PLANE WAVES in LOSSY MEDIA

Perfect dielectrics (insulators) have zero conductivity, $\sigma = 0$

Perfect conductors have infinitely large conductivity, $\sigma \to \infty$

All the other media (in between these extreme cases) are classified using the ratio $\left(\frac{\sigma}{\omega\epsilon}\right)$ called "Loss Tangent"

Consider the Maxwell's Eqn. $\bar{\nabla} \times \bar{\mathcal{H}} = \bar{\mathcal{J}} + \frac{\partial \bar{\mathcal{D}}}{\partial t}$

For a linear medium, in phasor domain: $\bar{\nabla} \times \bar{H} = \bar{J} + j\omega\epsilon\bar{E}$
$(\bar{D} = \epsilon\bar{E})$

Let the medium is simple, source-free but lossy,

where $\sigma \neq 0$, $\rho_v = 0$, $\bar{J}_{imp} = 0$ $\Rightarrow$ $\bar{J} = \sigma\bar{E}$

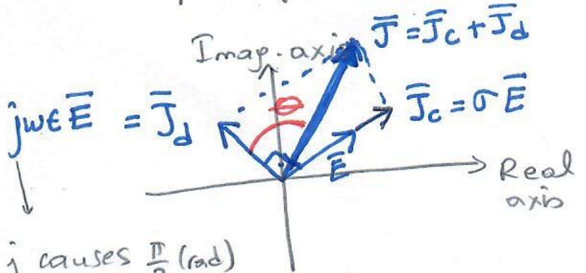
$$\Rightarrow \boxed{\bar{\nabla} \times \bar{H} = \underbrace{\sigma\bar{E}}_{\bar{J}_c} + \underbrace{j\omega\epsilon\bar{E}}_{\bar{J}_d}}$$

(in phasor domain)
($\bar{E}$ is a phasor vector)

$\bar{J}_c$: Conduction current density phasor

$\bar{J}_d$: Displacement current density phasor

$$\left|\frac{\text{Conduction current density}, \bar{J}_c}{\text{Displacement current density}, \bar{J}_d}\right| = \left|\frac{\sigma\bar{E}}{j\omega\epsilon\bar{E}}\right| = \frac{\sigma|\cancel{\bar{E}}|}{\omega\epsilon|\cancel{\bar{E}}|} = \frac{\sigma}{\omega\epsilon}$$



$$\boxed{\tan\theta = \frac{|\bar{J}_c|}{|\bar{J}_d|} = \frac{\sigma}{\omega\epsilon} = \text{Loss Tangent}}$$

j causes $\frac{\pi}{2}$ (rad) phase difference between $\bar{J}_c$ and $\bar{J}_d$ such that $\bar{J}_c \perp \bar{J}_d$.

If $\frac{\sigma}{\omega\epsilon} \gg 1$ ----→ medium is called good conductor

If $\frac{\sigma}{\omega\epsilon} \ll 1$ ---→ " " " good insulator

If $\frac{\sigma}{\omega\epsilon} \approx 1$ ---→ " " " { Poor insulator or poor conductor

(*) Note that behavior of a given medium depends on the frequency ($\omega = 2\pi f$) of the electromagnetic wave propagation as well as the medium parameters σ and ϵ. It means that the same material can behave as a good conductor at one frequency and as a good insulator at another frequency!!!

Now, remember the wave equation derived for a simple, source-free, lossy medium in time-domain:

$$\boxed{\nabla^2 \bar{E} - \mu\sigma \frac{\partial \bar{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

