METU

Department of Electrical-Electronics Engineering

EE303 Fall 2020-Solutions for HW-6

Question:1

a) From Snell's Law:

$$n_1 sin\theta_i = n_2 sin\theta_t$$
$$sin\theta_t = \frac{n_1}{n_2} sin\theta_i$$
$$sin\theta_t = 2$$

$$sin^2\theta_t + cos^2\theta_t = 1$$
$$cos\theta_t = -j\sqrt{3}$$

$\star$ "-" sign is chosen for a realistic, physical solution.

$$sin\theta_c = \sqrt{\frac{\mu_0\varepsilon_0}{\mu_0 16\varepsilon_0}} = \frac{1}{4}$$
$$\rightarrow \theta_c \approx 14.48^\circ$$
$$\rightarrow \theta_c < \theta_i$$

θ_i is greater than the critical angle. As expected $sin\theta_i$ is larger than 1 and $cos\theta_i$ is purely imaginary.

b)

$$\Gamma_{\perp} = \frac{\eta_2 cos\theta_i - \eta_1 cos\theta_t}{\eta_2 cos\theta_i + \eta_1 cos\theta_t}$$
$$\Gamma_{\perp} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{4}j\sqrt{3}}{\frac{\sqrt{3}}{2} - \frac{1}{4}j\sqrt{3}}$$
$$\Gamma_{\perp} = \frac{2\sqrt{3} + j\sqrt{3}}{2\sqrt{3} - j\sqrt{3}}$$
$$\Gamma_{\perp} = \frac{2+j}{2-j}$$
$$\Gamma_{\perp} = \frac{(2+j)^2}{5}$$
$$\Gamma_{\perp} = \frac{3+4j}{5}$$

$$T_{\perp} = 1 + \Gamma_{\perp}$$
$$T_{\perp} = \frac{8+4j}{5}$$

c)

$$\bar{E}_r = 10\Gamma_{\perp}\hat{a}_y e^{-j\bar{k}_r \cdot \bar{r}}$$

$$\hat{n}_r = sin\theta_i \hat{a}_x - cos\theta_i \hat{a}_z$$
$$\hat{n}_r = \frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z = \frac{\bar{k}_r}{|\bar{k}_r|} \text{ where } |\bar{k}_r| = \frac{4\omega}{c}$$
$$\rightarrow \bar{k}_r = \frac{4\omega}{c}\left(\frac{1}{2}\hat{a}_x - \frac{\sqrt{3}}{2}\hat{a}_z\right)$$
$$\rightarrow \bar{k}_r \cdot \bar{r} = \left(\frac{2\omega}{c}x - \frac{2\sqrt{3}\omega}{c}z\right)$$

$$\bar{E}_r = \hat{a}_y (6+8j)\, e^{-j\left(\frac{2\omega}{c}x - \frac{2\sqrt{3}\omega}{c}z\right)} \; V/m$$

d)

$$\bar{E}_r(\bar{r},t) = Re\left\{\bar{E}_r(\bar{r})\,e^{j\omega t}\right\}$$
$$\bar{E}_r(\bar{r},t) = Re\left\{\hat{a}_y\,(6+8j)\,e^{-j\left(\frac{2\omega}{c}x-\frac{2\sqrt{3}\omega}{c}z\right)}e^{j\omega t}\right\}$$
$$\bar{E}_r(\bar{r},t) = \hat{a}_y\left[6cos\left(\omega t-\frac{\omega}{c}\left(2x-2\sqrt{3}z\right)\right)-8sin\left(\omega t-\frac{\omega}{c}\left(2x-2\sqrt{3}z\right)\right)\right]\ V/m$$

e)

$$\bar{E}_t = 10T_{\perp}\hat{a}_y e^{-j\bar{k}_t\cdot\bar{r}}$$

$$\hat{n}_t = sin\theta_t\hat{a}_x + cos\theta_t\hat{a}_z$$
$$\hat{n}_t = 2\hat{a}_x - j\sqrt{3}\hat{a}_z = \frac{\bar{k}_t}{|\bar{k}_t|}\text{ where }|\bar{k}_t| = \frac{\omega}{c}$$
$$\rightarrow \bar{k}_t = \frac{\omega}{c}\left(2\hat{a}_x - j\sqrt{3}\hat{a}_z\right)$$
$$\rightarrow \bar{k}_t\cdot\bar{r} = \left(\frac{2\omega}{c}x - \frac{j\sqrt{3}\omega}{c}z\right)$$

$$\bar{E}_t = \hat{a}_y\,(16+8j)\,e^{-j\left(\frac{2\omega}{c}x-\frac{j\sqrt{3}\omega}{c}z\right)}$$
$$\bar{E}_t = \hat{a}_y\,(16+8j)\,e^{-j\frac{2\omega}{c}x}e^{-\frac{\sqrt{3}\omega}{c}z}\ V/m$$

f)

