Scalar and Vector Potential Functions in Time-varying EM

Remember,

$$\bar{\nabla} \cdot \bar{B} = 0 \quad \text{(4}^{th} \text{ eqn. of M.E.s)}$$

$$\nabla \cdot (\bar{\nabla} \times \vec{U}) \equiv 0 \quad \text{(null identity)}$$

any vector field!

$$\Rightarrow \boxed{\bar{B} = \bar{\nabla} \times \bar{A}}$$

where $\bar{A}$ is the vector potential function.

Also,

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\frac{\partial (\bar{\nabla} \times \bar{A})}{\partial t} = -\bar{\nabla} \times \left(\frac{\partial \bar{A}}{\partial t}\right)$$

$$\Rightarrow \bar{\nabla} \times \bar{E} + \bar{\nabla} \times \frac{\partial \bar{A}}{\partial t} = 0$$

$$\Rightarrow \bar{\nabla} \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0$$

(null identity) $\bar{\nabla} \times (\bar{\nabla} \phi) \equiv 0$

any scalar field!

$$\Rightarrow \boxed{\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla V}$$

where V is the scalar potential function.

In other words, $\bar{E}$ and $\bar{B}$ fields can be computed from the scalar and vector potential fields as follows:

$$\boxed{\begin{aligned} \bar{B} &= \bar{\nabla} \times \bar{A} \\ \bar{E} &= -\bar{\nabla} V - \frac{\partial \bar{A}}{\partial t} \end{aligned}}$$

(Note that in static problems we have $\frac{\partial}{\partial t} \equiv 0 \Rightarrow \bar{E} = -\bar{\nabla} V$.)

Now, let's obtain the partial differential equations for the potential fields $\bar{A}$ and V in a simple medium:

$$\begin{pmatrix}3^{rd}\text{ law}\\ \text{of M.E.S}\end{pmatrix}\quad \left.\begin{array}{l}\bar{\nabla}\cdot\bar{D}=\rho_v\\ \bar{D}=\underbrace{\epsilon}_{\text{a constant}}\bar{E}\end{array}\right\}\quad \bar{\nabla}\cdot\underbrace{\bar{E}}_{(-\bar{\nabla}V-\frac{\partial\bar{A}}{\partial t})}=\frac{\rho_v}{\epsilon}\ \Rightarrow\ \boxed{\nabla^2 V+\frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A})=-\frac{\rho_v}{\epsilon}}\quad (*)$$

$$\begin{pmatrix}2^{nd}\text{ law}\\ \text{of M.E.s}\end{pmatrix}\quad \bar{\nabla}\times\underbrace{\bar{H}}_{\frac{1}{\mu}\bar{B}}=\bar{J}+\frac{\partial\overbrace{\bar{D}}^{\epsilon\bar{E}}}{\partial t}\ \Longrightarrow\ \frac{1}{\mu}\nabla\times\underbrace{\bar{B}}_{\bar{\nabla}\times\bar{A}}=\bar{J}+\epsilon\frac{\partial\overbrace{\bar{E}}^{-\bar{\nabla}V-\frac{\partial\bar{A}}{\partial t}}}{\partial t}$$

$$\Longrightarrow\ \underbrace{\bar{\nabla}\times(\bar{\nabla}\times\bar{A})}_{\bar{\nabla}(\bar{\nabla}\cdot\bar{A})-\nabla^2\bar{A}}=\mu\bar{J}-\mu\epsilon\underbrace{\frac{\partial}{\partial t}\left(\bar{\nabla}V+\frac{\partial\bar{A}}{\partial t}\right)}_{\bar{\nabla}\left(\frac{\partial V}{\partial t}\right)+\frac{\partial^2\bar{A}}{\partial t^2}}$$

$$\Longrightarrow\ \boxed{\nabla^2\bar{A}-\mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2}-\bar{\nabla}\left[\bar{\nabla}\cdot\bar{A}+\mu\epsilon\frac{\partial V}{\partial t}\right]=-\mu\bar{J}}\quad (**)$$

Partial differential equations (pde's) (*) and (**) are difficult to solve as each of them contains both V and $\bar{A}$!

To decouple V and $\bar{A}$, we may set the term

$$\boxed{\bar{\nabla}\cdot\bar{A}+\mu\epsilon\frac{\partial V}{\partial t}=0}$$ Lorentz Condition (Lorentz Gauge)

which turns out to be consistent with the rest of the EM theory. Remember, curl of $\bar{A}$ is specified by $\bar{B}=\bar{\nabla}\times\bar{A}$ eqn. Now, divergence of $\bar{A}$ is specified to be $\bar{\nabla}\cdot\bar{A}=-\mu\epsilon\frac{\partial V}{\partial t}$ by the Lorentz condition (remember the Helmholtz Theorem!)