($\bar{H}$ satisfies the same eqn. also in time domain)

$$\Rightarrow \nabla^2 \bar{E} - \mu\sigma (j\omega) \bar{E} - \mu\epsilon (j\omega)^2 \bar{E} = 0$$ (in phasor domain)

$$\nabla^2 \bar{E} \underbrace{- j\omega\mu\sigma \bar{E} - j\omega\mu (j\omega\epsilon) \bar{E}}_{-j\omega\mu(\sigma + j\omega\epsilon)\bar{E}} = 0$$

$$\Rightarrow \boxed{\nabla^2 \bar{E} - \underbrace{j\omega\mu(\sigma + j\omega\epsilon)}_{\gamma^2} \bar{E} = 0}$$

let $\boxed{\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)}$

$$\Rightarrow \boxed{\begin{aligned} &\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0 \\ \text{Similarly, } &\nabla^2 \bar{H} - \gamma^2 \bar{H} = 0 \end{aligned}}$$

with $\boxed{\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}}$

must be solved in phasor domain for $\left\{\begin{array}{l}\text{simple} \\ \text{source-free} \\ \text{lossy}\end{array}\right.$ medium.

$$\boxed{\gamma = \sqrt{j\omega\epsilon(\sigma + j\omega\epsilon)} = \alpha + j\beta}$$

γ: Propagation constant

$\boxed{\alpha = Re\{\gamma\}}$: Attenuation constant (neper/meter)

$\boxed{\beta = Im\{\gamma\}}$: phase constant (radian/meter)

Solution of the $\bar{E}$ phasor for a simple case

where we assume $\boxed{\bar{E} = E_x(z)\,\hat{a}_x}$

$$\left.\begin{array}{l}\nabla^2 \bar{E} - \gamma^2 \bar{E} = 0 \\ \bar{E} = E_x(z)\,\hat{a}_x\end{array}\right\} \Rightarrow \frac{d^2 E_x(z)}{dz^2} - \gamma^2 E_x(z) = 0$$

$$s^2 - \gamma^2 = 0 \Rightarrow s_{1,2} = \mp\gamma$$

$$\Rightarrow E_x(z) = A\, e^{-\gamma z} + B\, e^{+\gamma z}$$

set $\gamma = \alpha + j\beta \Rightarrow E_x(z) = A\, e^{-(\alpha + j\beta) z} + B\, e^{(\alpha + j\beta) z}$

$$\Rightarrow \boxed{E_x(z) = A\, e^{-\alpha z}\, e^{-j\beta z} + B\, e^{\alpha z}\, e^{j\beta z}}$$

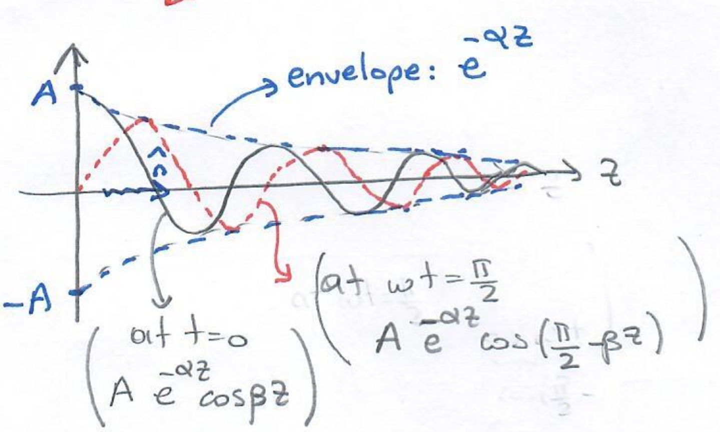
where A and B are arbitrary constants.

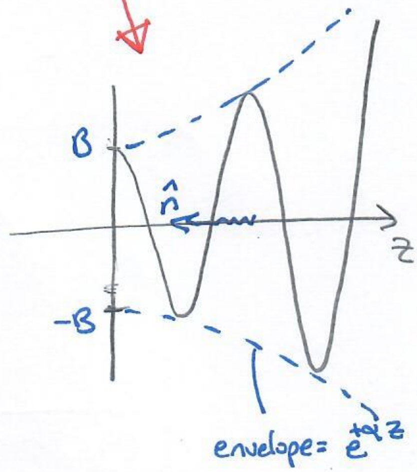
Assuming A and B as <u>real</u> constants, without any loss of generality, and using $\bar{E}(z,t) = Re\{\bar{E}(z)e^{j\omega t}\}$, we'll get

$$\boxed{E_x(z,t) = \underbrace{A\,e^{-\alpha z}\cos(\omega t-\beta z)}_{} + \underbrace{B\,e^{\alpha z}\cos(\omega t+\beta z)}_{}}$$

