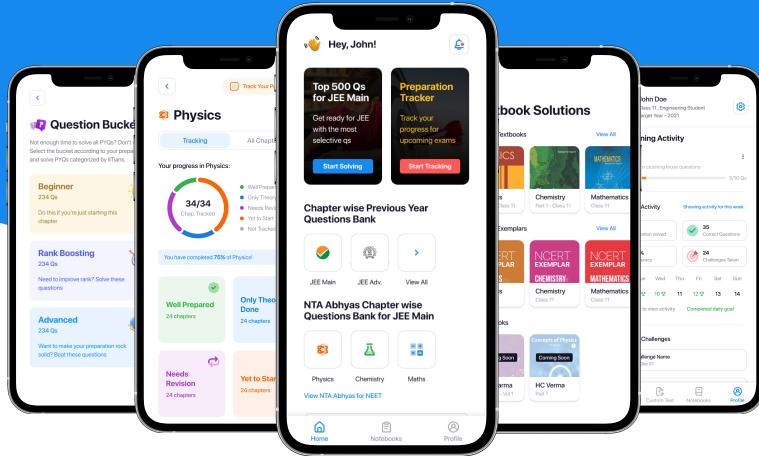




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DEFINITE INTEGRALS

DEFINITE INTEGRATION

Definition :

If $\int f(x)dx = F(x)$ i.e. $F(x)$ be an integral of $f(x)$, then $F(b) - F(a)$ is called the definite integral of $f(x)$ between the limits a and b and in symbols it is written as $\int_a^b f(x)dx$ or, $[F(x)]_a^b$.

Thus if $\int_a^b f(x)dx = F(x)$ then by definition

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

a is called the lower limit or inferior limit and b is called the upper limit or superior limit.

It is clear that value of a definite integral of a function is unique and it does not depend on different forms of indefinite integral. For if

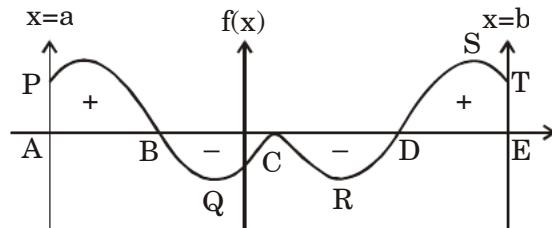
$$\int_a^b f(x)dx = [F(x) + c]_a^b = \{F(b) + c\} - \{F(a) + c\} = F(b) - F(a)$$

Thus the value of $\int_a^b f(x)dx$ is same as when we take $\int_a^b f(x)dx = F(x)$.

Geometrical Interpretation of Definite Integral

If $f(x) > 0$ for all $x \in [a, b]$; then $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x -axis and the straight lines $x = a$ and $x = b$

In general $\int_a^b f(x)dx$ represents algebraic sum of the areas of the figures bounded by the curve $y = f(x)$, the x -axis and the straight lines $x = a$ and $x = b$. The areas above x -axis are taken plus sign and the areas below x -axis are taken with minus sign i.e.,



i.e., $\int_a^b f(x)dx = \text{area APB} - \text{area BQC} - \text{area CRD} + \text{area DSTE}$

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Illustration 1

Find $\int_0^1 (4x^3 + 3x^2 - 2x + 1)dx$.

Solution :

$$\int_0^1 (4x^3 + 3x^2 - 2x + 1)dx = 4 \cdot \frac{x^4}{3} - 2 \cdot \frac{x^2}{2} + x = x^4 + x^3 - x^2 + x$$

$$\begin{aligned}\therefore \int_0^1 (4x^3 + 3x^2 - 2x + 1)dx \\ &= (1^4 + 1^3 - 1^2 + 1) - (0 + 0 - 0 + 0) = 2 - 0 = 2\end{aligned}$$

Illustration 2

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos 3x + 2\cos x} dx.$$

Solution :

$$\begin{aligned}\int \frac{\sin x}{\cos 3x + 3\cos x} dx &= \int \frac{\sin x}{(4\cos^3 x - 3\cos x) + 3\cos x} dx \\ &= \int \frac{\sin x}{4\cos^3 x} dx = \frac{1}{4} \int \tan x \sec^2 x dx = \frac{1}{8} \tan^2 x [\text{Put } z = \tan x]\end{aligned}$$

$$\begin{aligned}\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos 3x + 2\cos x} dx &= \frac{1}{8} [\tan^2 x]_0^{\frac{\pi}{4}} \\ &= \frac{1}{8} \left(\tan^2 \frac{\pi}{4} - \tan^2 0 \right) = \frac{1}{8} (1 - 0) = \frac{1}{8}\end{aligned}$$

Illustration 3

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$$

$$\text{Solution : } \int \frac{dx}{1 + \sin x} = \int \frac{dx}{1 + \cos \left(\frac{\pi}{2} - x \right)} = \int \frac{dx}{2 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right)}$$

$$= \frac{1}{2} \int \sec^2 \left(\frac{\pi}{4} - \frac{x}{2} \right) dx = \frac{1}{2} \cdot \frac{\tan \left(\frac{\pi}{4} - \frac{x}{2} \right)}{-\frac{1}{2}} = -\tan \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x} = - \left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \right]_0^{\frac{\pi}{2}}$$

$$= \left[\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) - \tan\frac{\pi}{4} \right] = -(\tan 0 - 1) = 1$$

2nd Method :

$$\int \frac{dx}{1 + \sin x} = \int \frac{1 - \sin x}{1 - \sin^2 x} dx = \int \frac{1 - \sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int (\sec^2 x - \tan x \sec x) dx = \tan x = \sec x.$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x} = [\tan x - \sec x]_0^{\frac{\pi}{2}}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^- 0} (\tan x - \sec x) - (\tan 0 - \sec 0) = \lim_{x \rightarrow \frac{\pi}{2}^- 0} \frac{\sin x - 1}{\cos x} - (0 - 1)$$

$\left[\because \tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are undefined $\right]$

Hence we can not take value of

$$\tan x - \sec x \text{ at } x = \frac{\pi}{2}$$

Here we take limit as $x \rightarrow \frac{\pi}{2} - 0$]

$$= \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{2} + h\right) - 1}{\cos\left(\frac{\pi}{2} + h\right)} + 1 \left[\text{putting } x = \frac{\pi}{2} + h \right]$$

$$= \lim_{h \rightarrow 0} \frac{\cosh - 1}{-\sin h} + 1 = \lim_{h \rightarrow 0} \frac{1 - \cosh}{\sin h} + 1 = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{h}{2}}{\sin h} + 1$$

$$= \lim_{h \rightarrow 0} \frac{2 \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \cdot \frac{h^2}{4}}{\frac{\sin h}{h} \cdot h} + 1 = 0 + 1 = 1$$

Note :

In the second method, numerator and denominator have been multiplied by $(1 - \sin x)$ and the value of $1 - \sin x$ is 0 when $x = \frac{\pi}{2}$ and hence when $x = \frac{\pi}{2}$ integrand is undefined.

Hence avoid multiplying numerator and denominator by an expression which becomes zero at any point of the interval $[a, b]$ where a and b are the lower and upper limits respectively of integration.

Problems in which integral can be found by Substitution method :

Working Rule :

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $z = \phi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be $\phi(a)$ and $\phi(b)$ respectively.

Illustration 4

Find the value of $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)^2} dx$.

Solution :

Let $z = 1 + \sin x$, then $dz = \cos x dx$

When $x = 0$, $z = 1 + \sin 0 = 1 + 0 = 1$

and when $x = \frac{\pi}{2}$, $z = 1 + \sin \frac{\pi}{2} = 1 + 1 = 2$

$$\text{Now } I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1 + \sin x)^2} dx = \int_1^2 \frac{dx}{z^2} = \int_1^2 z^{-2} dz$$

$$= \left[\frac{z^{-1}}{-1} \right]_1^2 = - \left[\frac{1}{z} \right]_1^2 = - \left[\frac{1}{2} - 1 \right] = \frac{1}{2}$$

Note : Only principal value of θ is taken. For example when $\sin \theta = 0$, $\theta = n\pi$ but principal value of θ is 0.

Illustration 5

Evaluate $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$.

Solution :

Let $x = \cos 2\theta$, then $dx = -2\sin 2\theta d\theta$.

When $x = 0, \cos 2\theta = 0 \therefore 2\theta = \frac{\pi}{2}$ or, $\theta = \frac{\pi}{4}$

$$\begin{aligned} \text{Now } I &= \int_0^1 \sqrt{\frac{1-x}{1+x}} dx = \int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} (-2\sin 2\theta) d\theta \\ &= \int_{\frac{\pi}{4}}^0 \sqrt{\frac{2\sin^2 \theta}{2\cos^2 \theta}} (-2\cdot 2\sin \theta \cos \theta) d\theta = -\int_{\frac{\pi}{4}}^0 4\sin^2 \theta d\theta \\ &= -4 \int_{\frac{\pi}{4}}^0 \frac{1-\cos 2\theta}{2} d\theta = -2 \int_{\frac{\pi}{4}}^0 (1-\cos 2\theta) d\theta \\ &= -2 \left[\theta - \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{4}}^0 = -[2\theta - \sin 2\theta]_{\frac{\pi}{4}}^0 \\ &= - \left[(0 - \sin 0) - \left(\frac{\pi}{2} - \sin \frac{\pi}{2} \right) \right] = - \left[(0 - \left(\frac{\pi}{2} - 1 \right)) \right] = \frac{\pi}{2} - 1 \end{aligned}$$

Illustration 6

Find $\int_a^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}.$

Solution :

Let $x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

Then $dx = (-2\alpha \cos \theta \sin \theta + 2\beta \sin \theta \cos \theta)d\theta = 2(\beta - \alpha) \sin \theta \cos \theta d\theta$

When $x = \alpha, \alpha = \alpha \cos^2 \theta + \beta \sin^2 \theta$

or, $(\alpha - \beta) \sin^2 \theta = 0$ or, $\sin^2 \theta = 0 \therefore \theta = 0$

when $x = \beta, \beta = \alpha \cos^2 \theta + \beta \sin^2 \theta$

or, $\beta \cos^2 \theta = \alpha \cos^2 \theta$ or, $(\beta - \alpha) \cos^2 \theta = 0$

or, $\cos^2 \theta = 0$ or, $\cos \theta = 0$ or, $\theta = \frac{\pi}{2}$

Now $I = \int_a^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

$$= \int_0^{\pi/2} \frac{2(\beta - \alpha) \sin \theta \cos \theta}{\sqrt{(\alpha \cos^2 \theta + \beta \sin^2 \theta - \alpha)(\beta - \alpha \cos^2 \theta - \beta \sin^2 \theta)}} d\theta$$

$$= \int_0^{\pi/2} \frac{2(\beta - \alpha) \sin \theta \cos \theta}{\sqrt{(\beta - \alpha) \sin^2 \theta (\beta - \alpha) \cos^2 \theta}} d\theta = 2 \int_0^{\pi/2} d\theta = 2 [\theta]_0^{\pi/2} = 2 \left[\frac{\pi}{2} - 0 \right] = \pi$$

Illustration 7

Find $\int_1^e \frac{e^x}{x} (1 + x \log x) dx.$

Solution :

$$\begin{aligned}\int \frac{e^x}{x} (1 + x \log x) dx &= \int e^x \left(\frac{1}{x} + \log x \right) dx \\ &= \int e^x [f'(x) + f(x)] dx, \text{ where } f(x) = \log x = e^x f(x) = e^x \log x\end{aligned}$$

$$\int_1^e \frac{e^x}{x} (1 + x \log x) dx = [e^x \log x]_1^e = e^e \log e - e \log 1 = e^e$$

Illustration 8

Evaluate $\int_{-2}^2 \frac{dx}{4+x^2}$ **directly as well as by the substitution** $x = 1/t$.

