EE303 HOMEWORK-4 SOLUTIONS

Solution of Problem-1:

$$\vec{E}(z,t) = \hat{a}_x E_o \exp(-\alpha z)\cos(\omega t - \beta z)$$

α: attenuation constant (Np/m)

β: phase constant (rad/m)

By looking at $z=0, t=0$ point, we can confirm that E_o is positive

- $|\vec{E}(z=0,t=0)| = |\hat{a}_x E_o| \approx 10$ V/m $\Rightarrow |E_o| = 10$ V/m $\boxed{E_o = 10 \text{ V/m}}$

$$|\vec{E}(z=0.157\text{m}, t=0)| = |\hat{a}_x E_o \exp(-0.157\alpha)\cos(-0.157\beta)| \approx 4.56 \text{ V/m}$$

Since we know that the amplitude of the electric field reaches a positive peak at $z = 0.157$m, $\cos(-0.157\beta) = 1$

$$0.157\beta = 2\pi \Rightarrow \boxed{\beta = \frac{2\pi}{0.157} \text{ rad/m} = 40.02 \text{ rad/m}} \approx 40 \text{ rad/m}$$

$$|\hat{a}_x E_o \exp(-0.157\alpha)\underbrace{\cos(-0.157\beta)}_{-1}| = |\hat{a}_x E_o \exp(-0.157\alpha)| \approx 4.56 \text{ V/m}$$

Then $|E_o \exp(-0.157\alpha)| \approx 4.56$ V/m $\Rightarrow \exp(-0.157\alpha) = \frac{4.56}{10} = 0.456$

$$\boxed{\alpha = \frac{-\ln(0.456)}{0.157} = 5 \text{ Np/m}}$$

$$\beta = \frac{2\pi}{\lambda} \Rightarrow \boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\frac{2\pi}{0.157}} = 0.157 \text{ m}}$$

- Low-Loss Dielectric Approximation:
  $\left(\frac{\sigma}{\omega\varepsilon} \ll 1\right)$

$\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ $\qquad \beta \cong k = \omega\sqrt{\mu\varepsilon}$

From the previous part we know that $\beta = \frac{2\pi}{0.157}$ rad/m

$\beta \cong \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu\varepsilon}$

It is given that the medium has vacuum permittivity and permeability $\Rightarrow$ $\varepsilon = \varepsilon_0$ $\quad \mu = \mu_0$

$\sqrt{\mu_0\varepsilon_0} = \frac{1}{c} = \frac{1}{3\times10^8}$ $\qquad \left(\varepsilon_0 = \frac{1}{36\pi}\times10^{-9}\ F/m,\ \mu_0 = 4\pi\times10^{-7}\ H/m\right)$

$\beta = \frac{2\pi}{0.157} = \frac{2\pi f}{3\times10^8}$ $\Rightarrow$ $\boxed{f = \frac{3\times10^8}{0.157} = 1.9108\ GHz}$

$\alpha = 5 \cong \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\sigma}{2}\,120\pi = 60\pi\sigma$ $\Rightarrow$ $\boxed{\sigma = \frac{1}{12\pi}\ \mho/m}$

$\boxed{\Rightarrow \sigma = 0.0265\ \mho/m}$

$$\boxed{\eta \approx \sqrt{\frac{\mu_0}{\varepsilon_0}}\left(1 + \frac{j\sigma}{2\omega\varepsilon_0}\right) = 120\pi\left(1 + \frac{j\,0.0265}{2(2\pi f\varepsilon_0)}\right) = 376.99 + j47.05\ \Omega}$$

$\sigma = 0.0265$ $\qquad \frac{\sigma}{\omega\varepsilon_0} = 0.2496$ which is smaller than 1 but not too much smaller than 1.

