STAT241/251 Lecture Notes
Chapter 4 Part B

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Ch 4 Part B

One minute recap :

We have learnt how to calculate the mean and variance of a discrete random variable.
The 'mean' is the same as asking for $E(X)$ where X is a random variable.

Example:

The manager of a stockroom in a factory has constructed the following probability distribution for the daily demand (number of times used) for a particular tool.

y	0	1	2
P(Y=y)	0.1	0.5	0.4

Find

(a) E(Y) $\Leftarrow$ *this is asking us to find the mean number of times used per day*

(b) Var(Y)

Solution:

$$\begin{aligned}
E(Y) =& (0 \times 0.1) + (1 \times 0.5) + (2 \times 0.4) \\
=& 1.3 \\
E(Y^2) =& (0^2 \times 0.1) + (1^2 \times 0.5) + (2^2 \times 0.4) \\
=& 2.1 \\
Var(Y) =& E(Y^2) - [E(Y)]^2 \\
=& 2.1 - 1.3^2 \\
=& 0.41
\end{aligned}$$

The above example calculates the mean and variance of a discrete random variable. We now continue with the study of continuous random variables.

There are many continuous random variables, but for STAT 241/251, we learn 3 types of continuous random variables thoroughly:

(a) Uniform distribution ← today's topic

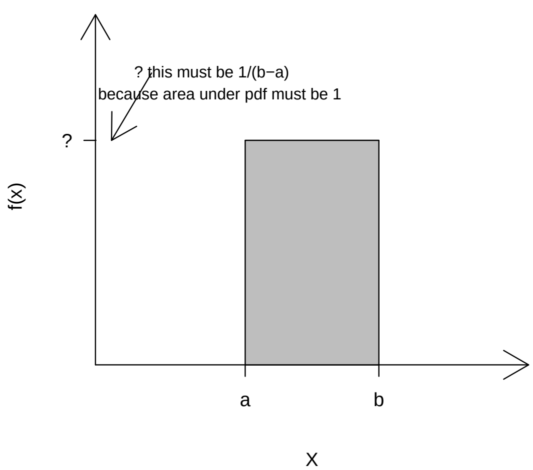
(b) Exponential distribution

(c) Normal distribution ← Chapter 5

By thorough, it means you are expected to know how to arrive at the mean, variance, cdf when given a uniform/exponential distribution.

Uniform Distribution

In general, this is the pdf of a random variable X which has the uniform distribution on the interval (a, b).

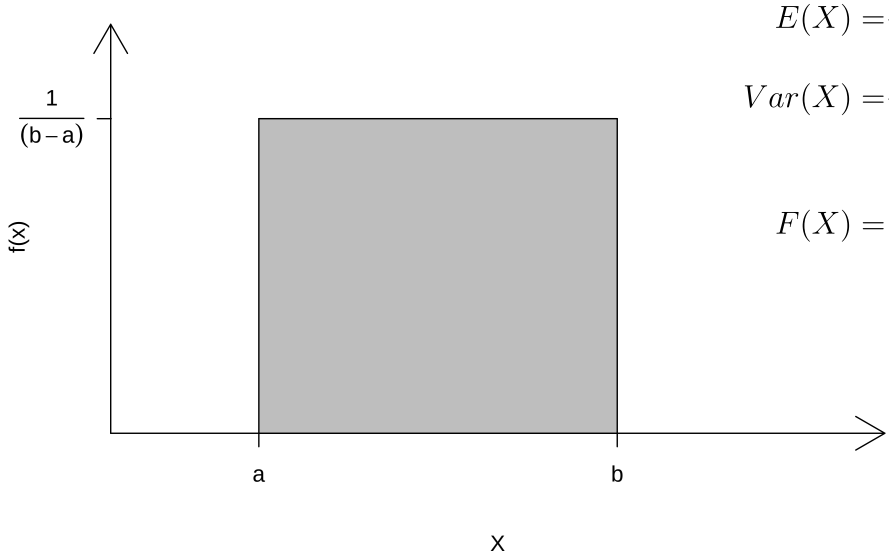


We can therefore write out the pdf of a uniform random variable when we know the 'x range' (e.g. $a \leq x \leq b$, the proper word for it is 'support of X').

