

作业二

姓名胡毅翔 学号 PB18000290 日期 2021年5月29日

第一题 本题考虑对于定义在 $[-1, 1]$ 上的一个光滑函数 $f(x)$ 的三次样条插值的使用。下面所说的误差都是指绝对误差。

- (a) (10分) 仿照课堂笔记或课本推导出关于额外给定边界点处 (即-1和1) 三次样条插值多项式的一次导数值时其在各插值点上的二次导数值应该满足的线性方程组。请给出推导过程。

解:

记 $S(x)$ 在区间 $[x_i, x_{i+1}]$ 上的表达式为 $S_i(x)$, $S(x)$ 是三次多项式, $S''(x)$ 是线性函数, 用插值点 $\{(x_i, S''(x_i)), (x_{i+1}, S''(x_{i+1}))\}$ 作线性插值, 记 $S''(x_i) = M_i$, $S''(x_{i+1}) = M_{i+1}$

$$S''_i(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} M_i + \frac{x - x_i}{x_{i+1} - x_i} M_{i+1}, \quad x_i \leq x \leq x_{i+1}$$

对 $S''(x)$ 积分两次, 记 $h_i = x_{i+1} - x_i$,

$$\begin{aligned} S(x) = S_i(x) &= \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + cx + d \\ &= \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1} + C(x_{i+1} - x) + D(x - x_i) \end{aligned}$$

将 $S(x_i) = y_i$, $S(x_{i+1}) = y_{i+1}$ 代入上式解出

$$C = \frac{y_i}{h_i} - \frac{h_i M_i}{6}, \quad D = \frac{y_{i+1}}{h_i} - \frac{h_i M_{i+1}}{6}$$

$$S(x) = \frac{(x_{i+1} - x)^3 M_i + (x - x_i)^3 M_{i+1}}{6h_i} + \frac{(x_{i+1} - x)y_i + (x - x_i)y_{i+1}}{h_i} - \frac{h_i}{6} [(x_{i+1} - x)M_i + (x - x_i)M_{i+1}], \quad x \in [x_i, x_{i+1}] \quad (1)$$

在内结点 x_i , 由 $S'_i(x_i) = S'_{i-1}(x_i)$ 可得到

$$f(x_i, x_{i+1}) - \frac{h_i}{3}M_i - \frac{h_i}{6}M_{i+1} = f(x_{i-1}, x_i) + \frac{h_{i-1}}{6}M_{i-1} + \frac{h_{i-1}}{3}M_i \quad (2)$$

整理后得到

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, 2, \dots, n-1 \quad (3)$$

其中

$$\lambda_i = \frac{h_i}{h_i + h_{i-1}}, \quad \mu_i = 1 - \lambda_i$$

$$d_i = \frac{6}{h_i + h_{i-1}} \left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}} \right) = 6f(x_{i-1}, x_i, x_{i+1})$$

式 2 称为样条插值的 M 关系方程组, 解方程组 2 得到 $\{M_i, i = 1, 2, \dots, M_{n-1}\}$, 再加上两个端点条件, 满足端点条件的样条插值函数 $S(x)$ 在 $[x_i, x_{i+1}]$ 上的表达就是式 1.

给定 $S'(x_0) = m_0, S'(x_n) = m_n$ 的值, 将 $S'(x_0) = m_0, S'(x_n) = m_n$ 的值分别代入 $S'(x)$ 在 $[x_0, x_1], [x_{n-1}, x_n]$ 中的表达式, 得到另外两个方程:

$$2M_0 + M_1 = \frac{6}{h_0} [f[x_0, x_1] - m_0] = d_0$$

$$M_{n-1} + 2M_n = \frac{6}{h_{n-1}} [m_n - f[x_{n-1}, x_n]] = d_n$$

得到 $n+1$ 个未知量, $n+1$ 个方程组

$$\begin{bmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-2} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

- (b) (10分) 令三次样条插值多项式在-1和1处的导数为0, 用Matlab基于上一问中的结果使用 $n = 2^4$ 个子区间插值一个定义在 $[-1, 1]$ 上的函数 $f(x) = \sin(4x^2) + \sin^2(4x)$ 并使用semilogy图通过在2000个等距点上取真实值画出你构造的三次样条插值的逐点误差。

解:

本题的MATLAB程序显示如下:

```

1  clear,clc
2  syms x;
3  F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
4  n = 2^4;
5  A = eye(n + 1);
6  A = 2 * A;
7  A(1, 2) = 1;
8  A(n + 1, n) = 1;
9  h = (1 - (-1)) / n;
10 lambda = 1/2;
11 mu = 1 - lambda;
12
13 for i = 2:n
14     A(i, i - 1) = mu;
15     A(i, i + 1) = lambda;
16 end
17
18 y = [0:1:n];
19 xx = [0:1:n];
20
21 for i = 1:n + 1
22     xx(i) = -1 + (i - 1) * h;
23     y(i) = F(xx(i));
24 end
25
26 d = [0:1:n];
27
28 for i = 2:n
29     d(i) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) / (h
        ^2);
30 end
31
32 d(1) = 6 / h * ((y(2) - y(1)) / h - 0);
33 d(n + 1) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
34 d = d';

```

```

35 M = A \ d;
36 k = 1;
37 xk_1 = xx(2);
38 xk = xx(1);
39 h_new = 2/2000;
40 t = -1;
41 C = y(1) / h - h * M(1) / 6;
42 D = y(2) / h - h * M(2) / 6;
43 error_delta = [0:1:n];
44
45 for i = 0:2000
46     t = -1 + i * h_new;
47
48     if (t >= xk && t <= xk_1) || t > 1
49         s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
           ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
           D * (t - xk);
50         error_delta(i + 1) = abs(s - F(t));
51
52     else
53         k = k + 1;
54         xk_1 = xx(k + 1);
55         xk = xx(k);
56         C = y(k) / h - h * M(k) / 6;
57         D = y(k + 1) / h - h * M(k + 1) / 6;
58         s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
           ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
           D * (t - xk);
59         error_delta(i + 1) = abs(s - F(t));
60     end
61
62 end
63
64 semilogy([-1:h_new:1], error_delta)

```

程序运行输出结果为：

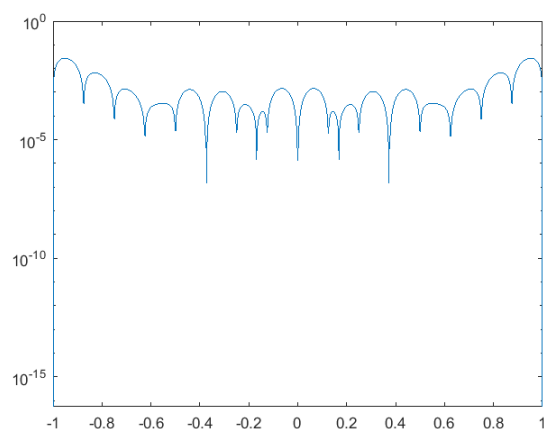


图 1: 三次样条插值的逐点误差

- (c) (15分) 使用不同的 n , 令 $n = 2^4, 2^5, \dots, 2^{10}$ 重复上一问, 取关于不同 n 的2000个等距点上的误差的最大值, 用loglog图描述插值区间上最大误差值随 n 变化的情况 (即横轴是 n)。

解:

本题的MATLAB程序显示如下:

```

1 clear, clc
2 syms x;
3 F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
4 err = [4:10];
5
6 for g = 4:10
7     n = 2^g;
8     A = eye(n + 1);
9     A = 2 * A;
10    A(1, 2) = 1;
11    A(n + 1, n) = 1;
12    h = (1 - (-1)) / n;
13    lambda = 1/2;
14    mu = 1 - lambda;
15
16    for i = 2:n
17        A(i, i - 1) = mu;
18        A(i, i + 1) = lambda;

```

```

19     end
20
21     y = [0:1:n];
22     xx = [0:1:n];
23
24     for i = 1:n + 1
25         xx(i) = -1 + (i - 1) * h;
26         y(i) = F(xx(i));
27     end
28
29     d = [0:1:n];
30
31     for i = 2:n
32         d(i) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) /
33             (h^2);
34
35     end
36
37     d(1) = 6 / h * ((y(2) - y(1)) / h - 0);
38     d(n + 1) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
39     d = d';
40     M = A \ d;
41     k = 1;
42     xk_1 = xx(2);
43     xk = xx(1);
44     h_new = 2/2000;
45     t = -1;
46     C = y(1) / h - h * M(1) / 6;
47     D = y(2) / h - h * M(2) / 6;
48     error_delta = [0:1:n];
49
50     for i = 0:2000
51         t = -1 + i * h_new;
52
53         if (t >= xk && t <= xk_1) || t > 1
54             s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -

```

```

53         xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
54         - t) + D * (t - xk);
55     error_delta(i + 1) = abs(s - F(t));
56
57     else
58         k = k + 1;
59         xk_1 = xx(k + 1);
60         xk = xx(k);
61         C = y(k) / h - h * M(k) / 6;
62         D = y(k + 1) / h - h * M(k + 1) / 6;
63         s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
64         xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
65         - t) + D * (t - xk);
66         error_delta(i + 1) = abs(s - F(t));
67     end
68 end
69 err(g - 3) = max(error_delta);
loglog([4:10], err);