$\Rightarrow$ Low-loss dielectric approximation is <u>not reasonable</u> because the condition that $\frac{\sigma}{\omega\varepsilon_0} \ll 1$ <u>is not satisfied.</u>

• Without any approximation

General equations: $\alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$\alpha + j\beta = 5 + j40$ (found in the first part)

$$\Rightarrow \sqrt{j2\pi f\mu_0(\sigma + j2\pi f\varepsilon_0)} = 5 + j40$$

Taking square of both sides:

$$j2\pi f\mu_0\sigma - (2\pi f)^2\mu_0\varepsilon_0 = -1575 + j400$$

$$\Rightarrow 2\pi f\mu_0\sigma = 400 \quad \text{and} \quad (2\pi f)^2\mu_0\varepsilon_0 = 1575$$

$$\mu_0\varepsilon_0 = \frac{1}{c^2} = \frac{1}{(3\times10^8)^2} \quad \Rightarrow (2\pi f)^2 = 1575\times(3\times10^8)^2$$

$$\Rightarrow \boxed{f = \frac{3\times10^8}{2\pi}\sqrt{1575} = 1.895 \text{ GHz}}$$

$$2\pi f\mu_0\sigma = 400 \quad \Rightarrow \boxed{\sigma = \frac{400}{2\pi f\mu_0} = 0.0267 \text{ ℧/m}}$$

$$\eta = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\varepsilon_0}} = \sqrt{\frac{j2\pi(1.895\times10^9)(4\pi\times10^{-7})}{0.0267 + j2\pi(1.895\times10^9)\left(\frac{1}{36\pi}\times10^{-9}\right)}}$$

$$\Rightarrow \boxed{\eta = 368.3 + j45.98\ \Omega}$$

$$\frac{\sigma}{\omega\varepsilon_0} = 0.2536 \sim 1$$

Comparing them with the results found by low-loss approximation, we can say that low-loss approximation can be used for fast calculations. However for more accurate results the general form of equations should be used considering the discrepancies between two solution sets.

Solution of Problem-2:

$$\sigma = \sigma_0 \left(\frac{d_0}{d}\right)^{\frac{3}{2}} \quad \text{here } \sigma_0 = 0.05 \text{ S/m} \quad \text{and } d_0 = 0.1\text{ m}$$

When $d = d_0$ (no stretch or compression)

$$\sigma = \sigma_0 = 0.05$$

$f = 100$ MHz $\Rightarrow$ Loss tangent: $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{\sigma}{2\pi f \varepsilon_0} = \dfrac{0.05}{2\pi \times 10^8 \times \left(\frac{1}{36\pi} \times 10^{-9}\right)}$

$\varepsilon = \varepsilon_0 = \dfrac{1}{36\pi} \times 10^{-9}$ F/m

(vacuum)

$$\Rightarrow \frac{\sigma}{\omega\varepsilon} = 9 \rightarrow \text{when } d = d_0$$

Magnitude of E-field: $|\vec{E}(z)| = |E_0 e^{-\alpha z} e^{-j\beta z}|$

at $z = 0^+$ $|\vec{E}(z=0)| = 1$ V/m $\Rightarrow |E_0| = 1$

at $z = d^-$ $|\vec{E}(z=d^-)| = |e^{-\alpha d^-} e^{-j\beta d^-}|$

$|e^{-j\beta z}| = 1$ for all values of z.

$\Big(e^{-j\beta z} = \cos(-\beta z) + j\sin(-\beta z) = \cos\beta z - j\sin\beta z$

and $|\cos\beta z - j\sin\beta z| = \sqrt{(\cos^2\beta z + \sin^2\beta z)} = 1\Big)$

The reason why $|e^{-j\beta z}| = 1$

$$\Rightarrow |E(z=d^-)| = |e^{-\alpha d^-}|$$

• The first case: $d \ll d_0$

If $d \ll d_0$ $\quad \sigma = \sigma_0 \left(\frac{d_0}{d}\right)^{\frac{3}{2}} = 0.05 \left(\frac{d_0}{d}\right)^{\frac{3}{2}} \gg \omega\varepsilon = 0.0056$

$\Rightarrow \frac{\sigma}{\omega\varepsilon} \gg 1$ (Good conductor)

Using good conductor approximation;

$$\alpha \cong \beta \cong \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{(2\pi 10^8)(4\pi 10^{-7})\,\sigma}{2}} = 4.44 \left(\frac{d_0}{d}\right)^{\frac{3}{4}}$$

$d_0 = 0.1\,m \Rightarrow \alpha = \frac{0.7901}{d^{3/4}}$

$\Rightarrow \boxed{|\vec{E}(z=d^-)| = e^{-\alpha d^-} = e^{-0.7901 d^{1/4}}\ V/m}$ → the amplitude of E-field