Examine as to why the answer do not tally ?

Solution :

$$\begin{aligned}I &= \int_{-2}^2 \frac{dx}{4+x^2} \\ &= \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_{-2}^2 = \frac{1}{2} [\tan^{-1}(1) - \tan^{-1}(-1)] = \frac{1}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}\end{aligned}$$

on the other hand; if $x = 1/t$ then,

$$\begin{aligned}I &= \int_{-2}^2 \frac{dx}{4+x^2} = - \int_{-1/2}^{1/2} \frac{dt}{t^2(4+1/t^2)} = - \int_{-1/2}^{1/2} \frac{dt}{4t^2+1} \\ &= \left[\frac{1}{2} \tan^{-1}(2t) \right]_{-1/2}^{1/2} = -\frac{1}{2} \tan^{-1}(1) - \left(-\frac{1}{2} \tan^{-1}(-1) \right) = -\frac{\pi}{8} - \frac{\pi}{8} = -\frac{\pi}{4} \\ \therefore I &= -\frac{\pi}{4} \text{ when } x = \frac{1}{t}\end{aligned}$$

In above two results $I = -\pi/4$, is wrong. Since the integrand $\frac{1}{4+x^2} > 0$ and therefore the definite integral of this function cannot be negative.

Since $x = 1/t$ is discontinuous at $t = 0$, the substitution is not valid ($I = \pi/4$).

Note : It is important the substitution must be continuous in the interval of integration.

PROPERTIES OF DEFINITE INTEGRALS

Property 1 :

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

i.e. integration is independent of change of variable.

Property 2 :

$$\int_a^b f(x)dx = - \int_a^b f(x)dx$$

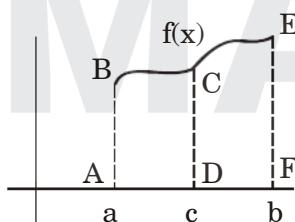
i.e. if the limits of a definite integral are interchangable then its value becomes negative of the earlier value.

Property 3 :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where $a < c < b$

Logic : We know that definite integral represents the area of the function between x-axis under the given limits.

Now $\int_a^b f(x)dx = \text{area of } f(x)$ i.e. area ABCEFI

= area ABCDA + area CEFDC

$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

You can prove all the above 3 properties by alzebraic method. We are leaving that part for you to do it yourself.

General form of Property-3

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \dots + \int_{c_{n-1}}^b f(x)dx$$

where $a < c_1 < c_2 < \dots < c_{n-1} < b$

Working Rule

This property is used when integrand is different in different intervals. This happens in the following cases.

1. function changes or is discontinuous at some points in $[a, b]$
2. Modulus function
3. Greatest integer function & fractional part.

In each of the 3 cases we find the point where the function is different & divide the interval accordingly using property-3.

Illustration 9

(a) Evaluate $\int_{-1}^1 f(x)dx$, where $f(x) = \begin{cases} 1-2x; & x \leq 0 \\ 1+2x; & x \geq 0 \end{cases}$

(b) Evaluate $\int_{-1}^1 f(x)dx$, where $f(x) = \begin{cases} 1-2x; & x \leq 0 \\ 1+2x; & x \geq 0 \end{cases}$

Solution :

- (a) The function is discontinuous at 0, at its value is changing. Hence we cannot integrate over $[-1, 1]$. So applying the rule.

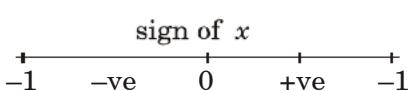
$$\int_{-1}^1 f(x)dx = \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx$$

$$= \int_{-1}^0 (1-2x)dx + \int_0^1 (1+2x)dx$$

$$= [x - x^2] \Big|_{-1}^0 + [x + x^2] \Big|_0^1$$

$$= [0 - (-1 - 1)] + [1 + 1 - 0] = 4$$

(b)



In case of modulus function, the value of function changes at the point where it becomes 0. Hence, breaking the interval

$$\text{Now } \int_{-1}^1 |x| dx = \int_{-1}^0 |x| dx + \int_0^1 |x| dx = \int_{-1}^0 -xdx + \int_0^1 xdx$$

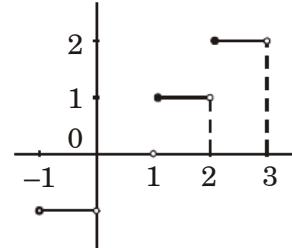
$[\because \text{when } -1 < x < 0, |x| = -x \text{ and when } 0 < x < 1, |x| = x]$

$$= -\left[\frac{x^2}{2} \right] \Big|_0^1 = -\left(0 - \frac{1}{2} \right) + \left(\frac{1}{2} - 0 \right) = 1$$

3. $\int_{-1}^3 [x] dx$

We know greatest integer function returns integral values only. So for every integral interval value will change.

$$\begin{aligned} &= \int_{-1}^0 [x] dx + \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx \\ &= -1 + 0 + 1 + 2 \\ &= 2 \end{aligned}$$



4. $\int_0^2 \{x\} dx$

For fractional part, let us draw the graph so for

$$\begin{aligned} 0 < x < 1 &\quad \{x\} = x \\ 1 < x < 2 &\quad \{x\} = x - 1 \end{aligned}$$

(this is the reason we did such graphs in functions chapter)

$$\begin{aligned} &= \int_0^1 \{x\} dx + \int_1^2 \{x\} dx = \int_0^1 x dx + \int_1^2 (x - 1) dx \\ &= \frac{x^2}{2} \Big|_0^1 + \frac{(x-1)^2}{2} \Big|_1^2 = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

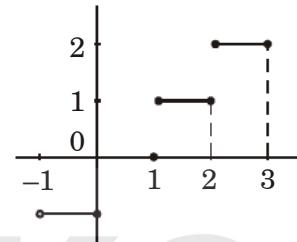
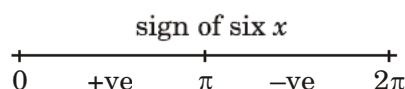


Illustration 10

(a) Find the value of $\int_0^{2\pi} |\sin x| dx$ (b) Evaluate $\int_0^2 |x^2 + 2x - 3| dx$

Solution :

(a) [When $\sin x = 0$, $x = n\pi$, where $n = 0, \pm 1, \pm 2, \dots$ out of which only p lies between lower and upper limits of integration].



$$\text{Now } I = \int_0^{2\pi} |\sin x| dx = \int_0^\pi |\sin x| dx + \int_\pi^{2\pi} |\sin x| dx$$

$$= \int_0^\pi \sin x dx - \int_\pi^{2\pi} \sin x dx = [-\cos x]_0^\pi - [-\cos x]_\pi^{2\pi}$$

$$= (\cos \pi - \cos 0) + (\cos 2\pi - \cos \pi)$$

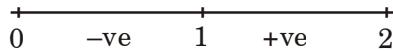
$$= -(-1 - 1) + [1 - (-1)] = 4$$

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$$(b) \quad x^2 + 2x - 3 = 0 \Rightarrow x = -3, 1$$

Sign scheme for $x^2 + 2x - 3$ in $[0, 2]$ is



$$\text{Now } I = \int_0^2 |x^2 + 2x - 3| dx$$

$$\begin{aligned} &= \int_0^1 |x^2 + 2x - 3| dx + \int_1^2 |x^2 + 2x - 3| dx = \int_0^1 -(x^2 + 2x - 3) dx + \int_1^2 (x^2 + 2x - 3) dx \\ &= -\left[\frac{x^3}{3} + x^2 - 3x \right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x \right]_1^2 \\ &= -\left[\left(\frac{1}{3} + 1 - 3 \right) - 0 \right] + \left[\left(\frac{8}{3} + 4 - 6 \right) - \left(\frac{1}{3} + 1 - 3 \right) \right] = \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4 \end{aligned}$$

Illustration 11

Find the value of

$$(a) \int_0^\pi |\cos x - \sin x| dx$$

$$(b) \int_0^4 \{\sqrt{x}\} dx$$

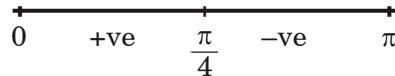
$$(c) \text{ Evaluate } \int_0^{3/2} |x \cos \pi x| dx$$

Solution :

$$(a) \quad \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 = \tan \frac{\pi}{4} \quad \therefore \quad x = n\pi + \frac{\pi}{4}$$

where $n = 0, \pm 1, \pm 2, \dots$ out of which only $\frac{\pi}{4}$ lies between lower and upper limits of definite integration.]

sign scheme for $\cos x - \sin x$



$$\text{Now } I = \int_0^\pi |\cos x - \sin x| dx$$

$$= \int_0^{\pi/4} |\cos x - \sin x| dx + \int_{\pi/4}^\pi |\cos x - \sin x| dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx - \int_{\pi/4}^\pi (\cos x - \sin x) dx = [\sin x + \cos x]_0^{\pi/4} - [\sin x + \cos x]_{\pi/4}^\pi$$

$$= \left[\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \right] - \left[\left(\sin \pi + \cos \pi \right) - \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) \right]$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right) \right] - \left[(0 - 1) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right] = \sqrt{2} - 1 - (-1 - \sqrt{2}) = \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}$$

$$(b) \int_0^4 \{ \sqrt{x} \} dx$$

Here also the value of fractional part will change at integral values

$$\begin{aligned}\sqrt{x} &= 1 && \text{at } x = 1 \\ \sqrt{x} &= 2 && \text{at } x = 4, \quad \text{which is the upper limit.}\end{aligned}$$

so value of

$$\begin{aligned}\{x\} &= \sqrt{x}, && 0 < x < 1 \\ &= \sqrt{x} - 1, && 1 < x < 4\end{aligned}$$

$$\int_0^4 \{ \sqrt{x} \} dx = \int_0^1 \sqrt{x} dx + \int_1^4 (\sqrt{x} - 1) dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^1 + \frac{2}{3} x^{3/2} \Big|_1^4 - x \Big|_1^4$$

$$\begin{aligned}&= \left(\frac{2}{3} \right) + \left(\frac{2}{3} \right) [(8-1)] - (4-1) \\ &= \frac{7}{3}\end{aligned}$$

$$(c) \quad x \cos \pi x = 0 \Rightarrow \begin{cases} x=0 \\ \cos n \pi x = 0 \text{ or } \pi x = (2n+1) \frac{\pi}{2}, n \in I \end{cases}$$

$$\Rightarrow \begin{cases} x=0 \\ x = \frac{1}{2}, \text{ between 0 and } \frac{3}{2} \end{cases}$$

sign scheme for $x \cos \pi x$ is

0	+ve	$\frac{1}{2}$	-ve	$\frac{3}{2}$

$$\text{Now } \int_0^{3/2} |x \cos \pi x| dx = \int_0^{1/2} |x \cos \pi x| dx + \int_{1/2}^{3/2} |x \cos \pi x| dx$$

$$= \int_0^{1/2} x \cos \pi x dx - \int_{1/2}^{3/2} x \cos \pi x dx$$

$$= \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_0^{1/2} - \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi} - \frac{1}{\pi^2} \right) - \left(\frac{3}{2\pi} - \frac{1}{2\pi} \right)$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{2}{\pi} = \frac{5}{2\pi} - \frac{1}{\pi^2}$$

Property 4 : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof. Put $a-x=t$ on R.H.S.

so lower limit becomes a

and upper limit becomes 0

& $-dx = dt$

$$\therefore \int_0^a f(a-x)dx = \int_a^0 f(t)(-dt) = - \int_a^0 f(t)dt$$

$$= \int_0^a f(t)dt \quad \text{[using Property-2]}$$

$$= \int_0^a f(x)dx \quad \text{[using Property-1]}$$

Usefulness

This property is useful to convert an indefinite integral to a more easily solvable integral. This property is specially very useful in trigonometric integrals. Let us see how.

