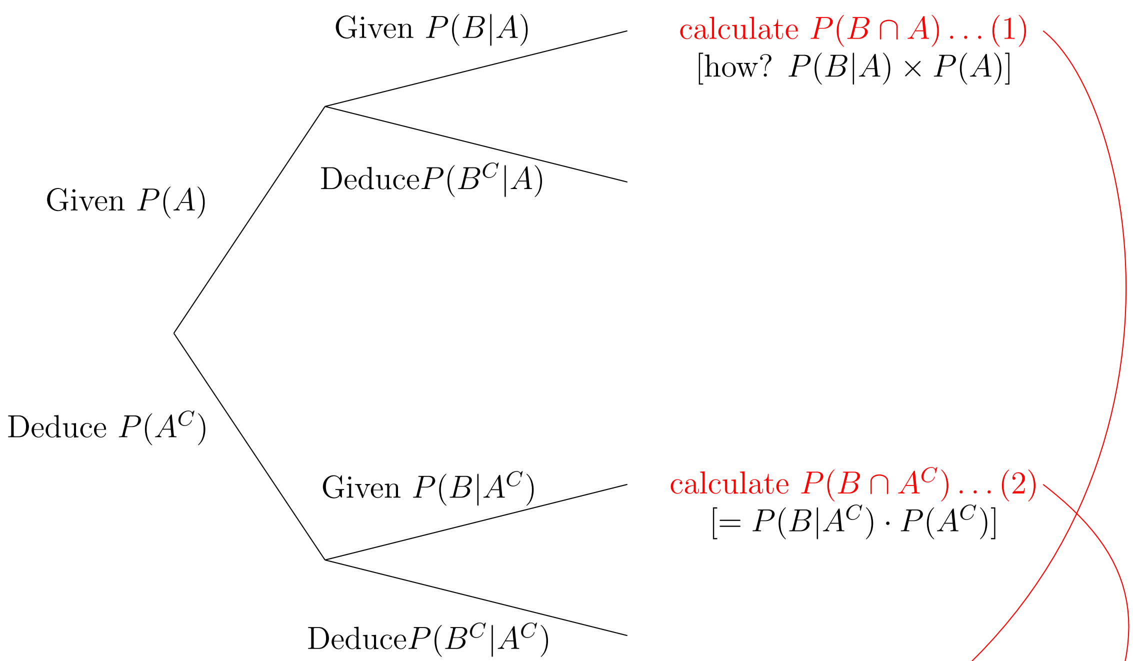
This is the general setup when you can use Bayes theorem (you need not use the theorem if you solve the problem using a tree diagram):



calculate P(B):

[how? Recall law of total probability says

$$P(B) = P(B|A) \cdot P(A) + P(B|A^C) \cdot P(A^C)]$$

you can now easily find $P(A|B)$ $\quad [= \frac{P(A \cap B)}{P(B)}]$

If you choose not to draw a tree diagram, you can use Bayes theorem to answer the previous example by

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1 \cap B)}{P(B)} \\ &= \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A^C) \cdot P(A^C)} \end{aligned}$$

It is easiest to see how to use Bayes theorem through an example (you need not use the theorem if you solve it using a tree diagram):

> A chain of computer stores sells three different brands of laptops. Of its laptop sales, 50% are brand 1 (least expensive), 30% are brand 2 and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labour. It is known that 25% of brand 1s laptops require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10% respectively.
>
> (a) What is the probability that a randomly selected purchaser has bought a brand 1 laptop that will need repair while under warranty?
>
> (b) What is the probability that a randomly selected purchaser has a laptop that will need repair while under warranty?
>
> (c) If a customer returns to the store with a laptop that needs warranty repair work, what is the probability that it is a brand 1 laptop? A brand 2 laptop? A brand 3 laptop?

The first stage of the problem involves a customer selecting one of the 3 brands of laptop. Let A_i={brand i is purchased}, for i=1,2 and 3.

$$P(A_1) = 0.5, \quad P(A_2) = 0.3, \quad P(A_3) = 0.2$$

Once a brand of laptop is selected, the second stage involves observing whether the selected laptop needs warranty repair. Let B={needs warranty repair} and B^C={doesn't need warranty repair}.

