STAT241/251 Lecture Notes
Chapter 6 Part 2

Yew-Wei Lim

Recall:

We learnt that the Bernoulli distribution has the following pmf:

$$P(X = x) = (1 - p)^{1-x}(p)^x, \text{x=0,1}$$

We also learnt that Bernoulli r.v., its

$$E(X) = p$$

$$Var(X) = p(1 - p)$$

We'll now learn about Binomial distribution. You'll see there is a relationship between Bernoulli and Binomial random variables.

Before we do that, let's learn about the notation: ${}^nC_r$

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Both notations OK. Mean the same thing

E.g.

$$\binom{6}{3} = \frac{6!}{(6-3)!3!} = \frac{\cancel{6} \times 5 \times 4 \times \cancel{3} \times \cancel{2} \times 1}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{3} \times \cancel{2} \times 1} = 20$$

Learn to use your calculator to get this!

But what's the point of learning about $\binom{n}{r}$? Very useful. Example:

(1) You've 1 A and 2 B's. How many ways can you arrange it?

Easy to count in this case:

$$\left.\begin{array}{l} A,B,B \\ B,A,B \\ B,B,A \end{array}\right\} \text{3 ways}$$

Note: ${}^3C_1 = {}^3C_{3-1} = {}^3C_2$

Quick answer: ${}^3C_1 = \binom{3}{1} = 3$. *Thought process: think of it as out of three boxes, choose 1 to place an A.*

(2) You've 2 A's and 2 B's. How many ways can you arrange it?

$$\left.\begin{array}{l} A,A,B,B \\ A,B,A,B \\ B,A,A,B \\ B,B,A,A \\ B,A,B,A \\ A,B,B,A \end{array}\right\} \text{6 ways. But it takes time and care to list them.}$$

Quick answer:

$$\begin{aligned} {}^4C_2 = \binom{4}{2} &= \frac{4 \times 3}{1 \times 2} \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

(3) What about 7 A's and 2 B's?

Quick answer:

${}^9C_2 = {}^9C_{9-2} = {}^9C_7$

$$\begin{aligned} {}^9C_2 = \binom{9}{2} &= \frac{9 \times 8}{1 \times 2} \\ &= 36 \end{aligned}$$

*In general, ${}^nC_r = {}^nC_{n-r}$

Binomial Distribution

If $Y_i \sim Bernoulli(p), i = 1, 2, \ldots, n,$ [you can think of this as a single toss of a coin where P(head) = p and P(tail) = 1-p]

then if X is the total number of success (eg heads) in n independent trials (tosses of coin),

$$\underbrace{X}_{Binomial} = \underbrace{Y_1 + Y_2 + \cdots + Y_n}_{Bernoulli}$$

The reason to set this up is because it provides an easy way to calculate E(X) and Var(X), where X is a Binomial random variable.

[Recall $E(Y_i) = p, Var(Y_i) = p(1-p)$]

Then,

$$\begin{aligned} E(X) =& E(Y_1 + Y_2 + \cdots + Y_n) \\ =& \underbrace{E(Y_1)}_{p} + \underbrace{E(Y_2)}_{p} + \cdots + \underbrace{E(Y_n)}_{p} \\ =& np \\ Var(X) =& Var(Y_1 + Y_2 + \cdots + Y_n) \\ =& \underbrace{Var(Y_1)}_{pq} + \underbrace{Var(Y_2)}_{pq} + \cdots + \underbrace{Var(Y_n)}_{pq} \qquad \textit{Since } Y_i' s \textit{ are independent} \\ =& npq \quad (\text{where } q = 1-p) \end{aligned}$$

Don't get parameters mixed up with Normal(μ, σ^2)

Very important:

probability of success

We write $X \sim Binomial(n, p)$,

fixed

$$E(X) = np$$

$$Var(X) = npq$$

Learn to recognize a Binomial situation:

Binomial situation arises when

- fixed number n of independent trials
- each trial has only two possible outcomes
- the probability of success, p is the same for each trial

$$Important : \begin{cases} X \sim Binomial(n, p) \\ P(X = x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x}, \quad \text{where} \quad x = 0, 1, 2, \ldots, n \end{cases}$$

Example:

The probability that Robin Hood hits the target is $\frac{3}{4}$,. If he makes 5 shots, what is the probability of

(a) 4 hits

(b) more that 2 hits ?

