Week - 1
FAQ
Activity Question
Mathematics for Data Science - 1

1. Suppose the relation $R = \{$(Snail, Frog), (Bird, Bird), (Fox, Frog), (Snail, Fox)$\}$ is defined on the set $S = \{$Snail, Fox, Bird, Frog$\}$.

   i. The relation R on the set S is(are)

   (a) Reflexive
   (b) Symmetric
   (c) Transitive
   (d) None of these.

   Solution:
   We have relation $R = \{$(Snail, Frog), (Bird, Bird), (Fox, Frog), (Snail, Fox)$\}$ defined on the set S = $\{$Snail, Fox, Bird, Frog$\}$.
   A relation R on a set A is said to be a reflexive relation if $(a, a) \in$ R, $\forall a \in A$. Snail $\in$ S but (Snail, Snail) $\notin R$, so R is not a reflexive relation.
   Hence, option (a) is not correct.
   A relation R on a set A is said to be a symmetric relation if $(a, b) \in R$ implies that (b, a) also belongs to R.
   Since (Snail, Frog) $\in R$ but (Frog, Snail) $\notin R$, R is not symmetric.
   So, option (b) is not correct.
   A relation R on a set A is said to be a transitive relation if $(a, b) \in R$ and $(b, c) \in R$ implies that (a, c) belongs to R.
   (Snail, Fox) $\in R$, (Fox, Frog) $\in R$ and (Snail, Frog) belongs to R. Similarly, (Bird, Bird) $\in R$. Hence R is a transitive relation and so option (c) is correct.

   ii. Let P be a subset of $[(S \times S) \setminus R]$. What is the set of possible cardinalities of P?

   (a) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
   (b) $\{\}$
   (c) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
   (d) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

   Solution:
   We have $|S| = 4$ and $|R| = 4$.
   So, cardinality of the set $S \times S, |S \times S| = |S| \times |S| = 4\times\ 4 = 16$.
   And $|(S \times S)\backslash R|$ = 16-4 = 12.
   Since P is a subset of $(S \times S)\backslash R$, so cardinality of P can be any natural number less than or equal to 12. So, possible cardinalities of set P is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1, 12\}$.
   Hence, option (d) is correct.

iii. What is the minimum number of elements of the Cartesian product $S \times S$ that need to be added to R such that the relation becomes symmetric?

(a) 4
(b) 8
(c) 2
(d) 3

Solution:
Given $R = \{$(Snail, Frog), (Bird, Bird), (Fox, Frog), (Snail, Fox)$\}$.
In order to make R a symmetric relation, we must do the following. Since (Snail, Frog) $\in R$, it is necessary that (Frog, Snail) must also be in R. So, add (Frog, Snail) to R. Similarly, corresponding to (Fox, Frog) and (Snail, Fox), we must add (Frog, Fox) and (Snail, Fox) respectively to R. Corresponding to (Bird, Bird) $\in R$, we need not add (Bird, Bird) again as we do not add duplicates to a set. So, the minimum number of elements of Cartesian product $S \times S$ that have to be added to R such that the relation becomes symmetric is three. Hence option (d) is correct.

2. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a function defined by $f(x) = \mathrm{A}x + \mathrm{B}$. For which of the following integer values of A and B, is the given function bijective?

(a) A = 0, B = $\{ z \mid z \in \mathbb{Z}\}$
(b) A = {-1, 1}, B = $\{z \mid z \in \mathbb{Z}\}$
(c) B = {-1, 1}, A = 0
(d) B = 0, A = $\{ z \mid z \in \mathbb{Z}\}$

Solution:
To solve this, let us go through each option.
Option (a): Suppose we substitute A $= 0$ and B with some element $b \in$ B in given function f.
Then $f(x) = b$. That is, f is a constant function. So, range of $f, R(f) = \{b\}$. $R(f) \neq$ codomain of f (i.e $\mathbb{Z}$). Hence, function f is not surjective. Since a function is bijective if and only if it is both injective as well as surjective, it follows that f is not bijective.
Now, consider option (b): In this case, if we substitute A $= -1$ and B with some $b \in$ B in f, then $f(x) = -x + b$. To check if it is injective, let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{Z}$ such that

$$f(x_1) = f(x_2) \implies -x_1 + b = -x_2 + b \implies x_1 = x_2.$$

From above, we see that as no two distinct elements in the domain will give the same image. Hence $x_1, x_2 \in \mathbb{Z}$ can not be distinct.
So, f is an injective function.
Now, to show that f is surjective, let us assume $y \in \mathbb{Z}$ (codomain of f) $\implies -y + b \in \mathbb{Z}$ which is given as domain of f.

