EE303 HW #5 Solutions

Problem 1

- Given that $\bar{E} = \bar{E}_o e^{-jk\hat{n}.\bar{r}}$ (general form of the $\bar{E}$ field phasor for u.p.w.)
- Medium: unbounded, lossless, source-free, simple, with ϵ & μ

a) Time-average Poynting's vector:

$$\bar{P}_{av}(\bar{r}) \triangleq \frac{1}{2} Re\{\bar{E} \times \bar{H}^*\} \qquad \left[\text{lossless medium} \Rightarrow \eta : \text{real}\right]$$

$$U.P.W \Rightarrow \quad \bar{E} \cdot \hat{n} = 0 \quad \& \quad \bar{H} = \frac{1}{\eta} \hat{n} \times \bar{E}$$

$$\bar{P}_{av}(\bar{r}) = \frac{1}{2} Re\left\{[\bar{E}_o e^{-jk\hat{n}.\bar{r}}] \times \left[\frac{1}{\eta} \hat{n} \times [\bar{E}_o^* e^{+jk\hat{n}.\bar{r}}]\right]\right\}$$

$$\left[\text{Remember: } \bar{A} \times \bar{B} \times \bar{C} = \bar{B}(\bar{A}.\bar{C}) - \bar{C}(\bar{A}.\bar{B})\right]$$

$$\bar{P}_{av}(\bar{r}) = \frac{1}{2} Re\left\{\frac{1}{\eta} \hat{n}\left([\bar{E}_o e^{-jk\hat{n}.\bar{r}}].[\bar{E}_o^* e^{+jk\hat{n}.\bar{r}}]\right) - [\bar{E}_o^* e^{+jk\hat{n}.\bar{r}}] \underbrace{\left([\bar{E}_o e^{-jk\hat{n}.\bar{r}}] \cdot \frac{1}{\eta}\hat{n}\right)}_{=0 \; (\bar{E}\cdot\hat{n}=0)}\right\}$$

$$\bar{P}_{av}(\bar{r}) = \frac{1}{2} Re\left\{\frac{1}{\eta} \hat{n} |\bar{E}_o|^2\right\}$$

$$\bar{P}_{av}(\bar{r}) = \hat{n} \frac{1}{2\eta} |\bar{E}_o|^2$$

b)
- medium: $\epsilon_o, \mu_o \Rightarrow k = w\sqrt{\epsilon_o \mu_o}$ & $\eta = \sqrt{\frac{\mu_o}{\epsilon_o}} = 120\pi\ \Omega$
- $f = 300\,MHz \Rightarrow w = 2\pi f = 6\pi \times 10^8\ rad/sec$ & $k = 2\pi$
- RHCP or LHCP
- $\bar{P}_{av} = \frac{4\pi}{30} \hat{a}_z\ W/m^2 \Rightarrow \hat{n} = \hat{a}_z$ & $\frac{|\bar{E}_o|^2}{2\eta} = \frac{4\pi}{30}$

$$|\bar{E}_o|^2 = \frac{4\pi}{30} 2\eta = \frac{4\pi}{30} \times 2 \times 120\pi = 32\pi^2 \Rightarrow |\bar{E}_o| = 4\sqrt{2}\pi$$

$$\bar{E}_o = 4\sqrt{2}\pi\left[\frac{\hat{a}_x}{\sqrt{2}} \mp j\frac{\hat{a}_y}{\sqrt{2}}\right]$$

$-$ → RHCP, $+$ → LHCP

$$\therefore \quad \bar{E} = 4\sqrt{2}\pi\left(\frac{\hat{a}_x}{\sqrt{2}} \mp j\frac{\hat{a}_y}{\sqrt{2}}\right) e^{-j2\pi z} \quad V/m$$

Problem 2

- $\bar{P}_{av}(\bar{r}) = \hat{n}\frac{1}{2\eta}|E_0|^2$ for u.p.w.

