2020-2021, EE303 – Recitation 08

Question 1

Consider a planar boundary at $z = 0$ that separates two lossless media defined with parameters $(\epsilon_1, \mu_1) = (4\epsilon_0, \mu_0)$ for $z < 0$ and $(\epsilon_2, \mu_2) = (2\epsilon_0, 3\mu_0)$ for $z > 0$. An incident electromagnetic wave exists in the first medium ($z < 0$) with electric field intensity given (in phasor domain) as

$$\bar{E}^{\text{inc}}(\bar{r}) = \hat{a}_y 2E_0 \cos[k_1(\sqrt{3}/2)x] \exp(-jk_1 z/2), \tag{1}$$

where $k_1 = \omega\sqrt{\mu_1\epsilon_1}$ and E_0 is a real constant. Find the expression for the transmitted wave in the second medium ($z > 0$).

Solution

The incident wave is a superposition of two UPWs:

$$\begin{aligned}
\bar{E}^{\text{inc}}(\bar{r}) &= \hat{a}_y E_0 \exp[-jk_1(\sqrt{3}/2)x] \exp(-jk_1 z/2) + \hat{a}_y E_0 \exp[jk_1(\sqrt{3}/2)x] \exp(-jk_1 z/2) && (2)\\
&= \hat{a}_y E_0 \exp[-jk_1(x\sqrt{3}/2 + z/2)] + \hat{a}_y E_0 \exp[-jk_1(-x\sqrt{3}/2 + z/2)] && (3)\\
&= \bar{E}_1^{\text{inc}}(\bar{r}) + \bar{E}_2^{\text{inc}}(\bar{r}) && (4)
\end{aligned}$$

For $\varphi_i = 60°$:

$$k_1 \sin\varphi_i = k_2 \sin\varphi_t \longrightarrow \sin\varphi_t = \sqrt{\frac{4\mu_0\epsilon_0}{6\mu_0\epsilon_0}}\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}} \longrightarrow \varphi_t = 45° \tag{5}$$

$$\tau_\perp = \frac{2\eta_2\cos\varphi_i}{\eta_2\cos\varphi_i + \eta_1\cos\varphi_t} = \frac{2\dfrac{\sqrt{3}}{\sqrt{2}}\dfrac{1}{2}}{\dfrac{\sqrt{3}}{\sqrt{2}}\dfrac{1}{2} + \dfrac{1}{2}\dfrac{\sqrt{2}}{2}} = \frac{\sqrt{3}}{1+\sqrt{3}} \approx 0.634 \tag{6}$$

Then, we obtain:

$$\begin{aligned}
\bar{E}_1^{\text{tra}}(\bar{r}) &\approx \hat{a}_y 0.634 E_0 \exp[-jk_2(\sqrt{2}/2)(x+z)] && (7)\\
\bar{E}_2^{\text{tra}}(\bar{r}) &\approx \hat{a}_y 0.634 E_0 \exp[-jk_2(\sqrt{2}/2)(-x+z)] && (8)\\
\bar{E}^{\text{tra}}(\bar{r}) &\approx \hat{a}_y 1.268 E_0 \cos[k_2(\sqrt{2}/2)x] \exp[-jk_2(\sqrt{2}/2)z] && (9)
\end{aligned}$$

Question 2

Consider a planar boundary at $z = 0$ that separates two lossless media. The first medium ($z < 0$) has fixed parameters $(\epsilon_1, \mu_1) = (6\epsilon_0, 5\mu_0)$. The second medium ($z > 0$) is nonmagnetic ($\mu_2 = \mu_0$); however, its permittivity changes with respect to frequency as

$$\epsilon_2 = 12\epsilon_0 - \frac{2\omega}{\omega_0}\epsilon_0, \tag{10}$$

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where ω_0 is a constant angular frequency. Two different electromagnetic waves are individually incident on the boundary from the first region. Electric field intensity expressions are given (in phasor domain) as

$$\bar{E}_1^{\mathrm{inc}}(\bar{r}) = \hat{a}_y E_0 \exp\left[-jk_0\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)\right] \tag{11}$$

$$\bar{E}_2^{\mathrm{inc}}(\bar{r}) = \hat{a}_y E_0 \exp\left[-j3k_0\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)\right], \tag{12}$$

where $k_0 = \omega_0\sqrt{\mu_1\epsilon_1}$ and E_0 is a real constant. Find the expressions for the reflected waves in the first medium ($z < 0$).

Solution
For both UPWs, $\varphi_i = 30°$, however their angular frequencies are ω_0 and $3\omega_0$, respectively. At these frequencies, the permittivity of the second medium becomes $\epsilon_2 = 10\epsilon_0$ and $\epsilon_2 = 6\epsilon_0$.

