How to find the pdf that represents the maximum of n independent random variables?

Setup of problem:

You are given pdf of n independent random variables $X_1, X_2, \ldots, X_n$.

Question: Find the pdf of the maximum of $X_1, X_2, \ldots, X_n$.

Solution:

Let $Y = max(X_1, X_2, \ldots, X_n)$

[Thought process: If y is the maximum of $X_1, X_2, \ldots, X_n$, then surely each of $X_1, X_2, \ldots, X_n$ must be less than or equal to y]

Remember: To find pdf of y, we start by first finding its cdf.

$$\begin{aligned} F_Y(y) =& P(Y \leq y) \\ =& P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y) \\ =& P(X_1 \leq y) \times P(X_2 \leq y) \times \cdots \times P(X_n \leq y) \\ & \text{(because } X_i's \text{ are independent)} \\ =& F_{X_1}(y) \times F_{X_2}(y) \times \ldots F_{X_n}(y) \end{aligned}$$

Furthermore, if all the $X_i's$ have the same pdf,

$$F_Y(y) = [F_X(y)]^n$$

How to get pdf of Y?

$$\begin{aligned} f_Y(y) =& F_Y'(y) \\ =& n[F_X(y)]^{n-1} \times f_X(y) \end{aligned}$$

Example: We'll work through Example 4.8 [in coursenotes, page 75] together in class.

How to find the pdf that represents the minimum of n independent random variable?

Setup of problem:

You are given the pdf of n independent random variables $X_1, X_2, \ldots, X_n$.

Question: Find the pdf of the minimum of $X_1, X_2, \ldots, X_n$..

Solution:

Let $Y = min(X_1, X_2, \ldots, X_n)$

Again, we need to first find the cdf of Y.

[Thought process: If y is the minimum of $X_1, X_2, \ldots, X_n$, then surely each $X_1, X_2, \ldots, X_n$ must be larger than or equal to y]

$$\begin{aligned}
F_Y(y) =& P(Y \leq y) \\
=& 1 - P(Y > y) \\
=& 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y) \\
=& 1 - \Big[P(X_1 > y) \times P(X_2 > y) \times \cdots \times P(X_n > y)\Big] \\
=& 1 - [1 - F_{X_1}(y)] \times [1 - F_{X_2}(y)] \times \cdots \times [1 - F_{X_n}(y)]
\end{aligned}$$

Furthermore, if all the $X_i's$ have the same pdf,

$$F_Y(y) = 1 - [1 - F_X(y)]^n$$

How to get pdf of Y?

$$\begin{aligned}
f_Y(y) =& F_Y'(y) \\
=& -n[1 - F_X(y)]^{n-1} \times (-f_X(y)) \\
=& n[1 - F_X(y)]^{n-1} \cdot f_X(y)
\end{aligned}$$

We will work on Example 4.9 (pg 81 coursenots) together in class.

Before doing Example 4.8, Let's work on a simple example first:

Electronic components of a certain type have a length of life Y (in hours), with probability density given by

$$f(y) = \begin{cases} \frac{1}{100} e^{\frac{-y}{100}}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Suppose two such components operate independently and in series in a certain system (hence, the system fails when either component fails). Find the pdf of X, the length of life of the system.

Solution:

$X = min(Y_1, Y_2)$ where Y_1 and Y_2 are independent random variables with the given pdf

Why minimum? Because the system fails at the first component failure

Remember: to find the pdf of min/max of random variable, first find its cdf.

$$\begin{aligned} F_X(x) =& P(X \leq x) \\ =& 1 - P(X > x) \\ =& 1 - P(Y_1 > x, Y_2 > x) \\ =& 1 - \Big[P(Y_1 > x) \times P(Y_2 > x)\Big] \\ & \text{(Since } Y_1 \text{ and } Y_2 \text{ are independent)} \\ =& 1 - \Big[1 - (1 - e^{-x/100})\Big]^2 \\ =& 1 - \Big[e^{-x/100}\Big]^2 \end{aligned}$$

We differentiate $F_X(x)$ to get pdf.

$$
\begin{aligned}
f_X(x) =& -2(e^{-x/100})(\frac{1}{100}e^{-x/100}) \\
=& \frac{1}{50}e^{-x/50}
\end{aligned}
$$

$$
\Rightarrow f_X(x) = \begin{cases} -\frac{1}{50}e^{\frac{-x}{50}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}
$$

Interesting: the minimum of two exponential distributed random variables turns out to have an exponential distribution as well (But you will see in the next example that the maximum of two exponential distributed random variables does <u>not</u> have an exponential distribution) Using the pdf, you can now answer questions such as the median life time, mean length etc of the system.

Interesting:

each component

$$E(Y) = 100 \text{ hrs}$$

$$E(X) = 50 \text{ hrs}$$

system connected in series

Next example: We now consider the 2 components when they operate in parallel (Hence, the system does not fail until both components fail). Find the pdf of X, the length of life of the system.

Solution:

$$X = max(Y_1, Y_2)$$

Why maximum? Because the system does not fail until both components fail

As usual, let's find the cdf of X first.

$$\begin{aligned} F_X(x) =& P(X \leq x) \\ =& P(Y_1 \leq x, Y_2 \leq x) \\ =& P(Y_1 \leq x) \times P(Y_2 \leq x) \\ & \textit{(Since } Y_1 \textit{ and } Y_2 \textit{ are independent)} \\ =& (1 - e^{-x/100})^2 \end{aligned}$$

means differentiate

$$\begin{aligned} f_X(x) =& F'_X(x) \\ =& 2(1 - e^{-x/100})(-e^{\frac{-x}{100}} \times -\frac{1}{100}) \\ =& \frac{1}{50}(e^{-x/100} - e^{-x/50}) \end{aligned}$$

this is not pdf of exponential random variable

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{50}(e^{-x/100} - e^{-x/50}), & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

We see that the maximum of 2 exponential random variables is not an exponential random variable.