```

程序运行输出结果为：

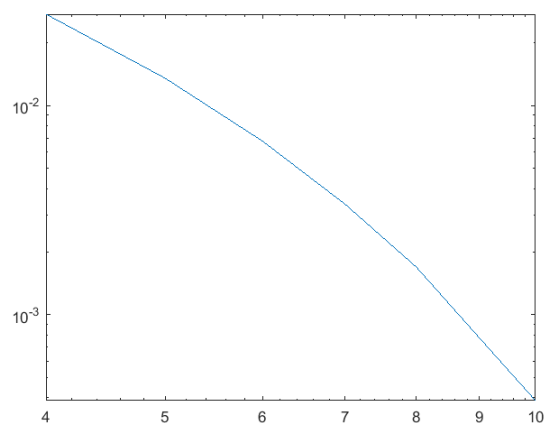


图 2: 三次样条插值区间上最大误差值随 n 变化的情况

- (d) (15分) 针对周期边界条件, 即假设三次样条函数满足 $S'(-1) = S'(1)$ 和 $S''(-1) = S''(1)$, 重复完成上面三问中的要求。

解:

在题设的边界条件下, 由 $S''(x_0 + 0) = S''(x_n - 0)$ 可得 $M_0 = M_n$ 。再由条件 $S'(x_0 + 0) = S'(x_n - 0)$ 可得

$$-M_0 \cdot \frac{h_1}{2} + \frac{y_1 - y_0}{h_1} - \frac{h_1}{6} (M_1 - M_0) = M_n \cdot \frac{h_n}{2} + \frac{y_n - y_{n-1}}{h_n} - \frac{h_n}{6} (M_n - M_{n-1})$$

只要注意到 $y_0 = y_n, M_0 = M_n$, 整理上式即得

$$\lambda_n M_1 + \mu_n M_{n-1} + 2M_n = \frac{6}{h_1 + h_n} \left(\frac{y_1 - y_0}{h_1} - \frac{y_n - y_{n-1}}{h_n} \right) \quad (4)$$

由式 3 和式 4, 可确定 M_1, M_2, \dots, M_n 的线性方程组为

$$\begin{bmatrix} 2 & \lambda_1 & & & \mu_1 \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ \lambda_n & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

本题 (重复(b)) 的MATLAB程序显示如下:

```
1 clear,clc
2 syms x;
3 F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
4 n = 2^4;
5 A = eye(n);
6 A = 2 * A;
7 h = (1 - (-1)) / n;
8 lambda = 1/2;
9 mu = 1 - lambda;
10 A(1, 2) = lambda;
11 A(n, 1) = lambda;
12 A(1, n) = mu;
13 A(n, n - 1) = mu;
14
15 for i = 2:n - 1
16     A(i, i - 1) = mu;
```



```

17     A(i, i + 1) = lambda;
18 end
19
20 y = [0:1:n];
21 xx = [0:1:n];
22
23 for i = 1:n + 1
24     xx(i) = -1 + (i - 1) * h;
25     y(i) = F(xx(i));
26 end
27
28 d = [1:1:n];
29
30 for i = 2:n
31     d(i - 1) = 3 * (y(i + 1) - 2 * y(i) + y(i - 1)) /
        (h^2);
32 end
33
34 d(n) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
35 d = d';
36 M = A \ d;
37 tmp = M;
38 M(1) = tmp(n);
39
40 for i = 1:n
41     M(i + 1) = tmp(i);
42 end
43
44 k = 1;
45 xk_1 = xx(2);
46 xk = xx(1);
47 h_new = 2/2000;
48 t = -1;
49 C = y(1) / h - h * M(1) / 6;
50 D = y(2) / h - h * M(2) / 6;

```

```

51 error_delta = [0:1:n];
52
53 for i = 0:2000
54     t = -1 + i * h_new;
55
56     if (t >= xk && t <= xk_1) || t > 1
57         s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
            ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
            D * (t - xk);
58         error_delta(i + 1) = abs(s - F(t));
59
60     else
61         k = k + 1;
62         xk_1 = xx(k + 1);
63         xk = xx(k);
64         C = y(k) / h - h * M(k) / 6;
65         D = y(k + 1) / h - h * M(k + 1) / 6;
66         s = (xk_1 - t)^3 / (6 * h) * M(k) + (t - xk)
            ^3 / (6 * h) * M(k + 1) + C * (xk_1 - t) +
            D * (t - xk);
67         error_delta(i + 1) = abs(s - F(t));
68     end
69
70 end
71
72 semilogy([-1:h_new:1], error_delta)