A plane wave travelling in $\boxed{\hat{n}=+\hat{a}_z}$ direction with a velocity $\boxed{v=\frac{\omega}{\beta}}$

A plane wave travelling in $\boxed{\hat{n}=-\hat{a}_z}$ direction with a velocity $\boxed{v=\frac{\omega}{\beta}}$





As the wave travels in $\hat{a}_z$ direction, its amplitude is attenuated exponentially at a rate α (Nep/m)

As the wave travels in $(-\hat{a}_z)$ direction, its amplitude decays exponentially at a rate α (Nep/m)

Consider argument $= \omega t - \beta z$ = constant

$\Rightarrow d(\omega t-\beta z) = 0$

$\omega dt - \beta dz = 0$

$\Rightarrow \boxed{\frac{dz}{dt} = \frac{\omega}{\beta} = v}$

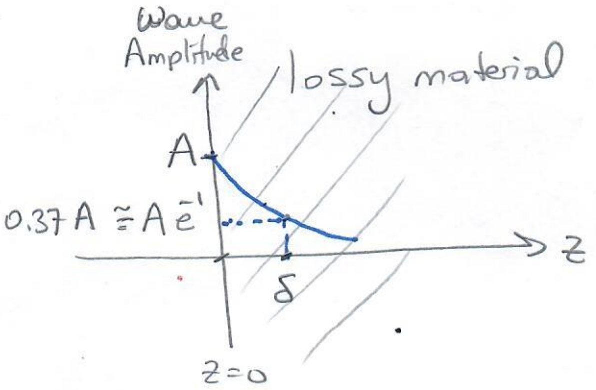
velocity of propagation

$\underbrace{d(\omega t)}_{2\pi} - \beta \underbrace{dz}_{\lambda} = 0$

$\Rightarrow \boxed{\lambda = \frac{2\pi}{\beta}}$

: wavelength

Definition: Skin depth (also called penetration depth) is the distance (along the direction of propagation) at which the amplitude of the wave decays to $\frac{1}{e} \approx 0.37$ of its initial value at $z=0$.



$$e^{-\alpha z}\Big|_{z=0} = 1$$

$$e^{-\alpha z}\Big|_{z=\delta} = e^{-\alpha\delta} = \frac{1}{e} = e^{-1}$$

$$\Rightarrow \alpha\delta = 1 \Rightarrow \boxed{\delta = \frac{1}{\alpha}}$$

Skin depth

Now, let's obtain the $\bar{H}$ phasor of the plane wave

for $\bar{E} = E_0 e^{-\gamma z}\hat{a}_x$ part where $\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

$$\bar{\nabla} \times \bar{E} = -j\omega\mu\bar{H} \Rightarrow \bar{H} = \frac{1}{-j\omega\mu}\bar{\nabla} \times \left(E_0 e^{-\gamma z}\hat{a}_x\right)$$

$$\bar{H} = \frac{1}{-j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 e^{-\gamma z} & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \bar{H} = \frac{1}{-j\omega\mu}(-\gamma)E_0 e^{-\gamma z}\hat{a}_y$$

$$\Rightarrow \boxed{\bar{H} = \frac{\gamma}{j\omega\mu} E_o e^{-\gamma z} \hat{a}_y} \quad \text{corresponding to} \quad \boxed{\bar{E} = \underbrace{E_o e^{-\gamma z}}_{E_x} \hat{a}_x}$$

$$\Downarrow$$

$$H_y = \frac{\gamma}{j\omega\mu} E_x$$

$$\Rightarrow \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\Rightarrow \boxed{\eta = \frac{E_x}{H_y} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}} \quad (\Omega)$$

ohm

Intrinsic impedance of the lossy medium (a complex quantity in general if $\sigma \neq 0$)

$$\left(\eta = \frac{E_x}{H_y} = |\eta| e^{j\phi} = \eta_r + j\eta_i \qquad \text{for } \sigma \neq 0 \text{ case} \right)$$

