Ch3 - Probability

Sample Space

The sample space of an experiment is the set of all possible outcomes of that experiment. If every outcome is equally likely, then the probability that event A will occur is $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ represents the number of elements in A and $n(S)$ represents the number of elements in the sample space.

Example:

A fair die is tossed once. Find the probability that the number is odd.

Solution:

$S = \{1, 2, 3, 4, 5, 6\}$

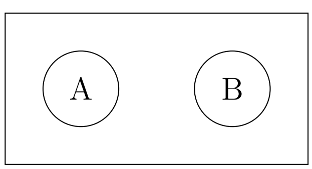
Let E be the event of obtaining an odd number.

P(number is odd)$= \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

Definition - mutually exclusive or disjoint events

When A and B have no outcomes in common, they are said to be mutually exclusive or disjoint events. [that is, $P(A \cap B) = 0$]

You can visualize mutually exclusive events using a Venn diagram.



A and B are mutually exclusive events.

Set Notation that you need to know

intersect

$$P(A \cap B) = P(A\, and\, B)$$

union

$$P(A \cup B) = P(A\, or\, B)$$

$$Important\ formula : \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$P(A^C) = 1 - P(A)$$

complement

Note that $0 \leq P(A) \leq 1$ for all A.

We know that if A and B are mutually exclusive events, then

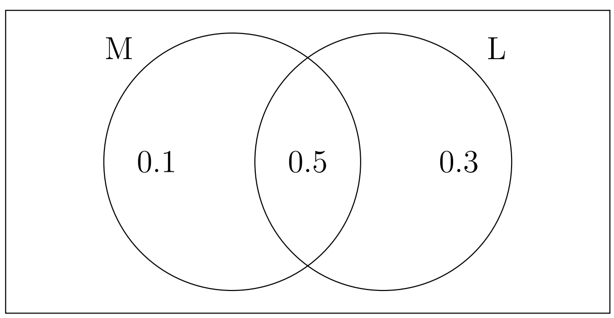
$$P(A \cap B) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$$

Example

In a certain residential suburb, 60% of all households subscribe to the metropolitan newspaper published in a nearby city, 80% subscribe to the local paper, and 50% of all households subscribe to both papers. If a household is selected at random, what is the probability that it subscribes to
(a) at least one of the two newspapers
(b) exactly one of the two newspapers?

Solution:

One strategy of solving probability questions is to use a Venn diagram:



M={ subscribe to metropolitan newspaper}

L={subscribe to local paper}

$$\begin{aligned}(a) \qquad P(M \; or \; L) &= P(M \cup L) \\ &= P(M) + P(L) - P(M \cap L) \\ &= 0.6 + 0.8 - 0.5 \\ &= 0.9\end{aligned}$$

$$(b) \qquad P(exactly\ 1\ newspaper) = P(M \cap L^C) + P(L \cap M^C)$$
$$= 0.1 + 0.3$$
$$= 0.4$$

Independence (part 1)

Two events A and B are independent if knowing that one occurs does not change the probability that the other occurs.

$$\boxed{\text{If A and B are independent, } P(A \cap B) = P(A) \times P(B)}$$

We do not use Venn diagram to visualize independence [if asked to show 2 events are independent, you should show $P(A \cap B) = P(A) \times P(B)$]

Conditional Probability

For any two events A and B with P(B) > 0, the conditional probability of A given that B has occurred is defined by

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Note that the definition of conditional probability yields the following result:

$$P(A \cap B) = P(A|B) \times P(B)$$

this formula is used quite often

Simple example on conditional probability

You toss a fair die. Find the probability of getting a "1", given that an odd number was obtained.

Solution:

A={observe a 1}, B={observe an odd number}

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Another example

The weather on any day is classified as wet or dry. If it is wet on any particular day, the probability that it will be wet the next day is 0.4. If it is dry on any particular day, the probability that it will be dry the next day is 0.7.