Using Lorentz Condition in pde's ✱ and ✱✱, we will get:

from ✱ $\longrightarrow \nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\epsilon \frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\epsilon}$

$$\Rightarrow \boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon}}$$

and from ✱✱ $\longrightarrow \boxed{\nabla^2 \bar{A} - \mu\epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}}$

} nonhomogeneous wave equations for $V(\bar{r},t)$ and $\bar{A}(\bar{r},t)$.

Note that in static problems $\frac{\partial}{\partial t} \equiv 0$, so these pde's reduce to

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}}$$

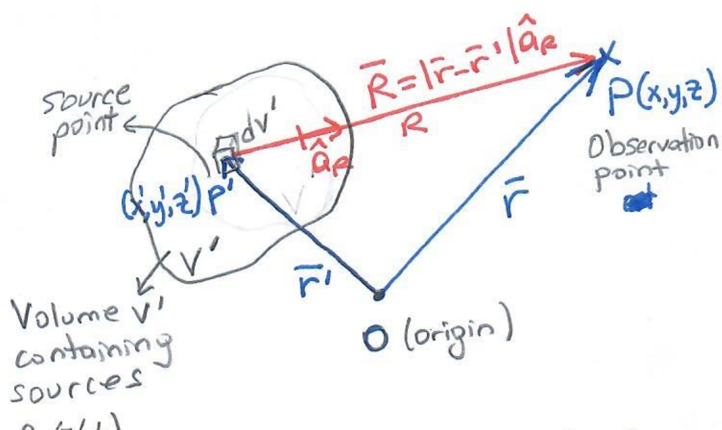
and

$$\boxed{\nabla^2 \bar{A} = -\mu \bar{J}}$$

} Poisson's equations with solutions $V(\bar{r})$ and $\bar{A}(\bar{r})$ $\Rightarrow$ (shown in EE 224 class)

$$\boxed{V(\bar{r}) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|}\,dv'}$$

$$\boxed{\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\int_{V'} \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\,dv'}$$



$\rho_v(\bar{r}',t)$

$\bar{J}(\bar{r}',t)$

$\bar{r}$: Position vector for P

$\bar{r}'$: Position vector for P'

$\bar{R} = \hat{a}_R R$ where $\begin{cases} R = |\bar{r}-\bar{r}'| = |\bar{R}| \\ \hat{a}_R = \frac{\bar{R}}{R} \end{cases}$

Time-varying potential function solutions can be shown to be "Retarded Scalar and Vector Potentials"

$$\boxed{V(\bar{r},t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho_v(\bar{r}', t-\frac{R}{u})}{R}\, dv'} \quad \text{(Volts)}$$

and

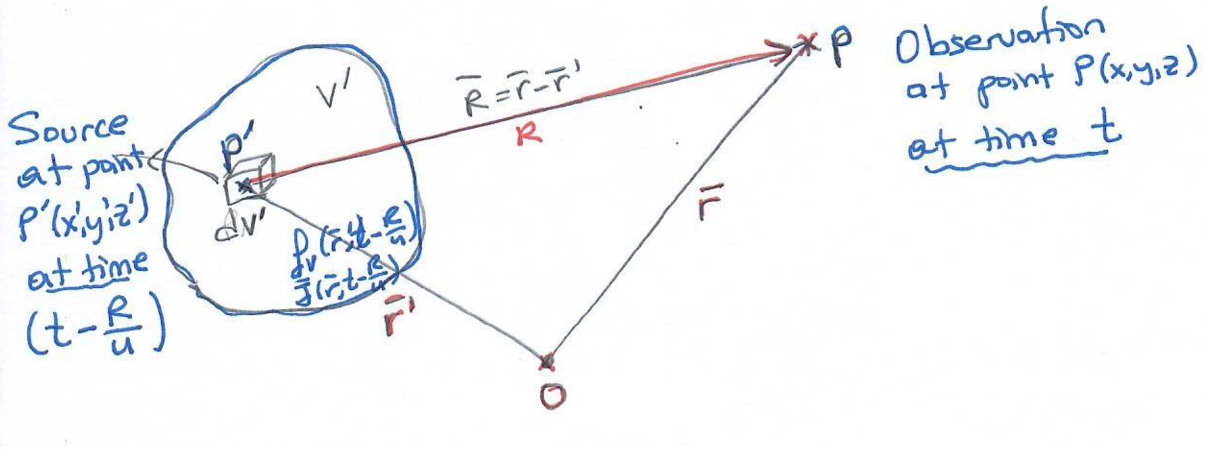
$$\boxed{\bar{A}(\bar{r},t) = \frac{\mu}{4\pi}\int_{V'} \frac{\bar{J}(\bar{r}', t-\frac{R}{u})}{R}\, dv'} \quad \text{(weber/m)}$$

which are in the same form as static $V(\bar{r})$ and $\bar{A}(\bar{r})$ solution except for the "time delay" term $\frac{R}{u}$ where

