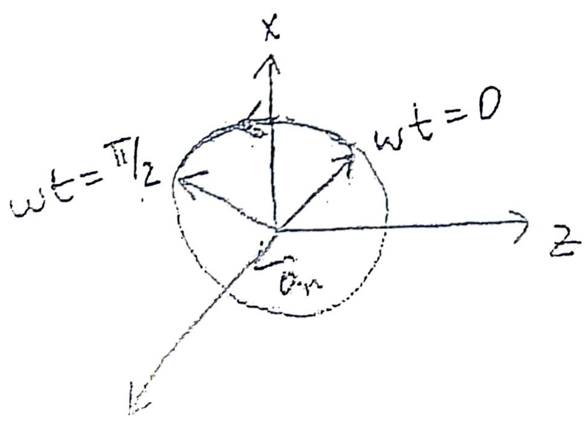
Left Hand

Elliptically Polarized

c) $\bar{E}(\bar{r},t)=\hat{a}_x\cos(\omega t-ky-\pi/4)+\hat{a}_z\cos(\omega t-ky+\pi/4)$

at $y=0$ $\omega t=0$ $\bar{E}=\frac{\sqrt{2}}{2}\hat{a}_x+\frac{\sqrt{2}}{2}\hat{a}_z$

$\omega t=\pi/2$ $\bar{E}=\frac{\sqrt{2}}{2}\hat{a}_x-\frac{\sqrt{2}}{2}\hat{a}_z$



Right Hand

Circularly Polarized

Q 2

The electric field intensity of a plane wave propagating in the $+\hat{a}_z$ direction is given as follows at z=0:

$$\vec{E}(z)\big|_{z=0} = \hat{a}_x 5 \text{ V/m}$$

The electrical properties of the lossy medium are as follows: $\varepsilon = 9\varepsilon_0$, $\mu = \mu_0$, $\sigma = 5$ S/m. The frequency of the wave is 10 GHz. (10^{10} Hz)

Determine:

a. The intrinsic impedance of the medium.
b. Complex propagation constant, phase velocity and wavelength of the wave.
c. Magnetic field intensity.
d. Average power flow vector.

HINT :

- *intrinsic impedance* $\eta = \sqrt{\dfrac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}}$, *propagation constant* $\gamma = j\omega\sqrt{\mu\left(\varepsilon - j\dfrac{\sigma}{\omega}\right)}$ *for lossy media*
- $(e^{j\theta})^{1/n} = e^{j\theta/n}$, $k = 1, 2, .. n-1$ *(not all mathematical answers are physical solutions)*

$\bar{E}(z) = \hat{a}_x 5 e^{-\alpha z} e^{-j\beta z}$, $\epsilon = 9\epsilon_0$, $\mu = \mu_0$, $\sigma = 5$, $f = 10$ GHz

$\frac{\sigma}{\omega\epsilon} = 1 \Rightarrow$ neither good conductor/nor lossy dielectric

a) $\eta = \sqrt{\dfrac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \dfrac{40\pi}{2^{1/4}} e^{j\pi/8} = 105.7 e^{j\pi/8}$ //

b) $\gamma = j\omega\sqrt{\mu(\epsilon - j\frac{\sigma}{\omega})} = 200\pi \cdot 2^{1/4} \cdot e^{-j\pi/8} \Rightarrow$ $\alpha = 91\pi = 286$ //
$\beta = 219.7\pi = 690$

$v_p = \dfrac{\omega}{\beta} = 91.6$ m/sec // , $\lambda = \dfrac{2\pi}{\beta} = 9.1$ mm //

c) $\bar{H} = \frac{1}{\eta} \hat{a}_z \times \bar{E} = \hat{a}_y \dfrac{5}{200\, 2^{-1/4}} e^{-j\pi/8} e^{-91\pi z} e^{-j219.7\pi z}$ //

d) $\bar{P}_{av} = \frac{1}{2} \text{Re}\{\bar{E} \times \bar{H}^*\} = \hat{a}_z\, 0.11\, e^{-2\alpha z} = \hat{a}_z\, 0.11\, e^{-182\pi z}$ //

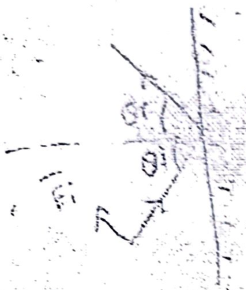
Q3

A uniform plane wave in air (ϵ_o, μ_o), represented by the phasor electric field,

$$\bar{E}_i = (\sqrt{3}\,\hat{a}_x - \hat{a}_z)\, e^{-j\pi\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)} \qquad z < 0$$

is obliquely incident upon a perfectly conducting wall at $z = 0$. Find,

a) the phasor expressions for the reflected electric and magnetic field, $\bar{E}_r$ and $\bar{H}_r$.
b) The surface current density, $\bar{J}_s$, at the interface.
c) Time average Poynting's vector of the total field in air.



$\hat{a}_{n_i} = \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z \qquad \theta_i = 30° = \theta_r$

$\hat{a}_{n_r} = \frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z$

$\bar{E}_i = (\sqrt{3}\,\hat{a}_x - \hat{a}_z)\, e^{-j\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$

$\bar{H}_i = \frac{1}{\eta_0}\hat{a}_{n_i} \times \bar{E}_i = 120\pi \left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right) \times (\sqrt{3}\,\hat{a}_x - \hat{a}_z)\, e$

$= \hat{a}_y \frac{1}{60\pi}\, e^{-j\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$

a) $\Gamma = -1, \quad \theta_r = 30°$

$\bar{E}_r = -(\sqrt{3}\,\hat{a}_x + \hat{a}_z)\, e^{-j\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)}$

$\bar{H}_r = \hat{a}_y \frac{1}{60\pi}\, e^{-j\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)}$

b) from B.C. on

$\bar{J}_s = -\hat{a}_z \times \bar{H}_1 \Big|_{z=0}$

$\bar{H}_1 = \bar{H}_i + \bar{H}_r = \hat{a}_y \frac{1}{60\pi}\left(e^{-j\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)} + e^{-j\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}\right)$

$= \hat{a}_y \frac{1}{30\pi}\, e^{-j\frac{1}{2}x}\, 2\cos\frac{\sqrt{3}}{2}z \Big|_{z=0}$

$\bar{J}_s = \hat{a}_x \frac{1}{30\pi}\, e^{-j\frac{1}{2}x} \cos\frac{\sqrt{3}}{2}z \Big|_{z=0} = \hat{a}_x \frac{e^{-j\frac{1}{2}x}}{30\pi}$

c) $P_{av} = \frac{1}{2}\,\mathrm{Re}\{\bar{E}_1 \times \bar{H}_1^*\}$

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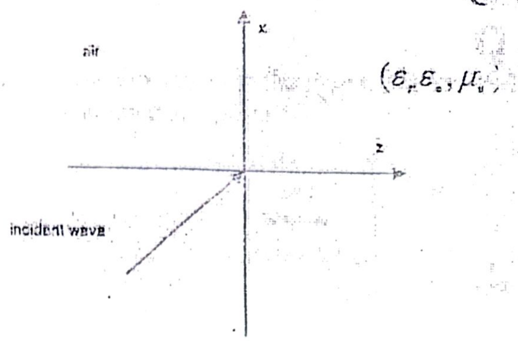
Q 4

A uniform plane wave is incident from air (ε_0, μ_0) onto a lossless dielectric medium with $\varepsilon_2 = \varepsilon_r \varepsilon_0$, $\mu_2 = \mu_0$ as shown in the figure. The phasor magnetic field intensities of the incident and transmitted waves are given by,



$$\vec{H}_i = \hat{a}_y e^{-j\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z\right)}$$

$$\vec{H}_t = \hat{a}_y H_{t0} e^{-j\beta_2\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$$

Find,

a) the angle of incidence, θ_i, the angle of reflection, θ_r, and the angle of refraction (angle of transmission), θ_t.

b) Relative dielectric constant of the dielectric medium, ε_r.

c) The reflected magnetic field intensity, $\vec{H}_r$, in the phasor domain.

a) $\bar{k}_i = \frac{1}{\sqrt{2}}\hat{a}_x + \frac{1}{\sqrt{2}}\hat{a}_z = k\,\hat{a}_{ni} \rightarrow \theta_i = 45°, \quad \theta_r = \theta_i = 45°$

$\hat{a}_{nt} = \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z \rightarrow \theta_t = \tan^{-1} \frac{1/2}{\sqrt{3}/2} \Rightarrow \theta_t = 30°$

b) From Snell's law $k_1 \sin\theta_i = k_2 \sin\theta_t \Rightarrow \sin\theta_i = \sqrt{\varepsilon_r}\sin\theta_t$

$\sqrt{\varepsilon_r} = \frac{1/\sqrt{2}}{1/2} = \sqrt{2} \Rightarrow \varepsilon_r = 2$

c) $H_r = -\hat{a}_y\, \Gamma_{\parallel}\begin{pmatrix}\theta_i = 45° \\ \theta_t = 30°\end{pmatrix} e^{-j\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}z\right)}$

$$\Gamma_{\parallel} = \frac{\eta_2\cos(30°) - \eta_1\cos(45°)}{\eta_2\cos(30°) + \eta_1\cos(45°)} = \frac{\frac{1}{\sqrt{2}}\eta_0\frac{\sqrt{3}}{2} - \eta_0\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\eta_0\frac{\sqrt{3}}{2} + \eta_0\frac{1}{\sqrt{2}}} = \frac{\frac{\sqrt{3}}{2} - 1}{\frac{\sqrt{3}}{2} + 1}$$

$$= \frac{\sqrt{3} - 2}{\sqrt{3} + 2}$$

$$\bar{E}_1 = (\sqrt{3}\,\hat{a}_x - \hat{a}_z)\, e^{-j\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)} + (-\sqrt{3}\,\hat{a}_x - \hat{a}_z)\, e^{-j\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)}$$

$$= e^{-j\frac{1}{2}x}\left(\sqrt{3}\,\hat{a}_x e^{-j\frac{\sqrt{3}}{2}z} - \hat{a}_z e^{-j\frac{\sqrt{3}}{2}z} - \sqrt{3}\,\hat{a}_x e^{j\frac{\sqrt{3}}{2}z} - \hat{a}_z e^{j\frac{\sqrt{3}}{2}z}\right)$$

$$= e^{-j\frac{1}{2}x}\left(\hat{a}_x\sqrt{3}\underbrace{\left(e^{-j\frac{\sqrt{3}}{2}z} - e^{j\frac{\sqrt{3}}{2}z}\right)}_{-j2\sin\frac{\sqrt{3}}{2}z} - \hat{a}_z\underbrace{\left(e^{-j\frac{\sqrt{3}}{2}z} + e^{j\frac{\sqrt{3}}{2}z}\right)}_{2\cos\frac{\sqrt{3}}{2}z}\right)$$

$$= \hat{a}_x\left(-j2\sqrt{3}\sin\frac{\sqrt{3}}{2}z\right)e^{-j\frac{1}{2}x} - \hat{a}_z\, 2\cos\frac{\sqrt{3}}{2}z\, e^{-j\frac{1}{2}x}\ldots$$

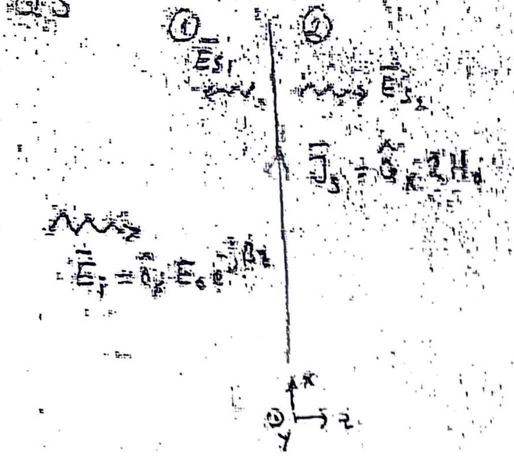
$$\bar{H}_1 = \hat{a}_y \frac{1}{30\pi}\cos\left(\frac{\sqrt{3}}{2}z\right)e^{-j\frac{1}{2}x}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\bar{E}_1 \times \bar{H}_1^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\left[\hat{a}_x(-j2\sqrt{3}\sin\frac{\sqrt{3}}{2}z - \hat{a}_z 2\cos\frac{\sqrt{3}}{2}z\right]e^{-j\frac{1}{2}x}\right.$$

$$\left.\times\ \hat{a}_y\frac{1}{30\pi}\cos\left(\frac{\sqrt{3}}{2}z\right)e^{j\frac{1}{2}x}\right]\Big\}$$

$$= \hat{a}_x \frac{\cancel{2}}{30\pi}\cos^2\left(\frac{\sqrt{3}}{2}z\right)\cdot\frac{1}{\cancel{2}}$$

Q3



In Region ②, incident fields and fields due to $\bar J_s$ exist such that the total field is zero.

i.e. $\bar E_i + \bar E_{s_2} = 0$ $(z>0$; in ②$)$

The fields due to $\bar J_s$ are plane waves.

$\bar E_{s1}, \bar H_{s1}$ are the fields due to $\bar J_s$ in ①:

$\bar E_1 = \bar E_i + \bar E_{s1}$, $\bar H_1 = \bar H_i + \bar H_{s1}$

$\bar E_{s1} = \hat a_x E_{s1} e^{+j\beta z}$, $\bar H_{s1} = \hat a_y H_{s1} e^{+j\beta z}$

$\bar H_{s1} = -\hat a_z \times \frac{\bar E_{s1}}{\eta_0}$

Boundary Condition $\hat a_{n_2} \times (\bar H_1 - \bar H_2)\Big|_{z=0} = \bar J_s$ $H_0 = \frac{E_0}{\eta_0}$

$-\hat a_z \times (\bar H_i(z=0) + \bar H_s(z=0)) = \hat a_x 2H_0$ i.e. $H_{s1} = 2H_0 - H_0 = H_0$

$\bar H_r = \bar H_{s1} = \hat a_y H_0 e^{+j\beta z} \rightarrow \bar E_r = \bar E_{s1} = -\hat a_x E_0 e^{+j\beta z}$

This problem is equivalent to the reflection of a wave normally incident onto a perfectly conducting boundary. The fields in Region 2 is zero due to the conductor. The impressed current $\bar J_s$ is the induced current on the conducting surface.

Q 4

A uniform plane wave with a magnetic field given by

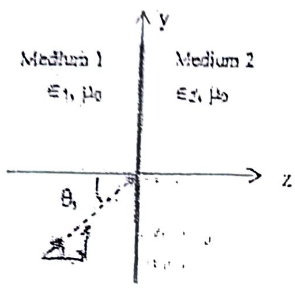
$$\overline{H}(y,z,t) = \hat{a}_x H_0 \cos\left[9\pi 10^9 t - 90\pi\left(y + \sqrt{3}z\right)\right] \quad \text{A/m}$$

is obliquely incident on an interface at z=0 separating two nonmagnetic ($\mu_1=\mu_2=\mu_0$) lossless media.

a) Calculate the relative permittivity of medium 1, ϵ_{1r} and the angle of incidence, θ_i.

b) Find the maximum value of the relative permittivity of medium 2, $\epsilon_{2r,max}$, so that total reflection occurs.

c) Is it possible to achieve total transmission by adjusting the incidence angle? Explain your reasoning.
If the answer is yes, use $\epsilon_{2r,max}$ found in part (b) to calculate the angle of incidence for total transmission.



a) $\omega = 9\pi 10^9$ $\qquad \bar{k}\cdot\bar{r} = 90\pi(y+\sqrt{3}z)$

$\bar{k} = \underbrace{90\cdot 2\pi}_{k_1}\left(\frac{1}{2}\hat{a}_y + \frac{\sqrt{3}}{2}\hat{a}_z\right)$ $\qquad k_1 = \omega\sqrt{\mu_0\epsilon_0\epsilon_{1r}} = \frac{9\pi 10^9}{3\cdot10^8}\sqrt{\epsilon_{1r}} = 180\pi$

$\sqrt{\epsilon_{1r}} = 6 \Rightarrow \epsilon_{1r} = 36$

$\tan\theta_i = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} \Rightarrow \theta_i = 30°$

b) $k_1 \sin\theta_i = k_2 \sin\theta_t$, $\theta_t = \frac{\pi}{2} \Rightarrow \omega\sqrt{\mu_0\epsilon_0\epsilon_{r2}} = \omega\sqrt{\mu_0\epsilon_0\epsilon_{r1}}\sin 30$

$\sqrt{\epsilon_{r2}} = 6\cdot\frac{1}{2} \Rightarrow \epsilon_{r2,max} = 9$

c) $\bar{E}_i$ // to plane of incidence

$$\sin\theta_B^{\parallel} = \sqrt{\frac{1-\epsilon_1/\epsilon_2}{1-(\epsilon_1/\epsilon_2)^2}} = \sqrt{\frac{1-36/9}{1-(36/9)^2}} = \sqrt{\frac{3}{15}} = \sqrt{\frac{1}{5}}$$

$$\theta_i = \theta_{B\parallel} = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) = 26.56°$$

$\vec{E}_{\hat{z}}, T$

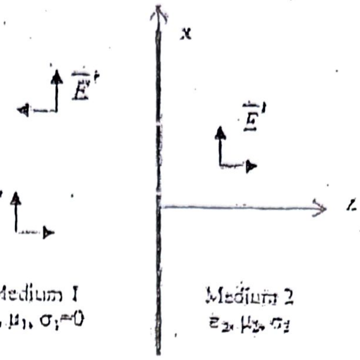
Q 5

Consider a uniform plane wave normally incident on a lossy medium (good conductor) as shown in the following figure. The phasor fields for incident, reflected and transmitted waves are given as follows:

$$\overline{E}^i = \bar{a}_x E_{i0} e^{-j\beta_1 z} \qquad \overline{H}^i = \bar{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

$$\overline{E}^r = \bar{a}_x \Gamma E_{i0} e^{j\beta_1 z} \qquad \overline{H}^r = -\bar{a}_y \frac{\Gamma E_{i0}}{\eta_1} e^{j\beta_1 z}$$

$$\overline{E}^t = \bar{a}_x T E_{i0} e^{-\gamma_2 z} \qquad \overline{H}^t = \bar{a}_y \frac{T E_{i0}}{\eta_2} e^{-\gamma_2 z}$$



where $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\theta_\Gamma}$,

$T = \dfrac{2\eta_2}{\eta_2 + \eta_1} = |T| e^{j\theta_T}$, E_{i0} is real.

Note that for a good conductor $\eta_2 = \dfrac{1}{\sigma_2 \delta}(1+j)$, $\gamma_2 = \dfrac{1}{\delta}(1+j)$, and δ is the skin depth.

a) Find the incident power density in medium 1. Express the time average Poynting's vector in medium 1 in terms of incident power density and the reflection coefficient.

b) Find the time average Poynting's vector in medium 2 at the boundary (z=0 plane).

c) Find the dissipated power density in medium 2.

(Hint: Power dissipated in volume v, $P_{dis} = \frac{1}{2}\int \overline{E} \cdot \overline{J}\, dv$).

d) Compare the power densities obtained in parts (a), (b) and (c) to each other. Comment on your comparison.

a) $\langle \overline{P} \rangle = \frac{1}{2} \mathrm{Re}\{\overline{E} \times \overline{H}^*\} \Rightarrow \langle \overline{P}_i \rangle = \hat{a}_z \dfrac{E_{i0}^2}{2\eta_1}$

$\langle \overline{P}_1 \rangle = \frac{1}{2} \mathrm{Re}\{(\overline{E}^i + \overline{E}^r) \times (\overline{H}^i + \overline{H}^r)^*\}$

$= \hat{a}_z \dfrac{E_{i0}^2}{2\eta_1}(1 - |\Gamma|^2) = \langle \overline{P}_i \rangle (1 - |\Gamma|^2)$

b) $\langle \overline{P}_2 \rangle \Big|_{z=0} = \frac{1}{2} \mathrm{Re}\{\overline{E}^t \times \overline{H}^{t*}\}\Big|_{z=0} = \dfrac{\hat{a}_z E_{i0}^2}{2} |T|^2 \mathrm{Re}\left\{\dfrac{1}{\eta_2^*}\right\}$

$= \hat{a}_z \dfrac{E_{i0}^2 |T|^2 \sigma_2 \delta}{4}$

c) $p_{dis} = \frac{1}{2} \mathrm{Re}\{\overline{E} \cdot \overline{J}^*\} = \frac{1}{2} \sigma_2 E_{i0}^2 |T|^2 e^{-\frac{2}{\delta} z}$

P_{dis} (per unit area) $= \displaystyle\int_{z=0}^{\infty} p_{dis}\, dz = \frac{1}{4} E_{i0}^2 |T|^2 \sigma_2 \delta$

To show

$$\langle \bar{P}_i \rangle \Big|_{z=0} = \langle \bar{P}_r \rangle \Big|_{z=0} + \langle \bar{P}_t \rangle \Big|_{z=0}$$

show

$$1 + |\Gamma|^2 = \eta_1 |T|^2 \operatorname{Re}\left\{ \frac{1}{\eta_2^*} \right\}$$

Question 1. (20 points)

A uniform plane wave is propagating in an unbounded simple (i.e. linear, homogeneous, isotropic and time-invariant) medium at the frequency $f = 8\ GHz = 8\times10^9\ Hz$. The permittivity of the medium is known to be $\varepsilon = 9\,\varepsilon_0$. The time-domain $\overline{E}$ and $\overline{H}$ field expressions of the uniform plane wave are given as

$$\overline{E}(\overline{r},t) = (-\sqrt{3}\,\hat{a}_x + \hat{a}_z)\,e^{-\alpha\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}z\right)}\cos\left(\omega t - \beta\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}z\right)\right) \qquad (V/m)$$

and

$$\overline{H}(\overline{r},t) = \hat{a}_y\left(-\frac{\sqrt{2}}{20\,\pi}\right)e^{-\alpha\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}z\right)}\cos\left(\omega t - \beta\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}z\right)-\frac{\pi}{6}\right) \qquad (A/m)$$

where α (attenuation constant) and β (phase constant) are yet to be determined.

a) Find the conductivity (σ) and the permeability ($\mu = \mu_r\,\mu_0$) of the medium.

b) For $\alpha = 80\sqrt{2}\,\pi$ (nep/m) and $\beta = 80\sqrt{6}\,\pi$ (rad/m), find

i) the velocity of propagation,

ii) the wavelength,

iii) the skin depth.

<u>DON'T forget to show your solution steps clearly and to give units wherever needed !</u>

<u>**SOLUTION:**</u>

a) For a u.p.w. propagating in a lossy medium, in general,

$$\overline{E} = \overline{E}_0\,e^{-\gamma\,\hat{n}\bullet\overline{r}} \text{ and } \overline{H} = \frac{1}{\eta}\hat{n}\times\overline{E} \text{ in phasor domain where } \gamma = \alpha + j\beta \text{ and}$$

$\eta = |\eta|\,e^{j\Phi} = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\varepsilon}}$. Then, obviously, $\dfrac{|\overline{E}|}{|\overline{H}|} = |\eta|$ and in phase, $\overline{H}$ lags $\overline{E}$ by Φ radians. Equivalently, in time-domain,

$$\overline{E}(\overline{r},t) = \overline{E}_0\,e^{-\alpha\,\hat{n}\bullet\overline{r}}\cos(\omega t - \beta\hat{n}\bullet\overline{r}) = \hat{u}_E\,|\overline{E}_0|\,e^{-\alpha\,\hat{n}\bullet\overline{r}}\cos(\omega t - \beta\hat{n}\bullet\overline{r}) \qquad \text{and}$$

$$\overline{H}(\overline{r},t) = \hat{u}_H\,|\overline{H}_0|\,e^{-\alpha\,\hat{n}\bullet\overline{r}}\cos(\omega t - \beta\hat{n}\bullet\overline{r} - \Phi)$$

where $|\overline{E}_0| = |\eta|\,|\overline{H}_0|$ with $\hat{u}_E$ and $\hat{u}_H$ being the unit vectors in the directions of $\overline{E}$ and $\overline{H}$ fields, respectively.

Here, in this problem,

$$|\eta| = \frac{|\bar{E}_0|}{|\bar{H}_0|} = \frac{|-\sqrt{3}\,\hat{a}_x + \hat{a}_z|}{\left|\dfrac{\sqrt{2}\,\hat{a}_y}{20\,\pi}\right|} = \frac{40\,\pi}{\sqrt{2}}\ (\Omega) \quad \text{and} \quad \Phi = \frac{\pi}{6}\ \text{radians} = 30\ \text{deg.}$$

Then,

$$\frac{j\omega\mu}{\sigma + j\omega\varepsilon} = \eta^2 = \left(\frac{40\,\pi}{\sqrt{2}}\,e^{j\frac{\pi}{6}}\right)^2 = 800\,\pi^2\,e^{j\frac{\pi}{3}} = 800\,\pi^2\,(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3})$$

$$\frac{j2\pi\times 8\times 10^9\times 4\pi\times 10^{-7}\,\mu_r}{\sigma + j2\pi\times 8\times 10^9\times 9\times\dfrac{10^{-9}}{36\,\pi}} = \frac{j\,6400\,\pi^2\,\mu_r}{\sigma + j4} = 800\,\pi^2\,\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\ j16\,\mu_r = (\sigma - 4\sqrt{3}) + j\,(4 + \sigma\sqrt{3}) \qquad \Rightarrow\ \sigma = 4\sqrt{3}\ \ (S/m)\ \ (\textit{from the real part})$$

$$\text{and}\ \ \mu_r = 1\ \Rightarrow\ \mu = \mu_0\ \ (H/m)\ \ (\textit{from the imaginary part})$$

b) For $\alpha = 80\sqrt{2}\,\pi\ \ (nep/m)$ and $\beta = 80\sqrt{6}\,\pi\ \ (rad/m)$, (actually, these values can be solved after answering part (a) above),

(i) The velocity of propagation is

$$v = \frac{\omega}{\beta} = \frac{2\pi\times 8\times 10^9}{80\sqrt{6}\,\pi} = \frac{2}{\sqrt{6}}10^8 \cong 0.816\times 10^8\ \ (m/s)$$

(ii) The wavelength is $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{80\sqrt{6}\,\pi} \cong 0.0102\ \ (m)$

(iii) The skin depth is $\delta = \dfrac{1}{\alpha} = \dfrac{1}{80\sqrt{2}\,\pi} \cong 2.81\times 10^{-3}\ \ (m)$

Q1

a) Consider the following circularly polarized uniform plane wave
$\overline{E}(z) = E_0(\hat{a}_x - j\hat{a}_y)e^{-jkz}$
Show that the <u>instantaneous</u> power density is constant (independent of time) and it is equal to
$\overline{\mathcal{P}}(z,t) = \dfrac{E_0^2}{\eta}\hat{a}_z$

b) The following uniform plane wave propagates in a dielectric medium with $\varepsilon=\varepsilon_r\varepsilon_0$ and $\mu=\mu_0$.

$$\overline{E}(z) = (6j\hat{a}_x + A\hat{a}_y + B\hat{a}_z)e^{-j(4\pi x+3\pi z)}$$

Find the constants A, B and the frequency of the wave if both of the following constraints are satisfied

i) the <u>instantaneous</u> power density is $\overline{\mathcal{P}}(z,t) = \dfrac{4\hat{a}_x + 3\hat{a}_z}{\pi}$ Watts/m² and

ii) time domain electric field intensity vector is in +y-direction at t=0.

a) $\overline{H} = \dfrac{E_0}{\eta}\,\hat{a}_z \times (\hat{a}_x - j\hat{a}_y)\,e^{-jkz} = \dfrac{E_0}{\eta}(j\hat{a}_x + \hat{a}_y)\,e^{-jkz}$

$\overline{\mathcal{H}}(z,t) = \dfrac{E_0}{\eta}\left(-\sin(\omega t - kz)\,\hat{a}_x + \cos(\omega t - kz)\,\hat{a}_y\right)$

$\overline{\mathcal{E}}(z,t) = E_0\left(\cos(\omega t - kz)\,\hat{a}_x + \sin(\omega t - kz)\,\hat{a}_y\right)$

$\overline{\mathcal{P}}(z,t) = \overline{\mathcal{E}} \times \overline{\mathcal{H}} = \dfrac{E_0^2}{\eta}\left(\sin^2(\omega t - kz)\,\hat{a}_z + \cos^2(\omega t - kz)\,\hat{a}_z\right) = \dfrac{E_0^2}{\eta}\hat{a}_z$

b) $\overline{k} = 4\pi\hat{a}_x + 3\pi\hat{a}_z \Rightarrow k = 5\pi \qquad \hat{a}_n = 0.8\hat{a}_x + 0.6\hat{a}_z$

Uniform plane wave $\Rightarrow \hat{a}_n \cdot \overline{E} = 0 \Rightarrow \boxed{B = -8j}$

$\overline{\mathcal{P}}(t)$ is constant $\Rightarrow$ circularly polarized.

$\Rightarrow |E_y| = |E_x\hat{a}_x + E_z\hat{a}_z| = \sqrt{6^2+8^2} = 10$

$\overline{\mathcal{E}}$ is in $+y$-direction at $t=0 \Rightarrow \boxed{A = +10}$

from part a): $\overline{\mathcal{P}} = \dfrac{E_0^2}{\eta}\hat{a}_n = \dfrac{10^2}{\eta}(0.8\hat{a}_x + 0.6\hat{a}_z) = \dfrac{4\hat{a}_x + 3\hat{a}_z}{\pi}$

$\eta = 20\pi \qquad \eta = \sqrt{\dfrac{\mu_0}{\varepsilon_0\varepsilon_r}} = \dfrac{120\pi}{\sqrt{\varepsilon_r}} = 20\pi \qquad \varepsilon_r = 36$

$k = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \dfrac{\omega\sqrt{\varepsilon_r}}{c} = 5\pi = \dfrac{2\pi f\, 6}{3\times10^8}$

$f = \dfrac{5\times10^8}{4} = 125\,\text{MHz}$

$\overline{k} = 4\pi\hat{a}_x + 3\pi\hat{a}_z \Rightarrow k = 5\pi \qquad \hat{a}_n = 0.8\hat{a}_x + 0.6\hat{a}_z$

Uniform plane wave $\Rightarrow \hat{a}_n \cdot \overline{E} = 0$

$(0.8\hat{a}_x + 0.6\hat{a}_z)\cdot(6j\hat{a}_x + A\hat{a}_y + B\hat{a}_z) = 0$

$4.8j + 0.6B = 0 \Rightarrow \boxed{B = -8j}$

$\overline{\mathcal{P}}(t)$ is constant $\Rightarrow$ cir

Q 2

Time-domain electromagnetic field expressions for a uniform plane wave are given as:

$$\bar{E}(\bar{r},t) = 2\, e^{-1.5\times10^4 z} \cos(2\pi\times10^6 t - 1.5\times10^4 z)\,\hat{a}_x \quad (V/m)$$

$$\bar{H}(\bar{r},t) = 5373.45\, e^{-1.5\times10^4 z} \cos(2\pi\times10^6 t - 1.5\times10^4 z - \frac{\pi}{4})\,\hat{a}_y \quad (A/m)$$

Answer the following questions including the units wherever applicable. Write down the formulas/expressions used in your calculations clearly.

a) Write down the attenuation constant, α.
$\alpha = 1.5\times10^4$ Neper/m
b) Write down the phase constant, β.
$\beta = 1.5\times10^4$ rad/m
c) Write down the frequency of oscillation, f.
$f = 10^6$ Hz $= 1$ MHz.
d) Classify the medium based on the given field expressions (i.e. Is it a perfect dielectric, good insulator, good conductor, perfect conductor, etc...?) Explain briefly but clearly, why.
e) Find the intrinsic impedance of the medium, η.
f) Find the wavelength, λ.
g) Find the velocity of propagation (i.e. the phase velocity), v_p.
h) Find the skin depth, δ.
i) Assume the medium is non-magnetic (i.e. $\mu = \mu_0$) and find its conductivity, σ.

d) The medium is a good conductor as $\alpha = \beta$.

e) $\eta = |\eta| e^{j\phi}$ where $\phi = \frac{\pi}{4}$ rad. and $|\eta| = \frac{2}{5373.45}$ Ω

$\Rightarrow \eta \cong 3.72\times10^{-4} \underbrace{e^{j\pi/4}}_{\frac{1}{\sqrt{2}}(1+j)} \cong 2.63\times10^{-4}(1+j)\ \Omega$

f) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1.5\times10^4} \cong 4.19\times10^{-4}$ m.

g) $v_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{2\pi\times10^6}{1.5\times10^4} \cong 418.9$ m/s

h) $\delta = \frac{1}{\alpha} = \frac{1}{1.5\times10^4} \cong 6.67\times10^{-5}$ m.

