STAT 241/251 ASSIGNMENT 2 SOLUTION KEYS

Problem 1.

Part(a)

Let X be the random variable representing the number of detective articles in the "first" sample.

$$X \sim \text{Binomial}(25, p)$$

Let Y be the random variable representing the number of detective articles in the "second" sample.

$$Y \sim \text{Binomial}(25, p)$$

The conditions for accepting the batch are
(i) $X \leq 1$
(ii) $X = 2$ and $Y = 0$

$P(\text{batch is accepted})$

$$\begin{aligned}
&= P(X \leq 1) + P(X = 2 \text{ and } Y = 0) \\
&= P(X \leq 1) + P(X = 2)P(Y = 0) \\
&[\text{assume events Y=0 and X=2 independent}] \\
&= P(X = 0) + P(X = 1) + P(X = 2)P(Y = 0) \\
&= (1-p)^{25} + \binom{25}{1}(1-p)^{24}p + \binom{25}{2}p^2(1-p)^{23}(1-p)^{25} \\
&= (1-p)^{24}[1 - p + 25p] + 300p^2(1-p)^{48} \\
&= (1 + 24p)(1-p)^{24} + 300p^2(1-p)^{48}
\end{aligned}$$

Part(b)

Since $p = 0.05$, here, we have

$$\begin{aligned}
P &= (1 + 24p)(1-p)^{24} + 300p^2(1-p)^{48} \\
&= 0.706
\end{aligned}$$

Part(c)

Let W be the number of batches accepted among the 100 batches.

$$W \sim \text{Binomial}(100, 0.706)$$

$$\text{It's OK to use normal approximation, since} \begin{cases} np = 100 \times 0.706 = 70.6 \geq 10 \\ nq = 100 \times 0.294 = 29.4 \geq 10 \end{cases}$$

$$W \overset{\text{appr.}}{\sim} N(70.6, npq)$$

$$W \overset{\text{appr.}}{\sim} N(70.6, 20.7564)$$

P(at least 75 batches are accepted)

$$\begin{aligned}
&= P(W \geq 75) \\
&= P(W \geq 74.5) \quad \text{[apply continuity correction]} \\
&= P(Z \geq \frac{74.5 - 70.6}{\sqrt{(20.7564)}}) \\
&= P(Z \geq 0.856) \\
&= 0.196 \quad \text{[used R, but answer should be close if you use tables]}
\end{aligned}$$

Problem 2.

Let X denote the random variable representing the number of games played. Let Y denote the random variable representing the total score. Note that

$$\begin{aligned}
Y &= 100 - 20(X - 1) \\
&= 100 - 20X + 20 \\
&= 120 - 20X
\end{aligned}$$

Also,

$$X \sim \text{Geometric}(\frac{1}{5})$$

Part(a)

$$\begin{aligned}
P(Y < 0) &= P(120 - 20X < 0) \\
&= P(120 < 20X) \\
&= P(X > 6)
\end{aligned}$$

In lecture, it was shown that if $W \sim \text{Geometric}(p)$, then, $F_w(w) = 1 - (1-p)^w$. Hence, for $X \sim \text{Geometric}(\frac{1}{5})$, we have

$$P(X \leq 6) = 1 - (1 - \frac{1}{5})^6$$

Hence,

$$\begin{aligned}
P(Y < 0) &= P(X > 6) \\
&= 1 - P(X \leq 6) \\
&= 1 - [1 - (1 - \frac{1}{5})^6] \\
&= (1 - \frac{1}{5})^6 \\
&= 0.262144
\end{aligned}$$

Part(b)

$$\begin{aligned} E(Y) &= E(120 - 20X) \\ &= 120 - 20E(X) \\ &= 120 - 20 \times 5 \text{ [for} X \sim \text{Geometrix}(\frac{1}{5}), E(X) = \frac{1}{\frac{1}{5}} = 5] \\ &= 120 - 100 \\ &= 20 \end{aligned}$$

Problem 3.

Part(a)

Let X represent the profit on investing $2000 in any particular stock.

X	400	200	0	-200
Prob	0.25	0.25	0.25	0.25

If you invest $2000 $\times$ 100=$200,000, then, the profit Y is given by

Y	400×100= 40,000	200×200= 20,000	0	-200×100= -20,000
Prob	0.25	0.25	0.25	0.25

$$\begin{aligned} P(\text{profit} \geq 15,000) &= P(Y \geq 15,000) \\ &= P(Y = 40,000) + P(Y = 20,000) \\ &= 0.25 + 0.25 \\ &= 0.5 \end{aligned}$$

Part(b)

The distribution of X where X is the profit on investing $2000:

X	400	200	0	-200
Prob	0.25	0.25	0.25	0.25

$$\begin{aligned} \mu = E(X) &= 400 \times \frac{1}{4} + 200 \times \frac{1}{4} + 0 \times \frac{1}{4} + (-200) \times \frac{1}{4} \\ &= 100 \end{aligned}$$

$$\begin{aligned} E(X^2) &= 400^2 \times \frac{1}{4} + 200^2 \times \frac{1}{4} + 0^2 \times \frac{1}{4} + (-200)^2 \times \frac{1}{4} \\ &= 60,000 \end{aligned}$$

$$\begin{aligned}\sigma^2 = \text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= 60,000 - (100)^2 \\ &= 50,000\end{aligned}$$

Let X_i be the profit of the i th stock, $i = 1, 2, \ldots, 100$. $n = 100$ is large and we can apply central limit theorem, let T be the sum of profit of all 100 stocks.

$$T = \sum_{i=1}^{100} X_i$$

$$T \overset{\text{appr.}}{\sim} N(n\mu, n\sigma^2)$$

$$T \overset{\text{appr.}}{\sim} N(10000, 5000, 000)$$

$$\begin{aligned}P(T > 15,000) &= P(Z > \frac{15,000 - 10,000}{\sqrt{5,000000}}) \\ &= P(Z > 2.24) \\ &= 1 - P(Z < 2.24) \\ &= 1 - 0.9875 \\ &= 0.0125\end{aligned}$$