$$H_y = \frac{1}{\eta} E_x = \frac{E_o e^{-\alpha z} e^{-j\beta z}}{|\eta| e^{j\phi}} = \frac{E_o}{|\eta|} e^{-\alpha z} e^{-j\beta z} e^{-j\phi}$$

$$\Rightarrow H_y(z) = \frac{E_o}{|\eta|} e^{-\alpha z} e^{j(-\beta z - \phi)} \quad \text{in phasor domain}$$

$$\Rightarrow \mathcal{H}_y(z,t) = Re\{H_y(z) e^{j\omega t}\} = Re\left\{\frac{E_o}{|\eta|} e^{-\alpha z} e^{j(-\beta z - \phi)} e^{j\omega t}\right\}$$

$$\boxed{\mathcal{H}_y(z,t) = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \phi)} \quad (\text{assuming } E_o \text{ is a real constant})$$

$$E_x(z) = E_0 e^{-\alpha z} e^{-j\beta z} \longleftrightarrow H_y(z) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{-j\beta z} \underbrace{e^{-j\phi}}_{\text{phase difference term!}}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\mathcal{E}_x(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \longleftrightarrow \mathcal{H}_y(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \phi)$$

There is a phase difference ϕ between the $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ fields! in a lossy medium with $\sigma \neq 0$

Special Case : If $\sigma = 0$ (i.e. lossless medium problem)

$$\eta = \left.\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}\,\right|_{\sigma=0} = \sqrt{\frac{j\omega\mu}{j\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \quad \text{: real in lossless media}$$

$$\Rightarrow \left.\begin{aligned} \mathcal{E}_x(z,t) &= E_0 \cos(\omega t - kz) \\ \mathcal{H}_y(z,t) &= \frac{E_0}{\eta} \cos(\omega t - kz) \end{aligned}\right\} \text{No phase difference between } \bar{\mathcal{E}} \text{ and } \bar{\mathcal{H}} \text{ fields in lossless media!}$$

$\therefore$ Conclusion: Presence of a phase difference between the $\bar{\mathcal{E}}$ and $\bar{\mathcal{H}}$ fields of a plane wave is the indication of a "lossy medium" propagation where η: intrinsic impedance is complex valued due to $\sigma \neq 0$.

Complex Permittivity $= \epsilon_c$ can be defined for a lossy medium where $\sigma \neq 0$.

$$\bar{\nabla} \times \bar{H} = \bar{J} + j\omega\epsilon \bar{E} = \sigma \bar{E} + j\omega\epsilon \bar{E}$$ in a simple, source-free, lossy medium.

$$\Rightarrow \quad \bar{\nabla} \times \bar{H} = (\sigma + j\omega\epsilon)\bar{E}$$

or $$\bar{\nabla} \times \bar{H} = j\omega\left(\frac{\sigma}{j\omega} + \epsilon\right)\bar{E}$$

$$\bar{\nabla} \times \bar{H} = j\omega \underbrace{\left(\epsilon - j\frac{\sigma}{\omega}\right)}_{\text{call } \epsilon_c} \bar{E}$$

$$\Rightarrow \quad \boxed{\bar{\nabla} \times \bar{H} = j\omega\epsilon_c \bar{E}} \quad \text{with} \quad \boxed{\epsilon_c = \epsilon - j\frac{\sigma}{\omega}}$$ Equivalent complex permittivity

$$\epsilon_c = \epsilon' - j\epsilon''$$

real part (ϵ'), imaginary part (ϵ'')

Note: Definition of ϵ_c is helpful to obtain the expressions in lossy medium problems just by replacing the ϵ term by replacing $\epsilon_c = \epsilon - j\frac{\sigma}{\omega}$!

