作业二

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第一题 本题考虑对于定义在 $[-1,1]$ 上的一个光滑函数 $f(x)$ 的三次样条插值的使用。下面所说的误差都是指绝对误差。

(a) （10分）仿照课堂笔记或课本推导出关于额外给定边界点处（即-1和1）三次样条插值多项式的一次导数值时其在各插值点上的二次导数值应该满足的线性方程组。请给出推导过程。

解:

记 $S(x)$ 在区间 $[x_i, x_{i+1}]$ 上的表达式为 $S_i(x), S(x)$ 是三次多项式, $S''(x)$ 是线性函数, 用插值点 $\{(x_i, S''(x_i)), (x_{i+1}, S''(x_{i+1}))\}$ 作线性插值, 记 $S''(x_i) = M_i$, $S''(x_{i+1}) = M_{i+1}$

$$S_i''(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} M_i + \frac{x - x_i}{x_{i+1} - x_i} M_{i+1}, \quad x_i \leqslant x \leqslant x_{i+1}$$

对 $S''(x)$ 积分两次, 记 $h_i = x_{i+1} - x_i$,

$$\begin{aligned} S(x) = S_i(x) &= \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + cx + d \\ &= \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + C(x_{i+1} - x) + D(x - x_i) \end{aligned}$$

将 $S(x_i) = y_i, S(x_{i+1}) = y_{i+1}$ 代入上式解出

$$C = \frac{y_i}{h_i} - \frac{h_i M_i}{6}, \quad D = \frac{y_{i+1}}{h_i} - \frac{h_i M_{i+1}}{6}$$

$$
\begin{aligned}
S(x)= & \frac{\left(x_{i+1}-x\right)^3 M_i+\left(x-x_i\right)^3 M_{i+1}}{6 h_i}+\frac{\left(x_{i+1}-x\right) y_i+\left(x-x_i\right) y_{i+1}}{h_i} \\
& -\frac{h_i}{6}\left[\left(x_{i+1}-x\right) M_i+\left(x-x_i\right) M_{i+1}\right], \quad x \in\left[x_i, x_{i+1}\right]
\end{aligned} \tag{1}
$$

在内结点 x_i, 由 $S_i^{\prime}\left(x_i\right)=S_{i-1}^{\prime}\left(x_i\right)$ 可得到

$$
f\left(x_i, x_{i+1}\right)-\frac{h_i}{3} M_i-\frac{h_i}{6} M_{i+1}=f\left(x_{i-1}, x_i\right)+\frac{h_{i-1}}{6} M_{i-1}+\frac{h_{i-1}}{3} M_i \tag{2}
$$

整理后得到

$$
\mu_i M_{i-1}+2 M_i+\lambda_i M_{i+1}=d_i, \quad i=1,2, \cdots, n-1 \tag{3}
$$

其中

$$
\begin{gathered}
\lambda_i=\tfrac{h_i}{h_i+h_{i-1}}, \quad \mu_i=1-\lambda_i \\
d_i=\tfrac{6}{h_i+h_{i-1}}\left(\tfrac{y_{i+1}-y_i}{h_i}-\tfrac{y_i-y_{i-1}}{h_{i-1}}\right)=6 f\left(x_{i-1}, x_i, x_{i+1}\right)
\end{gathered}
$$

式 2 称为样条插值的 M 关系方程组，解方程组 2 得到 $\{M_i, i=1,2,\cdots,M_{n-1}\}$, 再加上两个端点条件, 满足端点条件的样条插值函数 $S(x)$ 在 $[x_i,, x_{i+1}]$ 上的表达就是式 1.

给定 $S'\left(x_0\right)=m_0, S'\left(x_n\right)=m_n$ 的值, 将 $S'\left(x_0\right)=m_0, S'\left(x_n\right)=m_n$ 的值分别代入 $S'(x)$ 在 $\left[x_0, x_1\right],\left[x_{n-1}, x_n\right]$ 中的表达式, 得到另外两个方程:

$$
\begin{gathered}
2 M_0+M_1=\tfrac{6}{h_0}\left[f\left[x_0, x_1\right]-m_0\right]=d_0 \\
M_{n-1}+2 M_n=\tfrac{6}{h_{n-1}}\left[m_n-f\left[x_{n-1}, x_n\right]\right]=d_n
\end{gathered}
$$

得到 $n+1$ 个未知量, $n+1$ 个方程组

$$
\left[\begin{array}{cccccc}
2 & 1 & & & & \\
\mu_1 & 2 & \lambda_1 & & & \\
& \mu_2 & 2 & \lambda_2 & & \\
& & \ddots & \ddots & \ddots & \\
& & & \mu_{n-2} & 2 & \lambda_{n-1} \\
& & & & 1 & 2
\end{array}\right]\left[\begin{array}{c}
M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n
\end{array}\right]=\left[\begin{array}{c}
d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n
\end{array}\right]
$$

(b) （10分）令三次样条插值多项式在−1和1处的导数为0, 用**Matlab**基于上一问中的结果使用 $n=2^4$ 个子区间插值一个定义在 $[-1,1]$ 上的函数 $f(x)=\sin\left(4x^2\right)+\sin^2(4x)$ 并使用semilogy图通过在2000个等距点上取真实值画出你构造的三次样条插值的逐点误差。

