CPSC 320: Intermediate Algorithm Design and Analysis
Assignment #4, due Thursday, February 16th, 2012 at 11:00

[8] 1. Kruskal's algorithm is not the only existing simple, greedy algorithm to find a minimum spanning tree of an undirected graph G. Another such algorithm is the Prim-Jarník algorithm. It is very similar to Dijkstra's algorithm, but instead of storing in `Cost(v)` the cost of the least costly path from s to v, we instead store the cost of the cheapest edge that connects s to an element of the tree T we have constructed so far. Here is most of the pseudo-code of this algorithm.

Note: You can find the pseudocode of Kruskal's and Dijkstra's algorithm in the appendix.

```
Algorithm Prim-Jarník(V, E, cost)

T ← ∅
Cost(V[0]) ← 0
Prev(V[0]) ← none

for i ← 1 to length[V] - 1 do
    Cost(V[i]) ← +∞
    Prev(V[i]) ← none

Build heap NotInTree from V using costs as keys

for i ← 1 to length[V] do
    u ← DeleteMin(NotInTree)
    add (u, Prev(u)) to T
    for each neighbor v of u do
        if (**********) then
            **********
            Prev(v) ← u

return T
```

[2] (a) What code should replace the $**********$?

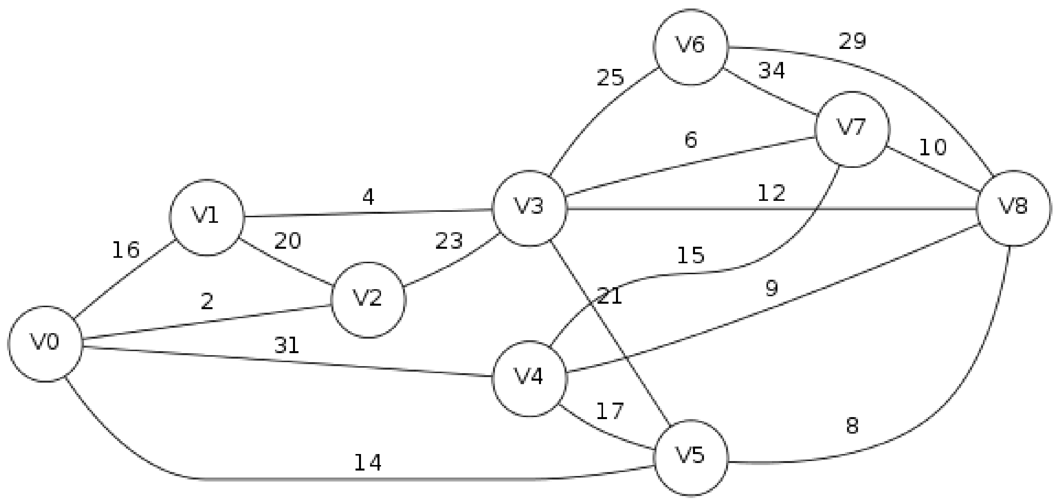
Solution: The `if` statement should be:

```
if (Cost(v) > cost(u,v)) then
    Cost(v) ← cost(u,v)
    Prev(v) ← u
```

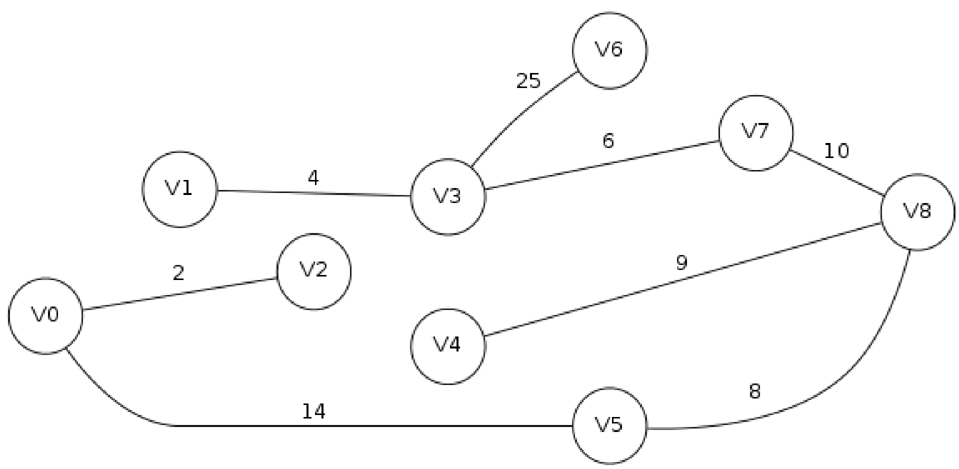
[2] (b) What is the worst-case running time of the Prim-Jarník algorithm, as a function of the number of nodes and edges of the graph? Justify your answer.

