STAT241/251 Lecture Notes

Chapter 6 Part 4

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Ch 6.3 Geometric distribution and Return Period

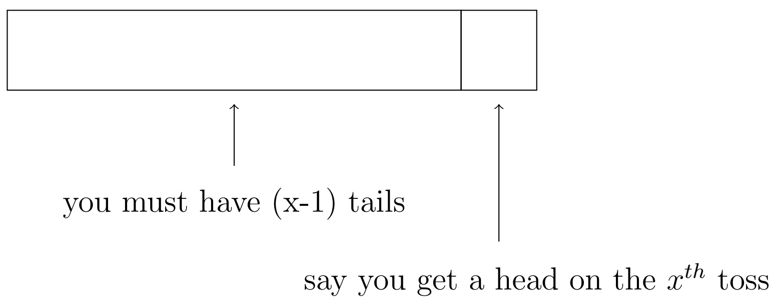
You count the number of independent coin tosses X you make until you get a head.

What kind of distribution does X take? It is not binomial (n is not fixed).

This is a Geometric distribution.

What pmf does a Geometric distribution take? We can picture it this way:

Let X be the count of the number of coin tosses until you get a head.



$$\therefore \quad P(X = x) = (1-p)^{x-1}p \qquad \text{x} = 1,2,3,\ldots$$

Note that $x = 1, 2, 3, ...$ in that it starts from 1, not 0 for this definition of $P(X = x)$

We write $X \sim Geometric(p)$ where p is the probability of success

What are the expected value $E(X)$ and variance $Var(X)$ of X when $X \sim Geo(p)$?

Ans:

$$E(X) = \frac{1}{p} \quad \text{(see page 107 for proof)}$$

$$Var(X) = \frac{1-p}{p^2} \quad \text{(Problem 6.6 in notes)}$$

Note: $E(X)$ is also called 'return period' [the number of trials before the first occurrence of a certain event]

Note that $E(X) = \frac{1}{p}$, so if p is known, its inverse is the return period (and vice versa).

Read Ex.6.2 on page 107 for an example of a question on return period.

Is it easy to calculate the cdf of a Geometric r.v.? Let's try. Say $X \sim Geometric(p)$.

$$
\begin{aligned}
F_X(x) =& P(X \le x) \\
=& \sum_{n=1}^{x} (1-p)^{n-1} p \\
=& p + (1-p)p + (1-p)^2 p + (1-p)^3 p + \cdots + (1-p)^{x-1} p \\
=& p[\underbrace{1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{x-1}}_{\text{Geometric series. Common ratio is } (1-p) < 1. \text{ Number of terms is x.}}]
\end{aligned}
$$

But we remember that the sum of

$$\textit{geometric series} = \frac{a(1-r^n)}{1-r}, \textit{ where a is the first term, r is the common ratio}$$

Hence,

$$
\begin{aligned}
F_X(x) =& p\left[\frac{1(1-(1-p)^x)}{1-(1-p)}\right] \\
=& p\left[\frac{1-(1-p)^x}{p}\right] \\
=& 1-(1-p)^x
\end{aligned}
$$

This result can be useful sometimes.

Example

14% of all drivers are uninsured. Traffic police establish a checkpoint and randomly stops cars to inspect proof of insurance.

(a) What is the probability the 3rd person stopped will be the first uninsured motorist?

Ans: Let X be the count of the number of cars until the first uninsured motorist is found.

$$X \sim Geometric(p = 0.14)$$

$$\begin{aligned} P(X = 3) =& (1 - p)^2 p \\ =& (1 - 0.14)^2 (0.14) \\ =& 0.1035 \end{aligned}$$

(b) What is the probability it will take at least 5 cars before the first uninsured motorist is found?

Ans: You wish to find $P(X = 5) + P(X = 6) + \dots$

$$
\begin{aligned}
P(X \geq 5) =& 1 - P(X < 5) \\
=& 1 - P(X \leq 4) \\
=& 1 - [1 - (1 - 0.14)^4] \quad (Use\, the\, result\, of\, cdf\, from\, two\, pages\, back) \\
=& 0.5470
\end{aligned}
$$