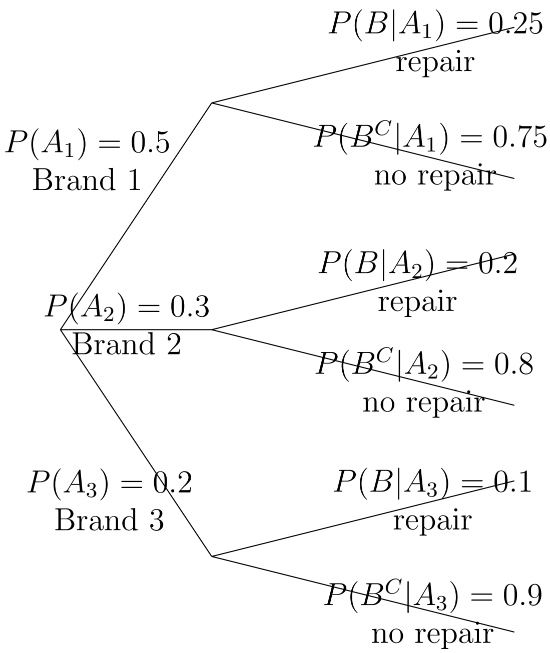
$$P(B|A_i) = 0.25, \quad P(B|A_2) = 0.2, \quad P(B|A_3) = 0.1$$

Before we draw a tree diagram, can you see that the 3 questions asked are translated in set notation as:

(a) $P(A_1 \cap B)$

(b) $P(B)$

(c) $P(A_1|B), P(A_2|B), P(A_3|B)$?



$P(B|A_1) \times P(A_1) = P(B \cap A_1) = 0.125$

$P(B|A_2) \times P(A_2) = P(B \cap A_2) = 0.06$

$P(B|A_3) \times P(A_3) = P(B \cap A_3) = 0.02$

$P(B) = 0.125 + 0.06 + 0.02 = 0.205$

From the tree diagram,

(a) $P(A_1 \cap B) = 0.125$

(b) $P(B) = 0.205$

(c) $P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.125}{0.205} = 0.61$

$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.06}{0.205} = 0.29$

$P(A_3|B) = 1 - P(A_1|B) - P(A_2|B) = 0.1$

In the example, note that

$$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) \\ &= P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3) \cdot P(A_3) \end{aligned}$$

This formula is used quite often and has a name: law of total probability.

Bayes theorem

$$\begin{aligned} P(A_j|B) &= \frac{P(A_j \cap B)}{P(B)} \\ &= \frac{P(B|A_j) \times P(A_j)}{\sum_{i=1}^{k} P(B|A_i) \times P(A_i)} \qquad (k = 3 \textit{ in our example}) \end{aligned}$$

Personally, I always draw a tree diagram and don't need to remember this formula but some students might like a formula to get a quick answer.

Test your knowledge

Suppose the probability that a student gets an A for STAT 251 is 0.1.

(a) If you pick 2 students at random, what is the probability both get A's?

Ans: P(both get A)=0.1$\times$ 0.1

(because we assume getting A's are independent events)

(b) If you pick 2 students at random, what is the probability one obtains an A but the other does not?

Hint: It is wrong if you answer 0.1$\times$0.9 $\leftarrow$ careless!

Ans: $2 \times 0.1 \times 0.9$

(how come? P(1^{st} student A, 2^{nd} student not A)+ P(1^{st} student not A, 2^{nd} student A)=2$\times$ 0.1 $\times$ 0.9)

Thought process: $P(A, A^C) + P(A^C, A)$

(c) Now that you have this concept, try this:

If you pick 4 students at random, what is the probability 2 of them get A's, while the other 2 do not?

Ans: $6 \times (0.1)^2 \times (0.9)^2$

How come?

It is tedious to list, but you can see by

$$A, A, A^C, A^C$$
$$A, A^C, A^C, A$$
$$A^C, A^C, A, A$$
$$A^C, A, A, A^C$$
$$A, A^C, A, A^C$$
$$A^C, A, A^C, A$$

This is tedious and prone to error.

Better method: $\binom{4}{2} = 6$

Teach students:

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

So

$$\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (2 \times 1)} = 15$$

and

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1) \times (2 \times 1)} = 6$$

Useful formula:

$$\binom{n}{r} = \binom{n}{n-r} \qquad eg. \binom{6}{4} = \binom{6}{2}$$

Let's try solving more problems.