$$\bar{E}_t(\bar{r},t) = Re\left\{\bar{E}_t(\bar{r})\,e^{j\omega t}\right\}$$
$$\bar{E}_t(\bar{r},t) = Re\left\{\hat{a}_y\,(16+8j)\,e^{-j\frac{2\omega}{c}x}e^{-\frac{\sqrt{3}\omega}{c}z}e^{j\omega t}\right\}$$
$$\bar{E}_t(\bar{r},t) = \hat{a}_y\left[16e^{-\frac{\sqrt{3}\omega}{c}z}cos\left(\omega t-\frac{2\omega}{c}x\right)-8e^{-\frac{\sqrt{3}\omega}{c}z}sin\left(\omega t-\frac{2\omega}{c}x\right)\right]\ V/m$$

g)

$$\begin{aligned}
&\bar{E}_t = \hat{a}_y\,(16+8j)\,e^{-j\frac{2\omega}{c}x}e^{-\frac{\sqrt{3}\omega}{c}z}\\
&|\bar{E}_t| = |16+8j||e^{-j\frac{2\omega}{c}x}||e^{-\frac{\sqrt{3}\omega}{c}z}|\\
&\rightarrow |16+8j| = \ constant\\
&\rightarrow |e^{-j\frac{2\omega}{c}x}| = 1
\end{aligned}$$

$|\bar{E}_t|$ is kept constant if z is constant. That means "constant amplitude" surfaces are z=constant planes.

$$\underline{/\bar{E}_t} = \frac{2\omega}{c}x$$

$\underline{/\bar{E}_t}$ is kept constant if x=constant. In other words, "constant phase" surfaces are x=constant planes.

⋆ Regarding the discussions made above, it is seen that transmitted wave is a <u>non-uniform plane wave.</u>

h)

$$\begin{aligned}
v_p &= \frac{\omega}{\beta} \text{ where } \beta = \frac{2\omega}{c}\\
v_p &= \frac{c}{2} = 1.5\times 10^8\ m/s
\end{aligned}$$

Knowing that this wave exists in air, calculated phase velocity also proves that transmitted wave is not a uniform plane wave since v_p is smaller than c.

<u>Note that</u> the propagation direction of the transmitted non-uniform plane wave in the second medium is parallel to the boundary (in + x direction) while its amplitude decays exponentially in the z-direction.

Question:2

Note that the 1st medium must be denser (i.e. $\mu\epsilon$ product of the incidence medium should be larger) for the existence of critical angle. Magnitude of the reflection coefficient will be unity when the incidence angle becomes larger than or equal to the critical angle.

Also note that Brewster angle (value of incidence angle that makes reflected field zero) exists only for parallel polarization if the permeabilities of both media are the same.

⋆ θ_c exists when the incident wave has higher $\mu\varepsilon$ product.
⋆ $\theta_B^{\perp}$ doesn't exist when $\mu_1 = \mu_2$.
⋆ When $\theta_i > \theta_c$, magnitude of the reflection coefficient equals to 1.

b) Figure 4 since:
⋆ θ_c exists when the incident wave has higher $\mu\varepsilon$ product.
⋆ $\theta_B^{//}$ exists.
⋆ When $\theta_i > \theta_c$, magnitude of the reflection coefficient equals to 1.

c) Figure 1 since:
⋆ θ_c doesn't exists when the incident wave has lower $\mu\varepsilon$ product.
⋆ $\theta_B^{\perp}$ doesn't exist when $\mu_1 = \mu_2$.

d) Brewster Angle since $|\Gamma| = 0$.

e) Critical Angle since $|\Gamma| = 1$ at this angle. Furthermore, $|\Gamma| = 1$ for the waves whose incoming angles are greater than B.

f) 0.5

$$\left|\frac{\eta_{TiO_2} - \eta_{Quartz}}{\eta_{TiO_2} + \eta_{Quartz}}\right| = \left|\frac{\eta_{Quartz} - \eta_{TiO_2}}{\eta_{Quartz} + \eta_{TiO_2}}\right|$$

g) 0.5

h) 0.5

Note: In answering parts (f), (g) and (h), note that under normal incidence (i.e. when $\theta_i = 0$), the expression for reflection coefficient is the same for both perpendicular and parallel polarizations.

