STAT241/251 Lecture Notes
Chapter 6 Ex 6.3

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From free coursenotes,
Example 6.3 (pg 109) & 6.3(e) - (h) (pg 112) are good questions on Poisson Process. Solutions are found in notes but they are provided here in case you need more detailed solutions.
Ex 6.3 In southern California, there is on average one earthquake per year with Richter magnitude 6.1 (big earthquake) or higher.

(a) What is the probability of having 3 or more big earthquakes in the next 5 years?

Let X be the number of earthquakes in the next 5 years.

Then

$$X \sim Poisson(5)$$

for 5 year period

$$\begin{aligned} P(X \geq 3) =& 1 - \Big[P(X=0) + P(X=1) + P(X=2) \Big] \\ =& 1 - \left[\frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} \right] \\ =& 0.875 \end{aligned}$$

(b) What is the most likely number of big earthquake in the next 15 months.

Let Y be the number of earthquakes in the next 15 months.

Then $Y \sim Poisson(1.25)$

[for 15 months]
12 mth $\rightarrow$ 1 earthquake
hence, 15mth $\rightarrow$ 1.25 earthquake

The question is not asking you for $E(Y)$. Rather, your answer must be an integer (whole number)

$$\left.\begin{array}{l} P(Y=0)= \\ P(Y=1)= \\ P(Y=2)= \\ \quad\vdots \end{array}\right\}$$ you wish to find the value of Y with the largest probability $P(Y=y)$

We first find

$$\frac{P(Y=y)}{P(Y=y-1)}$$

so that we can find the relationship between $P(Y=y)$ and $P(Y=y-1)$

$$\frac{P(Y=y)}{P(Y=y-1)} = \frac{\frac{e^{-\lambda}\lambda^y}{y!}}{\frac{e^{-\lambda}\lambda^{y-1}}{(y-1)!}} = \frac{\lambda^{y-(y-1)}}{y} = \frac{\lambda}{y} = \frac{1.25}{y}$$

Hence, $P(Y=y) = \frac{1.25}{y}P(Y=y-1)$

$$\begin{aligned}
P(Y=0) &= \frac{e^{-1.25}(1.25)^0}{0!} = e^{-1.25} \\
P(Y=1) &= \frac{1.25}{1}P(Y=0) = 1.25(e^{-1.25}) > P(Y=0) \\
P(Y=2) &= \boxed{\tfrac{1.25}{2}}P(Y=1) < P(Y=1) \\
\text{Note } \frac{1.25}{2} < 1 \quad P(Y=3) &= \boxed{\tfrac{1.25}{3}}P(Y=2) < P(Y=2)
\end{aligned}$$

< 1

We see that $P(Y=4)$ is going to be even smaller than $P(Y=3)$ and so on $\cdots$.

Hence, $P(Y=1)$ is the largest probability among all $P(Y=y)$, and therefore, the most likely number of big earthquakes is 1.

(c) What is the probability of having a period of 15 months without a big earthquake?

2 ways to answer this question:

(i)

no earthquake

$\vdash\!\!\!\longrightarrow\!\!\dashv$

15 mth

In 15 months,

$$Y \sim Poisson(1.25) \quad \text{[for 15 months]}$$

$$P(Y = 0) = \frac{e^{-1.25}(1.25)^0}{0!} = e^{-1.25}$$

(ii) Or you can think in terms of time between consecutive events exceeding 15 months.

Let T be the time between consecutive occurrences of big earthquakes.

$$\text{Then} \qquad T \sim Exponential(Mean = \frac{1}{1} = 1),$$

in years

$$\left.\begin{aligned} \text{Then} \quad f_T(t) =& e^{-t}, \quad t > 0 \\ \text{and} \quad F_T(t) =& 1 - e^{-t}, \end{aligned}\right\} \text{knowledge from Chapter 4}$$

$$\begin{aligned} P(T > 1.25) =& 1 - F_T(1.25) \\ =& 1 - \left[1 - e^{-1.25}\right] \\ =& e^{-1.25} \end{aligned}$$

Convert 15 mths to years

(d) What is the probability of having to wait more than 3.5 years until the occurence of the next 4 big earthquakes?

