Student Information

Full Name : Alperen OVAK
Id Number : 2580801

Answer 1

(a) Given $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{C}$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i \mathbf{x}_i \geq 0$ for all $i = 1, \ldots, m$, we want to show that $\sum_{i=1}^m \lambda_i \mathbf{x}_i \in \mathbb{C}$.

Now, since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are in $\mathbb{C}$, by the definition of a convex set, for any $t \in [0, 1]$, we have $t\mathbf{x}_k + (1-t)\mathbf{x}_l \in \mathbb{C}$.

Consider the convex combination:

$$\sum_{i=1}^m \lambda_i \left(t\mathbf{x}_k + (1-t)\mathbf{x}_l\right) = \sum_{i=1}^m \lambda_i t\mathbf{x}_k + \sum_{i=1}^m \lambda_i (1-t)\mathbf{x}_l \tag{1}$$

$$= t\sum_{i=1}^m \lambda_i \mathbf{x}_k + (1-t)\sum_{i=1}^m \lambda_i \mathbf{x}_j \tag{2}$$

Now, due to the linearity of the sum, we can factor out t and $(1-t)$:

$$= t\mathbf{x}_i \sum_{i=1}^m \lambda_i + (1-t)\mathbf{x}_j \sum_{i=1}^m \lambda_i$$

Since $\sum_{i=1}^m \lambda_i = 1$, we get:

$$= t\mathbf{x}_i + (1-t)\mathbf{x}_j$$

This expression is a convex combination of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, and by the convexity of $\mathbb{C}$, it is also in $\mathbb{C}$.

(b) counterexample to illustrate that the composition of convex functions is not always convex:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$, which is a convex function.

Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x^2 - 1$, which is also convex.

Then we have $h = f \circ g$, where $h(x) = f(g(x))$ be defined as $h(x) = x^4 - 2x^2 + 1$.

Let's choose two points $\mathbf{x}_1 = -1$ and $\mathbf{x}_2 = 1$, and $\mathbf{t} = 0.3$ in the range $[0, 1]$.

$$h(0.3(-1) + (0.7)1) = h(0.4) = 0.7056$$

$$0.3h(-1) + (0.7)h(1) = 0$$

$$0.7056 > 0$$

Therefore $h(x) = x^4 - 2x^2 + 1$ is not convex, so we cannot say $f \circ g$ is convex if g and f are convex functions.

(c) Implication 1:

Assume $f()$ is a convex function. We should show that S is a convex set, and $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$.

1. **Convexity of S:** Since $f()$ is defined on S, and $f()$ is convex, it implies that S must be a convex set. This is because the domain of a convex function is always convex.

2. **Convexity of $g(t) = f(x + tv)$:** Let $y_1 = x + t_1 v$ and $y_2 = x + t_2 v$ be two points in S where t_1, t_2 are such that $x + t_1 v, x + t_2 v \in S$.
   Now, consider $z = \lambda y_1 + (1 - \lambda) y_2$, where λ is a convex combination coefficient $(0 \le \lambda \le 1)$.
   $z = \lambda(x + t_1 v) + (1 - \lambda)(x + t_2 v)$ and $z = x + (\lambda t_1 + (1 - \lambda)t_2)v$.
   Since S is convex, $x + (\lambda t_1 + (1 - \lambda)t_2)v \in S$, and by the convexity of $f()$, $g(t) = f(x + tv)$ is convex.

Therefore, the first implication holds.

Implication 2:

Assume S is a convex set, and $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$. We want to show that $f()$ is a convex function.

To show that $f()$ is convex, we need to consider two arbitrary points x_1, x_2 in the domain of $f()$ and show that $f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda) f(x_2)$ for all λ in $[0, 1]$.

Consider x_1, x_2 in the domain of $f()$. Let λ be a convex combination coefficient $(0 \le \lambda \le 1)$. Now, consider $z = \lambda x_1 + (1 - \lambda)x_2$. Since S is convex, z is also in S. Therefore, we can use the convexity of $g(t) = f(x + tv)$ for t such that $x + tv = z$.

$$g(t) = f(x + tv) = f(\lambda x_1 + (1 - \lambda)x_2)$$

By the convexity of $g(t)$:

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Therefore, the second implication holds.

Since both implications hold, we can conclude that a function $f()$ is convex if and only if S is a convex set, and the function $g(t) = f(x + tv)$ is convex for all t such that $x + tv \in S$.

