Plane Electromagnetic Waves

Plane Waves in Lossless, Source-free, Simple Media:

In such a medium we have:

$$\nabla^2 \bar{E} - \epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \quad \longleftrightarrow \quad \nabla^2 \bar{E} + k^2 \bar{E} = 0 \quad \text{etc.}$$

$$\left(\text{where } k = \omega\sqrt{\mu\epsilon} \quad \text{and} \quad \omega = 2\pi f\right)$$

Let's solve the phasor field $\bar{E}(\bar{r})$ in phasor domain by solving the "homogeneous Helmholtz eqn." for a special case $\boxed{\bar{E} = \hat{a}_x E_x(z)}$ for simplicity.

$$\left.\begin{array}{l} \nabla^2 \bar{E} + k^2 \bar{E} = 0 \\ \bar{E} = \hat{a}_x E_x(z) \end{array}\right\} \quad \cancel{\frac{\partial^2 \hat{a}_x E_x(z)}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 \hat{a}_x E_x(z)}{\partial y^2}}^{\,0} + \underbrace{\frac{\partial^2 \hat{a}_x E_x(z)}{\partial z^2}}_{\hat{a}_x \frac{\partial^2 E_x(z)}{\partial z^2}} + k^2 \hat{a}_x E_x(z) = 0$$

$$\Rightarrow \boxed{\frac{d^2 E_x(z)}{dz^2} + k^2 E_x(z) = 0} \quad \text{solve for } E_x(z)\,!$$

Characteristic eqn: $s^2 + k^2 \underbrace{s^0}_{1} = 0 \Rightarrow s^2 = -k^2$

$$\Rightarrow s_{1,2} = \mp\sqrt{-k^2} = \mp jk$$

$$\Rightarrow \boxed{E_x(z) = A\, e^{-jkz} + B\, e^{+jkz}}$$

is the general soln.
A, B: some arbitrary constants.

(in phasor domain)

Find the corresponding time-domain solution from:

$$\mathcal{E}_x(z,t) = Re\left\{ E_x(z)\, e^{j\omega t} \right\}$$

$$= Re\left\{ \left[A\, e^{-jkz} + B\, e^{+jkz} \right] e^{j\omega t} \right\}$$

$$= Re\left\{ A\, e^{j(\omega t - kz)} + B\, e^{j(\omega t + kz)} \right\}$$

$$\boxed{\mathcal{E}_x(z,t) = A\cos(\omega t - kz) + B\cos(\omega t + kz)}$$

in time domain (assuming A, B real valued for simplicity)

In vector form:

$$\bar{\mathcal{E}}(z,t) = \hat{a}_x \left[\underbrace{A\cos(\omega t - kz)}_{\substack{\text{traveling wave} \\ \text{in } (+z) \text{ direction}}} + \underbrace{B\cos(\omega t + kz)}_{\substack{\text{traveling wave} \\ \text{in } (-z) \text{ direction}}} \right]$$

where the arguments of the solutions can be written as:

$$\omega t - \underbrace{kz}_{\omega\sqrt{\mu\epsilon} = \frac{\omega}{v}} = \omega t - \frac{\omega}{v} z = \omega\left(t - \frac{z}{v}\right) = -\frac{\omega}{v}(z - vt)$$

$$= -k(z - vt)$$

or

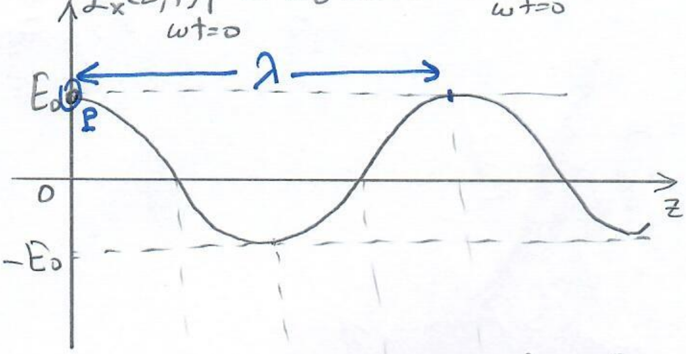
$$\omega t + kz = \omega\left(t + \frac{z}{v}\right) = \underbrace{\frac{\omega}{v}}_{k}(z + vt) = k(z + vt)$$

