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Answer 1

(a) Let $x_1, x_2, \ldots, x_m$ be m points in $\mathbb{C}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be non-negative coefficients summing up to 1 ($\sum_{i=1}^{m} \lambda_i = 1$). Consider the linear combination

$$\sum_{i=1}^{m} \lambda_i x_i.$$

Since $\mathbb{C}$ is convex, any convex combination of points in $\mathbb{C}$ is also in $\mathbb{C}$. Therefore,

$$\sum_{i=1}^{m} \lambda_i x_i \in \mathbb{C}.$$

(b) counterexample to illustrate that the composition of convex functions is not always convex:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$, which is a convex function.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x^2 - 1$, which is also convex.

Then we have $h = f \circ g$, where $h(x) = f(g(x))$ be defined as $h(x) = x^4 - 2x^2 + 1$.

Since $h(x) = x^4 - 2x^2 + 1$ is not convex, we cannot say $f \circ g$ is convex if g and f are convex functions.

(c) **Implication 1:**
Assume $f()$ is a convex function. We should show that S is a convex set, and $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$.

1. **Convexity of S:** Since $f()$ is defined on S, and $f()$ is convex, it implies that S must be a convex set. This is because the domain of a convex function is always convex.
2. **Convexity of $g(t) = f(x + tv)$:** Let $y_1 = x + t_1 v$ and $y_2 = x + t_2 v$ be two points in S where t_1, t_2 are such that $x + t_1 v, x + t_2 v \in S$.
   Now, consider $z = \lambda y_1 + (1 - \lambda) y_2$, where λ is a convex combination coefficient ($0 \leq \lambda \leq 1$).
   $z = \lambda(x + t_1 v) + (1 - \lambda)(x + t_2 v)$ and $z = x + (\lambda t_1 + (1 - \lambda) t_2) v$.
   Since S is convex, $x + (\lambda t_1 + (1 - \lambda) t_2) v \in S$, and by the convexity of $f()$, $g(t) = f(x + tv)$ is convex.

Therefore, the first implication holds.

Implication 2:

Assume S is a convex set, and $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$. We want to show that $f()$ is a convex function.

To show that $f()$ is convex, we need to consider two arbitrary points x_1, x_2 in the domain of $f()$ and show that $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$ for all λ in $[0, 1]$.

Consider x_1, x_2 in the domain of $f()$. Let λ be a convex combination coefficient ($0 \leq \lambda \leq 1$). Now, consider $z = \lambda x_1 + (1 - \lambda)x_2$. Since S is convex, z is also in S. Therefore, we can use the convexity of $g(t) = f(x + tv)$ for t such that $x + tv = z$.

$$g(t) = f(x + tv) = f(\lambda x_1 + (1 - \lambda)x_2)$$

By the convexity of $g(t)$:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Therefore, the second implication holds.

Since both implications hold, we can conclude that a function $f()$ is convex if and only if S is a convex set, and the function $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$.

Answer 2

(a)

(i) if X is uncountable The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{R}$ and $U_1 = \mathbb{R} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{R} - \{1\}$, which is infinite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(ii) if X is countable infinite The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{Z}$ and $U_1 = \mathbb{Z} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{Z} - \{1\}$, which is infinite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an countable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(iii) if X is finite

The set in question must contain the empty set $\emptyset$. This property is satisfied because $X - X = \emptyset$, and $\emptyset$ itself is also part of the set.

Since every U is finite where all $U \subseteq X$, $X - U$ is also finite and this satisfies the condiciton.

Therefore, if this set is denoted by Σ, then $X - U$ must be in this set.

Since $X - (X - U) = U$ is finite, this satisfies the condiciton.

Since each U where $U \subseteq X$ satisfies the condiciton, the set of all $U \subseteq X$ is $P(X)$.

Since $\Sigma \subseteq P(X)$, Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is a σ-algebra on X.

(b)

(i) if X is uncountable The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{R}$ and $U_1 = \mathbb{R} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{R} - \{1\}$, which is uncountable, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(ii) if X is countable infinite

The set in question must contain X. This property is satisfied because $X - \emptyset = X$, and X itself is also part of the set.

Since every U is countable where $U \subseteq X$, $X - U$ is also uncountable and this satisfies the condiciton.

Therefore, if this set is denoted by Σ, then $X - U$ must be in this set.

Since $X - (X - U) = U$ is countable, this satisfies the condiciton.

Since each U where $U \subseteq X$ satisfies the condiciton, the set of all $U \subseteq X$ is $P(X)$.

Since $\Sigma \subseteq P(X)$, Therefore, if X is an countable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either countable or is all of X is a σ-algebra on X.

(iii) if X is finite

(c)

(i) if X is uncountable

(ii) if X is countable infinite

(iii) if X is finite

Answer 3

(a) Let's consider the congruence $ax \equiv b \pmod{p}$. If there exists an integer solution $x = x_0$, then $ax_0 \equiv b \pmod{p}$. So $ax_0 - b$ is divisible by p. This implies there exists $y \in \mathbb{Z}$ such that $ax_0 - yp = b$.

Let $d = \gcd(a, p)$ (by Bezout's identity, there exist integers x_1 and y_1 such that $ax_1 + py_1 = d$). This implies:

$$\left(\frac{b}{d}\right) ax_1 + \left(\frac{b}{d}\right) py_1 = b$$

Then, we have:

$$ax + py = b$$

where x, y are integers. Since $d = \gcd(a, p)$ divides both p and a, it divides b too.

Therefore, $\gcd(a, p) \mid b$ is a must.

(b)

(c) The Chinese Remainder Theorem (CRT) asks for a (common) solution x to a system of congruences

$$x \equiv \begin{cases} a_1 \pmod{p_1} \\ a_2 \pmod{p_2} \\ a_3 \pmod{p_3} \\ \vdots \\ a_k \pmod{p_k} \end{cases}$$

with $\gcd(p_i, p_j) = 1$ for $i \neq j$. The theorem states that there are infinitely many solutions, and any two differ by a multiple of $\operatorname{lcm}(p_1, p_2, p_3, \ldots, p_k)$.

Answer 4

(a) Let's denote this set by X^ω. Then we will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

Let's denote this set by X^ω. We will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

For a function g defined as $g(n) = (x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots)$ where each x_{ij} belongs to the set $X = \{a, b, \ldots, z\}$, consider the element $y = (y_1, y_2, \ldots) \in X^\omega$ given by:

$$y_n = \begin{cases} x_{nn} & \text{if } x_{nn} \neq a \\ b & \text{if } x_{nn} = a \end{cases}$$

In other words, y is constructed such that it differs from each $g(n)$ by at least one coordinate. This means that y is not mapped to by g, and therefore, g cannot be surjective.

This argument generalizes to any countable product of a set X with $|X| > 1$. If X has $|X| = k$ elements, then there are $k^{\mathbb{N}}$ distinct sequences in the countable product X^ω, making it uncountable.

(b) Let $Y = \bigcup_{i \in \mathbb{N}} Y_i$.
The sets Y_i are countable; therefore, there exist surjective functions $f_i : \mathbb{N} \to Y_i$.
By Cantor's first diagonal argument, it is known that $\mathbb{N} \times \mathbb{N}$ is countable.
So let's define:

$$F : \mathbb{N} \times \mathbb{N} \to Y$$

$$(i, x) \mapsto f_i(x)$$

Per the definition of the union, this mapping is surjective.
So, Y is indeed countable.