Answer 2

(a)

(i) if X is uncountable The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{R}$ and $U_1 = \mathbb{R} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{R} - \{1\}$, which is infinite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(ii) if X is countable infinite The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{Z}$ and $U_1 = \mathbb{Z} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{Z} - \{1\}$, which is infinite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an countable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(iii) if X is finite

The set in question must contain the empty set $\emptyset$. This property is satisfied because $X - X = \emptyset$, and $\emptyset$ itself is also part of the set.

Since every U is finite where all $U \subseteq X$, $X - U$ is also finite and this satisfies the condiciton.

Therefore, if this set is denoted by Σ, then $X - U$ must be in this set.

Since $X - (X - U) = U$ is finite, this satisfies the condiciton.

Since each U where $U \subseteq X$ satisfies the condiciton, the set of all $U \subseteq X$ is $P(X)$.

Since $\Sigma \subseteq P(X)$, Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is a σ-algebra on X.

(b)

(i) if X is uncountable The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{R}$ and $U_1 = \mathbb{R} - \{1\}$.

Since $X - U_1 = \{1\}$ satisfies the condition U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \mathbb{R} - \{1\}$, which is uncountable, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either finite or is $\emptyset$ is **not** a σ-algebra on X.

(ii) if X is countable infinite

The set in question must contain X. This property is satisfied because $X - \emptyset = X$, and X itself is also part of the set.

Since every U is countable where $U \subseteq X$, $X - U$ is also uncountable and this satisfies the condiciton.

Therefore, if this set is denoted by Σ, then $X - U$ must be in this set.

Since $X - (X - U) = U$ is countable, this satisfies the condiciton.

Since each U where $U \subseteq X$ satisfies the condiciton, the set of all $U \subseteq X$ is $P(X)$.

Since $\Sigma \subseteq P(X)$, Therefore, if X is an countable infinite set, the set of all $U \subseteq X$ such that $X - U$ is either countable or is all of X is a σ-algebra on X.

(iii) if X is finite X is in Σ:

Since $X - X = \emptyset$ (the empty set) is finite, X is in Σ.

Σ is closed under complementation:

If U is in Σ, then $X - U$ is finite or all of X. The complement of U is $X - U$. If $X - U$ is finite, then U is in Σ. If $X - U$ is all of X, then U is also in Σ. Therefore, Σ is closed under complementation.

Σ is closed under finite unions:

Let $A_1, A_2, \ldots$ be sets in Σ. This means that for each A_i, $X - A_i$ is either finite or all of X.

Consider the union $A = A_1 \cup A_2 \cup \ldots$. The complement of A is

$$X - A = (X - A_1) \cap (X - A_2) \cap \ldots.$$

If for each i, $X - A_i$ is finite, then $X - A$ is also finite (finite union of finite sets is finite). If for each i, $X - A_i$ is all of X, then $X - A$ is also all of X.

Therefore, Σ is closed under finite unions.

Since the set Σ satisfies all three properties, it is a σ-algebra on the finite set X.

(c)

(i) if X is uncountable The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{R}$ and $U_1 = \{1\}$.

Since $X - U_1 = \mathbb{R} - \{1\}$ satisfies the condition. U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \{1\}$, which is finite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an uncountable infinite set, the set of all $U \subseteq X$ such that $X - U$ is infinite or $\emptyset$ or X is **not** a σ-algebra on X.

(ii) if X is countable infinite The set of all $U \subseteq X$ is not a σ-algebra on X

Let's show a counterexample:

Let $X = \mathbb{Z}$ and $U_1 = \{1\}$.

Since $X - U_1 = \mathbb{Z} - \{1\}$ satisfies the condition. U_1 must be in this set where $U \subseteq X$.

If this set is denoted by Σ, then from property (2) $X - U_1 = U_2$ must be in this set.

However, since $X - U_2 = \{1\}$, which is finite, U_2 cannot be in this set. This leads to a contradiction.

Therefore, if X is an countable infinite set, the set of all $U \subseteq X$ such that $X - U$ is infinite or $\emptyset$ or X is **not** a σ-algebra on X.

(iii) if X is finite

1. X is in Σ:

In this case, X itself is in the set because $X - X = \emptyset$ is in the set. So, the first property is satisfied.

2. Σ is closed under complementation:

If A is in Σ, then $X - A$ is infinite or $\emptyset$ or X. Let's consider each case:

- If A is finite where $A \neq X$ and $A \neq \emptyset$, then $X - A$ is also finite, which is not in the set.
- If A is $\emptyset$, then $X - A$ is X, which is in the set.
- If A is X, then $X - A$ is $\emptyset$, which is in the set.