Consider the "forward traveling wave" in $(+z)$ direction:

$$\mathcal{E}_x(z,t)\Big|_{\omega t=0} = E_0\cos(\omega t - kz)\Big|_{\omega t=0} = E_0\cos(kz)$$

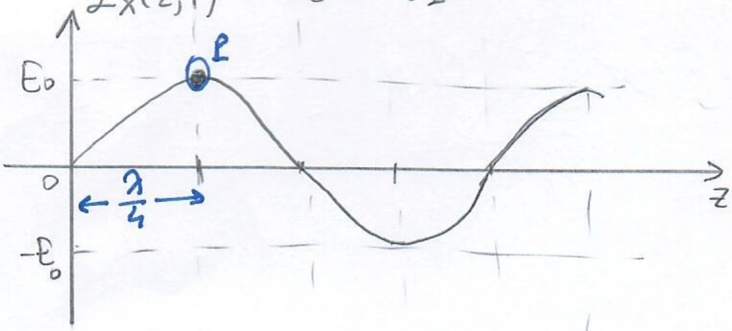
($E_0 = A$: assume a real constant)

at $\omega t = 0$



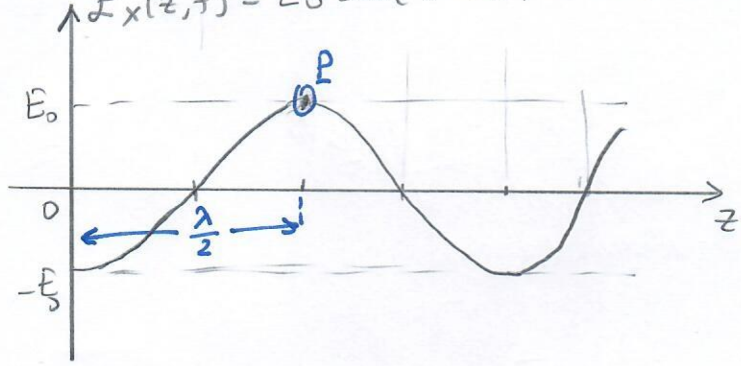
$$\mathcal{E}_x(z,t) = E_0\cos\left(\frac{\pi}{2} - kz\right) = E_0\sin(kz)$$

at $\omega t = \frac{\pi}{2}$



$$\mathcal{E}_x(z,t) = E_0\cos(\pi - kz) = -E_0\cos(kz)$$

at $\omega t = \pi$



P.S. when $\omega t = 2\pi$, $\mathcal{E}_x(z,t)\Big|_{\omega t=2\pi} = E_0\cos(2\pi - kz) = E_0\cos(kz) = \mathcal{E}_x(z,t)\Big|_{\omega t=0}$

(wave solution repeats itself when $\omega t = n\,2\pi$ (n: integer))

$$\boxed{\text{Argument} = s = \omega t - kz}$$

$\omega t - kz = \text{constant}$

$\Delta(\omega t - kz) = \underbrace{\Delta(\text{constant})}_{0}$

$\omega\Delta t - k\Delta z = 0$

$\Rightarrow \dfrac{\Delta z}{\Delta t} = \dfrac{\omega}{k} = \dfrac{\omega}{\omega\sqrt{\epsilon\mu}} = \dfrac{1}{\sqrt{\epsilon\mu}} = v$ velocity of prop.

or $\omega t - kz = \text{constant}$

$\Delta(\omega t - kz) = \underbrace{\Delta(\text{constant})}_{0}$

$\underbrace{\Delta(\omega t)}_{2\pi} - k\underbrace{\Delta z}_{\lambda\text{: wavelength}} = 0$

$\Rightarrow \boxed{\lambda = \dfrac{2\pi}{k}}$ wavelength

Wavelength : "λ" is the distance along the propagation direction at which the wave solution repeats itself.