Example:

$$\eta_{lossless} = \sqrt{\frac{\mu}{\epsilon}}$$ in lossless medium problems

Replace ϵ above by $\epsilon_c = \epsilon - j\frac{\sigma}{\omega}$ to get η_{lossy}

$$\eta_{lossy} = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{\epsilon + \frac{\sigma}{j\omega}}} = \sqrt{\frac{\mu}{\frac{j\omega\epsilon + \sigma}{j\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

as obtained earlier ✓

Example:

in lossless case, we have $e^{-jkz} = e^{-j\omega\sqrt{\mu\epsilon}\,z}$ as $k=\omega\sqrt{\mu\epsilon}$

replace ϵ by ϵ_c :

$$e^{-j\omega\sqrt{\mu\epsilon_c}\,z} = e^{-j\omega\sqrt{\mu(\epsilon - j\frac{\sigma}{\omega})}\,z} = \exp\left\{-\sqrt{(j\omega)^2\mu(\epsilon+\frac{\sigma}{j\omega})}\;z\right\}$$

$$= \exp\left\{-\underbrace{\sqrt{j\omega\mu(j\omega\epsilon+\sigma)}}_{\gamma}\;z\right\} = e^{-\gamma z}$$ as obtained before. In lossy medium.

e^{-jkz} (lossless) $\longleftrightarrow$ $e^{-\gamma z}$ (lossy) $\Longrightarrow$

$jk \longleftrightarrow \gamma$
$\epsilon \longleftrightarrow \epsilon_c$

These substitutions can be used to obtain a lossy case expression from the corresponding lossless medium expression.

Exercise: Show that for a plane wave propagating in an arbitrary direction $\hat{n}$ in a lossy medium has the exponential term $e^{-\gamma\hat{n}\cdot\bar{r}}$

Remember, we have $e^{-j\bar{k}\cdot\bar{r}} = e^{-jk\hat{n}\cdot\bar{r}}$ in lossless media $(\bar{k}=k\hat{n})$

Change (jk) by γ where $\gamma=\alpha+j\beta$

$\Longrightarrow$ $e^{-\gamma\hat{n}\cdot\bar{r}} = e^{-(\alpha+j\beta)\hat{n}\cdot\bar{r}}$ must appear in lossy medium solutions for a plane wave.

Behavior of Plane Waves in Media with Different Levels of Conductivity

Remember,

$$\text{Loss Tangent} = \left|\frac{\text{Cond. curr. density}}{\text{Disp. curr. density}}\right| = \frac{\sigma}{\omega\epsilon}$$

- $\frac{\sigma}{\omega\epsilon} = 0$ perfect dielectric (lossless, $\sigma = 0$) insulator
- $\frac{\sigma}{\omega\epsilon} \ll 1$ good insulator
- $\frac{\sigma}{\omega\epsilon} \sim 1$ (roughly between 0.1 and 10)
- $\frac{\sigma}{\omega\epsilon} \gg 1$ good conductor
- $\frac{\sigma}{\omega\epsilon} \to \infty$ perfect conductor $\sigma \to \infty$

In a lossy medium, we have in general:

$$\underline{\underline{\gamma}} = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta \quad \text{(Propagation constant)}$$

$$\underline{\underline{\eta}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta| e^{j\phi} \quad \text{(Intrinsic impedance)}$$

$$\underline{\underline{v}} = \frac{\omega}{Im\{\gamma\}} = \frac{\omega}{\beta} \quad \text{(phase velocity, velocity of propagation)}$$

$$\underline{\underline{\lambda}} = \frac{2\pi}{Im\{\gamma\}} = \frac{2\pi}{\beta} \quad \text{(wavelength)}$$

$$\underline{\underline{\delta}} = \frac{1}{Re\{\gamma\}} = \frac{1}{\alpha} \quad \text{(skin depth)}$$

(i) Perfect Dielectric (Insulator) Case: $(\sigma=0)$

$$\gamma = \sqrt{j\omega\mu(\underset{\substack{\downarrow\\0}}{\sigma}+j\omega\epsilon)} = \sqrt{(j\omega\mu)(j\omega\epsilon)} = j\omega\sqrt{\mu\epsilon} \quad \left.\right\} \begin{array}{l}\alpha=0\\ \beta=\omega\sqrt{\mu\epsilon}\end{array}$$

we know $\gamma = \alpha + j\beta$

$$\Rightarrow \boxed{\alpha = 0}\ , \quad \boxed{\beta = \omega\sqrt{\mu\epsilon} = k} \Rightarrow$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{\delta = \frac{1}{\alpha} \to \infty} \qquad \boxed{v = \frac{\omega}{\beta} = \frac{\omega}{k}} \text{ and } \boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{k}}$$