Illustration 12

$$(a) \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$(b) \int_0^1 x(1-x)^{99} dx$$

Solution :

$$(a) I = \int_0^{\pi/2} \frac{\sqrt{\sin x} dx}{\sqrt{\sin x} + \sqrt{\cos x}} \quad \dots(i)$$

see now if you solve this without any use of definite properties, as a normal indefinite question it will become a very lengthy problem.

Let us see how property comes handy in this case.

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}} dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\sqrt{\cos(x)}}{\sqrt{\cos x + \sqrt{\sin x}}} \quad \dots(ii)$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_0^{\pi/2} \frac{\sqrt{\sin x} dx}{\sqrt{\cos x + \sqrt{\sin x}}} + \int_0^{\pi/2} \frac{\sqrt{\cos x} dx}{\sqrt{\cos x + \sqrt{\sin x}}} \\ &= \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x + \sqrt{\sin x}}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x + \sqrt{\cos x}}} \right) dx \\ &= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x + \sqrt{\cos x}}} dx = \int_0^{\pi/2} dx \\ &= x \Big|_0^{\pi/2} = \frac{\pi}{2} \end{aligned}$$

$$\therefore 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

So you can notice the usefulness of this property. When $\sin x$ & $\cos x$ are interchangeable by this property, use it to reduce the integral.

$$(b) \int_0^1 x(1-x)^{99} dx$$

Though we can solve this question by first using by parts and then applying the limits, but here we will use this property to prevent that long step.

$$I = \int_0^1 x(1-x)^{99} dx \quad \dots(i)$$

applying Property-4

$$\begin{aligned} I &= \int_0^1 (1-x)[1-(1-x)]^{99} \\ &= \int_0^1 (1-x)x^{99} dx \\ &= \int_0^1 (x^{99} - x^{100}) dx \end{aligned}$$

Now solve simply as integral of 2 functions (no need of using by parts)

$$= \frac{x^{100}}{100} \Big|_0^1 - \frac{x^{101}}{101} \Big|_0^1 = \frac{1}{100} - \frac{1}{101} = \frac{1}{10100}$$

Illustration 13

(a) $\int_0^{\pi/2} \log \tan x dx$

(b) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

(c) $\int_0^{\pi/4} \log(1 + \tan x) dx$

(d) $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

Solution :

(a) $I = \int_0^{\pi/2} \log \tan x dx$

...(i)

applying Property-4

$$I = \int_0^{\pi/2} \log \tan \left(\frac{\pi}{2} - x \right) dx$$

$$= \int_0^{\pi/2} \log \cot x dx \quad \text{(ii)}$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_0^{\pi/2} \log \tan x dx + \int_0^{\pi/2} \log \cot x dx \\ &= \int_0^{\pi/2} (\log(\tan x) + \log \cot x) dx \end{aligned}$$

Using the log property, $\log a + \log b = \log ab$

$$\begin{aligned} &= \int_0^{\pi/2} \log(\tan x \times \cot x) dx \\ &= \int_0^{\pi/2} \log 1 dx = \int_0^{\pi/2} 0 dx = 0 \quad [\text{as } \log 1 = 0] \end{aligned}$$

Tip. Why I thought of using this property ?

1. Using by parts , is a very long process

2. Most importantly, upper limit is $\frac{\pi}{2}$ and every trigonometric, function gives itopposite pair at $\frac{\pi}{2} - x$, hence purpose solved.

(b) $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \quad \text{...(1)}$

Similarly in this question, a bell should ring that upper limit is $\pi / 2$ and function comprises

of $\sin x$ & $\cos x$ which can be interchanged.

Hence applying Property-4.

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin(\pi/2 - x) - \cos(\pi/2 - x)}{1 + \sin(\pi/2 - x)\cos(\pi/2 - x)} dx \\ &= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \quad \dots(ii) \end{aligned}$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \sin x \cos x} \right) dx \\ &= \int_0^{\pi/2} 0 \cdot dx = 0 \\ \therefore I &= 0 \end{aligned}$$

(c) $I = \int_0^{\pi/4} \log(1 + \tan x) dx$

applying Property-4

$$\begin{aligned} &= \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx = \int_0^{\pi/4} \log\left(1 + \frac{\tan\pi/4 - \tan x}{1 + \tan\pi/4 \tan x}\right) dx \\ &= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx \quad \dots(ii) \\ &= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx \end{aligned}$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_0^{\pi/4} \left(\log(1 + \tan x) + \log\left(\frac{2}{1 + \tan x}\right) \right) dx \\ &= \int_0^{\pi/4} \log\left((1 + \tan x) \frac{2}{1 + \tan x}\right) dx \\ &= \int_0^{\pi/4} (\log 2) dx \end{aligned}$$

$$\Rightarrow 2I = \log 2 \int_0^{\pi/4} dx = \log 2 \left. x \right|_0^{\pi/4} = \frac{\pi}{4} \log 2$$

$$\Rightarrow I = \frac{\pi}{8} \log 2$$

Note. Another common mistake is the last step. Students forget that on L.H.S. it is $2I$, and they have to divide by 2 to get the answer. So keep this in mind.

$$(d) \quad \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$$

$$\text{Let } f(x) = \frac{x}{\sin x + \cos x} \quad (i)$$

$$\text{Then } f\left(\frac{\pi}{2} - x\right) = \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$$

$$\text{or } f\left(\frac{\pi}{2} - x\right) = \frac{\frac{\pi}{2} - x}{\cos x + \sin x}$$

$$(1) + (2) \Rightarrow f(x) + f\left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} \frac{1}{\cos x + \sin x}$$

$$= \frac{\pi}{2\sqrt{2} \cos\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{\pi}{2\sqrt{2}} \sec\left(x - \frac{\pi}{4}\right)$$

$$\text{Now } I = \frac{1}{2} \int_0^{\pi/2} \left[f(x) + f\left(\frac{\pi}{2} - x\right) \right] dx$$

$$= \frac{1}{2} \cdot \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| \right]_0^{\pi/2}$$

$$\begin{aligned}
 &= \frac{\pi}{2\sqrt{2}} \left[\log \left(\cosec \frac{\pi}{4} + \cot \frac{\pi}{4} \right) - \log \left| \sec \frac{\pi}{4} - \tan \frac{\pi}{4} \right| \right] \\
 &= \frac{\pi}{4\sqrt{2}} \left[\log(\sqrt{2}+1) - \log(\sqrt{2}-1) \right] \\
 &= \frac{\pi}{4\sqrt{2}} \log \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) = \frac{\pi}{4\sqrt{2}} \log (\sqrt{2}+1)^2 \\
 &= \frac{\pi}{2\sqrt{2}} \log (\sqrt{2}+1)
 \end{aligned}$$

Property-5 $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

Property-4 is a special case of Property-5 when $a=0$ & $b=a$

Proof. Let $a+b-t=x$

$$\Rightarrow -dt = dx$$

$$\begin{array}{lll}
 \text{& at} & x=a & t=b \\
 & x=b & t=a
 \end{array}$$

$$\therefore I = \int_b^a f(t)(-dt)$$

by using Property-2

$$I = \int_a^b f(t) dt = \int_a^b f(x) dx$$

Illustration 14

$$(a) \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} dx$$

$$(b) \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}} dx$$

Solution :

$$(a) I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} dx$$

If we apply Property-5 in this integral then the denominator remains the same, which gives encouragement to use the same.

\therefore applying Property-5.

$$\begin{aligned} I &= \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)} + \sqrt{3-x}} dx \\ &= \int_1^2 \frac{\sqrt{3-x} dx}{\sqrt{x} + \sqrt{3-x}} \end{aligned} \quad (\text{ii})$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_1^2 \left(\frac{\sqrt{3-x}}{\sqrt{x} + \sqrt{3-x}} + \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} \right) dx \\ &= \int_1^2 \left(\frac{\sqrt{3-x} + \sqrt{x}}{\sqrt{3-x} + \sqrt{x}} \right) dx \\ &= \int_1^2 dx = x \Big|_1^2 = 2 - 1 = 1 \end{aligned}$$

$$\Rightarrow I = \frac{1}{2}.$$

$$(b) \quad \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}} dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Earlier we have seen that in case of $\left(\frac{\pi}{2} - x\right)$ transformation in trigonometric integrals, the integral was reduced to a very simple one. See, here also it is happening.

$$\text{Now, } \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

& property 5 replaces x by $(a + b - x)$ i.e. $\left(\frac{\pi}{2} - x\right)$.

Hence our purpose is solved.

$$\therefore I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x + \sqrt{\sin x}}} dx \quad (\text{ii})$$

Now I hope you understand what we are trying to do & what we will do next.
adding (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} dx = \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{6}$$

$$\Rightarrow I = \frac{\pi}{12}$$

Property-6 $\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$

Proof. From Property-3, we get

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad (\text{i})$$

$$\text{Now } \int_{-a}^0 f(x) dx = \int_{-a}^0 f(-t)(-dt) \quad (\text{put } x = -t)$$

$$= \int_0^a f(-t) dt = \int_0^a f(-x) dx$$

$$= \begin{cases} \int_0^a f(x) dx, & \text{if } f(x) \text{ is an even function} \\ - \int_0^a f(x) dx, & \text{if } f(x) \text{ is an odd function} \end{cases} \quad (\text{ii})$$

Thus, when $f(x)$ is an even function from (i) & (ii)

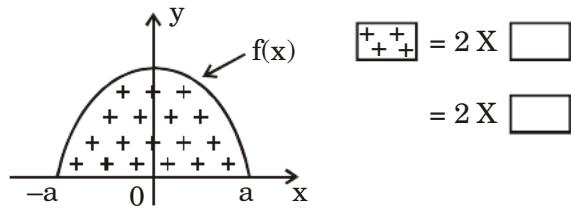
$$\Rightarrow \int_{-a}^0 f(x) dx = \int_0^a f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx$$

and when $f(x)$ is an odd function, from (i) & (ii)

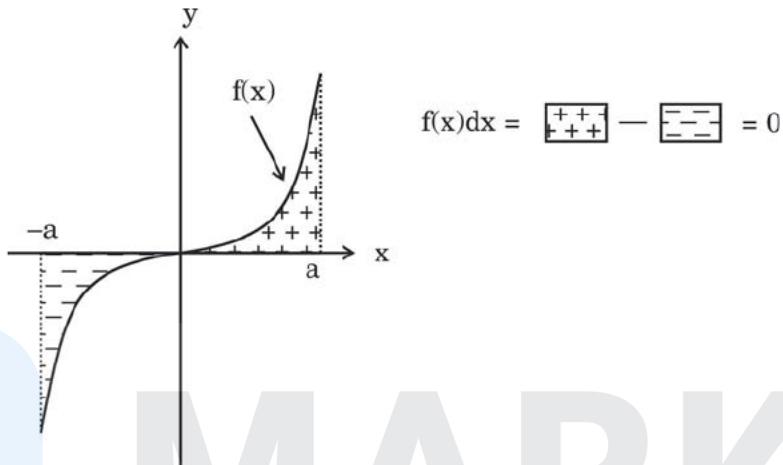
$$\Rightarrow \int_{-a}^0 f(x) dx = - \int_0^a f(x) dx + \int_0^a f(x) dx = 0$$

Geometrical Proof.