```

程序运行输出结果为：

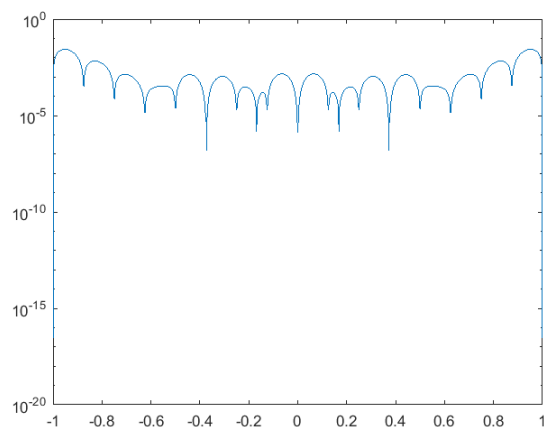


图 3: 三次样条插值的逐点误差(周期边界条件)

本题的MATLAB程序显示如下:

```

1 clear,clc
2 syms x;
3 F = @(x) sin(4 * (x^2)) + (sin(4 * x))^2;
4 err = [4:10];
5
6 for g = 4:10
7     n = 2^g;
8     A = eye(n);
9     A = 2 * A;
10    h = (1 - (-1)) / n;
11    lambda = 1/2;
12    mu = 1 - lambda;
13    A(1, 2) = lambda;
14    A(n, 1) = lambda;
15    A(1, n) = mu;
16    A(n, n - 1) = mu;
17
18    for i = 2:n - 1
19        A(i, i - 1) = mu;
20        A(i, i + 1) = lambda;
21    end
22

```

```

23     y = [0:1:n];
24     xx = [0:1:n];
25
26     for i = 1:n + 1
27         xx(i) = -1 + (i - 1) * h;
28         y(i) = F(xx(i));
29     end
30
31     d = [1:1:n];
32
33     for i = 2:n
34         d(i - 1) = 3 * (y(i + 1) - 2 * y(i) + y(i -
35             1)) / (h^2);
36
37     end
38
39     d(n) = 6 / h * (0 - (y(n + 1) - y(n)) / h);
40     d = d';
41     M = A \ d;
42     tmp = M;
43     M(1) = tmp(n);
44
45     for i = 1:n
46         M(i + 1) = tmp(i);
47     end
48
49     k = 1;
50     xk_1 = xx(2);
51     xk = xx(1);
52     h_new = 2/2000;
53     t = -1;
54     C = y(1) / h - h * M(1) / 6;
55     D = y(2) / h - h * M(2) / 6;
56     error_delta = [0:1:n];
57
58     for i = 0:2000

```

```

57         t = -1 + i * h_new;
58
59         if (t >= xk && t <= xk_1) || t > 1
60             s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
                xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                - t) + D * (t - xk);
61             error_delta(i + 1) = abs(s - F(t));
62
63         else
64             k = k + 1;
65             xk_1 = xx(k + 1);
66             xk = xx(k);
67             C = y(k) / h - h * M(k) / 6;
68             D = y(k + 1) / h - h * M(k + 1) / 6;
69             s = (xk_1 - t)^3 / (6 * h) * M(k) + (t -
                xk)^3 / (6 * h) * M(k + 1) + C * (xk_1
                - t) + D * (t - xk);
70             error_delta(i + 1) = abs(s - F(t));
71         end
72
73     end
74
75     err(g - 3) = max(error_delta);
76 end
77
78 loglog([4:10], err);

```

程序运行输出结果为：

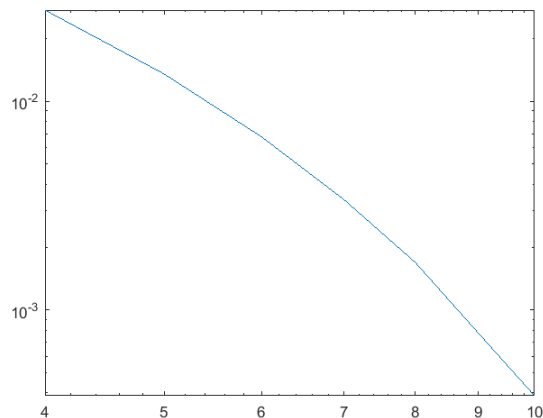


图 4: 三次样条插值区间上最大误差值随 n 变化的情况 (周期边界条件)

第二题 本题深入讨论Newton插值公式的性质。

- (a) (15分) 对于一个光滑函数 $f(x)$, 证明若 $\{i_0, i_1, \dots, i_k\}$ 是 $\{0, 1, \dots, k\}$ 的任意一个排列, 则

$$f[x_0, x_1, \dots, x_k] = f[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$$

