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Answer 1

(a) Let $x_1, x_2, \ldots, x_m$ be m points in $\mathbb{C}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be non-negative coefficients summing up to 1 ($\sum_{i=1}^{m} \lambda_i = 1$). Consider the linear combination

$$\sum_{i=1}^{m} \lambda_i x_i.$$

Since $\mathbb{C}$ is convex, any convex combination of points in $\mathbb{C}$ is also in $\mathbb{C}$. Therefore,

$$\sum_{i=1}^{m} \lambda_i x_i \in \mathbb{C}.$$

(b) Suppose $f \circ g$ is not convex. This means there exist points x_1 and x_2 in the domain of $f \circ g$ and $t \in [0, 1]$ such that

$$f \circ g\left((1-t)x_1 + tx_2\right) > (1-t)\left(f \circ g(x_1)\right) + t\left(f \circ g(x_2)\right).$$

Now, because f is convex,

$$f\left((1-t)g(x_1) + tg(x_2)\right) \leq (1-t)f(g(x_1)) + tf(g(x_2)),$$

contradicting the assumption. Therefore, $f \circ g$ must be convex.

(c)

Answer 2

(a) X is in this set because $X - X = \emptyset$.

If A is in this set, then $X - A$ is either finite or $\emptyset$. However, the complement of A, $X - A$, is not necessarily in this set because if $X - A$ is finite, then A might be infinite.

This set is not closed under countable unions. If we take countable unions of sets where $X - U$ is finite, the result could be a set where $X - U$ is infinite.

So, this set is not a σ-algebra on X.

(b)

(c) X is in this set because $X - X = \emptyset$.

If A is in this set, then $X - A$ is either infinite, $\emptyset$, or X. However, the complement of A, $X - A$, is not necessarily in this set because if $X - A$ is infinite, then A might be finite.

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So, this set is not a σ-algebra on X.

Answer 3

(a) Let's consider the congruence $ax \equiv b \pmod{p}$. If there exists an integer solution $x = x_0$, then $ax_0 \equiv b \pmod{p}$. So $ax_0 - b$ is divisible by p. This implies there exists $y \in \mathbb{Z}$ such that $ax_0 - yp = b$.

Let $d = \gcd(a, p)$ (by Bezout's identity, there exist integers x_1 and y_1 such that $ax_1 + py_1 = d$). This implies:

$$\left(\frac{b}{d}\right) ax_1 + \left(\frac{b}{d}\right) py_1 = b$$

Then, we have:

$$ax + py = b$$

where x, y are integers. Since $d = \gcd(a, p)$ divides both p and a, it divides b too.

Therefore, $\gcd(a, p) \mid b$ is a must.

(b)

(c)

Answer 4

(a) Let $Y = \bigcup_{i \in \mathbb{N}} Y_i$.
The sets Y_i are countable; therefore, there exist surjective functions $f_i : \mathbb{N} \to Y_i$.
By Cantor's first diagonal argument, it is known that $\mathbb{N} \times \mathbb{N}$ is countable.
So let's define:

$$F : \mathbb{N} \times \mathbb{N} \to Y$$

$$(i, x) \mapsto f_i(x)$$

Per the definition of the union, this mapping is surjective.

So, Y is indeed countable.

(b) Let's denote this set by X^ω. Then we will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

Let's denote this set by X^ω. We will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

For a function g defined as $g(n) = (x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots)$ where each x_{ij} belongs to the set $X = \{a, b, \ldots, z\}$, consider the element $y = (y_1, y_2, \ldots) \in X^\omega$ given by:

$$y_n = \begin{cases} x_{nn} & \text{if } x_{nn} \neq a \\ b & \text{if } x_{nn} = a \end{cases}$$

In other words, y is constructed such that it differs from each $g(n)$ by at least one coordinate. This means that y is not mapped to by g, and therefore, g cannot be surjective.

This argument generalizes to any countable product of a set X with $|X| > 1$. If X has $|X| = k$ elements, then there are $k^{\mathbb{N}}$ distinct sequences in the countable product X^ω, making it uncountable.