STAT241/251 Lecture Notes
Chapter 4 Part C

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For Science and Engineering, we encounter exponential distributions quite frequently. In general, the pdf of an exponential distribution is represented in one of 2 ways, depending on which textbook you use, (but they are really the same).

some books use this

(including your coursenotes)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Other books use this

$$f(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \lambda, \quad Var(X) = \lambda^2$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{\lambda}}, & x \geq 0 \end{cases}$$

If you look carefully, they are really the same. I mention it here so you'll be careful when reading other textbooks or looking up sources from the internet.

So that there is no ambiguity, I always either tell you the pdf of the exponential distribution (so no confusion) or tell you the mean (E(X)) of the exponential distribution (so you can derive the pdf yourself). For example, if I tell you the mean = 2 for an exponential distribution, the pdf is

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Now that you've learnt how to calculate $E(X), Var(X)$ and $F(X)$ of a continuous random variable, make sure you calculate them for the exponential distribution to understand how they are derived (note: requires integration by parts).

Something special about Exponential distribution: memoryless property.

The exponential distribution is often useful in modelling the length of life of electronic components. Suppose the length of time a component already has operated does not affect its chance of operating for at least b additional time units. That is, the probability that the component will operate for more than a+b time units, given that it has already operated for at least a time units, is the same as the probability that a new component will operate for at least b time units if the new component is put into service at time 0. A fuse is an example of a component for which this assumption is often reasonable.

Example

The lifetime (in hours) Y of an electronic component is a random variable with pdf given as:

$$f(y) = \begin{cases} \frac{1}{100}e^{-y/100} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment will operate for at least 200 hours without failure.

Solution:

Let us first deal with a single component.

$$Y \sim Exp(mean = 100)$$

$$P(Y > 200) = \int_{200}^{\infty} \frac{1}{100} e^{-y/100} dy \Leftarrow \text{Evaluate this}$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

OR

shortcut:

$$\begin{aligned} P(Y > 200) =& 1 - P(Y < 200) \\ =& 1 - F(200) \\ =& 1 - (1 - e^{-200/100}) \\ =& e^{-2} \end{aligned}$$

Now

$$P(\text{equipment operates for at least 200 hours})$$
$$= P(\text{at least 2 of the 3 components operate for at least 200 hours})$$

How?? Thought process:

By listing all the cases $\Rightarrow$

$$LLL^c \quad LL^cL \quad L^cLL \quad LLL$$

'L' means component lasts longer than 200 hours.

Stronger students may see that this is the same as

$$\binom{3}{2}(P(Y > 200))^2(P(Y \leq 200)) + (P(Y > 200))^3$$

Answers: $3 \times (e^{-2})^2(1 - e^{-2}) + (e^{-2})^3 = 0.04999 \approx 0.05$

We now work on Problem 4.13 of your course-notes. Part (d) in particular requires special attention. You need to learn the technique to solve part (d) - which is our next topic. Idea:

(1) You have pdf of X.

(2) You are told $Y = \sqrt{X}$ (or some other function of X)

(3) You are asked to find the pdf of Y.

Steps:

(1) Because you now have X and Y, we need to distinguish between

$F_X(x)$ and $F_Y(y)$

refers to the cdf of X *refers to the cdf of Y*

(2) Often, it helps to first find $F_X(x)$ from pdf of X.

(3) Next: to find pdf of Y, remember first find cdf of Y $\leftarrow$ *key point*

$$\text{That is, } F_Y(y) = P(Y \le y) = P(\quad \le y)$$

convert Y to use X

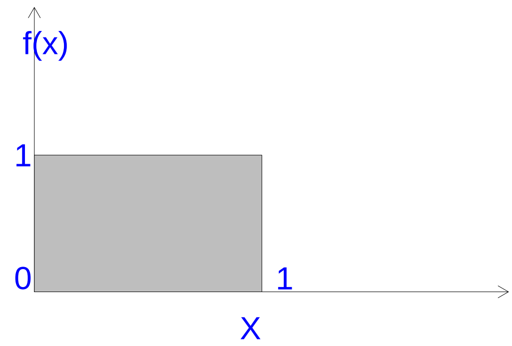
think of this small-letter y as a constant

Strategy: Construct it in such a way that you can make use of $F_x(x)$ which is known.