- medium: lossless (η: real) & simple

a) $\bar{E}_1 = (\sqrt{3}\,\hat{a}_x - \hat{a}_z)e^{-jk\left(\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$ V/m

$\Rightarrow |E_1|^2 = 4$, $\hat{n}_1 = \left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right)$

$$\bar{P}_{av,1} = \left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right)\frac{2}{\eta} \quad \text{W/m}^2$$

b) $\bar{E}_2 = (-\sqrt{3}\,\hat{a}_x - \hat{a}_z)e^{-jk\left(-\frac{1}{2}x + \frac{\sqrt{3}}{2}z\right)}$ V/m

$\Rightarrow |\bar{E}_2|^2 = 4$, $\hat{n}_2 = \left(-\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right)$

$$\bar{P}_{av,2} = \left(-\frac{1}{2}\hat{a}_x + \frac{\sqrt{3}}{2}\hat{a}_z\right)\frac{2}{\eta} \quad \text{W/m}^2$$

c) $\bar{E}_3 = \bar{E}_1 + \bar{E}_2 = e^{-jk\frac{\sqrt{3}}{2}z}\left[(\sqrt{3}\,\hat{a}_x - \hat{a}_z)e^{-jk\frac{1}{2}x} + (-\sqrt{3}\,\hat{a}_x - \hat{a}_z)e^{+jk\frac{1}{2}x}\right]$

$$= e^{-jk\frac{\sqrt{3}}{2}z}\left[\sqrt{3}\,\hat{a}_x\left(e^{-jk\frac{1}{2}x} - e^{jk\frac{1}{2}x}\right) - \hat{a}_z\left(e^{-jk\frac{1}{2}x} + e^{jk\frac{1}{2}x}\right)\right]$$

$$= e^{-jk\frac{\sqrt{3}}{2}z}\left[\hat{a}_x\sqrt{3}\left(-2j\sin\left(\frac{kx}{2}\right)\right) - \hat{a}_z\left(2\cos\left(\frac{kx}{2}\right)\right)\right]$$

$$\bar{E}_3 = \left[-\hat{a}_x\, j\, 2\sqrt{3}\sin\left(\frac{kx}{2}\right) - \hat{a}_z\, 2\cos\left(\frac{kx}{2}\right)\right]e^{-jk\frac{\sqrt{3}}{2}z} \quad \text{V/m}$$

$$\bar{\nabla}\times\bar{E}_3 = -\hat{a}_y\left[+k\sin\left(\frac{kx}{2}\right)e^{-jk\frac{\sqrt{3}}{2}z} + 3k\sin\left(\frac{kx}{2}\right)e^{-jk\frac{\sqrt{3}}{2}z}\right]$$

$$= -\hat{a}_y\, 4k\sin\left(\frac{kx}{2}\right)e^{-jk\frac{\sqrt{3}}{2}z}$$

$$\Rightarrow \bar{H}_3 = \frac{\bar{\nabla}\times\bar{E}_3}{-j\omega\mu} = -\hat{a}_y\, j\frac{4}{\eta}\sin\left(\frac{kx}{2}\right)e^{-jk\frac{\sqrt{3}}{2}z} \quad \text{A/m}$$

$$\Rightarrow \bar{P}_{av,3} = \frac{1}{2}\mathrm{Re}\{\bar{E}_3\times\bar{H}_3^*\} = \frac{1}{2}\mathrm{Re}\left\{\hat{a}_z\frac{8\sqrt{3}}{\eta}\sin^2\left(\frac{kx}{2}\right) + \hat{a}_x j\frac{4}{\eta}\sin(kx)\right\}$$

$$\bar{P}_{av,3} = \hat{a}_z\frac{4\sqrt{3}}{\eta}\sin^2\left(\frac{kx}{2}\right) \quad \text{W/m}^2$$

d) $\bar{P}_{av,1} + \bar{P}_{av,2} = \hat{a}_z \dfrac{2\sqrt{3}}{\eta}$ W/m^2

$\Rightarrow \bar{P}_{av,1} + \bar{P}_{av,2} \neq \bar{P}_{av,3}$

$\bar{P}_{av,3} = \hat{a}_z \dfrac{4\sqrt{3}}{\eta} \sin^2\left(\dfrac{kx}{2}\right)$ W/m^2

Superposition is valid for the linear systems (i.e. fields) However, power is by definition non-linear($\bar{E} \times \bar{H}$).

