Final Examination
CS/MATH 113 Discrete Mathematics

Habib University — Spring 2023

2 May, 2023. 1630-1900h.

Enter your name and ID below and at the top of all the subsequent pages.

Student ID: ______________________________

Student Name: ______________________________

Multiple choice problems

There are 20 problems below. Enter their solutions in the grid at the bottom of the next side. Note the following conventions: p, q, and r are propositions; R, S, and T are finite sets; f is a function; and $G = (V, E)$ is a graph.

1. $\boxed{\text{1 point}}$ Which of the following is equivalent to $p \iff q$?

   A. $p \iff \neg q$ B. $\neg p \iff q$ **C.** $\neg p \iff \neg q$ D. None of the mentioned

2. $\boxed{\text{1 point}}$ $p \implies (p \vee q)$ is a tautology.

   A. True B. False

3. $\boxed{\text{1 point}}$ Which of the following is *not* equivalent to the statement, $(p \wedge q) \implies (q \vee r)$?

   A. $(p \implies q) \vee (q \implies r)$ B. $(p \implies r) \vee (q \implies q)$ **C.** $\neg(p \wedge q) \implies \neg(q \vee r)$
   D. True

4. $\boxed{\text{1 point}}$ The premises $(p \wedge q) \vee r$ and $r \implies s$ lead to which statement?

   A. $p \wedge r$ **B.** $p \vee s$ C. $p \vee q$ D. $r \wedge s$

5. $\boxed{\text{1 point}}$ The predicate logic statement corresponding to, "The product of two negative real numbers is positive.", assuming a domain of $\mathbb{R}$, is

   A. $\exists x \forall y((x < 0) \wedge (y < 0) \implies (xy > 0))$ B. $\exists x \exists y((x < 0) \wedge (y < 0) \wedge (xy > 0))$
   C. $\forall x \exists y((x < 0) \wedge (y < 0) \wedge (xy > 0))$ **D.** $\forall x \forall y((x < 0) \wedge (y < 0) \implies (xy > 0))$

6. $\boxed{\text{1 point}}$ In a direct proof of the statement, "If n is an odd integer then n^2 is an odd integer.", using the predicate $P(n)$ for "n is an odd integer", we need to show that

   A. $\exists n(P(n) \implies P(n^2))$ B. $\forall n(P(n)) \implies \forall n(P(n^2))$ C. $\forall n(P(n)) \implies \exists n(P(n^2))$
   D. $\forall n(P(n) \implies P(n^2))$

7. $\boxed{\text{1 point}}$ Which of the following is *not* a subset of $S = \{\}$?

   A. S B. $\{\}$ C. $\emptyset$ **D.** $\{\emptyset\}$

8. $\boxed{\text{1 point}}$ $S \times T = T \times S$.

   A. True **B. False**

9. $\boxed{\text{1 point}}$ If $R \subseteq S$, then $R \times T \subseteq S \times T$.

   A. True B. False

10. $\boxed{\text{1 point}}$ Given that $|R \cup S| = |R \cap S|$, which of the following *need not* be true?

    A. $R \subseteq S$ B. $|R| = |S|$ **C.** $S = \emptyset$ D. $R - S = \emptyset$

11. $\boxed{\text{1 point}}$ Let $S_1, S_2, S_3, \ldots, S_{100}$ be 100 sets such that $|S_i| = i$ and $S_i \subseteq S_{i+1}$ for $1 \leq i \leq 99$. What is the cardinality of $\bigcup_{i=1}^{100} S_i$?

    A. 99 **B.** 100 C. 101 D. 102

12. $\boxed{\text{1 point}}$ Given $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 3$, the range of f is

    A. $\mathbb{Z}$ B. $\mathbb{Z}^+$ **C.** $\mathbb{Z}^+ - \{0, 1, 2\}$ D. $\mathbb{Z}^+ \cup \{3\}$

13. $\boxed{\text{1 point}}$ Given $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x–5$, what is a possible expression for $f^{-1}(x)$?