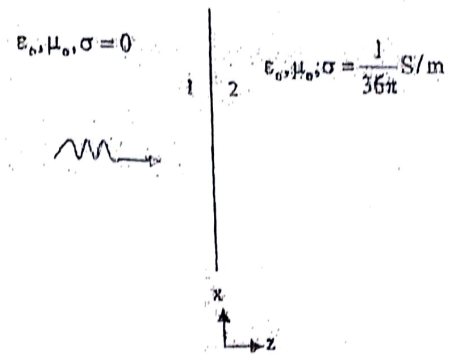
i) We know for a good conductor that $\alpha \cong \sqrt{\pi f \mu_0 \sigma}$

$\Rightarrow \sigma = \frac{\alpha^2}{\pi f \mu_0} = \frac{(1.5\times10^4)^2}{\pi\times10^6\times4\pi\times10^{-7}} \cong 5.7\times10^7$ S/m

Q2

A plane wave is normally incident to a planar dielectric boundary. The expression of the electric field intensity of the wave is:

$$\vec{E} = E_{oi}\hat{a}_x e^{-j\frac{10}{3}z} \text{ V/m}$$



The first medium is free-space ($\varepsilon_o, \mu_o, \sigma = 0$ S/m). The second medium is a lossy medium ($\varepsilon_o, \mu_o, \sigma = \frac{1}{36\pi}$ S/m). Both media are simple (linear, isotropic, homogenous and time-invariant)

a) Calculate the intrinsic impedances of the both media.
b) Determine the electric field intensity of the reflected and the transmitted waves.
c) Write down the total magnetic field ($\vec{H}_1 = \vec{H}_i + \vec{H}_r$) in region 1.

a) $\epsilon_2 = \epsilon_o - j\frac{\sigma}{\omega} = \epsilon_o - j\frac{1}{36\pi}10^{-9} = (1-j)\,\epsilon_o$

$\eta_1 = 120\pi \ \Omega$

$\beta = \frac{10}{3} = \omega\sqrt{\mu_o \epsilon_o}$

$\boxed{\omega = 10^9 \text{ rad/sec}}$

$\eta_2 = \left[\frac{\mu_o}{\epsilon_o(1-j)}\right]^{1/2} = 120\pi\left(\frac{1}{1-j}\right)^{1/2}$

$\eta_2 = 120\pi\left(\frac{1}{2}(1+j1)\right)^{1/2} = 293 + j121 \ \Omega$

b) $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{293 + j121 - 377}{293 + j121 + 377} = -0.090 + j0.197 = 0.216\, e^{j114°}$

$T = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \times (293 + j121)}{293 + j121 + 377} = 0.91 + j0.197 = 0.931\, e^{j12.2°}$

$\bar{E}_r = 0.216\, E_{oi}\, \hat{a}_x\, e^{+j\frac{10}{3}z}\, e^{j114°}$

$\bar{E}_t = 0.931\, E_{oi}\, \hat{a}_x\, e^{+j12.2°}\, e^{-\gamma z}$

$\gamma = j\omega\sqrt{\mu_o \epsilon_o (1-j)} = j\frac{10}{3}\, 1.19\, e^{-22.5°} = 3.96\, e^{+j67}$

$\gamma = 1.52 + j3.66$

$= 0.931\, E_{oi}\, \hat{a}_x\, e^{+j12.2}\, e^{-1.52z}\, e^{-j3.66z}$

c) $\vec{H}_1 = \frac{1}{120\pi}\left(\hat{a}_z \times \bar{E}_i + (-\hat{a}_z) \times \bar{E}_r\right) = \frac{1}{120\pi}\, \hat{a}_y\, E_{oi}\left(e^{-j\frac{10}{3}z} - 0.216\, e^{j114}\, e^{+j\frac{10}{3}z}\right)$

Here, in this problem,

$$|\eta| = \frac{|\overline{E}_0|}{|\overline{H}_0|} = \frac{\left|-\sqrt{3}\,\hat{a}_x + \hat{a}_z\right|}{\left|-\frac{\sqrt{2}\,\hat{a}_y}{20\,\pi}\right|} = \frac{40\,\pi}{\sqrt{2}}\ (\Omega) \quad \text{and} \quad \Phi = \frac{\pi}{6} \text{ radians} = 30 \text{ deg.}$$

Then,

$$\frac{j\omega\mu}{\sigma + j\omega\varepsilon} = \eta^2 = \left(\frac{40\,\pi}{\sqrt{2}}\, e^{j\frac{\pi}{6}}\right)^2 = 800\,\pi^2\, e^{j\frac{\pi}{3}} = 800\,\pi^2\,(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3})$$

$$\frac{j2\pi\times 8\times 10^9\times 4\pi\times 10^{-7}\,\mu_r}{\sigma + j2\pi\times 8\times 10^9\times 9\times\frac{10^{-9}}{36\,\pi}} = \frac{j6400\pi^2\,\mu_r}{\sigma + j4} = 800\,\pi^2\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\ j16\,\mu_r = (\sigma - 4\sqrt{3}) + j(4 + \sigma\sqrt{3}) \qquad \Rightarrow\ \sigma = 4\sqrt{3}\quad (S/m)\quad (\textit{from the real part})$$

and $\mu_r = 1 \ \Rightarrow\ \mu = \mu_0 \quad (H/m) \quad$ (*from the imaginary part*)

b) For $\alpha = 80\sqrt{2}\,\pi$ (nep/m) and $\beta = 80\sqrt{6}\,\pi$ (rad/m), (actually, these values can be solved after answering part (a) above),

(i) The velocity of propagation is

$$v = \frac{\omega}{\beta} = \frac{2\pi\times 8\times 10^9}{80\sqrt{6}\,\pi} = \frac{2}{\sqrt{6}}10^8 \cong 0.816\times 10^8 \quad (m/s)$$

(ii) The wavelength is $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{80\sqrt{6}\,\pi} \cong 0.0102 \quad (m)$

(iii) The skin depth is $\delta = \dfrac{1}{\alpha} = \dfrac{1}{80\sqrt{2}\,\pi} \cong 2.81\times 10^{-3} \quad (m)$

Q 3.

Solution

(a) $\beta_1 = \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r} \to \varepsilon_r = 1 \to \boxed{\varepsilon_1 = \varepsilon_0}$

$\beta_2 = \frac{2\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r} \to \varepsilon_r = 4 \to \boxed{\varepsilon_2 = 4\varepsilon_0}$

(b) Note that impedances of the two media are

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 = 120\pi\ \Omega \text{ and } \eta_2 = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = \frac{\eta_0}{2} = 60\pi\ \Omega$$

Wave 1 : $\vec{E}_1 = \hat{a}_x e^{-jz\omega/c}$ and $\vec{H}_1 = \hat{a}_y \frac{e^{-jz\omega/c}}{\eta_0}$ where $z < 0$

Wave 2: $\vec{E}_2 = 2\hat{a}_x e^{j2z\omega/c}$ and $\vec{H}_1 = -\hat{a}_y 2\frac{e^{j2z\omega/c}}{\eta_0/2}$ where $z > 0$

Wave 3 $\vec{E}_3 = \hat{a}_x Ae^{jz\omega/c}$ and $\vec{H}_3 = -\hat{a}_y \frac{Ae^{jz\omega/c}}{\eta_0}$ where $z < 0$

Wave 4 $\vec{E}_4 = \hat{a}_x Be^{-j2z\omega/c}$ and $\vec{H}_4 = \hat{a}_y \frac{Be^{-j2z\omega/c}}{\eta_0/2}$ where $z > 0$

$E_1(0) + E_3(0) = E_2(0) + E_4(0) \to 1 + A = 2 + B \to A = B + 1$

$H_1(0) + H_3(0) = H_2(0) + H_4(0) \to \frac{1 - A}{\eta_0} = \frac{-2 + B}{\eta_0/2} \to 1 - A = -4 + 2B \to A = 5 - 2B$

$B + 1 = 5 - 2B \to 3B = 4 \to B = 4/3 \to A = 1 + 4/3 = 7/3$

(b) $\boxed{\vec{E}_3 = \hat{a}_x \frac{7}{3} e^{jz\omega/c}}$ where $z < 0$

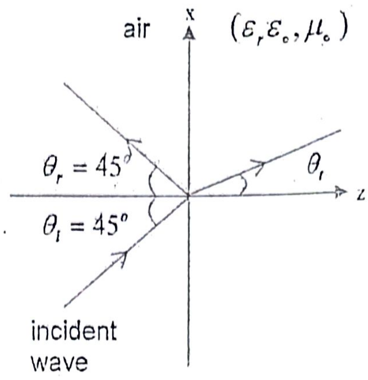
(c) $\boxed{\vec{E}_4 = \hat{a}_x \frac{4}{3} e^{-j2z\omega/c}}$ where $z > 0$

Q 4

There is a planar boundary at z=0 plane between free space and a lossless ($\sigma = 0$), non-magnetic ($\mu = \mu_o$) dielectric medium. A uniform plane wave is incident from air region to this boundary. Electric field intensities of the incident and the reflected waves are given as follows:

$$\bar{E}_i = \hat{a}_y\, 4\, e^{-j10\sqrt{2}\pi(x+z)} \text{ V/m}$$

$$\bar{E}_r = -\hat{a}_y\, 2.44\, e^{-j10\sqrt{2}\pi(x-z)} \text{ V/m}$$



It is known that angle of incidence is 45° and the frequency is 3 GHz (3×10^9 Hz).

Determine the electric and magnetic field intensities, $\bar{E}_t$ and $\bar{H}_t$, of the transmitted wave to the dielectric medium and the angle of transmission, θ_t.

Hint: Use boundary conditions.

$\hat{a}_{ni} = \frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_z)$ $\quad \bar{H}_i = \frac{1}{\eta_o}\hat{a}_{ni} \times \bar{E}_i = \frac{4}{120\pi}\frac{1}{\sqrt{2}}(-\hat{a}_x + \hat{a}_z)\, e^{-j\bar{k}_i\cdot\bar{r}}$ $\quad \bar{k}_i = 20\pi\, \hat{a}_{ni}$

$\hat{a}_{nr} = \frac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_z)$ $\quad \bar{H}_r = \frac{2.44}{120\pi}\frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_z)\, e^{-j\bar{k}_r\cdot\bar{r}}$ $\quad \bar{k}_r = 20\pi\, \hat{a}_{nr}$

BC: $E_{tang,1} = E_{tang,2}$, $\quad H_{tang,1} = H_{tang,2}$ and $B_{n,1} = B_{n,2} \Rightarrow H_{n,1} = H_{n,2}$

$\bar{H}_1 = \bar{H}_i + \bar{H}_r$, $\quad \bar{H}_2 = \bar{H}_t$; $\quad \bar{E}_1 = \bar{E}_i + \bar{E}_r$, $\quad \bar{E}_2 = \bar{E}_t$

$\therefore\ \bar{E}_t(z=0) = \hat{a}_y\, 1.56\, e^{-j\bar{k}_2\cdot\bar{r}}$

$\therefore\ \bar{H}_t(z=0) = (-12.08\,\hat{a}_x + 2.93\,\hat{a}_z)\, 10^{-3}\, e^{-j\bar{k}_2\cdot\bar{r}}$

$\frac{|E_t|}{|H_t|} = \eta_2 = \frac{1}{\sqrt{\varepsilon_r}}120\pi = \frac{1.56}{0.01243} = 125.50 \rightarrow \varepsilon_r = 9$ $\qquad k_2 = \sqrt{\varepsilon_r}\, k_1 = 60\pi$

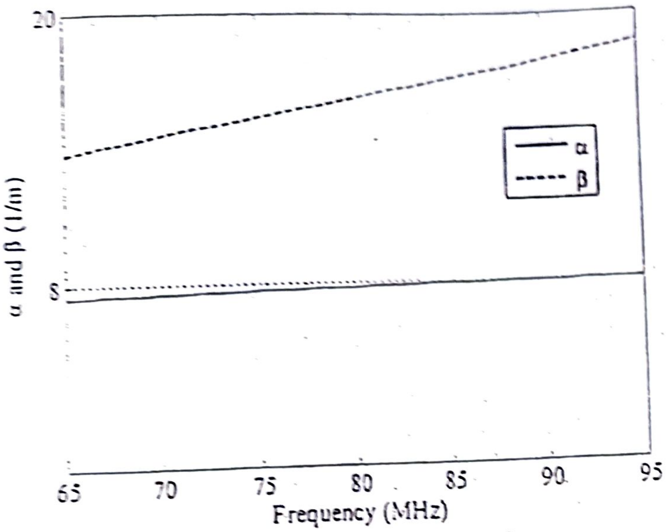
$\theta_t = \arcsin\left(\frac{n_1}{n_2}\sin\theta_i\right) = \arcsin\left(\frac{1}{3}\frac{1}{\sqrt{2}}\right) = \underline{13.63°}$ $\qquad \bar{k}_2 = 60\pi\,(\sin 13.6°\,\hat{a}_x + \cos 13.6°\,\hat{a}_z)$

$$\underline{\bar{E}_t} = 1.56\,\hat{a}_y\, e^{-j60\pi(0.236x + 0.972z)}$$

$$\underline{\bar{H}_t} = \frac{1.56}{60\pi}(-0.972\,\hat{a}_x + 0.236\,\hat{a}_z)\, e^{-j60\pi(0.236x + 0.972z)}$$

Q1

For a non-magnetic ($\mu=\mu_0$) medium, the variation of attenuation and phase constants with respect to frequency are shown in the below figures for three different frequency ranges.

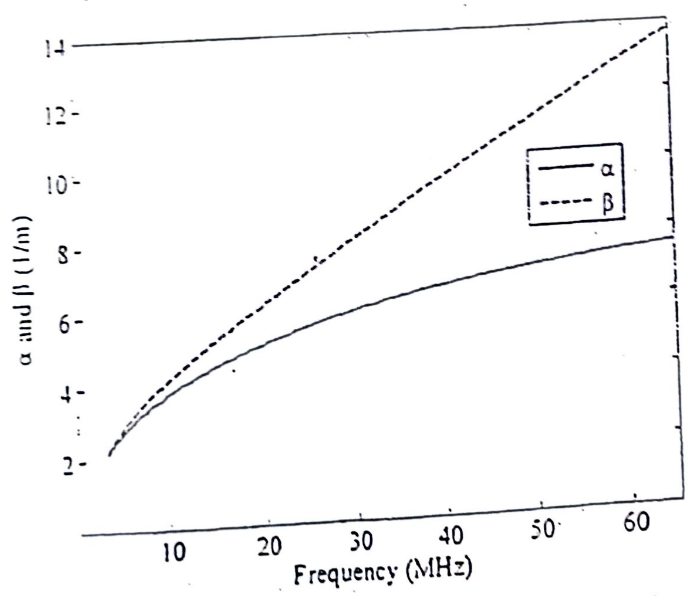


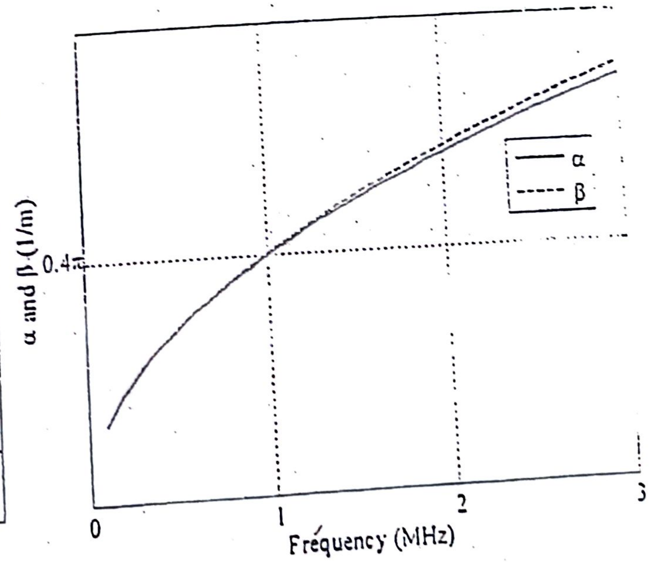
For each frequency range, classify the type of the medium (i.e. low-loss dielectric, good conductor or neither of them) and comment.

Find
a) the conductivity of the medium.
b) the relative permittivity of the medium.
c) the skin depth at 2 MHz.
d) the phase velocity and the wavelength at 80 MHz.

Give the units of all calculated quantities.

Classification: Low-loss dielectric
Why? α is independent of frequency. β is a linear function of frequency





Classification: Neither
Why? α is not equal to β. α is not constant, β is not a linear function of f

Classification: Good conductor
Why? α is equal to β

a) For a good conductor

$$\alpha = \beta = \sqrt{\pi f \mu \sigma}$$

at 1MHz $\quad \alpha = \beta = 0.4\pi$, $\mu = \mu_0$

$$0.4\pi = \sqrt{\pi \, 1\times10^6 \times 4\pi\times10^{-7}\,\sigma} = \pi\sqrt{0.4\sigma}$$

$\sigma = 0.4$ S/m or 0.4 $\mho$/m

b) For a low-loss dielectric

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{\sqrt{\epsilon_r}} = \frac{\sigma}{2}\,\eta_0\,\frac{1}{\sqrt{\epsilon_r}}$$

For this medium $\alpha = 8$ in low-loss dielectric frequency range

$$8 = \frac{0.4}{2}\,120\pi\,\frac{1}{\sqrt{\epsilon_r}} \Rightarrow \epsilon_r = 88.82$$

c) $\delta = \frac{1}{\alpha}$, 2MHz is in the good conductor frequency range

$\alpha_1 = 0.4\pi$ at $f_1 = 1$MHz

$\alpha_2 = \sqrt{\frac{f_2}{f_1}}\,\alpha_1$ at $f_2 = 2$MHz

$$\alpha_2 = \sqrt{\frac{2}{1}}\,0.4\pi = 1.777 \text{ nepers/m}$$

$\delta = 0.5627$ m

d) 80 MHz is in the low-loss dielectric frequency range

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\epsilon_r} = \frac{2\pi\times80\times10^6}{3\times10^8}\sqrt{88.82} = 15.79 \text{ rad/m}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\epsilon_r}} = 3.18\times10^7 \text{ m/sec}$$

$$\lambda = \frac{2\pi}{\beta} = 0.397 \text{ m}$$

Q2

[*Parts a) and b) of this question are not related*]

a) The time-domain electric and magnetic field intensities of an electromagnetic wave are given by:

$$\bar{E}(z,t) = 100\sin(5z)\sin(\omega t)\,\hat{a}_x \quad , \qquad \bar{H}(z,t) = 0.5\cos(5z)\cos(\omega t)\,\hat{a}_y \; .$$

Find the time-average Poynting's vector
i) using the given time-domain fields,
ii) using the corresponding phasor fields.

b) A sinusoidally varying electromagnetic wave of angular frequency ω exists in a certain region occupied by air (ε_o, μ_o). The phasor electric field intensity is given by:

$$\bar{E}(\bar{r}) = E_o \sin(\frac{\pi x}{a})\, e^{-j\beta z}\,\hat{a}_y$$

where E_o : a complex constant (in *V/m*)
a, β : real, positive constants (in *m* and *rad/m*, respectively).

i) find the corresponding phasor $\bar{H}$,
ii) find the time-average power (in *Watts*) passing through the rectangular surface defined by $\{0 \le x \le a, 0 \le y \le b, z = c : \text{constant}\}$.

a) i) $\bar{P}(t) = \bar{E}(t) \times \bar{H}(t) = 50 \sin 5z \cos 5z \sin\omega t \cos\omega t\; \hat{a}_z$

$$\bar{P}_{av} = \frac{1}{T}\int_0^T \bar{P}(t)\,dt = \frac{1}{T} 50 \sin 5z \cos 5z \underbrace{\left[\int_0^T \sin\omega t \cos\omega t\, dt\right]}_{0} \hat{a}_z$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\boxed{\bar{P}_{av} = 0}$$

ii) phasor fields:

$\sin\omega t = \cos(\omega t - \frac{\pi}{2})$

$$\bar{E} = 100 \sin 5z\; e^{-j\pi/2}\hat{a}_x = -j100 \sin 5z\, \hat{a}_x$$

$$\bar{H} = 0.5 \cos 5z\; \hat{a}_y$$

$$\bar{P}_{av} = \frac{1}{2}\,\mathrm{Re}\{\bar{E} \times \bar{H}^*\} = \frac{1}{2}\,\mathrm{Re}\{\underbrace{-j50 \sin 5z \cos 5z\; \hat{a}_z}_{\text{purely imaginary}}\}$$

$$= \boxed{0}$$

b) i) $\nabla \times \overline{E} = -j\omega\mu_0 \overline{H}$

$$\nabla \times \overline{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \cancel{\frac{\partial}{\partial y}} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \hat{a}_x\left(-\frac{\partial E_y}{\partial z}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x}\right)$$

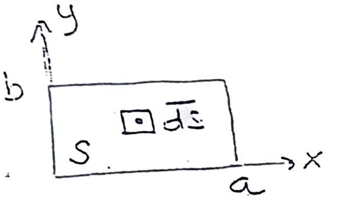
$$= \hat{a}_x\left(j\beta E_0 \sin\frac{\pi x}{a} e^{-j\beta z}\right) + \hat{a}_z\left(\frac{\pi}{a}E_0 \cos\frac{\pi x}{a} e^{-j\beta z}\right)$$

$$\overline{H} = \frac{\nabla \times \overline{E}}{-j\omega\mu_0} = \hat{a}_x \underbrace{\left(-\frac{\beta}{\omega\mu_0}E_0 \sin\frac{\pi x}{a} e^{-j\beta z}\right)}_{H_x} + \hat{a}_z \underbrace{\left(j\frac{\pi/a}{\omega\mu_0}E_0 \cos\frac{\pi x}{a} e^{-j\beta z}\right)}_{H_z = \,..}$$

<u>Note</u> that this is <u>not</u> a TEM wave

ii) $\overline{P}_{av} = \frac{1}{2}\mathrm{Re}\{\overline{E} \times \overline{H}^*\}$

$$\overline{E} \times \overline{H}^* = E_y \hat{a}_y \times (H_x^* \hat{a}_x + H_z^* \hat{a}_z) = (-E_y H_x^* \hat{a}_z + E_y H_z^* \hat{a}_x)$$



$$P_{av} = \int_S \overline{P}_{av} \cdot \underbrace{\overline{ds}}_{ds\,\hat{a}_z} = \int_S (\overline{P}_{av})_z \, ds$$

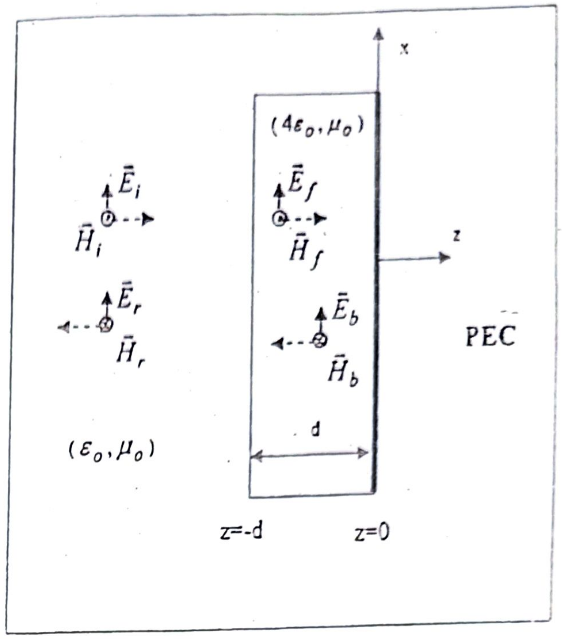
$$(\overline{P}_{av})_z = \frac{1}{2}\mathrm{Re}\left\{-E_y H_x^* = -\left(E_0 \sin\frac{\pi x}{a} e^{-j\beta z}\right)\left(-\frac{\beta}{\omega\mu_0}E_0^* \sin\frac{\pi x}{a} \cancel{e^{j\beta z}}\right)\right\}$$

$$= \frac{\beta}{2\omega\mu_0}|E_0|^2 \sin^2\frac{\pi x}{a} \quad \text{Watts/m}^2$$

$$P_{av} = \frac{\beta}{2\omega\mu_0}|E_0|^2 \underbrace{\int_0^b\int_0^a \sin^2\frac{\pi x}{a}\,dx\,dy}_{\frac{a}{2}\cdot b} = \boxed{\frac{\beta}{4\omega\mu_0}|E_0|^2 ab \ \text{Watts}}$$

Q3

An infinitely large perfectly conducting (PEC) plane is coated by a dielectric material with $\varepsilon = 4\varepsilon_o$, $\mu = \mu_o$ and thickness d as shown in the figure. A plane electromagnetic wave, whose phasor domain electric field intensity is given by $\bar{E}_i = \hat{a}_x E_o e^{-jk_o z}$, is normally incident upon this plane from the air.



Find:

a) The expressions for the reflected electric field in the air region ($\bar{E}_r$) and forward and backward travelling electric fields ($\bar{E}_f$ and $\bar{E}_b$) inside the dielectric coating if thickness of the coating is given by $d = \lambda_d/4$, where λ_d is the wavelength in the dielectric region.

b) Surface current density, $\bar{J}_s$, on the surface of the conducting plane.

a)

$\bar{E}_i = E_{io}\hat{a}_y e^{-jk_o z}$ $\qquad$ $\bar{E}_f = E_{fo}\hat{a}_y e^{-jk_d z}$ $\qquad$ $k_o = \omega\sqrt{\varepsilon_o\mu_o}$

$\bar{E}_r = E_{ro}\hat{a}_x e^{jk_o z}$ $\qquad$ $\bar{E}_b = E_{bo}\hat{a}_x e^{jk_d z}$ $\qquad$ $k_d = 2\omega\sqrt{\varepsilon_o\mu_o} = 2k_o$

$-k_1 d = \frac{2\pi}{\lambda_d}\frac{\lambda_d}{4} = \frac{\pi}{2}$

B.C. at $z = 0 \Rightarrow \boxed{E_{fo} = -E_{bo}}$

$-k_o d = \frac{2\pi}{\cancel{\lambda_o}}\frac{\lambda_d}{4}\frac{\cancel{\lambda_o}}{2} = \frac{\pi}{4}$

B.C. at $z = -d \Rightarrow E_{io}e^{j\frac{\pi}{4}} + E_{ro}e^{-j\frac{\pi}{4}} = E_{fo}\left(e^{+j\frac{\pi}{2}} - e^{-j\pi/2}\right)$

$\frac{E_{io}}{\eta_o}e^{j\frac{\pi}{4}} - \frac{E_{ro}}{\eta_o}e^{-j\pi/4} = \frac{E_{fo}}{\eta_1}\underbrace{\left(e^{-j\pi/2} + e^{j\pi/2}\right)}_{2\cos\pi/2} \Rightarrow$

$\Rightarrow E_{ro} = E_{io}e^{j\pi/2} = jE_{io}$

$E_{io}e^{j\pi/4} + \underbrace{E_{io}e^{j\pi/2}}_{E_{ro}}e^{-j\pi/4} = E_{fo}(j2\sin\overset{1}{\pi/2}) \Rightarrow E_{io}(2e^{j\pi/4}) = -2E_{fo}e^{j\pi/2}$

$\Rightarrow \boxed{E_{fo} = E_{io}e^{-j\pi/4}}$

$\Rightarrow$

$\bar{E}_r = \hat{a}_x E_{io}e^{jk_o z}e^{j\pi/2}$

$\bar{E}_f = \hat{a}_y E_{io}e^{-jk_d z}e^{-j\pi/4}$

$\bar{E}_b = -\hat{a}_y E_{io}e^{+jk_d z}e^{-j\pi/4}$

b)

$$\bar{H}_f = \hat{a}_y \frac{E_{fo}}{\eta_d} e^{-jk_d z}$$

$$\bar{H}_b = -\hat{a}_y \frac{E_{fo}}{\eta_d} e^{jk_d z}$$

$$\bar{H}_{total} = \hat{a}_y \frac{E_{fo}}{\eta_d} \left(e^{-jk_d z} + e^{jk_d z} \right)$$

$$\bar{J}_s = \hat{a}_n \times \bar{H}_{total}\Big|_{z=0} = (\hat{a}_z \times \hat{a}_y) \frac{2E_{fo}}{\eta_d} = \boxed{\hat{a}_x \frac{2E_{fo}}{\eta_d}}$$

$$= \hat{a}_x 2 \frac{E_{io}}{\eta_d} e^{-j\pi/4}$$

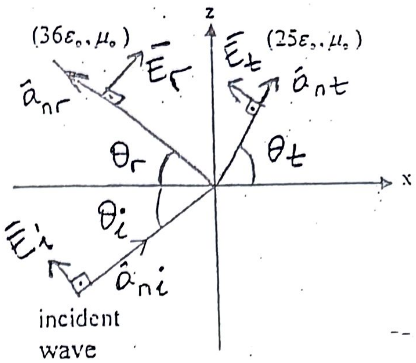
where $\eta_d = \sqrt{\frac{\mu_0}{4\epsilon_0}} = \frac{\eta_0}{2} = 60\pi \ \Omega$

Q 4

A uniform plane wave having the phasor electric field intensity

$$\bar{E} = E_0\left(\frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z\right)e^{-j10\pi\left(\sqrt{3}x+z\right)}$$

is incident upon the interface (at $x = 0$ plane) between two lossless dielectric mediums as shown in the figure.



a) Obtain the expression for the reflected wave electric field.
b) Obtain the expression for the transmitted wave electric field.

$$\bar{E}_i = E_{i0}\left(-\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i\right) e^{-jk_1(\cos\theta_i x + \sin\theta_i z)}$$

$$\theta_i = 30^\circ \Rightarrow \theta_r = 30^\circ$$

$$E_{i0} = -E_0 \qquad \tan\theta_i = \frac{1}{\sqrt{3}}$$

$$\bar{k}_1 = 10\pi(\sqrt{3}\hat{a}_x + \hat{a}_z) \qquad k_1 = |\bar{k}_1| = 20\pi \text{ rad/m}$$

$$k_2 = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, k_1 = \sqrt{\frac{25}{36}}\, 20\pi = \frac{50}{3}\pi \text{ rad/m}$$

From Snell's Law $k_1 \sin\theta_i = k_2 \sin\theta_t$

$$20\pi\,\frac{1}{2} = \frac{50\pi}{3}\sin\theta_t, \quad \sin\theta_t = \frac{3}{5} \Rightarrow \theta_t = 36.86^\circ$$

For parallel polarization

$$\Gamma_{//} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_0}{\sqrt{25}}\,\frac{4}{5} - \frac{\eta_0}{\sqrt{36}}\,\frac{\sqrt{3}}{2}}{\frac{\eta_0}{\sqrt{25}}\,\frac{4}{5} + \frac{\eta_0}{\sqrt{36}}\,\frac{\sqrt{3}}{2}}$$

$$\Gamma_{//} = 0.051$$

$$\tau_{//} = \frac{\cos\theta_i}{\cos\theta_t}(1+\Gamma_{//}) = \frac{\sqrt{3}/2}{4/5}(1+0.051) = 1.082$$

a) $\bar{E}_r = E_{i0}\,\Gamma_{\parallel}\,(\hat{a}_x \sin\theta_r + \hat{a}_z \cos\theta_r)\, e^{-jk_1(-\cos\theta_r x + \sin\theta_r z)}$

$= -E_0\, 0.051 \left(\hat{a}_x \frac{1}{2} + \hat{a}_z \frac{\sqrt{3}}{2}\right) e^{-j10\pi(-\sqrt{3}x + z)}$

b) $\bar{E}_t = E_{i0}\, Z_{\parallel}\,(-\hat{a}_x \sin\theta_t + \hat{a}_z \cos\theta_t)\, e^{-jk_2(\cos\theta_t x + \sin\theta_t z)}$

$= -E_0\, 1.082 \left(-\hat{a}_x \frac{3}{5} + \hat{a}_z \frac{4}{5}\right) e^{-j\frac{10\pi}{3}(4x + 3z)}$

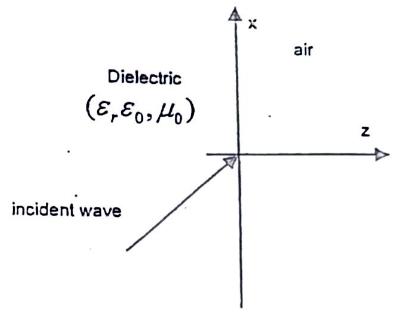
Q 5

Consider the plane wave dielectric to air reflection problem shown in the figure. The incident, reflected and transmitted phasor electric fields are given by expressions

$$\bar{E}_{inc}(x,y,z) = \hat{a}_y\, e^{-j(x+z)} \qquad z < 0$$

$$\bar{E}_{ref}(x,y,z) = \hat{a}_y\, e^{j\psi}\, e^{-j(x-z)} \qquad z < 0$$

$$\bar{E}_{trans}(x,y,z) = \hat{a}_y\, (1+e^{j\psi})\, e^{-jx}\, e^{-z/\sqrt{3}} \qquad z > 0$$



Air can be assumed to have the properties of free space

(a) Calculate the wavelength λ of the transmitted wave (m)

(b) Calculate the wave velocity v of the transmitted wave (m/sec)

(c) Calculate relative permittivity ε_r of the dielectric medium

(d) Calculate the values of the angle of incidence θ_{inc} and the critical angle θ_c. Show that $\theta_{inc} > \theta_c$.

(e) Calculate the reflected signal phase angle ψ (radians)

Solution :

(a) $\beta_x = \dfrac{2\pi}{\lambda} = 1 \rightarrow \lambda = 2\pi \; meters$

(b)

$$\beta_x^{\,2} - \alpha_z^{\,2} = 1 - \frac{1}{3} = \frac{2}{3} = \frac{\omega^2}{c^2} \rightarrow 2\pi f = \sqrt{2/3}\cdot 3\cdot 10^8 \rightarrow v = f\lambda = \sqrt{2/3}\cdot 10^8 \; m/\sec$$

(c)

$$\beta_x^2 + \beta_z^2 = 1 + 1 = 2 = \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \qquad \beta_x^{\,2} - \alpha_z^{\,2} = 2/3 = \omega^2 \varepsilon_0 \mu_0 \rightarrow \varepsilon_r = \frac{2}{2/3} = 3$$

(d) $\theta_{inc} = tg^{-1}(\beta_x / \beta_z) = 45^0 \; whereas \; \theta_c = \sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = 35.26438968^0 < 45^0$

(e)

Solution via Snell and Fresnel Formulas:

$$\sqrt{3}\sin\theta_{inc} = \sin\theta_{tr} \rightarrow \sin\theta_{tr} = \sqrt{3}/2 \rightarrow \cos\theta_{tr} = \sqrt{1-3/2} = j\frac{1}{\sqrt{2}}$$

$$\frac{E_{0ref}}{E_{0inc}} = e^{j\psi} = \frac{\dfrac{1}{\cos\theta_{tr}} - \dfrac{1}{\sqrt{3}\cos\theta_{inc}}}{\dfrac{1}{\cos\theta_{tr}} + \dfrac{1}{\sqrt{3}\cos\theta_{inc}}} = \frac{-j\sqrt{2} - \sqrt{\dfrac{2}{3}}}{-j\sqrt{2} + \sqrt{\dfrac{2}{3}}} = \frac{1+j\sqrt{3}}{1-j\sqrt{3}} = \frac{-2+j2\sqrt{3}}{4} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$\psi = \tan^{-1}(-\sqrt{3}/2) = -\frac{\pi}{3} \quad radians$$

Solution via H fields :

$$\vec{H}^0_{inc} = \frac{(\hat{a}_x + \hat{a}_z)\times \hat{a}_y}{\omega\mu_0} = -\hat{a}_x\frac{1}{\omega\mu_0} + \hat{a}_z\frac{1}{\omega\mu_0}$$

$$\vec{H}^0_{ref} = \frac{(\hat{a}_x - \hat{a}_z)\times \hat{a}_y e^{j\psi}}{\omega\mu_0} = \hat{a}_x\frac{e^{j\psi}}{\omega\mu_0} + \hat{a}_z\frac{e^{j\psi}}{\omega\mu_0}$$

$$\vec{H}^0_{trans} = \frac{\left(\hat{a}_x - j\dfrac{1}{\sqrt{3}}\hat{a}_z\right)\times \hat{a}_y(1+e^{j\psi})}{\omega\mu_0} = \hat{a}_x j\frac{(1+e^{j\psi})}{\omega\mu_0\sqrt{3}} + \hat{a}_z\frac{(1+e^{j\psi})}{\omega\mu_0}$$

Tangential (x - component) H - field continuity yields

$$j\frac{(1+e^{j\psi})}{\omega\mu_0\sqrt{3}} = -\frac{1}{\omega\mu_0} + \frac{e^{j\psi}}{\omega\mu_0} \rightarrow e^{j\psi} = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \rightarrow \psi = \tan^{-1}(-\sqrt{3}/2) = -\frac{\pi}{3} \quad radians$$

EE 303 Fall 2009 Mt 2 Solutions

Q1. (25points) [Parts a) and b) of this question are not related, part b) is on the next page]

a) (13 pnts)

The phasor electric field intensity of a uniform plane wave propagating in a lossy medium is given by:

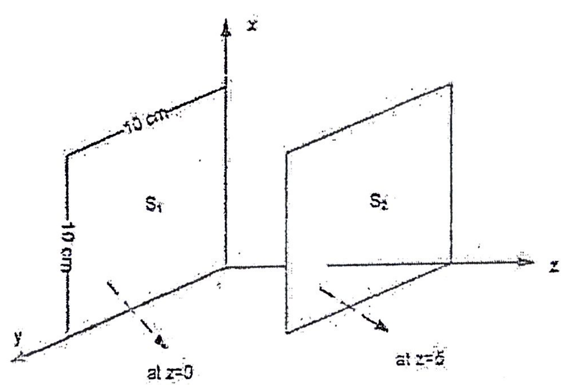
$$\bar{E}(z) = 10(\hat{a}_x + j2\hat{a}_y)e^{-\alpha z}e^{-j\beta z} \quad (Volts/m)$$

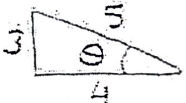
The intrinsic impedence of the medium at the frequency of this wave is $\eta = 8 + j6 \quad (\Omega)$.