$u = \frac{1}{\sqrt{\mu\epsilon}} \triangleq \frac{dR}{dt}$ is the velocity of wave propagation in a simple lossless medium with parameters (ϵ, μ).

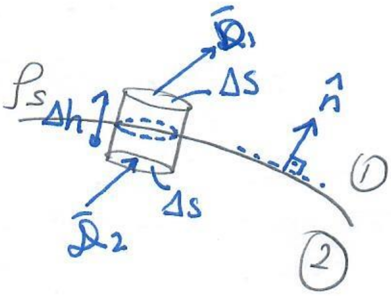
$R = |\bar{r} - \bar{r}'|$: Distance between the "source" and "observation" points.

$\frac{R}{u}$: Time duration needed for the propagating electromagnetic wave to travel from source point to observation point.



Boundary Conditions for Time-Varying Electromagnetic Fields

(i) B.C.'s for normal components:



$\hat{n}$: Unit normal vector at the boundary pointing from medium (2) to medium (1)

Start with $\bar{\nabla} \cdot \bar{D} = \rho_v \iff \oint_S \bar{D} \cdot d\bar{S} = Q_{\substack{enclosed \\ free \\ within\ S}}$

$$\lim_{\Delta h \to 0} \oint_S \bar{D} \cdot d\bar{S} = \underbrace{\hat{n} \cdot \bar{D}_1}_{D_{1n}} \cancel{\Delta S} + \underbrace{(-\hat{n}) \cdot \bar{D}_2}_{-D_{2n}} \cancel{\Delta S} = \rho_s \cancel{\Delta S}$$

ρ_s ↳ free surface charge density at the boundary

$\Rightarrow \boxed{D_{1n} - D_{2n} = \rho_s}$ or vectorially $\boxed{\hat{n} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s}$

i.e. D_{normal} is discontinuous at the boundary by the surface charge density ρ_s. ($D_{1n} = D_{2n}$ if $\rho_s = 0$ at the surface)

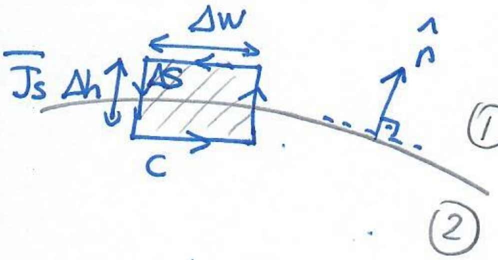
Similarly, starting with $\bar{\nabla} \cdot \bar{B} = 0 \iff \oint_S \bar{B} \cdot d\bar{S} = 0$

we can obtain

$\boxed{B_{1n} = B_{2n}}$ or vectorially $\boxed{\hat{n} \cdot (\bar{B}_1 - \bar{B}_2) = 0}$

B_{normal} is <u>always</u> continuous across a boundary.

(ii) B.C.'s for tangential components:



Start with

$$\bar{\nabla} \times \bar{\mathcal{H}} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$ express it in integral form

$$\oint_C \bar{\mathcal{H}} \cdot d\bar{l} = \int_{\Delta S} \bar{J} \cdot d\bar{S} + \int_{\Delta S} \frac{\partial \bar{D}}{\partial t} \cdot d\bar{S}$$

C : rectangular contour

$d\bar{l}$: tangent to C

ΔS : rectangular area enclosed by C.