Also, $\eta = \sqrt{\dfrac{j\omega\mu}{\underset{\substack{\swarrow\\0}}{\sigma}+j\omega\epsilon}} = \sqrt{\dfrac{j\omega\mu}{j\omega\epsilon}} = \sqrt{\dfrac{\mu}{\epsilon}} \Rightarrow \boxed{\eta = \sqrt{\dfrac{\mu}{\epsilon}}}$

This is the lossless medium case examined earlier.

(ii) Good Insulator (Low-Loss Dielectric) Case: $\left(\frac{\sigma}{\omega\epsilon} \ll 1\right)$

$$\gamma = \sqrt{j\omega\mu(\sigma+j\omega\epsilon)} = \sqrt{j\omega\mu \cdot j\omega\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}$$

$$= j\omega\sqrt{\mu\epsilon}\sqrt{1-j\frac{\sigma}{\omega\epsilon}}$$ where $\frac{\sigma}{\omega\epsilon} \ll 1$ in this case!

Remember the Binomial Theorem

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \quad \text{for } |x|<1$$

$$\cong 1 + \frac{x}{2} \quad \text{if } |x| \ll 1$$

In our problem x stands for $\left(-j\frac{\sigma}{\omega\epsilon}\right)$ resulting:

$$\gamma \cong j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{2\omega\epsilon}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon}$$

we know $\gamma = \alpha + j\beta$

$\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$

$\beta \approx \omega\sqrt{\mu\epsilon} \cong k$

(nep/m) $\boxed{\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$: Attenuation constant $\Rightarrow$ $\boxed{\delta = \frac{1}{\alpha} \cong \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}$: Skin depth (meter)

(rad/m) $\boxed{\beta \cong k = \omega\sqrt{\mu\epsilon}}$: Phase constant $\Rightarrow$

$\boxed{v = \frac{\omega}{\beta} \approx \frac{\omega}{k}}$ phase velocity (meter/sec)

$\boxed{\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{k}}$ wavelength (meter)

Note that v and λ are approximately the same as in the lossless case (since $\mathrm{Re}\{\gamma\} = \beta \cong k$).

Also,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{\sqrt{1 - j\frac{\sigma}{\omega\epsilon}}}$$

where $\sqrt{1 - j\frac{\sigma}{\omega\epsilon}} \approx 1 - j\frac{\sigma}{2\omega\epsilon}$

$$\Rightarrow \eta \approx \sqrt{\frac{\mu}{\epsilon}}\frac{1}{1 - j\frac{\sigma}{2\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}\frac{1 + j\frac{\sigma}{2\omega\epsilon}}{1 + \underbrace{\left(\frac{\sigma}{2\omega\epsilon}\right)^2}_{\ll 1}} \approx \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{\sigma}{2\omega\epsilon}\right)$$

$$\Rightarrow \boxed{\eta \approx \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{\sigma}{2\omega\epsilon}\right)}$$ intrinsic impedance (Ω)

(in addition to the lossless case value $\sqrt{\frac{\mu}{\epsilon}}$, there is also a very small reactive part resulting from nonzero σ.)