    A. $\frac{1}{3x-5}$ **B.** $\frac{x+5}{3}$ C. does not exist since f is not a bijection D. none of the mentioned

14. $\boxed{\text{1 point}}$ The value of $\sum_{i=1}^{100}(-1)^i$ is

    A. -1 **B.** 0 C. 1 D. 2

15. $\boxed{\text{1 point}}$ The value of $\sum_{i=1}^{100}(f(i) - f(i-1))$ is

    A. $f(0)$ B. $f(1)$ C. $f(100)$ **D.** $f(100) - f(0)$

16. $\boxed{\text{1 point}}$ In an inductive proof of the statement, "$P(n) : \sum_{i=1}^{n} 2^i = f(n)$", the inductive step will have to show that

    A. $f(k+1) = f(k) + 2^k$ **B.** $f(k+1) = f(k) + 2^{k+1}$ C. $f(k+1) = f(k+1) + 2$ D. $f(k+1) = \sum_{i=1}^{k} 2^i + \sum_{i=1}^{k+1} 2^i$

17. $\boxed{\text{1 point}}$ If G is finite, then the number of elements of V that have odd degree is

    A. even B. odd C. indeterminate - depends on G D. infinite

18. $\boxed{\text{1 point}}$ G is an undirected graph with 26 edges and with the degree of each vertex equal to 4 or more. What is the maximum possible value of $|V|$?

    A. 7 B. 10 **C.** 13 D. 43

19. $\boxed{\text{1 point}}$ If G is a simple graph with n vertices, the minimum possible value of $|E|$ is

    A. 0 B. 1 C. $n-1$ D. $\frac{n(n-1)}{2}$

20. $\boxed{\text{1 point}}$ If G is a simple graph with n vertices, the maximum possible value of $|E|$ is

    A. 0 B. 1 C. $n-1$ **D.** $\frac{n(n-1)}{2}$

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**	**9.**	**10.**
11.	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**	**18.**	**19.**	**20.**

Written problems

Attempt the following 4 problems in the answer sheet. Submit the entirety of this problem sheet with the answer sheet when done.

1. Cardinality [15 points]

For each of the following cases, provide example sets A and B that are uncountable and show the value of $A \cap B$.

(a) 5 points $A \cap B$ is finite.

Solution: Let $A = [0,1], B = [1,2]$. Then $A \cap B = \{1\}$.

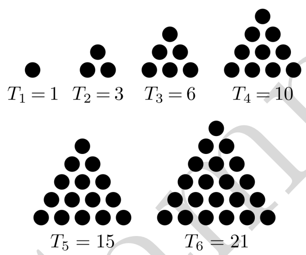
(b) 5 points $A \cap B$ is countably infinite.

Solution: Let $A = [0,1] \cup \mathbb{Z}, B = [1,2] \cup \mathbb{Z}$. Then $A \cap B = \mathbb{Z}$.

(c) 5 points $A \cap B$ is uncountable

Solution: Let $A = B = [0,1]$. Then $A \cap B = [0,1]$.

2. Sequences and Summation [15 points]



The first six terms of the sequence $[T_n]$ are pictured in the figure. Find

(a) 5 points a recurrence relation for the sequence.

(b) 5 points a closed form for the sequence, which does not contain a summation symbol.

(c) 5 points the sum of the first k terms of the sequence, where k is a positive integer.

Solution:

(a) $T_n = T_{n-1} + n,\ T_1 = 1$

(b) We see that $T_1 = 1, T_2 = 1+2, T_3 = 1+2+3, T_4 = 1+2+3+4, \ldots$
Thus, $T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(c)

$$\begin{aligned}
\sum_{i=1}^{k} T_i &= \sum_{i=1}^{k} \frac{i(i+1)}{2} \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} i^2 + \sum_{i=1}^{k} i\right) \\
&= \frac{1}{2}\left(\frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right) \\
&= \frac{1}{12}(k(k+1)(2k+1) + 3k(k+1)) \\
&= \frac{k(k+1)(2k+4)}{12}
\end{aligned}$$

3. Induction

[10 points]

(a) $\boxed{\text{5 points}}$ Use mathematical induction to prove that if a set, A, has n elements, where n is an integer greater than or equal to 2, then it has $\frac{n(n-1)}{2}$ subsets of cardinality 2. For example, the set $A = \{x, y, z\}$ has $\frac{3\cdot 2}{2} = 3$ subsets of cardinality 2, which are: $\{x, y\}, \{x, z\}, \{y, z\}$.

Solution: For ease of notation, let us denote the number of subsets of A that have cardinality 2 as s_n where $|A| = n$.
Then, $P(n) : s_n = \frac{n(n-1)}{2}$, where $n \geq 2$.