(In the present case, we have a $\cos(\omega t - kz)$ type wave solution. When "ωt" changes by 2π, the wave repeats itself and in the meantime "z" changes by "λ" to keep the argument constant)

$$\left.\begin{aligned} \lambda &= \frac{2\pi}{k} \\ k &= \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} = \frac{2\pi f}{v} \end{aligned}\right\} \quad \lambda = \frac{\cancel{2\pi}}{\cancel{2\pi} f / v} = \frac{v}{f}$$

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{v}{f}}$$

$\left(\text{where } v = \frac{1}{\sqrt{\mu\epsilon}}\right)$

Note that in free space

$$v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \Rightarrow \underbrace{\lambda_0}_{\text{wavelength in free space!}} = \frac{c}{f}$$

Show that $\cos(\omega t - k(z+\lambda)) = \cos(\omega t - kz)$

$$\cos(\omega t - k(z+\lambda)) = \cos(\omega t - kz - k\underbrace{\lambda}_{2\pi/k}) = \cos(\omega t - kz - 2\pi)$$
$$= \cos(\omega t - kz) \checkmark$$

∴ The wave solution $E_0 \cos(\omega t - kz)$ is periodic both in time (repeats itself for $\Delta(\omega t) = 2\pi$) and in distance (repeats itself for $\Delta z = \lambda = \frac{2\pi}{k} = \frac{v}{f}$).

Note: Period in time $= T$ such that $\Delta(\underset{\downarrow}{\omega} t) = \Delta(\cancel{2\pi} f t) = \cancel{2\pi}$
$2\pi f$

$$\Rightarrow f(\underbrace{\Delta t}_{T}) = 1 \Rightarrow \boxed{T = \frac{1}{f}}$$

Note: For $\Delta(\omega t) = \frac{\pi}{2}$, for instance:

$$E_x(z,t) = E_0 \cos\left(\frac{\pi}{2} - kz\right) = E_0 \cos\left(-k\left[z - \frac{\pi}{2k}\right]\right)$$

$k = \frac{2\pi}{\lambda}$

$$= E_0 \cos\left(-k\left[z - \frac{\pi}{4\pi/\lambda}\right]\right) = E_0\left(k\left(z - \frac{\lambda}{4}\right)\right]$$

when "ωt" changes by $\frac{\pi}{2}$, cosine solution shift along $(+z)$ by $\frac{\lambda}{4}$ in $(+z)$ dir.

Exercise: Show that for $\Delta(\omega t) = \pi$,

$$E_x(z,t) = E_0 \cos(\pi - kz) = E_0 \cos\left(k\left[\underbrace{z - \frac{\lambda}{2}}\right]\right)$$

i.e., wave propagates by "half wavelength" along $(+z)$ direction.

Exercise: Also show that for $\Delta(\omega t) = 2\pi$,

$$E_x(z,t) = E_0 \cos(2\pi - kz) = E_0 \cos\left(k\left[\underbrace{z - \lambda}\right]\right)$$

i.e wave propagates by a "full wavelength", hence it repeats itself!

$\Delta(\omega t)$		$\Delta(z)$
$\pi/2$	$\longleftrightarrow$	$\lambda/4$
π	$\longleftrightarrow$	$\lambda/2$
2π	$\longleftrightarrow$	λ

etc.

Computation of $\bar{\mathcal{H}}$-field solution corresponding to $\bar{\mathcal{E}}$-field:

given as:

$$\bar{\mathcal{E}} = \hat{a}_x \mathcal{E}_x(z,t) = \hat{a}_x \underbrace{E_0}_{\text{amplitude}} \cos(\omega t \underbrace{-kz}_{\text{phase}}) \longleftrightarrow \boxed{\bar{E} = \hat{a}_x E_0 e^{-jkz}}$$

forward traveling plane wave solution in phasor domain!

