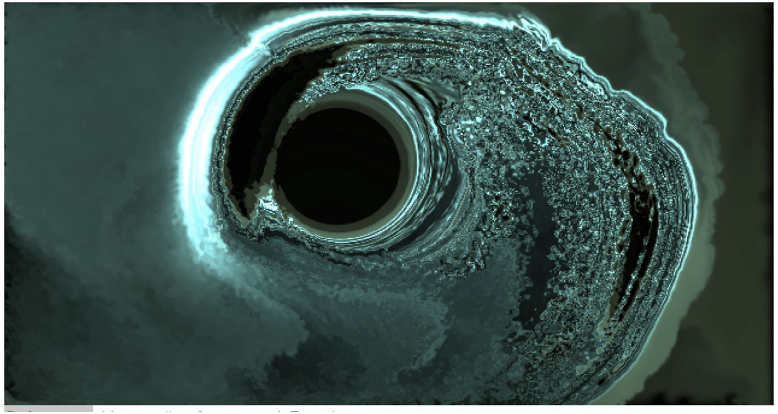
Research of the one-component 2D Navier-Stokes (NS) modeling - transitions to finer meshes and accelerations

Introduction

Aims:

- Select the best interpolation method for (1) Turbulent and (2) Laminar flows by comparing several interpolation methods: Nearest-Neighbour, Linear, Cubic , and Piecewise Cubic Hermite Interpolating Polynomial (PCHIP).
- Analyze sparse matrix application to improve computational speed up for:
    - Laplacian operator
    - First order derivative
    - Navier-Stokes equation

The Navier–Stokes equations



The case of incompressible fluid

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = -(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} - \frac{1}{\rho}\nabla \mathbf{p} + \nu \nabla^2 \tilde{\mathbf{u}} + \tilde{\mathbf{F}}$$

Transformation of the viscosity equation to apply the Jacobian method:

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} = \nu \nabla^2 \tilde{\mathbf{u}} \xrightarrow[\text{iterative}]{} u(\tilde{\mathbf{x}}, t + \delta t) = u(\tilde{\mathbf{x}}, t) + \nu \delta t \nabla^2 \tilde{\mathbf{u}}$$

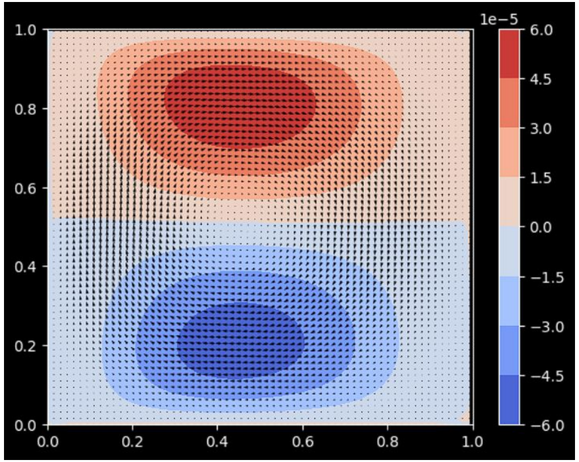
$$\downarrow \quad \mathbf{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$$

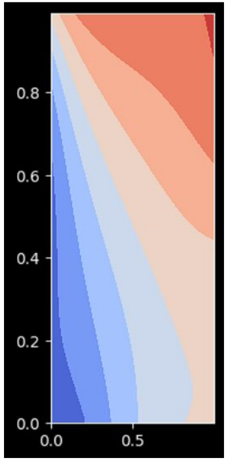
$$(\mathbf{I} - \nu \delta t \nabla^2) u(\tilde{\mathbf{x}}, t + \delta t) = u(\tilde{\mathbf{x}}, t)$$

Force:

$$\vec{\mathbf{F}} = \vec{\mathbf{G}} \delta t \mathbf{exp} \left(- \frac{(x - x_p)^2 + (y - y_p)^2}{r} \right)$$