If $f(x)$ is EVEN



If $f(x)$ is ODD



This property should be used only when limits are equal and opposite and the function which is to be integrated is either odd or even.

Illustration 15

(a) Find $\int_{-1}^1 x^3 e^{x^4} dx$

(b) Find $\int_{-1}^1 x|x| dx$

(c) Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx$

Solution :

(a) Let $f(x) = x^3 e^{x^4}$

Then $f(-x) = (-x)^3 e^{(-x)^4} = -x^3 e^{x^4} = -f(x)$

Hence $f(x)$ is an odd function.

$\therefore \int_{-1}^1 f(x) dx = 0$ or $\int_{-1}^1 x^3 e^{x^4} dx = 0$

(b) Let $f(x) = x|x|$

Then $f(-x) = -x|-x| = -x|x| = -f(x)$ [$\because |x| = |-x|$]

Hence $f(x)$ is an odd function.

$$\int_{-1}^1 f(x)dx = 0 \quad \text{or} \quad \int_{-1}^1 \frac{|x|}{x} dx = 0$$

(c) Let $f(x) = x^3 \sin^4 x$

$$\text{Then } f(-x) = (-x)^3 \sin^4(-x) = -x^3(-\sin x)^4$$

$$= -x^3 \sin^4 x = -f(x).$$

Hence $f(x)$ is an odd function

$$\therefore \int_{-\pi/4}^{\pi/4} f(x)dx = 0$$

Illustration 16

(a) Show that $\int_{-a}^a f(x^2)dx = 2 \int_0^a f(x^2)dx$

(b) Evaluate $\int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$

Solution :

(a) $f[(-x)^2] = f(x^2)$. Hence $f(x^2)$ is an even function.

$$\therefore \int_{-a}^a f(x^2)dx = 2 \int_0^a f(x^2)dx$$

(b) $I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx = \int_{-a}^a \frac{a-x}{\sqrt{a^2-x^2}} dx$

$$= a \int_{-a}^a \frac{dx}{\sqrt{a^2-x^2}} - \int_{-a}^a \frac{a-x}{\sqrt{a^2-x^2}} dx$$

$$= a \cdot 2 \int_0^a \frac{dx}{\sqrt{a^2-x^2}} - 0 \quad [\because \frac{x}{\sqrt{a^2-x^2}} \text{ is an odd function}]$$

$$= 2a \left[\sin^{-1} \frac{x}{a} \right]_0^a = 2a [\sin^{-1}(1) - \sin^{-1} 0]$$

$$= 2a \left(\frac{\pi}{2} - 0 \right) = \pi a$$

Property-7

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx; & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Proof. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

Put $x = 2a - t$ in 2nd integral ($dx = -dt$) when $x = a$ then $t = a$

when $x = 2a$ then $t = 0$

$$\therefore \int_a^{2a} f(x)dx = - \int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(2a-x)dx$$

If $f(2a-x) = f(x)$

then $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2 \int_0^a f(x)dx$

and if $f(2a-x) = -f(x)$

then $\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$

Illustration 17**Evaluate**

(a) $\int_0^{2\pi} \cos^5 x dx$

(b) $\int_0^\pi \frac{x dx}{1 + \cos^2 x}$

Solution :

(a) We will first check for the property-7 conditions. For that let

$$f(x) = \cos^5 x$$

then $f(2\pi - x) = \cos^5(2\pi - x)$

$$= \cos^5 x = f(x)$$

$$\therefore \int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx \quad \text{if } f(2a-x) = f(x)$$

applying the rule.

$$I = 2 \int_0^\pi \cos^5 x dx$$

... (i)

Now applying property-4

$$\begin{aligned}
 I &= 2 \int_0^\pi \cos^5(\pi - x) dx \\
 &= 2 \int_0^\pi -\cos^5 x dx \\
 &= -2 \int_0^\pi \cos^5 x dx
 \end{aligned} \tag{ii}$$

adding (i) and (ii)

$$\begin{aligned}
 2I &= 0 \\
 \Rightarrow I &= 0
 \end{aligned}$$

$$(b) \quad I = \int_0^\pi \frac{x dx}{1 + \cos^2 x} \tag{i}$$

This is an interesting problem, because here Property-7 is not visible at first. So the tip that we can derive from this question is that approach the question as given rather than going by a fixed mind.

Here we can see that property-4 is applicable so without thinking anything else I will use it first

$$\therefore I = \int_0^\pi \frac{(\pi - x) dx}{1 + \cos^2(\pi - x)} = \int_0^\pi \frac{(\pi - x) dx}{1 + \cos^2 x} \tag{ii}$$

adding (i) and (ii)

$$\begin{aligned}
 2I &= \int_0^\pi \frac{x dx}{1 + \cos^2 x} + \int_0^\pi \frac{\pi - x}{1 + \cos^2 x} dx \\
 &= \int_0^\pi \frac{\pi}{1 + \cos^2 x} dx
 \end{aligned}$$

Now if I apply property-4 back then I will have no advantage as I will get the same integral. Hence no use. But if I apply Property-7.

$$\begin{aligned}
 I &= \frac{1}{2} \left[2 \int_0^{\pi/2} \frac{\pi dx}{1 + \cos^2 x} \right] \quad \text{as } f(2a - x) = f(x) \\
 \therefore I &= \int_0^{\pi/2} \frac{\pi dx}{1 + \cos^2 x} \\
 \Rightarrow I &= \frac{1}{2} \int_0^\pi \frac{\pi dx}{1 + \cos^2 x}
 \end{aligned}$$

MISTAKE : Common mistake at this step, is to take it as a normal substitution integral.

$$\text{i.e. } I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 x}{\sec^2 x + 1} dx \quad (\text{dividing by } \cos^2 x)$$

Now the common substitution

$$x = \tan x$$

$$dx = \sec^2 x dx$$

But wait this is a wrong step, as per the rule of substitution the function which is substituted should be continuous in the interval.

But there for $x = \tan x$, $\tan x$ is not continuous over interval $[0, \pi]$. It is discontinuous at $x = \frac{\pi}{2}$.

Therefore, it is not possible to substitute $\tan x$ in the interval $[0, \pi]$.

So next thought should be to break the interval so that we can apply the transformation.

Illustration 18

$$\text{Prove } \int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx = -\frac{\pi}{2} \log 2$$

$$\text{Solution : Let } I = \int_0^{\pi/2} \log \sin x dx \quad \dots(i)$$

clearly property-4 is applicable here,

$$\therefore I = \int_0^{\pi/2} \log \left(\frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \log \cos x dx \quad \dots(ii)$$

adding (i) and (ii)

$$\begin{aligned} 2I &= \int_0^{\pi/2} (\log \sin x + \log \cos x) dx \\ &= \int_0^{\pi/2} (\log \sin x \cos x) dx = \int_0^{\pi/2} \log \left(\frac{\sin 2x}{2} \right) dx \\ &= \int_0^{\pi/2} (\log \sin 2x - \log 2) dx \\ &= \int_0^{\pi/2} \log \sin 2x dx - \int_0^{\pi/2} \log 2 dx \\ &= \int_0^{\pi/2} \log \sin 2x dx - \log 2 [x]_0^{\pi/2} \\ \Rightarrow 2I &= \int_0^{\pi/2} \log \sin 2x dx - (\log 2) \left(\frac{\pi}{2} \right) \end{aligned} \quad \dots(iii)$$

Now let us solve the integral part separately.

$$I' = \int_0^{\pi/2} (\log \sin 2x) dx$$

See if I apply property-4 again here I will again get I' , which becomes futile as I am struck here. So what should I do ?

There are only 2 options

1. I' gives a definite value, which does not seem to be the case here.
2. express I' in terms of I to solve the question.

Now, let

$$\begin{aligned} 2x &= t \\ \Rightarrow 2dx &= dt \\ \therefore I' &= \int_0^{\pi/2} \frac{1}{2} \log(\sin t) dt \end{aligned}$$
... (iv)

Now (iv) is almost similar to I with the only difference being in the upper limit.

\therefore applying property -

$$I' = \left(\frac{1}{2} \right) 2 \int_0^{\pi/2} \log \sin t dt = \int_0^{\pi/2} \log(\sin t) dt = I$$

Putting this value back in (iii)

$$\begin{aligned} 2I &= I - \frac{\pi}{2} \log 2 \\ \Rightarrow I &= \int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \cos x dx = -\frac{\pi}{2} \log 2 \end{aligned}$$

Now solving it as we would have done in indefinite integral case.

$$\begin{aligned} I &= \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{1 + \sec^2 x} \\ &= \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{2 + \tan^2 x} \end{aligned}$$

Now obviously we will substitute $\tan x = t$

$$\text{as } x = 0 \quad \tan x = t \rightarrow 0$$

$$x = \frac{\pi}{2} \quad \tan x = t \rightarrow \infty$$

$$\begin{aligned} I &= \int_0^\infty \frac{\pi dt}{2+t^2} = \frac{\pi}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \Big|_0^\infty \\ &= \frac{\pi}{\sqrt{2}} \left(\tan^{-1} \infty - \tan^{-1} 0 \right) = \frac{\pi}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2\sqrt{2}} \end{aligned}$$

Property 8 $\int_0^a f(x) dx = \int_0^{a/2} f(x) dx + \int_0^{a/2} f(a-x) dx$

Geometrical Proof :

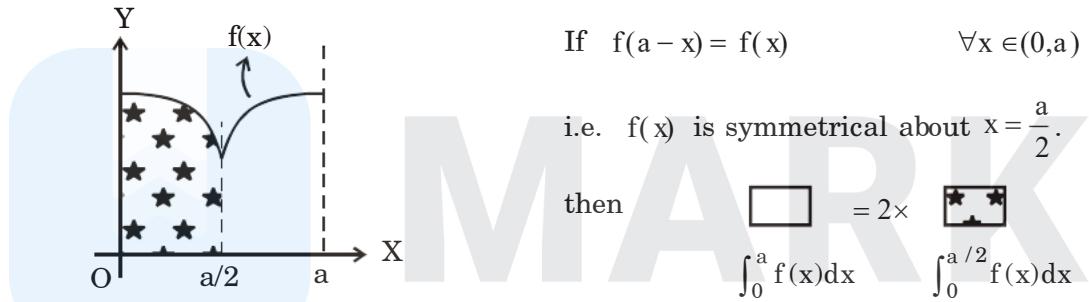
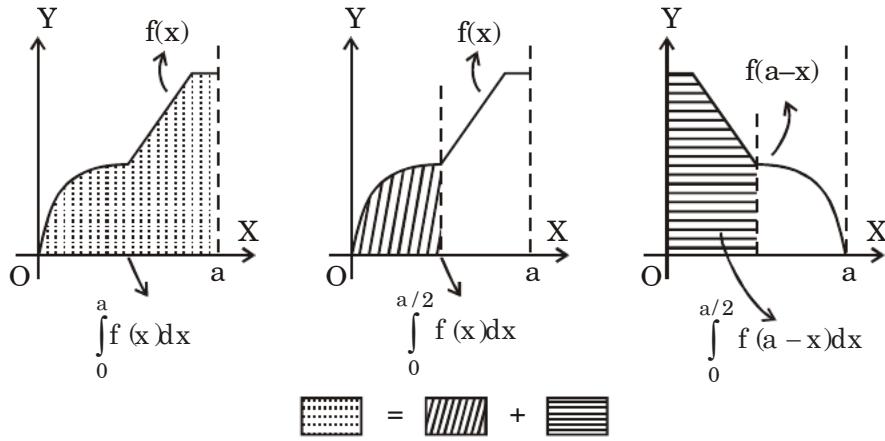