解:

本题的**MATLAB**程序显示如下:

```
clear,clc
syms x;
F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
n = 2^4;
A = eye(n + 1);
A = 2 * A;
A(1, 2) = 1;
A(n + 1, n) = 1;
h = (1 - (-1)) / n;
lambda = 1/2;
mu = 1 - lambda;

for i = 2:n
    A(i, i - 1) = mu;
    A(i, i + 1) = lambda;
end

y = [0:1:n];
xx = [0:1:n];

for i = 1:n + 1
    xx(i) = -1 + (i - 1) * h;
    y(i) = F(xx(i));
end

d = [0:1:n];

for i = 2:n
    d(i) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) / (h
       ^2);
end

d(1) = 6 / h * ((y(2) - y(1)) / h - 0);
d(n + 1) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
d = d';
```

```
M = A \ d;
k = 1;
xk_1 = xx(2);
xk = xx(1);
h_new = 2/2000;
t = -1;
C = y(1) / h - h * M(1) / 6;
D = y(2) / h - h * M(2) / 6;
error_delta = [0:1:n];

for i = 0:2000
    t = -1 + i * h_new;

    if (t >= xk && t <= xk_1) || t > 1
        s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) + D * (t - xk);
        error_delta(i + 1) = abs(s - F(t));

    else
        k = k + 1;
        xk_1 = xx(k + 1);
        xk = xx(k);
        C = y(k) / h - h * M(k) / 6;
        D = y(k + 1) / h - h * M(k + 1) / 6;
        s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) + D * (t - xk);
        error_delta(i + 1) = abs(s - F(t));
    end

end

semilogy([-1:h_new:1], error_delta)
```

程序运行输出结果为:

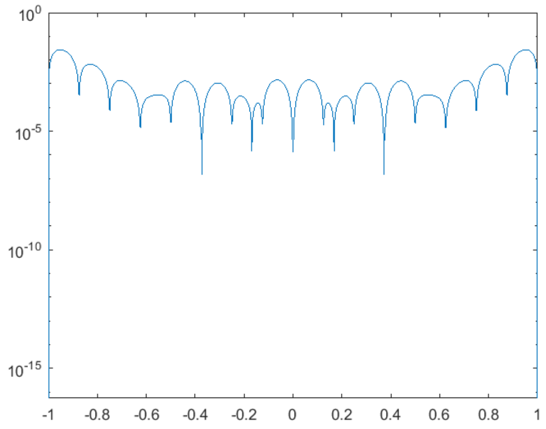

图 1: 三次样条插值的逐点误差

(c) （15分）使用不同的 n，令 $n = 2^4, 2^5, \ldots, 2^{10}$ 重复上一问，取关于不同 n 的2000个等距点上的误差的最大值，用loglog图描述插值区间上最大误差值随 n 变化的情况（即横轴是 n ）。

解：

本题的**MATLAB**程序显示如下：

```
clear,clc
syms x;
F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
err = [4:10];

for g = 4:10
    n = 2^g;
    A = eye(n + 1);
    A = 2 * A;
    A(1, 2) = 1;
    A(n + 1, n) = 1;
    h = (1 - (-1)) / n;
    lambda = 1/2;
    mu = 1 - lambda;

    for i = 2:n
        A(i, i - 1) = mu;
        A(i, i + 1) = lambda;
```

```
    end

    y = [0:1:n];
    xx = [0:1:n];

    for i = 1:n + 1
        xx(i) = -1 + (i - 1) * h;
        y(i) = F(xx(i));
    end

    d = [0:1:n];

    for i = 2:n
        d(i) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) / (h^2);
    end

    d(1) = 6 / h * ((y(2) - y(1)) / h - 0);
    d(n + 1) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
    d = d';
    M = A \ d;
    k = 1;
    xk_1 = xx(2);
    xk = xx(1);
    h_new = 2/2000;
    t = -1;
    C = y(1) / h - h * M(1) / 6;
    D = y(2) / h - h * M(2) / 6;
    error_delta = [0:1:n];

    for i = 0:2000
        t = -1 + i * h_new;

        if (t >= xk && t <= xk_1) || t > 1
            s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
```

```
                xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                 - t) + D * (t - xk);
            error_delta(i + 1) = abs(s - F(t));

        else
            k = k + 1;
            xk_1 = xx(k + 1);
            xk = xx(k);
            C = y(k) / h - h * M(k) / 6;
            D = y(k + 1) / h - h * M(k + 1) / 6;
            s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
                xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                 - t) + D * (t - xk);
            error_delta(i + 1) = abs(s - F(t));
        end

    end

    err(g - 3) = max(error_delta);
end
loglog([4:10], err);
```

程序运行输出结果为:

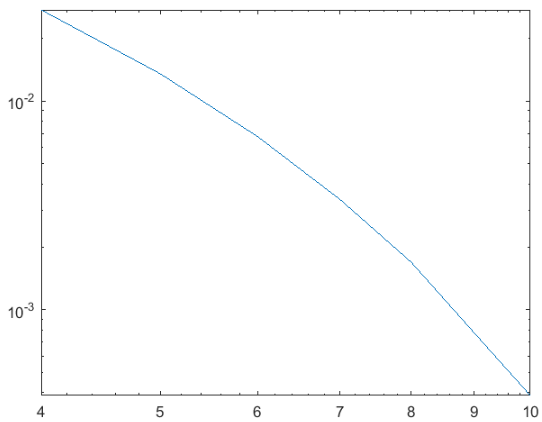

图 2: 三次样条插值区间上最大误差值随 n 变化的情况

(d) （15分）针对周期边界条件，即假设三次样条函数满足 $S'(-1)=S'(1)$ 和 $S''(-1)=S''(1)$, 重复完成上面三问中的要求。

解：

在题设的边界条件下， 由 $S''(x_0+0)=S''(x_n-0)$ 可得 $M_0=M_n$。再由条件 $S'(x_0+0)=S'(x_n-0)$ 可得

$$-M_0\cdot\frac{h_1}{2}+\frac{y_1-y_0}{h_1}-\frac{h_1}{6}(M_1-M_0)=M_n\cdot\frac{h_n}{2}+\frac{y_n-y_{n-1}}{h_n}-\frac{h_n}{6}(M_n-M_{n-1})$$

只要注意到 $y_0=y_n, M_0=M_n$, 整理上式即得

$$\lambda_n M_1+\mu_n M_{n-1}+2M_n=\frac{6}{h_1+h_n}\left(\frac{y_1-y_0}{h_1}-\frac{y_n-y_{n-1}}{h_n}\right) \tag{4}$$

由式 3 和式 4, 可确定 $M_1,M_2,\cdots,M_n$ 的线性方程组为

$$\begin{bmatrix} 2 & \lambda_1 & & & \mu_1 \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ \lambda_n & & & \mu_n & 2 \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

本题（重复(b)）的**MATLAB**程序显示如下：

```
clear,clc
syms x;
F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
n = 2^4;
A = eye(n);
A = 2 * A;
h = (1 - (-1)) / n;
lambda = 1/2;
mu = 1 - lambda;
A(1, 2) = lambda;
A(n, 1) = lambda;
A(1, n) = mu;
A(n, n - 1) = mu;

for i = 2:n - 1
    A(i, i - 1) = mu;
```

```
    A(i, i + 1) = lambda;
end

y = [0:1:n];
xx = [0:1:n];

for i = 1:n + 1
    xx(i) = -1 + (i - 1) * h;
    y(i) = F(xx(i));
end

d = [1:1:n];

for i = 2:n
    d(i - 1) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) /
        (h^2);
end

d(n) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
d = d';
M = A \ d;
tmp = M;
M(1) = tmp(n);

for i = 1:n
    M(i + 1) = tmp(i);
end

k = 1;
xk_1 = xx(2);
xk = xx(1);
h_new = 2/2000;
t = -1;
C = y(1) / h - h * M(1) / 6;
D = y(2) / h - h * M(2) / 6;
```

```
error_delta = [0:1:n];

for i = 0:2000
    t = -1 + i * h_new;

    if (t >= xk && t <= xk_1) || t > 1
        s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
           ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
            D * (t - xk);
        error_delta(i + 1) = abs(s - F(t));

    else
        k = k + 1;
        xk_1 = xx(k + 1);
        xk = xx(k);
        C = y(k) / h - h * M(k) / 6;
        D = y(k + 1) / h - h * M(k + 1) / 6;
        s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
           ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
            D * (t - xk);
        error_delta(i + 1) = abs(s - F(t));
    end

end

semilogy([-1:h_new:1], error_delta)
```

程序运行输出结果为：

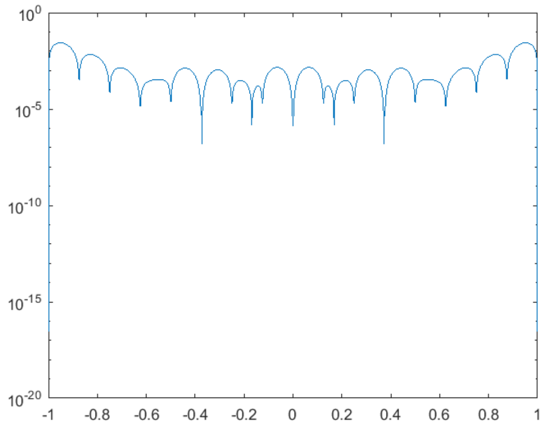

图 3: 三次样条插值的逐点误差(周期边界条件)

本题的MATLAB程序显示如下:

```
clear,clc
syms x;
F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
err = [4:10];

for g = 4:10
    n = 2^g;
    A = eye(n);
    A = 2 * A;
    h = (1 - (-1)) / n;
    lambda = 1/2;
    mu = 1 - lambda;
    A(1, 2) = lambda;
    A(n, 1) = lambda;
    A(1, n) = mu;
    A(n, n - 1) = mu;

    for i = 2:n - 1
        A(i, i - 1) = mu;
        A(i, i + 1) = lambda;
    end

