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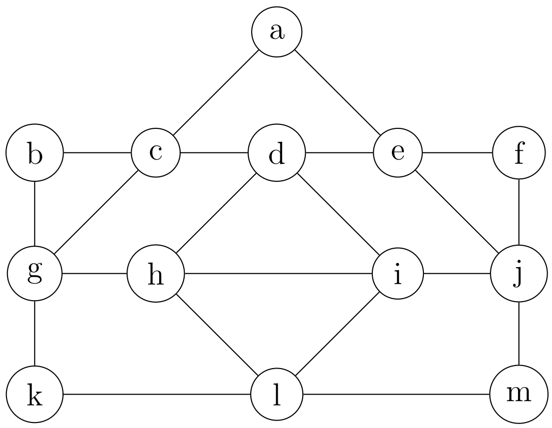
Answer 1



a)

For a graph to have an Eulerian circuit, each vertex must have an even degree (number of edges incident to it), and the graph must be connected.

From the graph G provided:

- Vertex a has degree 2.
- Vertex b has degree 2.
- Vertex c has degree 4.
- Vertex d has degree 4.
- Vertex e has degree 4.
- Vertex f has degree 2.
- Vertex g has degree 4.
- Vertex h has degree 4.
- Vertex i has degree 4.
- Vertex j has degree 4.
- Vertex k has degree 2.

- Vertex l has degree 4.
- Vertex m has degree 2.

All vertices have even degrees, which means graph G meets the first criterion.

The graph G is also connected as you can see in the graph, as we can trace a path from any vertex to any other vertex in the graph by following the edges.

Therefore, graph G does have an Eulerian circuit.

b)

For an Eulerian path (that is not a circuit) to exist, the number of vertices which have odd degree must be 0 or 2. Additionally, the graph must be connected.

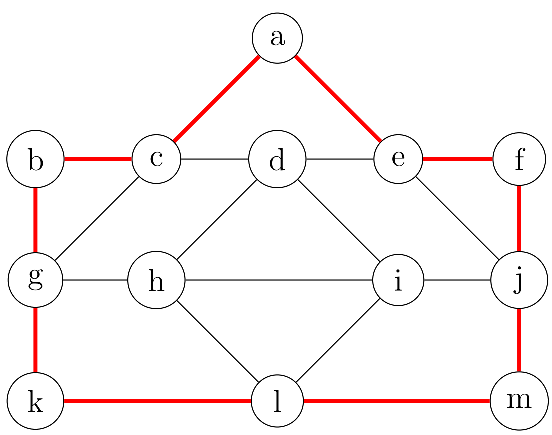
Since there is no odd degreed vertices, and the graph is connected, we can say that there is an Eulerian path.

c)

To determine if graph G has a Hamiltonian circuit, we must find a closed loop that visits each vertex exactly once and returns to the starting vertex.

Observations and Analysis:

- **Vertex Degrees:** The vertices a, b, f, k, and m have a degree of 2. In a Hamiltonian circuit, we must use all edges connected to vertices of degree 2, entering and exiting each of these vertices.
- **Outer Circle Formation:** When we consider the edges connected to vertices of degree 2 and ensure that we enter and exit each of these vertices, we form an outer circle or loop. This outer loop includes edges connected to vertices a, c, b, g, k, l, m, j, f and e.



- **Inner Vertices:** The vertices which are not visited in the outer loop are d, h and i. To form a Hamiltonian circuit, we must also visit these inner vertices exactly once.

- **Revisiting Vertices:** The challenge arises when we try to connect the outer loop with the inner vertices without revisiting any vertex. Due to the connectivity of graph G, it is not possible to visit all inner vertices without revisiting at least one vertex, breaking the condition for a Hamiltonian circuit.

Based on the structure of the graph G and the analysis above, we can conclude that there is no Hamiltonian circuit in graph G. The inability to connect the outer loop with the inner vertices without revisiting a vertex confirms the absence of a Hamiltonian circuit in this graph.

d)

Yes there is a Hamilton path in G: $b - c - a - e - f - j - m - l - i - d - h - g - k$

e)

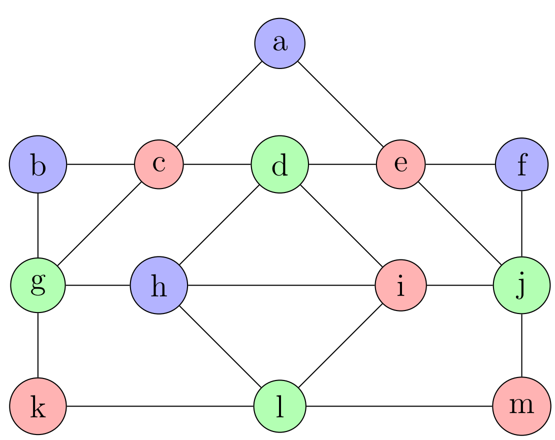
To determine the chromatic number $\chi(G)$ of the graph G:

1. **Subgraph Analysis:** Consider the subgraph of G with the maximum size that forms a complete graph, K_3 (a triangle). This subgraph consists of vertices a, b, and c. Given that K_3 requires 3 different colors to ensure no adjacent vertices share the same color, we know that at least 3 colors are necessary.