6 i) Find the time-average Poynting's vector (simplify the expression as much as possible, but do not attempt to find numerical values of α and β).

4 ii) Find, separately, the time average power passing through the $100\ cm^2$ square surfaces S_1 and S_2 shown in the figure. (δ denotes the depth of penetration in *meters*).

3 iii) What is the average dissipated power in the volume $\{0 \le x \le 10\ cm,\ 0 \le y \le 10\ cm,\ 0 \le z \le \delta\}$? Explain.





i) upw $\Rightarrow$ $\bar{H} = \frac{1}{\eta}\hat{a}_z \times \bar{E}$

$\eta = 8 + j6 = |\eta|e^{j\theta}$ $\quad |\eta| = 10, \quad \theta = \tan^{-1}\frac{3}{4} = 36.87^\circ$

$$\bar{H} = \frac{10}{10e^{j\theta}}\hat{a}_z \times (\hat{a}_x + j2\hat{a}_y)e^{-\alpha z}e^{-j\beta z}$$

$$\bar{H} = (\hat{a}_y - j2\hat{a}_x)e^{-j\theta}e^{-\alpha z}e^{-j\beta z}$$

$$\bar{P}_{av} = \frac{1}{2}\mathrm{Re}\{\bar{E} \times \bar{H}^*\}$$

$$= \frac{1}{2}\mathrm{Re}\Big\{\underbrace{10(\hat{a}_x + j2\hat{a}_y) \times (\hat{a}_y + j2\hat{a}_x)}_{5\hat{a}_z} e^{-\alpha z}e^{-j\beta z}e^{j\theta}e^{-\alpha z}e^{j\beta z}\Big\}$$

Q.1 a) continued:

$$\bar{P}_{av} = \frac{1}{2} \text{Re}\left\{ 50 e^{-2\alpha z} e^{j\theta} \hat{a}_z \right\}$$

$$= 25 e^{-2\alpha z} \underbrace{\cos\theta}_{4/5} \hat{a}_z$$

$$\bar{P}_{av} = \boxed{20 e^{-2\alpha z} \hat{a}_z} \text{ Watts/m}^2$$

ii) $$P_1 = \int_{S_1} \bar{P}_{av} \cdot d\bar{S} = P_{av_z}\Big|_{z=0} S_1 = 20 S_1$$

$$P_1 = 20(100)\times 10^{-4} = 0.2 \text{ Watts} = \boxed{200 \text{ mWatts}}$$

$$P_2 = P_{av_z}\Big|_{z=\delta=\frac{1}{\alpha}} S_2 = 20 e^{-2\alpha \frac{1}{\alpha}} S_2$$

$$P_2 = 20 e^{-2} S_2 = e^{-2} P_1 = 0.2 \underbrace{e^{-2}}_{0.135} \text{ Watts}$$

$$P_2 \cong \boxed{27 \text{ mWatts}}$$

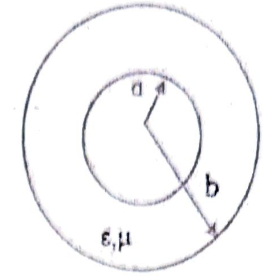
iii) P_1 enters the volume, P_2 leaves. (Note that there is no power flow to the region through the remaining 4 surfaces, since $\bar{P}_{av}$ is in $\hat{a}_z$ direction). So, from <u>conservation of power</u>

dissipated power $= P_1 - P_2 \approx \boxed{173 \text{ mWatts}}$

Q1. continued

b) (12 pnts)

The figure below shows the cross-section of a lossless coaxial cable which is infinitely long in the z-direction. The phasor magnetic field intensity in the region $a<r<b$ is given by



$$\bar{H} = \hat{a}_\phi \frac{I_o}{2\pi r} e^{-jkz} \quad (Amps/m)$$

where I_o is a costant and $k = \omega\sqrt{\mu\varepsilon}$.

4 i) Using Maxwell's equation find phasor $\bar{E}$.

8 ii) Find the time average power flowing through the cable.

i) $\bar{\nabla}\times\bar{H} = j\omega\varepsilon\bar{E} \quad (\sigma = 0 \text{ in } a<r<b)$

$$\bar{\nabla}\times\bar{H} = -\frac{\partial H_\phi}{\partial z}\hat{a}_r + \frac{1}{r}\underbrace{\frac{\partial}{\partial r}(rH_\phi)}_{0}\hat{a}_z$$

4 $$= -\frac{I_o}{2\pi r}(-jk)e^{-jkz}\,\hat{a}_r = j\omega\varepsilon\bar{E}$$

$$\therefore \quad \bar{E} = \frac{kI_o}{2\pi\omega\varepsilon}\frac{e^{-jkz}}{r}\hat{a}_r = \boxed{\underbrace{\sqrt{\frac{\mu}{\varepsilon}}}_{\eta}\frac{I_o\,e^{-jkz}}{2\pi r}\hat{a}_r} \quad V/m$$

ii) $$\bar{P}_{av} = \frac{1}{2}Re\{\bar{E}\times\bar{H}^*\}$$

$$= \frac{1}{2}Re\left\{\eta\frac{I_o}{2\pi r}e^{-jkz}\frac{I_o^*}{2\pi r}e^{jkz}\underbrace{\hat{a}_r\times\hat{a}_\phi}_{\hat{a}_z}\right\}$$

4 $$= \boxed{\frac{1}{2}\frac{\eta|I_o|^2}{(2\pi)^2}\frac{1}{r^2}\hat{a}_z} \quad Watts/m^2$$

$$P_{av} = \int_S \bar{P}_{av}\cdot\bar{dS} = \int_0^{2\pi}\int_a^b \frac{\eta|I_o|^2}{2(2\pi)^2}\frac{1}{r^2}\hat{a}_z\cdot r\,dr\,d\phi\,\hat{a}_z$$

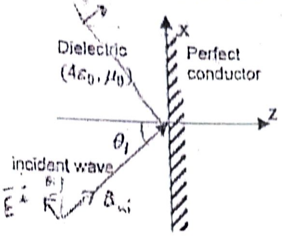
4 $$= \frac{\eta|I_o|^2}{2(2\pi)^2}(2\pi)\underbrace{\int_a^b\frac{dr}{r}}_{\ln\frac{b}{a}} = \boxed{\frac{\eta|I_o|^2}{4\pi}\ln\frac{b}{a}} \quad Watts$$

Q.2

Question (25 pts)

A parallel polarized uniform plane wave in a dielectric medium ($\mu=\mu_0$, $\varepsilon=4\varepsilon_0$) is incident upon a perfectly conducting half space which occupies $z \ge 0$ as shown in the Figure. The angle of incidence is $\theta_i=60°$. The phasor surface current density induced by this incident field at the dielectric- conductor interface (z=0 surface) is

$\bar{J}_s = \hat{a}_x 2e^{-j2\sqrt{3}x}$ A/m. Find the expression for the phasor electric field $\bar{E}^i$ of the incident wave.



$$k = \omega\sqrt{\mu_0 4\varepsilon_0} = 2k_0$$

$$\bar{E}^i = E_{i0}(\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)\, e^{-jk(x\sin\theta_i + z\cos\theta_i)}$$

$$\bar{H}^i = \frac{1}{\eta}\hat{a}_{ni} \times \bar{E}^i = \frac{1}{\eta}(\hat{a}_x \sin\theta_i + \hat{a}_z\cos\theta_i) \times E_{i0}$$

$$= \frac{E_{i0}}{\eta}\hat{a}_y \underbrace{(\sin^2\theta_i + \cos^2\theta_i)}_{1}\, e^{-jk(x\sin\theta_i + z\cos\theta_i)}$$

$$\bar{E}^r = \underbrace{E_{r0}}_{-E_{i0}}(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)\, e^{-jk(x\sin\theta_i - z\cos\theta_i)}$$

$$\bar{H}^r = \frac{1}{\eta}(\hat{a}_x\sin\theta_i - \hat{a}_z\cos\theta_i) \times \bar{E}^r$$

$$= \frac{1}{\eta}\hat{a}_y(\sin^2\theta_i + \cos^2\theta_i)\, E_{i0}\, e^{-jk(x\sin\theta_i - z\cos\theta_i)}$$

$$\bar{H} = \bar{H}^i + \bar{H}^r = \hat{a}_y\frac{2E_{i0}}{\eta}\cos(kz\cos\theta_i)\, e^{-jkx\sin\theta_i}$$

$$\bar{J}_s = (-\hat{a}_z) \times \bar{H}\Big|_{z=0} = \hat{a}_x\frac{2E_{i0}}{\eta}e^{-jkx\sin\theta_i} = \hat{a}_x 2e^{-j2\sqrt{3}\pi x}$$

$$k \sin\theta = k\frac{\sqrt{3}}{2} = 2\sqrt{3}\pi \Rightarrow k = 4\pi$$

$$\frac{2E_{io}}{\eta} = 2 \Rightarrow E_{io} = \eta = \frac{1}{2}\ 120\pi = 60\pi$$

$$\bar{E}^i = 60\pi\left(\hat{a}_x\frac{1}{2} - \hat{a}_z\frac{\sqrt{3}}{2}\right) e^{-j4\pi\left(x\frac{\sqrt{3}}{2} + z\frac{1}{2}\right)} \ \text{V/m.}$$

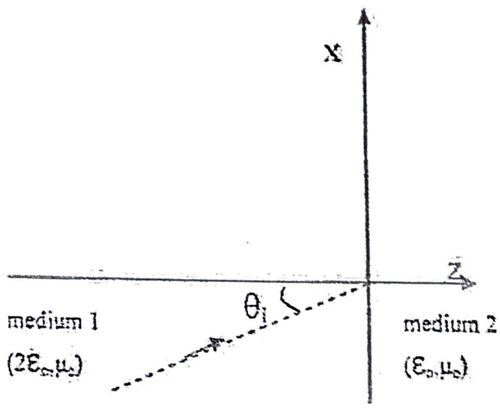
Q3. (25 points)

A uniform plane wave is incident from a dielectric medium with ($\varepsilon = 2\varepsilon_o, \mu = \mu_o$) onto air ($\varepsilon = \varepsilon_o, \mu = \mu_o$) as shown in the figure where the planar boundary is placed at $z=0$. The phasor electric field intensity vector of the incident wave is given by,

$$\bar{E}_i = (3\sqrt{3}\,\hat{a}_x - 3\hat{a}_z)e^{-j(2x+2\sqrt{3}z)} \ V/m.$$

Find,

a) Angle of incidence, angle of reflection and angle of transmission.
b) The unit vector in the direction of propagation for the incident wave.
c) Phasor electric field intensity vector of the reflected wave.
d) Phasor electric field intensity vector of the transmitted wave.
e) The range of θ_i (angle of incidence) values so that "total internal reflection" occurs.



a) $k_1 \hat{a}_{n_i} \cdot \bar{r} = 2x + 2\sqrt{3}z = 4\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)$

$\Rightarrow \hat{a}_{n_i} = \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z$, $\theta_i = \tan^{-1}\frac{1/2}{\sqrt{3}/2} = 30° = \theta_r$ //

$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \Rightarrow \sin\theta_t = \frac{1}{\sqrt{2}} \Rightarrow \theta_t = 45°$ //

b) $\hat{a}_{n_i} = \frac{1}{2}\hat{a}_y + \frac{\sqrt{3}}{2}\hat{a}_z$ //

c) $\Gamma_{\parallel}(\theta_i = 30°) = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = 0.072$

where $\eta_1 = \sqrt{\frac{\mu_0}{2\epsilon_0}}$, $\eta_2 = \sqrt{\frac{\mu_0}{\epsilon_0}}$

MIDDLE EAST TECHNICAL UNIVERSITY
ELECTRICAL AND ELECTRONICS ENGINEERING

EEE-303 MIDTERM-2 **21.05.2010**

Name, Surname:

Duration: 100 Min.

Q-1:
Q-2:
Q-3:
Q-4:

Total:

1) (25 points) Propagation constant, characteristic impedance and electrical length of a transmission line can be defined through the following expressions:

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{LC} = \frac{2\pi}{\lambda} \text{ (rad/m)}, \quad \eta = \sqrt{\frac{L}{C}}, \quad \theta = \beta d \text{ (rad)}$$

$$\mu_o = 4\pi\, 10^{-7} \text{ (H/m)} \qquad \varepsilon_o = \frac{10^{-9}}{36\pi} \text{ (F/m)} \qquad c = 3\text{x}\, 10^{8} \text{ (m/s)}$$

where L and C are capacitance and inductance per unit length of the line, d is the physical length. It is given that $\mu=\mu o$ and $\varepsilon = \varepsilon_r\, \varepsilon_o$.

a) Write the expression for θ in terms of frequency ω, length d, relative dielectric constant ε_r and velocity of light c.

b) Find electrical length for a transmission line which is quarter wavelength long ($d=\lambda/4$) at the operating frequency.

c) The frequency at which the physical length d of a line is a quarter wavelength ($\lambda/4$) is defined as quarter wavelength frequency f_q. Show that the expression for f_q can be written as:

$$f_q = \frac{c}{4d\sqrt{\varepsilon_r}}$$

d) Show that the expression for θ can also be written in terms of the signal frequency f and quarter wavelength frequency f_q as

$$\theta = \frac{\pi}{2}\frac{f}{f_q}$$

... length of a line with $\mu=\mu o$, $\varepsilon = 9\varepsilon_o$ is given as $C=10^{-10}$ (F/m).

2) (25 points) The expression for the input impedance Zi of a transmission line of characteristic impedance Zo, phase constant β and length d loaded by impedance Z_L is. as follows:

$$Z_i = Z_o \frac{Z_L + jZ_o \operatorname{Tan} \beta d}{Z_o + jZ_L \operatorname{Tan} \beta d}$$

a) Given Z_L=2 ohms, Zo= 50 ohms, find Zi for $d=\lambda/8$, $d=\lambda/4$ and $d=\lambda/2$.
b) A quarter wavelength transmission line is to be used for impedance matching between a source of impedance Zs=50 ohms and a load of impedance Z_L=2 ohms at frequency $f=10^6$ Hz. Find the characteristic impedance Zo and length d of the transmission line with parameters $\mu=\mu_o$ and $\varepsilon=4\varepsilon_o$.

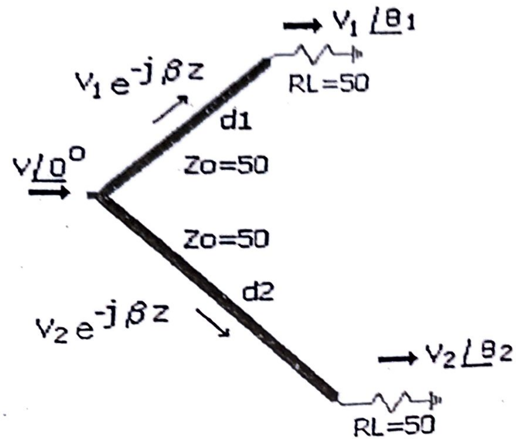
3) (25 points) Given a transmission line of length $d=0.01$ m, $\mu=\mu_o$ and $\varepsilon=9\varepsilon_o$, with characteristic impedance $Z_o= 50$ ohms.

a) If this line is open circuited at its far end ($Z_L=\infty$) then we can use its input impedance as a capacitive reactance or if it is short circuited at its far end ($Z_L=0$) then we can use it as an inductive reactance if its length is less than quarter wavelength at the operating frequency. Show that the frequency range in which this piece of line can be used for such purpoeses is $0< f < 2.5$ GHz.

b) Calculate the equivalent lumped capacitance at frequency $f=100$ MHz if it is open circuited.

c) Calculate the equivalent lumped inductance at frequency $f=100$ MHz if it is open circuited.

4) (25 points) A voltage wave is split into two parts and fed into two transmission lines of characteristic impedance Zo=50 ohms, $\mu=\mu o$, $\varepsilon r = 9$, with lengths d1 and d2, terminated in matched loads as shown below.



a) The phase angle of the input voltage is taken as zero degrees and the phase angles of the voltage waves at the outputs of line-1 and line-2 are denoted as θ_1 and θ_2. Given d1=0.5 m, find length d2 so that the phase difference between the two voltages will be $\Delta\theta = \theta_2 - \theta_1 = \pi/2$ at frequency f=100 MHz.

b) The parallelled two lines are connected to a third line as load (see Figure ... the third line is Zo=100 ohms. Find the

MIDDLE EAST TECHNICAL UNIVERSITY
Department of Electrical and Electronic Engineering

Midterm 2 - EE 303
2010-2011

Q.1. (25 points)
The phasor electric field intensity of a uniform plane wave in air $\varepsilon = \varepsilon_0,\ \mu = \mu_0$ is given by

$$\vec{E}(\bar{r}) = \left(\sqrt{3}\hat{a}_x + \hat{a}_y + jA\hat{a}_z\right)e^{-j\pi(x-\sqrt{3}y)} \quad V/m$$

where A is a real constant.

a) Find the angular frequency (ω), and the wavelength (λ).

b) Write down the expression for the time-domain electric field intensity $\vec{E}(\bar{r}, t)$.

c) Find the unit vector in the direction of propagation.

d) Determine the polarization (type and sense-when applicable) for the following cases:

(I) $A = 0$
(II) $A = 1$
(III) $A = 2$

Justify your answers with sketches and/or explanations.

a) $\bar{E}(\bar{r}) = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$ $\quad \bar{k} = \pi(\hat{a}_x - \sqrt{3}\hat{a}_y)$

$$k = |\bar{k}| = \pi\sqrt{1+3} = \boxed{2\pi}$$

$$k = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} \Rightarrow \omega = kc = 2\pi \times 3\times10^8$$

$$\boxed{\omega = 6\pi\times10^8 \text{ rad/sec}}$$ 2

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{2\pi} = \boxed{1 \text{ m}}$$ 2

b) $$\bar{E}(\bar{r},t) = (\sqrt{3}\hat{a}_x + \hat{a}_y)\cos[\omega t - \pi(x-\sqrt{3}y)] + A\hat{a}_z \underbrace{\cos\left[\omega t - \pi(x-\sqrt{3}y) + \frac{\pi}{2}\right]}_{-\sin[\omega t - \pi(x-\sqrt{3}y)]}$$ 4

c) $$\hat{n} = \frac{\bar{k}}{k} = \frac{\pi(\hat{a}_x - \sqrt{3}\hat{a}_y)}{2\pi} = \boxed{\frac{1}{2}(\hat{a}_x - \sqrt{3}\hat{a}_y)}$$

4

Q.1 continued:

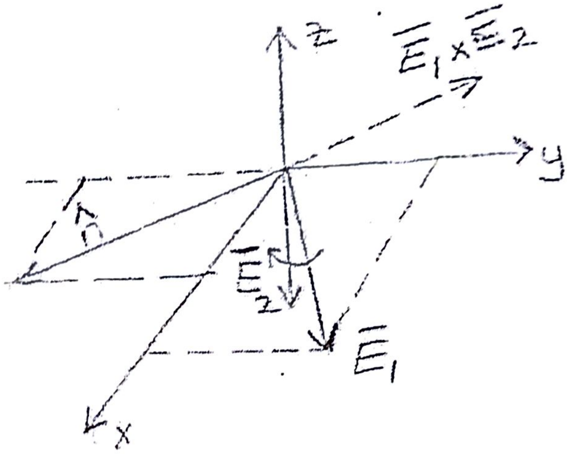
d) $\bar{E}(0,t) = (\sqrt{3}\,\hat{a}_x + \hat{a}_y)\cos\omega t - A\hat{a}_z \sin\omega t$

i) $A = 0$

(4) $\bar{E}(0,t) = \underbrace{(\sqrt{3}\,\hat{a}_x + \hat{a}_y)}_{\text{constant direction for all } t}\cos\omega t$

$\Rightarrow$ Linear Polarization (LP)

ii) $A = 1$



(4.5) $\bar{E}\big|_{t=0} = \sqrt{3}\,\hat{a}_x + \hat{a}_y \overset{\text{call}}{=} \bar{E}_1$

$\bar{E}\big|_{\omega t = \frac{\pi}{2}} = -\hat{a}_z \overset{\text{call}}{=} \bar{E}_2$

$\bar{E}_1 \times \bar{E}_2 = (\sqrt{3}\,\hat{a}_x + \hat{a}_y) \times (-\hat{a}_z)$

$= -(\hat{a}_x - \sqrt{3}\,\hat{a}_y)$: is opposite to $\hat{n}$

$|\bar{E}_1| = 2$ $\quad$ $|\bar{E}(t)|$ is <u>not</u> constant

$|\bar{E}_2| = 1$

$\Rightarrow$ Left Hand Elliptic Polarization (LHEP)

iii) $A = 2$

(4.5) $\bar{E}_1 = \sqrt{3}\,\hat{a}_x + \hat{a}_y$ as above

$\bar{E}_2 = -2\hat{a}_z$

$|\bar{E}(t)|$ is constant $\Rightarrow$ Left Hand Circular Polarization (LHCP)

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Department of Electrical and Electronic Engineering

Q.2 (25 pts)
A uniform plane wave propagates in a lossy, nonmagnetic ($\mu = \mu_0$) medium along the x-direction. The electric field intensity vector is in the y-direction, and its value at $x = 0$ is given as $\vec{E}(x=0,t) = 10\cos(10^8 t)\,\hat{a}_y$ V/m.
The wave has an attenuation constant of π Nepers/m, and a phase constant of 3π rad/m.

a) Write down the the expression for the phasor electric field intensity vector, $\vec{E}$ (as a function of space coordinates).
b) Find the corresponding magnetic field intensity vector, $\vec{H}$ (without any unknowns).
c) Find the numerical value of intrinsic impedance, η. (Hint: Consider $\gamma\eta$ product).
d) Find the phase velocity and the depth of penetration (skin depth).

a) $\bar{E}(\bar{r}) = \hat{a}_y\, 10\, e^{-\pi x}\, e^{-j3\pi x} = \hat{a}_y\, 10\, e^{-\underbrace{(\pi + j3\pi)}_{\gamma} x}$

b) $\bar{\nabla}\times\bar{E} = -j\omega\mu_0\bar{H} \Rightarrow \bar{H} = -\dfrac{1}{j\omega\mu_0}\,\bar{\nabla}\times\bar{E}$

$$\bar{\nabla}\times\bar{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \hat{a}_z \frac{\partial E_y}{\partial x} = \hat{a}_z 10(-\pi - j3\pi)\, e^{-\pi x}\, e^{-j3\pi x}$$

$$\Rightarrow \bar{H} = \hat{a}_z \frac{1}{j\cancel{10^8} \times 4\pi\times\cancel{10^{-7}}}\, \cancel{10}\underbrace{(-\pi - j3\pi)}_{-\gamma} e^{-\pi x} e^{-j3\pi x}$$

$$= \hat{a}_z \frac{(1+j3)}{j4} e^{-(\ )} = \frac{3-j}{4} e^{-\pi x} e^{-j3\pi x}\, \hat{a}_z$$

c) $\gamma\eta = j\omega\mu \Rightarrow \eta = \dfrac{j\omega\mu}{\gamma} = \dfrac{j\times 10^8 \times 4\pi\times 10^{-7}}{\pi(1+j3)}$

$$= \frac{j40}{1+j3} = \frac{j40(1-j3)}{10}$$

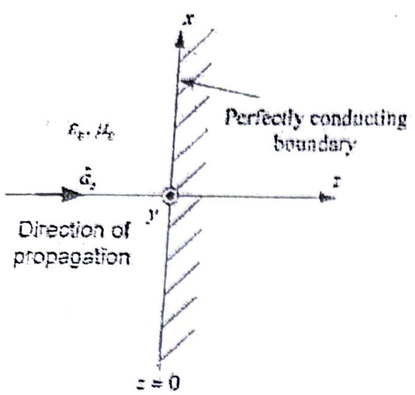
$$= 12 + j4$$

OR $\eta = \dfrac{E_y}{H_z} = \ldots$

d) $v = \dfrac{\omega}{\beta} = \dfrac{10^8}{3\pi} = 1.06\times 10^7$ m/sec.

$\delta = \dfrac{1}{\alpha} = \dfrac{1}{\pi} = 0.32$ m.

Q.3 (25 pts)
A uniform plane wave is normally-incident upon a perfectly conducting planar boundary as shown in the figure given below:



The incident electric field intensity is given as $\overline{E}_i(z) = (2\hat{a}_x + j\hat{a}_y)e^{-jkz}$, where $k = \omega\sqrt{\varepsilon_0\mu_0}$.

a) Determine the polarization of the incident electric field intensity.

b) Evaluate the reflected electric field intensity and determine its polarization. Compare the polarizations of the incident and reflected electric field intensities and comment on your results.

c) Evaluate the time-average Poynting's vectors for the incident, reflected and total fields. Comment on your results.

a) In time domain, $\overline{E}_i(z,t) = Re\left\{\overline{E}_i(z)e^{j\omega t}\right\}$

$$\overline{E}_i(z,t) = Re\left\{(2\hat{a}_x + j\hat{a}_y)\underbrace{e^{-jkz}\,e^{j\omega t}}_{e^{j(\omega t - kz)}}\right\} = Re\left\{(2\hat{a}_x + j\hat{a}_y)e^{j(\omega t - kz)}\right\}$$

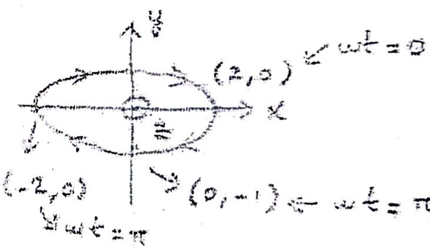
$$e^{j(\omega t - kz)} = \cos(\omega t - kz) + j\sin(\omega t - kz)$$

$$\overline{E}_i(z,t) = 2\cos(\omega t - kz)\,\hat{a}_x - \sin(\omega t - kz)\,\hat{a}_y$$

To determine the polarization, let $z = 0$.

$$\overline{E}_i(0,t) = 2\cos\omega t\,\hat{a}_x - \sin\omega t\,\hat{a}_y$$

plot



The locus is an ellipse. The propagation direction is $\hat{a}_z$. So, the wave is LHEP.

b) The reflection coefficient for a perfectly conducting boundary is $\Gamma = -1$. Therefore, in phasor domain:

$$\overline{E}_r(z) = (-2\hat{a}_x - j\hat{a}_y)e^{jkz}$$

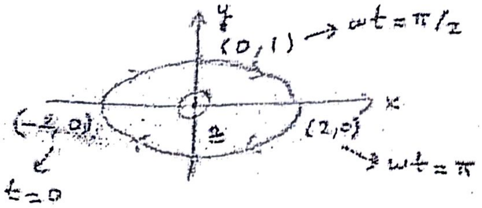
$$\overline{E}_r(z,t) = Re\left\{(-2\hat{a}_x - j\hat{a}_y)e^{j(\omega t + kz)}\right\}$$

$$= -2\cos(\omega t + kz)\hat{a}_x + \sin(\omega t + kz)\hat{a}_y$$

$$\overline{E}_r(0,t) = -2\cos\omega t\,\hat{a}_x + \sin\omega t\,\hat{a}_y$$

8.3 (continued)

Let us plot the locus of the tip of $\bar{E}_r(0,t)$:



This wave is RHEP. (Thumb of the right hand along $(-\hat{a}_z)$, and four fingers along the elliptic locus)

Comment: Reflection changes the sense of polarization, That is, "left-handed" becomes "right-handed" and vice versa, and note that this is a result of the reflection coeff Γ... multiplying the components of the field $\bar{E}_i$

c) $\bar{E}_i(z) = (2\hat{a}_x + j\hat{a}_y)e^{-jkz}$, $\bar{H}_i(z) = \frac{1}{\eta_0}\hat{a}_z \times \bar{E}_i(z)$

$$\bar{H}_i(z) = \frac{1}{\eta_0}[2\hat{a}_y - j\hat{a}_x]e^{-jkz}$$

$$\bar{P}_{av,i} = \frac{1}{2}Re(\bar{E}_i(z) \times \bar{H}_i^*(z)) = \frac{1}{2\eta_0}Re\{(2\hat{a}_x + j\hat{a}_y) \times (2\hat{a}_y + j\hat{a}_x)\}$$

$$= \frac{1}{2\eta_0}Re\{4\hat{a}_z + \hat{a}_z\} = \frac{5}{2\eta_0}\hat{a}_z$$

Also, note that $\bar{P}_{av,i} = \frac{1}{2\eta_0}(\bar{E}_i \cdot \bar{E}_i^*)\hat{a}_z$

$$\bar{E}_r(z) = (-2\hat{a}_x - j\hat{a}_y)e^{jkz}, \quad \bar{H}_r(z) = \frac{1}{\eta_0}(-\hat{a}_z) \times \bar{E}_r(z)$$

Then: $\bar{P}_{av,r} = \frac{1}{2}Re\{\bar{E}_r(z) \times \bar{H}_r^*(z)\}$

$$\bar{P}_{av,r} = \frac{1}{2\eta_0}Re\{(-2\hat{a}_x - j\hat{a}_y) \times (2\hat{a}_y + j\hat{a}_x)\}$$

$$= \frac{1}{2\eta_0}Re\{-4\hat{a}_z - \hat{a}_z\} = \frac{5}{2\eta_0}(-\hat{a}_z)$$

Q.3 (continued):

Now, let us evaluate the total fields:

$$\overline{E}_{total}(z) = \overline{E}_i(z) + \overline{E}_r(z)$$

$$= (2\hat{a}_x + j\hat{a}_y)e^{-jkz} + (-2\hat{a}_x - j\hat{a}_y)e^{jkz}$$

$$= 2(e^{-jkz} - e^{jkz})\hat{a}_x + j(e^{-jkz} - e^{jkz})\hat{a}_y$$

$$= -4j\sin kz\,\hat{a}_x + 2\sin kz\,\hat{a}_y$$

$$\overline{H}_{total}(z) = \overline{H}_i(z) + \overline{H}_r(z)$$

$$= \frac{1}{\eta_o}[2\hat{a}_y - j\hat{a}_x]e^{-jkz} + \frac{1}{\eta_o}[2\hat{a}_y - j\hat{a}_x]e^{jkz}$$

$$= \frac{1}{\eta_o}2(e^{-jkz} + e^{jkz})\hat{a}_y - \frac{j}{\eta_o}(e^{-jkz} + e^{jkz})\hat{a}_x$$

$$= \frac{1}{\eta_o}\left[4\cos kz\,\hat{a}_y - 2j\cos kz\,\hat{a}_x\right]$$

$$\overline{P}_{av,total} = \frac{1}{2}\,Re\{\overline{E}_{total} \times \overline{H}^*_{total}\}$$

$$= \frac{1}{2\eta_o}Re\{-16j\sin kz\cos kz\,\hat{a}_z + 4j\sin kz\cos kz\,\hat{a}_z\}$$

$$= 0$$

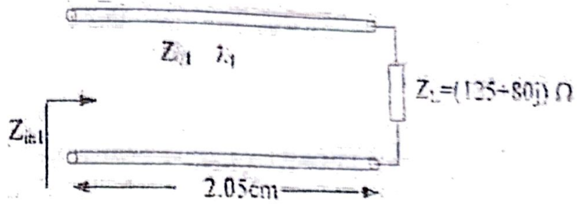
Comment: Note that:

$\overline{P}_{av,total} = 0$, because $\overline{P}_{av,i}$ and $\overline{P}_{av,r}$ are equal in magnitude, but opposite in direction.

Q4. A load impedance (Z_L=125+80j Ω) is connected at the end of a 2.05 cm long transmission line. The inductance per unit length of the transmission line is 500 nH/m and the phase velocity along the transmission line is 1.2×10^8 m/sec.

(a) Find the characteristic impedance (Z_{01}) and the wavelength (λ_1) on the transmission line at 600MHz.

(b) Find the input impedance (Z_{in1}) seen at the sending end of the transmission line at 600 MHz.



a) $v_p = \frac{1}{\sqrt{LC}}$ $\quad Z_o = \sqrt{\frac{L}{C}} = L v_p = 500\times10^{-9}\times1.2\times10^{8}$

$= 60\Omega$

$\lambda = \frac{v_p}{f} = \frac{1.2\times10^{8}}{600\times10^{6}} = 0.2m = 20cm$

b) $\beta l = \frac{2\pi}{\lambda}\cdot l = \frac{2\pi}{20}\times 2.05 = 0.64$

$\tan\beta l = 0.75$

$Z_{in1} = 60 \; \frac{125+80j + j\,60\times0.75}{60 + j(125+80j)\times0.75}$

$= 60 \; \frac{125+125j}{93.75j} = 80(1-j)\sqrt{2}$

$$\bar{E}_i = 6\left(\frac{\sqrt{3}}{2}\hat{a}_x - \frac{1}{2}\hat{a}_z\right)$$

$$\bar{E}_r = 6\,\Gamma_{\parallel}\,(\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r)\,e^{-j4(x\sin\theta_r - z\cos\theta_r)}$$

$$= 0.43\left(\hat{a}_x\frac{\sqrt{3}}{2} + \hat{a}_z\frac{1}{2}\right)e^{-j4\left(x\cdot\frac{1}{2} - z\frac{\sqrt{3}}{2}\right)}$$

//

d) $\tau_{\parallel} = (1+\Gamma_{\parallel})\dfrac{\cos\theta_i}{\cos\theta_t} = 1.32$

$$\left.\begin{aligned} k_1 &= \omega\sqrt{2\varepsilon_0\mu_0} \\ k_2 &= \omega\sqrt{\varepsilon_0\mu_0}\end{aligned}\right\} \Rightarrow k_2 = \frac{k_1}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$\bar{E}_t = 6\cdot\tau_{\parallel}\left(\frac{1}{\sqrt{2}}\hat{a}_x - \frac{1}{\sqrt{2}}\hat{a}_z\right)e^{-jk_2\left(\frac{1}{\sqrt{2}}\hat{a}_x + \frac{1}{\sqrt{2}}\hat{a}_z\right)}$$

$$= 7.9\left(\frac{1}{\sqrt{2}}\hat{a}_x - \frac{1}{\sqrt{2}}\hat{a}_z\right)e^{-j2(\hat{a}_x + \hat{a}_z)}$$

//

e) θ_c: critical angle

$$\frac{\sin\theta_t}{\sin\theta_c} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{2} \Rightarrow \sin\theta_c = \frac{1}{\sqrt{2}} \Rightarrow \theta_c = 45^\circ$$

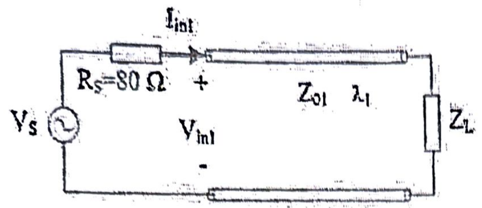
$\Rightarrow \theta_i > 45^\circ$ results in total internal reflection

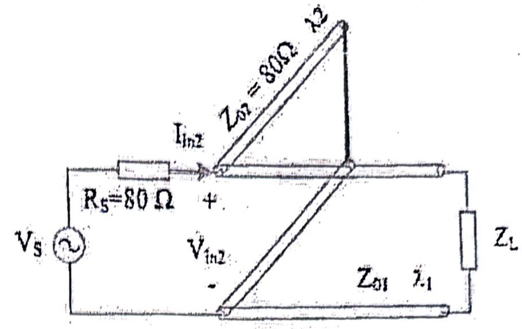
//

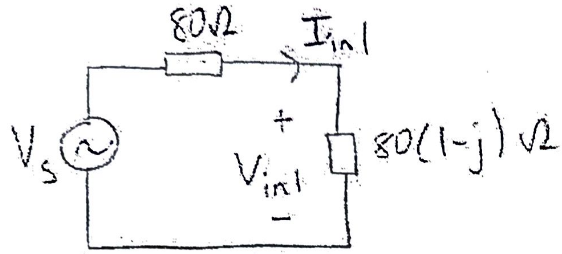
Q.4 continued

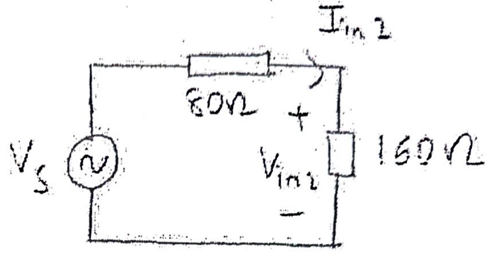
(e) A voltage source ($v_s(t) = 400\cos(2\pi \times 600\times10^6 t)\,\mathrm{mV}$) with internal resistance of 80Ω is connected to the circuits defined in parts (b) and (d). Find the time averaged input powers P_{in1} and P_{in2}.