```

```
    y = [0:1:n];
    xx = [0:1:n];

    for i = 1:n + 1
        xx(i) = -1 + (i - 1) * h;
        y(i) = F(xx(i));
    end

    d = [1:1:n];

    for i = 2:n
        d(i - 1) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) / (h^2);
    end

    d(n) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
    d = d';
    M = A \ d;
    tmp = M;
    M(1) = tmp(n);

    for i = 1:n
        M(i + 1) = tmp(i);
    end

    k = 1;
    xk_1 = xx(2);
    xk = xx(1);
    h_new = 2/2000;
    t = -1;
    C = y(1) / h - h * M(1) / 6;
    D = y(2) / h - h * M(2) / 6;
    error_delta = [0:1:n];

    for i = 0:2000
```

```
            t = -1 + i * h_new;

            if (t >= xk && t <= xk_1) || t > 1
                s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
                    xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                     - t) + D * (t - xk);
                error_delta(i + 1) = abs(s - F(t));

            else
                k = k + 1;
                xk_1 = xx(k + 1);
                xk = xx(k);
                C = y(k) / h - h * M(k) / 6;
                D = y(k + 1) / h - h * M(k + 1) / 6;
                s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
                    xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                     - t) + D * (t - xk);
                error_delta(i + 1) = abs(s - F(t));
            end

        end

        err(g - 3) = max(error_delta);
end

loglog([4:10], err);
```

程序运行输出结果为:

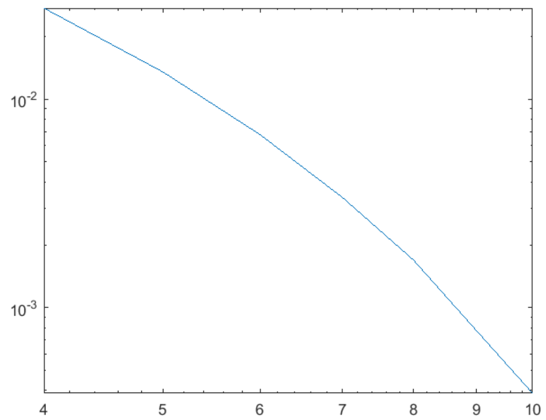

图 4: 三次样条插值区间上最大误差值随 n 变化的情况（周期边界条件）

第二题 本题深入讨论Newton插值公式的性质。

(a) （15分）对于一个光滑函数 $f(x)$, 证明若 $\{i_0, i_1, \ldots, i_k\}$ 是 $\{0, 1, \ldots, k\}$ 的任意一个排列，则

$$f\left[x_0, x_1, \ldots, x_k\right]=f\left[x_{i_0}, x_{i_1}, \ldots, x_{i_k}\right]$$

证：

先证引理：k 阶差商 $f\left[x_0, x_1, \cdots, x_k\right]$ 是由函数值 $f\left(x_0\right), f\left(x_1\right), \cdots, f\left(x_k\right)$的线性组合而成.

$$f\left[x_0, x_1, \cdots, x_k\right]=\sum_{i=0}^k \frac{1}{\left(x_i-x_0\right) \cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right) \cdots\left(x_i-x_k\right)} f\left(x_i\right)$$

用归纳法可以证明这一引理。

显然，当$k=1$时，引理成立。

假设当$k=n-1$时，引理成立,故有：

$$f\left[x_0, x_1, \cdots, x_{n-1}\right]=\sum_{i=0}^{n-1} \frac{1}{\left(x_i-x_0\right) \cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right) \cdots\left(x_i-x_{n-1}\right)} f\left(x_i\right)$$

$$f\left[x_1, x_2, \cdots, x_n\right]=\sum_{i=1}^n \frac{1}{\left(x_i-x_1\right) \cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right) \cdots\left(x_i-x_n\right)} f\left(x_i\right)$$

则$k=n$时：

$$\begin{aligned} f\left[x_0, x_1, \cdots, x_n\right] & =\frac{f\left[x_0, x_1, \cdots, x_{n-1}\right]-f\left[x_1, x_2, \cdots, x_n\right]}{x_0-x_n} \\ & =\sum_{i=0}^k \frac{1}{\left(x_i-x_0\right) \cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right) \cdots\left(x_i-x_n\right)} f\left(x_i\right)\end{aligned}$$

引理证毕，故有：

$$f\left[x_0, x_1, \cdots, x_k\right]=\sum_{i=0}^{k} \frac{1}{\left(x_i-x_0\right) \cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right) \cdots\left(x_i-x_k\right)} f\left(x_i\right)$$

$$f\left[x_{i_0}, x_{i_1}, \cdots, x_{i_k}\right]=\sum_{j=0}^{k} \frac{1}{\left(x_{i_j}-x_{i_0}\right) \cdots\left(x_{i_j}-x_{i_{j-1}}\right)\left(x_{i_j}-x_{i_{j+1}}\right) \cdots\left(x_{i_j}-x_{i_k}\right)} f\left(x_{i_j}\right)$$