2. **Degree and Chromatic Number Relation:** There's a relationship between the maximum degree of a graph and its chromatic number. Specifically, for a graph with maximum degree k, it is $(k+1)$-colorable. In G, the maximum degree is 4. Hence, by this relationship, G is $(4+1)$-colorable, implying G is 5-colorable.

3. **Narrowing Down the Options:** Given the insights from the subgraph and the degree-chromatic number relationship, we know that $3 \leq \chi(G) \leq 5$. To find the exact chromatic number, we can attempt to color the graph with 3 colors.

4. **Verification:** By trying to color the graph G using only 3 colors, we can verify if it's possible to ensure no adjacent vertices share the same color. If successful, then $\chi(G) = 3$. If not, then $\chi(G)$ would be 4 or 5.



Conclusion:

After attempting to color the graph G with 3 colors, we find that it is indeed possible to do so without any adjacent vertices sharing the same color. Thus, the chromatic number $\chi(G)$ of the graph G is 3.

f)

Chromatic Number and Bipartite Graphs: It's correct that the chromatic number of a bipartite graph is 2. Since $\chi(G) = 3$ for G, it confirms that G cannot be bipartite.

Making G Bipartite: To make G bipartite, we would aim to decrease its chromatic number to 2. This can be achieved by removing or "breaking" cycles of odd length in G, as any such cycle in a graph prevents it from being bipartite.

Removing K_3 Subgraphs: You correctly identified the K_3 subgraphs in G. To make G bipartite by removing edges, we would focus on the K_3 subgraphs.

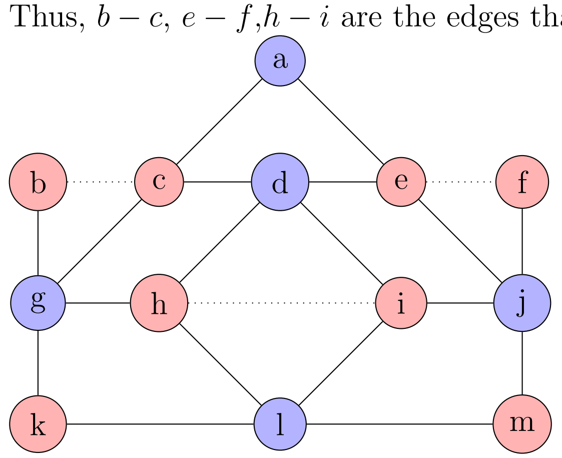
- For K_3 subgraphs where all vertices are pairwise connected, removing any single edge would suffice.

- For K_3 subgraphs where two K_3 subgraphs share a common edge (forming a 4-clique or K_4), removing just one edge would break both K_3 subgraphs into separate components.

Note that we need to create odd cycle. The steps are so simple: Select a random node to be the first object in set A. Choose one node to be the first in set B if there are any connected to it. If not, select any other node to be the set B's initial member.

Right now, A and B are our two sets of nodes. Select a node that is not in either set repeatedly. Determine how many edges connect that node to the nodes in A and B. Remove the edges connecting it to nodes in set B and place it in set B if there are further edges connecting it to set A. If not, insert it into set A and remove the edges connecting it to the nodes there.

Thus, $b - c$, $e - f$,$h - i$ are the edges that should be deleted to get bipartite graph.



g)

Given the graph G with chromatic number $\chi(G) = 3$, it indicates that no set of four vertices in G can form a complete subgraph K_4 since a K_4 would require 4 distinct colors.

Upon examining the graph:

1. Vertices d, h, and i form a complete subgraph K_3.

To introduce a complete subgraph K_4 into G, we need to add a vertex connected to d, h, i, and l.

The edge between vertices d and l can be added to G to ensure that the vertices d, h, i, and l together form a complete subgraph K_4.

Thus, to incorporate a complete subgraph K_4 into G, the edge connecting d and l should be added.

Answer 2

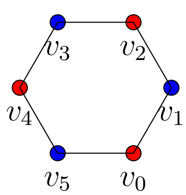
Answer 3

a)

The chromatic number $\chi(C_n)$ of the cycle graph C_n is 2 if n is even, and 3 if n is odd.

When n is even: For n even, we can see that the vertices of C_n can be alternately colored using two colors. To illustrate, consider C_6:

- Start with vertex v_0 and color it red.
- Proceeding clockwise, the next vertex v_1 is colored blue.
- Continue this pattern, alternating between red and blue, until v_5, which is adjacent to v_0, is colored blue.