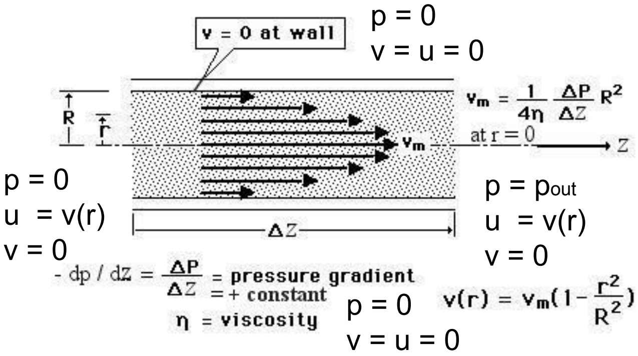
The Navier–Stokes equations

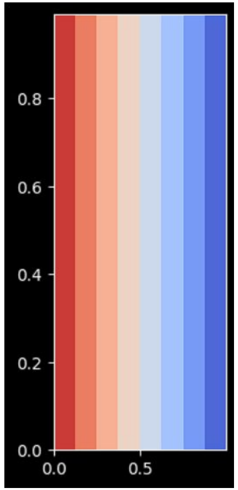


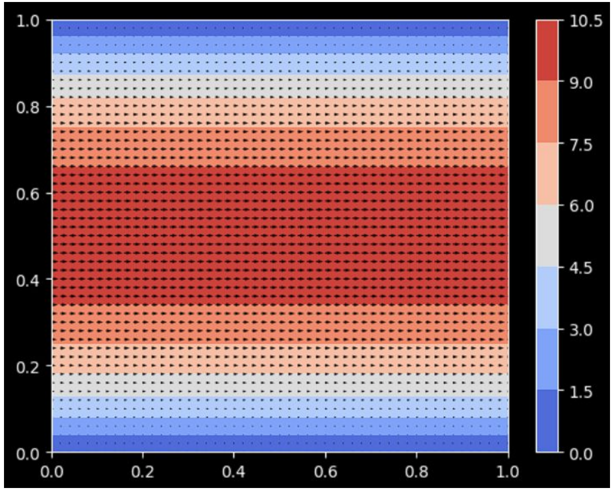


$$\frac{\tilde{\mathbf{u}}_{0,j} + \tilde{\mathbf{u}}_{1,j}}{2\delta y} = 0, \quad \frac{\tilde{\mathbf{u}}_{i,0} + \tilde{\mathbf{u}}_{i,1}}{2\delta x} = 0$$

$$\frac{\mathbf{p}_{0,j} - \mathbf{p}_{1,j}}{\delta x} = 0, \quad \frac{\mathbf{p}_{i,0} - \mathbf{p}_{i,1}}{\delta y} = 0$$



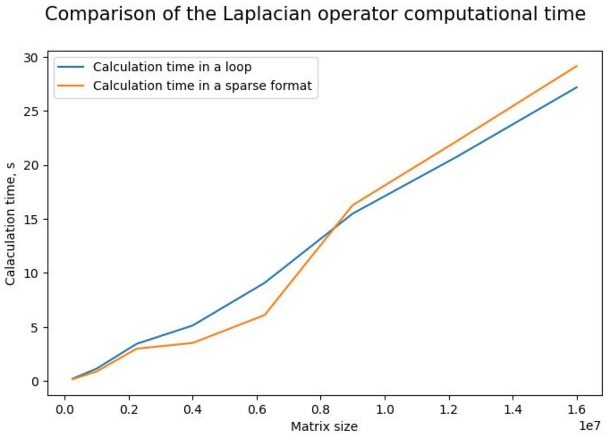




Implementation of the sparse matrices

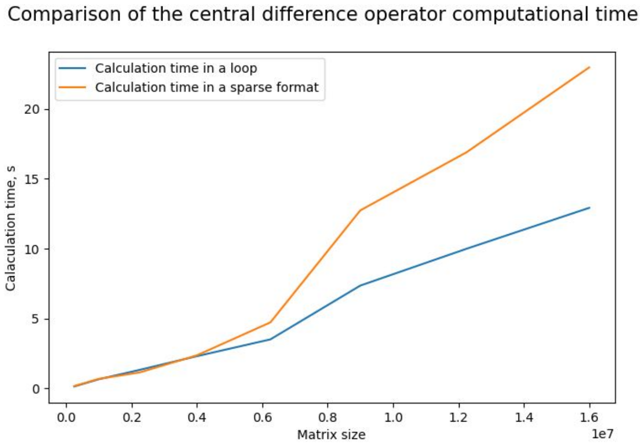
Laplacian operator

$$\Delta = \frac{1}{h^2}\begin{bmatrix} 4 & 1 & 0 & 1 & 0 \\ 1 & -4 & 1 & 0 & 1 \\ 0 & 1 & -4 & 1 & 0 \\ 1 & 0 & 1 & -4 & 1 \\ 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