Example, pdf of a random variable which has a uniform distribution on $(1, 5)$ is:

$$f(x) = \begin{cases} \frac{1}{4}, & \boxed{1 \leq x \leq 5}, \leftarrow \text{ support of } X \\ 0 & \text{otherwise} \end{cases}$$

In general,



$$
\begin{aligned}
E(X) &= \frac{a+b}{2} \\
Var(X) &= \frac{(b-a)^2}{12} \\
F(X) &= \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \le x \le b, \\ 1 & x > b \end{cases}
\end{aligned}
$$

Proof:

(a) Let us show that $E(X) = \frac{a+b}{2}$

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\
&= \int_{a}^{b} x \cdot \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\
&= \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2} \right) \\
&= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right) \\
&= \frac{1}{\cancel{b-a}} \frac{(b+a)\cancel{(b-a)}}{2} \\
&= \frac{b+a}{2}
\end{aligned}
$$

(b) Proof that $Var(X) = \frac{(b-a)^2}{12}$ when $\underline{X \sim Uniform(a,b)}$

⋆ Note: This is how we write out the distribution of a random variable. $X \sim Uniform(a,b)$ [your coursenotes use $X \sim U(\underbrace{-\frac{1}{2}}_{a}, \underbrace{\frac{4}{5}}_{b})$ which is fine too]

To find $Var(X)$, we first find $E(X^2)$.

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 \cdot f(x)dx \\
&= \int_a^b x^2 \cdot \frac{1}{b-a}dx \\
&= \frac{1}{b-a}[\frac{x^3}{3}]_a^b \\
&= \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) \\
&= \frac{1}{3(b-a)}(b^3 - a^3) \\
&= \frac{1}{3\cancel{(b-a)}}\cancel{(b-a)}(b^2 + ab + a^2)
\end{aligned}
$$

I'll show how you get this under ⋆⋆

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{1}{3}(b^2 + ab + a^2) - (\frac{a+b}{2})^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

$\star\star$ To show how I obtain $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$

Let $f(b) = b^3 - a^3, then f(a) = a^3 - a^3 = 0 \Rightarrow b - a$ is a factor,[use factor theorem]

$$\begin{array}{r l}
 & \quad b^2 \quad +ab \quad +a^2 \\ \cline{2-2}
b-a) & \quad b^3 \qquad\qquad\qquad -a^3 \\
 & -(b^3 - ab^2) \\ \cline{2-2}
 & \qquad ab^2 \\
 & \qquad -(ab^2 - a^2b) \\ \cline{2-2}
 & \qquad\qquad a^2b - a^3 \\
 & \qquad\qquad -(a^2b - a^3) \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}$$

$$\Rightarrow b^3 - a^3 = (b-a)(b^2 + ab + a^2)$$

(c) Show that $F(x)$ is $\frac{(x-a)}{b-a}$ when $X \sim Uniform(a,b)$

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \\ &= \int_{a}^{x} \frac{1}{b-a}dt \\ &= \frac{1}{b-a}[t]_a^x \\ &= \frac{x-a}{b-a} \\ \Rightarrow F(x) &= \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1 & x > b \end{cases} \end{aligned}$$

Make sure you either memorize these results or have them on your cheatsheet:

$$X \sim Uniform(a,b)$$

$$\begin{cases} E(X) = \frac{a+b}{2} \\ Var(X) = \frac{(b-a)^2}{12} \\ F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases} \end{cases}$$

Simple Example: $X \sim Uniform(-2,3)$,then

$$f(x) = \begin{cases} \frac{1}{5} & -2 \le x << 3 \\ 0 & \text{otherwise} \end{cases}$$

$E(X) = \frac{-2+3}{2} = 0.5, Var(X) = \frac{[3-(-2)]^2}{12} = \frac{25}{12}$

$$F(x) = \begin{cases} 0 & x < -2 \\ \frac{x+2}{5} & -2 \le x \le 3 \\ 1 & x > 3 \end{cases}$$