Illustration 19

Show that $\int_0^{\pi/2} f(\sin 2x) \sin x dx = \int_0^{\pi/2} f(\sin 2x) \cos x dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x dx$

Solution :

$$\text{Let } I = \int_0^{\pi/2} f(\sin 2x) \sin x dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi/2} f\left[\sin 2\left(\frac{\pi}{2}-x\right)\right] \sin\left(\frac{\pi}{2}-x\right) dx \quad (\text{using property-4})$$

$$\Rightarrow I = \int_0^{\pi/2} f[\sin(\pi-2x)] \cos x dx$$

$$\Rightarrow I = \int_0^{\pi/2} f(\sin 2x) \cos x dx \quad \dots(ii)$$

Hence the first part is proved.

$$I = \int_0^{\pi/2} f(\sin 2x) \sin x dx$$

$$\begin{aligned}
 &= \int_0^{\pi/4} f(\sin 2x) \sin x dx + \int_0^{\pi/4} f\left[\sin 2\left(\frac{\pi}{2} - x\right)\right] \sin\left(\frac{\pi}{2} - x\right) dx \quad (\text{using Property-5}) \\
 &= \int_0^{\pi/4} f(\sin 2x) \sin x dx + \int_0^{\pi/4} f(\sin 2x) \cos x dx \\
 &= \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x) dx \\
 &= \int_0^{\pi/4} f\left[\sin 2\left(\frac{\pi}{4} - x\right)\right] \left[\sin\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - x\right)x \right] dx \quad (\text{using property-4}) \\
 &= \int_0^{\pi/4} f(\cos 2x) \left[\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x \right] dx \\
 &= \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x dx
 \end{aligned}$$

Hence the second part is also proved.

Some more algebraic properties :

Property-9 $\int_a^b f(x) dx = (b-a) \int_0^1 f((b-a)x+a) dx$

for the proof of this property we will approach from R.H.S.

Put $z = (b-a)x + a$ in R.H.S.

$$\Rightarrow dz = (b-a)dx$$

$$\begin{aligned}
 &\text{& when } x=0, & z=a \\
 &x=1, & z=b
 \end{aligned}$$

\therefore new integral becomes

$$\int_a^b (b-a) f(z) \frac{dz}{(b-a)} = \int_a^b f(x) dx = \text{L.H.S.}$$

some other properties.

$$1. \quad \int_a^b f(x) dx = \int_{a+c}^{b+c} f(x) dx$$

$$2. \quad \int_a^b f(x) dx = \int_{a-c}^{b-c} f(x+c) dx$$

$$\text{or } \int_a^b f(x) dx = \int_{a+c}^{b-c} f(x-c) dx$$

$$3. \quad \int_a^b f(x) dx = \frac{1}{c} \int_a^b f\left(\frac{x}{c}\right) dx$$

Illustration 20

Evaluate $\int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$

Solution :

Note : Here we know $\int e^{x^2} dx$ cannot be evaluated by indefinite integral

$$\text{Thus, } I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx$$

$$= (-5 + 4) \int_0^1 e^{((x-4)+5)^2} dx$$

$$\therefore I_1 = - \int_0^1 e^{(x-1)^2} dx \quad \dots(i)$$

again, let

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$$

$$= \left(\frac{2}{3} - \frac{1}{3} \right) \int_0^1 e^{9 \left[\left(\frac{2}{3} - \frac{1}{2} \right)x + \frac{1}{3} - \frac{2}{3} \right]^2} dx$$

$$= \frac{1}{3} \int_0^1 e^{(x-1)^2} dx$$

$$= \frac{1}{3} (-I_1) \quad \dots(ii)$$

$$\text{where, } I = I_1 + 3I_2$$

$$= I_1 + 3 \left(-\frac{I_1}{3} \right)$$

$$= I_1 - I_1$$

$$I = 0$$

$$\therefore \int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx = 0$$

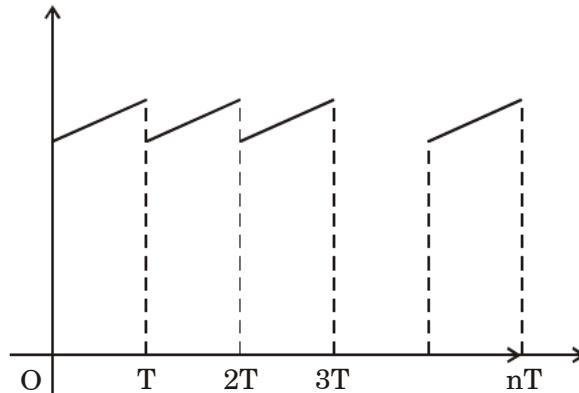
PROPERTIES RELATED TO PERIODICITY

Property-10 : If $f(x)$ is a periodic function with period T then $\int_0^{nT} f(x) dx = n \int_0^T f(x) dx$

The proof of this property is really easy one.

Geometrical Proof.

If $f(x)$ is periodic then it will repeat (the curve also) after an interval of T .



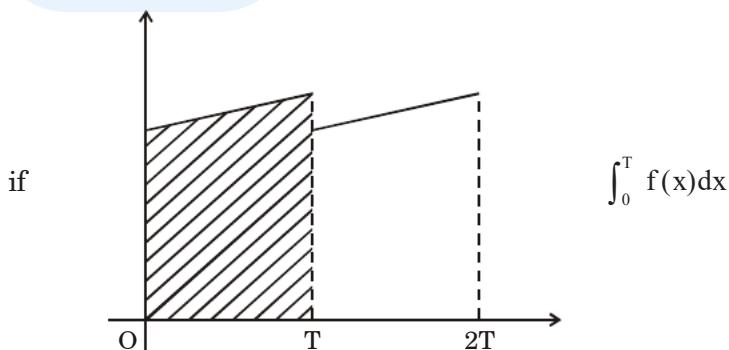
Since the area under the curve for an interval of T is same everytime.

$$\text{Total area} = n \times (\text{area under one interval})$$

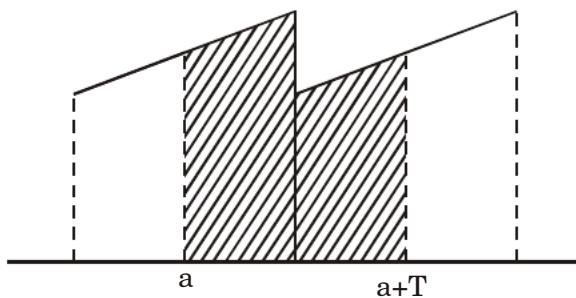
$$\therefore \int_0^{nT} f(x) dx = n \int_0^T f(x) dx$$

Property-11 $\boxed{\int_a^{a+T} f(x) dx = \int_0^T f(x) dx}$

Let us see the proof using geometry



now



so observation only we can say

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$$

Property-12 : Generalization of the above property is

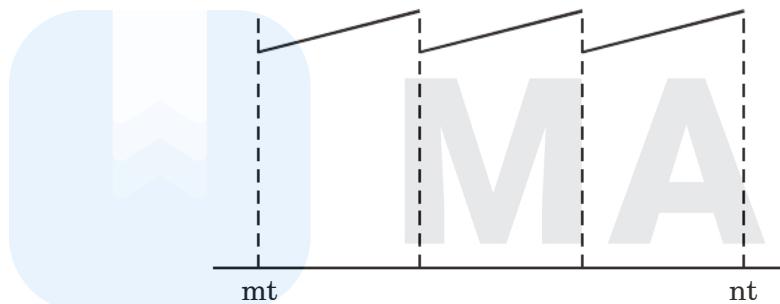
$$\boxed{\int_a^{a+nT} f(x)dx = n \int_0^T f(x)dx}$$

$$\int_a^{a+nT} f(x)dx = \int_0^{nT} f(x)dx = n \int_0^T f(x)dx$$

We can also use geometrical proof here.

Property-13 : $\boxed{\int_{mT}^{nT} f(x)dx = (n-m) \int_0^T f(x)dx, \quad n, m \in I}$

Geometrical Proof.



We can write,

$$\int_{mT}^{nT} f(x)dx = \int_0^{nT} f(x)dx - \int_0^{mT} f(x)dx$$

i.e. area of curve between = area of curve upto nT - area of curve upto mT

$$\begin{aligned} \therefore \int_{mT}^{nT} f(x)dx &= n \int_0^T f(x)dx - m \int_0^T f(x)dx \\ &= (n-m) \int_0^T f(x)dx \end{aligned}$$

Some other properties deduced from earlier properties

$$1. \quad \int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx$$

$$2. \quad \int_{a+mT}^{a+nT} f(x)dx = \int_{mT}^{nT} f(x)dx$$

$$3. \quad \int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$$

Illustration 21

(a) Prove that $\int_0^{10} (x - [x]) dx = 5$. (b) $\int_0^{100} e^{x-[x]} dx = 100(e-1)$ (c) $\int_0^{400\pi} \sqrt{1-\cos 2x} dx = 800\sqrt{2}$

Solution :

(a) Since $x - [x]$ is a periodic function with period one unit. Therefore

$$\int_0^{10} (x - [x]) dx = 10 \int_0^1 (x - [x]) dx = 10 \left[\int_0^1 x dx - \int_0^1 [x] dx \right]$$

$$= 10 \left[\left[\frac{x^2}{2} \right]_0^1 - 0 \right] = \frac{10}{2} = 5$$

(b) Since $x - [x]$ is a periodic function with period one unit, therefore so is $e^{x-[x]}$, and hence

$$\int_0^{100} e^x - [x] dx = 100 \int_0^1 e^{x-[x]} dx = 100 \int_0^1 e^{x-0} dx$$

$$= 100 \int_0^1 e^x dx = 100(e-1)$$

$$(c) \int_0^{400\pi} \sqrt{1-\cos 2x} dx = \int_0^{400\pi} \sqrt{2} |\sin x| dx$$

$$= \sqrt{2} \times 400 \int_0^\pi |\sin x| dx \quad [\because |\sin x| \text{ is periodic with period } \pi]$$

$$= 400\sqrt{2} \int_0^\pi \sin x dx = 400\sqrt{2} [-\cos x]_0^\pi = 800\sqrt{2}$$

Illustration 22

(a) Evaluate $\int_0^{4\pi} |\cos x| dx$ (b) Evaluate $\int_0^{32\pi/3} \sqrt{1+\cos 2x} dx$

Solution :

(a) Note that $|\cos x|$ is a periodic with period π .