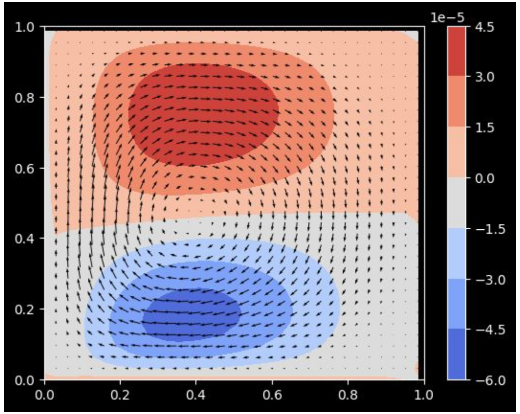


First order derivative

$$D = \frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$



Application to the N-S modeling

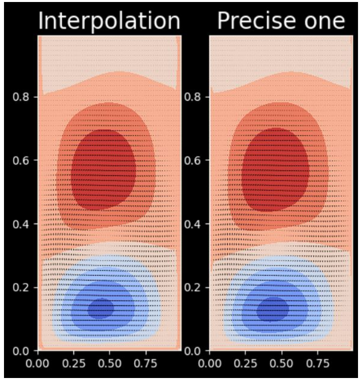


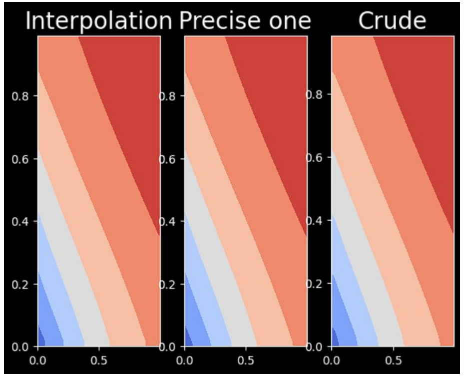
Interpolation of structured grid [1]

Velocities

Pressure

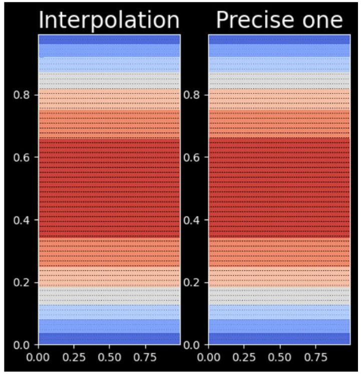
Nearest-Neighbour

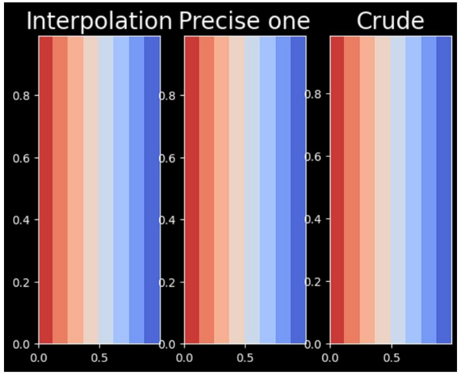




Nearest-neighbour interpolation - method selects the value of the nearest data point and assign that value

Linear





Polynomial interpolation when the interpolation constructed is a polynomial of degree 1. To construct linear interpolation, the equation is solving

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

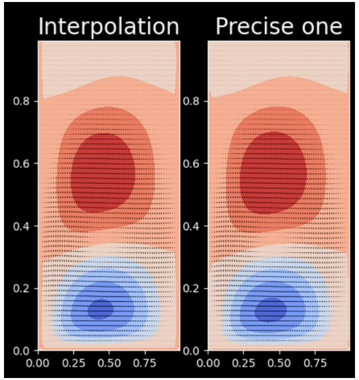
Interpolation of structured grid [2]

Cubic

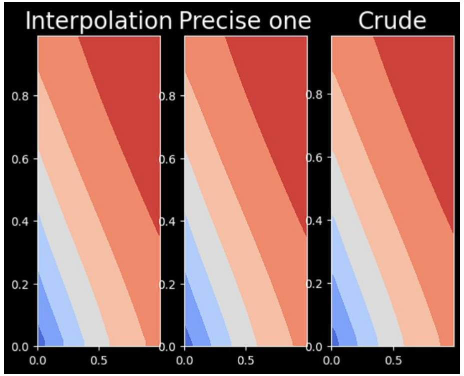
Interpolation using a third-degree polynomial

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

Whirlpool velocities

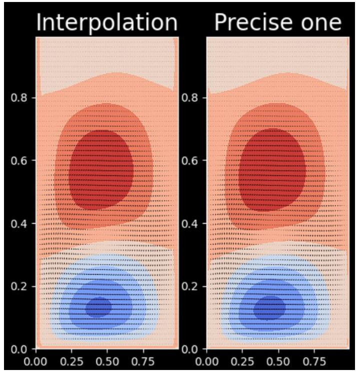


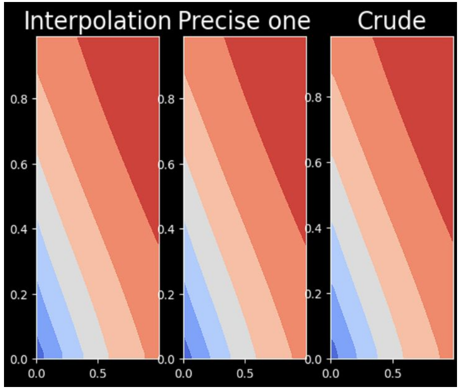
Whirlpool pressure



PCHIP (Piecewise Cubic Hermite Interpolating Polynomial)

