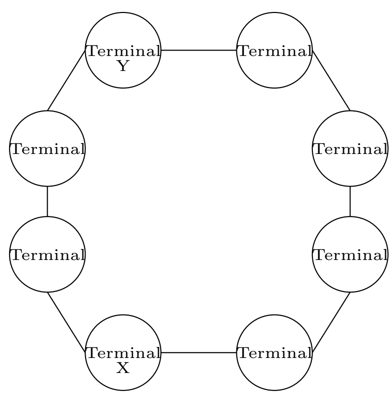
STAT 241/251 Assignment 1

You must use the pre-formatted cover sheet when you hand in the assignment.

Date Due: Monday 15 Oct by 5 pm in STAT 241/251 dropbox (dropbox located on first floor of ESB building, just outside ESB 1041.)

Total Marks: 34 (1 bonus mark given for using cover sheet)

1. The following diagram shows a ring network consisting of eight links. There are two paths connecting any two terminals for data transmission. Assume that links fail independently with probability 0.2. A network packet is transmitted from Terminal X to Terminal Y. A network packet is simply data transmitted along a network. Find the probability of successful transmission of a network packet from Terminal X to Terminal Y. Note that Terminal X transmits the packet of data in both directions on the ring. Also, Terminal Y discards the packet of data from the ring upon reception. As long as Terminal Y received the packet of data from either direction, it is considered a successful transmission. [5 marks]



2. The distance by road from UBC Vancouver to UBC Okanagan is 400 km. When I travel by car, my average speed is V km/hr, where V is uniformly distributed over the interval $60 \leq V \leq 80$.

   (a) Write down the cumulative distribution function of V. [2 marks]

   (b) Obtain the cumulative distribution function of T, the time in hours for the journey from UBC Vancouver to UBC Okanagan. Hence, obtain the probability density function of T. [4 marks]

   (c) Calculate $E(T)$ and $var(T)$. [2 marks]

   (d) I have to attend a meeting at UBC Okanagan which starts at 2 pm. Find the latest time I can leave UBC Vancouver in order that the probability of my arriving in time for the meeting should be at least 80%. [4 marks]

3. The yields of two varieties of corn (in bushels) are independent random variables each with probability density function (pdf) given by

$$f(x) = \begin{cases} kx & 0 \leq x \leq 2 \\ k(4-x) & 2 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that $k = \frac{1}{4}$. [1 mark]

   (b) Find the expected value and variance of the yield of any variety of corn. [4 marks]

   (c) Find the expected value and the variance of the total amount of corn produced. [3 marks]

   (d) Find the probability $P(|X - \mu| < 1)$ where μ is the expected value of the yield of any variety of corn. [3 marks]

   (e) Find the probability that one variety of corn has a yield that exceeds 1 bushel, while the other does not. [3 marks]

   (f) Find the value a such that $P(X > a) = \frac{3}{5}$, giving your answer to 1 decimal place. [2 marks]

Solution to STAT 241/251 Assignment 1

1. Let the probability of a link working be p.
$p = 0.8$ (given in question)

<u>Method 1</u>

Let $A = \{\text{successful transmission clockwise direction}\}$
$B = \{\text{successful transmission anticlockwise direction}\}$
P (successful transimission from Terminal $X \rightarrow$ Terminal Y) $= P\ (A \cup B)$

$$\begin{aligned} \text{But } (A \cup B) &= P(A) + P(B) - P(A \cap B) \quad \text{means "and"} \\ &= p^3 + p^5 - p^8 \\ \text{Hence, required probability } &= (0.8)^3 + (0.8)^5 - (0.8)^8 \\ &= 0.6719 \end{aligned}$$

<u>Method 2</u>

$$\begin{aligned} P(\text{successful transmission}) &= 1 - P(\text{unsuccessful transmission}) \\ &= 1 - P(\text{at least 1 broken link clockwise } \underline{\underline{\text{and}}} \text{ at least 1 broken link anticlockwise}) \\ &= 1 - \big[P(\text{at least 1 broken link clockwise}) \times P(\text{at least 1 broken link anticlockwise})\big] \\ &\quad \text{(since direction of transmission independent)} \\ &= 1 - \big[(1 - P(\text{no broken link clockwise})) \times (1 - P(\text{no broken link anticlockwise}))\big] \\ &= 1 - \big[(1 - p^3)(1 - p^5)\big] \\ &= 1 - \big[1 - p^5 - p^3 + p^8\big] \\ &= p^5 + p^3 - p^8 \end{aligned}$$

Hence, required probability $= (0.8)^5 + (0.8)^3 - (0.8)^8 = 0.6719$

<u>Method 3</u>

Very tedious but you can list out all $2^8 = 256$ combinations of links working/failing and adding up all the probabilities where transmission is successful.