$d\bar{S}$: normal to surface S

($d\bar{S}$ and $d\bar{l}$ are related by the Right Hand Rule)

as $\Delta h \to 0 \Rightarrow \Delta S \to 0$

as $\frac{\partial \bar{D}}{\partial t}$ term in the integrand remains finite (as physically expected)

$$\Rightarrow \lim_{\substack{\Delta h \to 0 \\ \Delta S \to 0}} \int_{\Delta S} \frac{\partial \bar{D}}{\partial t} \cdot d\bar{S} \to 0 \; !$$

$$\Rightarrow \lim_{\Delta h \to 0} \oint_C \bar{\mathcal{H}} \cdot d\bar{l} = \bar{\mathcal{H}}_1 \cdot \bar{\Delta w} + \bar{\mathcal{H}}_2 \cdot (-\bar{\Delta w}) = J_{sn} \Delta w + 0$$

direction of contours on the edges (ab) and (cd) are opposite to each other

Component of the surface current density vector $\bar{J}_s$, which is perpendicular to area ΔS. (Due to $\bar{J}_s \cdot d\bar{S}$ term)

$$\Rightarrow \underbrace{\bar{\mathcal{H}}_1 \cdot \hat{l}}_{\mathcal{H}_{1tan}} \cancel{\Delta w} + \underbrace{\bar{\mathcal{H}}_2 \cdot (-\hat{l})}_{-\mathcal{H}_{2tan}} \cancel{\Delta w} = J_{sn} \cancel{\Delta w}$$

$$\Rightarrow \boxed{\mathcal{H}_{1tan} - \mathcal{H}_{2tan} = J_{sn}} \quad (\text{Amp}/\text{m})$$

or vectorially $\boxed{\hat{n} \times (\bar{\mathcal{H}}_1 - \bar{\mathcal{H}}_2) = \bar{J}_s}$

i.e. the tangential component of $\bar{\mathcal{H}}$ field is discontinuous by the current density component J_{sn}.

(If $J_{sn} = 0 \Rightarrow \mathcal{H}_{1tan} = \mathcal{H}_{2tan}$)

Similarly, starting with $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ having the integral form $\oint_C \bar{E} \cdot d\bar{l} = -\int_{\Delta S} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$

and letting $\Delta h \to 0 \Rightarrow \Delta S \to 0 \Rightarrow \int_{\Delta S} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} \to 0$

for finite valued integrand,

we can obtain

$\boxed{E_{1_{tan}} = E_{2_{tan}}}$ or vectorially $\boxed{\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0}$

i.e., tangential component of $\bar{E}$ field is always continuous across the boundary.

Note: The B.C.'s obtained above have the same form as the B.C's of the static problems. However, only two of those four boundary conditions are independent.

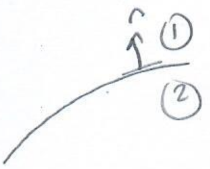
$\boxed{E_{1tan} = E_{2tan}}$ is equivalent to $\boxed{B_{1n} = B_{2n}}$

and $\boxed{\hat{a}_n \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s}$ is equivalent to $\boxed{\hat{n} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s}$

Boundary Conditions in Two Important Special Cases

Case I: Boundary between lossless ($\sigma_1 = \sigma_2 = 0$) linear media with parameters (ϵ_1, μ_1) and (ϵ_2, μ_2).

At such boundaries, $\rho_s = 0$, $\bar{J}_s = 0$ (free sources are zero!)



$$\Longrightarrow \quad E_{1tan} = E_{2tan} \Longrightarrow \left(\frac{D_{1tan}}{\epsilon_1} = \frac{D_{2tan}}{\epsilon_2} \right)$$

$$H_{1tan} = H_{2tan} \Longrightarrow \left(\frac{B_{1tan}}{\mu_1} = \frac{B_{2tan}}{\mu_2} \right)$$

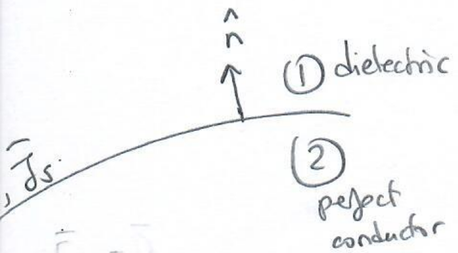
$$D_{1n} = D_{2n} \Longrightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

for linear media.

$$B_{1n} = B_{2n} \Longrightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Case II: Boundary between a lossless dielectric $(\mu_1, \epsilon_1, \sigma_1 = 0)$ and a perfect conductor $(\mu_2, \epsilon_2, \sigma_2 \to \infty)$

We know that all fields must be zero within a perfectly conducting medium. However, non-zero free sources ρ_s and $\bar{J}_s$ can be maintained at the boundary!



$\bar{E}_2 = \bar{D}_2 = \bar{B}_2 = \bar{H}_2 = 0$

On the dielectric side of the boundary	On the perfect conductor side of the boundary
$E_{1tan} = 0$	$E_{2tan} = 0$
$\hat{n} \times \bar{H}_1 = \bar{J}_s$	$H_{2tan} = 0$
$\hat{n} \cdot \bar{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$