Example 1

The Italian Aspide missile has a sophisticated homing guidance system with a single-shot hit probability of 0.8. Suppose an enemy plane is within range of three missile firing stations. All three fire an Aspide surface-to-air missile, and the missiles operate independently.

(a) What is the probability the plane is hit?

(b) What is the probability all three missiles miss?

(c) How many missiles would have to be fired at the plane in order to be 99.99% sure it would be hit?

Solution:

1.(a) It is incorrect to answer $(0.8)^3$

$$\begin{aligned} P(plane\ is\ hit) &= 1 - P(plane\ is\ not\ hit) \\ &= 1 - [P(missile\ missed)]^3 \quad (since\ all\ 3\ missiles\ are\ independent) \\ &= 1 - (0.2)^3 \\ &= 0.992 \end{aligned}$$

Another way (but tedious):

$$\begin{aligned} P(plane\ is\ hit) &= P(1\ out\ of\ 3\ missiles\ hit\ the\ plane) \\ &\quad + P(2\ out\ of\ 3\ missiles\ hit\ the\ plane) \\ &\quad + P(all\ 3\ missiles\ hit\ the\ plane) \\ &= \binom{3}{1}(0.8)(0.2)^2 + \binom{3}{2}(0.8)^2(0.2) + (0.8)^3 \\ &= 0.096 + 0.384 + 0.512 \\ &= 0.992 \qquad (agree\ with\ prev\ answer) \end{aligned}$$

(b) $P(all\ 3\ missiles\ missed) = 0.2^3 = 0.008$

(c) $P(plane\, is\, hit) > 0.9999$

it is important to use an inequality sign.(because you need to know to round up or down later)

$$
\begin{aligned}
1 - P(plane\ is\ not\ hit) &> 0.9999 \\
P(plane\ is\ not\ hit) &< 0.0001 \\
(0.2)^n &< 0.0001 \qquad where\ n\ is\ the\ number\ of\ missiles
\end{aligned}
$$

Take ln both sides

$$n \ln 0.2 < \ln 0.0001$$

Note that $\ln 0.2 < 0 \Rightarrow n > \frac{\ln 0.0001}{\ln 0.2}$ (we switch the sign from $<$ to $>$ as we divide by a negative number)

$\Rightarrow n > 5.7$

$\Rightarrow n = 6$ since n must be a whole number

Example 2

A chemical engineer is interested in determining whether a certain trace impurity is present in a product. An experiment has a probability of 0.8 of detecting the impurity if it is present. The probability of not detecting the impurity if it is absent is 0.9. The prior probabilities of the impurity being present and being absent are 0.4 and 0.6 respectively. Three separate experiments result in only two detections. What is the posterior probability that the impurity is present? (I will explain what prior and posterior probabilities mean in class, but you can probably guess from the question.)

Solution (impurity of product problem):

We should write down the events and use set notation:

Let A={impurity present in product}

A^C={impurity not present in product}

$+ve$ ={test detected impurity}

$-ve$ ={test did not detect impurity}

From question:

$$P(A) = 0.4, \quad P(A^C) = 0.6, \quad P(+ve|A) = 0.8, \quad P(-ve|A^C) = 0.9$$

$$\begin{aligned}
&P(A|3\ experiments\ result\ in\ 2\ +ve's) \\
=&\frac{P(A \cap 3\ experiments\ result\ in\ 2\ +ve's)}{P(3\ experiments\ result\ in\ 2\ +ve's)} \\
=&\frac{P(A \cap B)}{P(B)} \qquad\qquad [\,Let\ B = \{3\ experiments\ result\ in\ 2\ +ve's\}\,] \\
=&\frac{P(B|A) \cdot \overset{=0.4}{P(A)}}{P(B|A^C) \cdot \underbrace{P(A^C)}_{0.6} + P(B|A) \cdot \underbrace{P(A)}_{0.4}} \\
=&\frac{\binom{3}{2}(0.8)^2(0.2) \times 0.4}{\binom{3}{2}(0.1)^2(0.9) \times 0.6 + \binom{3}{2}(0.8)^2(0.2) \times 0.4} \\
=&\frac{0.1536}{0.0162 + 0.1536} \\
\approx&0.905
\end{aligned}$$