Proof. **Basis Step** We consider $P(2)$.
When A has 2 elements, it has only 1 subset of cardinality 2, i.e. itself.
According to $P(2)$, $s_2 = \frac{2\cdot 1}{2} = 1$.
$\therefore$ the base case holds.

Inductive Step We consider $P(k+1)$ which claims that $s_{k+1} = \frac{(k+1)k}{2}$.
From the IH, we have that $s_k = \frac{k(k-1)}{2}$.
Consider a set, A with k elements and a set, $B = A \cup \{e\}$ where $e \notin A$.
Then $|B| = k + 1$. Let us consider its subsets that have cardinality 2.
All the subsets of A are also subsets of B.
So, s_{k+1} is s_k plus the new formed by adding e.
The new subsets of cardinality 2 will be those that contain e and each of the k elements of A.
Thus, $s_{k+1} = \frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$.

$\square$

(b) $\boxed{\text{5 points}}$ Let A be a set of ordered pairs of integers defined recursively as follows.

Basis step $(0, 0) \in A$

Recursive step If $(a, b) \in A$, then the following also belong to A: $(a, b+1)$, $(a+1, b+1)$, and $(a+2, b+1)$

Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in A$.

Solution: $P(n) : (a, b) \in A \implies a \leq 2b$

Proof. **Basis Step** $(a, b) = (0, 0)$ and $0 \leq 2 \cdot 0$.
$\therefore$ the base case holds.

Recursive Step Assume that $(a, b) \in A$ and $a \leq 2b$.
We obtain (a', b') from (a, b) and show that $a' \leq 2b'$.
There are 3 cases for (a', b').

Case 1: $(a', b') = (a, b + 1)$
We know that $a \leq 2b$.
Then $a \leq 2b + 2 = 2(b + 1)$
$\therefore a' \leq 2b'$

Case 2: $(a', b') = (a + 1, b + 1)$
We know that $a \leq 2b$.
Then $a + 1 \leq 2b + 2 = 2(b + 1)$
$\therefore a' \leq 2b'$

Case 3: $(a', b') = (a + 2, b + 1)$
We know that $a \leq 2b$.
Then $a + 2 \leq 2b + 2 = 2(b + 1)$
$\therefore a' \leq 2b'$

$\square$

4. Graphs [5 points]

Show that in a simple graph with at least two vertices, there must be two or more vertices of the same degree. For example, in the complete graph, K_5, the degrees are $4, 4, 4, 4$, and 4, and there are 5 vertices with the same degree.

Solution: We prove the statement through contradiction. We show that the negation, i.e., all vertices in a simple graph have distinct degrees, leads to a contradiction.

Proof. In a simple graph with n vertices, the minimum and maximum possible degrees are 0 and $(n - 1)$.
Consider an assignment (a bijection) between the n vertices and the possible degrees.
Let v_{min} be the vertex with degree 0, and v_{max} be the vertex with degree $(n - 1)$.
v_{min} is not connected to any vertex.
v_{max} is connected to every vertex except itself, i.e., it is connected to v_{min}. $\perp$

$\square$

Good luck!

1 Some useful formulas and definitions

Definition 1.1 (Finite Set)**.** Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Definition 1.2 (Equality of Cardinality)**.** The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

Definition 1.3 (Countable Set)**.** A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable.

Definition 1.4 (Mathematical Induction)**.** To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:
Basis Step: We verify that $P(1)$ is true.
Inductive Step: We show that the statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Definition 1.5 (Structural Induction)**.** To prove that $P(n)$ is true for a recursively defined structure, we complete two steps:
Basis Step: Show that $P(n)$ holds for all elements specified in the basis step of the recursive definition.
Recursive Step: Show that if $P(n)$ holds for each of the elements used to construct new elements in the recursive step of the definition, then $P(n)$ holds for these new elements.

Definition 1.6 (Graph)**.** A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Definition 1.7 (Simple Graph)**.** A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Definition 1.8 (Degree)**.** The degree of a vertex in a graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Closed Forms of some Common Summations

Sum	$\sum_{k=0}^{n} ar^k (r \neq 0)$	$\sum_{k=1}^{n} k$	$\sum_{k=1}^{n} k^2$	$\sum_{k=1}^{n} k^3$	$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$
Formula	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$	$\frac{n(n+1)}{2}$	$\frac{n(n+1)(2n+1)}{6}$	$\frac{n^2(n+1)^2}{4}$	$\frac{1}{1-x}$	$\frac{1}{(1-x)^2}$