(iii) Good Conductor Case: $\left(\frac{\sigma}{\omega\epsilon} \gg 1\right)$ Conductors with finite conductivity

$$\gamma = \sqrt{j\omega\mu(\sigma + \underbrace{j\omega\epsilon}_{\text{negligible}})} \quad \text{where} \quad \sigma \gg \omega\epsilon$$

$$\Rightarrow \gamma \cong \sqrt{j\omega\mu\sigma} = \sqrt{j}\,\sqrt{\omega\mu\sigma}$$

where $\sqrt{j} = \sqrt{e^{j\pi/2}} = e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4}$

$$\Rightarrow \sqrt{j} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+j)$$

$$\Rightarrow \gamma = \frac{1}{\sqrt{2}}(1+j)\sqrt{\omega\mu\sigma}$$

$$\Rightarrow \boxed{\gamma = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j)}$$

Also $\gamma = \alpha + j\beta$

$$\boxed{\alpha \cong \beta \cong \sqrt{\frac{\omega\mu\sigma}{2}}}$$

or let $\omega = 2\pi f$

$$\boxed{\alpha \cong \beta \cong \sqrt{\pi f \mu\sigma}}$$ for good conductors

skin depth $$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}}$$

$$\boxed{v = \frac{\omega}{\beta} \cong \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}}$$ phase velocity

and

$$\boxed{\lambda = \frac{2\pi}{\beta} \cong \frac{2\pi}{\alpha} = 2\pi\delta}$$ wavelength

Also, $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}} \approx \sqrt{\dfrac{j\omega\mu}{\sigma}} = \underbrace{\sqrt{j}}_{\frac{1}{\sqrt{2}}(1+j)} \sqrt{\dfrac{\omega\mu}{\sigma}}$

$$\Rightarrow \boxed{\eta \cong \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j)}$$ intrinsic impedance

It can also be shown that $\boxed{\eta = \dfrac{1}{\sigma\delta}(1+j)}$

Surface Resistance : $\boxed{R_s = \mathrm{Re}\{\eta\} = \dfrac{1}{\delta\sigma}}$

Note that v, λ and η have very small values in good conductors.

Also, as $f\uparrow \Rightarrow \alpha\uparrow, \beta\uparrow, v\uparrow, \delta\downarrow, \lambda\downarrow$

($\uparrow$: increases, $\downarrow$: decreases)

(iv) Poor Conductor / Poor Insulator Case : $(\sigma \approx \omega\epsilon)$

(Dielectrics with high losses)

No approximations apply! Use the original expressions to compute $\gamma, \alpha, \beta, \eta, v, \lambda$ without any simplification!

(v) Perfect Conductor Case: $(\sigma \rightarrow \infty)$

$\bar{E} = \bar{D} = \bar{B} = \bar{H} = 0$ inside a perfect conductor!

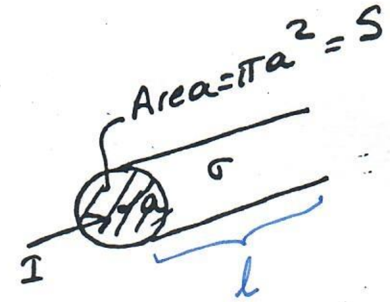
As no fields can penetrate into a perfect conductor $\Rightarrow \delta \approx 0$

A surface current density $\bar{J}_s$ and a surface charge density ρ_s may exist on the surface of a perfect conductor.

AC and DC resistances of a conductor

Consider a solid metallic conductor of radius (a) and conductivity (σ).

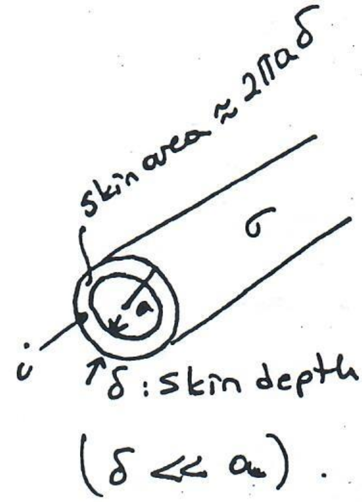
DC-Case



at $f=0$, current is uniformly distributed over the cross-section of area (πa^2).
Then for a length of $l=1$ meter of the conductor, the DC-resistance R_{DC} is computed as

$$R_{DC} = \frac{l}{\sigma S} = \frac{1}{\sigma \pi a^2} \ \Omega/m.$$

AC-Case



for $f>0$, AC current is concentrated over a "skin" area which is a thin cylindrical shell close to the surface of the conductor.
Again for $l=1$ meter, the AC-resistance, R_{AC} is computed as

$$R_{AC} \cong \frac{l}{\sigma S} \cong \frac{1}{\sigma(2\pi a \delta)} \ \Omega/m$$

Note that $\quad \dfrac{R_{AC}}{R_{DC}} = \dfrac{\frac{1}{\sigma 2\pi a\delta}}{\frac{1}{\sigma \pi a^2}} = \dfrac{a}{2\delta} \gg 1 \Rightarrow R_{AC} \gg R_{DC}$

Also note that $\quad R_{AC} \downarrow$ as $a\uparrow$ and $\sigma\uparrow$.