(c) at least 3 misses ?

Solution:

Let X be the number of hits.

$$X \sim Binomial(\underbrace{5}_{n}, \overbrace{\frac{3}{4}}^{p})$$

(a)

You may use calculator to evaluate $^5C_4$

$$\begin{aligned} P(X=4) &= {}^5C_4 \cdot \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^{5-4} \\ &= 5 \cdot \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 \\ &= 0.3955 \end{aligned}$$

[Thought Process: Using the above formula to calculate binomial probabilities provides a fast way to get the answer.

If you want to do this on your own without formula, it will be slow.

In 5 shots, H = hit, M=miss. To get 4 hits:

$$5\,ways \begin{cases} HHHHM \leftarrow (\frac{3}{4})(\frac{3}{4})(\frac{3}{4})(\frac{3}{4})(\frac{1}{4}) \\ HHHMH \leftarrow (\frac{3}{4})(\frac{3}{4})(\frac{3}{4})(\frac{1}{4})(\frac{3}{4}) \\ HHMHH \leftarrow (\frac{3}{4})(\frac{3}{4})(\frac{1}{4})(\frac{3}{4})(\frac{3}{4}) \\ HMHHH \leftarrow (\frac{3}{4})(\frac{1}{4})(\frac{3}{4})(\frac{3}{4})(\frac{3}{4}) \\ MHHHH \leftarrow (\frac{1}{4})(\frac{3}{4})(\frac{3}{4})(\frac{3}{4})(\frac{3}{4}) \end{cases}$$

You can see now that the answer agrees with $^5C_4 \cdot (\frac{3}{4})^4(\frac{1}{4})^1$

(b)

$$
\begin{aligned}
&P(X>2) \quad \textit{careful with sign for discrete r.v.!}\\
=&P(X=3)+P(X=4)+P(X=5)\\
=&{}^5C_3(\frac{3}{4})^3(\frac{1}{4})^2+{}^5C_4(\frac{3}{4})^4(\frac{1}{4})^1+\underbrace{{}^5C_5}_{1}(\frac{3}{4})^5(\frac{1}{4})^0\\
=&0.8965
\end{aligned}
$$

(c)

[Think: 3 misses + 4 misses + 5 misses]

$$
\begin{aligned}
&P(\text{at least 3 misses})\\
=&P(\text{at most 2 hits})\\
=&1-P(X>2)\\
=&1-0.8965\\
=&0.1035
\end{aligned}
$$

[Think: 2 hits+1 hits+0 hit]

[It will be slower, but you can verify this other way will get the same answer too.

$$
\begin{aligned}
&P(\text{at most 2 hits})\\
=&\underbrace{P(X=0)}_{\text{Don't miss out this case}}+P(X=1)+P(X=2)\\
=&{}^5C_0(\frac{3}{4})^0(\frac{1}{4})^5+{}^5C_1(\frac{3}{4})^1(\frac{1}{4})^4+{}^5C_2(\frac{3}{4})^2(\frac{1}{4})^3
\end{aligned}
$$

Try it. Should work out to be 0.1035]

You need to know expectation and variance of Binomial distribution well.

Example. The random variable X has a binomial distribution with mean 12 and variance 8. Find $P(X = 12)$.

Solution:

$$X \sim Binomial(n, p)$$

$$\begin{aligned} E(X) &= np = 12 && (1) \\ Var(X) &= npq = np(1-p) = 8 && (2) \end{aligned}$$

Substitute (1) into (2)

$$\begin{aligned} 12(1-p) &= 8 \\ 1 - p &= \frac{8}{12} = \frac{2}{3} \\ p &= 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

From (1),

$$\begin{aligned} np &= 12 \\ n(\frac{1}{3}) &= 12 \\ &\Rightarrow n = 36 \\ P(X = 12) &= {}^{36}C_{12}(\frac{1}{3})^{12}(\frac{2}{3})^{24} \\ &= 0.140 \end{aligned}$$