So, the set is closed under complementation.

3. Σ is closed under countable unions:

Let $A_1, A_2, \ldots$ be sets in Σ. We want to show that their union,

$$A = A_1 \cup A_2 \cup \ldots,$$

is also in Σ.

- Since we have only two elements in this set, which is $\emptyset$ and X, then $A = X \cup \emptyset = X$ is also in the set.

Since all three properties are satisfied, the given set is a σ-algebra on the finite set X.

Answer 3

(a) **Suppose** $ax \equiv b \pmod{p}$ **has a solution.**

This implies that there exists an integer x such that $ax - b = p \cdot q$ for some integer q.

Take $d = \gcd(a, p)$. We have:

$$a = a \cdot t + p \cdot r$$

where t and r are integers.

Since $d = \gcd(a, p)$, it follows that d divides both a and p. Therefore, we can express a and n as:

$$a = d \cdot q_1 \quad \text{where} \quad q_1 \in \mathbb{Z}$$

$$p = d \cdot q_2 \quad \text{where} \quad q_2 \in \mathbb{Z}$$

Lastly, from the earlier expression $ax - b = p \cdot q$, we can substitute a and n using the above expressions:

$$b = a \cdot x - p \cdot q = (d \cdot q_1) \cdot x - (d \cdot q_2) \cdot q = d \cdot (q_1 \cdot x - q_2 \cdot q)$$

Let $c = q_1 \cdot x - q_2 \cdot q$, then $b = d \cdot c$.

Thus, $d = \gcd(a, p)$ divides b, indicating that if there is a solution to $ax \equiv b \pmod{n}$, then $d = \gcd(a, p)$ divides b.

Suppose $d = \gcd(a, p)$ **and** $d \mid b$**.**

Then we have

$$\begin{cases} d = a \cdot t + p \cdot r, \quad t, r \in \mathbb{Z} \\ d \mid b \end{cases}$$

From these equations, it follows that

$$b = d \cdot c = (a \cdot t + p \cdot r) \cdot c = atc + prc \implies b - a \cdot (tc) = p \cdot (rc) \implies a \cdot (tc) = b \pmod{p}$$

This implies that $ax \equiv b \pmod{p}$ has a solution, where $x = t \cdot c$.

Therefore,the congruence $ax \equiv b \pmod{p}$ has a solution for x if and only if $\gcd(a, p) \mid b$.

(b)

(c) The Chinese Remainder Theorem (CRT) asks for a (common) solution x to a system of congruences

$$x \equiv \begin{cases} a_1 & \pmod{p_1} \\ a_2 & \pmod{p_2} \\ a_3 & \pmod{p_3} \\ \vdots \\ a_k & \pmod{p_k} \end{cases}$$

with $\gcd(p_i, p_j) = 1$ for $i \neq j$. The theorem states that there are infinitely many solutions, and any two differ by a multiple of $\text{lcm}(p_1, p_2, p_3, \ldots, p_k)$.

Answer 4

(a) Let's denote this set by X^ω. Then we will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

Let's denote this set by X^ω. We will show that a function $g : \mathbb{Z}^+ \to X^\omega$ cannot be surjective to prove the uncountability of this set.

For a function g defined as $g(n) = (x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots)$ where each x_{ij} belongs to the set $X = \{a, b, \ldots, z\}$, consider the element $y = (y_1, y_2, \ldots) \in X^\omega$ given by:

$$y_n = \begin{cases} x_{nn} & \text{if } x_{nn} \neq a \\ b & \text{if } x_{nn} = a \end{cases}$$

In other words, y is constructed such that it differs from each $g(n)$ by at least one coordinate. This means that y is not mapped to by g, and therefore, g cannot be surjective.

This argument generalizes to any countable product of a set X with $|X| > 1$. If X has $|X| = k$ elements, then there are $k^{\mathbb{N}}$ distinct sequences in the countable product X^ω, making it uncountable.

(b) Let $Y = \bigcup_{i \in \mathbb{N}} Y_i$.

The sets Y_i are countable; therefore, there exist surjective functions $f_i : \mathbb{N} \to Y_i$.

By Cantor's first diagonal argument, it is known that $\mathbb{N} \times \mathbb{N}$ is countable.

So let's define:

$$F : \mathbb{N} \times \mathbb{N} \to Y$$

$$(i, x) \mapsto f_i(x)$$

Per the definition of the union, this mapping is surjective.

So, Y is indeed countable.