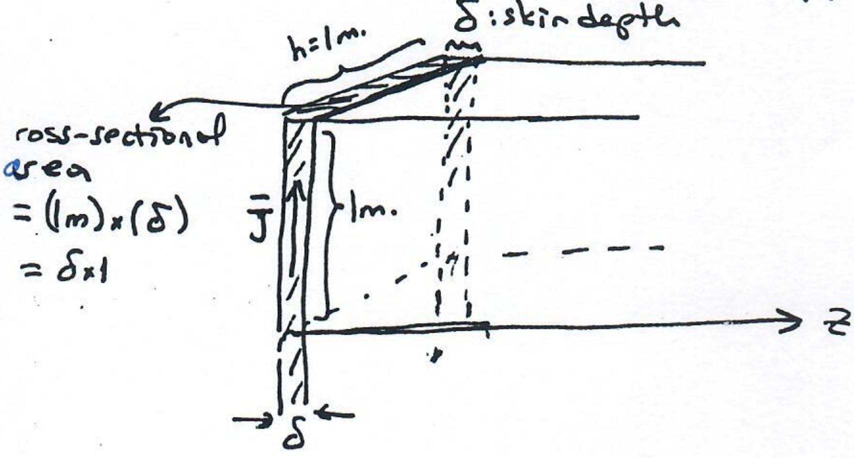
Surface resistance in good conductors

$$\eta = \frac{1+j}{\sigma\delta} \quad \text{in good conductors}$$

$$R_s = \mathrm{Re}\{\eta\} = \mathrm{Re}\left\{\frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta}\right\} = \frac{1}{\sigma\delta}$$

$$\boxed{R_s = \frac{1}{\sigma\delta} \quad \Omega/m^2} \quad \text{Surface resistance.}$$

Consider the following good conductor block, only a skin close to surface supports most of the current flow



$$R = \frac{l}{\sigma(h\times\delta)} \quad \text{where } l = 1\,m. \text{ let } h = 1\,m.$$

$$R = \frac{1}{\sigma(1\times\delta)} = \frac{1}{\sigma\delta} \quad (\Omega/m^2)$$

Skin effect and current density in good conductors

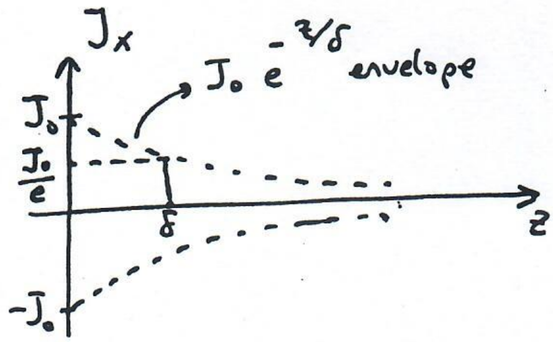
$\bar{J} = \sigma\bar{E}$ when $\sigma \neq 0$

let $\bar{E} = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\,\hat{a}_x$ for example,

with $\alpha = \beta \cong \frac{1}{\delta}$ in a good conductor

$$\Rightarrow J_x = \sigma E_0 e^{-\alpha z}\cos(\omega t - \beta z)$$

$$= \underbrace{\sigma E_0}_{\text{let } J_0} e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta}\right)$$



$$\Rightarrow J_x = J_0 e^{-z/\delta}\cos\left(\omega t - \frac{z}{\delta}\right)$$

for $0 \leq z < \infty$ with $\delta = \frac{\lambda}{2\pi}$