又因$\{x_0, x_1, \cdots, x_k\}$与$\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\}$之间存在双射,易得:

$$f\left[x_0, x_1, \ldots, x_k\right]=f\left[x_{i_0}, x_{i_1}, \ldots, x_{i_k}\right]$$

(b) （10分）课堂上我们提到了Chebyshev点

$$x_j=\cos(j\pi/n) \quad j=0,1,\ldots,n$$

以及使用Chebyshev点可以有效地克服Runge现象。写一个**MATLAB**程序，令 $n=2^2,2^3,2^4,\ldots,2^7$, 按照从右到左的顺序（即 j 从小到大的顺序）使用对应的 $n+1$ 个Chebyshev点对定义在 $[-1,1]$ 上的Runge函数

$$f(x)=\frac{1}{1+25x^2}$$

进行插值，并取2000个等距点上的误差的最大值，用semilogy图描述插值区间上最大误差值随 n 变化的情况（即横轴是 n）。

解:

本题的**MATLAB**程序显示如下:

```
clear,clc

syms x;
F = @(x) 1 / (1 + 25 * x^2);
y = [0:1:2000];

for i = 0:2000
    y(i + 1) = F(-1 + i * 2/2000);
end

result = [0:1:2000];
delta_result = [2:1:7];

```

```
for n = 2:7
    count = 2^n;
    chebyshev = [0:1:count];

    for j = 0:1:count
        chebyshev(j + 1) = cos(j * pi / (count));
    end

    d_f = [1:1:count];
    g = [0:1:count];

    for i = 1:count + 1
        g(i) = F(chebyshev(i));
    end

    for k = 2:count + 1

        for i = count + 1:-1:k
            g(i) = (g(i) - g(i - 1)) / (chebyshev(i)
               - chebyshev(i - k + 1));
        end

    end

    for i = 0:2000
        a = -1 + i * 2/2000;
        result(i + 1) = F(chebyshev(1));

        for j = 1:count
            tmp = 1;

            for k = 1:j
                tmp = tmp * (a - chebyshev(k));
            end

```

```
            result(i + 1) = result(i + 1) + tmp * g(j + 1);
        end

    end

    tmp = [0:1:2000];

    for i = 0:2000
        tmp(i + 1) = abs(result(i + 1) - y(i + 1));
    end

    delta_result(n - 1) = max(tmp);
end

semilogy([2:7], delta_result);
```

程序运行输出结果为:

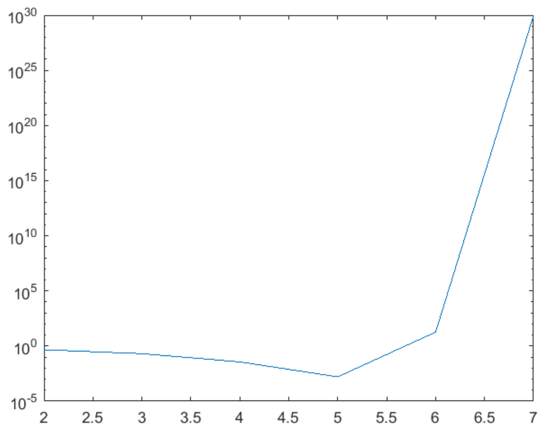

图 5: Newton插值区间上误差值随n变化的情况（Chebyshev点按从左到右顺序使用）

(c) (10分) 重复上一问，但使用随机数种子rng(22)和randperm函数来随机计算差商时插值点的使用顺序，取关于不同 n 的2000个等距点上的误差的最大值，用semilogy图描述插值区间上最大误差值随 n 变化的情况（即横轴是 n）。

解:

本题的MATLAB程序显示如下:

```
clear,clc

syms x;
F = @(x) 1 / (1 + 25 * x^2);
y = [0:1:2000];

for i = 0:2000
    y(i + 1) = F(-1 + i * 2/2000);
end

result = [0:1:2000];
delta_result = [2:1:7];

for n = 2:7
    count = 2^n;
    chebyshev = [0:1:count];
    rng(22);
    r = randperm(count + 1);

    for j = 1:1:count + 1
        chebyshev(j) = cos((r(j) - 1) * pi / (count))
            ;
    end

    d_f = [1:1:count];
    g = [0:1:count];

    for i = 1:count + 1
        g(i) = F(chebyshev(i));
    end

    for k = 2:count + 1

        for i = count + 1:-1:k
```

```
                g(i) = (g(i) - g(i - 1)) / (chebyshev(i) - chebyshev(i - k + 1));
            end

    end

    for i = 0:2000
        a = -1 + i * 2/2000;
        result(i + 1) = F(chebyshev(1));

        for j = 1:count
            tmp = 1;

            for k = 1:j
                tmp = tmp * (a - chebyshev(k));
            end

            result(i + 1) = result(i + 1) + tmp * g(j + 1);
        end

    end

    tmp = [0:1:2000];

    for i = 0:2000
        tmp(i + 1) = abs(result(i + 1) - y(i + 1));
    end

    delta_result(n - 1) = max(tmp);
end

semilogy([2:7], delta_result);
```

程序运行输出结果为:

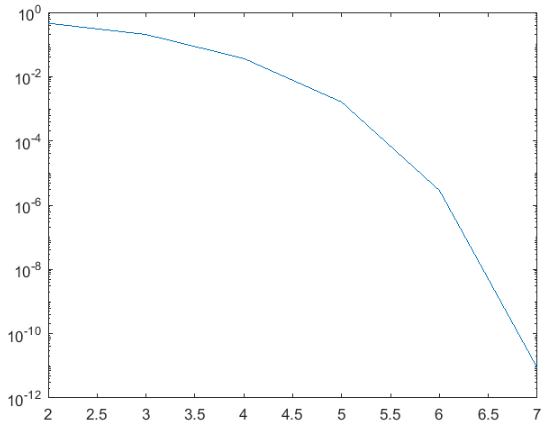

图 6: Newton插值区间上误差值随n变化的情况（Chebyshev点按随机顺序使用）

(d) （10分）试着解释上面两小问中你观察到的不同现象产生的原因。注：此问答不出来也无妨。

解：

首先观察Chebyshev点按从左到右顺序使用时，拟合结果的semilogy图：

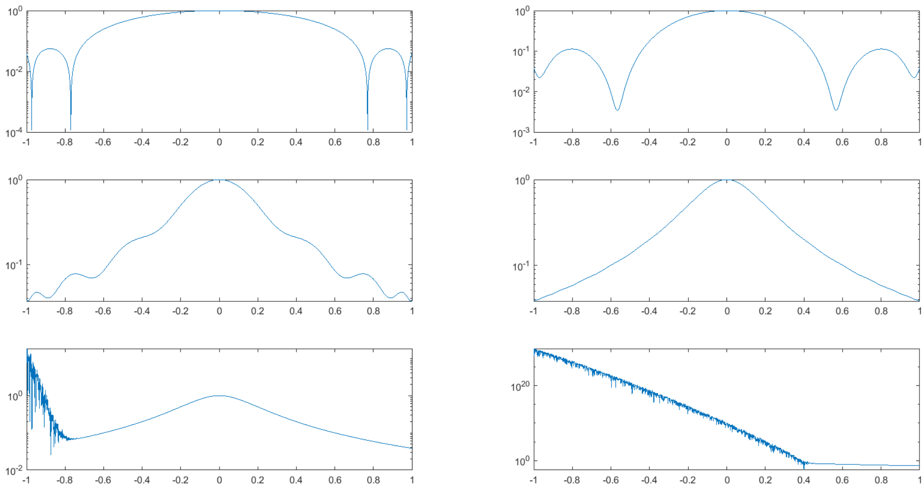

图 7: 拟合结果的semilogy图（n从左向右，从上到下递增）

可以发现，在$n = 6, 7$时，当x趋于-1时，拟合结果的数量级逐渐增大。

下面单独取出$x = -1$进行分析，进行数值实验。

由$Newton$插值多项式：

$$N(x) = f\left[x_0\right] + \sum_{k=1}^{n} f\left[x_0, x_1, \cdots, x_k\right]\left(x - x_0\right)\left(x - x_1\right)\cdots\left(x - x_{k-1}\right)$$