证:

先证引理: k 阶差商 $f[x_0, x_1, \dots, x_k]$ 是由函数值 $f(x_0), f(x_1), \dots, f(x_k)$ 的线性组合而成.

$$f[x_0, x_1, \dots, x_k] = \sum_{i=0}^k \frac{1}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_k)} f(x_i)$$

用归纳法可以证明这一引理。

显然, 当 $k=1$ 时, 引理成立。

假设当 $k=n-1$ 时, 引理成立, 故有:

$$f[x_0, x_1, \dots, x_{n-1}] = \sum_{i=0}^{n-1} \frac{1}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_{n-1})} f(x_i)$$

$$f[x_1, x_2, \dots, x_n] = \sum_{i=1}^n \frac{1}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} f(x_i)$$

则 $k=n$ 时:

$$\begin{aligned} f[x_0, x_1, \dots, x_n] &= \frac{f[x_0, x_1, \dots, x_{n-1}] - f[x_1, x_2, \dots, x_n]}{x_0 - x_n} \\ &= \sum_{i=0}^k \frac{1}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} f(x_i) \end{aligned}$$

引理证毕，故有：

$$f[x_0, x_1, \dots, x_k] = \sum_{i=0}^k \frac{1}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_k)} f(x_i)$$

$$f[x_{i_0}, x_{i_1}, \dots, x_{i_k}] = \sum_{j=0}^k \frac{1}{(x_{i_j} - x_{i_0}) \cdots (x_{i_j} - x_{i_{j-1}})(x_{i_j} - x_{i_{j+1}}) \cdots (x_{i_j} - x_{i_k})} f(x_{i_j})$$

又因 $\{x_0, x_1, \dots, x_k\}$ 与 $\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\}$ 之间存在双射, 易得:

$$f[x_0, x_1, \dots, x_k] = f[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$$

(b) (10分) 课堂上我们提到了Chebyshev点

$$x_j = \cos(j\pi/n) \quad j = 0, 1, \dots, n$$

以及使用Chebyshev点可以有效地克服Runge现象。写一个MATLAB程序, 令 $n = 2^2, 2^3, 2^4, \dots, 2^7$, 按照从右到左的顺序(即 j 从小到大的顺序) 使用对应的 $n+1$ 个Chebyshev点对定义在 $[-1, 1]$ 上的Runge函数

$$f(x) = \frac{1}{1 + 25x^2}$$

进行插值, 并取2000个等距点上的误差的最大值, 用semilogy图描述插值区间上最大误差值随 n 变化的情况(即横轴是 n)。

解:

本题的MATLAB程序显示如下:

```
1 clear, clc
2
3 syms x;
4 F = @(x) 1 / (1 + 25 * x^2);
5 y = [0:1:2000];
6
7 for i = 0:2000
8     y(i + 1) = F(-1 + i * 2/2000);
9 end
10
11 result = [0:1:2000];
12 delta_result = [2:1:7];
13
```

```

14 for n = 2:7
15     count = 2^n;
16     chebyshev = [0:1:count];
17
18     for j = 0:1:count
19         chebyshev(j + 1) = cos(j * pi / (count));
20     end
21
22     d_f = [1:1:count];
23     g = [0:1:count];
24
25     for i = 1:count + 1
26         g(i) = F(chebyshev(i));
27     end
28
29     for k = 2:count + 1
30
31         for i = count + 1:-1:k
32             g(i) = (g(i) - g(i - 1)) / (chebyshev(i)
33                 - chebyshev(i - k + 1));
34         end
35     end
36
37     for i = 0:2000
38         a = -1 + i * 2/2000;
39         result(i + 1) = F(chebyshev(1));
40
41         for j = 1:count
42             tmp = 1;
43
44             for k = 1:j
45                 tmp = tmp * (a - chebyshev(k));
46             end
47

```



```

48         result(i + 1) = result(i + 1) + tmp * g(j
           + 1);
49     end
50
51 end
52
53 tmp = [0:1:2000];
54
55 for i = 0:2000
56     tmp(i + 1) = abs(result(i + 1) - y(i + 1));
57 end
58
59 delta_result(n - 1) = max(tmp);
60 end
61
62 semilogy([2:7], delta_result);