Hint: Time averaged power $P_{in} = \frac{1}{2}\mathrm{Re}\{V_{in} I_{in}^*\}$









$$I_{in1} = \frac{V_s}{80 + 80(1-j)} = \frac{V_s}{160-80j} = \frac{V_s}{80(2-j)} = \frac{400(2+j)}{80 \cdot 5} = (2+j)\,\mathrm{mA}$$

$$V_{in1} = \frac{V_s\, 80(1-j)}{160-80j} = \frac{400 \cdot 80(1-j)}{80(2-j)} = 400\frac{(1-j)}{(2-j)}\,\mathrm{mV}$$

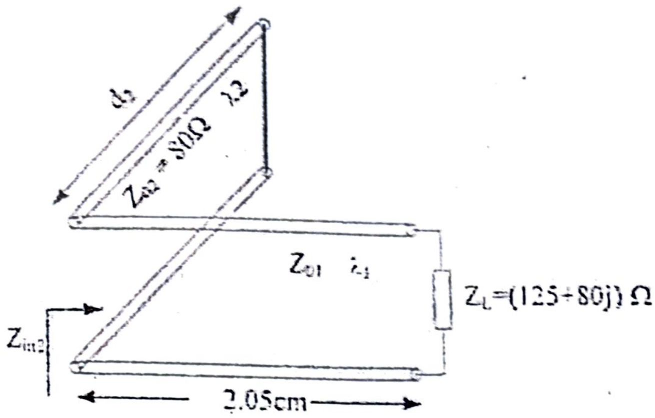
$$P_{in1} = \frac{1}{2}\mathrm{Re}\left\{400\frac{(1-j)}{\cancel{(2-j)}}\cancel{(2-j)}\right\} = 200\,\mu W$$

$$I_{in2} = \frac{V_s}{240} = \frac{40}{24}\,\mathrm{mA} \qquad V_{in2} = \frac{V_s \cdot 16}{24} = \frac{400 \cdot 16}{24}\,\mathrm{mV}$$

$$P_{in2} = \frac{1}{2}\mathrm{Re}\left\{400 \cdot \frac{\overset{4}{\cancel{16}}}{\underset{6}{\cancel{24}}} \cdot \frac{\overset{10}{\cancel{40}}}{\underset{6}{\cancel{24}}}\right\} = \frac{2000}{9} = 222\,\mu W$$

(c) A short circuited transmission line with 80 Ω characteristic impedance is connected in parellel to Z_{in1}. The inductance per unit length of this transmission line is also 500 nH/m. Find the wavelength (λ_2) on the second transmission line at 600 MHz.

(d) Find the minimum length (d_2) of the second transmission line so that the input impedance (Z_{in2}) seen at the sending end is purely resistive at 600 MHz.



c) $v_{p2} = \dfrac{Z_{02}}{L} = \dfrac{80}{500\times10^{-9}} = 1.6\times10^{8}\ m/sec$

$\lambda_2 = \dfrac{1.6\times10^{8}\ m/sec}{600\times10^{6}\ Hz} = 0.266\,m = 26.66\,cm$

$Y_{in1} = \dfrac{1}{80(1-j)} = \dfrac{1+j}{160}\ \mho$

Z_{in2} real $\Rightarrow Y_{stub} = \dfrac{-j}{160}$ $\quad Z_{stub} = 160j\ \Omega$

$Z_{stub} = 80\,j\tan\beta_2 d_2 = 160j \quad \Rightarrow \tan\beta d_2 = 2$

$\beta_2 d_2 = 63.43°$

$\dfrac{2\pi}{26.66}\, d_2\, \dfrac{180°}{\pi} = 63.43° \qquad \ell = 4.697\,cm$

EE303 MT2 Solutions, Fall 2011

QUESTION1

Assume the presence of an electromagnetic wave with phasor electric field intensity described in cartesian (x,y,z) coordinates as

$$\vec{E}(x,y,z) = \hat{a}_y e^{-4x} e^{-5jz} \quad (V/m)$$

Note that this is not a uniform plane wave. It is not even a TEM wave. The medium is free space with permeability μ_0 and permittivity ϵ_0.

(a) What is the value of angular frequency ω ?

For an arbitrary point (x,y,z) find the full explicit expressions of

(b) The time domain electric field $\vec{E}(x,y,z,t)$.

(c) The phasor magnetic field intensity $\vec{H}(x,y,z)$

(d) The time averaged Poyntings vector $\vec{P}_{av}(x,y,z)$

(e) Find the wavelength and velocity of this wave in the direction of propagation. Compare the velocity with c, freespace velocity of light.

SOLUTION

(a) $\nabla^2\vec{E} + \omega^2\mu_0\,\epsilon_0\,\vec{E} = 0$

$$(4^2 - 5^2 + \omega^2\mu_0\,\epsilon_0)\vec{E} = 0 \rightarrow \omega^2\mu_0\,\epsilon_0 = 9 \rightarrow \quad \omega = 3c = 9\times 10^8 \text{ rad/s}$$

(b) $\vec{E}(x,y,z,t) = \hat{a}_y\, e^{-4x} cos(3ct - 5z)$

(c) $\nabla \times E = -j\omega\mu H$

$$\vec{H}(x,y,z) = \frac{j}{\omega\mu_0}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & e^{-4x}e^{-5jz} & 0 \end{vmatrix} = \frac{j}{\omega\mu_0}\left(\hat{a}_x\, 5je^{-4x}e^{-5jz} - \hat{a}_z 4e^{-4x}e^{-5jz}\right)$$

$$\omega\mu_0 = 3\mu_0\frac{1}{\sqrt{\mu_0\,\epsilon_0}} = 3\sqrt{\frac{\mu_0}{\epsilon_0}} = 360\pi \rightarrow \vec{H}(x,y,z) = \left(-\hat{a}_x\frac{1}{72\pi}e^{-4x}e^{-5jz} - \hat{a}_z\frac{1}{90\pi}je^{-4x}e^{-5jz}\right)$$

$$\vec{P}_{av}(x,y,z) = \Re e\left\{\frac{1}{2}\vec{E}\times\vec{H}^*\right\} = \frac{1}{2}\Re e\left\{\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & e^{-4x}e^{-5jz} & 0 \\ -\frac{1}{72\pi}e^{-4x}e^{5jz} & 0 & \hat{a}_z\frac{1}{90\pi}je^{-4x}e^{5jz} \end{vmatrix}\right\}$$

$$\vec{P}_{av}(x,y,z) = \hat{a}_z\frac{1}{144\pi}e^{-8x}\ Watt/m^2$$

(d) $5 = \frac{2\pi}{\lambda} \rightarrow \lambda = \frac{2\pi}{5}\ (m)\quad \omega = 3c \rightarrow f = \frac{3c}{2\pi} \rightarrow v_p = f\lambda = \frac{3}{5}c$ Less than c.

Q. 2. (25 pts) A uniform plane wave, which is linearly polarized along the z axis, propagates in $\hat{n} = \hat{a}_x$ direction in a non-magnetic (i.e. having $\mu = \mu_o$) lossy medium. Using time-domain electric field measurements, it is found that this uniform plane wave oscillates at $f = 400\ MHz$ with wavelength $\lambda = 5\ cm$ and skin depth $\delta = 1\ cm$ within this lossy medium with a maximum amplitude of 1 Volt/m.

Determine:

(5 pts) a) the complex propagation constant $\gamma = \alpha + j\beta$,
(4 pts) b) the relative permittivity ε_r and the conductivity σ,
(3 pt) c) the type of given medium (i.e. is it a good conductor or good dielectric or neither of those?)
(2.5 pts) d) the propagation velocity (i.e. the phase velocity) v_p of the uniform plane wave,
(2 pts) e) the intrinsic impedance η (write down the mathematical expression needed to compute η, insert the values of parameters with correct units and state the proper unit for η. You do NOT have to obtain the numerical result for this complex computation. If the value of η is needed in the following parts, you may simply express it symbolically in the polar form as $\eta = |\eta| e^{j\theta}$),
(3.5 pts) f) the mathematical expression of time-domain $\vec{E}(\vec{r},t)$ vector without any unknowns.
(5 pts) g) the mathematical expression of time-domain $\vec{H}(\vec{r},t)$ vector without any unknowns.

Important notes:

- Do NOT forget to specify units (You'll loose 0.5 points for each missing/incorrect unit).
- Clearly write down the formulas you used and show your solution steps.

Hint: In a lossy medium, $\gamma = \alpha + j\beta = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$

leading to $\gamma^2 = -\omega^2\mu\varepsilon + j\omega\mu\sigma = (\alpha^2 - \beta^2) + j2\alpha\beta$.

Solution:

a) $\alpha = \frac{1}{\delta} = \frac{1}{0.01(m)} = 100$ Neper/m , $\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.05(m)} = 40\pi$ rad/m $\cong 125.7$ rad/m

$\Rightarrow \boxed{\gamma = \alpha + j\beta = 100 + j40\pi \ (m^{-1})}$

b) Using the hint:

$2\alpha\beta = \omega\mu_o\sigma \Rightarrow 2 \times 100 \times 40\pi = 2\pi \times 4\times10^8 \times 4\pi\times10^{-7}\sigma \Rightarrow \boxed{\sigma \cong 7.96\ (S/m)}$

$\beta^2 - \alpha^2 = \omega^2\mu_o\varepsilon_o\varepsilon_r \Rightarrow (40\pi)^2 - (100)^2 = (2\pi)^2(4\times10^8)^2 \times 4\pi\times10^{-7} \times \frac{10^{-9}}{36\pi}\varepsilon_r \Rightarrow \boxed{\varepsilon_r = 82.5}$

c) $\left|\frac{J_c}{J_d}\right| = \frac{\sigma}{\omega\varepsilon} = \frac{7.96}{2\pi \times 4 \times 10^8 \times \frac{10^{-9}}{36\pi} \times 82.5} \cong 4.35 \Rightarrow$ Medium is neither good conductor nor good dielectric!

d) $v_p = \frac{\omega}{Im\{\gamma\}} = \frac{\omega}{\beta} = \frac{2\pi \times 4 \times 10^8}{40\pi} = 0.2 \times 10^8$ m/sec (Or, you may use $v_p = \lambda f$)

e) $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{j2\pi \times 4\times10^8 \times 4\pi\times10^{-7}}{7.96 + j2\pi \times 4\times10^8 \times 82.5 \times \frac{10^{-9}}{36\pi}}}$ $(\Omega) = |\eta| e^{j\theta}$ (Ω)

(You are not asked to obtain the final result but it is found as
$\eta \cong 19.6\ e^{j38.5°}$ (Ω))

f) As $\bar{E}(\bar{r}) = \hat{a}_z\ e^{-\gamma \hat{n}\cdot\bar{r}}$ with $\hat{n} = \hat{a}_x$, $\gamma = \alpha + j\beta$ and $|\bar{E}|_{max} = 1$ V/m

$\Rightarrow \bar{E}(\bar{r}) = \hat{a}_z\ e^{-100x}\ e^{-j40\pi x}$ (V/m) in phasor domain.

$$\Rightarrow \boxed{\bar{\mathcal{E}}(\bar{r},t) = Re\{\bar{E}(\bar{r})\, e^{j\omega t}\} = \hat{a}_z\, e^{-100x} \cos(8\pi\, 10^8 t - 40\pi x)\ \text{V/m}}$$

(where $\omega = 2\pi f = 8\pi\, 10^8$ rad/sec)

g) $\bar{H}(\bar{r}) = \frac{1}{\eta}\hat{n} \times \bar{E}(\bar{r}) = \frac{1}{|\eta| e^{j\theta}} \underbrace{\hat{a}_x \times \hat{a}_z}_{-\hat{a}_y}\ e^{-100x}\ e^{-j40\pi x}$

$\bar{H}(\bar{r}) = -\hat{a}_y \frac{1}{|\eta|}\ e^{-100x}\ e^{-j40\pi x}\ e^{-j\theta}$ (A/m)

$$\boxed{\bar{\mathcal{H}}(\bar{r},t) = Re\{\bar{H}(\bar{r})\, e^{j\omega t}\} = -\hat{a}_y \frac{e^{-100x}}{|\eta|} \cos(8\pi\, 10^8 t - 40\pi x - \theta)\ \text{A/m}}$$

(If the numerical value for $\eta = |\eta| e^{j\theta}$ is inserted:

$\bar{\mathcal{H}}(\bar{r},t) = -\hat{a}_y \frac{e^{-100x}}{19.6} \cos(8\pi\, 10^8 t - 40\pi x - \underbrace{0.214\pi}_{\text{(radian value of } \theta = 38.5°)})$ A/m)

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Q.3 (25 points) A uniform plane wave of phasor electric field intensity

$$\bar{E}^i = E_o \hat{a}_y e^{-j(x+2z)} \quad (V/m)$$

is incident from air onto a perfectly conducting half-space as shown in the figure.

(4) a) Find the values of $\sin\theta_i$ and $\cos\theta_i$, θ_i being the angle of incidence. (Note: you won't need the value of θ_i itself)

(8) b) Find the phasor expressions for $\bar{H}^i$ of the incident wave, and $\bar{E}^r$, $\bar{H}^r$ of the reflected wave (Simplify the expressions as much as possible).

(5) c) Find the phasor surface current density induced on the boundary of the perfect conductor. Give also its unit.

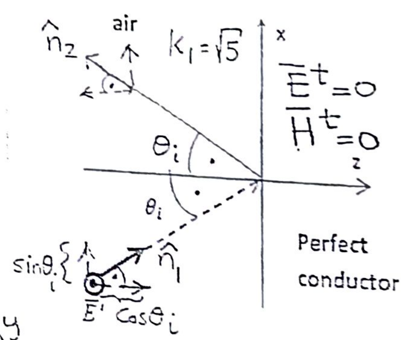
(3) d) Considerig the total (i.e incident + reflected) wave existing in air, determine the time-average Poynting's vector. Does this vector have any z-component? Comment on the result.

a) $\bar{E}^i = E_o \hat{a}_y e^{-j\bar{k}_1 \cdot \bar{r}}$

$\bar{k}_1 = \hat{a}_x + 2\hat{a}_z$

$\hat{n}_1 = \dfrac{\bar{k}_1}{|\bar{k}_1|} = \boxed{\dfrac{\hat{a}_x + 2\hat{a}_z}{\sqrt{5}}}$

$\boxed{\sin\theta_i = \dfrac{1}{\sqrt{5}}}$, $\boxed{\cos\theta_i = \dfrac{2}{\sqrt{5}}}$



b) $\bar{H}^i = \dfrac{1}{\eta}\hat{n}_1 \times \bar{E}^i = \dfrac{1}{\eta}\dfrac{1}{\sqrt{5}}(\hat{a}_x + 2\hat{a}_z) \times E_o \hat{a}_y e^{-j(x+2z)}$

$\boxed{\eta = 120\pi \ \Omega}$

$$\boxed{\bar{H}^i = \frac{E_o}{\eta\sqrt{5}}(\hat{a}_z - 2\hat{a}_x) e^{-j(x+2z)}} \ (A/m)$$

perfect conductor $\Rightarrow \Gamma = -1$

$\therefore \boxed{\bar{E}^r = -E_o \hat{a}_y e^{-j(x-2z)}}$ $\quad \boxed{\hat{n}_2 = \dfrac{1}{\sqrt{5}}(\hat{a}_x - 2\hat{a}_z)}$

$\bar{H}^r = \dfrac{1}{\eta}\hat{n}_2 \times \bar{E}^r = \dfrac{1}{\eta}\dfrac{1}{\sqrt{5}}(\hat{a}_x - 2\hat{a}_z) \times (-E_o\hat{a}_y) e^{-j(x-2z)}$

$$\boxed{\bar{H}^r = \frac{E_o}{\eta\sqrt{5}}(-\hat{a}_z - 2\hat{a}_x) e^{-j(x-2z)}}$$

(Q.3 cont'd)

c) $\bar{J}_s = -\hat{a}_z \times (\bar{H}^i + \bar{H}^r)\Big|_{z=0}$

$$= -\hat{a}_z \times \frac{E_0}{\eta\sqrt{5}}\Big[\underbrace{(\cancel{\hat{a}_z} - 2\hat{a}_x)\, e^{-jx} + (-\cancel{\hat{a}_z} - 2\hat{a}_x)\, e^{-jx}}_{-4\hat{a}_x e^{-jx}}\Big]$$

$$\boxed{\bar{J}_s = \frac{E_0}{\eta\sqrt{5}} 4 e^{-jx}\, \hat{a}_y} \quad \boxed{(A/m)}$$

---

d) $\bar{P}_{av} = \frac{1}{2} \text{Re}\{(\bar{E}^i + \bar{E}^r) \times (\bar{H}^i + \bar{H}^r)^*\}$

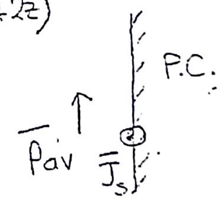
$$= \underbrace{\frac{1}{2}\text{Re}\{\bar{E}^i \times \bar{H}^{i*}\}}_{\frac{|E_0|^2}{2\eta}\hat{n}_1 \text{ : purely real}} + \underbrace{\frac{1}{2}\text{Re}\{\bar{E}^i \times \bar{H}^{r*} + \bar{E}^r \times \bar{H}^{i*}\}}_{\text{call } A} + \underbrace{\frac{1}{2}\text{Re}\{\bar{E}^r \times \bar{H}^{r*}\}}_{\text{purely real : } \frac{|E_0|^2}{2\eta}\hat{n}_2}$$

$$\bar{E}^i \times \bar{H}^{r*} = E_0 \hat{a}_y e^{-j(\cancel{x}+2z)} \times \frac{E_0^*}{\eta\sqrt{5}}(-\hat{a}_z - 2\hat{a}_x) e^{j(\cancel{x}-2z)}$$

$$= \frac{|E_0|^2}{\eta\sqrt{5}}(-\hat{a}_x + 2\hat{a}_z)\, e^{-j4z}$$

$$\bar{E}^r \times \bar{H}^{i*} = -E_0 \hat{a}_y e^{-j(\cancel{x}-2z)} \times \frac{E_0^*}{\eta\sqrt{5}}(\hat{a}_z - 2\hat{a}_x) e^{j(\cancel{x}+2z)}$$

$$= \frac{|E_0|^2}{\eta\sqrt{5}}(-\hat{a}_x - 2\hat{a}_z)\, e^{j4z}$$



$$\therefore\ A = \frac{|E_0|^2}{2\eta\sqrt{5}}(-2\hat{a}_x)\cos(4z)$$

1st & 4th terms: $\frac{|E_0|^2}{2\eta}(\hat{n}_1 + \hat{n}_2) = \frac{|E_0|^2}{2\eta}\frac{1}{\sqrt{5}}(2\hat{a}_x)$

$$\therefore\ \boxed{\bar{P}_{av} = \frac{|E_0|^2}{\eta\sqrt{5}}(1 - \cos 4z)\,\hat{a}_x} \quad (\text{Watts}/m^2)$$

has no z-component. A nonzero z-component would mean a net power flow into the perfect conductor. But $\bar{P}_{av}$ must be zero in the perfect conductor since there is no wave.

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Q.4 (25pts) A uniform plane wave is incident from a medium with $(3\varepsilon_0, \mu_0)$ onto air with (ε_0, μ_0) as shown in the figure below. If the electric fields of the incident and reflected waves are given by,

$$\vec{E}_i(x,z) = \underbrace{\left(\tfrac{3}{2}\hat{a}_x + 2\hat{a}_y + A\hat{a}_z\right)}_{\bar{E}_{io}} e^{-jk_1\hat{a}_{n_i}\cdot\vec{r}} \ V/m \quad \text{and}$$

$$\vec{E}_r(x,z) = E_{r_0}\hat{a}_y e^{-jk_1\hat{a}_{n_r}\cdot\vec{r}} \ V/m$$

Where A is a constant, $\hat{a}_{n_i}$ and $\hat{a}_{n_r}$ are unit vectors in the direction of propagation for incident and reflected fields, respectively, $\vec{r}$ is the position vector. Find,

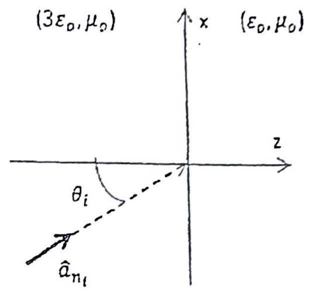
a) the angles of incidence, reflection and transmission,

b) $\hat{a}_{n_i}$ and $\hat{a}_{n_r}$, the unit vectors in the direction of propagation for incident and reflected fields, respectively,

c) the constant, A,

d) the amplitude of the reflected electric field, E_{r_0},

e) the electric field of the transmitted field, $\vec{E}_t(x,z)$.



reflected field has no // polarized component $\Rightarrow \theta_i = \theta_{B_{//}}$

$$\sin\theta_{B_{//}} = \sqrt{\frac{\epsilon_0}{\epsilon_0 + 3\epsilon_0}} = 1/2 \Rightarrow \theta_i = 30° = \theta_B$$

$\theta_r = \theta_i = 30°$,

$\theta_B + \theta_t = 90° \Rightarrow \theta_t = 60°$

b) $\hat{a}_{n_i} = \sin\theta_i \hat{a}_x + \cos\theta_i \hat{a}_z$
$= \frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z$

$\hat{a}_{n_r} = \sin\theta_r \hat{a}_x - \cos\theta_r \hat{a}_z$
$= \frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z$

c) $\bar{E}_{io}\cdot\hat{a}_{n_i} = 0 \Rightarrow \bar{A} = -\frac{\sqrt{3}}{2}$

d) $\Gamma_\perp(\theta = 30°) = \frac{1}{2} \Rightarrow E_{ro} = 1$

e) $\tau_\perp = \frac{3}{2} \Rightarrow \bar{E}_{t_\perp} = 3e^{-jk_0(\frac{\sqrt{3}}{2}x + \frac{1}{2}z)}$

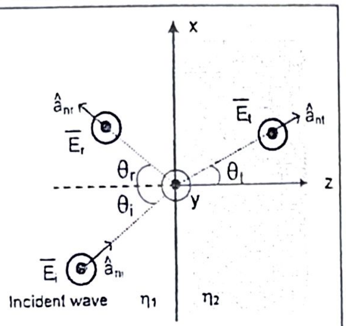
$\tau_{//} = \frac{\cos\theta_i}{\cos\theta_t}(1 + \underbrace{\Gamma_{//}}_{0}) = \sqrt{3}$

$\Rightarrow \bar{E}_{t_{//}} = \underbrace{\sqrt{3}\cdot\sqrt{\frac{9}{4}+\frac{3}{4}}}_{3}\left(\hat{a}_x\frac{1}{2} - \hat{a}_z\frac{\sqrt{3}}{2}\right) \cdot e^{-jk_0(\frac{\sqrt{3}}{2}x + \frac{1}{2}z)}$ //

Middle East Technical University
Department of Electrical & Electronics Engineering

EE 303Midterm Examination 2

Fresnel Formulas :

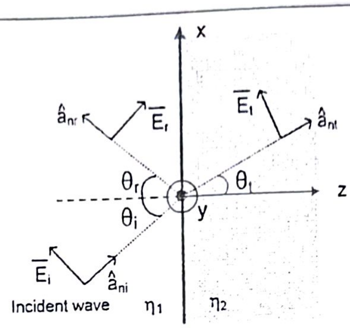


E ⊥ Plane of Incidence Case

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$T_{\perp} = 1 + \Gamma_{\perp}$$

$$\sin\theta_B^{\perp} = \sqrt{\frac{1-(\mu_1\varepsilon_2/\varepsilon_1\mu_2)}{1-(\mu_1/\mu_2)^2}}$$



E // Plane of Incidence Case

$$\Gamma_{//} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$T_{//} = \frac{\cos\theta_i}{\cos\theta_t}(1+\Gamma_{//})$$

$$\sin\theta_B^{//} = \sqrt{\frac{1-(\varepsilon_1\mu_2/\mu_1\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)^2}}$$

Propagation constant and intrinsic impedance for lossy medium

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

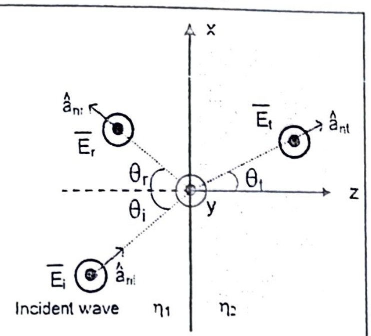
$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

For air and free space :

$$\mu_0 = 4\pi\, 10^{-7}\ H/m, \qquad \varepsilon_0 = \frac{10^{-9}}{36\pi}\ F/m$$

Only simple calculators are allowed.

Fresnel Formulas :

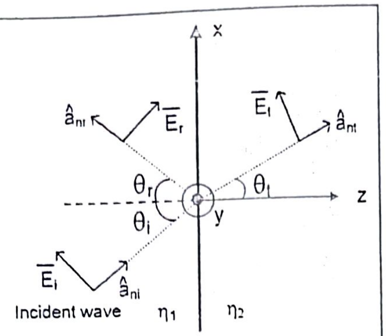


E ⊥ Plane of Incidence Case

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$T_{\perp} = 1 + \Gamma_{\perp}$$

$$\sin\theta_B^{\perp} = \sqrt{\frac{1-(\mu_1\varepsilon_2/\varepsilon_1\mu_2)}{1-(\mu_1/\mu_2)^2}}$$



E // Plane of Incidence Case

$$\Gamma_{//} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$T_{//} = \frac{\cos\theta_i}{\cos\theta_t}\left(1 + \Gamma_{//}\right)$$

$$\sin\theta_B^{//} = \sqrt{\frac{1-(\varepsilon_1\mu_2/\mu_1\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)^2}}$$

Propagation constant and intrinsic impedance of lossy medium

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

For air and free space:

$\mu_0 = 4\pi\, 10^{-7}$ H/m, $\varepsilon_0 = \frac{10^{-9}}{36\pi}$ F/m

Only simple calculators are allowed.
Keep your mobile phones turned OFF!
Give your answers neatly showing all necessary DETAILS!
Use proper VECTOR notations! Provide UNITs!
If your solution continues in ANOTHER PAGE, indicate it!

HINTS (You may or may not need)

Constitutive parameters of free space

$$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} \quad F/m \qquad \mu_0 = 4\pi \times 10^{-7} \quad H/m$$

Vector identities:

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} = \overline{\nabla}\left(\overline{\nabla} \cdot \overline{A}\right) - \nabla^2 \overline{A}$$

$$\overline{\nabla} \cdot \left(\alpha \overline{A}\right) = \overline{\nabla}(\alpha) \cdot \overline{A} + \alpha \overline{\nabla} \cdot \overline{A}$$

$$\overline{\nabla} \times \left(\alpha \overline{A}\right) = \overline{\nabla}(\alpha) \times \overline{A} + \alpha \overline{\nabla} \times \overline{A}$$

Lorentz condition in time domain

$$\overline{\nabla} \cdot \overline{A} = -\mu \in \frac{\partial V}{\partial t}$$

Propagation constant and intrinsic impedance of lossy medium

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega \in)} \qquad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

EE303 MT2 Solutions, Fall 2012

Q1. (25 pts) A uniform plane wave is <u>normally incident</u> onto a planar boundary at $z = 0$ which separates a <u>*lossless medium 1*</u> (in region $z < 0$ with η_1 and $\gamma_1 = jk_1$) from a <u>*lossy medium 2*</u> (in region $z > 0$ with $\eta_2 = |\eta_2|e^{j\theta}$ and $\gamma_2 = \alpha + j\beta$) as shown in the figure.

Assume that the electric field phasors $\bar{E}^i$, $\bar{E}^r$ and $\bar{E}^t$ are in the $(+\hat{a}_x)$ direction, and then choose the directions of $\bar{H}^i$, $\bar{H}^r$ and $\bar{H}^t$ consistently. <u>State your solution steps clearly</u>. Your answers can contain <u>only those parameters defined in the question statement</u>.

a) Obtain the mathematical expressions for the incident field phasors $\bar{E}^i$ and $\bar{H}^i$. Assume that $(\bar{E}^i)_{amplitude} = E_1$ which may be a complex constant in general.

b) Obtain the mathematical expression for the incident time-average Poynting's vector $\bar{P}^i_{av}$.

c) State the mathematical expressions for the <u>reflection coefficient</u> Γ and the <u>transmission coefficient</u> T in terms of intrinsic impedances. In the rest of your solution, whenever these coefficients are needed, leave them compactly as $\Gamma = |\Gamma|e^{j\theta_\Gamma}$ and $T = |T|e^{j\theta_T}$.

d) Obtain the mathematical expressions for the reflected field phasors $\bar{E}^r$ and $\bar{H}^r$.

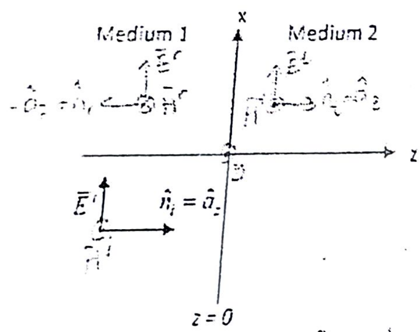
e) Obtain the mathematical expression for the reflected time-average Poynting's vector $\bar{P}^r_{av}$.

f) Obtain the mathematical expressions for the transmitted field phasors $\bar{E}^t$ and $\bar{H}^t$.

g) Obtain the mathematical expression for the transmitted time-average Poynting's vector $\bar{P}^t_{av}$.

h) Now consider the case where *Medium 2* approaches to a perfect conductor with $\sigma_2 \to \infty$. Write down η_2, Γ, T, $\bar{P}^r_{av}$ *and* $\bar{P}^t_{av}$ for this condition.

<u>SOLUTION:</u>



(a) $\bar E^i = \hat a_x E_1 e^{-jk_1 z}$, $\bar H^i = \frac{\hat a_z \times \bar E^i}{\eta_1} = \hat a_y \frac{E_1}{\eta_1} e^{-jk_1 z}$

(b) $\bar P^i_{av} = \frac12 Re\{\bar E^i \times (\bar H^i)^*\} = \frac12 Re\{E_1 e^{-jk_1 z}\, \hat a_x \times \hat a_y \frac{E_1^*}{\eta_1} e^{+jk_1 z}\}$

$\boxed{\bar P^i_{av} = \hat a_z \frac{|E_1|^2}{2\eta_1}}$

(c) $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = |\Gamma| e^{j\theta_\Gamma}$, $T = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_1} = |T| e^{j\theta_T}$

(d) $\bar E^r = \hat a_x (E_1 \Gamma) e^{jk_1 z} = \hat a_x E_1 |\Gamma| e^{j\theta_\Gamma} e^{+jk_1 z}$

$\bar H^r = \frac{-\hat a_z \times \bar E^r}{\eta_1} = -\hat a_y \frac{E_1 |\Gamma| e^{j\theta_\Gamma}}{\eta_1} e^{jk_1 z}$

(e) $\bar P^r_{av} = \frac12 Re\{\bar E^r \times (\bar H^r)^*\} = \boxed{\frac{1}{2}\frac{|E_1|^2 |\Gamma|^2}{\eta_1}(-\hat a_z)}$

(f) $\bar E^t = \hat a_x (E_1 T) e^{-\gamma_2 z} = \hat a_x E_1 |T| e^{j\theta_T} e^{-\alpha z} e^{-j\beta z}$

$\bar H^t = \frac{\hat a_z \times \bar E^t}{\eta_2} = \hat a_y \frac{E_1 |T|}{|\eta_2| e^{j\theta}} e^{j\theta_T} e^{-\alpha z} e^{-j\beta z}$

g) $\bar P^t_{av} = \frac12 Re\{\bar E^t \times (\bar H^t)^*\}$

$= \hat a_z \frac{|E_1|^2 |T|^2}{2|\eta_2|} e^{-2\alpha z} Re\{e^{j\theta}\}$

$\boxed{\bar P^t_{av} = \hat a_z \frac{|E_1|^2 |T|^2}{2|\eta_2|} e^{-2\alpha z} \cos\theta}$

h) $\lim_{\sigma_2 \to \infty} \eta_2 = \lim_{\sigma_2 \to \infty} \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \boxed{0 = \eta_2}$

$\boxed{\Gamma = -1}$, $\boxed{T = 0}$

$\boxed{\bar P^t_{av} = 0}$, $\boxed{\bar P^r_{av} = \frac{|E_1|^2}{2\eta_1}(-\hat a_z)}$

Q2. (25 pts) The region $z>0$ is a lossless dielectric with parameters $\varepsilon = 2\varepsilon_o$, $\mu = \mu_o$, and contains no sources. An evanescent wave of angular frequency ω exists in this region, with the phasor electric field intensity

$$\overline{E} = E_o \hat{a}_y e^{-\alpha z} e^{-j\beta x} \qquad (V/m)$$

where E_0 is a complex constant, α and β are positive real constants.

3 a) Is this wave a uniform plane wave? Explain.

5 b) Find the expression for the phasor magnetic field intensity, $\overline{H}$.

5 c) Determine the time-average Poynting vector.

7 d) Evaluate, separately, the time-average power through each face of the cube shown in the figure. What is the net time-average power leaving the cubic volume? Comment on your result.