$f(-y+b) = -(-y+b)+b = y$.
That is, we have an element $-y+b$ in $\mathbb{Z}$ (domain of f) such that $f(-y+b) = y$.
Hence, function f is surjective.
So, when A = -1, and B is $b \in$ B, function f is bijective.
Similarly for 1 $\in$ A and any element b $\in$ B, function f is bijective. So, for otion (b), function f is bijective .
Now, for option (c): If A = 0 and for any element of B, it is same case as option (a).
Hence, for option (c), function f is not bijective.
Now, for option (d): It is given that B = 0 and A is set of integers. So we choose 0 from A and substitute A = 0 $\in \mathbb{Z}$ in the given function f.
Then $f(x) = 0$ which is again a constant function and it is the same case as option (a).
So, function f is not surjective.
So, for option (d), function f is again not a bijective function.
Hence, the only correct option is option (b).

3. Which of the following intervals are subsets of [2, 3]?

   (a) ((0, 2.3] ∪ (2.3 , 3]) ∩ (0, 2.3)

   (b) ((2, 2.5] ∪ (2.5 , 4]) \ (0, 2.3)

   (c) ((2, 2.5] ∪ (2.5 , 4]) ∩ (0, 3)

   (d) ((0, 2.3] ∪ (2.2 , 3]) \ (0, 2.3)

   Solution:
   Let us go through the options:
   Option (a): ((0, 2.3] ∪ (2.3 , 3]) ∩ (0, 2.3) = (0 , 3] ∩ (0, 2.3) = (0, 2.3). And 1 ∈ (0, 2.3) but 1 ∉ [2, 3].
   Hence, (0, 2.3) can not be a subset of [2, 3]. So, this option is not correct.

   Option (b): ((2, 2.5] ∪ (2.5, 4]) \ (0, 2.3) = (2, 4] \ (0, 2.3) = [2.3, 4]. And 4 ∈ [2.3, 4] but 4 ∉ [2, 3].
   Hence, [2.3, 4] can not be a subset of [2, 3]. So, this option is not correct.

   Option (c): ((2, 2.5] ∪ (2.5 , 4]) ∩ (0, 3) = (2, 4]∩ (0, 3) = (2, 3) ⊂ [2, 3]. So, this option is correct.

   Option (d): ((0, 2.3] ∪ (2.2 , 3]) \ (0, 2.3) = (0, 3] \ (0, 2.3) = [2.3, 3] ⊂ [2, 3]. So, this option is correct.

4. The domain and range of the function that is shown in the Figure 1.1 are $[m, x]$ and $[n, y]$ respectively, where $m, n, x, y \in \mathbb{R}$.

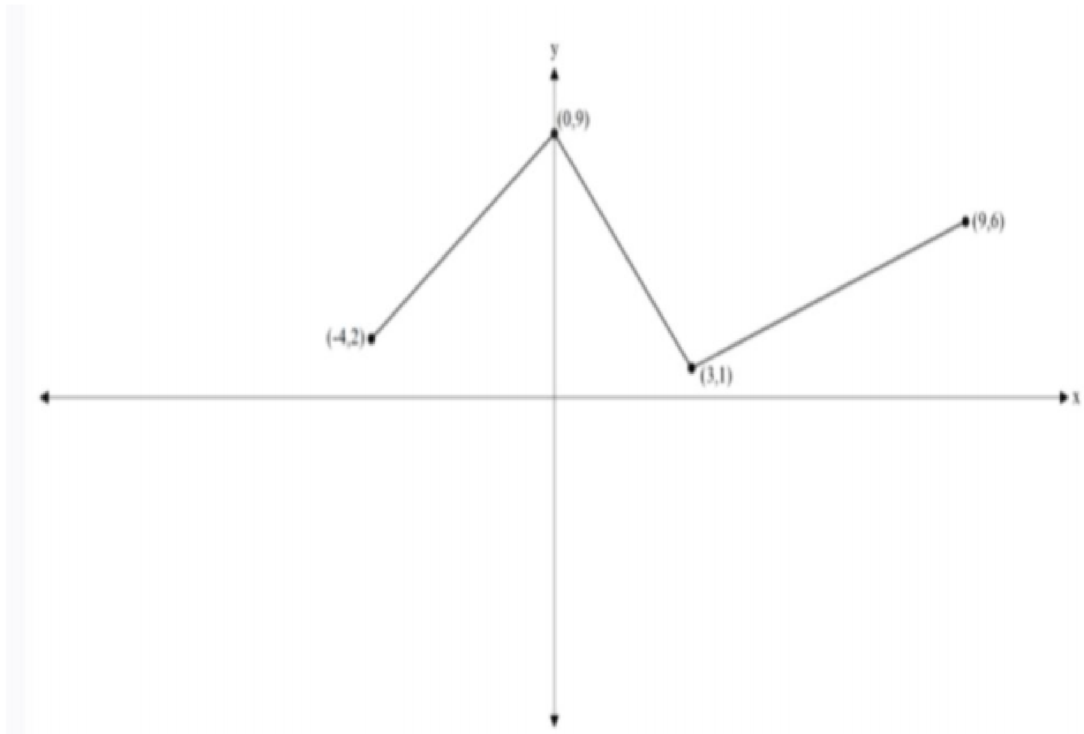

Figure 1.1

i. What are the values of m, n respectively?
   (a) 0, 3
   (b) -4, 1
   (c) -4, 9
   (d) 2, 6

ii. What are the values of x, y respectively?
   (a) 9, 9
   (b) -4, 9
   (c) 1, 9
   (d) 2, 6