逐一分析累加项的各项的数量级，当Chebyshev点按从左到右顺序使用时,有：

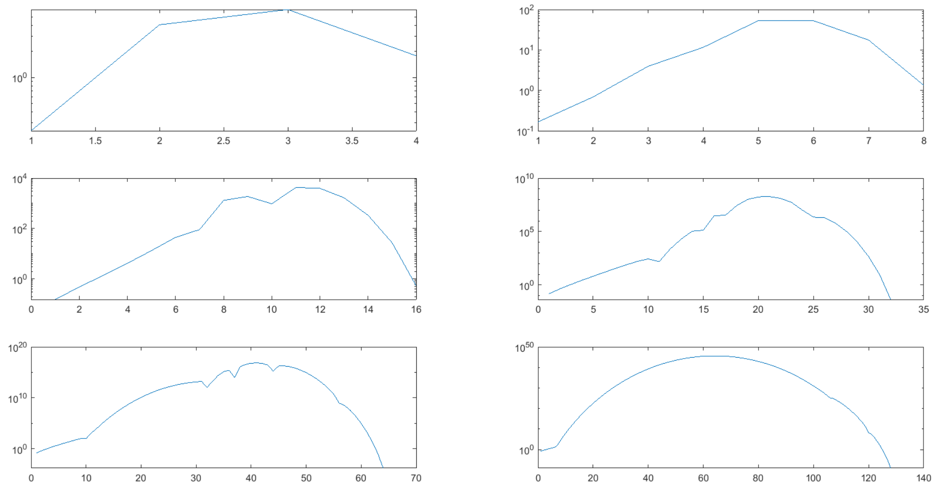

图 8: 累加项的各项的数量级（Chebyshev点顺序使用）（n从左向右，从上到下递增）

当Chebyshev点按随机顺序使用时,有：

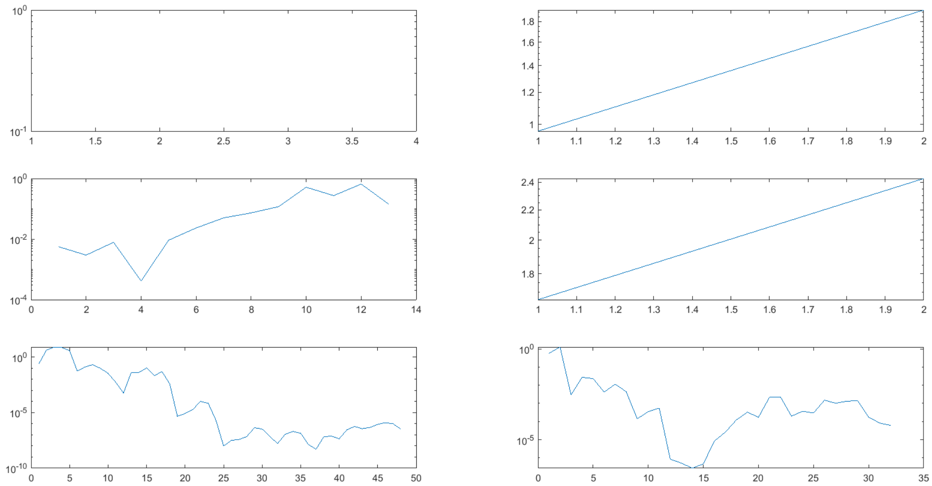

图 9: 累加项的各项的数量级（Chebyshev点乱序使用）（n从左向右，从上到下递增）

图0d中未标出的点表示该项为0。

第三题 本题用于讨论周期函数的Lagrange插值方法。对于周期函数而言，多项式不再是最有效的基函数，而等距插值点也不再会出现Runge现象。逼近周期函数的基函

数通常选用三角函数或者复指数。同时注意对于周期函数而言，插值点数量和子区间个数相等。

(a) （10分）在 $[0,1]$ 上关于周期函数的基于等间距插值点 $x_j=\frac{j}{n}, j=0,1,\ldots,n-1$ 的Lagrange插值基函数为

$$\ell_k(x)=\begin{cases}\frac{(-1)^k}{n}\sin(n\pi x)\csc\left(\pi\left(x-x_k\right)\right) & \text{若 } n \text{ 为奇数}\\ \frac{(-1)^k}{n}\sin(n\pi x)\cot\left(\pi\left(x-x_k\right)\right) & \text{若 } n \text{ 为偶数}\end{cases}$$

证明对于 n 分别为奇数和偶数的情况下

$$\ell_k\left(x_j\right)=\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}$$

证：

当n 为奇数时：

$$\ell_k\left(x_j\right)=\frac{(-1)^k}{n}\sin(n\pi x_j)\csc\left(\pi\left(x_j-x_k\right)\right)$$

将$x_j=\frac{j}{n}, x_k=\frac{k}{n}$代入得：

$$\ell_k\left(x_j\right)=\frac{(-1)^k}{n}\sin(j\pi)\csc\left(\frac{\pi}{n}\left(j-k\right)\right)$$

当$k\neq j$时，$\sin(j\pi)=0, \csc\left(\frac{\pi}{n}\left(j-k\right)\right)\neq 0$，故得：

$$\ell_k\left(x_j\right)=0, k\neq j, n \text{ 为奇数}$$

当$k=j$时，$\sin(j\pi)=0, \sin\left(\frac{\pi}{n}\left(j-k\right)\right)=0$,故有：

$$\begin{aligned}\ell_k\left(x_j\right)&=\frac{(-1)^k}{n}\sin(j\pi)\csc\left(\frac{\pi}{n}\left(j-k\right)\right)\\&=\lim_{x\to x_k}\frac{(-1)^k}{n}\sin(n\pi x)\csc\left(\pi\left(x-x_k\right)\right)\\&=\lim_{j\to k}\frac{(-1)^k}{n}\sin(j\pi)\csc\left(\frac{\pi}{n}\left(j-k\right)\right)\\&=\lim_{j\to k}\frac{(-1)^k}{n}\frac{\sin(j\pi)}{\sin\left(\frac{\pi}{n}\left(j-k\right)\right)}\\&=\frac{(-1)^k}{n}\lim_{j\to k}\frac{\pi\cos(j\pi)}{\frac{\pi}{n}\cos\left(\frac{\pi}{n}\left(j-k\right)\right)}\\&=\frac{(-1)^k}{n}\frac{\pi\cos(k\pi)}{\frac{\pi}{n}\cos\left(0\right)}\\&=1\end{aligned}$$