5 e) Propose a configuration to create such a wave in the region $z>0$. (Hint: you can adjust the parameters of the other half-space, $z<0$, and use a proper uniform plane wave there). Give a qualitative but complete answer.

a) It is not a uniform plane wave, since the amplitude ($E_o e^{-\alpha z}$) doesn't stay constant on constant phase surfaces (which are defined by $x = $ constant)

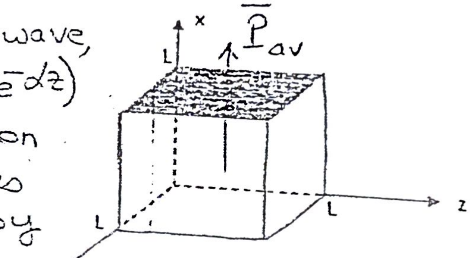


Figure for part d)

b) $\nabla \times \overline{E} = -j\omega\mu_o \overline{H}$

$$\nabla \times \overline{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & E_y & 0 \end{vmatrix} = \hat{a}_x\left(-\frac{\partial E_y}{\partial z}\right) + \hat{a}_z \frac{\partial E_y}{\partial x}$$

$$= (\hat{a}_x \alpha - \hat{a}_z j\beta) E_o e^{-\alpha z} e^{-j\beta x}$$

$$\overline{H} = \frac{\nabla \times \overline{E}}{-j\omega\mu_o} = \boxed{\frac{E_o}{\omega\mu_o}(j\alpha \hat{a}_x + \beta \hat{a}_z) e^{-\alpha z} e^{-j\beta x}} \quad A/m$$

Note that $\overline{H}$ has a ...

Q.2 Solution (cont'd – 1)

c) $$\bar{E}\times\bar{H}^* = \frac{E_o E_o^*}{\omega\mu_o} e^{-\alpha z} \cancel{e^{-j\beta x}}\, \hat{a}_y \times (-j\alpha\hat{a}_x + \beta\hat{a}_z) e^{-\alpha z} \cancel{e^{j\beta x}}$$

$$= \frac{|E_o|^2}{\omega\mu_o} e^{-2\alpha z} (j\alpha\hat{a}_z + \beta\hat{a}_x)$$

$$\bar{P}_{av} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \boxed{\frac{|E_o|^2}{2\omega\mu_o}\beta e^{-2\alpha z}\hat{a}_x \quad \text{Watts}/m^2}$$

d) $P_{av} = \int_S \bar{P}_{av}\cdot\bar{ds}$, $\bar{P}_{av} \perp \bar{ds}$ on the four side surfaces

i.e no t. av. power passes through the four side surfaces

$$P_{top} = \int_{S_{top}} \bar{P}_{av}\cdot\bar{ds} = \int_{S_{top}} P_x \Big|_{x=L} ds = \frac{|E_o|^2\beta}{2\omega\mu_o}\int_0^L\int_0^L e^{-2\alpha z} dy\, dz$$

$\bar{ds} = ds\,\hat{a}_x$

$$= \frac{|E_o|^2\beta}{2\omega\mu_o}(L)\,\frac{e^{-2\alpha z}}{-2\alpha}\Big|_0^L$$

$$= \frac{|E_o|^2\beta L}{2\omega\mu_o}\frac{1}{2\alpha}(1-e^{-2\alpha L}) \quad \text{Watts}$$

similarly

$P_{bottom} = \int_{S_{bott.}} \bar{P}_{av}\cdot\bar{ds}$ = gives the same result

$ds\,\hat{a}_x$

i.e all of the time average power entering the cube from bottom surface leaves through the top surface

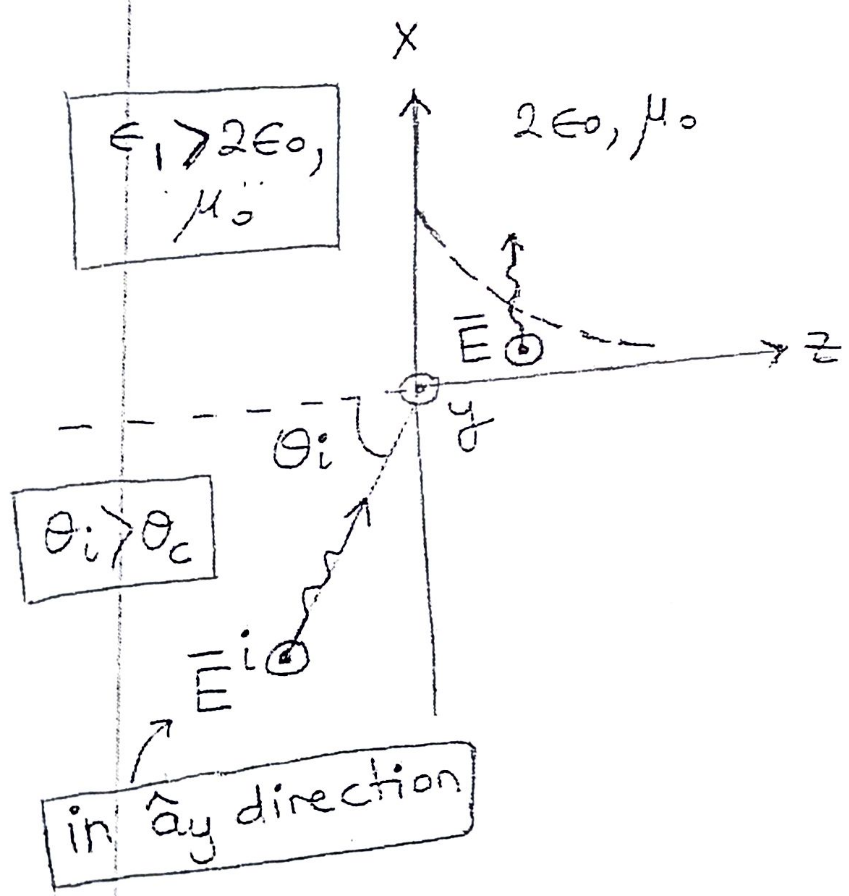
$\oint_{S_{cube}} \bar{P}_{av}\cdot\bar{ds}$ = net t. av. power <u>leaving</u> the volume

$$= P_{top} - P_{bott} = \boxed{0}$$

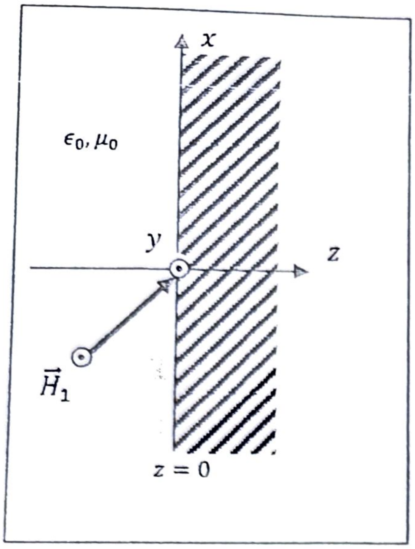
This result must be expected, since there is no loss (i.e no use of real power) inside the cube (...

Q.2 solution (cont'd - 2)

e) We can create such an evanescent wave in the region $z>0$, by putting a denser medium (i.e with permittivity $\epsilon_1 > 2\epsilon_0$) in the other half-space; and by sending a u.p.w with an incidence angle greater than the critical angle. The electric field of the incident wave must be polarized in $\hat{a}_y$ direction.



Question 3



A plane wave traveling in free space has the phasor magnetic field given as

$$\vec{H}_{inc}(x,y,z) = \hat{\mathbf{a}}_y \frac{1}{24\pi} e^{-j(3x+4z)}\ ,\ \ z<0$$

is incident upon a perfectly conducting half space ($z \geq 0$). Find the expressions of

(a) The incident wave electric field $\vec{E}_{inc}(x,y,z),\ z<0$
(b) The reflected wave electric field $\vec{E}_{ref}(x,y,z),\ z<0$
(c) The reflected wave magnetic field $\vec{H}_{ref}(x,y,z),\ z<0$
(d) The surface current density $\vec{J}_s(x,y)$ on the conducting surface at $z=0$
(e) The surface charge density $\rho_s(x,y)$ on the conducting surface at $z=0$

Answer

(a)

$$\vec{E}_{inc_0} = -\eta\, \hat{\mathbf{a}}_k \times \vec{H}_{inc_0} = -120\,\pi \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ {}^3/_5 & 0 & {}^4/_5 \\ 0 & \frac{1}{24\pi} & 0 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ {}^3/_5 & 0 & {}^4/_5 \\ 0 & -5 & 0 \end{vmatrix} = 4\hat{\mathbf{a}}_x - 3\hat{\mathbf{a}}_z$$

$$\vec{E}_{inc} = (4\hat{\mathbf{a}}_x - 3\hat{\mathbf{a}}_z)e^{-j(3x+4z)}\ ,\ \ z<0$$

(b) Using the boundary condition for total tangential E field

$$\vec{E}_{ref} = (-4\hat{\mathbf{a}}_x - 3\hat{\mathbf{a}}_z)e^{-j(3x-4z)}\ ,\ \ z<0$$

(c)

$$\vec{H}_{ref_0} = \frac{\hat{\mathbf{a}}_k \times \vec{H}_{inc_0}}{\eta} = \frac{1}{120\pi}\begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ {}^3/_5 & 0 & -{}^4/_5 \\ -4 & 0 & -3 \end{vmatrix} = \hat{\mathbf{a}}_y \frac{\left(\frac{9}{5}+\frac{16}{5}\right)}{120\pi} = \hat{\mathbf{a}}_y \frac{1}{24\pi}$$

$$\vec{H}_{ref} = \hat{\mathbf{a}}_y \frac{1}{24\pi} e^{-j(3x-4z)}\ ,\ \ z<0$$

(d)

$$\vec{J}_s(x,y) = (-\hat{\mathbf{a}}_z) \times \vec{H}_{TOT}(x,y,z=0) = (-\hat{\mathbf{a}}_z) \times (\hat{\mathbf{a}}_y)\frac{1}{12\pi}e^{-j3x} = \hat{\mathbf{a}}_x \frac{1}{12\pi}e^{-j3x}$$

(e)

$$\rho_s(x,y) = (-\hat{\mathbf{a}}_z)\cdot \vec{D}_{TOT}(x,y,z=0) = 6\epsilon_0 e^{-j3x}$$

Check via 2D continuity equation

$$\nabla\cdot\vec{J}_s(x,y) = -j\frac{3}{12\pi}e^{-j3x} = -j\frac{1}{4\pi}e^{-j3x}$$

$$k = 5 = \omega\sqrt{\mu_0\epsilon_0} \rightarrow j\omega\rho_s(x,y) = j\frac{5}{\sqrt{\mu_0\epsilon_0}}6\epsilon_0 e^{-j3x} = j\frac{30}{120\pi}e^{-j3x} = j\frac{1}{4\pi}e^{-j3x}$$

$$\nabla\cdot\vec{J}_s(x,y) + j\omega\rho_s(x,y) = 0$$

Q4. (25 pts) A circularly polarized uniform plane wave of phasor electric field intensity

$\bar{E}^i = \left(A\hat{a}_x + j\sqrt{5}\hat{a}_y - B\hat{a}_z\right)e^{-j\bar{k}_i\cdot\bar{r}}$ V/m, where A and B are real constants, and $f = 3\times10^9$ Hz

is obliquely incident from a dielectric medium with $(3\varepsilon_0, \mu_0)$ onto another dielectric with $(2\varepsilon_0, \mu_0)$ as shown in the figure.

5 a) Determine the angle of incidence θ_i, so that reflected field is linearly polarized. Explain your reasoning. *You don't have to find the numerical value of θ_i, you can just give $\sin\theta_i$ and $\cos\theta_i$.*

For the questions asked in parts b, c, d, and e, use the incidence angle found in part a.

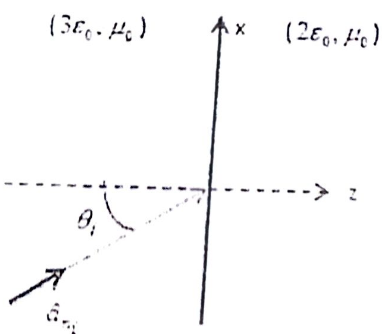
4 b) Find the propagation vectors of the incident and the reflected waves, $\bar{k}_i, \bar{k}_r$.

4 c) Find the numerical values of A and B.

4 d) Find the expression of the reflected electric field.

4 e) Comment on the polarization of the transmitted wave without actually computing fields.

4 f) Find the range of incidence angles, so that incident field is totally reflected. *You don't have to find the numerical values of θ_i, you can just give $\sin\theta_i$ and $\cos\theta_i$.*



a) if $\theta_i = \theta_{B//} \Rightarrow \bar{E}^r = \Gamma_\perp j\sqrt{5}\,\hat{a}_y e^{-j\bar{k}_r\cdot\bar{r}}$

$\theta_{B//} = \sin^{-1}\dfrac{1}{\sqrt{1+\epsilon_1/\epsilon_2}}$ $\quad \sin\theta_{B//} = \sqrt{\dfrac{2}{5}}$

$\cos\theta_i = \sqrt{1-\dfrac{2}{3}} = \sqrt{\dfrac{3}{5}}$

b) $k_1 = \omega\sqrt{\mu_0 3\epsilon_0} = \dfrac{2\pi\cdot 3\cdot10^9\sqrt{3}}{3\times10^8} = 20\sqrt{3}\pi$

$\bar{k}_i = k_1(\cos\theta_i\,\hat{a}_z + \sin\theta_i\,\hat{a}_x) = 20\sqrt{3}\pi\left(\sqrt{\tfrac{2}{5}}\,\hat{a}_x + \sqrt{\tfrac{3}{5}}\,\hat{a}_z\right)$

$\bar{k}_r = k_1(\sin\theta_i\,\hat{a}_x - \cos\theta_i\,\hat{a}_z) = 20\sqrt{3}\pi\left(\sqrt{\tfrac{2}{5}}\,\hat{a}_x - \sqrt{\tfrac{2}{5}}\,\hat{a}_z\right)$

c) $\bar{E}^i_{//} = (A\hat{a}_x - B\hat{a}_z)e^{-j\bar{k}_i\cdot\bar{r}} = E^i_{0//}(\cos\theta_i\,\hat{a}_x - \sin\theta_i\,\hat{a}_z)e^{-j\bar{k}_i\cdot\bar{r}}$

since $\bar{E}^i$ is CP, $E^i_{0//} = \sqrt{A^2+B^2} = \sqrt{5} = E^i_{0\perp}$

$A = E^i_{0//}\cos\theta_i = \sqrt{3}$, $\quad B = E^i_{0//}\sin\theta_i = \sqrt{2}$

or $(A\hat{a}_x - B\hat{a}_z)\cdot\hat{a}_{ni} = 0 \Rightarrow A\sin\theta_i - B\cos\theta_i = 0$ & $\sqrt{A^2+B^2} = \sqrt{5}$

$\Rightarrow A = \sqrt{3},\ B = \sqrt{2}$

d)

$$E^r = \Gamma_\perp \, j\sqrt{5}\, \hat{a}_y \, e^{-j \bar{k}_r \cdot \bar{r}}$$

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad \eta_1 = \frac{1}{\sqrt{3}}\eta_0 \; , \; \eta_2 = \frac{1}{\sqrt{2}}\eta_0$$

$$\theta_B + \theta_t = \frac{\pi}{2} \qquad \theta_t = \frac{\pi}{2} - \theta_{B\parallel} \qquad \cos(\theta_t) = \cos\left(\frac{\pi}{2} - \theta_{B\parallel}\right) = \sin\theta_{B\parallel}$$

$$= \frac{\sqrt{2}}{\sqrt{5}}$$

$$\Gamma_\perp = \frac{\frac{1}{\sqrt{2}}\sqrt{\frac{3}{5}} - \frac{1}{\sqrt{3}}\sqrt{\frac{2}{5}}}{\frac{1}{\sqrt{2}}\sqrt{\frac{3}{5}} + \frac{1}{\sqrt{3}}\sqrt{\frac{2}{5}}} = \frac{3-2}{3+2} = \frac{1}{5}$$

$$E^r = j\frac{\sqrt{5}}{5}\hat{a}_y \, e^{-j 20\sqrt{3}\pi \left(-\sqrt{\frac{3}{5}} z + \sqrt{\frac{2}{5}} x\right)} \quad V/m$$

e) $E_\parallel^i$ is completely transmitted
$E_\perp^i$ is partially transmitted

so that magnitude of $E_\parallel^t$ and $E_\perp^t$ will be different $\Rightarrow$ Elliptically Pol.

f) $\theta_i > \theta_c$

EE 303 Solutions of MT#2 Fall 2013

Middle East Technical University
Department of Electrical & Electronics Engineering

Q.1 (25 pts)

a) Show that the propagation constant (γ), and the intrinsic impedance (η) of a uniform plane wave propagating in a lossy medium are related by $\gamma\eta = j\omega\mu$.

b) The phasor fields of a uniform plane wave of angular frequency $\omega = 1.6\times 10^9$ (rad/sec) in a nonmagnetic (i.e. $\mu = \mu_o$), lossy medium are given by

$$\bar{E}(z) = 100\, e^{-\alpha z} e^{-j\beta z}\, \hat{a}_x \quad \text{(Volts/m)}$$

$$\bar{H}(z) = 0.5\, e^{-j\pi/8} e^{-\alpha z} e^{-j\beta z}\, \hat{a}_y \quad \text{(Amps/m)}$$

Determine:

i. the intrinsic impedance,
ii. the numerical values of α and β,
iii. the phase velocity and the depth of penetration,
iv. the time-domain fields, $\bar{E}(z,t)$, $\bar{H}(z,t)$,
v. the time-average Poynting's vector associated with this wave,
vi. the distance at which the power carried by this wave decays to 1% of its value at $z = 0$.

a) (3) $$\gamma\eta = \sqrt{j\omega\mu(\cancel{\sigma+j\omega\varepsilon})}\sqrt{\frac{j\omega\mu}{\cancel{\sigma+j\omega\varepsilon}}} = j\omega\mu$$

b) i) for a u.p.w propagating in +z direction.

(3) $$\eta = \frac{E_x}{H_y} = \frac{100}{0.5e^{-j\pi/8}} = 200e^{j\pi/8}\ \Omega$$

ii) $\gamma\eta = j\omega\mu_o$ $\quad \gamma = \frac{j\omega\mu_o}{\eta} = \frac{j1.6\times10^9\times4\pi\times10^{-7}}{200e^{j\pi/8}}$

(4) $$\gamma = 3.2\pi e^{j3\pi/8} \qquad \left(\frac{3\pi}{8}\to 67.5^\circ\right)$$
$$= 10.05(0.383+j0.924)$$
$$= \underbrace{3.846}_{\substack{\alpha \\ (Np/m)}} + j\underbrace{9.285}_{\beta\,(rad/m)}$$

iii) $$v = \frac{\omega}{\beta} = \frac{1.6\times10^9}{9.285} \cong 1.72\times10^8 \text{ m/sec}$$

(4) $$\delta = \frac{1}{\alpha} = \frac{1}{3.846} \approx 0.26 \text{ m.}$$

Continues!

iv) $\overline{E}(z,t) = 100\, e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$

3 $\overline{H}(z,t) = 0.5\, e^{-\alpha z} \cos\left(\omega t - \beta z - \frac{\pi}{8}\right) \hat{a}_y$

ω, α, β as given above

v) $\overline{P}_{av} = \frac{1}{2} \mathrm{Re}\left\{ \overline{E}(z) \times \overline{H}(z)^* \right\}$

$= \frac{1}{2} \mathrm{Re}\left\{ 100\, e^{-\alpha z} \cancel{e^{-j\beta z}}\, 0.5 e^{j\pi/8} e^{-\alpha z} \cancel{e^{j\beta z}}\, \underbrace{\hat{a}_x \times \hat{a}_y}_{\hat{a}_z} \right\}$

$= \frac{1}{2} \mathrm{Re}\left\{ 50\, e^{-2\alpha z} e^{j\pi/8}\, \hat{a}_z \right\}$

4 $= 25\, e^{-2\alpha z} \underbrace{\cos\left(\frac{\pi}{8}\right)}_{0.924} \hat{a}_z \qquad \left(\frac{\pi}{8} \rightarrow 22.5^\circ\right)$

$= 23.1\, e^{-2\alpha z}\, \hat{a}_z$ Watts/m^2

vi) $e^{-2\alpha z} = 0.01$

$-2\alpha z = \ln(0.01) = -4.605$

4 $z = \dfrac{4.605}{2\alpha} = \dfrac{4.605}{2 \times 3.846} \approx 0.6$ m.

Q.2 (25 pts)

Consider a finite section of an infinitely long conducting pipe (also called as waveguide) of rectangular cross section as shown in the figure. The walls of the pipe are perfect conductor and the pipe is filled with free space.

The electric field intensity inside this source-free conducting pipe is given as:

$$\bar{E}(x,y,z;t) = E_0 \sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta z)\hat{a}_y \quad (Volts/m)$$

a) Find the time-domain magnetic field intensity inside the waveguide.

b) Find the instantaneous Poynting's vector.

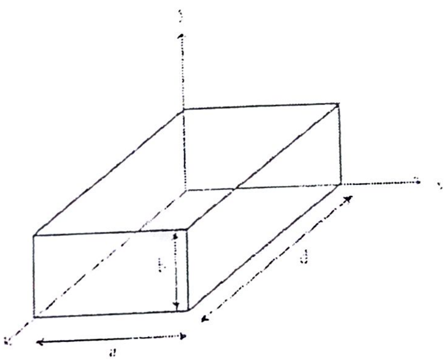
c) Find the time averaged power crossing the left and right walls of the pipe at x=0 and x=a, respectively.

d) Find the time averaged power crossing the bottom and top walls of the pipe at y=0 and y=b, respectively.

e) Find the time averaged power crossing the plane at z=0.

f) By using the results of part (c)-(e), find the total average power leaving the closed surface formed by closing the ends of the pipe at z=0 and z=d planes. Comment on your result by referring to Poynting's theorem.

Hint: $\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{a}{2}$



a) $\nabla \times \bar{E} = -\mu_0 \frac{\partial \bar{H}}{\partial t}$

$$\nabla \times \bar{E} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -a_x \frac{\partial E_y}{\partial z} + a_z \frac{\partial E_y}{\partial x}$$

b) $\bar{P} = \bar{E} \times \bar{H}$

$= E_y \hat{a}_y \times (H_x \hat{a}_x + H_z \hat{a}_z)$

$= - E_y H_x \hat{a}_z + E_y H_z \hat{a}_x$

$= E_0^2 \sin^2 \frac{\pi}{a} x \frac{\beta}{\omega \mu_0} \cos^2(\omega t - \beta z) \hat{a}_z$

$- E_0^2 \sin \frac{\pi}{a} x \cos \frac{\pi}{a} x \frac{\pi}{a \omega \mu_0} \cdot \sin(\omega t - \beta z) \cos(\omega t - \beta z) \hat{a}_x$

c) Time averaged Poynting's vector is

$\bar{P}_{av} = \frac{E_0^2}{2} \sin^2 \frac{\pi}{a} x \frac{\beta}{\omega \mu_0} \hat{a}_z$ (The time average of x-component is zero)

Power crossing the walls at $x=0$ and $x=a$

$$\int \bar{P}_{av} \cdot \bar{ds} \Big|_{x=0} = \int \bar{P}_{av} \cdot \bar{ds} \Big|_{x=a} = 0$$

d) Similarly $\int \bar{P}_{av} \cdot \bar{ds} \Big|_{y=0} = \int \bar{P}_{av} \cdot \bar{ds} \Big|_{y=b} = 0$

e) At $z=0$ $\int_0^b \int_0^a \frac{E_0^2}{2} \frac{\beta}{\omega \mu_0} \sin^2 \frac{\pi x}{a} \, dx \, dy$

$$= \frac{E_0^2}{2} \frac{\beta}{\omega \mu_0} \frac{a}{2} \cdot b = \frac{E_0^2}{4} \frac{a b \beta}{\omega \mu_0} = P_1$$

f) at $z = d$ $\bar{ds}$ is in $(-\hat{a}_z)$ direction

$$\Rightarrow P_2 = \int_0^b \int_0^a \bar{P}_{av} \cdot \bar{ds} \Big|_{z=d} = -P_1$$

$\Rightarrow$ Total power leaving the closed surface is $P_2 - P_1 = 0$

The walls of the pipe are perfect conductor so there isn't any ohmic loss inside the closed surface. From Poynting's theorem this implies that the time rate of change of stored magnetic and electric energies are zero

Alternate solution for part c)

Phasor domain $\bar{E} = E_0 \sin\left(\frac{\pi}{a}x\right) e^{-j\beta z} \hat{a}_y$

$$\bar{H} = -\hat{a}_x \frac{E_0 \beta}{\omega\mu_0} \sin\left(\frac{\pi}{a}x\right) e^{-j\beta z}$$

$$+ \hat{a}_z \frac{j E_0}{\omega\mu_0} \cos\left(\frac{\pi}{a}x\right) e^{-j\beta z}$$

$\bar{S}_{av} = \frac{1}{2} \mathrm{Re}\{\bar{E} \times \bar{H}^*\} = \frac{E_0^2 \beta}{2\omega\mu_0}$ [illegible] $\left(\frac{\pi}{a}x\right) \hat{a}_z$

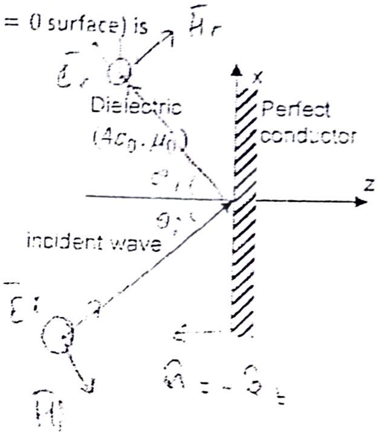
[illegible]

Q.3 (25 pts)

Perpendicularly polarized uniform plane wave in a dielectric medium ($\mu=\mu_0$, $\epsilon=4\epsilon_0$) is incident upon a perfectly conducting half space which occupies $z \geq 0$. The incident electric field expression is given as
$\vec{E} = \hat{a}_y E_0 \cos\left(\omega t - k_0\left(x + \sqrt{3}z\right)\right)$ V/m

where E_0 is the amplitude of the field, $k_0 = \omega\sqrt{\mu_0 \epsilon_0}$, and ω is the angular frequency.
The phasor surface current density at the dielectric- conductor interface ($z = 0$ surface) is
$\vec{J}_s = \hat{a}_y 3\sqrt{3}e^{-jk_0 x}$ A/m.

a) Find the angle of incidence.
b) Find E_0.



a) $\bar{k}_i \cdot \bar{r} = k_0 (x + \sqrt{3} z)$

$\bar{k}_i = 2k_0 (\sin\theta_i \hat{a}_x + \cos\theta_i \hat{a}_z)$

$\tan\theta_i = \frac{1}{\sqrt{3}} \Rightarrow \theta_i = 30°$

b) $\bar{E}_i = \hat{a}_y E_0 e^{-jk_0(x+\sqrt{3}z)}$

$\bar{H}_i = \frac{1}{\eta}\hat{a}_{n_i}\times\bar{E}_i$ $\quad \hat{a}_{n_i} = \sin\theta_i \hat{a}_x + \cos\theta_i \hat{a}_z$, $\quad \eta = \sqrt{\frac{\mu_0}{4\epsilon_0}} = \frac{\eta_0}{2}$

$\bar{H}_i = \frac{1}{\eta}E_0(-\hat{a}_x\cos\theta_i + \hat{a}_z \sin\theta_i) e^{-jk_0(x+\sqrt{3}z)}$

$\bar{H}_r = \frac{1}{\eta}\Gamma E_0(\hat{a}_x \cos\theta_i + \hat{a}_z\sin\theta_i) e^{-j2k_0(x\sin\theta_i - z\cos\theta_i)}$

$\Gamma = -1$

$\bar{J}_s = \hat{n} \times (\bar{H}_i + \bar{H}_r)\Big|_{z=0} = -\hat{a}_z \times \left(-\hat{a}_x \frac{2E_0\cos\theta_i}{\eta} e^{-jk_0 x}\right)$

$= \hat{a}_y \frac{2E_0}{\eta}\cos\theta_i e^{-jk_0 x} = \hat{a}_y 3\sqrt{3} e^{-jk_0 x}$

$\frac{2 \cdot 2E_0}{\eta_0}\frac{\sqrt{3}}{2} = 3\sqrt{3} \Rightarrow E_0 = \frac{3}{2}\eta_0 = \frac{3}{2}120\pi = 180\pi$

Q.4 (25 pts)

A Uniform Plane Wave propagating in a lossless dielectric medium of ($\varepsilon = 4\varepsilon_o, \mu = \mu_o$) is obliquely incident to the interface of this medium with air as shown. The phasor electric field of the incident wave is given by,

$$\vec{E}_i = \overbrace{(\hat{a}_x + 2\hat{a}_y + A\hat{a}_z)}^{\vec{E}_o} e^{-jk_1 \hat{a}_{n_i} \cdot \vec{r}}$$

Where, $\hat{a}_{n_i}$ is the unit vector in the direction of propagation wave of incident wave, $\vec{r}$ is the position vector and k_1 is the propagation constant of *medium 1*. If the reflected wave is y-polarized , ($\vec{E}_r = \hat{a}_y E_r$), find:

a) 5 i) The angle of incidence, θ_i, angle of reflection, θ_r, and the angle of transmission, θ_t.

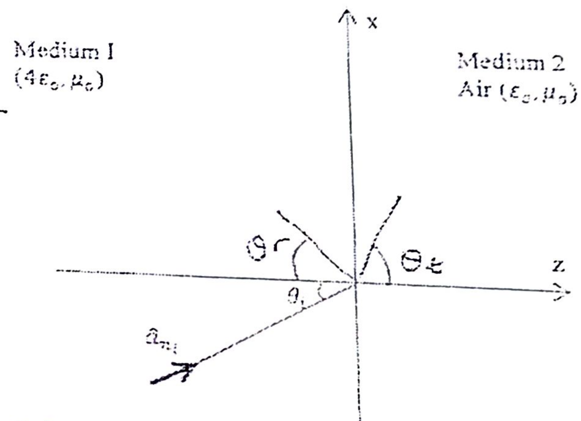
4 ii) The unit vector, $\hat{a}_{n_i}$, and the unknown constant A.

6 b) The reflected electric field expression.

10 c) The transmitted electric field expression.

a)

i) Only $\perp$ polarized wave is reflected $\Rightarrow \theta_i = \theta_B$ (Brewster Angle)



$\theta_i = \theta_{B_{//}} = \sin^{-1} \dfrac{1}{\sqrt{\varepsilon_1 + \varepsilon_2}} = \dfrac{1}{\sqrt{5}}$

(since $\mu_1 = \mu_2 = \mu_0$)

$\Rightarrow \theta_i = \theta_r = 26.57°$ //

$\theta_i + \theta_t = 90° \Rightarrow \theta_t = 63.43°$ //

ii) $\hat{a}_{n_i} = \hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i = \dfrac{1}{\sqrt{5}}\hat{a}_x + \dfrac{2}{\sqrt{5}}\hat{a}_z$ //

$\hat{a}_{n_i} \perp \vec{E}_o \Rightarrow A = -1/2$

c) $\tau_\perp = 1 + \Gamma_\perp = 1.6$

$$\tau_{//} = \frac{\cos\theta_i}{\cos\theta_t}(1 + \Gamma_{//}^{\,0}) = 2$$

$$\hat{a}_{n_t} = \frac{2}{\sqrt{5}}\hat{a}_x + \frac{1}{\sqrt{5}}\hat{a}_z$$

$$\bar{E}_{t\perp} = \hat{a}_y\, 2 . \Gamma_\perp\, e^{-jk_2\, \hat{a}_{n_t}\cdot \bar{r}} = \hat{a}_y\, 3.2\, e^{-jk_0\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z\right)}$$

$$E_{t\,//} = |E_{i_{//}}| . \tau_{//}\left(\frac{1}{\sqrt{5}}\hat{a}_x - \frac{2}{\sqrt{5}}\hat{a}_z\right) e^{-jk_2\, \hat{a}_{n_t}\cdot \bar{r}}$$

$$= (\hat{a}_x - 2\hat{a}_z)\, e^{-jk_0\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z\right)}$$

$$\Rightarrow \bar{E}_t = \bar{E}_{t\perp} + E_{t//} = (\hat{a}_x + 3.2\,\hat{a}_y - 2\hat{a}_z)\, e^{-jk_0\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}z\right)}$$

$$k_0 = \omega\sqrt{\epsilon_0\mu_0} = \frac{\omega}{c}$$

7

Middle East Technical University
Department of Electrical & Electronics Engineering

Q1. (30 pts.)

Propagation constant of a uniform plane wave (u.p.w.) in a lossy medium is expressed as $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$ in terms of given medium parameters $(\sigma, \mu, \varepsilon)$ at a given frequency $\omega = 2\pi f$.

(15 pt) a) Using a proper approximation, derive the expressions for attenuation constant (α) and phase constant (β) for a **good dielectric** material first. Then, Derivation: 7 pts.

- (5 pt) i. assume that a u.p.w. is propagating in a good dielectric material with $\mu_r = 1$ and $\varepsilon_r = 4$ at $f = 5$ MHz. Find the wavelength (λ) and phase velocity (v).
- (3 pt) ii. If $\alpha = 1.5\pi \times 10^{-3}$ $(Neper/m)$ is given, calculate the conductivity (σ) for this good dielectric material.

(15 pt) b) Using a proper approximation, derive the expressions for attenuation constant (α) and phase constant (β) for a **good conductor** material first. Then, Derivation: 5 pts.

- (5 pt) i. find the wavelength (λ) and phase velocity (v) of a u.p.w oscillating at $f = 1\ MHz$ if it propagates in copper which has $\mu_r = 1, \varepsilon_r = 1$ and $\sigma = 5.8 \times 10^7\ (S/m)$.
- (5 pt) ii. To shield a box (containing electronic equipment) against electromagnetic interference over the frequency range $10\ KHz \le f \le 100\ MHz$, the box needs to be enclosed within a layer of copper which is at least five skin depths thick. What should be the minimum thickness of copper layer in millimeters?

Note: Be sure to provide units with your answers. NO points will be given for directly written memorized answers for the "derivation" parts of this question. Clearly indicate derivation steps.

Hint: You may find the following approximation useful: $(1+x)^{1/2} \approx 1 + \frac{x}{2}$ if $|x| << 1$.

Solution:

a) $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ where $(\sigma/\omega\varepsilon) << 1$ for good dielectrics.

$$= \sqrt{j\omega\mu(j\omega\varepsilon)\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)} = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

let $x = -j\frac{\sigma}{\omega\varepsilon} \Rightarrow |x| << 1 \Rightarrow \left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{1/2} \approx 1 - j\frac{\sigma}{2\omega\varepsilon}$

$$\Rightarrow \gamma \cong j\omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right) = j\omega\sqrt{\mu\varepsilon} + \frac{\omega\sqrt{\mu\varepsilon}\,\sigma}{2\omega\varepsilon}$$

$$\gamma \cong \underbrace{\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}_{\alpha} + j\underbrace{\omega\sqrt{\mu\varepsilon}}_{\beta} = \alpha + j\beta \Rightarrow \boxed{\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}},\ \beta \cong \omega\sqrt{\mu\varepsilon}}$$

a-i) $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2\pi f\sqrt{4\varepsilon_0\mu_0}} = \frac{c}{2f} = \frac{3\times10^8}{2\times5\times10^6} = \boxed{30\ m = \lambda}$$

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{4\mu_0\varepsilon_0}} = \frac{c}{2} = \boxed{1.5\times10^8\ m/sec = v}$$ $\left(\lambda = \frac{v}{f}\right.$ here! $\left.\right)$

(a-ii) $\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = 1.5\pi\times10^{-3} \cong \frac{\sigma}{2}\sqrt{\frac{\mu_o}{4\epsilon_o}} = \frac{\sigma}{2}\,\frac{120\pi}{2} = \sigma\, 30\pi$

$\sigma = \frac{1.5\cancel{\pi}\times10^{-3}}{30\cancel{\pi}} = \frac{\cancel{150}^{5}\times10^{-5}}{\cancel{30}} = \boxed{5\times10^{-5}\ S/m = \sigma}$

(b) $\gamma = \sqrt{j\omega\mu(\sigma + \underbrace{j\omega\epsilon}_{\text{negligible}})}$ where $(\sigma/\omega\epsilon) >> 1$ for a good conductor

$\cong \sqrt{j\omega\mu\sigma} = \sqrt{j}\,\sqrt{\omega\mu\sigma} = \underbrace{\left(e^{j\pi/2}\right)^{1/2}}_{e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)}\sqrt{\omega\mu\sigma}$

$\cong (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$

$\Rightarrow \boxed{\alpha \approx \beta \approx \sqrt{\pi f\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}}$

(b-i) $\mu_r = \epsilon_r = 1$, $\sigma = 5.8\times10^{7}\ S/m$, $f = 1\ MHz = 10^{6}\ Hz$.