$\boxed{\lim_{d \to 0} e^{-0.7901 d^{1/4}} = 1\ V/m}$ (as expected)

• The second case: $d \gg d_0$

For $d \gg d_0$ $\quad \sigma = \sigma_0 \left(\frac{d_0}{d}\right)^{\frac{3}{2}} = 0.05 \left(\frac{d_0}{d}\right)^{\frac{3}{2}} \ll \omega\varepsilon = 0.0056$ (very small number)

$\Rightarrow \frac{\sigma}{\omega\varepsilon} \ll 1$ (Good insulator)

Using good insulator approximation;

$$\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma_0}{2}\left(\frac{d_0}{d}\right)^{\frac{3}{2}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = 9.425\left(\frac{d_0}{d}\right)^{\frac{3}{2}}$$

$d_0 = 0.1\,m \Rightarrow \alpha = \frac{0.298}{d^{3/2}} \Rightarrow \boxed{|E(z=d^-)| = e^{-\alpha d^-} = e^{-\frac{0.298}{d^{1/2}}}\ V/m}$

the amplitude of E-field

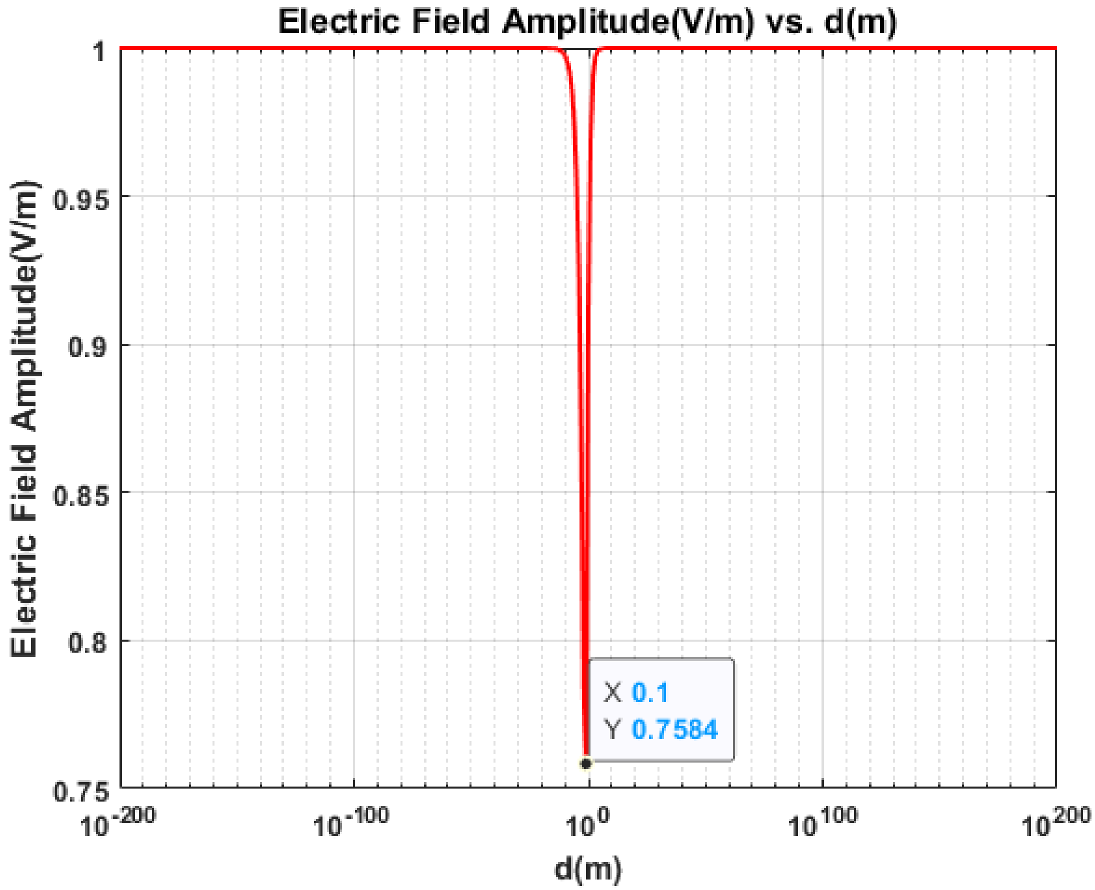
$$\boxed{\lim_{d\to\infty} e^{-\frac{0.298}{d^{1/2}}} = 1 \text{ V/m}}$$

```
close all
clear all
clc

mu=4*pi*1e-7; % H/m
epsilon=(1/(36*pi))*1e-9; % F/m
frequency=100*1e6; % Hz
omega=2*pi*frequency; % rad/s
sigma0=0.05;
d0=0.1; % m
d=logspace(-200,200,401); % m
% d is starting from 10^(-200) ending at 10^(200) which means
% approximately from 0 to infinity
sigma=sigma0*((d0./d).^(3/2)); % Siemens/m

alpha=real(sqrt(i*omega*mu*(sigma+i*omega*epsilon))); % Np/m, general eqn. of attenuation constant

E0=1; %V/m
Efield_amplitude=E0*exp(-alpha.*d); % V/m, The amplitude of E field
%Plotting
figure
s=semilogx(d,Efield_amplitude,'r','Linewidth',1.5);
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontWeight','bold','FontSize',10);
grid on
xlabel('d(m)')
ylabel('Electric Field Amplitude(V/m)','Fontweight','bold','Fontsize',12)
title('Electric Field Amplitude(V/m) vs. d(m)','Fontweight','bold','Fontsize',12)
ind_min=find(Efield_amplitude==min(Efield_amplitude)); % finding the indice of point where E field is minimum
datatip(s,d(ind_min),Efield_amplitude(ind_min))
```