Using $\bar{\nabla} \times \bar{E} = -j\omega \bar{B} = -j\omega\mu \bar{H}$ in phasor domain

$$\bar{H} = \frac{1}{-j\omega\mu} \bar{\nabla} \times \bar{E} = j \frac{1}{\omega\mu} \bar{\nabla} \times \left(E_0 e^{-jkz} \hat{a}_x\right)$$

$$\bar{H} = j \frac{E_0}{\omega\mu} \bar{\nabla} \times \left(e^{-jkz} \hat{a}_x\right) = j \frac{E_0}{\omega\mu} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$\bar{H} = j \frac{E_0}{\omega\mu} \left[\hat{a}_x(0-0) - \hat{a}_y\left(0 - \underbrace{\frac{\partial}{\partial z} e^{-jkz}}_{-jk e^{-jkz}}\right) + \hat{a}_z\left(0 - \underbrace{\cancel{\frac{\partial}{\partial y} e^{-jkz}}}_{0}\right)\right]$$

$$\bar{H} = j \frac{E_0}{\omega\mu} (-\hat{a}_y)\, j \underbrace{k}_{\omega\sqrt{\mu\epsilon}} e^{-jkz} = -\hat{a}_y \underbrace{(j)^2}_{-1} \frac{E_0}{\omega\mu} \omega\sqrt{\mu\epsilon}\, e^{-jkz}$$

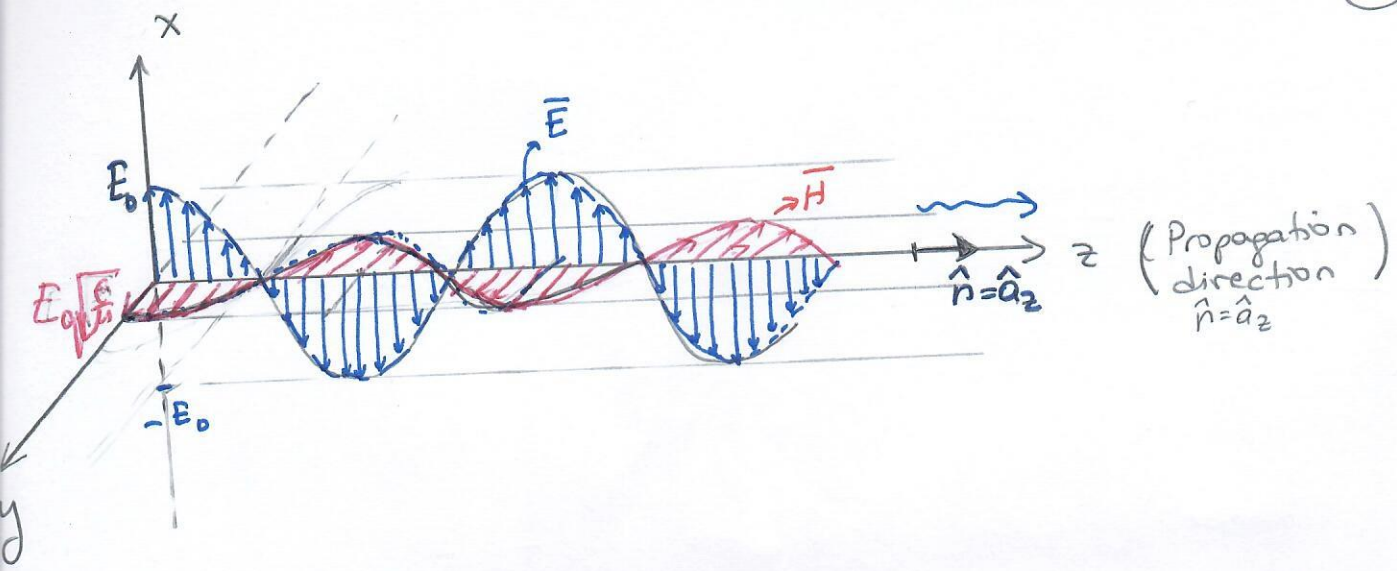
In phasor domain $\boxed{\bar{H} = \hat{a}_y \sqrt{\frac{\epsilon}{\mu}}\, E_0 e^{-jkz}}$ corresponding to $\boxed{\bar{E} = \hat{a}_x E_0 e^{-jkz}}$