Hence the given integral,

$$I = 4 \int_0^\pi |\cos x| dx$$

$$= 4 \left\{ \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^\pi \cos x dx \right\}$$

$$= 4 \left\{ (\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^\pi \right\} = 4(1+1) = 8.$$

$$\begin{aligned}
 (b) \quad & \int_0^{32\pi/3} \sqrt{1+\cos 2x} dx = \sqrt{2} \int_0^{10\pi} |\cos x| dx + \sqrt{2} \int_{10\pi}^{32\pi/3} |\cos x| dx \\
 &= 10\sqrt{2} \int_0^\pi |\cos x| dx + \sqrt{2} \int_0^{2\pi/3} |\cos x| dx \\
 &= 10\sqrt{2} \left[\int_0^{\pi/2} \cos x dx - \int_{\pi/2}^\pi \cos x dx \right] + \sqrt{2} \left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{2\pi/3} -\cos x dx \right] \\
 &= 10\sqrt{2}[1+1] + \sqrt{2} \left[1 - \frac{\sqrt{3}}{2} + 1 \right] \\
 &= 20\sqrt{2} + \sqrt{2} \left(2 - \frac{\sqrt{3}}{2} \right) = 22\sqrt{2} - \sqrt{\frac{3}{2}}
 \end{aligned}$$

Illustration 23

Show $\int_0^{n\pi+V} |\sin x| dx = (2n+1) - \cos V$, where n is positive integer. and $0 \leq V < \pi$. [IIT-1994]

Solution :

$$\begin{aligned}
 \int_0^{n\pi+V} |\sin x| dx &= \int_0^V |\sin x| dx + \int_V^{n\pi+V} |\sin x| dx \\
 &= \int_0^V \sin x dx + n \int_0^\pi |\sin x| dx \quad (\text{Using Property-IX}) \\
 &= (-\cos x)_0^V + n \int_0^\pi \sin x dx
 \end{aligned}$$

$$= (-\cos V + 1) + n(-\cos x)_0^\pi$$

$$= -(\cos V) + 1 + n(1+1)$$

$$= (2n+1) - \cos V$$

$$\Rightarrow \int_0^{n\pi+V} |\sin x| dx = (2n+1) - \cos V$$

where n is positive integer and $0 \leq V < \pi$.

PROPERTIES INCLUDING INEQUALITIES :

1. If $f(x) \leq 0$ on an interval $[a,b]$, then

$$\int_a^b f(x) dx \leq 0$$

$$\text{or if } f(x) \geq 0, \text{ then } \int_a^b f(x) dx \geq 0$$

The reason is very obvious because if $f(x) \geq 0$ the area will be above x-axis i.e. positive & for $f(x) < 0$ it will be negative.

2. Property-14 :

If $f(x) \leq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

The proof is similar to the reasoning given in the above case.

3. Property-15 : If m and M are the smallest & largest values of function $f(x)$ defined on an interval $[a,b]$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Proof. It is given that

$$m \leq f(x) \leq M$$

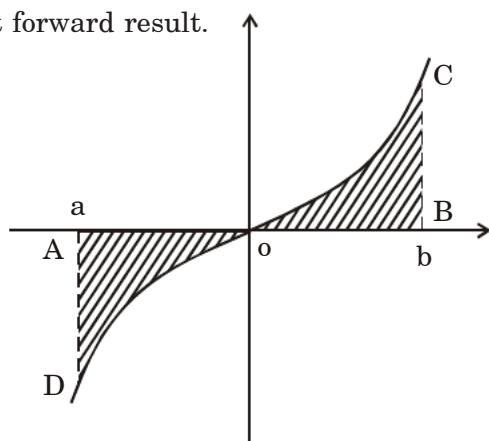
integrating both sides

$$\begin{aligned} \int_a^b m dx &\leq \int_a^b f(x)dx \leq \int_a^b M dx \\ \Rightarrow m(b-a) &\leq \int_a^b f(x)dx \leq M(b-a) \end{aligned}$$

Property-16 : If $f(x)$ is defined over $[a,b]$ then

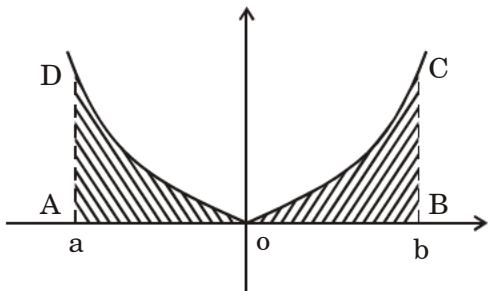
$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

This is very straight forward result.



$$\text{So } \left| \int_a^b f(x)dx \right| = |\text{area OBC} - \text{area OAD}| \quad \dots(i)$$

whereas



$$\therefore \int_a^b |f(x)| dx = \text{area OBC} + \text{area OAD} \quad \dots \text{(ii)}$$

compare (i) and (ii) to get to the result.

Property-17 : If $f^2(x)$ & $g^2(x)$ are integrable over $[a, b]$ then

$$\left| \int_a^b f(x)g(x)dx \right| \leq \left(\int_a^b f^2(x)dx \right) \left(\int_a^b g^2(x)dx \right)^{1/2}$$

Illustration 24

(a) Show that $\left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| < \frac{1}{10^7}$

(b) Prove that $\frac{\pi}{6} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2}-x} \leq \frac{\pi}{4\sqrt{2}}$

Solution :

$$(a) \left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| \leq \int_{10}^{19} \left| \frac{\sin x}{1+x^8} \right| dx \quad [\because \left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx]$$

$$= \int_{10}^{19} \frac{|\sin x|}{1+x^8} dx$$

$$\leq \int_{10}^{19} \frac{1}{1+x^8} dx \leq \int_{10}^{19} \frac{1}{x^8} dx \quad [\because |\sin x| \leq 1]$$

$$= \left[-\frac{1}{7x^7} \right]_{10}^{19} = \frac{1}{7.(10)^7} - \frac{1}{7.(19)^7} < \frac{1}{7.(10)^7} < \frac{1}{10^7}$$

(b) $0 \leq x \leq 1$

$$\therefore 4-x^2 \geq 4-x^2-x^3 \geq 4-x^2-x^2 \quad [\because x^2 > x^3]$$

$$\Rightarrow 4-x^2 \geq 4-x^2-x^3 \geq 4-2x^2 > 1$$

$$\Rightarrow 4-x^2 \geq 4-x^2-x^3 \geq \sqrt{4-2x^2}$$

$$\Rightarrow \sqrt{4-x^2} \geq \sqrt{4-x^2-x^3} > \sqrt{4-2x^2}$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}} \leq \frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{4-2x^2}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{2-x^2}}$$

$$\Rightarrow \left[\sin^{-1} \frac{x}{2} \right]_0^1 \leq I \leq \frac{1}{\sqrt{2}} \left[\sin^{-1} \frac{x}{\sqrt{2}} \right]_0^1$$

$$\Rightarrow \frac{\pi}{6} \leq I \leq \frac{\pi}{4\sqrt{2}}$$

Illustration 25

(a) Prove that $4 \leq \int_1^3 \sqrt{3+x^3} dx \leq 2\sqrt{30}$ (b) Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)} dx \leq \sqrt{\frac{15}{8}}$.

Solution :

(a) Let $y = \sqrt{3+x^3}$, then $\frac{dy}{dx} = \frac{-3x^2}{2\sqrt{3+x^3}} > 0$

∴ y is an increasing function

$$\therefore 1 \leq x \leq 3$$

$$\Rightarrow \sqrt{3+1^3} \leq \sqrt{3+x^3} \leq \sqrt{3+3^3}$$

$$\Rightarrow 2 \leq \sqrt{3+x^3} \leq \sqrt{30}$$

$$\Rightarrow \int_1^3 2dx \leq \int_1^3 \sqrt{3+x^3} dx \leq \sqrt{30} \int_1^3 dx$$

$$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^3} dx \leq 2\sqrt{30}$$

(b) Let $f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{1+x^3}$

If $f^2(x)$ and $g^2(x)$ and $f(x)g(x)$ are integrable functions on $[a,b]$, then

$$\left| \int_a^b f(x)g(x) dx \right| \leq \sqrt{\left(\int_a^b f^2(x) dx \right) \left(\int_a^b g^2(x) dx \right)}$$

$$\therefore \int_0^1 \sqrt{(1+x)(1+x^3)} dx \leq \sqrt{\left(\int_0^1 (1+x) dx \int_0^1 (1+x^3) dx \right)}$$

$$= \sqrt{\left[x + \frac{x^2}{2} \right]_0^1 \left[x + \frac{x^4}{4} \right]_0^1}$$

$$= \sqrt{\frac{3}{2} \cdot \frac{5}{4}} = \sqrt{\frac{15}{8}}$$

Thus, $\int_0^1 \sqrt{(1+x)(1+x^3)} dx \leq \sqrt{\frac{15}{8}}$.

Leibnitz's Rule for differentiation

Case-I. If the limits are function of the variable whose derivative is taken.

i.e. $\frac{d}{dx} \left[\underbrace{\int_{g(x)}^{h(x)} f(t) dt}_{\text{independent of } x} \right] = f[h(x)] \times h'(x) - f[g(x)] \times g'(x)$

a very common case is

$$\frac{d}{dx} \int_0^x f(t) dt = f(x)$$

Case-2. If the integrand is the function of variable whose derivative is taken, i.e.

$$\frac{d}{dx} \left[\int_a^b f(x,t) dt \right] = \int_a^b \frac{d}{dx} f(x,t) dt$$

taking t as a constant while differentiating.

Case-3. General Case :

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t,x) dt = \int_{g(x)}^{h(x)} \frac{d}{dx} f(t,x) dt + f[h(x)]h'(x) - f[g(x)]g'(x)$$

You can see that its a combination of both the results.

TIP :

Learn the formulas by variable of differentiation, integrand & limit of the integral not by cases.

Property-18. Let a function $f(x,a)$ be continuous for $a \leq x \leq b$ and $c \leq \alpha \leq d$. Then for any $\alpha \in [c,d]$,

if $I(\alpha) = \int_a^b f(x,\alpha) dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial(f(x,\alpha))}{\partial \alpha} dx$$

Illustration 26

Differentiate the following w.r.t. x

$$(a) \int_0^{x^2} (\cos t^2) dt$$

$$(b) \int_{1/x}^{\sqrt{x}} \sin t^2 dt$$

Solution :

$$(a) \text{ We have to find } I = \frac{d}{dx} \left(\int_0^{x^2} \cos t^2 dt \right)$$

This is an example of case-1 where only limits are a function of x.

$$\therefore I = (\cos(x^2)^2) \times \frac{d}{dx}(x^2) - \cos(0).0$$

$$\Rightarrow I = 2x \cos x^4$$

$$(b) I = \frac{d}{dx} \int_{1/x}^{\sqrt{x}} \sin t^2 dt$$

This is again an example of case-1

$$\begin{aligned} I &= \left(\sin(\sqrt{x})^2 \right) \times \frac{d}{dx}(\sqrt{x}) - \left(\sin\left(\frac{1}{x}\right)^2 \right) \times \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \sin x \times \frac{1}{2\sqrt{x}} - \sin\left(\frac{1}{x^2}\right) \left(-\frac{1}{x^2}\right) \\ &= \frac{1}{2\sqrt{x}} \sin x + \frac{1}{x^2} \sin\left(\frac{1}{x^2}\right) \end{aligned}$$

Illustration 27

Find the points of maxima / minima of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$.

Solution. We will apply the normal rules of maxima/minima & for maxima/minima we differentiate

$$\therefore \text{ if } f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$$

$$\text{then } f'(x) = \frac{d}{dx} \int_0^{x^2} \left(\frac{t^2 - 5t + 4}{2 + e^t} \right) dt$$

again case-1 example, only limit is a function of x.

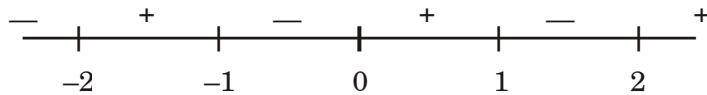
$$\therefore f'(x) = \frac{(x^2)^2 - 5(x^2) + 4}{2 + e^{x^2}} \cdot \frac{d}{dx}(x^2) - 0$$

$$= 2x \left[\frac{(x^2)^2 - 5(x^2) + 4}{2 + e^{x^2}} \right]$$

$$= \frac{2x(x^2 - 4)(x^2 - 1)}{2 + e^{x^2}}$$

$$= \frac{2x(x-2)(x+2)(x-1)(x+1)}{2 + e^{x^2}}$$

set equality it to zero.



