STAT241/251 Lecture Notes
Chapter 4 Part D

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Previously, we went through Problem 4.13 in coursenotes.
We work on another example today.
You've pdf of X. Let $Y = X^2$. You wish to find pdf of Y. How?
Remember: To find pdf of Y, first find cdf of Y.

$$\begin{aligned} F_Y(y) =& P(Y \le y) \\ =& P(X^2 \le y) \\ =& P(-\sqrt{y} \le X \le \sqrt{y}) \\ =& F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

This step requires extra care. You should look at support of X to decide the correct form of this expression. In this generic version, X is assumed to take on both negative and positive values. If that is not the case, you have to modify accordingly. (Do problem 4.18 to see a harder example)

After getting $F_Y(y)$, differentiate $F_Y(y)$ to get $f_Y(y)$ [or use $f_Y(y) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y})+f_X(-\sqrt{y})]$]

Example:

$$f_Y(y) = \begin{cases} \frac{y+1}{2} & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find pdf of $U = Y^2$. Since

$$-1 \leq y \leq 1$$

$$0 \leq y^2 \leq 1$$

$$\Rightarrow 0 \leq u \leq 1$$

Find the support of U

To find pdf, first find cdf.

$$\begin{aligned} F_U(u) =& P(U \leq u) \\ =& P(Y^2 \leq u) \\ =& P(-\sqrt{u} \leq Y \leq \sqrt{u}) \\ =& \sqrt{u} \, (obtained\, via\, Method\, A\, or\, Method B) \end{aligned}$$

Either use $F_Y(y)$(next page, Method A)

OR use integration directly (next page, Method B)

$$F_U(u) = \begin{cases} 0 & u < 0 \\ \sqrt{u} & 0 \leq u \leq 1 \\ 1 & u > 1 \end{cases}$$

$$f_U(u) = \begin{cases} \frac{1}{2} u^{-\frac{1}{2}} & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Method A

$$
\begin{aligned}
F_Y(y) =& P(Y \le y) \\
=& \int_{-1}^{y} (\frac{x+1}{2}) dx \\
=& \frac{1}{2}[\frac{x^2}{2} + x]_{-1}^{y} \\
=& \frac{1}{2}[\frac{y^2}{2} + y - (\frac{1}{2} - 1)] \\
=& \frac{1}{2}[\frac{y^2}{2} + y - \frac{1}{2} + 1)] \\
=& \frac{1}{2}(\frac{y^2}{2} + y + \frac{1}{2}) \\
=& \frac{1}{2}(\frac{y^2 + 2y + 1}{2}) \\
=& \frac{y^2 + 2y + 1}{4}
\end{aligned}
$$

$$
\begin{aligned}
P(-\sqrt{u} \le Y \le \sqrt{u}) =& F(\sqrt{u}) - F(-\sqrt{u}) \\
=& \frac{u + 2\sqrt{u} + 1}{4} - [\frac{u - 2\sqrt{u} + 1}{4}] \\
=& \frac{u + 2\sqrt{u} + 1 - u + 2\sqrt{u} - 1}{4} = \frac{4\sqrt{u}}{4} = \sqrt{u}
\end{aligned}
$$

Method B

$$
\begin{aligned}
\textstyle\int_{-\sqrt{u}}^{\sqrt{u}} f(y) dy = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{y+1}{2} dy = \frac{1}{2}[\frac{y^2}{2} + y]_{-\sqrt{u}}^{\sqrt{u}} \\
= \frac{1}{2}[\frac{u}{2} + \sqrt{u} - (\frac{u}{2} - \sqrt{u})] = \frac{1}{2}[\frac{u}{2} + \sqrt{u} - \frac{u}{2} + \sqrt{u}] = \sqrt{u}
\end{aligned}
$$

Method A and Method B produce same result. That is, $P(-\sqrt{u} \le Y \le \sqrt{u}) = \sqrt{u}$.

Test your knowledge:
If the pdf of X is:

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of $Y = X^3$.
If you can do this, you've mastered the technique!
Answer:

$$f_Y(y) = \begin{cases} 2(y^{-\frac{1}{3}} - 1), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Properties of the mean and variance

random variable

Property 1:

$E(aX+b) = aE(X)+b$ where a and b are constants.
Eg. $E(X) = 3$, then $E(2X+5) = 2E(X)+5 = 2\cdot 3+5 = 11$

Property 2:

X and Y are random variables.
$E(X+Y) = E(X)+E(Y)$

Property 3:

If X and Y are independent random variables,
$E(XY) = E(X)E(Y)$

Property 4:

$$Var(aX+b) = a^2 Var(X), \text{where a and b are constants}$$

Property 5:

If X and Y are independent random variables,

$$Var(X+Y) = Var(X)+Var(Y)$$

$$Var(X-Y) = Var(X)+Var(Y)$$

Yes! + sign. Not a mistake!

Sum and Average of independent random variables

If $X_1, X_2, \ldots, X_n$ are n independent random variables, and $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$, where $a_1, a_2, \ldots, a_n$ are constants,

$$E(Y) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n)$$

$$Var(Y) = a_1^2Var(X_1) + a_2^2Var(X_2) + \cdots + a_n^2Var(X_n)$$

If $X_1, X_2, \ldots, X_n$ are n independent random variables, and $\bar{X} = \frac{X_1+X_2+\cdots+X_n}{n}$

$$\begin{aligned} E(\bar{X}) &= E(\frac{X_1 + X_2 + \cdots + X_n}{n}) \\ &= \frac{1}{n}[E(X_1) + E(X_2) + \cdots + E(X_n)] \\ Var(\bar{X}) &= Var(\frac{X_1 + X_2 + \cdots + X_n}{n}) \\ &= \frac{1}{n^2}[Var(X_1) + Var(X_2) + \cdots + Var(X_n)] \end{aligned}$$

Some simple examples

1. If $E(X) = 3, Var(X) = 2$, what is

$$E(2X+3)? \quad \text{Ans: } 2E(X)+3=9$$

$$Var(\frac{1}{2}X)? \quad \text{Ans: } \frac{1}{4}Var(X)=\frac{1}{2}.$$

2. X and Y are independent random variables. Given

$$E(X) = 3, Var(X) = 2$$

$$E(Y) = 5, Var(Y) = 1$$

What is

$$E(X-Y)? \quad \text{Ans: } E(X)-E(Y)=3-5=-2$$

$$Var(X-Y)? \quad \text{Ans: } Var(X)+Var(Y)=2+1=3$$

Important concept

Let's say $X \sim Uniform(1,2)$. You take 3 independent observations of X and sum them up. Let's call this sum Y (it is also a random variable). What is $Var(Y)$?
Many students new to Statistics will calculate $Var(Y)$ the following way which is incorrect:

$$\begin{aligned} X \sim & Uniform(1,2) \\ E(X) = & \frac{1+2}{2} = 1.5 \\ Var(X) = & \frac{(2-1)^2}{12} = \frac{1}{12} \\ Y = & 3X \longleftarrow \text{INCORRECT} \\ \Rightarrow Var(Y) = & Var(3X) \\ = & 9Var(X) \\ \times \qquad = & 9 \times \frac{1}{12} \\ = & \frac{9}{12} = \frac{3}{4} \end{aligned}$$

Correct Answer:

$$Y = X_1 + X_2 + X_3 \checkmark$$

$$\begin{aligned} Var(Y) = & Var(X_1 + X_2 + X_3) \\ = & Var(X_1) + Var(X_2) + Var(X_3), \text{ (since } X_1, X_2, X_3 \text{ are independent)} \\ = & \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4} \checkmark \end{aligned}$$