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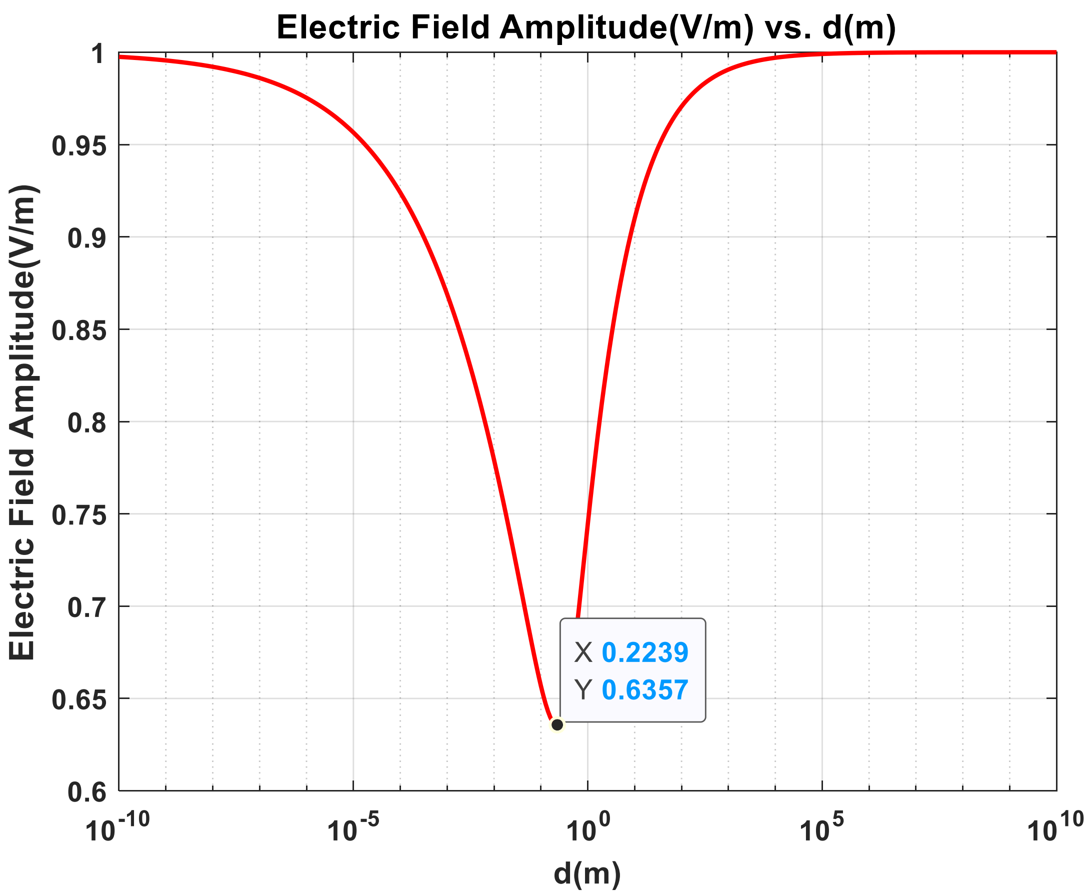
Note: z=d⁻ for this plot