Given that in a particular week Monday is dry, find the probability that

(a) both Tuesday and Wednesday of that week will be dry,

(b) Wednesday of that week will be dry.

Given also that Wednesday of that week is dry,

(c) find the probability that Tuesday of that week was also dry.

(a) P(Tues dry $\cap$ Wed dry)=0.7×0.7=0.49

[you can think of this as $P(A \cap B) = P(A) \times P(B|A)$]

(b) P(Wednesday dry)= P(Tues dry $\cap$ Wed dry)+P(Tues wet $\cap$ Wed dry)

$= (0.7 \times 0.7) + (0.3 \times 0.6)$

we are listing all possible cases to get to our event of interest. This strategy is quite common

$= 0.49 + 0.18$

$= 0.67$

part a's answer

(c) P(Tues dry | Wed dry)=$\frac{P(Tues\ dry \cap Wed\ dry)}{P(Wed\ dry)} = \frac{0.49}{0.67} = \frac{49}{67}$

part b's answer

Back to Independent (part 2)

We know that when 2 events are independent,

$$P(A \cap B) = P(A) \times P(B)$$

Note that (when A and B are independent)

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{P(A) \times P(B)}{P(B)} \\
&= P(A)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P(B|A) &= \frac{P(B \cap A)}{P(A)} \\
&= \frac{P(B) \times P(A)}{P(A)} \\
&= P(B)
\end{aligned}
$$

When you are asked to show 2 events are independent, you need to show any one of the following:

$$
\left.\begin{array}{c}
P(A \cap B) = P(A) \times P(B) \\
P(A|B) = P(A) \\
P(B|A) = P(B)
\end{array}\right\} \begin{array}{l}
\text{any 1 will do, but} \\
P(A \cap B) = P(A) \times P(B) \\
\text{is commonly used}
\end{array}
$$

<u>more on independence</u>

The following facts might help you get a quick answer sometimes:

If events A and B are independent, then

$\rightarrow$ (a) A^C and B are also independent

$\rightarrow$ (b) A and B^C are also independent

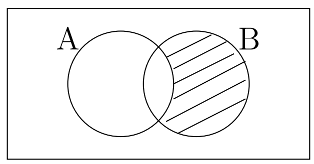
$\rightarrow$ (c) A^C and B^C are also independent

Proof for (a):

Since A and B are independent, $P(A \cap B) = P(A) \times P(B)$

Aim: Show that $P(A^C \cap B) = P(A^C) \times P(B)$

$$\begin{aligned} RHS &= P(A^C) \times P(B) \\ &= [1 - P(A)]P(B) \\ &= P(B) - P(A) \times P(B) \\ &= P(B) - P(A \cap B) \qquad (\textit{since A and B are independent}) \\ &= P(A^C \cap B) \end{aligned}$$



Another way to prove:

$$\begin{aligned} P(A|B) + P(A^C|B) &= 1 \\ P(A^C|B) &= 1 - P(A|B) \\ &= 1 - P(A) \; (\textit{ since} \\ &\textit{A and B are independent}) \\ &= P(A^C) \\ &\Rightarrow A^C \textit{ and B are independent.} \end{aligned}$$

You can use similar logic to show (b) and (c).

Note: If you use these facts in assignments, MT or exam, you should prove them as you should assume the marker does not know these facts.

Example on when these facts are useful:

Question: The independent events A and B are such that P(A)=0.6 and P(B)=0.3. Find $P(A \cap B^C)$

<u>method A:</u>

Use the fact that since A and B are independent, then A and B^C are also independent.

$$P(A \cap B^C) = P(A) \times P(B^C) = 0.6 \times 0.7 = 0.42$$

<u>method B:</u>

Another student does not know this fact but she can still produce a correct answer as follows:

$$\begin{aligned} P(A \cap B^C) &= P(A) - P(A \cap B) \\ &= P(A) - P(A) \times P(B) \qquad [\textit{because A and B are independent}] \\ &= 0.6 - (0.6 \times 0.3) \\ &= 0.42 \end{aligned}$$

But this student has to be pretty good at visualizing to come up with this

<u>Example</u>

A and B are two events such that P(A)=1/3, P(B)=2/5 and P(A ∪ B)=17/30.