$\lambda = \frac{2\pi}{\beta} = \frac{2\cancel{\pi}}{\sqrt{\cancel{\pi}\times10^{6}\times4\pi\times10^{-7}\times5.8\times10^{7}}} = \frac{1}{10^{3}\sqrt{5.8}} = \frac{10^{-3}}{\sqrt{5.8}}\ m. \cong \underline{\underline{4.15\times10^{-4}\ m}}$

$v = \frac{\omega}{\beta} = \frac{2\pi f}{\beta\lambda} = \lambda f = 4.15\times10^{-4}\times10^{6} \cong \underline{\underline{415\ m/sec}}$

(b-ii) $\delta = \frac{1}{\alpha}\Big|_{\substack{f=10kHz.\\=10^{4}Hz.}} = \frac{1}{\sqrt{\pi\times10^{4}\times4\pi\times10^{-7}\times5.8\times10^{7}}} = \frac{1}{\underbrace{2\pi\times10^{2}\sqrt{5.8}}_{1.513\times10^{3}}} \approx \underbrace{6.6\times10^{-4}\ m.}_{\text{max skin depth at 10kHz}}$

$d_{min} = 5\delta\Big|_{10kHz} \cong 3.3\times10^{-3}\ m \cong \underline{\underline{3.3\ mm.}}$

at $f = 100\ MHz = 10^{8}\ Hz$ $\quad \delta = 6.6\times10^{-6}\ m. \Rightarrow d_{min} = 5\delta = 33$ micron

which does not protect the electronic circuit at lower frequencies!

Q.2 (20 pts)

a) Consider the following circularly polarized uniform plane wave

$$\vec{E} = E_o(\hat{a}_x - j\hat{a}_y)e^{-jkz} \quad V/m$$

Show that the instantaneous power density is constant (independent of time) and it is equal to

$$\vec{\mathcal{P}}(z,t) = \frac{E_o^2}{\eta}\hat{a}_z \ \text{Watts/m}^2$$

b) The following uniform plane wave propagates in a dielectric medium with $\epsilon = \epsilon_r\epsilon_o$ and $\mu = \mu_o$

$$\vec{E} = (6j\hat{a}_x + A\hat{a}_y + B\hat{a}_z)e^{-j(4\pi y+3\pi z)} \quad V/m.$$

Find the constants A, B and frequency of the wave so that both of the constraints given below are satisfied.

i. the instantaneous power density is $\vec{\mathcal{P}}(z,t) = \frac{4\hat{a}_y + 3\hat{a}_z}{\pi}$ Watts/m^2, and
ii. time domain electric field intensity vector is in +y direction at $t = 0$.

a) $\bar{H} = \frac{1}{\eta}\hat{a}_z \times \bar{E} = \frac{E_o}{\eta}(j\hat{a}_x + \hat{a}_y)e^{-jkz}$

$\bar{H}(z,t) = Re\{\bar{H}(z)e^{j\omega t}\} = \frac{E_o}{\eta}(-\sin(\omega t - kz)\,\hat{a}_x + \sin(\omega t - kz)\,\hat{a}_y$

$\bar{E}(z,t) = E_o(\cos(\omega t - kz)\,\hat{a}_x + \sin(\omega t - kz)\,\hat{a}_y)$

$\bar{P}(z,t) = \bar{E}(z,t) \times \bar{H}(z,t) = \frac{E_o^2}{\eta}\hat{a}_z\underbrace{(\sin^2(\omega t - kz) + \cos^2(\omega t - kz))}_{1}$

$= \frac{E_o^2}{\eta}\hat{a}_z$ //

b) $\bar{k} = k\hat{a}_n = 4\pi\,\hat{a}_y + 3\pi\,\hat{a}_z$

$\Rightarrow k = 5\pi \quad \hat{a}_n = 0.8\,\hat{a}_y + 0.6\,\hat{a}_z$

U.P.W $\Rightarrow \hat{a}_n \perp \bar{E}_o \Rightarrow \hat{a}_n \cdot \bar{E}_o = 0 \Rightarrow B = -8j$ //

$\bar{P}(z,t)$ is constant $\Rightarrow$ circularly polarized (CP)

$|E_y| = |E_y\hat{a}_y + E_z\hat{a}_z| = 10 \Rightarrow A = 10$ //

$\bar{P}(z,t) = \frac{E_o^2}{\eta}\hat{a}_n = \frac{100}{\eta}(0.6\,\hat{a}_y + 0.8\,\hat{a}_z) = \frac{1}{\pi}(4\,\hat{a}_y + 3\,\hat{a}_z)$

$\eta = 20\pi = \sqrt{\frac{\mu_o}{\epsilon_o}}\frac{1}{\sqrt{\epsilon_r}} \Rightarrow \epsilon_r = 36$

$k = \omega\sqrt{\epsilon\mu} = \omega\sqrt{\epsilon_o\mu_o}\sqrt{\epsilon_r} = \frac{\omega}{c}\sqrt{\epsilon_r} = \frac{2\pi f}{c}\sqrt{\epsilon_r} = 5\pi$

$\Rightarrow f = 125$ MHz //

Q.3 (25pts)

A uniform plane wave of phasor electric field intensity

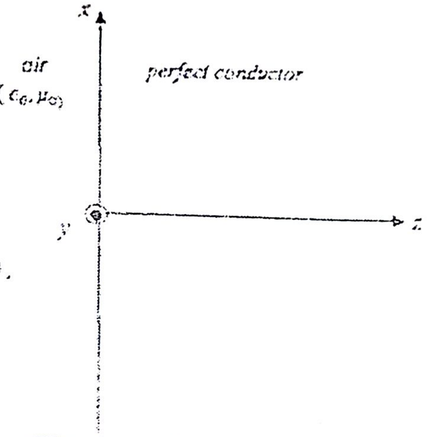
$$\bar{E}^i(z) = (3\hat{a}_x + j\hat{a}_y)\,e^{-jkz} \qquad (\text{Volts/m}),\quad k = \omega\sqrt{\epsilon_0\mu_0}$$

in air ($z<0$) is normally incident upon a perfect conductor of planar boundary at $z=0$ as shown in the figure.

5 a) Find the polarization (type and sense) of the incident wave.
5 b) Find the expression of the phasor electric field intensity of the reflected wave.
5 c) Find the polarization (type and sense) of the reflected wave. Compare with part a).
6 d) Evaluate the time-average Poynting vector in air using the total (i.e *incident + reflected*) fields.
4 e) Evaluate the time-average Poynting vectors for the incident and reflected waves separately. Comment on your results.

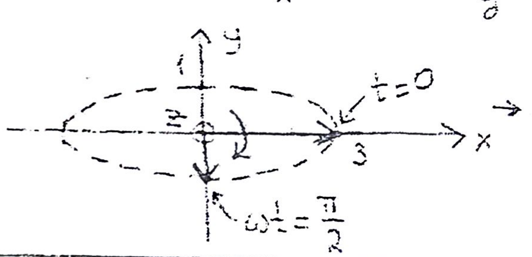
a)

$$\bar{E}^i(z,t) = 3\cos(\omega t - kz)\,\hat{a}_x + \underbrace{\cos\left(\omega t - kz + \frac{\pi}{2}\right)}_{\sin(\omega t - kz)}\hat{a}_y$$



To determine polarization, let $z=0$.

$$\bar{E}^i(0,t) = 3\cos\omega t\,\hat{a}_x - \sin\omega t\,\hat{a}_y$$



$\boxed{\text{Left Hand Elliptic Polarization (LHEP)}}$

b)

The reflection coefficient is $\Gamma = -1$ for both components of the electric field

$$\therefore \quad \bar{E}^r(z) = -(3\hat{a}_x + j\hat{a}_y)\,e^{jkz}$$

c) $\bar{E}^r(0,t) = -3\cos\omega t\,\hat{a}_x + \sin\omega t\,\hat{a}_y$

$\boxed{\text{RHEP}}$ since this wave propagates in $-z$ direction (into the paper)

So, the reflection by a perfect conductor doesn't change the 'type' of polarization, but the 'sense' is <u>reversed</u>.

<u>Q.3 cont'd</u>

d)

$\eta = 120\pi\ \Omega$

$$\overline{E}^i = (3\hat{a}_x + j\hat{a}_y)e^{-jkz} \longrightarrow \overline{H}^i = \frac{1}{\eta}\hat{a}_z \times \overline{E}^i$$

$$= \frac{1}{\eta}(3\hat{a}_y - j\hat{a}_x)e^{-jkz}$$

$$\overline{E}^r = -(3\hat{a}_x + j\hat{a}_y)e^{jkz} \longrightarrow \overline{H}^r = \frac{1}{\eta}(-\hat{a}_z) \times \overline{E}^r$$

$$= \frac{1}{\eta}(3\hat{a}_y - j\hat{a}_x)e^{jkz}$$

Total fields in air:

$$\overline{E} = \overline{E}^i + \overline{E}^r = 3(e^{-jkz} - e^{jkz})\hat{a}_x + j\underbrace{(e^{-jkz} - e^{jkz})}_{-2j\sin kz}\hat{a}_y$$

$$\boxed{\overline{E} = (-6j\hat{a}_x + 2\hat{a}_y)\sin kz}$$

$$\overline{H} = \overline{H}^i + \overline{H}^r = \frac{1}{\eta}\left[3(e^{-jkz} + e^{jkz})\hat{a}_y - j\underbrace{(e^{-jkz} + e^{jkz})}_{2\cos kz}\hat{a}_x\right]$$

$$\boxed{\overline{H} = \frac{1}{\eta}(6\hat{a}_y - j2\hat{a}_x)\cos kz}$$

$$\overline{P}_{av} = \frac{1}{2}\text{Re}\{\overline{E} \times \overline{H}^*\} = \frac{1}{2}\text{Re}\left\{(-6j\hat{a}_x + 2\hat{a}_y) \times (6\hat{a}_y + j2\hat{a}_x)\frac{1}{\eta}\sin kz\cos kz\right\}$$

$$= \frac{1}{2}\text{Re}\left\{\underbrace{[-36j\hat{a}_z + 4j(-\hat{a}_z)]}_{-40j\hat{a}_z}\frac{\sin kz\cos kz}{\eta}\right\} = \boxed{0}$$

purely imaginary

The <u>net</u> time-average power flow towards the boundary is zero. This should be expected from the conservation of real power, since $\overline{P}_{av}$ is identically zero in the perfect conductor!

Q.3 cont'd

e) 3

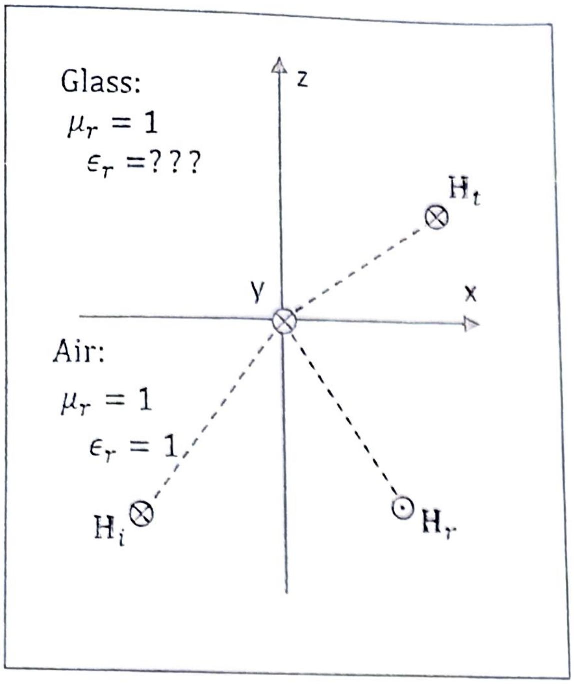
$$\bar{P}_{av}^{i} = \frac{1}{2} Re\{\bar{E}^{i} \times \bar{H}^{i*}\} = \frac{1}{2\eta} |\bar{E}^{i}|^2 \hat{a}_z = \frac{1}{2\eta} (\bar{E}^{i} \cdot \bar{E}^{i*}) \hat{a}_z$$

$$= \frac{1}{2\eta} [3(3) + j(-j)] \hat{a}_z = \frac{10}{2\eta} \hat{a}_z = \boxed{\frac{1}{24\pi} \hat{a}_z} \text{ Watts}/m^2$$

$$\bar{P}_{av}^{r} = \frac{1}{2} Re\{\bar{E}^{r} \times \bar{H}^{r*}\} = \frac{1}{2\eta} |\bar{E}^{r}|^2 (-\hat{a}_z)$$

$$= \frac{1}{2\eta} [(-3)(-3) + (-j)(j)] (-\hat{a}_z) = \frac{10}{2\eta} (-\hat{a}_z) = \boxed{\frac{1}{24\pi} (-\hat{a}_z)}$$

i and r waves carry equal time-average powers, but in opposite directions. This result is consistent with part d).



Q.4.(25pts) A uniform plane wave in air ($z < 0$) is incident upon glass ($z > 0$) at $= 0$. The incident, reflected and transmitted phasor magnetic field expressions are given as

$$\mathbf{H}_i = 0.05\, \hat{\mathbf{a}}_y \exp\left(-j0.2\pi\left(\frac{x\sqrt{3}}{2} + \frac{z}{2}\right)\right), \quad z < 0$$

$$\mathbf{H}_r = -A\, \hat{\mathbf{a}}_y \exp\left(-j0.2\pi\left(\frac{x\sqrt{3}}{2} - \frac{z}{2}\right)\right), \quad z < 0$$

$$\mathbf{H}_t = B\, \hat{\mathbf{a}}_y \exp\left(-j\,C\left(\frac{x}{2} + \frac{z\sqrt{3}}{2}\right)\right), \quad z > 0$$

Calculate the numerical values of the constants A, B, C and ϵ_r (the relative permittivity of glass).

Solution :

$$\sin\theta_t = \cos\theta_i = \frac{1}{2}, \qquad \cos\theta_t = \sin\theta_i = \frac{\sqrt{3}}{2}, \qquad check: \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\text{Snell Law of Transmission} \rightarrow k_{ix} = k_{tx} \rightarrow 0.2\pi\frac{\sqrt{3}}{2} = \frac{1}{2}C$$

$$\rightarrow \boxed{C = 0.2\pi\sqrt{3} = 1.0887}$$

$$\underbrace{C}_{0.2\pi\sqrt{3}} = \omega\sqrt{\mu\epsilon} = \underbrace{\omega\sqrt{\mu_0\epsilon_0}}_{0.2\,\pi}\sqrt{\epsilon_{r_{glass}}} \rightarrow \boxed{\epsilon_{r_{glass}} = 3}$$

$$\left[\frac{E_{r0}}{E_{i0}}\right] = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} + \frac{1}{2}} = 0 \rightarrow H_r = 0 \rightarrow \boxed{A = 0}$$

$$H_{tx} = H_{ix} - H_{rx} \rightarrow \boxed{B = 0.05}$$

B,C,ϵ_r 5 Points Each and A is 10 P

EE303 Midterm Examination 2 - Solutions

Middle East Technical University
Department of Electrical & Electronics Engineering

Q.1 (30 pts)

A uniform plane wave of frequency $f = 4\ GHz$ is propagating in a lossless and non-magnetic medium of unknown permittivity $\varepsilon = \varepsilon_o \varepsilon_r$. The phasor domain expression for the electric field intensity vector is given as

$\bar{E} = (j\hat{a}_x + A\hat{a}_y - j\hat{a}_z)e^{-j40\sqrt{2}\pi(x+z)}$ (V/m) where A is known to be a real constant.

a) (8 pts) Obtain the phasor domain expression for the magnetic field intensity vector $\bar{H}$ in terms of A. (i.e., you must compute all relevant parameters needed for $\bar{H}$ and simplify the result; only A will show up in your expression as an unspecified parameter).

b) (8 pts) Obtain the time-average Poynting's vector $\bar{P}_{avg}$ (in terms of A) using phasor domain fields.

c) (12 pts) Find the type and sense of polarization of this uniform plane wave for the following cases:

i. $A = -1$

ii. $A = \sqrt{2}$

d) (2 pts) Find the value of A needed for LHCP (left-hand circular polarization).

Please provide units, necessary formulations and show your solution steps.

Solution:

(a) $\bar{E} = \underbrace{(j\hat{a}_x + A\hat{a}_y - j\hat{a}_z)}_{\bar{E}_o} e^{-j40\sqrt{2}\pi(x+z)} = \bar{E}_o e^{-jk\hat{n}\cdot\bar{r}} = \bar{E}_o e^{-j\bar{k}\cdot\bar{r}}$

$k\hat{n}\cdot\bar{r} = 40\sqrt{2}\pi(x+z) \Rightarrow k_x = k_z = 40\sqrt{2}\pi,\ k_y = 0 \Rightarrow \boxed{k = |\bar{k}| = 80\pi \text{ rad/m}}$

$\boxed{\hat{n} = \frac{\bar{k}}{k} = \frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_y)}$

$k = 2\pi f\sqrt{\varepsilon\mu} = 2\pi \times 4\times 10^9 \sqrt{\varepsilon_o\mu_o}\sqrt{\varepsilon_r} = 80\pi$

$\Rightarrow \boxed{\varepsilon_r = 9} \Rightarrow \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{\sqrt{\varepsilon_r}} = \frac{120\pi}{3} = 40\pi\ (\Omega)$

For a u.p.w., $\bar{H} = \frac{1}{\eta}\hat{n}\times\bar{E} = \frac{1}{40\pi}\frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_z)\times(j\hat{a}_x + A\hat{a}_y - j\hat{a}_z)e^{-j\bar{k}\cdot\bar{r}}$

$\Rightarrow \boxed{\bar{H} = \frac{1}{40\sqrt{2}\pi}(-A\hat{a}_x + 2j\hat{a}_y + A\hat{a}_z)\, e^{-j40\sqrt{2}\pi(x+z)} \quad (A/m)}$

(b) $\bar{P}_{avg} = \frac{1}{2}Re\{\bar{E}\times\bar{H}^*\} = \frac{1}{2}Re\left\{(j\hat{a}_x + A\hat{a}_y - j\hat{a}_z)e^{-j\bar{k}\cdot\bar{r}} \times e^{+j\bar{k}\cdot\bar{r}}\frac{(-A\hat{a}_x - j2\hat{a}_y + A\hat{a}_z)}{40\sqrt{2}\pi}\right\}$

$\Rightarrow \boxed{\bar{P}_{avg} = \frac{1}{80\sqrt{2}\pi}(A^2+2)(\hat{a}_x + \hat{a}_z)\ (W/m^2)}$ ($\bar{P}_{avg} \parallel \hat{n}$ as expected)

(Or, you may use $\bar{P}_{avg} = \frac{1}{2\eta}|E|^2\hat{n}$ for a u.p.w. in a lossless medium) (skin depth) is measured as $\delta = 8\ cm$. Determine the conductivity ... velocity.

b) At $f = 10\ MHz$, the lake water behaves as a *good dielectric*, and the depth of penetration is measured as $\delta = 12\ m$. Determine the conductivity, and the phase velocity.

Make all reasonable approximations.

a) $\gamma = \sqrt{j\omega\mu(\sigma + \underbrace{j\omega\epsilon}_{\text{neglect this term } (\sigma \gg \omega\epsilon)})} \approx \sqrt{j}\sqrt{\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}(1+j) = \alpha + j\beta$

i.e $\boxed{\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma}}$

$\boxed{\delta = \frac{1}{\alpha}}$ $\quad \delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$ $\quad \delta^2 = \frac{1}{\pi f \mu \sigma}$

$$\sigma = \frac{1}{\delta^2 \pi f \mu} = \frac{1}{(8\times10^{-2})^2 \pi \times \cancel{10^{7}} \times 4\pi \times \cancel{10^{-7}}}$$

$$\sigma = \frac{10^4}{256\pi^2} \boxed{\simeq 3.96\ \mho/m}$$

$\boxed{V = \frac{\omega}{\beta}} \approx \frac{\omega}{\alpha} = \omega\delta = 2\pi f \delta = 2\pi\times10^{7}\times 8\times10^{-2} = 16\pi\times10^{5}$

$\simeq 5.03\times10^{6}\ m/sec$

b) $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = j\omega\sqrt{\mu\epsilon}\underbrace{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}_{\approx 1 - j\frac{\sigma}{2\omega\epsilon}} = \alpha + j\beta$

$\approx 1 - j\frac{\sigma}{2\omega\epsilon}$ since $\frac{\sigma}{\omega\epsilon} \ll 1$

$\boxed{\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$, $\boxed{\beta \approx \omega\sqrt{\mu\epsilon} = k}$

$$\sigma = 2\left(\frac{1}{\delta}\right)\Big/\sqrt{\frac{\mu_0}{81\epsilon_0}} = 2\left(\frac{1}{12}\right)\frac{9}{120\pi} = \frac{1}{80\pi} \simeq 3.98\times10^{-3}\ \mho/m$$

$$V = \frac{\omega}{\beta} \approx \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0 81\epsilon_0}} = \frac{c}{9} \boxed{\simeq 3.33\times10^{7}\ m/sec}$$

Q.3 (30 pts)

A uniform plane wave of 100 MHz propagating in air (medium 1: $z \le 0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$) is normally incident upon a good conducting half space, (medium 2: $z > 0$, $\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$, $\sigma = 10$) Phasor electric field of the incident wave is given by,

$$\vec{E} = \vec{a}_x 10 e^{-jk_0 z} \text{ V/m}$$

E_{i0}

Determine,

2 a) numerical value of the propagation constants of both medium 1 and medium 2,

12 b) reflected and transmitted Electric fields in the phasor domain,

5 c) transmitted Electric field in the time domain,

5 d) magnitude of the Electric field intensity at $z = 0.1$ mm, and $z = 2$cm. Comment on the numerical values you obtained.

a) $k_0 = \omega\sqrt{\varepsilon_0\mu_0} = \frac{\omega}{c} = \frac{2\pi}{3}$ rad/m //

$\gamma = \alpha + j\beta$, $\frac{\sigma}{\omega\varepsilon} = 450 >> 1$ good conductor ✓

$\alpha = \beta = \sqrt{\pi f \mu \sigma} = 20\pi \Rightarrow \gamma = 20\pi(1+j)$ //

b) $\Gamma = \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0}$, $\eta_0 = 120\pi\ \Omega$, $\eta_2 = \sqrt{\frac{\mu}{\varepsilon_c}} \simeq \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} = 2\pi(1+j)$

$\Gamma = \frac{2\pi + j2\pi - 120\pi}{2\pi + j2\pi + 120\pi} = \frac{-59+j}{59+j} = 0.967 e^{j\phi_r}$ // $\phi_r = 178° = 0.989\pi$ rad. //

$\tau = 1 + \Gamma = 0.031 + j0.034 = 0.46 e^{j\phi_\tau}$ $\phi_\tau = 47°.64 = 0.26\pi$ rad. //

$\bar{E}_r = \hat{a}_x \Gamma E_{i0} e^{jk_0 z} = \hat{a}_x 0.97 e^{j\frac{2\pi}{3}z} e^{j\pi} = \hat{a}_x 0.97 e^{j(\frac{2\pi z}{3} + 0.989\pi)}$ //

$E_t = \hat{a}_x \tau E_{i0} e^{-\alpha z} e^{-j\beta z} = \hat{a}_x 0.46 e^{-20\pi z} e^{-j(20\pi z - 0.26\pi)}$ //

c) $\bar{E}_t(z) = \hat{a}_x 0.46 e^{-20\pi z} \cos(20\pi z - 0.26\pi)$ // V/m

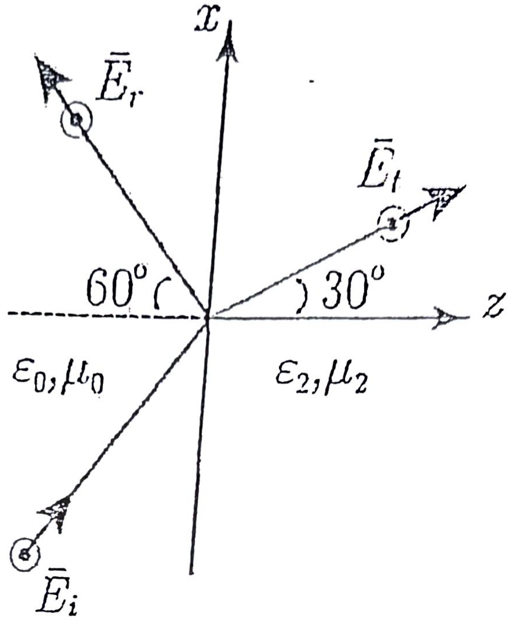
d) $z = 0.1$ mm $= 10^{-4}$ m $\Rightarrow 0.46 e^{-20\pi z} = 0.457$,

$z = 2$ cm $= 10^{-2}$ m $\Rightarrow 0.46 e^{-20\pi z} = 0.131$

skin depth $\delta = \frac{1}{\alpha} \cong 16$ mm.

20 mm is beyond δ (and close to δ). at $z = \delta$ field decays to $\frac{1}{e}$ of its maximum value $(\frac{1}{e} = 0.3679)$

Q.4 (20 pts)



An engineer performs a single experiment in order to find the electrical properties (ϵ_2 and μ_2) of a lossless material. As depicted in the figure, an oblique incidence of a perpendicularly polarized plane wave is considered, where the left half-space is vacuum, while the right half-space is filled with the material. When the angle of incidence is 60°, it is measured that the angle of refraction is 30°, while the amplitude of the electric field is reduced by 66.7%, i.e., $|\bar{E}_t| = |\bar{E}_i|/3$. Find the permittivity and permeability of the material in terms of ϵ_0 and μ_0.

$$k_1 \sin\theta_i = k_2 \sin\theta_t \rightarrow \cancel{\sqrt{\mu_0\epsilon_0}} \sin\theta_i = \cancel{\sqrt{\mu_0\epsilon_0}} \sqrt{\mu_r\epsilon_r} \sin\theta_t$$

$$\frac{\sqrt{3}}{2} = \sqrt{\mu_r\epsilon_r}\,\frac{1}{2} \rightarrow \boxed{\mu_r\epsilon_r = 3}$$

$$\tau_\perp = \frac{1}{3} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \rightarrow 5\eta_2\cos\theta_i = \eta_1\cos\theta_t$$

$$\rightarrow 5\cancel{\sqrt{\frac{\mu_0}{\epsilon_0}}}\sqrt{\frac{\mu_r}{\epsilon_r}}\,\frac{1}{2} = \cancel{\sqrt{\frac{\mu_0}{\epsilon_0}}}\,\frac{\sqrt{3}}{2}$$

$$\rightarrow 25\frac{\mu_r}{\epsilon_r} = 3 \rightarrow \boxed{25\mu_r = 3\epsilon_r}$$

Then, $\epsilon_r = \frac{25\mu_r}{3} \rightarrow \mu_r^2\frac{25}{3} = 3 \rightarrow \boxed{\mu_r = 3/5} \rightarrow \boxed{\mu_2 = \mu_0\frac{3}{5}}$

$$\epsilon_r = \frac{25}{3}\,\frac{3}{5} = \boxed{5} \rightarrow \boxed{\epsilon_2 = 5\epsilon_0}$$

Middle East Technical University
Department of Electrical & Electronics Engineering

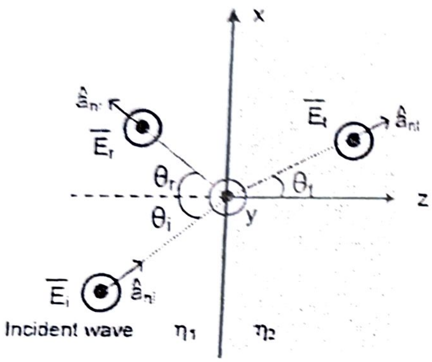
EE 303 Midterm Examination 2

December 22, 2016

Only simple calculators are allowed.
Mobile Phones must be turned off during the exam.
Give your answers neatly showing all necessary DETAILS.
Use proper VECTOR notations! Provide UNITs!

HINTS (You may or may not need)

Fresnel Formulas:

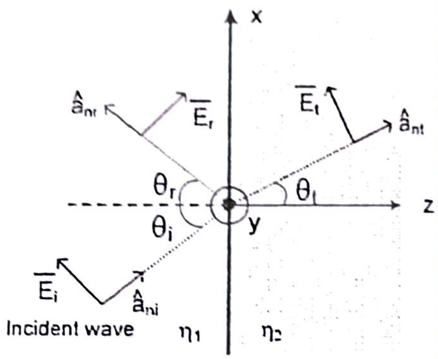


E ⊥ Plane of Incidence Case

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$T_\perp = 1 + \Gamma_\perp$$

$$\sin\theta_B^\perp = \sqrt{\frac{1-(\mu_1\varepsilon_2/\varepsilon_1\mu_2)}{1-(\mu_1/\mu_2)^2}}$$



E // Plane of Incidence Case

$$\Gamma_{//} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$T_{//} = \frac{\cos\theta_i}{\cos\theta_t}\left(1 + \Gamma_{//}\right)$$

$$\sin\theta_B^{//} = \sqrt{\frac{1-(\varepsilon_1\mu_2/\mu_1\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)^2}}$$

Propagation constant and intrinsic impedance of lossy medium

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}, \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

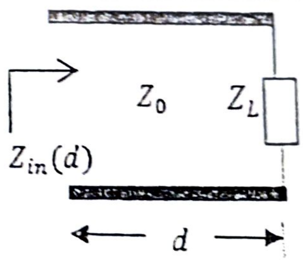
for $\frac{\sigma}{\omega\varepsilon} << 1$

$$\gamma \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}, \quad \eta \approx \sqrt{\frac{\mu}{\varepsilon}}$$

for $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma \cong \sqrt{\pi f \mu\sigma} + j\sqrt{\pi f \mu \sigma}, \quad \eta \approx (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

TL Input Impedance



$$Z_{in}(d) = Z_0 \frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta d}$$

For air and free space:

$$\mu_0 = 4\pi\, 10^{-7} \quad H/m, \qquad \varepsilon_0 = \frac{10^{-9}}{36\pi} \quad F/m$$

EE 303 Fall 2016-2017 Solutions to Midterm Examination 2

Question 1 (25 Points)

The electric field intensity in free space (ϵ_0, μ_0) is defined as

$$\bar{E} = \hat{a}_z E_0 \cos(by)\cos(\omega t),$$

where t represents the time, E_0 and b constants, and ω is the angular frequency. Note that the medium is source free.

(a) Find the magnetic field intensity $\bar{H}$ everywhere.

(b) Find b in terms of ω and medium parameters, and express both $\bar{E}$ and $\bar{H}$ in terms of medium parameters.

(c) Consider a unit cube defined as $x \in [0, 1]$, $y \in [0, 1]$, and $z \in [0, 1]$. Find the electric potential energy stored inside the cube with respect to time.

(d) Find the magnetic potential energy stored inside the same unit cube with respect to time.

(e) Find the Poynting vector everywhere.

(f) Integrate the Poynting vector on the surface of the unit cube and clearly show the conservation of energy.

(a) $\vec{\nabla}\times\vec{E} = \hat{a}_y \times \hat{a}_z (-b) \overset{E_0}{} \sin(by)\cos(\omega t) = -\mu \frac{\partial}{\partial t}\vec{H} \rightarrow \boxed{\vec{H} = \frac{bE_0}{\mu\omega}\hat{a}_x \sin(by)\sin(\omega t)}$

(b) $\vec{\nabla}\times\vec{H} = -\hat{a}_z \frac{b^2 E_0}{\mu\omega}\cos(by)\sin(\omega t) = \epsilon\frac{\partial \vec{E}}{\partial t} = -\hat{a}_z \epsilon\omega \hat{E}_0 \cos(by)\sin(\omega t) \rightarrow b^2 = \omega^2 \mu\epsilon$

$\boxed{b = \omega\sqrt{\mu\epsilon} = k}$

$\boxed{\vec{E} = \hat{a}_z E_0 \cos(ky)\cos(\omega t)}$ $\boxed{\vec{H} = \frac{kE_0}{\mu\omega}\hat{a}_x \sin(ky)\sin(\omega t)}$

(c) $w_E = \frac{\epsilon}{2}\int_V |E|^2 dv = \frac{\epsilon}{2}E_0^2\cos^2(\omega t)\int_0^1 \cos^2(ky)\,dy = E_0^2\frac{\epsilon}{2}\cos^2(\omega t)\int_0^1\left(\frac{1}{2}+\frac{\cos(2ky)}{2}\right)dy$

$= \boxed{\frac{\epsilon E_0^2}{2}\cos^2(\omega t)\left[\frac{1}{2}+\frac{\sin(2k)}{4k}\right]}$

(d) $w_H = \frac{\mu}{2}\int_V |H|^2 dv = \frac{\mu}{2}E_0^2\sin^2(\omega t)\frac{b^2}{\mu^2\omega^2}\int_0^1\left(\frac{1}{2}-\frac{\cos(2ky)}{2}\right)dy$ $(b=k)$

$= \boxed{\frac{\epsilon}{2}E_0^2\sin^2(\omega t)\left[\frac{1}{2}-\frac{\sin(2k)}{4k}\right]}$

(e) $\vec{S} = \vec{E}\times\vec{H} = \hat{a}_y \frac{E_0^2 b}{\mu\omega}\sin(\omega t)\cos(\omega t)\sin(ky)\cos(ky) = \hat{a}_y \frac{E_0^2 b}{4\mu\omega}\sin(2\omega t)\sin(2ky)$

$= \boxed{\hat{a}_y \frac{E_0^2}{4\eta}\sin(2\omega t)\sin(2ky)}$

... is $y=0$ surface $\boxed{\oint S \cdot d\vec{s} = \frac{E_0^2}{4\eta}\sin(2\omega t)\sin(2k)}$

Question 2 (25 points)

A uniform plane wave of phasor electric field intensity, $\vec{E} = \hat{a}_y E_o e^{-jk_o z}$ in air $(z < 0)$ is normally incident upon a planar boundary as shown in the figure.