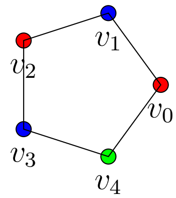


Thus, C_6 is bipartite and can be colored using two colors, indicating that the chromatic number is 2 amd bipartite when n is even.

When n is odd: For n odd, alternating colors while traversing C_n clockwise will eventually lead to a vertex (e.g., v_5 in C_5) that is adjacent to both the starting vertex v_0 and the vertex before the starting vertex. This situation requires a third color to ensure no adjacent vertices share the same color.

For instance, in C_5:

- Start with v_0 colored red.
- Proceeding clockwise, v_1 is blue, v_2 is red, v_3 is blue, but when reaching v_4, which is adjacent to both v_3 and v_0, a third color (say green) is needed.



Thus, C_5 requires three colors, indicating that the chromatic number is 3 and it is not bipartite when n is odd.

b)

The chromatic number of the hypercube Q_n is 2. This can be understood by observing the binary nature of the vertex labels. We can partition the vertices into two sets based on the parity of the number of 1s in their binary labels:

- One set contains vertices labeled with sequences having an even number of 1s.
- The other set contains vertices labeled with sequences having an odd number of 1s.

Within each set, vertices are not connected by edges because the Hamming distance between them would always be even. On the other hand, there will never be an edge between two sequences with the same parity (i.e., either both sequences have an even number of 1s or both have an odd number of 1s) because edges in Q_n exist only between sequences with binary labels that differ in one position (leading to a change in parity).

As we can understand that Q_n is a bipartite.

Thus, the vertices of Q_n can be colored with two colors in such a way that no two adjacent vertices have the same color, indicating that the chromatic number of Q_n is 2.

Answer 4

a)

b)

c)

Answer 5

a)

Base Case (d = 1):

Consider a perfect binary tree B_1 with depth 1. This tree consists of a single node (the root). The number of nodes N_1 is 1, and the number of leaves L_1 is also 1.

From the provided information:

$$N_1 = 2^1 - 1 = 2 - 1 = 1$$
$$L_1 = \frac{N_1 + 1}{2} = \frac{2}{2} = 1$$

Both the number of nodes and leaves satisfy the given formulas for the base case.

Inductive Step (d → d+1):

Assume the statements hold for some arbitrary depth d. That is:

$$N_d = 2^d - 1$$
$$L_d = \frac{N_d + 1}{2}$$

Now, consider a perfect binary tree B_{d+1} of depth $d+1$ with B_d as its last layer. Each leaf of B_d will become two leaves in B_{d+1}.

1. **Nodes:** For each leaf in B_d, we add 2 nodes in B_{d+1}. So, the new number of nodes N_{d+1} is:

$$\begin{aligned}
N_{d+1} &= N_d + 2L_d \\
&= (2^d - 1) + 2\left(\frac{2^d - 1 + 1}{2}\right) \\
&= 2^d - 1 + 2^d - 1 + 1 \\
&= 2(2^d) - 1 \\
&= 2^{d+1} - 1
\end{aligned}$$

This matches the formula for N_{d+1}.

2. **Leaves:** Each leaf in B_d transforms into two leaves in B_{d+1}. Thus, the number of leaves L_{d+1} becomes:

$$\begin{aligned}
L_{d+1} &= 2L_d \\
&= 2\left(\frac{N_d + 1}{2}\right) \\
&= 2\left(\frac{2^d - 1 + 1}{2}\right) \\
&= 2^d - 1 + 1 \\
&= \frac{2^{d+1}}{2} \\
&= \frac{2^{d+1} - 1 + 1}{2} \\
&= \frac{N_{d+1} + 1}{2}
\end{aligned}$$

This matches the formula for L_{d+1}.

Thus, both the number of nodes and leaves in the tree of depth $d+1$ satisfy the given formulas based on the induction hypothesis. This completes the inductive step.

By the principle of mathematical induction, the statements are true for all positive integers d.

b)

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G such that no two adjacent vertices have the same color.

For a tree T, the chromatic number $\chi(T)$ is 2. Here's why:

1. A tree is a connected graph with no cycles.
2. Consider any vertex v of the tree. Since the tree is connected and has no cycles, there is exactly one path between any two vertices in the tree. Therefore, if we start coloring the tree from vertex v and proceed along the tree, we can ensure that adjacent vertices get different colors.
3. Because of the above property, we only need two colors to color a tree such that no two adjacent vertices share the same color: one for v and its immediate neighbors and another for the neighbors of those neighbors, and so on.

Hence, the chromatic number of a tree $\chi(T) = 2$.

c)

The tree is called a full m-ary tree if every internal vertex has exactly m children.(Definition 3 - 11.1)

To get upper bound for height, we need to just add m children to one of the nodes in each level.

Therefore, we have m nodes in every level, except level-1. This means that the number of levels which have m many nodes equals to height of the tree.

So, we have total $1 + mh = n$ nodes.

It means that the upper bound of height is $h = \frac{n-1}{m}$.