Solution:

Given that $f : [m, x] \to [n, y]$.
From the Figure 1.1, on X-axis the input set for the function f is the closed interval [-4,

9] and the output set for the function f is the close interval [1, 9].
So, let us compare $[m, x]$ with input set interval [-4, 9] and $[n, y]$ with output set interval [1, 9].
We will get $m = -4, x = 9, n = 1, y = 9$.
Hence, in question (i), the second option is correct and in question (ii), the first option is correct.

5. Let $B = \{$Anil, Ramu, Suraj$\}$ and $G = \{$Neha, Keerthi$\}$ be the sets of boys and girls respectively. Ramu is brother of Neha, Anil is brother of Keerthi, Suraj is brother of Neha and Keerthi.
Let us define a relation R as follows,
$R = \{(a, b) \mid (a, b) \in B \times G, a \text{ is brother of } b\}$. Which of the following will be R?

   (a) {(Anil, Keerthi), (Suraj, Neha), (Ramu, Neha), (Suraj, Keerthi)}

   (b) {(Anil, Keerthi), (Ramu, Keerthi), (Ramu, Neha), (Suraj, Neha)}

   (c) {(Anil, Neha), (Ramu, Keerthi), (Ramu, Neha), (Suraj, Keerthi)}

   (d) {(Anil, Keerthi), (Ramu, Neha), (Suraj, Neha), (Ramu, Keerthi), (Anil, Neha), (Suraj, Keerthi)}

Solution:
Given $B = \{$Anil, Ramu, Suraj$\}$ and $G = \{$Neha, Keerthi$\}$. Ramu is brother of Neha, Anil is brother of Keerthi, Suraj is brother of Neha and Keerthi. Hence, they are all siblings.
$R = \{(a, b) \mid (a, b) \in B \times G, a \text{ is brother of } b\}$.
Therefore, R is the set $B \times G$ and elements of R correspond to those given in option d.
Hence, option (d) is correct.

6. Let $x \in \mathbb{R}$. Which of the following functions is(are) injective?

   (a) $f(x) = \sqrt{10 - x}$

   (b) $f(x) = \frac{7x+6}{3x}$

   (c) $f(x) = 2x + 9$

   (d) $f(x) = \frac{(5x+4)(2x-3)}{2}$

Solution:
For each function, we have to take those x from the set $\mathbb{R}$ such that the function is well defined.
Now, let us go through the options:
Option (a): $f(x) = \sqrt{10 - x}$.
Observe that, domain$(f) = \{\ x \in \mathbb{R} | 10 - x \geq 0 \ \ i.e \ \ x \leq 10\}$
Remark: Square root of any real number is positive. So, this function is a well defined function.

To check if the function is injective, let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) = f(x_2)$$

$$\implies \sqrt{10 - x_1} = \sqrt{10 - x_2}$$

Squaring both sides, we get,

$$10 - x_1 = 10 - x_2$$

$$\implies x_1 = x_2.$$

It follows that no two distinct elements in the domain will give the same image. Hence $x_1, x_2 \in \mathbb{R}$ are not distinct. Therefore, this function is injective.

Option (b): $f(x) = \frac{7x+6}{3x}$, $x \neq 0$
Observe that , domain$(f) = \mathbb{R} \setminus \{0\}$
Again, let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ such that

$$f(x_1) = f(x_2)$$

$$\implies \frac{7x_1 + 6}{3x_1} = \frac{7x_2 + 6}{3x_2}$$

Since $x_1, x_2 \neq 0$, after re-arranging, we get,

$$7x_1x_2 + 6x_2 = 7x_1x_2 + 6x_1$$

$$\implies x_1 = x_2$$

Using the same reasoning as in option (a), this function is also injective.

Option (c): $f(x) = 2x + 9$
Let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) = f(x_2)$$

$$\implies 2x_1 + 9 = 2x_2 + 9$$

$$\implies x_1 = x_2$$

Using the same reasoning as in option (a), it follows that this function is also injective.

Option (d): $f(x) = \frac{(5x+4)(2x-3)}{2}$.

Substitute $x = -\frac{4}{5}$ and $x = \frac{3}{2}$ in the given function. Then, we get $f(x) = 0$ in both cases. For two different inputs, we get same output. So, this function is not injective.