Interpolation using monotonic cubic splines to find the value of new points





Interpolation precision

$$\frac{\|\hat{V} - V\|_2}{\|V\|_2}$$

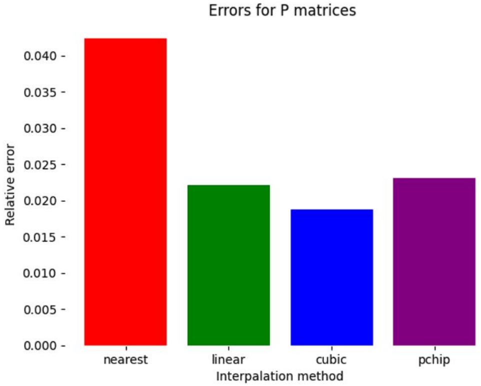
$$\frac{\|\hat{U} - U\|_2}{\|U\|_2}$$

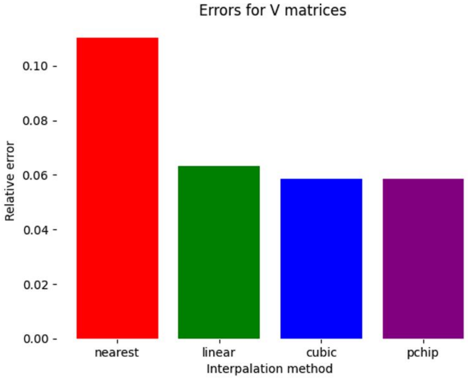
$$\frac{\|\hat{P} - P\|_2}{\|P\|_2}$$

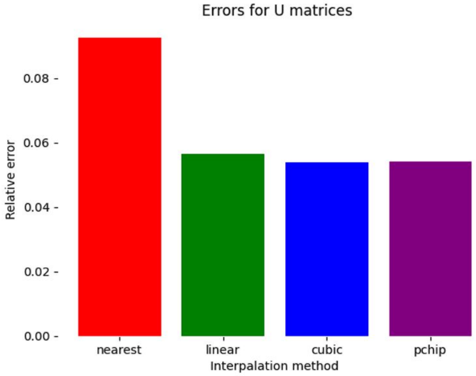
P, V, U - Matrices obtained by numerical method

$\hat{P}, \hat{V}, \hat{U}$ - Matrices obtained by interpolation

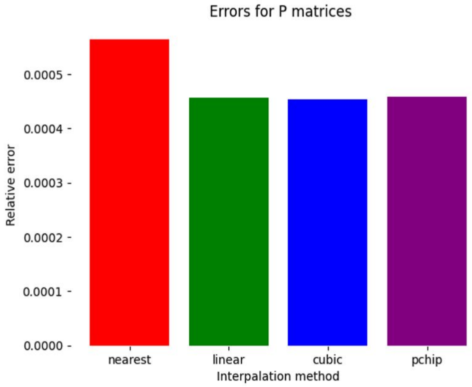
Whirlpool flow

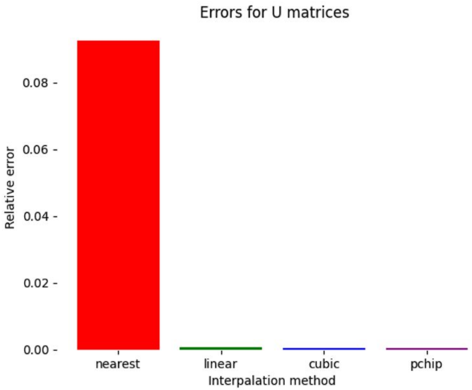






Pipe flow





Conclusions

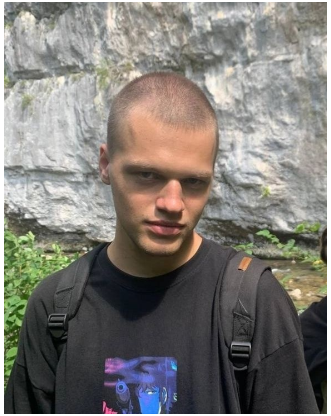
- The best interpolation method for whirlpool flow is cubic method
- The best interpolation method for pipe flow is linear method
- Sparse matrix speed up calculation of Navier-Stokes equation up to 50%

Eigenjoy Enthusiasts team

Determination of turbulent flow boundary conditions and their realisation in the Navier - Stokes equation. Calculations Jacobian method

Determination of laminar flow boundary conditions and their realisation in the Navier - Stokes equation. Calculations with sparse matrices.

Searching for Information interpolations of matrices Output results



Searching for Information. interpolations of matrices Output results

[NLA_project/intorpalation_of_whirlwind.ipynb at main · PavelMuravskii/NLA_project (github.com)]

Thank for your attention!

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