即：

$$\ell_k(x_j)=1, k=j, n \text{ 为奇数}$$

当n 为偶数时：

$$\ell_k(x_j)=\frac{(-1)^k}{n}\sin(n\pi x_j)\cot(\pi(x_j-x_k))$$

将$x_j=\frac{j}{n}, x_k=\frac{k}{n}$代入得：

$$\ell_k(x_j)=\frac{(-1)^k}{n}\sin(j\pi)\cot\left(\frac{\pi}{n}(j-k)\right)$$

当$k\neq j$时，$\sin(j\pi)=0, \cot\left(\frac{\pi}{n}(j-k)\right)\neq 0$，故得：

$$\ell_k(x_j)=0, k\neq j, n \text{ 为偶数}$$

当$k=j$时，$\sin(j\pi)=0, \sin\left(\frac{\pi}{n}(j-k)\right)=0$,故有：

$$\begin{aligned}
\ell_k(x_j) &= \frac{(-1)^k}{n}\sin(j\pi)\cot\left(\frac{\pi}{n}(j-k)\right)\\
&= \lim_{x\to x_k}\frac{(-1)^k}{n}\sin(n\pi x)\cot(\pi(x-x_k))\\
&= \lim_{j\to k}\frac{(-1)^k}{n}\sin(j\pi)\cot\left(\frac{\pi}{n}(j-k)\right)\\
&= \lim_{j\to k}\frac{(-1)^k}{n}\cos\left(\frac{\pi}{n}(j-k)\right)\frac{\sin(j\pi)}{\sin\left(\frac{\pi}{n}(j-k)\right)}\\
&= \frac{(-1)^k}{n}\lim_{j\to k}\cos\left(\frac{\pi}{n}(j-k)\right)\lim_{j\to k}\frac{\pi\cos(j\pi)}{\frac{\pi}{n}\cos\left(\frac{\pi}{n}(j-k)\right)}\\
&= \frac{(-1)^k}{n}\frac{\pi\cos(k\pi)}{\frac{\pi}{n}\cos(0)}\\
&= 1
\end{aligned}$$

即：

$$\ell_k(x_j)=1, k=j, n \text{ 为偶数}$$

综上：

$$\ell_k(x_j)=\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}$$

(b) （10分）用上述对应于 n 为偶数的Lagrange基函数构造Lagrange插值多项式. 并用 $n=2^6$ 个点对周期函数 $f(x)=\sin(2\pi x)e^{\cos(2\pi x)}$ 在 $[0,1]$ 上进行插值。取1000个等距点上的误差，用semilogy图描述插值区间上误差值随 x 变化的情况（即横轴是 x ）。

解:

本题的MATLAB程序显示如下:

```
clear,clc
syms x;
syms k;
n = 2^6;
F = @(x) sin(2 * pi * x) * exp(cos(2 * pi * x));
l = @(x, k) ((-1)^k) / n * sin(n * pi * x) * cot(pi *
    (x - k / n));
result=[0:1:1999];
y=[0:1:1999];
delta_y=[0:1:1999];
for i=0:1999
    y(i+1)=F(i/2000);
    tmp=0;

    for j=0:n-1
        tmp=tmp+l(i/2000,j)*F(j/n);
    end
    result(i+1)=tmp;
    delta_y(i+1)=abs(result(i+1)-y(i+1));
end
x=[0:1/2000:1-1/2000];
semilogy(x,delta_y)
```

程序运行输出结果为:

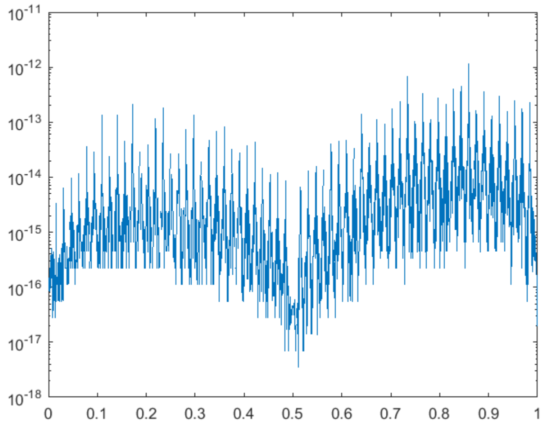

图 10: Lagrange插值区间上误差值随x变化的情况

第四题 （10分）写程序完成课本 59 页第7题，并计算出你的拟合函数对比所给数据点的误差的2-范数。

解:

本题的**MATLAB**程序显示如下:

```
clc, clear
x = [2.1, 2.5, 2.8, 3.2];
y = [0.6087, 0.6849, 0.7368, 0.8111];
a_1 = 0;
b_1 = 0;
x_y = x .* y;
c_11 = sum(x .* x);
c_12 = -sum(x .* x_y);
c_21 = -c_12;
c_22 = -sum(x_y .* x_y);
B_1 = sum(x .* y);
B_2 = sum(x_y .* y);
C = [c_11, c_12; c_21, c_22];
B = [B_1; B_2];
tmp = C \ B;
a_1 = tmp(1);
b_1 = tmp(2);
a = 1 / a_1;
b = b_1 / a_1;
```

```
y_bar = [Phi(x(1), a, b), Phi(x(2), a, b), Phi(x(3), a, b), Phi(x(4), a, b)];
delta_y = y - y_bar;
X = sprintf('The 2-norm of the error of the fitting function compared to the given data point is %.15f', norm(delta_y, 2));
disp(X)

function value = Phi(x, a, b)
    value = x / (a + b * x);
end
```

程序运行输出结果为：

```
The 2-norm of the error of the fitting function compared to the given data point is 0.005738349475781
```