$$\bar{\mathcal{H}}(\bar{r},t) = Re\left\{\bar{H}(\bar{r}) e^{j\omega t}\right\} \qquad \bar{\mathcal{E}}(\bar{r},t) = Re\left\{\bar{E}(\bar{r}) e^{j\omega t}\right\}$$

$$\Downarrow \qquad\qquad \Downarrow$$

In time domain $\boxed{\bar{\mathcal{H}} = \hat{a}_y \sqrt{\frac{\epsilon}{\mu}}\, E_0 \cos(\omega t - kz)}$ corresponding to $\boxed{\bar{\mathcal{E}} = \hat{a}_x E_0 \cos(\omega t - kz)}$

where $\boxed{k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} = \frac{2\pi}{\lambda}}$



Conclusions: For the "uniform plane wave" solution obtained in this simplified case:

(i) Direction of propagation = $+z$ ($\hat{n}=\hat{a}_z$: unit vector in the dir. of propagation)

$k=\omega\sqrt{\mu\epsilon}$: wave number (propagation constant)

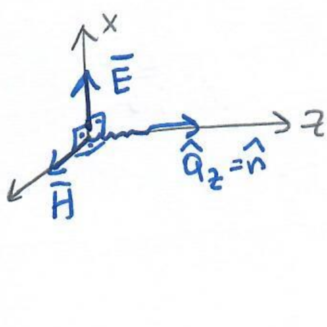
$v=\dfrac{1}{\sqrt{\mu\epsilon}}$: velocity of propagation

$\lambda=\dfrac{v}{f}=\dfrac{2\pi}{k}$: wavelength of propagation

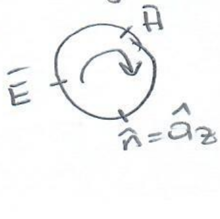
(ii) $\bar{E}\perp\bar{H}$ because $\underset{\text{in } \hat{a}_x \text{ dir.}}{\bar{E}}\cdot\underset{\text{in } \hat{a}_y \text{ dir.}}{\bar{H}}=0$

(iii) $\bar{E}\perp\hat{a}_z$ and $\bar{H}\perp\hat{a}_z$ $\Rightarrow$ both $\bar{E}$ and $\bar{H}$ fields are perpendicular to the dir. of propagation.

$\Rightarrow$ $\bar{E}\times\bar{H}\rightarrow\hat{a}_x\times\hat{a}_y=\hat{a}_z$ gives the direction of propagation!



$\therefore$ The vectors $\bar{E}$, $\bar{H}$ and $\hat{n}=\hat{a}_z$ are mutually perpendicular satisfying a Right Hand Cyclical Relation.



(iv) $$\frac{E_x}{H_y} = \frac{E_0\, e^{-jkz}}{\sqrt{\frac{\epsilon}{\mu}}\, E_0\, e^{-jkz}} = \sqrt{\frac{\mu}{\epsilon}} = \eta \quad \text{(eta)}$$

where $\boxed{\eta = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}}}$ is the "intrinsic impedance" of the medium.

$$\eta = \frac{E_x}{H_y} \propto \frac{\text{Volts/meter}}{\text{Amp/meter}} = \frac{\text{Volts}}{\text{Amp}} = \text{Ohm } (\Omega)$$ ✓ Indeed unit of impedance!

(*) Note that if η is real, then $\bar{E}$ and $\mathcal{H}$ fields are in phase!

(v) For free space (ϵ_0, μ_0); $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \approx 377\ (\Omega)$

(vi) Combine conclusions (iii) and (iv) to write:

$$\boxed{\bar{H} = \frac{1}{\eta}\, \hat{a}_z \times \bar{E}}$$ in vector notation!

Exercise: Repeat the whole solution by assuming
$$\bar{E} = \hat{a}_y E_y(z) = \hat{a}_y E_0 e^{-jkz}$$ this time.

Show that $\bar{H} = \hat{a}_x H_x(z) = -\sqrt{\frac{\epsilon}{\mu}}\, E_0\, e^{-jkz}\, \hat{a}_x$

i.e. $$\frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\epsilon}} = -\eta$$