2.(a)

$$F_V(v) = \begin{cases} 0 & v < 60 \\ \frac{v-60}{20} & 60 \leq v \leq 80 \\ 1 & v > 80 \end{cases}$$

(b)

$$\text{Note that} \quad V = \frac{400}{T} \quad (\text{speed} = \frac{\text{Distance}}{\text{Time}})$$

$$\Rightarrow \quad T = \frac{400}{V}$$

$$\begin{aligned} F_T(t) &= P(T \leq t) \\ &= P\big(\frac{400}{V} \leq t\big) \quad \textcolor{red}{\text{note sign change}} \\ &= P\big(V \geq \frac{400}{t}\big) \\ &= 1 - P\big(V \leq \frac{400}{t}\big) \\ &= 1 - F_V\big(\frac{400}{t}\big) \\ &= 1 - \Big(\frac{\frac{400}{t} - 60}{20}\Big) \\ &= 1 - \frac{1}{20}\big(\frac{400 - 60t}{t}\big) \\ &= 1 - \big(\frac{20 - 3t}{t}\big) \\ &= \frac{t - 20 + 3t}{t} \\ &= 4 - \frac{20}{t} \end{aligned}$$

Note that

$$60 \leq V \leq 80$$

$$5 \leq \frac{400}{V} \leq 6\frac{2}{3}$$

$$\Rightarrow 5 \leq T \leq 6\frac{2}{3}$$

Hence,

$$F_T(t) = \begin{cases} 0 & t < 5 \\ 4 - \frac{20}{t} & 5 \leq t \leq 6\frac{2}{3} \\ 1 & t > \frac{2}{3} \end{cases}$$

$$f_T(t) = \begin{cases} 20t^{-2} & 5 \leq t \leq 6\frac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$\begin{aligned} E(T) &= \int_5^{6\frac{2}{3}} t \cdot 20t^{-2} dt \\ &= 20\Big[ln\, t\Big]_5^{6\frac{2}{3}} \\ &= 5.754 \end{aligned}$$

$$Var(T) = E(T^2) - [E(T)]^2$$

$$\begin{aligned} E(T^2) &= \int_5^{6\frac{2}{3}} t^2 \cdot 20t^{-2} dt \\ &= 20\Big[t\Big]_5^{6\frac{2}{3}} \\ &= 33.3333 \end{aligned}$$

$$\begin{aligned} Var(T) &= 33.3333 - (5.754)^2 \\ &= 0.2248 \end{aligned}$$

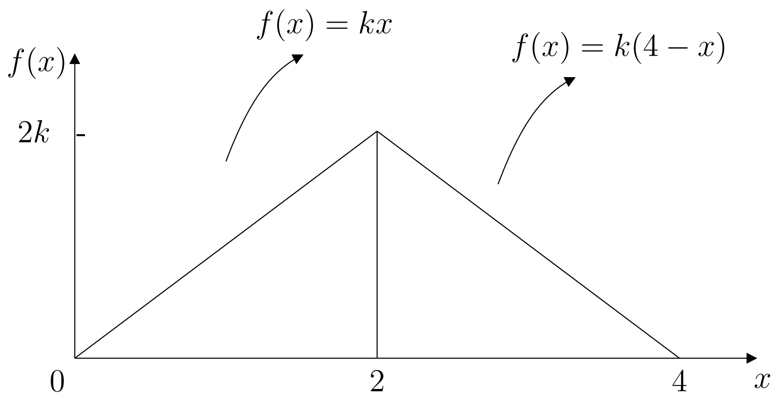
(d)

$$\begin{aligned} P(T \leq t) &\geq 0.8 \\ F_T(t) &\geq 0.8 \\ 4 - \frac{20}{t} &\geq 0.8 \\ \frac{20}{t} &\leq 3.2 \\ 3.2t &\geq 20 \\ t &\geq 6.25 \end{aligned}$$

Lastest time to leave UBC Vancouver is 2 pm minus 6.25 hrs (=6 hrs 15 mins) $\approx 7:45$ am.

3.(a) There are 2 ways to find k

Graphical method:



When $x = 2$

$f(2) = 2k$

$f(0) = f(4) = 0$

$$\begin{aligned} \text{Area of} \quad \triangle &= 1 \\ \Rightarrow \quad \frac{1}{2} \times 4 \times 2k &= 1 \\ 4k &= 1 \\ \Rightarrow \quad k &= \frac{1}{4} \end{aligned}$$

<u>Integration method:</u>

$$\begin{aligned} \int_0^2 kx dx + \int_2^4 k(4-k)dx &= 1 \\ k\left[\frac{x^2}{2}\right]_0^2 + k\left[4x - \frac{x^2}{2}\right]_2^4 &= 1 \\ 2k + k\Big[(16-8) - (8-2)\Big] &= 1 \\ 4k &= 1 \\ \Rightarrow k &= \frac{1}{4} \end{aligned}$$

(b) 2 ways to find $E(x)$:
<u>Usual way</u>

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot f(x)dx \\
&= \int_{0}^{2} x \cdot \frac{1}{4}xdx + \int_{2}^{4} x \cdot \frac{1}{4}(4-x)dx \\
&= \frac{1}{4}\Big[\frac{x^3}{3}\Big]_0^2 + \int_2^4 xdx - \frac{1}{4}\int_2^4 x^2dx \\
&= \frac{1}{4}\Big[\frac{8}{3}\Big] + \Big[\frac{x^2}{2}\Big]_2^4 - \frac{1}{4}\Big[\frac{x^3}{3}\Big]_2^4 \\
&= \frac{2}{3} + (8-2) - \frac{1}{4}[\frac{64}{3} - \frac{8}{3}] \\
&= \frac{2}{3} + 6 - \frac{1}{4} \cdot \frac{56}{3} \\
&= 2
\end{aligned}
$$

Fast lazy way (but smart!)