Solution: The algorithm is identical to Dijkstra's algorithm, except for the details of the computation in the inner loop. Thus its running time is the same, that is in $O((|V| + |E|) \log |V|)$.

[4] (c) Execute the Prim-Jarník algorithm on the following graph, starting from node v_3, and draw the final tree.



Solution: This is the final tree:



[15] 2. Consider an undirected graph $G = (V, E)$ with positive edge weights defined by the function cost : $E \to \mathbf{R}^+$. Assume furthermore than no two edges have the same weight. For each of the following statements about G, either prove that the statement is true, or give a counter-example that shows that it is false (hint: think of the algorithms we discussed in class).

[5] a. Given a node s of G, the tree of shortest paths from s and a minimum spanning tree of G must share at least one edge.

Solution: This is true. Consider the edge e out of s with the smallest weight:

- If we run Dijkstra's algorithm starting from s, then e will be the first edge added to the tree of shortest paths from s.
- The edge e will also be the first of the edges incident upon s that Kruskal's algorithm will consider, and hence it will be added to the minimum spanning tree.

[5] b. For every connected subgraph H of G, and minimum spanning tree T of G, $T \cap H$ is contained in a minimum spanning tree of H.

Solution: This is true. Consider the execution of Kruskal's algorithm on G, and let G_i be the subgraph obtained after the ith iteration of the `while` loop. We prove by induction of i that $G_i \cap H$ is contained in a minimum spanning tree of H.

- When $i = 0$, $G_i \cap H$ consists of the nodes of H, with no edge, and hence $G_i \cap H$ is a subset of every minimum spanning tree of H.
- Consider now an arbitrary $i > 0$. By the induction hypothesis, we can assume that $G_{i-1} \cap H$ is a subset of a minimum spanning tree of H. Let e be the edge of G considered during the ith iteration of the `while` loop.
  - If e is not an edge of H, then $G_i \cap H = G_{i-1} \cap H$, and so $G_i \cap H$ is a subset of a minimum spanning tree of H.
  - If e is an edge of H, and Kruskal's algorithm does not add e to the tree it is building, then $G_i = G_{i-1}$ and hence $G_i \cap H$ is a subset of a minimum spanning tree of H.
  - If e is an edge of H that is selected by Kruskal's algorithm, then its endpoints belong to different connected components of G_{i-1}, and hence to different connected components of $G_{i-1} \cap H$. Since e is the lowest-cost edge that connects these two connected components, it follows that $(G_{i-1} \cap H) \cup \{e\}$ is contained in a minimum spanning tree of H.

[5] c. There is a minimum spanning tree of G that contains, for every node v of G, the least-cost edge incident upon v.

Solution: This is also true: let e be the least-cost edge incident upon an arbitrary node v of G. Because no two edges of G have the same weight, e is the first edge incident upon v that is considered by Kruskal's algorithm. At this point, v is still an isolated node, and hence is not in the same connected component as the other endpoint of e. Therefore e will be added to the minimum spanning tree.

Minimum Spanning Trees

```
Algorithm Kruskal(V, E, cost)

T ← ∅
H ← heap with elements of E using costs as keys
for each vertex v ∈ V do
    set C(v) to { v }

while T has fewer than |V| - 1 edges do
  (u,v) ← deleteMin(H)
  if C(u) ≠ C(v) then
    add (u,v) to T
    merge C(u) and C(v) into one cluster

return T
```

Shortest Paths

```
Algorithm Dijkstra(V, E, cost, s)

T ← ∅
Cost(V[s]) ← 0
Prev(V[s]) ← none

for i ← 0 to length[V] - 1 do
  if (i ≠ s) then
    Cost(V[i]) ← +∞
    Prev(V[i]) ← none

Build heap NotInTree from V

for i ← 1 to length[V] do
  u ← DeleteMin(NotInTree)
  add (u, Prev(u)) to T
  for each neighbor v of u do
    if (Cost(v) > Cost(u) + cost(u,v)) then
        Cost(v) ← Cost(u) + cost(u,v)
        Prev(v) ← u

return T
```