```

程序运行输出结果为：

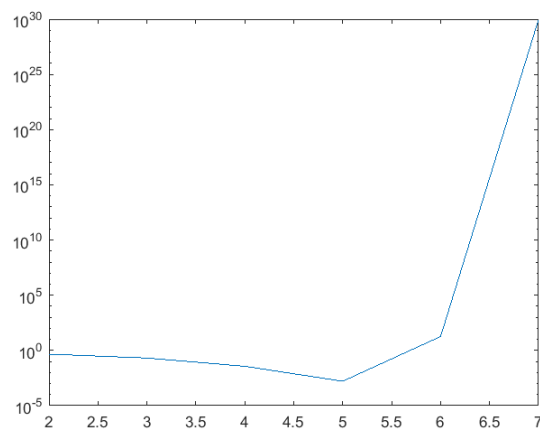


图 5: Newton插值区间上误差值随 n 变化的情况（Chebyshev点按从左到右顺序使用）

- (c) (10分) 重复上一问，但使用随机数种子`rng(22)`和`randperm`函数来随机计算差商时插值点的使用顺序，取关于不同 n 的2000个等距点上的误差的最大值，用`semilogy`图描述插值区间上最大误差值随 n 变化的情况（即横轴是 n ）。

解：

本题的MATLAB程序显示如下:

```
1 clear,clc
2
3 syms x;
4 F = @(x) 1 / (1 + 25 * x^2);
5 y = [0:1:2000];
6
7 for i = 0:2000
8     y(i + 1) = F(-1 + i * 2/2000);
9 end
10
11 result = [0:1:2000];
12 delta_result = [2:1:7];
13
14 for n = 2:7
15     count = 2^n;
16     chebyshev = [0:1:count];
17     rng(22);
18     r = randperm(count + 1);
19
20     for j = 1:1:count + 1
21         chebyshev(j) = cos((r(j) - 1) * pi / (count))
22         ;
23     end
24
25     d_f = [1:1:count];
26     g = [0:1:count];
27
28     for i = 1:count + 1
29         g(i) = F(chebyshev(i));
30     end
31
32     for k = 2:count + 1
33         for i = count + 1:-1:k
```

```

34         g(i) = (g(i) - g(i - 1)) / (chebyshev(i)
35             - chebyshev(i - k + 1));
36     end
37 end
38
39 for i = 0:2000
40     a = -1 + i * 2/2000;
41     result(i + 1) = F(chebyshev(1));
42
43     for j = 1:count
44         tmp = 1;
45
46         for k = 1:j
47             tmp = tmp * (a - chebyshev(k));
48         end
49
50         result(i + 1) = result(i + 1) + tmp * g(j
51             + 1);
52     end
53 end
54
55 tmp = [0:1:2000];
56
57 for i = 0:2000
58     tmp(i + 1) = abs(result(i + 1) - y(i + 1));
59 end
60
61 delta_result(n - 1) = max(tmp);
62 end
63
64 semilogy([2:7], delta_result);

```

程序运行输出结果为：

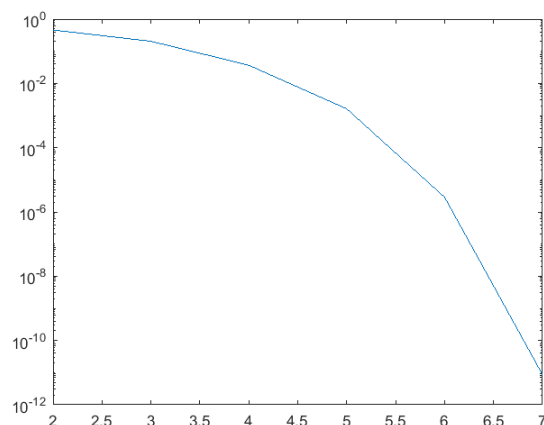


图 6: Newton插值区间上误差值随n变化的情况 (Chebyshev点按随机顺序使用)

- (d) (10分) 试着解释上面两小问中你观察到的不同现象产生的原因。注: 此问答不出来也无妨。

解:

由Newton插值多项式的余项公式:

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) = f[x, x_0, \dots, x_n] \prod_{i=0}^n (x - x_i)$$

结合公式:

$$f[x_0, x_1, \dots, x_k] = f[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$$

可以推断, Newton插值的余项大小与插值点的使用顺序无关。

但在本题中, Chebyshev点分别按顺序使用和乱序使用, 得到的Newton插值结果差异较大。可能的原因是: 在计算差商的过程中, 需要反复除以区间长度。Chebyshev点间的区间长度随着n的增大, 而不断减小。在顺序使用的过程中, 除以的区间长度也不断减小。但随着区间长度变小, 误差随之增大, 且多次计算后, 误差累积放大。而乱序使用Chebyshev点, 则能够减少区间长度过小的情况出现, 减小了因为机器精度等因素带来的影响。

第三题 本题用于讨论周期函数的Lagrange插值方法。对于周期函数而言, 多项式不再是最有效的基函数, 而等距插值点也不再会出现Runge现象。逼近周期函数的基函数通常选用三角函数或者复指数。同时注意对于周期函数而言, 插值点数量和子区间个数相等。

