Math Week 1: Problem Solving Tips
Mathematics for Data Science - 1

- **Natural numbers and integers**
  - The set of natural numbers : $\{0, 1, 2, 3, 4, \ldots\}$,i.e. 0 is a natural number.
  - The set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
  - In general when we talk about prime numbers we generally consider set of natural numbers as our universal set.
  - Observe that 0 and 1 are neither prime nor composite.
  - To check whether a natural number n is prime or not, it is enough to check whether it is divisible by any natural number less than or equal to $\sqrt{n}$. If it is divisible by some natural number less than or equal to $\sqrt{n}$, other than 1, then the number n is not prime.
    - Suppose we want to check whether 11 is prime or not. At first observe that $\sqrt{11}$ is a number between 3 and 4, as $3^2 = 9$ and $4^2 = 16$. Hence, we have to check whether 11 is divisible by $0, 1, 2, 3$.
      0 cannot be a divisor of any non-zero number. 1 is divisor of any natural number. 2 and 3 do not divide 11. Hence, there is only one divisor (i.e. 1) of 11 which is less than $\sqrt{11}$. Hence, we can conclude 11 is prime.
  - To list out all the primes less than some fixed natural number "Sieve of Eratosthenes" is an efficient algorithm.
  - Finding the *gcd* (greatest common divisor) of finitely many natural numbers:
    - Factorise all the given numbers.
    - List out all the common factors.
    - Find the greatest one among those common factors.
  - $gcd(0, a) = a = gcd(a, 0)$, for any non-zero natural number a.
  - $gcd(0, 0)$ is undefined.
- **Rational and irrational numbers**
  - Learn to identify perfect squares, perfect cubes etc
    - Squares: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$ etc
    - Cubes: $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$ etc
  - Learn to identify numbers that are not perfect squares cubes etc
    - Non-squares: $2, 3, 5, 6, 7, 8, 10$
    - Non-cubes: $2, 3, 4, 5, 6, 7, 9, 10$
  - **Square root of a non-square is irrational**

- Operations with rational and irrational numbers
  * rational + rational = rational
  * rational × rational = rational
  * rational + irrational = irrational
  * rational (except 0) × irrational = irrational
  * irrational + irrational = can be rational or irrational
  * irrational × irrational = can be rational or irrational
- Learn to deal with fractions that have a numerator and a denominator that could be rational or irrational
- Rationalising the denominator:
  * multiply both top and bottom by square root of a non-square
  * multiply both top and bottom by conjugate of the denominator
- Learn to deal with square roots of an irrational number

- **How to check two sets A and B are equal**
  Let $A = \{x \mid x \text{ satisfies } \textit{condition } A\}$ and $B = \{y \mid y \text{ satisfies } \textit{condition } B\}$ be the set-builder form of sets A and B.

Procedure 1: If possible try to list out all the elements of A and B and then check all the elements of A and B are same. This is possible if the sets are finite and cardinalities of these two sets are small.

Procedure 2: In some cases it is easier to use Venn-diagram to compare two sets.

Procedure 3: If *condition* A can be derived from *condition* B, then $B \subseteq A$. Similarly, if *condition* B can be derived from *condition* A, then $A \subseteq B$. These two together implies $A = B$.

- **Some equalities when $A = B$:**
  * $A \cup B = A = B$
  * $A \cap B = A = B$

- **Relations**
  - Observe that relations are subsets of cartesian products of sets.
  - Suppose a relation R is defined on a set A. Then $R \subseteq A \times A$.
    * If for each $a \in A$, $(a, a) \in R$ then R is reflexive.
    * If for each pair $(a, b) \in R$, the pair (b, a) is also in R then R is symmetric.
    * If for pairs $(a, b) \in R$ and $(b, c) \in R$, the pair (a, c) is also in R then R is transitive.
    * If $(b, a) \notin R$, whenever $(a, b) \in R$ (together with $a \neq b$) then R is anti-symmetric.
  - Reflexive + Symmetric + Transitive = Equivalence relation

- **Functions**
  - Observe that all functions are relations but all relations are not functions.
  - An ordered pair can be represented as (input, output). A function is a relation which describes that there should be only one output for each input.
    * The sets from which inputs are taken is called 'Domain'.
    * The set into which all of the output of the function is constrained to fall is called 'Codomain'.
    * The set of outputs are called 'Range'.
    * If for a function f, $f(a) = b$, then a is called a preimage of b.
  - Domain of some function with special form:
    * If $f(x) = \sqrt{g(x)}$, then for the function $f(x)$ to be real valued we must have $g(x) \geq 0$. Hence, domain of $f(x)$ is $\{x \mid g(x) \geq 0\}$.
    * If $f(x) = \frac{g(x)}{h(x)}$, then for the function $f(x)$ to be defined we must have $h(x) \neq 0$. Domain of $f(x)$ is
      $\{x \mid h(x) \neq 0$ and x is in the intersection of domain of $g(x)$ and domain of $h(x)\}$
  - Learn to deal with different types of functions:
    * For one to one (injective) function two different inputs cannot give same output.
    * For onto (surjective) function codomain=range, i.e. for each element in the codomain there exists a preimage in the domain.