Therefore, the time-averaged-Poynting's vector must be calculated from total field.

Note:
$$\bar{P}_{av,3} = \frac{1}{2} Re\{\bar{E}_3 \times \bar{H}_3^*\} = \frac{1}{2} Re\{(\bar{E}_1 + \bar{E}_2) \times (\bar{H}_1 + \bar{H}_2)^*\}$$

$$= \underbrace{\frac{1}{2} Re\{\bar{E}_1 \times \bar{H}_1^*\}}_{\bar{P}_{av,1}} + \underbrace{\frac{1}{2} Re\{\bar{E}_2 \times \bar{H}_2^*\}}_{\bar{P}_{av,2}} + \underbrace{\frac{1}{2} Re\{\bar{E}_1 \times \bar{H}_2^* + \bar{E}_2 \times \bar{H}_1^*\}}_{\text{cross-terms}}$$

$\therefore$ Due to nonzero cross-terms (resulting from the nonlinear nature of power computations), $\bar{P}_{av,3} \neq \bar{P}_{av,1} + \bar{P}_{av,2}$

$$\bar{P}_{av,3} = \hat{a}_z \frac{4\sqrt{3}}{\eta} \sin^2\left(\frac{kx}{2}\right) \quad \text{W/m}^2$$

$$= \hat{a}_z \frac{4\sqrt{3}}{\eta} \left[\frac{1-\cos(kx)}{2}\right] \quad \text{W/m}^2$$

$$= \underbrace{\hat{a}_z \frac{2\sqrt{3}}{\eta}}_{\bar{P}_{av,1} + \bar{P}_{av,2}} - \underbrace{\hat{a}_z \frac{2\sqrt{3}}{\eta} \cos(kx)}_{\text{cross terms}} \quad \text{W/m}^2$$

Problem 3

$$\bar{E}(z,t) = \hat{a}_x \; 10 \sin(kz) \sin(wt) \quad V/m$$

$$\bar{H}(z,t) = \hat{a}_y \; 0.1 \cos(kz) \cos(wt) \quad A/m$$

a) $\bar{P}(\bar{r},t) \triangleq \bar{E}(\bar{r},t) \times \bar{H}(\bar{r},t)$: Instantaneous Poynting's vector

$$\bar{P}(z,t) = \hat{a}_z \; \frac{\sin(2kz)\sin(2wt)}{4} \quad W/m^2$$

b) $\bar{P}_{av}(z) \triangleq \frac{1}{T} \int_{t=0}^{t=T} \bar{P}(z,t)\,dt$ where $T = \frac{1}{f} = \frac{2\pi}{w}$

$$\bar{P}_{av}(z) = \frac{w}{2\pi} \int_{t=0}^{\frac{2\pi}{w}} \hat{a}_z \frac{\sin(2kz)\sin(2wt)}{4}\,dt$$

$$= \hat{a}_z \frac{w}{2\pi} \frac{\sin(2kz)}{4} \left[-\frac{\cos(2wt)}{2w} \Bigg|_{t=0}^{t=\frac{2\pi}{w}} \right]$$

$$= \hat{a}_z \frac{w}{2\pi} \frac{\sin(2kz)}{4} \left[0 - \cancel{0} \right] = 0$$

$$\bar{P}_{av}(z) = 0$$

$\therefore$ There is no net energy transfer.

c) It is a standing wave, and it does not propagate. Waves that are travelling in opposite sides form standing wave by cancelling each other (with same amplitude & frequency). Standing waves fluctuates in amplitude without propagating.