• When $d \cong 0.1$, the minimum amplitude is reached according to the plot above. However, for a more accurate result we should plot in smaller steps and range of d which is given below. According to this plot, when $d \cong 0.22$, the amplitude of E-field is minimum.

$$\boxed{E(z=0.22) = 0.636 \text{ V/m}}$$



Note: $z=d^-$ for this plot

Solution of Problem-3

For both $\vec{E}_1(\vec{r},t) = \hat{a}_y E_0 \cos(\omega t - kx)$ and $\vec{E}_2(\vec{r},t) = \hat{a}_y E_0 \cos(\omega t + kx)$ since $\alpha = 0$ we can say that the medium is lossless.

$\Rightarrow \sigma = 0 \Rightarrow \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ which is a real number

- $\vec{E}_1(\vec{r},t) = \hat{a}_y E_0 \cos(\omega t - kx)$ V/m

$\Rightarrow \vec{H}_1(\vec{r},t) = \hat{a}_z \frac{E_0}{\eta} \cos(\omega t - kx)$ A/m

$\vec{P} = \vec{E} \times \vec{H}^*$ $\quad \vec{P}$: Complex Poynting's vector

$$\boxed{\vec{P}_1 = \left(\hat{a}_y E_0 e^{-jkx}\right) \times \left(\hat{a}_z \frac{E_0}{\eta} e^{+jkx}\right) = \hat{a}_x \frac{E_0^2}{\eta} \text{ Watt/m}^2}$$

$$\boxed{= \hat{a}_x \frac{E_0^2}{\sqrt{\frac{\mu}{\varepsilon}}} \text{ Watt/m}^2}$$

- $\vec{E}_2(\vec{r},t) = \hat{a}_y E_0 \cos(\omega t + kx)$ V/m

$\Rightarrow \vec{H}_2(\vec{r},t) = -\hat{a}_z \frac{E_0}{\eta} \cos(\omega t + kx)$ A/m

$$\Rightarrow \boxed{\vec{P}_2 = \vec{E} \times \vec{H}^* = -\hat{a}_x \frac{E_0^2}{\sqrt{\frac{\mu}{\varepsilon}}} \text{ Watt/m}^2}$$

$$\bullet\ \vec{E}_3(\vec{r},t) = \vec{E}_1(r,t) + \vec{E}_2(r,t)$$

$$\Rightarrow \vec{E}_3(\vec{r},t) = \hat{a}_y E_0 \left(\cos(\omega t - kx) + \cos(\omega t + kx)\right) \quad V/m$$

$$\vec{H}_3(\vec{r},t) = \hat{a}_z \frac{E_0}{2} \left(\cos(\omega t - kx - \cos(\omega t + kx)\right) \quad A/m$$

$$\vec{P}_3 = \vec{E}_3 \times H_3^* = \hat{a}_x \frac{E_0^2}{2} \left(e^{-jkx} + e^{jkx}\right)\left(e^{jkx} - e^{-jkx}\right)$$

$$\Rightarrow \boxed{\vec{P}_3 = \hat{a}_x \frac{E_0^2}{\sqrt{\frac{\mu}{\varepsilon}}} \left(e^{j2kx} - e^{-j2kx}\right) \quad Watt/m^2}$$

$$\left(e^{j2kx} - e^{-j2kx} = j2\sin 2kx\right)$$

$$\Rightarrow \boxed{\vec{P}_3 = \hat{a}_x \left(j2\sin(2kx)\right) \frac{E_0^2}{\sqrt{\frac{\mu}{\varepsilon}}} \quad Watt/m^2}$$