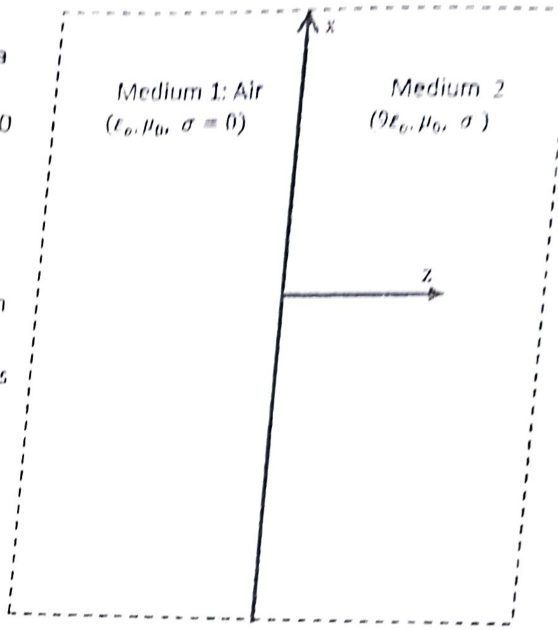
a) Assuming $\sigma = 10$ and frequency of the wave is 100 MHz, determine:

   i. Whether Medium 2 is a "good dielectric" or "good conductor" or "neither good conductor nor good dielectric". Explain.

   ii. Propagation constant and depth of penetration of medium 2.

   iii. Reflected and transmitted electric fields. Express your results in the time domain.

b) Assume σ is infinitely large ($\sigma \to \infty$).Find the surface current density at the interface.



a) $\sigma = 10 \quad f = 100 \times 10^6\,Hz = 10^8\,Hz$

i) $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{10}{2\pi \times 10^8 \times \frac{9}{36\pi} \times 10^{-9}} = 200 \gg 1$ Good Conductor

ii) $\gamma = \alpha + j\beta \qquad \alpha = \beta = \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 10^8 \times 4\pi \times 10^{-7} \times 10} = 20\pi$

$\delta = \dfrac{1}{\alpha} = \dfrac{1}{20\pi}$

iii) Since $\dfrac{\sigma}{\omega\varepsilon} \gg 1$, $\eta_2 \cong (1+j)\sqrt{\dfrac{\pi f \mu}{\sigma}} = (1+j)2\pi$

$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 0.97 \angle -1.94 \qquad \tau = \dfrac{2\eta_2}{\eta_2 + \eta_0} = 0.023 \angle 45^\circ$

$\bar{E}_r = \hat{a}_y E_o\, 0.97\, e^{+jk_o z} e^{-j1.94}$, $\quad \bar{E}_r(z,t) = \hat{a}_y\, 0.97\, E_o \cos(\omega t + k_o z - 1.94)$

$\bar{E}_t = \hat{a}_y E_o\, 0.023\, e^{j45} e^{-\gamma z}$ $\quad \bar{E}_t(z,t) = \hat{a}_y\, 0.023 e^{-20\pi z} E_o \cos(\omega t - 20\pi z + 45^\circ)$

b) $\bar{H}_i = \dfrac{1}{\eta_o} \hat{a}_z \times \bar{E}_i = -\hat{a}_x \dfrac{1}{\eta_o} E_o e^{-jk_o z}$

$\bar{H}_r = -\hat{a}_y \dfrac{E_o}{\eta_o} e^{jk_o z}$

$\bar{H}_1 = \bar{H}_i + \bar{H}_r$

$\bar{J}_s = \hat{a}_n \times \bar{H}_1 \Big| = \hat{a}_y \dfrac{2}{\eta_o} E_o$

Question 3 (25 points)

In medium 1 ($z < 0$) a uniform plane wave with the following phasor electric field intensity vector is incident upon a planar boundary with incidence angle θ_i as shown in the figure.

$\vec{E}_i = (A\hat{a}_x + 7\hat{a}_y - 8\hat{a}_z)e^{-jk_1\hat{a}_{ni}\cdot\vec{R}}$

The reflected phasor electric field intensity in medium 1 is:

$$\vec{E}_r = 2\hat{a}_y e^{-jk_1\hat{a}_{nr}\cdot\vec{R}}$$

a) What are the values of reflection coefficients?
b) Find the ratio of wave impedances (η_2/η_1).
c) Find the relative permeability of medium 1 (μ_{r1}).
d) Find the value of the constant A. (Hint: First find the angle of incidence, θ_i)
e) Find the transmitted phasor electric field intensity vector in medium 2. Write the exponent as a known function of x, y, z and free space wave number, k_0.

a) $\Gamma_\perp = \dfrac{2}{7}$ $\qquad \Gamma_{//} = 0$

b) $\Gamma_{//} = 0 \Rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_i$

$$\Gamma_\perp = \frac{2}{7} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} =$$

$$2\eta_2^2 + 2\eta_1^2 = 7\eta_2^2 - 7\eta_1^2 \Rightarrow 9\eta_1^2$$

c) $$\sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{\frac{\epsilon_2}{\mu_2}} = \sqrt{\frac{\mu_{r1}}{3}\frac{1}{1.5}} = \frac{\sqrt{5}}{3}$$

$$\mu_{r1} = \frac{5}{9} \times 4.5 = 2.5$$

$$\bar{E}_i^{\parallel} = (A\,\hat{a}_x - 8\,\hat{a}_z)\, e^{-jk_1 \hat{a}_{ni}\cdot\bar{R}}$$

$$= E_{i0}^{\parallel}\,(\cos\theta_i\,\hat{a}_x - \sin\theta_i\,\hat{a}_z)\, e^{-jk\,\hat{a}_{ni}\cdot\bar{R}}$$

$$\sin\theta_i = 0.8 \Rightarrow E_{i0}^{\parallel} = 10 \Rightarrow A = 10\cos\theta_i = 6$$

e) $\dfrac{\cos\theta_t}{\cos\theta_i} = \dfrac{\sqrt{5}}{3}$ from part a) $\cos\theta_t = 0.6 \times \dfrac{\sqrt{5}}{3} = \sqrt{0.2} = 0.44$

$\sin\theta_t = \sqrt{0.8} = 0.89$

$$T_{\perp} = 1 + \Gamma_{\perp} = 1 + \frac{2}{7} = \frac{9}{7}$$

$$T_{\parallel} = \frac{3}{\sqrt{5}}(1+0) = \frac{3}{\sqrt{5}} \qquad E_{t0}^{\parallel} = E_{i0}^{\parallel} \times \frac{3}{\sqrt{5}} = \frac{30}{\sqrt{5}}$$

$$k_2 = \omega\sqrt{\varepsilon_0 3\mu_0} = \sqrt{3}\,k_0 \qquad \hat{a}_{nt} = \sqrt{0.8}\,\hat{a}_x + \sqrt{0.2}\,\hat{a}_z$$

$$\bar{E}_t = \left(\frac{30}{\sqrt{5}} \times \sqrt{0.2}\,\hat{a}_x + 7 \times \frac{9}{7}\,\hat{a}_y - \frac{30}{\sqrt{5}} \times \sqrt{0.8}\,\hat{a}_z\right) e^{-jk_2 \hat{a}_{nt}\cdot\bar{R}}$$

$$= (6\,\hat{a}_x + 9\,\hat{a}_y - 12\,\hat{a}_z)\, e^{-jk_0(1.55x + 0.77z)}$$

Note that $\theta_B + \theta_t = 90^\circ$ only when $\mu_1 = \mu_2$

But $\mu_1 \neq \mu_2$ in this problem so

$\cos\theta_t \neq \sin\theta_B$

Question 4 (25 points)

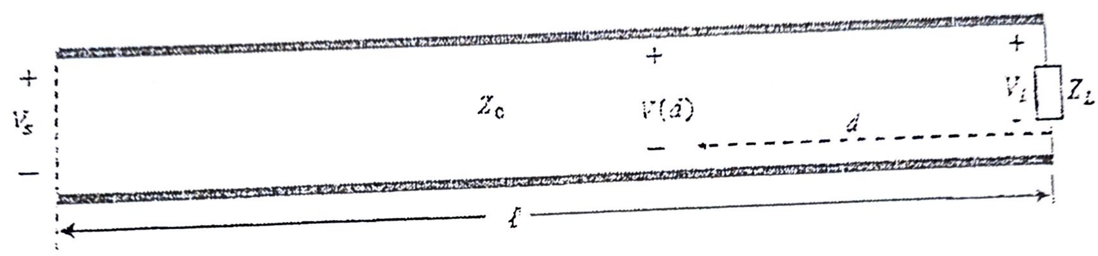
The figure below shows a lossless transmission line of characteristic impedance $Z_o = 50\ \Omega$, and length $l = 2.25\ m$. The transmission line is excited by a sinusoidal source of frequency $f = 50\ MHz$, and the load impedance is $Z_L = (40 - j30)\ \Omega$. The phasor voltage along the line is given as

$$V(d) = 30e^{j\pi d} + V_o e^{-j\pi d} \quad (Volts),$$

where d is the distance measured from the load end, and V_o is a complex constant to be found.

Determine:

a) The wavelength and the phase velocity.
b) The distributed parameters L and C of the transmission line.
c) The reflection coefficient at the load.
d) The phasor voltage at the load, V_L.
e) The phasor voltage at the input of the line, V_S.



a) (5) $\beta = \pi$ $\quad \lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{\pi} = \boxed{2\ m.}$

$v = \lambda f = 2 \times 50 \times 10^6 = \boxed{10^8\ m/sec}$

b) (5) $Z_o = \sqrt{\dfrac{L}{C}}$ $\quad v = \dfrac{1}{\sqrt{LC}}$ $\Rightarrow$ $Z_o v = \dfrac{1}{C}$, $\dfrac{Z_o}{v} = L$

$\therefore$ $C = \dfrac{1}{Z_o v} = \dfrac{1}{50 \times 10^8} = \boxed{2 \times 10^{-10}\ F/m}$

$L = \dfrac{Z_o}{v} = \dfrac{50}{10^8} = \boxed{5 \times 10^{-7}\ H/m}$

c) (5) $\Gamma_L = \dfrac{Z_L - Z_o}{Z_L + Z_o} = \dfrac{40 - j30 - 50}{40 - j30 + 50} = \dfrac{-1 - j3}{9 - j3} = -\dfrac{1}{3}\dfrac{1 + j3}{3 - j}$

$= -\dfrac{1}{3}\dfrac{(1 + j3)(3 + j)}{10} = -\dfrac{1}{3}\dfrac{j10}{10} = \boxed{-j\dfrac{1}{3}}$

Q.4 cont'd:

d) $V_L = V(d)\Big|_{d=0} = 30 + V_0$

5

$\frac{V_0}{30} = \Gamma_L \Rightarrow V_0 = 30\Gamma_L = -j10$ ($\Gamma_L = -j\frac{1}{3}$)

$\therefore \quad V_L = \boxed{30 - j10}$ (Volts)

e) $V_s = V(d)\Big|_{d=\ell} = 30\, e^{j\pi(2.25)} - j10\, e^{-j\pi(2.25)}$

5

$\left[2.25\pi \equiv \frac{\pi}{4}\right]$

$= 30\,\frac{1+j}{\sqrt{2}} - j10\,\frac{1-j}{\sqrt{2}} = \boxed{\sqrt{2}\,10\,(1+j) = 20 \angle 45^\circ}$

(Volts)

Question 1. (25 points)

A uniform plane wave, propagating in a simple, source-free and lossy medium is specified as

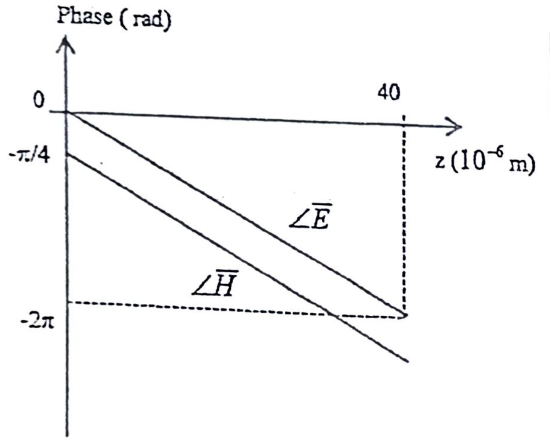
i. Having E field linearly polarized along the x-axis,

ii. propagating in $(+\hat{a}_z)$ direction,

iii. oscillating at the frequency, $f = 0.1\,GHz = 10^8\ Hz$,

iv. having $\overline{P}_{av}(z=0) = 4\ \hat{a}_z\ (Watt/m^2)$ at z=0 where $\overline{P}_{av}$ is the time-average Poynting's vector,

v. having the phase plots. ($\angle\overline{E}$) and ($\angle\overline{H}$) versus propagation distance (z), for the E-field and H-field phasors as shown in the figure below.



Answer the following questions :

a) Based on the provided information, write down the mathematical expressions for the E-phasor and H-phasor vectors in terms of α, β, η and E_o where E_o is a real positive constant being equal to $|\overline{E}(z)|$ at z=0.

b) Using the results of part (a), derive the expression for the time-average Poynting's vector, $\overline{P}_{av}(z)$.

c) What is the type of propagation medium? Why? Explain clearly.

d) Find the wavelength of propagation (λ).

e) Find the phase constant (β).

f) Find the attenuation constant (α).

g) Find the skin depth (δ).

h) Find the phase velocity (v_p).

i) Find the value of conductivity (σ) of the medium if it is non-magnetic (i.e. $\mu = \mu_o$).

j) Find the intrinsic impedance (η).

k) Find E_o.

Solutions:

(a) In phasor domain,

$$\bar{E}(z) = \hat{a}_x E_o e^{-\gamma z} = \hat{a}_x E_o e^{-\alpha z} e^{-j\beta z} \quad (V/m)$$

$$\bar{H}(z) = \frac{\hat{a}_z \times \bar{E}}{\eta} = \hat{a}_y \frac{E_o e^{-\alpha z} e^{-j\beta z}}{|\eta| e^{j\theta}} \quad (A/m)$$

(b) $\bar{P}_{av}(z) = \frac{1}{2} Re\{\bar{E}(z) \times \bar{H}^*(z)\}$

$$= \frac{1}{2} Re\left\{E_o e^{-\alpha z} \cancel{e^{-j\beta z}}\, \hat{a}_x \times \hat{a}_y \frac{E_o}{|\eta|} e^{-\alpha z} \cancel{e^{+j\beta z}} e^{j\theta}\right\}$$

$$= \frac{1}{2} E_o^2 \frac{1}{|\eta|} e^{-2\alpha z} \hat{a}_z \underbrace{Re\{e^{j\theta}\}}_{\cos\theta}$$

$$\boxed{\bar{P}_{av}(z) = \hat{a}_z \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos\theta \quad (W/m^2)}$$

(c) As $\theta = \frac{\pi}{4}$ rad. (from the given graph)

$$\eta = |\eta| e^{j\theta} = |\eta| e^{j\pi/4} = \frac{1+j}{\sqrt{2}} |\eta|$$

(having equal real and imaginary parts, which is possible only for good conductor medium.)

Or, you may use the following reasoning:

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \Rightarrow \underbrace{\angle\eta}_{=\theta=\pi/4} = \frac{1}{2}\left[\underbrace{\angle j\omega\mu}_{\pi/2} - \angle\sigma + j\omega\varepsilon\right]$$

$$\Rightarrow \angle\sigma + j\omega\varepsilon = 0 \Rightarrow \tan^{-1}\frac{\omega\varepsilon}{\sigma} \to 0 \text{ or } \frac{\omega\varepsilon}{\sigma} \to 0$$

This implies $\sigma \gg \omega\varepsilon$ as in the good conductors. (excluding the perfect conductor case)

Q.1 - Soln. - Continue...

(d) Wavelength is the minimum distance along the propagation direction taken for a phase change of 2π. Therefore, it is seen from the given graph that $\boxed{\lambda = 40\times10^{-6}\ m = 40\ \mu m.}$

(e) $\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{40\times10^{-6}} = 5\pi\times10^{4}\ rad/m \cong \boxed{1.57\times10^{5}\ rad/m. = \beta}$

(or it can be found from the slope of phase lines)

(f) For a good conductor ($\frac{\sigma}{\omega\epsilon} >> 1$), $\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

can be simplified to $\gamma \cong (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = (1+j)\sqrt{\pi f\mu\sigma}$

Namely, $\alpha \approx \beta \Rightarrow \alpha \cong 1.57\times10^{5}\ nep/m. = \boxed{5\pi\times10^{4}\ nep/m \cong \alpha}$

(g) $\delta = \frac{1}{\alpha} = \frac{1}{5\pi\times10^{4}} = \frac{1}{1.57\times10^{5}} \cong 6.366\times10^{-6}\ m. = \boxed{6.366\ \mu m = \delta}$

(or $\delta = \frac{1}{\alpha} \cong \frac{1}{\beta} = \frac{\lambda}{2\pi}$ can be used)

(h) $v_p = \frac{\omega}{\beta} = \frac{2\pi f}{5\pi\times10^{4}} = \frac{2\cancel{\pi}\times10^{8}}{5\cancel{\pi}\times10^{4}} = 0.4\times10^{4} = \boxed{4000\ m/s = v_p}$

(i) $\alpha \approx \beta \approx \sqrt{\pi f\mu\sigma} \Rightarrow \alpha^2 = \pi f\mu_0\sigma \Rightarrow \sigma = \frac{(5\cancel{\pi}\times10^{4})^2}{\cancel{\pi}\times10^{8}\times4\cancel{\pi}\times10^{-7}}$

$\boxed{\sigma = 6.25\times10^{7}\ S/m\ (\text{or } \mho/m)}$

(j) $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = e^{j\pi/2}\sqrt{\frac{\omega\mu}{\sigma}}$ (for good conductor)

$\eta = e^{j\pi/2}\sqrt{\frac{2\pi\times10^{8}\times4\pi\times10^{-7}}{6.25\times10^{7}}} \cong 1.13\pi\times10^{-3}\ e^{-j\pi/2} \cong 0.00355\ e^{-j\pi/2}\ (\Omega)$

$\boxed{\eta \cong 0.0025\,(1+j)\ (\Omega)}$

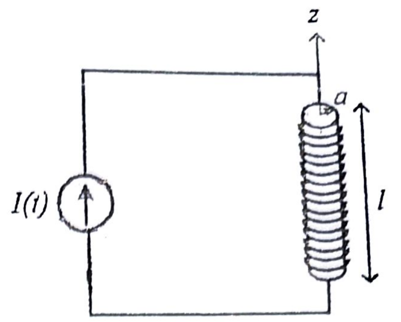
(or you may use $\eta = (1+j)\frac{\alpha}{\sigma} = (1+j)\frac{1}{\delta\sigma}$ for good conductor)

Question 2 (25 points)

A long and thin solenoid with radius a and length l ($a<<l$) is wound around a magnetic material of permeability μ and permittivity ε. A generator with a current which increases linearly with time is connected to the solenoid as shown in the figure, creating a magnetic flux density:

$\overline{B} = \hat{a}_z Kt$ $\quad$ K is a constant

inside the solenoid.



a) By using integral form of Faraday's law, show that the electric field intensity inside the solenoid is:

$$\overline{E} = -\hat{a}_\phi \frac{Kr}{2} \quad \text{for } r \leq a$$

where r is the distance measured from the z-axis.

b) By assuming that the fields outside the solenoid ($r > a$) are zero, find the stored electric energy and stored magnetic energy in the solenoid. Find the time rate of change of the stored energies.

c) Use Poynting's vector to find the power entering the structure through the side surface ($r=a$).

d) Comment on the results of (b) and (c) considering Poynting's Theorem.

a) $\overline{\nabla} \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$ $\qquad$ $\displaystyle\int_S \overline{\nabla} \times \overline{E} \cdot \overline{ds} = -\int \frac{\partial \overline{B}}{\partial t} \cdot \overline{ds}$

Stoke's Theorem

$$\oint \overline{E} \cdot \overline{dl} = -\int \frac{\partial \overline{B}}{\partial t} \cdot \overline{ds}$$

$$\int_0^{2\pi} \overline{E} \cdot r' d\phi \, \hat{a}_\phi = -\int_0^{2\pi}\int_0^{r'} K \hat{a}_z \cdot r dr d\phi \, \hat{a}_z$$

$$E_\phi r' 2\pi = -K \pi r'^2 \Rightarrow E_\phi = -\frac{Kr'}{2}$$

b) $$W_e = \frac{\varepsilon}{2}\int_V |\overline{E}|^2 dv = \frac{\varepsilon}{2} \int_0^{l}\int_0^{2\pi}\int_0^{a} \frac{K^2 r^2}{4} r \, dr \, d\phi \, dz$$

$$= \frac{\varepsilon K^2 a^4 \pi l}{16} \qquad \frac{dW_e}{dt} = 0$$

$$W_m = \frac{\mu}{2}\int |\bar{H}|^2 \, dv = \frac{\mu}{2}\int_0^{\ell}\int_0^{2\pi}\int_0^{a} \frac{K^2 t^2}{\mu^2} \, r \, dr \, d\phi \, dz$$

$$= \frac{K^2 t^2 a^2 \pi \ell}{2\mu}$$

$$\frac{dW_m}{dt} = \frac{K^2 a^2 \pi \ell t}{\mu}$$

c) Power entering $-\int_S \bar{P} \cdot \bar{ds}$

$$\bar{P} = \bar{E} \times \bar{H} = -\frac{Kr}{2}\hat{a}_\phi \times \frac{Kt}{\mu}\hat{a}_z = -\frac{K^2 r t}{2\mu}\hat{a}_r$$

$$-\iint -\frac{K^2 a t}{2\mu}\hat{a}_r \cdot a \, d\phi \, dz \, (\hat{a}_r) = \frac{K^2 a^2 t \pi \ell}{\mu}$$

d) Poynting's Theorem

$$\underbrace{-\oint_S \bar{P} \cdot \bar{ds}}_{(c)} = \underbrace{\frac{dW_e}{dt}}_{=0} + \underbrace{\frac{dW_m}{dt}}_{(b)} + \int_v \sigma |\bar{E}|^2 \, dv$$

Since the power entering to the solenoid is equal to the time rate of increase of the energies, it can be concluded that the ohmic losses are zero and all the power is stored as magnetic energy.

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Question 3. (25 points)

A uniform plane wave propagating in air (medium 1: $z \leq 0$, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$) is normally incident upon a dielectric half space, (medium 2: $z > 0$, $\mu = \mu_0$, $\varepsilon = 9\varepsilon_0$)

Phasor electric field of the incident wave is given by,

$$\vec{E}_i = \hat{a}_x 6 e^{-jk_0 z} \quad \text{V/m}$$

At the interface between two media (at z=0 plane) there is an externally placed (impressed) infinite current sheet with a uniform current density phasor

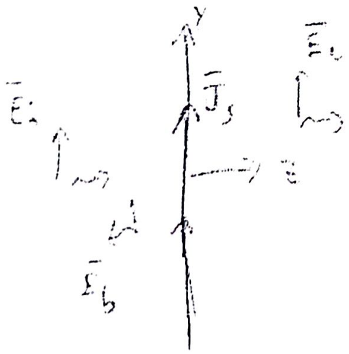
$$\vec{J}_s = \hat{a}_x \frac{4}{\eta_0} \quad \text{A/m.}$$

where η_0 is the intrinsic impedance of air.

An infinite current sheet at z=0 plane generates a uniform plane wave propagating in +z-direction for z>0 and a uniform plane wave propagating in −z direction for z<0 with an electric field vector parallel to the surface current density.

a) Determine the total electric field in medium 1.
b) Determine the total electric field in medium 2.

Hint: You may find it useful to express the field in medium 1 as a sum of two uniform planes, one is propagating in +z-direction, the other one is propagating in −z direction and the field in medium 2 as a uniform plane wave propagating in +z-direction.



$\bar E_1 = \bar E_i + \bar E_b \qquad z<0$

$\bar E_b = \hat a_x E_{0b}\, e^{jk_0 z}$

$\bar E_2 = \hat a_x E_{20}\, e^{-jk_2 z} \qquad z>0 \qquad k_2 = \omega\sqrt{\mu_0 9\varepsilon_0} = 3k_0$

$\bar H_i = \frac{1}{\eta_0}\hat a_z \times \hat a_x 6 e^{-jk_0 z} = \hat a_y \frac{6}{\eta_0} e^{-jk_0 z}$

$\bar H_b = \frac{1}{\eta_0}(-\hat a_z)\times \hat a_x E_{0b} e^{jk_0 z} = -\hat a_y \frac{E_{0b}}{\eta_0} e^{jk_0 z}$

$\bar H_2 = \frac{1}{\eta_2}\hat a_z \times \hat a_x E_{20} e^{-j3k_0 z} = \hat a_y \frac{E_{20}}{\eta_2} e^{-j3k_0 z}, \quad \eta_2 = \sqrt{\frac{\mu_0}{9\varepsilon_0}} = \frac{\eta_0}{3}$

B.C. at $z=0$: $\bar E_i + \bar E_b = \bar E_2 \Rightarrow 6 + E_{0b} = E_{20}$ (1)

$-\hat a_z \times (\bar H_1 - \bar H_2)|_{z=0} = -\hat a_z \times \hat a_y \left(\frac{6}{\eta_0} - \frac{E_{0b}}{\eta_0} - \frac{E_{20}}{\eta_2} \right) = \hat a_x \frac{4}{\eta_0}$

$$6 - E_{0b} - 3E_{20} = 4 \qquad (2)$$

$$6 + E_{0b} - E_{20} = 0 \qquad (1)$$

$$12 - 4 = 4E_{20} \Rightarrow E_{20} = 2$$

$$E_{0b} = E_{20} - 6 = -4$$

$$\bar{E}_1 = \hat{a}_x \left(6 e^{-jk_0 z} - 4 e^{jk_0 z} \right)$$

$$\bar{E}_2 = \hat{a}_x \, 2 e^{-j3k_0 z}$$

EE 303 Midterm Examination #2

Solutions

Q 1 (20 Points)

N. Günalp

Consider the propagation of uniform plane waves in wet soil, which is a non-magnetic material (i.e $\mu = \mu_o$). Wet soil behaves as a *good conductor* at $f = 100\ kHz$, and the depth of penetration (skin depth) is measured as $\delta = 16\ m$ at this frequency. The same wet soil behaves as a *good dielectric* at $f = 1\ GHz$, where the depth of penetration is $\delta = 2\ m$.

a) (10pts) Determine the conductivity (σ), and the relative permittivity (ϵ_r) of wet soil.

b) (4pts) Verify that wet soil behaves as a *good dielectric* at $f = 1\ GHz$.

c) (6pts) Determine the phase velocity of the uniform plane wave at $100\ kHz$ and $1\ GHz$.

Make all reasonable approximations.

a) $f = 100\ kHz$, $\delta = 16\ m.$, good conductor:

$$\alpha \approx \beta \approx \sqrt{\omega \mu_0 \sigma / 2} = \sqrt{\pi f \mu_0 \sigma}, \quad \alpha = \frac{1}{\delta}$$

$$\alpha = \sqrt{\pi \times 10^5 \times 4\pi \times 10^{-7}\,\sigma} = \frac{1}{16} \Rightarrow \sqrt{\sigma} = \frac{10}{32\pi} = 0.0995$$

$$\boxed{\sigma = 0.989 \times 10^{-2}\ S/m}$$

$f = 1\ GHz$, $\delta = 2\ m.$, good dielectric:

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\epsilon}} = \frac{\sigma}{2}\frac{120\pi}{\sqrt{\epsilon_r}} = \frac{1}{\delta} = \frac{1}{2}$$

$$\sqrt{\epsilon_r} = 120\pi\sigma \cong 3.73 \Rightarrow \boxed{\epsilon_r \approx 13.9}$$

b) $$\frac{\sigma}{\omega\epsilon} = \frac{0.989 \times 10^{-2}}{2\pi \times 10^9 \times 13.9 \times \frac{10^{-9}}{36\pi}} = \frac{0.989 \times 10^{-2}}{0.772} = \boxed{1.29 \times 10^{-2}}$$

$\frac{\sigma}{\omega\epsilon} \ll 1$, i.e $\sigma \ll \omega\epsilon$ (displacement current dominates) $\Rightarrow$ good dielectric ✓

c) $100\ kHz$: $\beta \approx \alpha = \frac{1}{\delta}$

$$v = \frac{\omega}{\beta} = \omega\delta = 2\pi \times 10^5 \times 16 \approx \boxed{10^7\ m/sec}$$

$1\ GHz$: $\beta \approx k = \omega\sqrt{\mu_0 \epsilon}$

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0 \epsilon}} = \frac{3 \times 10^8}{\sqrt{\epsilon_r}} \approx \boxed{8 \times 10^7\ m/sec}$$

Q.2 (15 points)

Electric field intensity and magnetic field intensity vectors of a uniform plane wave propagating in a lossy medium ($\sigma = 2\ S/m$) are given as

$$\vec{E}(z,t) = \hat{a}_x E_0 e^{-4z}\cos(\omega t - \beta z), \qquad V/m$$

$$\vec{H}(z,t) = \hat{a}_y \frac{E_0}{2} e^{-4z}\cos\left(\omega t - \beta z + \frac{\pi}{3}\right), \qquad A/m$$

4 a) Find time average Poynting's vector.

Consider the following hypotetical cylindrical volume

$$V: \{0 \le r \le a, 0 \le \emptyset \le 2\pi, 0 \le z \le d\}$$

b) Calculate the time average power entering this volume.

c) Calculate the time average power leaving this volume.

2 d) Calculate the time average dissipated power in this volume using the formula given below

$$P_d = \frac{1}{2}\iiint_V \sigma|\vec{E}|^2 dv, \text{ where } \vec{E} \text{ is phasor electric field intensity vector}$$

3 e) Compare your answer in part d with the ones in part b and c and comment.

Note that all answers will be in terms of $E_0, \beta, a,$ and d

a) $\bar{P}_{av} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\}$

$\bar{E} = \hat{a}_x E_0 e^{-4z} e^{-j\beta z}$ $\qquad$ $\bar{H} = \hat{a}_y \frac{E_0}{2} e^{-4z} e^{-j\beta z} e^{j\frac{\pi}{3}}$

$\bar{P}_{av} = \hat{a}_z \frac{1}{2}\mathrm{Re}\left\{\frac{E_0^2}{2} e^{-8z} e^{-j\frac{\pi}{3}}\right\} = \hat{a}_z \frac{E_0^2}{4} e^{-8z}\cos\frac{\pi}{3}$

b) $P_e = \iint \frac{E_0^2}{8} e^{-8z}\Big|_{z=0} r\,dr\,d\phi = \frac{E_0^2}{8}\pi a^2$

c) $P_{l.} = \iint \frac{E_0^2}{8} e^{-8d} r\,dr\,d\phi = \frac{E_0^2}{8} e^{-8d}\pi a^2$

d) $P_d = \frac{1}{2}2\iiint E_0^2 e^{-8z} r\,dr\,d\phi\,dz = E_0^2 \frac{1}{-8} e^{-8z}\Big|_0^d \pi a^2$

$= \frac{E_0^2}{8}\left(1 - e^{-8d}\right)\pi a^2$

e) $P_e - P_l = P_d$

Question 4. (25 points)

A uniform plane wave of time-domain magnetic field intensity

$$\overline{H}^i(x,y,z,t) = \frac{1}{40\pi}\hat{a}_y \cos\left[\omega t - 10(\sqrt{5}x + 2z)\right] \qquad (A/m)$$

is incident from air onto a lossless dielectric half-space of relative permittivity $\varepsilon_r = 4$, as shown in the figure.

a) Find the phasor expressions for the magnetic field intensity and electric field intensity of the incident wave.