Notice in the diagram that the pdf is symmetric about x=2. Hence, E(X) is 2.

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
E(X^2) &= \int_{-\infty}^{\infty} x^2 \cdot f(x)dx \\
&= \int_{0}^{2} x^2 \cdot \frac{1}{4}xdx + \int_{2}^{4} x^2 \cdot \frac{1}{4}(4-x)dx \\
&= \frac{1}{4}\Big[\frac{x^4}{4}\Big]_0^2 + \int_2^4 x^2dx - \frac{1}{4}\int_2^4 x^3dx \\
&= \frac{1}{4}\Big[4\Big] + \Big[\frac{x^3}{3}\Big]_2^4 - \frac{1}{4}\Big[\frac{x^4}{4}\Big]_2^4 \\
&= 1 + (\frac{64}{3} - \frac{8}{3}) - \frac{1}{4}(64-4) \\
&= 1 + \frac{56}{3} - \frac{60}{4} = \frac{12+224-180}{12} \\
&= \frac{14}{3} \\
Var(X) &= E(X^2) - \big[E(X)\big]^2 \\
&= \frac{14}{3} - 4 \\
&= \frac{2}{3}
\end{aligned}
$$

(c) Let X_1 and X_2 denote the two yields

$$Y = X_1 + X_2$$

$$
\begin{aligned}
E(Y) &= E(X_1) = E(X_2) \\
&= 2 + 2 = 4
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= Var(X_1) + Var(X_2) \\
&= \frac{2}{3} + \frac{2}{3} \qquad \text{Since } X_1, X_2 \text{ are indenpendent} \\
&= \frac{4}{3}
\end{aligned}
$$

(d)

$$
\begin{aligned}
P(|X-2| < 1) &= P(-1 < X - 2 < 1) \\
&= P(1 < X < 3) \\
&= \int_1^3 f(x)dx \\
&= \int_1^2 \frac{1}{4}xdx + \int_2^3 \frac{1}{4}(4-x)dx \\
&= \frac{1}{4}\Big[\frac{x^2}{2}\Big]_1^2 + \Big[x\Big]_2^3 - \frac{1}{4}\Big[\frac{x^2}{2}\Big]_2^3 \\
&= \frac{1}{4}(2 - \frac{1}{2}) + 1 - \frac{1}{4}[\frac{9}{2} - 2] \\
&= (\frac{1}{4} \times \frac{3}{2}) + 1 - (\frac{1}{4} \times \frac{5}{2}) \\
&= \frac{3}{8} + 1 - \frac{5}{8} \\
&= \frac{3}{4}
\end{aligned}
$$

(or use graphical method to find the area)

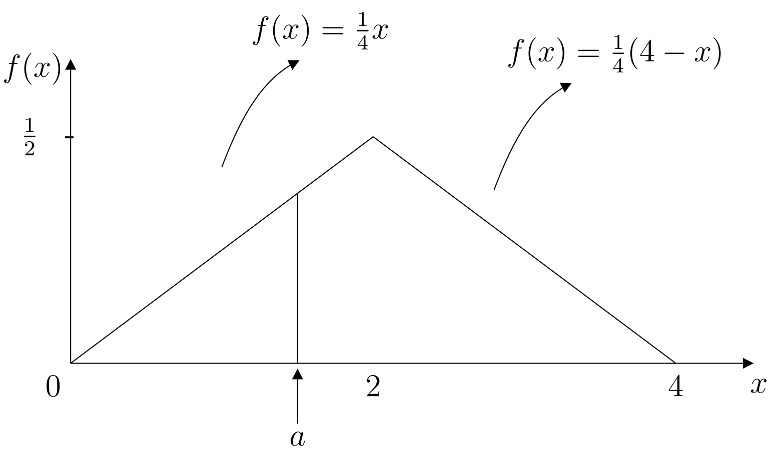
(e)

$$
\begin{aligned}
P(X > 1) &= 1 - P(X < 1) \\
&= 1 - [\frac{1}{2} \times 1 \times \frac{1}{4}] \\
&= 1 - \frac{1}{8} \\
&= \frac{7}{8}
\end{aligned}
$$

$$
\begin{aligned}
\text{Required probability} &= P(X_1 > 1, X_2 < 1) + P(X_1 < 1, X_2 > 1) \\
&= 2 \times \frac{7}{8} \times \frac{1}{8} \\
&= 0.21875
\end{aligned}
$$

(f)

$$P(X > a) = \frac{3}{5}$$



a must be between 0 and 2

$$P(X < a) = \tfrac{2}{5}$$

$$\tfrac{1}{2} \times a \times \tfrac{1}{4}a = \tfrac{2}{5}$$

$$a^2 = \tfrac{16}{5}$$

$$a = \sqrt{3.2} = 1.8$$