- (a) (10分) 在 $[0, 1]$ 上关于周期函数的基于等间距插值点 $x_j = \frac{j}{n}, j = 0, 1, \dots,$

$n-1$ 的Lagrange插值基函数为

$$\ell_k(x) = \begin{cases} \frac{(-1)^k}{n} \sin(n\pi x) \csc(\pi(x-x_k)) & \text{若 } n \text{ 为奇数} \\ \frac{(-1)^k}{n} \sin(n\pi x) \cot(\pi(x-x_k)) & \text{若 } n \text{ 为偶数} \end{cases}$$

证明对于 n 分别为奇数和偶数的情况下

$$\ell_k(x_j) = \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases}$$

证:

当 n 为奇数时:

$$\ell_k(x_j) = \frac{(-1)^k}{n} \sin(n\pi x_j) \csc(\pi(x_j - x_k))$$

将 $x_j = \frac{j}{n}, x_k = \frac{k}{n}$ 代入得:

$$\ell_k(x_j) = \frac{(-1)^k}{n} \sin(j\pi) \csc\left(\frac{\pi}{n}(j-k)\right)$$

当 $k \neq j$ 时, $\sin(j\pi) = 0, \csc\left(\frac{\pi}{n}(j-k)\right) \neq 0$, 故得:

$$\ell_k(x_j) = 0, k \neq j, n \text{ 为奇数}$$

当 $k = j$ 时, $\sin(j\pi) = 0, \sin\left(\frac{\pi}{n}(j-k)\right) = 0$, 故有:

$$\begin{aligned} \ell_k(x_j) &= \frac{(-1)^k}{n} \sin(j\pi) \csc\left(\frac{\pi}{n}(j-k)\right) \\ &= \lim_{x \rightarrow x_k} \frac{(-1)^k}{n} \sin(n\pi x) \csc(\pi(x-x_k)) \\ &= \lim_{j \rightarrow k} \frac{(-1)^k}{n} \sin(j\pi) \csc\left(\frac{\pi}{n}(j-k)\right) \\ &= \lim_{j \rightarrow k} \frac{(-1)^k}{n} \frac{\sin(j\pi)}{\sin\left(\frac{\pi}{n}(j-k)\right)} \\ &= \frac{(-1)^k}{n} \lim_{j \rightarrow k} \frac{\pi \cos(j\pi)}{\frac{\pi}{n} \cos\left(\frac{\pi}{n}(j-k)\right)} \\ &= \frac{(-1)^k}{n} \frac{\pi \cos(k\pi)}{\frac{\pi}{n} \cos(0)} \\ &= 1 \end{aligned}$$

即:

$$\ell_k(x_j) = 1, k = j, n \text{ 为奇数}$$

当 n 为偶数时:

$$\ell_k(x_j) = \frac{(-1)^k}{n} \sin(n\pi x_j) \cot(\pi(x_j - x_k))$$

将 $x_j = \frac{j}{n}, x_k = \frac{k}{n}$ 代入得:

$$\ell_k(x_j) = \frac{(-1)^k}{n} \sin(j\pi) \cot\left(\frac{\pi}{n}(j - k)\right)$$

当 $k \neq j$ 时, $\sin(j\pi) = 0, \cot\left(\frac{\pi}{n}(j - k)\right) \neq 0$, 故得:

$$\ell_k(x_j) = 0, k \neq j, n \text{ 为偶数}$$

当 $k = j$ 时, $\sin(j\pi) = 0, \sin\left(\frac{\pi}{n}(j - k)\right) = 0$, 故有:

$$\begin{aligned} \ell_k(x_j) &= \frac{(-1)^k}{n} \sin(j\pi) \cot\left(\frac{\pi}{n}(j - k)\right) \\ &= \lim_{x \rightarrow x_k} \frac{(-1)^k}{n} \sin(n\pi x) \cot(\pi(x - x_k)) \\ &= \lim_{j \rightarrow k} \frac{(-1)^k}{n} \sin(j\pi) \cot\left(\frac{\pi}{n}(j - k)\right) \\ &= \lim_{j \rightarrow k} \frac{(-1)^k}{n} \cos\left(\frac{\pi}{n}(j - k)\right) \frac{\sin(j\pi)}{\sin\left(\frac{\pi}{n}(j - k)\right)} \\ &= \frac{(-1)^k}{n} \lim_{j \rightarrow k} \cos\left(\frac{\pi}{n}(j - k)\right) \lim_{j \rightarrow k} \frac{\pi \cos(j\pi)}{\frac{\pi}{n} \cos\left(\frac{\pi}{n}(j - k)\right)} \\ &= \frac{(-1)^k}{n} \frac{\pi \cos(k\pi)}{\frac{\pi}{n} \cos(0)} \\ &= 1 \end{aligned}$$