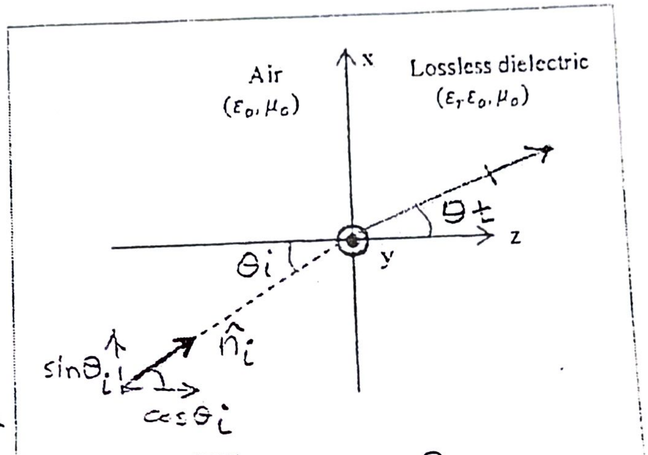
b) Find the phasor electric field intensity of the reflected wave.

c) Find the phasor electric field intensity of the transmitted wave.

a) $\boxed{\overline{H}^i = \frac{1}{40\pi}\hat{a}_y e^{-j10(\sqrt{5}x+2z)}}$

$$\hat{n}_i = \frac{\sqrt{5}\hat{a}_x + 2\hat{a}_z}{\sqrt{5+4}}$$

$$\hat{n}_i = \frac{1}{3}(\sqrt{5}\hat{a}_x + 2\hat{a}_z)$$



$$\sin\theta_i = \frac{\sqrt{5}}{3}, \quad \cos\theta_i = \frac{2}{3}$$

$$\overline{E}^i = \eta\,\overline{H}^i \times \hat{n}_i$$

$$= 120\pi\frac{1}{40\pi}\frac{1}{3}(\sqrt{5}\underbrace{\hat{a}_y\times\hat{a}_x}_{-\hat{a}_z} + 2\underbrace{\hat{a}_y\times\hat{a}_z}_{\hat{a}_x})\,e^{-j\cdots}$$

$$\boxed{\overline{E}^i = (2\hat{a}_x - \sqrt{5}\hat{a}_z)\,e^{-j10(\sqrt{5}x+2z)}}$$

b) Snell's Law: $k_1\sin\theta_i = k_2\sin\theta_t$

$$\sin\theta_i = \sqrt{\varepsilon_r}\sin\theta_t \Rightarrow \sin\theta_t = \frac{\sin\theta_i}{\sqrt{\varepsilon_r}}$$

$$\sin\theta_t = \frac{\sqrt{5}/3}{2} = \boxed{\frac{\sqrt{5}}{6}}$$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{5}{36}} = \boxed{\frac{\sqrt{31}}{6}}$$

Q.4 cont'd:

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{+} = \frac{\cos\theta_t - \sqrt{\epsilon_r}\cos\theta_i}{+}$$

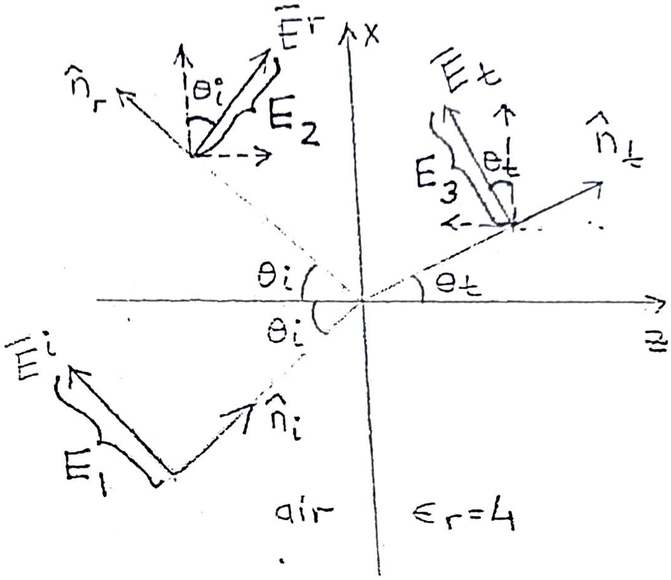
since $\eta_1 = \sqrt{\frac{\mu_o}{\epsilon_o}}$

$\eta_2 = \sqrt{\frac{\mu_o}{\epsilon_r \epsilon_o}}$

$$= \frac{\frac{\sqrt{31}}{6} - 2\,\frac{2}{3}}{+} = \frac{\sqrt{31} - 8}{\sqrt{31} + 8} = \boxed{-0.179}$$

$E_1 = \left| 2\hat{a}_x - \sqrt{5}\,\hat{a}_z \right| = 3$

$E_2 = \Gamma_{\parallel} E_1$



$\bar{E}r = E_2(\cos\theta_i\,\hat{a}_x + \sin\theta_i\,\hat{a}_z)$

$e^{-j\bar{k}_r \cdot \bar{r}}$

$= \Gamma_{\parallel}(3)\left(\frac{2}{3}\hat{a}_x + \frac{\sqrt{5}}{3}\hat{a}_z\right)$

$$\boxed{\bar{E}r = \underbrace{\Gamma_{\parallel}}_{-0.179}\left(2\hat{a}_x + \sqrt{5}\,\hat{a}_z\right) e^{-j10(\sqrt{5}x - 2z)}}$$

c) $T_{\parallel} = \frac{\cos\theta i}{\cos\theta t}(1 + \Gamma_{\parallel}) = \frac{2/3}{\sqrt{31}/6}(1 - 0.179) = \frac{4(0.821)}{\sqrt{31}} \approx \boxed{0.59}$

$E_3 = T_{\parallel} E_1$, $\hat{n}_t = \sin\theta_t\,\hat{a}_x + \cos\theta_t\,\hat{a}_z$

$k_1 = |\bar{k}_1| = 10\sqrt{5+4} = \boxed{30}$, $k_2 = \sqrt{\epsilon_r}\,k_1 = 2k_1 = \boxed{60}$

$\bar{E}t = E_3(\cos\theta_t\,\hat{a}_x - \sin\theta_t\,\hat{a}_z)\; e^{-jk_2 \hat{n}_t \cdot \bar{r}}$

$= T_{\parallel}(3)\left(\frac{\sqrt{31}}{6}\hat{a}_x - \frac{\sqrt{5}}{6}\hat{a}_z\right) e^{-j60\left(\frac{\sqrt{5}}{6}x + \frac{\sqrt{31}}{6}z\right)}$

$$\boxed{\bar{E}t = \underbrace{T_{\parallel}}_{0.59}\left(\frac{1}{2}\right)\left(\sqrt{31}\,\hat{a}_x - \sqrt{5}\,\hat{a}_z\right) e^{-j10(\sqrt{5}x + \sqrt{31}z)}}$$

Q.3 (20 pts)

A lossless dielectric material (Medium #1) with parameters $\mu = \mu_o$ and $\varepsilon = \varepsilon_o \varepsilon_r$ is separated from a perfect conductor (Medium #2) by an infinitely large planar boundary at $z = 0$ as shown in the figure. A uniform plane wave (u.p.w.) propagating in the first medium with the electric field intensity phasor vector $\bar{E}^i(z) = E_o e^{-jk_1 z}\hat{a}_x$ is normally incident upon this boundary. Assume that E_o is a real [redacted] constant.

a) **(8 points)** Derive the expressions for the total time domain fields $\bar{E}_1(z,t)$ and $\bar{H}_1(z,t)$ in the lossless dielectric medium in terms of E_o, k_1, η_1 and ω at a given frequency f.

b) **(12 points)** Now, assume that the u.p.w. oscillates at a frequency $f = 300\ MHz$. Using the standing wave plots drawn in Medium #1 at a fixed time $t_0 = T/6$ (remember $T = 1/f$ is the period of the u.p.w. in time) determine

i. λ_1, the wavelength in Medium 1,

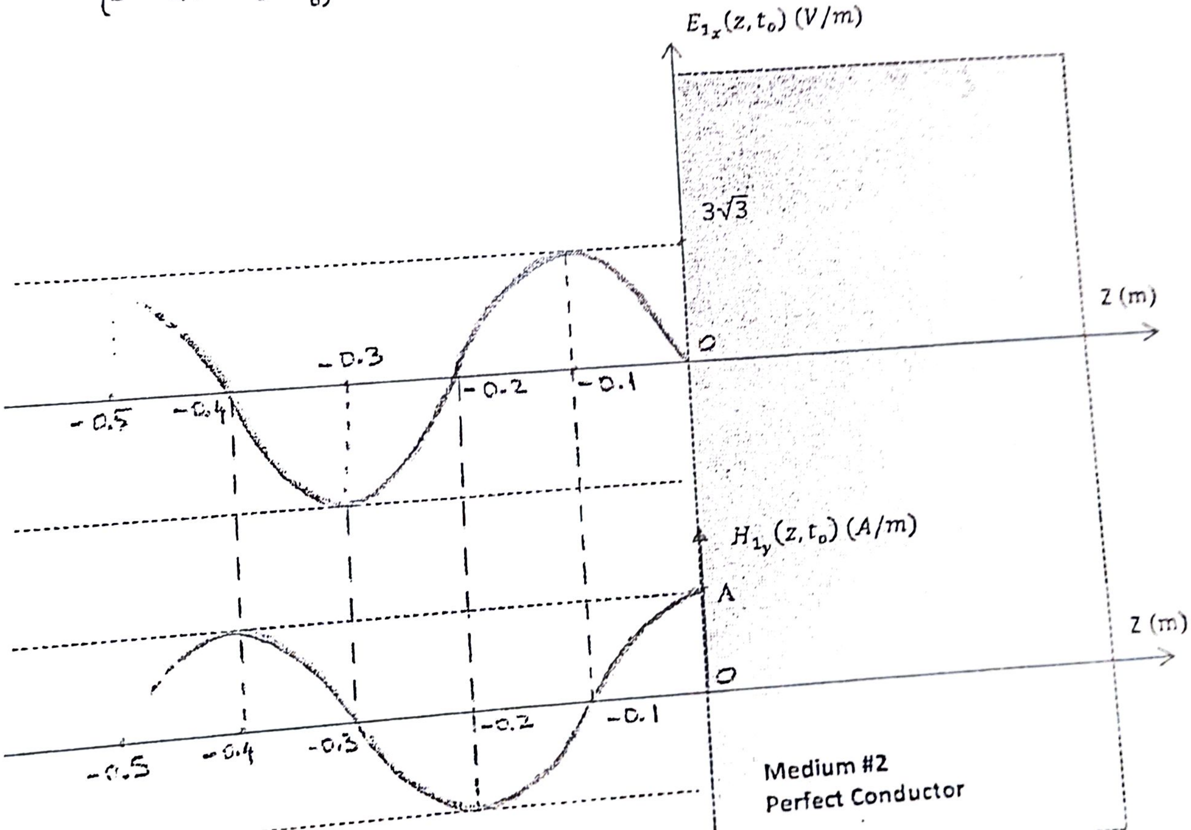
ii. k_1, the wavenumber in Medium 1,

iii. ε_r, the relative permittivity in Medium 1,

iv. η_1, the intrinsic impedance in Medium 1,

v. E_o, the magnitude of the incident electric field phasor,

vi. The value of the total time domain magnetic field intensity $A = H_{1_y}(0, \frac{T}{6})$ at $\{z = 0, t = t_o = \frac{T}{6}\}$.



Q.3 Solution

(a) $\bar{E}_1(z) = \bar{E}^i(z) + \bar{E}^r(z) = E_o e^{-jk_1 z}\hat{a}_x + (\Gamma E_o)e^{+jk_1 z}\hat{a}_x$

where $\Gamma = \left.\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right|_{\eta_2 = 0} = -1$

$\bar{E}_1(z) = E_o\left(e^{-jk_1 z} - e^{+jk_1 z}\right)\hat{a}_x = -2jE_o \sin(k_1 z)\hat{a}_x$

$\bar{\mathcal{E}}_1(z,t) = Re\left\{\bar{E}_1(z)e^{j\omega t}\right\} = Re\left\{-2jE_o\sin(k_1 z)(\cos\omega t + j\sin\omega t)\hat{a}_x\right\}$

$$\boxed{\bar{\mathcal{E}}_1(z,t) = \hat{a}_x\, 2E_o \sin(k_1 z)\sin(\omega t)}$$

Using $\bar{H}^i(z) = \frac{1}{\eta_1}\hat{a}_z \times \bar{E}^i = \frac{E_o}{\eta_1}e^{-jk_1 z}\hat{a}_y$

and $\bar{H}^r(z) = \frac{1}{\eta_1}(-\hat{a}_z)\times\bar{E}^r = \frac{E_o}{\eta_1}e^{+jk_1 z}\hat{a}_y$

$\bar{H}_1(z) = \bar{H}^i(z) + \bar{H}^r(z)$

$\bar{H}_1(z) = \frac{E_o}{\eta_1}2\cos(k_1 z)\hat{a}_y \Rightarrow \bar{\mathcal{H}}_1(z,t) = Re\left\{\bar{H}_1(z)e^{j\omega t}\right\}$

$$\boxed{\bar{\mathcal{H}}_1(z,t) = \hat{a}_y \frac{2E_o}{\eta_1}\cos(k_1 z)\cos(\omega t)}$$

(Result is a standing wave)

(b) (i) $\boxed{\lambda_1 = 0.4\ m}$ (From the figure, $\frac{\lambda}{4} = 0.1\ m$)

(ii) $k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{0.4} = 5\pi \Rightarrow \boxed{k_1 = 5\pi\ rad/m}$

(iii) $k_1 = 2\pi f\sqrt{\mu_o\epsilon_o\epsilon_r} = \frac{2\pi f}{c}\sqrt{\epsilon_r} = \frac{2\pi \times 3\times 10^8}{3\times 10^8}\sqrt{\epsilon_r} = 5\pi \Rightarrow \sqrt{\epsilon_r} = 2.5$

$\Rightarrow \boxed{\epsilon_r = 6.25}$

(iv) $\eta_1 = \sqrt{\frac{\mu_o}{\epsilon_o\epsilon_r}} = \frac{\eta_o}{\sqrt{\epsilon_r}} = \frac{120\pi}{2.5} \Rightarrow \boxed{\eta_1 = 48\pi\ \Omega \cong 151\ \Omega}$

(v) $E_{1x}(z = -0.1, t = \frac{T}{\ }) = 2E_o \sin(\frac{\pi}{\ })\sin(\frac{\pi}{\ }) \ldots$

Q.4 (25 pts)

A uniform plane wave is incident from a dielectric medium with ($\varepsilon_1 = 4\varepsilon_0, \mu_1 = \mu_0$) onto another dielectric region with ($\varepsilon_2 = 9\varepsilon_0, \mu_2 = \mu_0$) as shown in the figure where the planar boundary is placed at $z=0$. The phasor electric field intensity vector of the incident wave is given by

$$\bar{E}_i(\bar{r},t) = (3\hat{a}_x - 6\hat{a}_z)e^{-j20\pi(4x+Az)} \quad V/m.$$

Find,

3 a) The value of unknown constant A,

2 b) The unit vector, $\hat{a}_{n_i}$, in the propagation direction of the incident electric field.

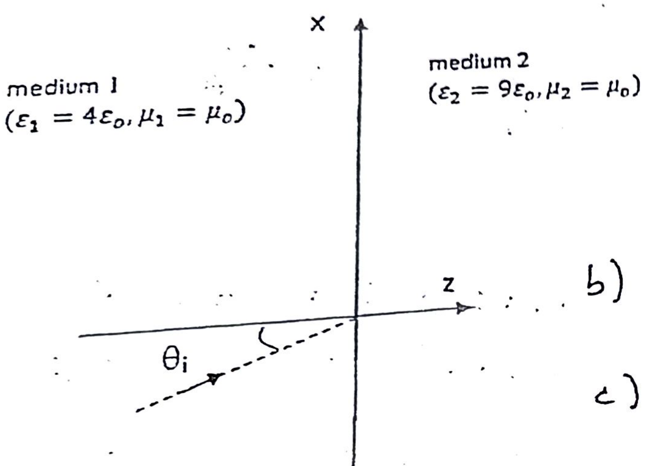
2 c) Find $\sin\theta_i$ and $\cos\theta_i$ of the angle of incidence, θ_i.

2 d) $\sin\theta_t$ and $\cos\theta_t$ of the angle of transmission, θ_t

8 e) Phasor electric field intensity vector of the reflected wave.

8 f) Phasor electric field intensity vector of the transmitted wave.

You may leave your answers as rational numbers ($\frac{a}{b}, \frac{\sqrt{a}}{b}$, e.t.c).



a) $\bar{E}_0 \perp k\hat{a}_{n_i}$

$(3\hat{a}_x - 6\hat{a}_z) \perp (4\hat{a}_x + 6\hat{a}_z)$

$(3\hat{a}_x - 6\hat{a}_z)\cdot(4\hat{a}_x + 6\hat{a}_z) = 0$

$\Rightarrow (3\hat{a}_x - 6\hat{a}_z$

$\boxed{A = 2}$

b) $\hat{a}_{n_i} = \dfrac{2\hat{a}_x + \hat{a}_z}{\sqrt{5}}$ //

c) $\sin\theta_i = \dfrac{2}{\sqrt{5}}$ 0.894 // $\quad \cos\theta_i = \dfrac{1}{\sqrt{5}}$ 0.447

d) $k_1 \sin\theta_i = k_2 \sin\theta_t$

$\Rightarrow \cancel{\omega}\sqrt{4}\,\cancel{\sqrt{\varepsilon_0\mu_0}}\sin\theta_i = \cancel{\omega}\sqrt{9}\,\cancel{\sqrt{\varepsilon_0\mu_0}}\sin\theta_t$

$\Rightarrow \sin\theta_t = \dfrac{4}{\sqrt{45}}$ 0.596 // $\quad \cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{\dfrac{29}{45}}$ 0.80 //

e) $\hat{a}_{n_r} = \dfrac{2 - \hat{a}_z}{\sqrt{5}}$ $\quad \Gamma_{\parallel} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$

$\eta_1 = \sqrt{\dfrac{\mu_0}{4\varepsilon_0}} = \dfrac{\eta_0}{2} = 60\pi$, $\quad \eta_2 = \sqrt{\dfrac{\mu_0}{9\varepsilon_0}} = \dfrac{\eta_0}{3} = 40\pi$

$$\Gamma_{\parallel} = \frac{\overset{2}{\cancel{40\pi}}\sqrt{\frac{29}{\cancel{45}\,9}} - \overset{3}{\cancel{60\pi}}\frac{1}{\cancel{\sqrt{5}}}}{\overset{2}{\cancel{40\pi}}\sqrt{\frac{29}{\cancel{45}\,9}} + \overset{3}{\cancel{60\pi}}\frac{1}{\cancel{\sqrt{5}}}} = \frac{\frac{2}{3}\sqrt{29} - 3}{\frac{2}{3}\sqrt{29} + 3} = 0.0896 \text{ //}$$

$$\bar{E}_i = \sqrt{45}\,(\cos\theta_i\,\hat{a}_x - \sin\theta_i\,\hat{a}_z)$$

$$\bar{E}_r = \sqrt{45}\,.\Gamma_{11}\,(\cos\theta_r\,\hat{a}_x + \sin\theta_r\,\hat{a}_z)\,e^{-j40\pi(2\hat{a}_x - \hat{a}_z)}$$

$$= \Gamma_{11}\sqrt{45}\left(\frac{1}{\sqrt{5}}\hat{a}_y + \frac{2}{\sqrt{5}}\hat{a}_z\right)e^{-j40\pi(2\hat{a}_y - \hat{a}_z)}$$

$$= \underbrace{3\Gamma_{11}}_{0.27}\,(\hat{a}_y + 2\hat{a}_z)\;(3)\;e^{-j40\pi(2\hat{a}_x - \hat{a}_z)}\;(3)$$

f) $$Z_{11} = \frac{\cos\theta_i}{\cos\theta_t}(1+\Gamma_{11}) = \frac{3}{\sqrt{29}} \times 1.089 \simeq 0.61 \quad (2)$$

$$\bar{E}_t = Z_{11}\,\cancel{\sqrt{45}}\,(\cos\theta_t\,\hat{a}_x - \sin\theta_t\,\hat{a}_z)\,e^{-jk_2\,\hat{a}_{n_t}\cdot\bar{r}}$$

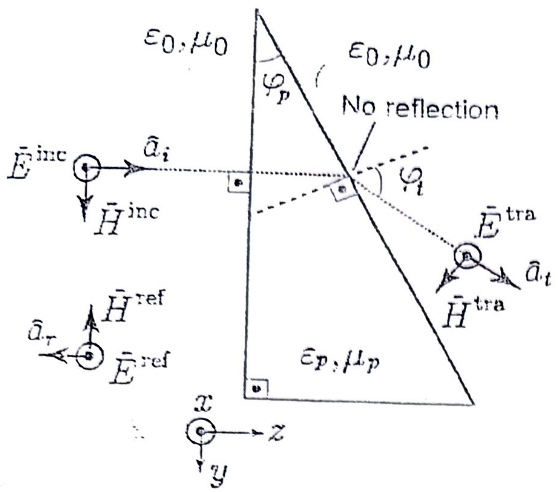
$$\cos\theta_t = \cancel{\sqrt{\tfrac{29}{45}}} \qquad \sin\theta_t = \cancel{\tfrac{4}{\sqrt{45}}}$$

$$\hat{a}_{n_t} = \cos\theta_t\,\hat{a}_z + \sin\theta_t\,\hat{a}_x \qquad k_1 = \cancel{3}\,\omega\sqrt{\epsilon_0\mu_0} = 20\pi$$

$$= \frac{4}{\sqrt{45}}\hat{a}_x + \sqrt{\frac{29}{45}}\,\hat{a}_z \qquad k_2 = 3\,\omega\sqrt{\epsilon_0\mu_0} = 30\pi$$

$$\bar{E}_t = 0.61\,(\sqrt{29}\,\hat{a}_x - 4\hat{a}_z)\,e^{-j30\pi\left(\frac{4}{\sqrt{45}}x + \sqrt{\frac{29}{45}}z\right)}$$

Q.5 (20 pts)



Consider a normal incidence of a plane wave on a triangular right prism with permittivity ε_p and permeability μ_p located in vacuum (ε_0, μ_0). Assume that all dimensions of the prism are very large in comparison to the wavelength and there is no any edge effect. In addition, there is no reflection when the plane wave is passing from the prism side to vacuum (see figure). The angle of the prism is

$$\varphi_p = \sin^{-1}(1/8).$$

Consider the directions defined in the figure, i.e., both incident and reflected electric field intensity is defined in the x direction. Then, it is given than $E^{\text{ref}} = \Gamma E^{\text{inc}} = E^{\text{inc}}/5$.

(a) [15 pts] Find the relative permittivity and the relative permeability of the prism.

(b) [5 pts] Find the angle φ_t, i.e., the angle between the transmitted wave and the normal of the prism. You can express the angle using $\sin^{-1}$.

$$\frac{\eta_p-\eta_0}{\eta_p+\eta_0} = \frac{1}{5} \Rightarrow 5\eta_p - 5\eta_0 = \eta_p+\eta_0 \Rightarrow 4\eta_p = 6\eta_0 \Rightarrow \boxed{\sqrt{\frac{\mu_{pr}}{\varepsilon_{pr}}} = \frac{3}{2}}$$

$$\sin\varphi_p = \left[\frac{1-\frac{\mu_p\varepsilon_0}{\varepsilon_p\mu_0}}{1-\left(\frac{\mu_p}{\mu_0}\right)^2}\right]^{1/2} = \frac{1}{8} \Rightarrow \left(1-\frac{\mu_{pr}}{\varepsilon_{pr}}\right)64 = 1-(\mu_{pr})^2$$

$$\Rightarrow \left(1-\frac{9}{4}\right)64 = 1-(\mu_{pr})^2 \Rightarrow -80 = +1-(\mu_{pr})^2 \Rightarrow \boxed{\mu_{pr}=9} \Rightarrow \boxed{\varepsilon_{pr}=4}$$

$$\mu_p = 9\mu_0 \text{ and } \varepsilon_p = 4\varepsilon_0$$

$$\sin\varphi_p\sqrt{\varepsilon_p\mu_p} = \sin\varphi_t\sqrt{\varepsilon_0\mu_0}$$

$$\Rightarrow \sin\varphi_t = \sin\varphi_p\sqrt{9\times4} = 6\sin\varphi_p = 3/4 \Rightarrow \boxed{\varphi_t = \sin^{-1}(3/4)}$$

EE303 Mt2 Solutions

Question: 1 (25 points)

A non-magnetic ($\mu = \mu_0$) simple material medium having an unknown conductivity (σ) and an unknown permittivity ($\varepsilon = \varepsilon_0\, \varepsilon_r$) is illuminated by an x-polarized uniform plane wave (u.p.w.) propagating in z-direction. Assume that conductivity, permittivity and permeability of the medium do not change with frequency.

i. Firstly, the u.p.w. is sent at a frequency $f = f_1$ with the resulting approximate expressions for the time-domain electric and magnetic fields as follows:

$$E_x(z,t) \cong e^{-2\pi z} \cos(2\pi \times 10^7\, t - 2\pi z) \quad V/m$$

fits to the general form

$$E_x(z,t) \cong e^{-\alpha z} \cos(2\pi f t - \beta z) \quad V/m$$

and

$$H_y(z,t) \cong \frac{1}{2\sqrt{2}\,\pi} e^{-2\pi z} \cos(2\pi \times 10^7\, t - 2\pi z - 0.25\,\pi) \quad A/m$$

fits to the general form

$$H_y(z,t) \cong \frac{1}{|\eta|} e^{-\alpha z} \cos(2\pi f t - \beta z - \theta) \quad A/m$$

where $\quad \eta = |\eta|\, e^{j\theta}$ is the intrinsic impedance of the lossy medium, in general.

For $\alpha \cong \beta$ and $\theta \cong \pi/4$ the medium is a good conductor for which $\sigma \gg \omega\varepsilon$.
Then, the propagation constant becomes $\gamma = \alpha + j\,\beta \cong \sqrt{j\,\omega\,\mu\,\sigma}$.
Therefore, you can answer the questions as follows:

Answer the following questions (including the units) **within the boxes** provided:

a) (1 pts) What is the frequency of propagation f_1?

f_1=10 MHz

b) (2 pts) What is α (attenuation coefficient) and β (phase coefficient) at f_1?

$\alpha=2\pi$ Neper/m. $\quad\quad$ $\beta=2\pi$ radian/m

c) (7 pts) Compute the conductivity of the medium, σ. Briefly but clearly state any assumption you make about the medium type, including your reasoning.

... is the phase difference between E and H fields it is

ii. Secondly, the u.p.w. is sent at a different frequency $f = f_2$ with the following approximate expressions for the time-domain electric and magnetic fields:

Similar to case (i),

$$E_x(z,t) \cong e^{-10\pi z} \cos(1.8\,\pi \times 10^{10}\, t - 360\,\pi\, z) \quad V/m$$

fits to the general form

$$E_x(z,t) \cong e^{-\alpha z} \cos(2\pi f t - \beta z) \quad V/m$$

and

$$H_y(z,t) \cong \frac{1}{20\,\pi} e^{-10\pi z} \cos(1.8\,\pi \times 10^{10}\, t - 360\,\pi\, z - 0.0088\,\pi) \quad A/m$$

fits to the general form

$$H_y(z,t) \cong \frac{1}{|\eta|} e^{-\alpha z} \cos(2\pi f t - \beta z - \theta) \quad A/m$$

where $\eta = |\eta|\, e^{j\theta}$ is the intrinsic impedance of the lossy medium, in general.

Therefore, in this case, $\alpha = 10\pi \ll \beta = 360\,\pi$. Also, $\theta = 0.0088\,\pi$ is a very small phase difference indicating the presence of a good dielectric (good insulator) medium for which $\sigma \ll \omega\varepsilon$.
For such a medium, we know that $\beta \cong k = \omega\sqrt{\varepsilon\mu}$.
Accordingly, you can answer the following questions as:

Answer the following questions (including the units) **within the boxes** provided:

a) (1 pts) What is the frequency of propagation f_2?

f_1=9 GHz

b) (2 pts) What is α (attenuation coefficient) and β (phase coefficient) at f_2?

α =10 π neper/m, β= 360 π radian/m

c) (7 pts) Compute the relative permittivity of the medium, ε_r. Briefly but clearly state any assumption you make about the medium type, including your reasoning.

As the phase of $\eta = 0.0088\,\pi$ (rad) $\cong 1.584\ degree$ is a very small angle (and also $\alpha \ll \beta$), the medium is a good insulator (good dielectric) for which $\beta \cong k = \omega\sqrt{\varepsilon\mu}$.

Then, $360\,\pi \cong 2\,\pi\, 9 \times 10^9 \sqrt{\varepsilon_r} / c$ where $c = 3 \times 10^8$ m/s

From here, $\sqrt{\varepsilon_r} = 6$ and $\varepsilon_r = 36$.

Also note that for this good dielectric medium, the intrinsic impedance can be computed as

$\eta \cong 20\,\pi\,(1 + j\,0.0277) = 1.00038 \times 20\,\pi\, e^{j0.0088\,\pi}$ ohms. As expected,

$|\eta_{good\ dielectric}| \cong |\eta_{lossless\ medium}| = 120\pi/\sqrt{\varepsilon_r} \cong 20\,\pi\ ohms$ leads to $\varepsilon_r = 36$ as well.

hat you do not need to use the attenuation constant (α) expression to answer question ii-c.

(5 pts) Verify the assumptions you made in parts (i-c) and (ii-c) by computing the Loss Tangent at frequencies f_1 and f_2. Comment on your results briefly but clearly.

Loss Tangent = $\sigma/(\omega\varepsilon)$

At $f = f_1$= 10 MHz,

Loss Tangent = $\frac{\sigma}{\omega\varepsilon} = 50 \gg 1$ Medium is GOOD CONDUCTOR as assumed in part i-c.

At $f = f_1$= 9 GHz,

Loss Tangent = $\frac{\sigma}{\omega\varepsilon} = 0.0555 \ll 1$ Medium is GOOD ISOLATOR as assumed in part ii-c.

The behavior of the lossy medium is not determined by only α, β and σ but it is a strong function of frequency also.

Question 2. (20 pts)

Consider the two electromagnetic waves expressed in phasor domain as

wavenumber $= k\sqrt{2}$

$$\vec{E}_1 = E_0 \hat{a}_z e^{-jk(x+y)}$$
$$\vec{E}_2 = E_0 \hat{a}_z e^{-jk(x-y)}$$

$\eta = \dfrac{k\sqrt{2}}{\omega\mu}$

The sum of these waves can be written as

$$2 E_0 \hat{a}_z \cos(ky) e^{-jkx}$$

a) Determine the time average Poynting vector for the two waves separately.
b) Determine the time average Poynting vector for the sum of the two waves.
c) Compare the two results and explain your result in one sentence.

a) Since these are uniform plane waves, we can write

$\bar{H} = \dfrac{1}{\eta} \hat{n} \times \bar{E}$ and $\bar{P} = \bar{E} \times \bar{H}^* = \dfrac{|\bar{E}|^2}{\eta} \hat{n}$; $\bar{P}_{av} = \dfrac{1}{2} \mathrm{Re}\{\bar{P}\}$

i) $\hat{n}_1 = \dfrac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_y)$; $\bar{P}_{1,av} = \dfrac{1}{2\sqrt{2}} \dfrac{|E_0|^2}{\eta} (\hat{a}_x + \hat{a}_y)$

ii) $\hat{n}_2 = \dfrac{1}{\sqrt{2}}(\hat{a}_x - \hat{a}_y)$; $\bar{P}_{2,av} = \dfrac{1}{2\sqrt{2}} \dfrac{|E_0|^2}{\eta} (\hat{a}_x - \hat{a}_y)$

b) The sum is <u>not</u> a UPW, we must find $\bar{H}$ first

$$\bar{H}_T = \frac{\nabla \times \bar{E}_T}{-j\omega\mu}; \quad \nabla \times \bar{E}_T = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & 2E_0 \cos ky \, e^{-jkx} \end{vmatrix}$$

$\Rightarrow \nabla \times \bar{E}_T = 2E_0 k(-\hat{a}_x \sin ky + j\hat{a}_y \cos ky) e^{-jkx}$

$\Rightarrow \bar{H}_T = \dfrac{\sqrt{2} E_0}{\eta}(-\hat{a}_x j \sin ky - \hat{a}_y \cos ky) e^{-jkx}$

$\bar{P}_{T,av} = \dfrac{1}{2} \mathrm{Re}\{\bar{E} \times \bar{H}^*\} = \dfrac{1}{2} \mathrm{Re}\left\{\dfrac{2\sqrt{2}|E_0|^2}{\eta}\left(\cos^2 ky \, \hat{a}_x + \dfrac{j}{2} \sin 2ky \, \hat{a}_y\right)\right\}$

$= \dfrac{\sqrt{2}|E_0|^2}{\eta} \cos^2 ky \, \hat{a}_x$

c) The components that propagate in y- direction form a standing wave and do not carry any power. The components in x- direction cannot be superposed since power is not linear.

Question 3. (25 pts)

A uniform plane wave is normally incident from a lossless dielectric medium with $\epsilon = \epsilon_r\epsilon_0$ and $\mu = \mu_0$ onto perfect conductor where the planar boundary is at $z = 0$. The incident electric field intensity vector is given as

$$\vec{E}_i = \hat{a}_x 60\pi e^{-jkz}, \text{ V/m}$$

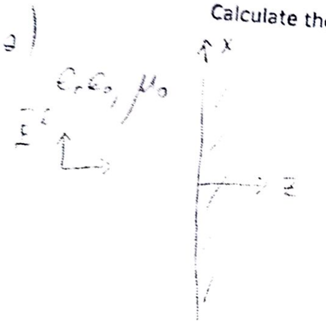
a) Derive the expressions of total electric field intensity vector, $\vec{E}_1$, and magnetic field intensity vector, $\vec{H}_1$ in the dielectric medium in <u>time domain</u> in terms of k, η, and ω.

b) The phasor surface current density induced on the boundary of the perfect conductor (z=0 plane) is given as

$$\vec{J}_s = 3\hat{a}_x, \text{ A/m.}$$

Calculate the dielectric constant, ϵ_r, of the dielectric medium.

a)



$$\bar{E}^i = \hat{a}_x 60\pi e^{-jkz}$$

$$\bar{E}_r = \hat{a}_x \Gamma 60\pi e^{+jkz} = -\hat{a}_x 60\pi e^{jkz}$$

$$\bar{E}_1 = \hat{a}_x 60\pi (e^{-jkz} - e^{jkz})$$

$$= -\hat{a}_x 60\pi 2j \sin kz$$

$$\bar{E}_1(z,t) = \hat{a}_x 120\pi \sin kz \sin\omega t$$

$$\bar{H}_i = \frac{1}{\eta} \hat{a}_z \times \hat{a}_x 60\pi e^{-jkz} = \hat{a}_y \frac{60\pi}{\eta} e^{-jkz}$$

$$\bar{H}_r = \frac{1}{\eta}(-\hat{a}_z \times \hat{a}_x \Gamma 60\pi e^{+jkz}) = -\hat{a}_y \frac{\Gamma 60\pi}{\eta} e^{jkz}$$

$$= \hat{a}_y \frac{60\pi}{\eta} e^{jkz}$$

$$\bar{H}_1 = \hat{a}_y \frac{60\pi}{\eta}(e^{-jkz} + e^{jkz}) = \hat{a}_y \frac{120\pi}{\eta} \cos kz$$

$$\bar{H}_1(z,t) = \hat{a}_y \frac{120\pi}{\eta} \cos kz \cos\omega t$$

b) J ...

Q. 4 (30 pts)

A uniform plane wave is incident from a medium with $(4\varepsilon_o, \mu_o)$ onto air with (ε_o, μ_o) as shown in the figure below. The phasor electric field of the transmitted wave in the second medium is given by,

$$\bar{E}_t = (\hat{a}_x 5 - \hat{a}_z 5\sqrt{3})e^{-jk_o(\frac{\sqrt{3}}{2}x + \frac{1}{2}z)} \ V/m.$$

(5) a) Find $\sin\theta_i$ and $\cos\theta_i$ where θ_i is the angle of incidence.

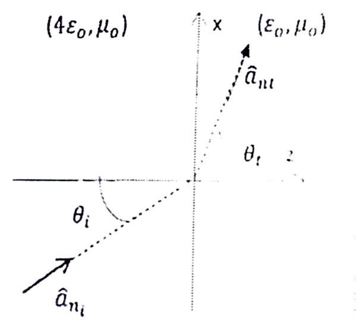
(20) b) Find the phasor electric field expressions of the incident and reflected fields.

(5) c) Find, θ_c, the angle of incidence which results in total internal reflection.

a) $\sin\theta_t = \frac{\sqrt{3}}{2}$, $\cos\theta_t = \frac{1}{2} \Rightarrow \theta_t = 60°$

$\underbrace{k_1}_{2k_o} \sin\theta_i = k_o \sin\theta_t \Rightarrow 2\cancel{k_o}\sin\theta_i = \cancel{k_o}\frac{\sqrt{3}}{2} \Rightarrow \sin\theta_i = \frac{\sqrt{3}}{4}$

$\cos\theta_i = \sqrt{1-\sin^2\theta_i} = \frac{\sqrt{13}}{4}$



b) $\bar{E}_i = \bar{E}_{io} e^{-j2k_o \hat{a}_{n_i} \cdot \bar{r}}$

$\hat{a}_{n_i} = \sin\theta_i \hat{a}_x + \cos\theta_i \hat{a}_z$

$= \frac{\sqrt{3}}{4}\hat{a}_x + \frac{\sqrt{13}}{4}\hat{a}_z$

$|E_{io}| = \frac{|E_{to}|}{\tau_{\parallel}}$, $E_{to} = 10$

$k_1 = 2k_o$ //

$$\Gamma_{\parallel} = \frac{\eta_o\cos\theta_t - \frac{\eta_o}{2}\cos\theta_i}{\eta_o\cos\theta_t + \frac{\eta_o}{2}\cos\theta_i} = 0.051$$

$$\tau_{\parallel} = (1+\Gamma_{\parallel})\frac{\cos\theta_i}{\cos\theta_t} = 1.9$$

$$\bar{E}_i = \frac{10}{1.9}(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i)e^{-j2k_o(x\sin\theta_i + z\cos\theta_i)}$$

$$= 5.26(\hat{a}_x\frac{\sqrt{13}}{4} - \hat{a}_z\frac{\sqrt{3}}{4})e^{-2k_o(x\frac{\sqrt{3}}{4} + z\frac{\sqrt{13}}{4})} \quad //$$

$$\bar{E}_r = |E_{io}|\Gamma_{\parallel}(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)e^{-j2k_o(x\sin\theta_i - z\cos\theta_i)}$$

$$= 0.268(\hat{a}_x\frac{\sqrt{13}}{4} + \hat{a}_z\frac{\sqrt{3}}{4})e^{-j2k_o(x\frac{\sqrt{3}}{4} - z\frac{\sqrt{13}}{4})}$$

b) $\underbrace{k_1}_{2k_0} \sin\theta_c = k_0 \underbrace{\sin\frac{\pi}{2}}_{1}$

$$\Rightarrow \sin\theta_c = \frac{1}{2} \quad \Rightarrow \quad \theta_c = 30^\circ$$