Habib University - City Campus
Course Title: Discrete Mathematics
Course Code: CS/MATH 113
Examination: Final — Exam Date: 12 May 2022

Total Marks: 110 — Duration: 150 minutes

Instructions:

- Please submit your device(s) including calculators in your bag at the front of the examination room. Possession of a device during the exam counts as unfair means.
- You may answer the questions in any order in the provided answer book provided the question number is clearly labeled in your solution.
- Do not use red ink.
- Work done in pencil will be considered rough work and ignored for grading.
- Communicating with other students, including asking for stationery, is not allowed and counts as unfair means.
- Keep your student ID card next to you on your desk at all times.
- The use of a "cheat sheet" meeting the previously shared conditions is allowed. Any other cheat sheet counts as unfair means. The conditions are repeated below.
  - It must be a single A4 sheet of paper, utilizing any number of its sides.
  - It must be handwritten by you.
  - It must not be in red ink, black ink, or pencil.
  - The top of each side must clearly indicate your name and student ID.
- Students found using unfair means will have their exam copy confiscated and be disqualified immediately. Other penalties may also apply in accordance with Habib University's disciplinary code.
- This question paper contains 17 questions on 3 pages including this one. Make sure that your question book is complete and to answer all the questions.

Good luck!

Sets

1. $\boxed{\text{5 points}}$ Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

   Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

2. $\boxed{\text{5 points}}$ Suppose that A, B, C, and D are sets. Prove or disprove that $(A - B) - (C - D) = (A - C) - (B - D)$

Logic

3. $\boxed{\text{5 points}}$ Express each of the statements below in formal notation using quantifiers and over the indicated domains.

   (a) No one has climbed every mountain in Pakistan (domain: people of Pakistan, mountains in Pakistan).

   (b) At least one email message can be saved if there is a disk with more than 10KB of free space (domain: email messages, disks).

   (c) There is somebody whom everyone loves (domain: people).

   (d) A person is popular only if they are cool or funny (domain: people).

   (e) You are either with us or against us (domain: people).

4. $\boxed{\text{10 points}}$ Prove the following statements using rules of inference

   (a) $((\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r) \rightarrow p$.

   (b) $((p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)) \rightarrow (\neg q \rightarrow s)$.

Relations

5. $\boxed{\text{5 points}}$ Let $S = \{3, 5, 9, 15, 24, 45\}$ and $R = \{(a, b) \mid (\text{ a divides b})\}$.

   (a) Draw the Hasse Diagram for (S, R)

   (b) Identify whether (S, R) is a poset or not?

   (c) Find the maximal elements.

   (d) Find the minimal elements.

   (e) Find the least element? If there is no least element then why?

6. $\boxed{\text{5 points}}$ Construct a relation on the set $\{a, b, c, d\}$ that is

   (a) reflexive, symmetric, but not transitive.

   (b) reflexive and transitive but neither symmetric nor antisymmetric

7. $\boxed{\text{5 points}}$ Let R be a reflexive relation on a set A. Show that $R \subset R^2$.

8. $\boxed{\text{10 points}}$ Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.

   (a) $R_1 \cup R_2$

   (b) $R_1 \cap R_2$

9. $\boxed{\text{10 points}}$ Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.

Functions

10. $\boxed{\text{5 points}}$ In the game of Hearts, four players are each dealt 13 cards from a deck of 52. Is this a function? If so, what sets make up the domain and co-domain, and is the function injective, surjective, bijective, or neither?

11. $\boxed{\text{5 points}}$ A function $f : A \to B$ is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

    Let $g : A \to B$ and let $f : B \to C$. The *composition* of the functions f and g, denoted for all $a \in A$ by$f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

    Given $g : A \to B$ and $f : B \to C$, show that if both f and g are onto functions, then $f \circ g$ is also onto.

12. $\boxed{\text{10 points}}$ A *bit-string* is a sequence of `0`s and `1`s. If the number of `0`s and `1`s in a bit-string is equal to a non-negative integer, n, then the bit-string is said to be *finite* and have *length*, n. For example, `0111001` is a bit-string of length 7. A bitstring that is not finite is said to be *infinite*.

    Prove that the set of all bit-strings is uncountable.

Proof by Induction

13. $\boxed{\text{10 points}}$ Use mathematical induction to prove that $1^2 + 2^2 + 3^3 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Graphs

14. $\boxed{\text{5 points}}$ Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

    (a) Model the possible marriages on the island using a bipartite graph.

    (b) Use Hall's theorem to determine whether there is a matching of the young men and young women on the island such that each young woman is matched with a young man she is willing to marry.

15. $\boxed{\text{5 points}}$ Draw a simple, connected graph, $G = (V, E)$, with $|V| > 5$ and $|E| > 10$ that does not contain an Euler cycle but contains an Euler path. Prove that no Euler cycle exists in this graph and that an Euler path exists.

16. $\boxed{\text{5 points}}$ A *subgraph* of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$. A *complete graph* on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.

    Prove that K_m is a subgraph of K_n for all integers $n \geq m \geq 1$.

17. $\boxed{\text{5 points}}$ A *tree* is a simple undirected graph that is connected and does not contain cycles. Prove by induction that a tree with n nodes has $(n-1)$ edges.