For the first UPW:

$$k_1 \sin\varphi_i = k_2 \sin\varphi_t \longrightarrow \omega_0\sqrt{30\mu_0\epsilon_0}\frac{1}{2} = \omega_0\sqrt{10\mu_0\epsilon_0}\sin\varphi_t \longrightarrow \sin\varphi_t = \frac{\sqrt{3}}{2} \tag{13}$$

$$\Gamma_\perp = \frac{\eta_2\cos\varphi_i - \eta_1\cos\varphi_t}{\eta_2\cos\varphi_i + \eta_1\cos\varphi_t} = \frac{\frac{1}{\sqrt{10}}\frac{\sqrt{3}}{2} - \frac{\sqrt{5}}{\sqrt{6}}\frac{1}{2}}{\frac{1}{\sqrt{10}}\frac{\sqrt{3}}{2} + \frac{\sqrt{5}}{\sqrt{6}}\frac{1}{2}} = -0.25 \tag{14}$$

$$\bar{E}_1^{\mathrm{ref}}(\bar{r}) = -\hat{a}_y 0.25 E_0 \exp\left[-jk_0\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)\right] \tag{15}$$

For the second UPW:

$$k_1 \sin\varphi_i = k_2 \sin\varphi_t \longrightarrow 3\omega_0\sqrt{30\mu_0\epsilon_0}\frac{1}{2} = 3\omega_0\sqrt{6\mu_0\epsilon_0}\sin\varphi_t \longrightarrow \sin\varphi_t = \frac{\sqrt{5}}{2} \tag{16}$$

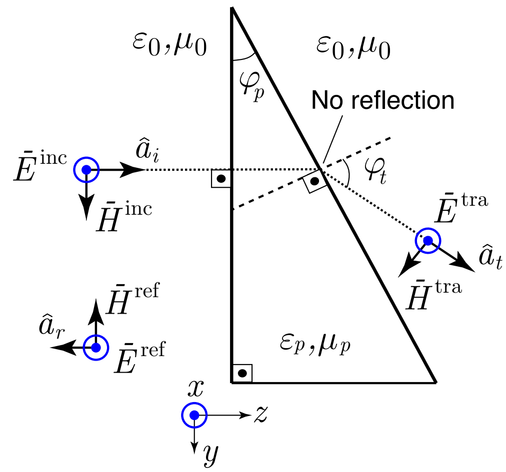
$$\longrightarrow \cos\varphi_t = -j/2 \qquad (\varphi_i \text{ is larger than the critical angle}) \tag{17}$$

$$\Gamma_\perp = \frac{\eta_2\cos\varphi_i - \eta_1\cos\varphi_t}{\eta_2\cos\varphi_i + \eta_1\cos\varphi_t} = \frac{\frac{1}{\sqrt{6}}\frac{\sqrt{3}}{2} + j\frac{\sqrt{5}}{\sqrt{6}}\frac{1}{2}}{\frac{1}{\sqrt{6}}\frac{\sqrt{3}}{2} - j\frac{\sqrt{5}}{\sqrt{6}}\frac{1}{2}} \approx -0.25 + 0.97j \tag{18}$$

$$\bar{E}_2^{\mathrm{ref}}(\bar{r}) \approx \hat{a}_y(-0.25 + 0.97j)E_0 \exp\left[-j3k_0\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}z\right)\right] \tag{19}$$

Question 3

Consider a normal incidence of a plane wave on a triangular right prism with permittivity ϵ_p and permeability μ_p located in vacuum (ϵ_0, μ_0). Assume that all dimensions of the prism are very large in comparison to the wavelength and there is no any edge effect. In addition, there



is no reflection when the plane wave is passing from the prism side to vacuum. The angle of the prism is $\varphi_p = \sin^{-1}(1/8)$. Considering the directions defined in the figure (i.e., both incident and reflected electric field intensities are defined in the x direction), it is given than $E^{\text{ref}} = \Gamma E^{\text{inc}} = E^{\text{inc}}/5$. Find the relative permittivity and the relative permeability of the prism, as well as the angle φ_t, i.e., the angle between the transmitted wave and the normal of the prism.

Solution

$$\sin\varphi_p = \sin\varphi_{B\perp} = \sqrt{\frac{1-\dfrac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}{1-\left(\dfrac{\mu_1}{\mu_2}\right)^2}} = \sqrt{\frac{1-\dfrac{\mu_p\epsilon_0}{\mu_0\epsilon_p}}{1-\left(\dfrac{\mu_p}{\mu_0}\right)^2}} = \sqrt{\frac{1-\dfrac{\mu_{pr}}{\epsilon_{pr}}}{1-(\mu_{pr})^2}}, \tag{20}$$

where $\epsilon_{pr} = \epsilon_p/\epsilon_0$ and $\mu_{pr} = \mu_p/\mu_0$. Since $\varphi_p = \sin^{-1}(1/8)$, we obtain

$$\frac{1}{8} = \sqrt{\frac{1-\dfrac{\mu_{pr}}{\epsilon_{pr}}}{1-(\mu_{pr})^2}} \longrightarrow 1-(\mu_{pr})^2 = 64 - 64\frac{\mu_{pr}}{\epsilon_{pr}}. \tag{21}$$

To derive another relationship between ϵ_{pr} and μ_{pr}, we can consider the boundary from vacuum to the prism. Given that $\Gamma = 1/5$, we have

$$\Gamma = \frac{1}{5} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\eta_p-\eta_0}{\eta_p+\eta_0} \longrightarrow 4\eta_p = 6\eta_0 \longrightarrow 4\eta_p^2 = 9\eta_0^2, \tag{22}$$

leading to $\mu_{pr}/\epsilon_{pr} = 9/4$. Inserting this relationship into (21), we obtain

$$\epsilon_{pr} = 4, \qquad \mu_{pr} = 9. \tag{23}$$

The transmission angle can be found via Snell's law as

$$\sqrt{\mu_p\epsilon_p}\sin\varphi_p = \sqrt{\mu_0\epsilon_0}\sin\varphi_t \longrightarrow \sin\varphi_t = \sqrt{\mu_{pr}\epsilon_{pr}}\sin\varphi_p = 6\times\frac{1}{8} = \frac{3}{4} \tag{24}$$

$$\longrightarrow \varphi_t = \sin^{-1}(3/4). \tag{25}$$