Hence the points of maxima (i.e. where sign = -1, 1 changes from +ve to -ve) & points of minima (where sign changes from -ve to +ve) = -2, 0, 2

Illustration 28

(a) If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, find $\frac{dy}{dx}$ at $x = \pi$.

(b) Evaluate $\int_0^{\pi/2} \log(1 + \sin \alpha \sin^2 x) \operatorname{cosec}^2 x dx$.

Solution :

(a) $y = \int_{\pi^2/16}^{x^2} (\cos x) \cdot \frac{\cos \sqrt{\theta}}{1 + \sin^2 \theta} d\theta$

here $\cos x$ is a constant in integration, so it can be moved out of integral & this is the trick here.

$$y = \cos x \cdot \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$$

now differentiate using product rule.

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + \cos x \frac{d}{dx} \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$$

the derivative of integral is an example of our case-1 i.e. integrand is not the function of x , only limits are

$$\frac{dy}{dx} = -\sin x \cdot \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta + (2x) \times (\cos x) \frac{\cos \sqrt{x^2}}{1 + \sin^2 \sqrt{x^2}} + 0$$

$$= \frac{2x \cos^2 x}{1 + \sin^2 x} - \sin x \int_{\pi/16}^{x^2} \frac{\cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta \quad (i)$$

now solving the integral by substituting

$$\theta = t^2$$

$$\Rightarrow d\theta = 2t dt$$

$$\text{& limits at } \theta = x^2 \quad t = x$$

$$\theta = \frac{\pi^2}{16} \quad t = \frac{\pi}{4}$$

$$\therefore \text{ Integral } I = \int_{\pi/4}^x \frac{\cos t}{1 + \sin t} 2t dt$$

But stop, we are solving in futile. We need to find the value of $\frac{dy}{dx}$ at $x = \pi$ & ahead of this integral is $\sin x$ which is 0 at $x = \pi$, so no need of solving the integral

$$\therefore \left. \frac{dy}{dx} \right|_{x=\pi} = \frac{2\pi \cos^2(\pi)}{1 + \sin^2(\pi)} = 2\pi \cos^2(\pi) = 2\pi$$

$$(b) \quad I = \int_0^{\pi/2} \log(1 + \sin \alpha \sin^2 x) \cosec^2 x dx$$

now this is question based on property-18.

Here I is a function of α , so according to property

$$\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{\partial}{\partial \alpha} (\log(1 + \sin \alpha \sin^2 x)) \cosec^2 x dx$$

∂ means differentiating the function containing α only & taking all other variables as constant while differentiating.

$$\begin{aligned} \therefore \frac{\partial I}{\partial \alpha} &= \int_0^{\pi/2} \frac{1}{(1 + \sin \alpha \sin^2 x)} \times \sin^2 x \cos \alpha \cosec^2 x dx \\ &= \int_0^{\pi/2} \frac{\cos \alpha dx}{(1 + \sin \alpha \sin^2 x)} = \int_0^{\pi/2} \frac{\cos \alpha \sec^2 x dx}{\cosec^2 x + \sin \alpha \tan^2 x} \\ &= \int_0^{\pi/2} \frac{\cos \alpha \sec^2 x dx}{1 + (\sin \alpha) \tan^2 x} \end{aligned}$$

Put $\tan x = t$

$$\sec^2 x dx = dt$$

& limits at $x = 0$ $t = 0$

$$x = \frac{\pi}{2} \quad t = \infty$$

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_0^\infty \frac{\cos \alpha dt}{1 + (+\sin \alpha)t^2} \\ &= \frac{\cos \alpha}{(1 + \sin \alpha)} \int_0^\infty \frac{dt}{t^2 + \frac{1}{\sin \alpha}} \\ &= \frac{\cos \alpha}{1 + \sin \alpha} \tan^{-1} t \sqrt{1 + \sin \alpha} \Big|_0^\infty \left\{ \sqrt{1 + \sin \alpha} \right\} \\ &= \frac{\cos \alpha}{1 + \sin \alpha} \left(\sqrt{1 + \sin \alpha} \right) \left(\frac{\pi}{2} - 0 \right)\end{aligned}$$

$$= \frac{\pi \cdot \cos \alpha}{2(1 + \sin \alpha)} \sqrt{1 + \sin \alpha}$$

$$= \frac{\pi}{2} \frac{\cos \alpha}{\sqrt{1 + \sin \alpha}}$$

$$= \frac{\pi}{2} \frac{(\cos^2 \alpha / 2 - \sin^2 \alpha / 2)}{\sin \alpha / 2 + \cos \alpha / 2}$$

$$\text{as } 1 + \sin \alpha = \left(\sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha}{2} + 2 \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} \right)$$

$$= \frac{\pi}{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\therefore \frac{dI(\alpha)}{d\alpha} = \frac{\pi}{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

now integrating to get the value of I

$$I = \frac{\pi}{2} \int \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) dx$$

$$I = \frac{\pi}{2} \left(2 \sin \frac{\alpha}{2} + 2 \cos \frac{\alpha}{2} \right) + C$$

$$I = \pi \left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right) + C \quad \dots(i)$$

for value of C put $\alpha = 0$ in the equation

$$\begin{aligned} I(0) &= \pi(\sin 0 + \cos 0) + C \\ &= \pi + C \end{aligned} \quad \dots(ii)$$

& $I(0)$ can be found out using the original function.

$$\& \quad I(a) = \int_0^{\pi/2} \log(1 + \sin \alpha \sin^2 x) \cos \sec^2 x dx$$

$$I(0) = \int_0^{\pi/2} (\log(1)).\cos \sec^2 x dx = 0$$

Putting this in (ii)

$$I(0) = \pi + C$$

$$\Rightarrow 0 = \pi + C$$

$$\Rightarrow C = -\pi$$

putting this value in (i)

$$I(\alpha) = \pi \left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right) - \pi$$

Property-19 : If $f(t)$ is an odd function, then $g(x) = \int_a^x f(t) dt$ is an even function.

Property-20 : If $f(t)$ is an even function, then $g(x) = \int_0^x f(t) dt$ is an odd function.

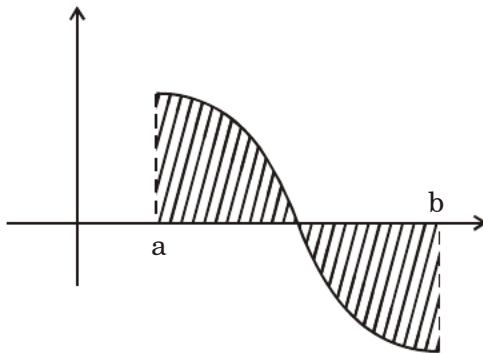
NOTE : For $\int_a^x f(t) dt$ might not be an odd function. So apply the above property taking special care of limits.

Property-21 : If $f(x)$ is a continuous function on $[a, b]$ then there exists a point $c \in (a, b)$ such

that $\int_a^b f(x) dx = f(c)(b-a)$. This is known as Mean Value Theorem of Integration.

Property-22 : If $f(x)$ is continuous in $[a, b]$ & $\int_a^b f(x) dx = 0$ then the equation $f(x) = 0$ has atleast one root in (a, b) .

Proof of this property is very simple.



The area can be zero only iff there is some part of $f(x)$ below the x-axis (i.e. negative area). And for that to happen for a continuous function $f(x)$, $f(x)$ must cross the $y=0$ line at atleast one point.

IMPROPER INTEGRAL

If $f(x)$ is continuous on $[a, \infty]$, then $\int_a^{\infty} f(x)dx$ is called as improper integral and

$$\int_a^{\infty} f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

If the Right Hand Limit of integral exists then the improper integral is convergent otherwise it is divergent.

GAMMA FUNCTION

It is defined by the improper integral, by $\int_0^{\infty} e^{-x} x^{n-1} dx$ and is denoted by Γn

$$\therefore \Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx \quad \text{where } x \text{ is a positive rational number.}$$

Properties of Gamma function

1. $\Gamma 1 = 1, \quad \Gamma 0 = \infty \quad \text{and} \quad \Gamma(n+1) = n\Gamma n$
2. if $n \in N, \quad \Gamma(n+1) = n!$
3. $\Gamma(1/2) = \sqrt{\pi}$

Useful extensions of gamma function :

$$1. \int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^n x dx$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \left(\frac{\pi}{2}\right), & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1, & \text{if } n \text{ is odd} \end{cases}$$

$$2. \int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx$$

$$= \frac{(m-1)(m-3)\dots(1)(n-1)(n-3)\dots(1)}{(m+n)(m+n-2)\dots2} \cdot \frac{\pi}{2} \quad \text{when both } m \text{ & } n \text{ belong to even integer}$$

$$= \frac{(m-1)(m-3)\dots(1 \text{ or } 2)(n-1)(n-3)\dots(1 \text{ or } 2)}{(m+n)(m+n-2)\dots1 \text{ or } 2} \quad \text{when either of } m \text{ or } n \text{ belong to odd integer}$$

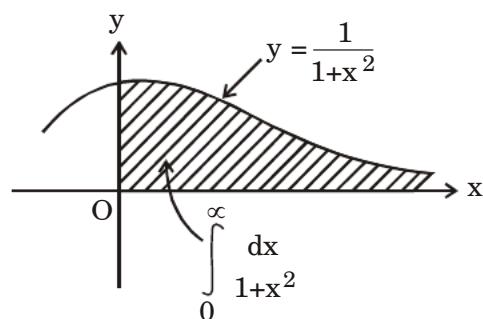
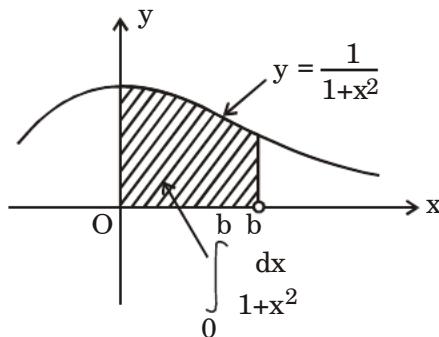
$$3. \int_0^\pi \sin^m x \cos^n x dx = 0, \quad \text{if } n \text{ is odd}$$

$$= 2 \int_0^{\pi/2} \sin^m x \cos^n x dx, \quad \text{if } n \text{ is even}$$

Illustration 29

Evaluate the integral $\int_0^{+\infty} \frac{dx}{1+x^2}$

Solution :



By the definition of an improper integral we find

$$\int_0^{+\infty} \frac{dx}{1+x^2} + \lim_{b \rightarrow +\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b \rightarrow +\infty} \tan^{-1} x \Big|_0^b = \lim_{b \rightarrow +\infty} \tan^{-1} b = \frac{\pi}{2}.$$

Illustration 30

Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Solution :

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2}$$

The second integral is equal to $\frac{\pi}{2}$. Compute the first integral :

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{\alpha \rightarrow -\infty} \int_{\alpha}^0 \frac{dx}{1+x^2} = \lim_{\alpha \rightarrow -\infty} \tan^{-1} x|_{\alpha}^0 = \lim_{\alpha \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} \alpha) = \frac{\pi}{2}$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

In many cases, it is sufficient to determine, whether the given intergal converges or diverges, and to estimate its value.