(i) Find $P(A \cap B)$

A student who answers that $P(A \cap B) = P(A) \times P(B)$ in this case is wrong because that formula works only when A and B are independent (we don't know if that is the case here).

The correct answer is to use

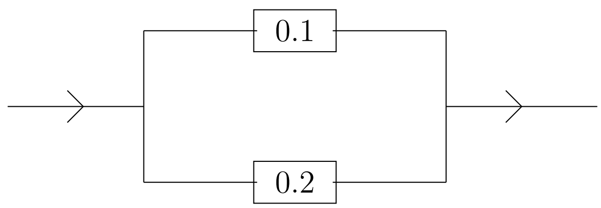
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \therefore P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 1/3 + 2/5 - 17/30 \\ &= 1/6 \end{aligned}$$

(ii) Show that A and B are <u>not</u> independent.

Since $P(A) \times P(B) = 2/15 \neq P(A \cap B)$, A and B are not independent.

<u>Example</u>

Consider the following electronic system which shows the probabilities of the system components operating properly. The entire system operates if at least one of the components operates. Assume that the components operate independently, what is the probability that the entire system operates? [Note: <u>reliability</u> of the system=probability of the system works]



Solution:

A={component A works }

B={component B works}

because as long as A or B works, system works

$$\begin{aligned} P(system\ works) &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= 0.1 + 0.2 - (0.1 \times 0.2) \qquad [because\ A\ and\ B\ are\ independent] \\ &= 0.28 \end{aligned}$$

must state this for assignments and exams

Idea: Sometimes, it is easier to think of the opposite of what the question asks for.

$$\begin{aligned} Eg. \qquad P(system\,works) &= 1 - P(system\,does\,not\,work) \\ &= 1 - P(A^C \cap B^C) \\ &= 1 - (0.9 \times 0.8) \\ &= 1 - 0.72 \\ &= 0.28 \end{aligned}$$

both do not work

3.3 EXERCISES

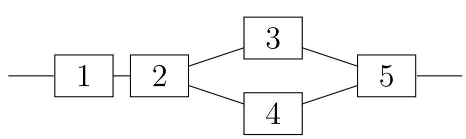
Problem 3.8 Suppose that the numbers 1 through 10 form the sample space of a random experiment, and assume that each number is equally likely. Define the following events: A_1, the number is even; A_2, the number is between 4 and 7, inclusive.
(a) Are A_1 and A_2 mutually exclusive events? Why?
(b) Calculate $P(A_1), P(A_2), P(A_1 \cap A_2)$, and $P(A_1 \cup A_2)$.
(c) Are A_1 and A_2 independent events? Why?

Problem 3.9 A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed three times,
(a) what is the sample space of the random experiment?
(b) what is the probability of getting exactly two tails?

Problem 3.10 Items in your inventory are produced at three different plants: 50 percent from plant A_1, 30 percent form plant A_2, and 20 percent from plant A_3. You are aware that your plants produce at different levels of quality: A_1 produces 5 percent defectives, A_2 produces 7 percent defective, and A_3 yields 8 percent defectives. You select an item from your inventory and it turns out to be defective. Which plant is the item most likely to have come from? Why does knowing the item is defective decrease the probability that it has come from plant A_1, and increase the probability that it has come form either of the other two plants?

Problem 3.11 Calculate the reliability of the system described in the following figure. The numbers beside each component represent the probabilities of failure for this component. Note that the components work independently of one another.

try as many questions
as you can to be
successful in this course



Problem 3.12 A system consists of two subsystems connected in series. Subsystem 1 has two components connected in parallel. Subsystem 2 has only one component. Suppose the three components work independently and each has probability of failure equal to 0.2. What is the probability that the system works?