即:

$$\ell_k(x_j) = 1, k = j, n \text{ 为偶数}$$

综上:

$$\ell_k(x_j) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

- (b) (10分) 用上述对应于 n 为偶数的Lagrange基函数构造Lagrange插值多项式. 并用 $n = 2^6$ 个点对周期函数 $f(x) = \sin(2\pi x)e^{\cos(2\pi x)}$ 在 $[0, 1]$ 上进行插值. 取1000个等距点上的误差, 用semilogy图描述插值区间上误差值随 x 变化的情况(即横轴是 x).

解:

本题的MATLAB程序显示如下:

```

1 clear,clc
2 syms x;
3 syms k;
4 n = 2^6;
5 F = @(x) sin(2 * pi * x) * exp(cos(2 * pi * x));
6 l = @(x, k) ((-1)^k) / n * sin(n * pi * x) * cot(pi *
    (x - k / n));
7 result=[0:1:1999];
8 y=[0:1:1999];
9 delta_y=[0:1:1999];
10 for i=0:1999
11     y(i+1)=F(i/2000);
12     tmp=0;
13
14     for j=0:n-1
15         tmp=tmp+l(i/2000,j)*F(j/n);
16     end
17     result(i+1)=tmp;
18     delta_y(i+1)=abs(result(i+1)-y(i+1));
19 end
20 x=[0:1/2000:1-1/2000];
21 semilogy(x,delta_y)

```

程序运行输出结果为：

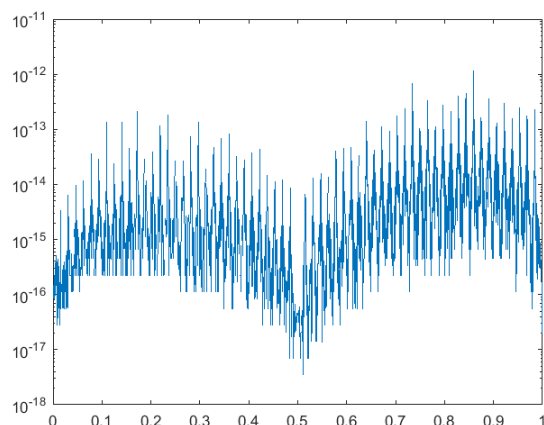


图 7: Lagrange插值区间上误差值随x变化的情况

第四题 (10分) 写程序完成课本 59 页第7题, 并计算出你的拟合函数对比所给数据点的误差的2-范数。

解:

作有理函数拟合 $\varphi(x) = \frac{a_0 + a_1x + \dots + a_px^p}{1 + b_1x + \dots + b_qx^q}$, 可令

$$Q = \sum_{i=1}^m (a_0 + a_1x_i + \dots + a_px_i^p - b_1x_iy_i - \dots - b_qx_i^qy_i - y_i)^2$$

本题中 $\varphi(x) = \frac{a_1}{1+b_1x}$, 故

$$Q = \sum_{i=1}^m (a_1x_i - b_1x_iy_i)^2$$

本题的MATLAB程序显示如下:

```

1 clc, clear
2 x = [2.1, 2.5, 2.8, 3.2];
3 y = [0.6087, 0.6849, 0.7368, 0.8111];
4 a_1 = 0;
5 b_1 = 0;
6 x_y = x .* y;
7 c_11 = sum(x .* x);
8 c_12 = -sum(x .* x_y);
9 c_21 = -c_12;
10 c_22 = -sum(x_y .* x_y);
11 B_1 = sum(x .* y);

```



```

12 B_2 = sum(x_y .* y);
13 C = [c_11, c_12; c_21, c_22];
14 B = [B_1; B_2];
15 tmp = C \ B;
16 a_1 = tmp(1);
17 b_1 = tmp(2);
18 a = 1 / a_1;
19 b = b_1 / a_1;
20 y_bar = [Phi(x(1), a, b), Phi(x(2), a, b), Phi(x(3), a, b
    ), Phi(x(4), a, b)];
21 delta_y = y - y_bar;
22 X = sprintf('The 2-norm of the error of the fitting
    function compared to the given data point is %.15f',
    norm(delta_y, 2));
23 disp(X)
24
25 function value = Phi(x, a, b)
26     value = x / (a + b * x);
27 end

```

程序运行输出结果为：

```

1 The 2-norm of the error of the fitting function compared
    to the given data point is 0.005738349475781

```