(4) After getting $F_Y(y)$, we can differenate $F_Y(y)$ to get $f_Y(y)$.

(Let us do 4.13 to see an example)

Ex 4.13 from coursenotes, $X \sim Uniform(0, 1)$



(a)

$$f(y) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \rightarrow F(x) = \begin{cases} 0, & x < 0 \\ \frac{x-0}{1-0}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

That is,

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x > 1 \end{cases}$$

(b)

$$E(X) = \frac{0+1}{2} = \frac{1}{2}, Var(X) = \frac{(1-0)^2}{12} = \frac{1}{12}$$

(c) $Y = \sqrt{X}, E(Y) =?$

Two ways:

(1) Find pdf of Y, then find $E(Y)$. But how to find pdf of Y? see part(d)

(2) Use the fact that $Y = \sqrt{X}$,

$E(Y) = E(\sqrt{X})$ *find this instead*

$$
\begin{aligned}
E(Y) =& E(\sqrt{X}) = \int_{-\infty}^{\infty} x^{\frac{1}{2}} \cdot f(x)\, dx \\
=& \int_0^1 x^{\frac{1}{2}} \cdot 1\, dx = \left[\frac{2}{3} x^{\frac{3}{2}}\right]_0^1 \\
=& \frac{2}{3} \\
Var(Y) =& E(Y^2) - [E(Y)]^2 = E((\sqrt{X})^2) - (\underbrace{E(\sqrt{X})}_{\frac{2}{3}})^2 \\
E(X) =& \frac{1}{2} \quad \text{from part b} \\
\Rightarrow Var(Y) =& \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{2} - \frac{4}{9} \\
=& \frac{9-8}{18} = \frac{1}{18}
\end{aligned}
$$

(d) $Y = \sqrt{X}$, find the pdf of Y.

How? Remember: first step is to start with finding the cdf of Y, even if the question asks for pdf.

treat as a number

$$\begin{aligned} F_Y(y) =& P(Y \le y) \quad \leftarrow \text{ memorize!} \\ =& P(\sqrt{X} \le y) \\ =& P(X \le y^2) \\ =& F_X(y^2) \\ =& y^2 [\text{how come? remember part a}] \end{aligned}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x > 1 \end{cases}$$

$F_X(x) = x$, so $F_X(y^2) = y^2$

Remember to consider the support of Y (it might be different from X).

$$\begin{aligned} 0 < x < 1 \quad & \leftarrow \text{ look at pdf of x} \\ \Rightarrow 0 < \sqrt{x} < 1 & \\ \Rightarrow 0 < y < 1 & \end{aligned}$$

Hence,

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y^2, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Next, differentiate to get pdf of Y.

$$f_Y(y) = F'_Y(y) = 2y$$

Hence,

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Note: To check that the answer has a good chance of being correct, a careful student will check that the area under this pdf of Y will integrate to 1.

Let's re-do part(c) using this pdf.

$$\begin{aligned}
E(Y) = &= \int_{-\infty}^{\infty} y \cdot f(y)\, dy \\
&= \int_0^1 y \cdot 2y\, dy = 2\int_0^1 y^2\, dy \\
&= 2\left[\frac{y^3}{3}\right]_0^1 \\
&= \frac{2}{3} \Leftarrow \text{ same answer as before} \\
Var(Y) &= E(Y^2) - [E(Y)]^2 \\
E(Y^2) &= \int_{-\infty}^{\infty} y^2 \cdot f(y)\, dy \\
&= \int_0^1 y^2 \cdot 2y\, dy \\
&= 2\left[\frac{y^4}{4}\right]_0^1 \\
&= \frac{1}{2} \\
Var(Y) &= \frac{1}{2} - (\frac{2}{3})^2 \\
&= \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \Leftarrow \text{ same answer as before}
\end{aligned}$$