i) From Figure 1:
@ $\theta_i = 0$:

$$0.5 = \left| \frac{\eta_{TiO_2} - \eta_{Quartz}}{\eta_{TiO_2} + \eta_{Quartz}} \right|$$

$$\eta_{TiO_2} = \frac{1}{3}\eta_{Quartz}$$
$$\sqrt{\frac{\mu_0}{\varepsilon_{TiO_2}}} = \frac{1}{3}\sqrt{\frac{\mu_0}{\varepsilon_{Quartz}}}$$
$$\sqrt{\frac{\varepsilon_{TiO_2}}{\varepsilon_{Quartz}}} = 3$$

$$\theta_B^{//} = tan^{-1}\sqrt{\frac{\varepsilon_{TiO_2}}{\varepsilon_{Quartz}}}$$
$$\theta_B^{//} \approx 71.56° = A$$
$$\theta_B^{//} \approx \frac{\pi}{2.52} = A$$

j)

$$\theta_c = sin^{-1}\sqrt{\frac{\varepsilon_{Quartz}}{\varepsilon_{TiO_2}}}$$
$$\theta_c \approx 19.47° = B$$
$$\theta_c \approx \frac{\pi}{9.24} = B$$

k)

$$\theta_B^{//} = tan^{-1}\sqrt{\frac{\varepsilon_{Quartz}}{\varepsilon_{TiO_2}}}$$
$$\theta_B^{//} \approx 18.43° = C$$
$$\theta_B^{//} \approx \frac{\pi}{9.76} = C$$

l)

$$\theta_c = sin^{-1}\sqrt{\frac{\varepsilon_{Quartz}}{\varepsilon_{TiO_2}}}$$
$$\theta_c \approx 19.47^\circ = B = D$$
$$\theta_c \approx \frac{\pi}{9.24} = B = D$$

m)

$$\frac{\lambda_{TiO_2}}{\lambda_{Quartz}} = \sqrt{\frac{\varepsilon_{Quartz}}{\varepsilon_{TiO_2}}}$$
$$\rightarrow \lambda_{Quartz} = \lambda_{TiO_2}\sqrt{\frac{\varepsilon_{TiO_2}}{\varepsilon_{Quartz}}}$$
$$\rightarrow \lambda_{Quartz} = 3\ mm$$

Question: 3

Brewster angle is a specific incidence angle, when no reflected wave occur.

- For // polarization,

$$\Gamma_{//} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \quad [1] \left.\right\} \text{ at Brewster angle } (\theta_i = \theta_B)\ \Gamma = 0 \text{ must be satisfied.}$$

$\downarrow$ eq. [1] reduces to

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_B \xrightarrow{\text{rearrange}} \sqrt{\varepsilon_1 \mu_2}\cos\theta_t = \sqrt{\varepsilon_2 \mu_1}\cos\theta_B \quad [2]$$

- Snell law;

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \quad , \quad \frac{\sin\theta_t}{\sin\theta_B} = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \quad [3]$$

Take square of [2] and [3]

$$\left.\begin{aligned} [2]^2 &\longrightarrow \cos^2\theta_t \cdot \varepsilon_1 \mu_2 = \cos^2\theta_B\ \varepsilon_2 \mu_1 \\ [3]^2 &\longrightarrow \sin^2\theta_t \cdot \varepsilon_2 \mu_2 = \sin^2\theta_B\ \varepsilon_1 \mu_1 \end{aligned}\right\} \text{multiply by } \varepsilon_2/\varepsilon_1 \text{ of both sides and sum two equation.}$$

$$\underbrace{(\sin^2\theta_t + \cos^2\theta_t)}_{1} \cdot \varepsilon_2 \mu_2 = \underbrace{\cos^2\theta_B}_{1-\sin^2\theta_B} \cdot \frac{\varepsilon_2^2 \cdot \mu_1}{\varepsilon_1} + \sin^2\theta_B\ \varepsilon_1 \mu_1$$

$$\varepsilon_2 \mu_2 = \frac{\varepsilon_2^2 \mu_1}{\varepsilon_1} + \sin^2\theta_B \left(\varepsilon_1 \mu_1 - \frac{\varepsilon_2^2 \cdot \mu_1}{\varepsilon_1} \right)$$

$$\sin^2\theta_B \cdot \frac{\mu_1}{\varepsilon_1}\left(\varepsilon_1^2 - \varepsilon_2^2\right) = \varepsilon_2\mu_2 - \frac{\varepsilon_2^2\mu_1}{\varepsilon_1}$$

$$\sin^2\theta_B \cdot \left(\varepsilon_2^2 - \varepsilon_1^2\right) = \varepsilon_2^2\left(\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}\right) \cdot \frac{\varepsilon_1}{\mu_1}$$

$$\sin^2\theta_B \cdot \frac{\left(\varepsilon_2^2 - \varepsilon_1^2\right)}{\varepsilon_2^2} = 1 - \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

$$\sin^2\theta_B \cdot \left[1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right] = 1 - \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

$$\sin^2\theta_B = \frac{1 - \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)}{1 - \left[\frac{\varepsilon_1}{\varepsilon_2}\right]^2}$$

$$\theta_B = \arcsin\left[\sqrt{\frac{1 - \left(\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)}{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}}\right]$$

for non magnetic case $\mu_1 = \mu_2 = \mu_0$

$$\theta_B = \arcsin\left(\sqrt{\frac{1 - \frac{\varepsilon_1}{\varepsilon_2}}{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}}\right) \longrightarrow \theta_B = \arcsin\left(\frac{1}{\sqrt{1 + \frac{\varepsilon_1}{\varepsilon_2}}}\right)$$