The Integral of a Discontinuous function :

A function $f(x)$ is defined and continuous when $a \leq x < c$, and either not defined or discontinuous when $x = c$. In this case, one cannot speak of the integral $\int_a^c f(x)dx$ as the limit of integral sums, because $f(x)$ is not continuous on the interval $[a, c]$, and for this reason the limit may not exist.

The integral $\int_a^c f(x)dx$ of the function $f(x)$ discontinuous at the point c is defined as follows :

$$\int_a^c f(x)dx = \lim_{b \rightarrow c-0} \int_a^b f(x)dx$$

If the limit on the right exist, the integral is called an important convergent integral, otherwise it is divergent.

If the function $f(x)$ is dicsontinuous at the left extremity of the interval $[a, c]$ (that is, for

$x = a$), then by defintion $\int_a^c f(x)dx = \lim_{b \rightarrow a+0} \int_b^c f(x)dx$

if the function $f(x)$ is discontinuous at some point $x = x_0$ inside the interval $[a, c]$, we put

$$\int_a^c f(x)dx = \int_a^{x_0} f(x)dx + \int_{x_0}^c f(x)dx$$

if both imporper integral on the right side of the equation exist.

Illustration 31

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

(b) Evaluate the integral $\int_{-1}^1 \frac{dx}{x^2}$

Solution :

$$\begin{aligned}(a) \quad \int_0^1 \frac{dx}{\sqrt{1-x}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} = -\lim_{b \rightarrow 1^-} 2\sqrt{1-x} \Big|_0^b \\ &= -\lim_{b \rightarrow 1^-} 2(\sqrt{1-b} - 1) = 2\end{aligned}$$

(b) Since inside the interval of integration there exists a point $x=0$ where the integrand is discontinuous, the integral must be represented sum of two terms :

$$\int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon_1 \rightarrow 0^-} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \rightarrow 0^+} \int_{\varepsilon_2}^1 \frac{dx}{x^2}$$

Calculate each limit separately :

$$\lim_{\varepsilon_1 \rightarrow 0^-} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} = -\lim_{\varepsilon_1 \rightarrow 0^-} \frac{1}{x} \Big|_{-1}^{\varepsilon_1} = -\lim_{\varepsilon_1 \rightarrow 0^-} \left(\frac{1}{\varepsilon_1} - \frac{1}{-1} \right) = \infty$$

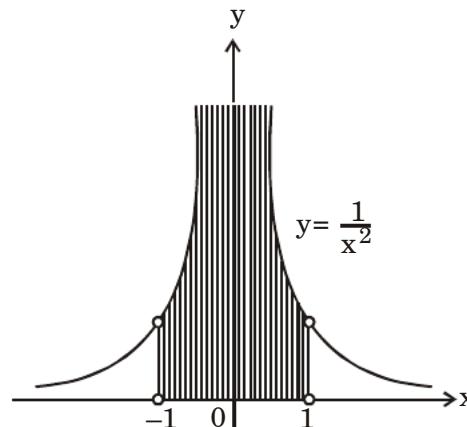
Thus, the integral diverges on the interval $[-1, 0]$

$$\lim_{\varepsilon_2 \rightarrow 0^+} \int_{\varepsilon_2}^1 \frac{dx}{x^2} = -\lim_{\varepsilon_2 \rightarrow 0^+} \left(1 - \frac{1}{\varepsilon_2} \right) = \infty$$

And this means that the integral also diverges on the interval $[0, 1]$. Hence, the given integral diverges on the entire interval $[-1, 1]$. It should be noted that if we had begun to evaluate the given integral without paying attention to the discontinuity of the integrand at the point $x=0$, the result would have been wrong.

$$\text{Indeed } \int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_{-1}^1 = -\left(\frac{1}{1} - \frac{1}{-1} \right) = -2$$

which is impossible (fig.)



SUMMATION OF SERIES USING DEFINITE INTEGRAL AS A LIMIT OF SUM

If $f(x)$ is an integrable function defined on $[a, b]$ then

$$\lim_{h \rightarrow 0} [h \{ f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h) \}]$$

is called the definite integral of $f(x)$ between limits a and b .

$$\therefore \int_a^b f(x) dx = \lim_{h \rightarrow 0} h \left[\{ f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h) \} \right]$$

$$= \lim_{h \rightarrow 0} \left| \sum_{r=0}^n f(a+rh) \right|$$

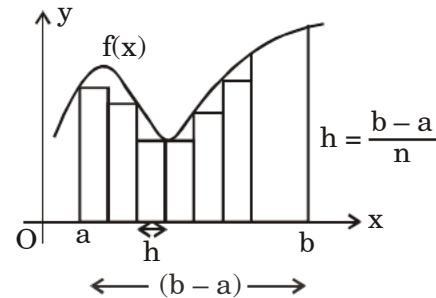
It should be noted that as $b \rightarrow 0, a \rightarrow \infty$

$$nh = b - a$$

Putting $a = 0, b = 1$, so that $h = \frac{1}{n}$

We get $\int_0^t f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

WORKING RULE



Step-1. Express the series in the form, $\lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum f\left(\frac{r}{n}\right) \right]$

Step-2. Replace \sum by \int ,

$\frac{r}{n}$ by x and

$\left(\frac{1}{n}\right)$ by dx

Step-3. Obtain the lower & upper limits of the integral by computing $\lim_{n \rightarrow \infty} \left(\frac{r}{n} \right)$ for the least & greatest value of r respectively i.e. put the starting & ending values of r to get the limits.

Illustration 32

Show that $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{6n} \right) = \log 6$

[IIT - 81]

Solution :

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{6n} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+5n} \right) \end{aligned}$$

[writing last term in the same form as the 1st, 2nd, 3rd, ... terms are]

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{r=1}^{5n} \frac{1}{n+r} = \lim_{n \rightarrow \infty} \sum_{r=1}^{5n} \frac{1}{n} \left(\frac{n}{n+r} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{5n} \frac{1}{1 + \frac{r}{n}} \end{aligned}$$

 \therefore lower limit of $r = 1$ \therefore lower limit of integration $= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) = 0$ \therefore upper limit of $r = 5n$ \therefore upper limit of integration $= \lim_{n \rightarrow \infty} \frac{5n}{n} = 5$

Hence from (i) required limit

$$\begin{aligned} &= \int_0^5 \frac{dx}{1+x} = \left[\log(1+x) \right]_0^5 \\ &= \log 6 - \log 1 = \log 6 \quad [\because \log 1 = 1] \end{aligned}$$

Illustration 33

Evaluate $\lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{1}{2n} \right)$

Solution : $\lim_{n \rightarrow \infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right)$

[Writing last term in the same form as first, 2nd, 3rd,... term are]

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{n}{n^2 + r^2} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{n^2}{n^2 + r^2} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{1 + \left(\frac{r}{n} \right)^2} \quad \dots(i)
 \end{aligned}$$

\therefore lower limit of $r = 1$

\therefore lower limit of integration $= \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

\therefore upper limit of $r = n$

\therefore upper limit of integration $= \lim_{n \rightarrow \infty} \frac{n}{n} = 1$

Hence from (i), required limit

$$\begin{aligned}
 &= \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x \right]_0^1 \\
 &= \tan^{-1}(1) - \tan^{-1}(0) \\
 &= \frac{\pi}{4} - 0 = \frac{\pi}{4}
 \end{aligned}$$

Illustration 34

Evaluate $\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \frac{1}{\sqrt{n^2 - 2^2}} + \dots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right]$

Solution :
$$\begin{aligned}
 &\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \frac{1}{\sqrt{n^2 - 2^2}} + \dots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{n}{\sqrt{n^2 - r^2}} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\sqrt{1 - \left(\frac{r}{n} \right)^2}}
 \end{aligned}$$

$$\because \text{lower limit of } r = 0 \quad \therefore \text{lower limit of integration} = \lim_{n \rightarrow \infty} \frac{y}{n} = 0$$

\therefore upper limit of $r = n - 1$

$$\therefore \text{upper limit of integration} = \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1$$

\therefore from (i), required limit

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x \right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Illustration 35

$$\text{Evaluate } \lim_{n \rightarrow \infty} \sum_{r=1}^{r=n} \frac{r^2}{r^3 + n^3}.$$

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^2}{r^3 + n^3} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{r^2 n}{r^3 + n^3} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3 + 1} \end{aligned} \quad \dots(i)$$

$$\text{lower limit of } r = 1 \quad \therefore \text{lower limit of integration} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\text{upper limit of } r = n \quad \therefore \text{upper limit of integration} = \lim_{n \rightarrow \infty} \frac{n}{n} = 1$$

$$\therefore \text{from (i), required limit} = \int_0^1 \frac{x^2}{x^3 + 1} dx \quad \dots(ii)$$

Let $z = x^3 + 1$, then $dz = 3x^2 dx$

when $x = 0, z = 1$ and when $x = 1, z = 2$

$$\therefore \text{from (ii), required limit} = \frac{1}{3} \int_1^2 \frac{dz}{z} = \frac{1}{3} [\log z]_1^2 = \frac{1}{3} (\log 2 - \log 1) = \frac{1}{3} \log 2.$$

$$\because \text{lower limit of } r = 0 \quad \therefore \text{lower limit of integration} = \lim_{n \rightarrow \infty} \frac{y}{n} = 0$$

\because upper limit of $r = n - 1$

$$\therefore \text{upper limit of integration} = \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} \right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1$$

\therefore from (i), required limit

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x \right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Illustration 35

Evaluate $\lim_{n \rightarrow \infty} \sum_{r=1}^{r=n} \frac{r^2}{r^3 + n^3}.$

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^2}{r^3 + n^3} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{r^2 n}{r^3 + n^3} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3 + 1} \end{aligned} \quad \dots(i)$$

$$\text{lower limit of } r = 1 \quad \therefore \text{lower limit of integration} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\text{upper limit of } r = n \quad \therefore \text{upper limit of integration} = \lim_{n \rightarrow \infty} \frac{n}{n} = 1$$

$$\therefore \text{from (i), required limit} = \int_0^1 \frac{x^2}{x^3 + 1} dx \quad \dots(ii)$$

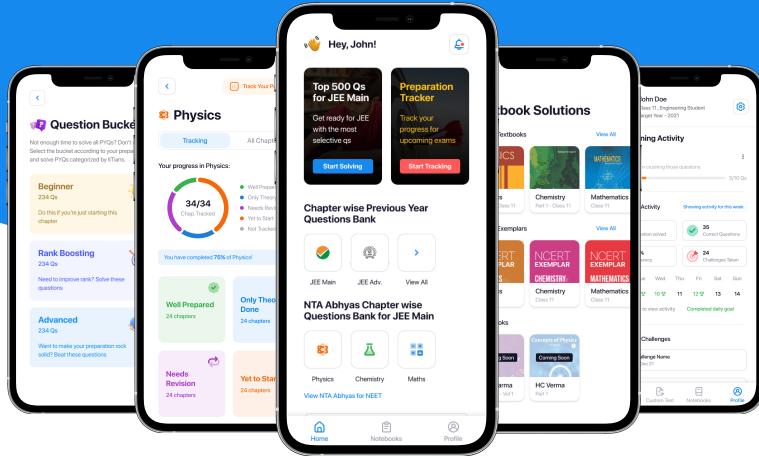
Let $z = x^3 + 1$, then $dz = 3x^2 dx$

when $x = 0, z = 1$ and when $x = 1, z = 2$

$$\therefore \text{from (ii), required limit} = \frac{1}{3} \int_1^2 \frac{dz}{z} = \frac{1}{3} [\log z]_1^2 = \frac{1}{3} (\log 2 - \log 1) = \frac{1}{3} \log 2.$$



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