Let Y be the number of earthquakes in 3.5 years.

Then $Y \sim Poisson(3.5)$ [in 3.5 yeras]

Think: Does the question want $P(Y > 4)$ or $P(Y < 4)$?

Ans:

$$
\begin{aligned}
P(Y < 4) =& P(Y \leq 3) \\
=& F_Y(3) \\
=& P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3) \\
=& \frac{e^{-3.5} 3.5^0}{0!} + \frac{e^{-3.5} 3.5^1}{1!} + \frac{e^{-3.5} 3.5^2}{2!} + \frac{e^{-3.5} 3.5^3}{3!} \\
=& e^{-3.5} \left[1 + 3.5 + \frac{3.5^2}{2!} + \frac{3.5^3}{3!} \right] \\
=& 0.5366
\end{aligned}
$$

Fewer than 4
in 3.5 years
$\Rightarrow$ need to wait more
than 3.5 yrs to see
4 big earthquakes

(e) [pg112] What is the expected number of big earthquakes in the next 5 years? 15 months? What are the corresponding stardard deviation?

Let X be the number of earthquakes in 5 years.

$X \sim Poisson(5)$, $E(X) = 5$ $Var(X) = 5$ $\Rightarrow$ standard dev $= \sqrt{5}$ → Properties of Poisson

Let Y be the number of earthquakes in 15 months.

$$X \sim Poisson(1.25), \quad E(X) = 1.25 \quad Var(X) = 1.25 \quad \Rightarrow \text{standard dev} = \sqrt{1.25}$$

(f) What is the expected waiting time (in years) between 2 consecutive big earthquakes?

Let T be the time in years between 2 consecutive big earthquakes.

$$T \sim Exponential(Mean = \frac{1}{1})$$

$E(T) = 1$ (that is, $\frac{1}{\lambda}$) → Properties of Exponential distribution See Ch4

[if you wish to find $Var(T)$, it is $\frac{1}{\lambda^2} = \frac{1}{1} = 1$]

this will have to match the λ rate in this question, λ is 1 per year, so T is in years

(g) What is the expected waiting time (in yrs) until the 25^{th} big earthquake? What is the standard deviation?

Let W be the waiting time in years, until the 25^{th} big earthquake.

Let T_i be the waiting time in years between $(i-1)^{th}$ and i^{th} big earthquake.

Each T_i is an exponential distribution (but note W is <u>not</u> exponential)

$$\begin{aligned}
W =& T_1 + T_2 + T_3 + \cdots + T_{25} \\
E(W) =& E(T_1 + T_2 + T_3 + \cdots + T_{25}) \\
=& 25E(T_1) \\
=& 25 \times 1 \\
=& 25 \\
Var(W) =& Var(T_1 + T_2 + T_3 + \cdots + T_{25}) \\
=& 25Var(T_1) \quad \text{because the } T_i's \text{ are independent} \\
=& 25 \times 1 \\
=& 25 \\
\Rightarrow& \quad \text{standard deviation of } W = \sqrt{25} = 5
\end{aligned}$$

(h) What is the approximate probability that the waiting time until the 25^{th} big earthquake exceed 27 years? (Solution requires using CLT see page 122-123 for solution).

From (g), we see that

$$W = T_1 + T_2 + T_3 + \cdots + T_{25}$$

We know that $T_1 \sim Exponential(Mean = 1)$, but we have no idea about the distribution of W [the sum of independent exponential r.v. is not exponential. If you take more advance Statistics courses, you'll learn that the sum of independent exponential r.v. is a gamma distribution. Since you know nothing about Gamma distribution, use CLT to answer this question since $n \geq 20$ which justifies using CLT]

not part of STAT 241/251

By CLT, $(n = 25 > 20)$,

$$W \overset{approx.}{\sim} N(\underbrace{25}_{E(\sum T)}, \underbrace{25}_{Var(\sum T)})$$

$$\begin{aligned}
P(W > 27) =& P(Z > \frac{27-25}{5}) \\
=& 1 - P(Z < 0.4) \\
=& 1 - 0.6554 \\
=& 0.3446
\end{aligned}$$