CS/Math 113 - Problem Set 7

Dead TAs Society
Habib University - Spring 2023

Week 09 - Week 10

Problems

Problem 1.[Chapter 2.4, Question 10] Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

(a) $a_n = -2a_{n-1}, a_0 = -1$
(b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$
(c) $a_n = 3a_{n-1}^2, a_0 = 1$
(d) $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
(e) $a_n = a_{n-1} - an - 2 + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$

Solution:

(a) $a_0 = -1, a_1 = -2a_0 = 2, a_2 = -2a_1 = -4, a_3 = -2a_2 = 8, a_4 = -2a_3 = -16, a_5 = -2a_4 = 32$

(b) $a_0 = 2, a_1 = -1, a_2 = a_1 - a_0 = -1 - 2 = -3, a_3 = a_2 - a_1 = -2, a_4 = a_3 - a_2 = 1, a_5 = a_4 - a_3 = 3$

(c) $a_0 = 1, a_1 = 3, a_2 = 27, a_3 = 2187, a_4 = 315, a_5 = 3^{31}$

(d) $a_0 = -1, a_1 = 0, a_2 = 1, a_3 = 3, a_4 = 13, a_5 = 74$

(e) $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 1, a_5 = 1$

Problem 2.[Chapter 2.4, Question 11] Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \cdots$

(a) Find a_0, a_1, a_2, a_3, and a_4
(b) Show that $a_2 = 5a_1 - 6a_0, a_3 = 5a_2 - 6a_1$ and $a_4 = 5a_3 - 6a_2$
(c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$

Solution:

(a) $a_0 = 6, a_1 = 17, a_2 = 49, a_3 = 143, a_4 = 421$

(b) $a_2 = 49 = 5 \cdot 17 - 6 \cdot 6 \implies a_2 = 5a_1 - 6a_0, a_3 = 143 = 5 \cdot 49 - 6 \cdot 17 \implies a_3 = 5a_2 - 6a_1, a_4 = 421 = 5 \cdot 143 - 6 \cdot 49 \implies a_4 = 5a_3 - 6a_2$

(c) $a_n = 5a_{n-1} - 6a_{n-2}$

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 2^{n-2}(10-6) + 3^{n-2}(75-30) \\ &= 2^{n-2} \cdot 4 + 3^{n-2}(9 \cdot 5) \\ &= 2^n + 3^n \cdot 5 = a_n \end{aligned}$$

Problem 3.[Chapter 2.4, Question 12] Show that the sequence a_n is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

(a) $a_n = 0$

(b) $a_n = 1$

(c) $a_n = (-4)^n$

(d) $a_n = 2(-4)^n + 3$

Solution:

(a) $-3(0) + 4(0) = 0 + 0 = 0 = a_n$. Hence, is a solution

(b) $-3(1) + 4(1) = -3 + 4 = 1 = a_n$. Hence, is a solution

(c) $-3((-4)^{n-1}) + 4((-4)^{n-2}) = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} = (-4)^{n-2} \cdot (-3 \cdot (-4) + 4) = (-4)^{n-2} \cdot (12 + 4) = (-4)^{n-2}(16) = (-4)^{n-2} \cdot (-4)^2 = (-4)^n = a_n$. Hence is a solution

Problem 4.[Chapter 2.4, Question 16] Find the solution to each of these recurrence relations with the given initial conditions.

(a) $a_n = -a_{n-1}, a_0 = 5$

(b) $a_n = a_{n-1} + 3, a_0 = 1$

(c) $a_n = a_{n-1} - n, a_0 = 4$

(d) $a_n = 2a_{n-1} - 3, a_0 = -1$

(e) $a_n = (n+1)a_{n-1}, a_0 = 2$

(f) $a_n = 2na_{n-1}, a_0 = 3$

(g) $a_n = -a_{n-1} + n - 1, a_0 = 7$

Solution:

(a) We notice that we get a pattern of 5s but with alternating signs - where n is even, we have positive 5, where n is odd, we have negative 5. Then our relation can be $a_n = (-1)^n \cdot 5$

(b) $a_1 = 1 + 3 = 4$
$a_2 = 4 + 3 = 1 + 3 + 3 = 7$
$a_3 = 7 + 3 = 1 + 3 + 3 + 3 = 10$
$\vdots$
$a_n = 1 + 3n$

Problem 5.[Chapter 2.4, Question 28] Let a_n be the n^{th} term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \cdots$, constructed by including the integer k exactly k times. Show that $a_n = \lceil \sqrt{2n} + \frac{1}{2} \rceil$

Solution:

Problem 6.[Chapter 2.4, Question 29] What are the values of these sums ?

1. $\sum_{k=1}^{5}(k+1)$
2. $\sum_{j=0}^{4}(-2)^j$
3. $\sum_{j=1}^{10} 3$
4. $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

Solution:

(a) $$\sum_{k=1}^{5}(k+1) = 2 + 3 + 4 + 5 + 6 = 20$$

(b) $$\sum_{j=0}^{4}(-2)^j = 1 - 2 + 4 - 8 + 16 = 11$$

Problem 7.[Chapter 2.4, Question 31] What is the value of each of these sums of terms of a geometric progression ?

(a) $\sum_{j=0}^{8} 3 \cdot 2^j$

(b) $\sum_{j=1}^{8} 2^j$

(c) $\sum_{j=2}^{8} (-3)^j$

(d) $\sum_{j=0}^{8} 2 \cdot (-3)^j$

Solution:

Sum of n terms for a Geometric Progression: $\mathcal{S}_n = \dfrac{a(1-r^n)}{1-r}$ for $r < 1$, and $\dfrac{a(r^n-1)}{r-1}$ for $r > 1$ where a is the first term of the series, r is the ratio, and n is the n^{th} term of the sequence.

(a) Using the above series, $a = 3$, $r = 2$, $n = 9$ [notice that n is 9 as 0 was our first term, hence 8 will be our 9th term], sum of the progression becomes $\mathcal{S}_9 = \dfrac{3(2^9-1)}{2-1} = 1533$

Same procedure for the remaining parts

Problem 8.[Chapter 2.4, Question 34] Compute each of these double sums

(a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i-j)$

(b) $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i+2j)$

(c) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$

(d) $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

Solution:

For computing double sums, j has the values for the limits of the sum for each i^{th} iteration of i [the inner sum has has limits for each iteration of the outer sum]

(a) $$\sum_{i=1}^{3} \sum_{j=1}^{2} = \underbrace{(1-1)}_{i=1,j=1} + \underbrace{(1-2)}_{i=1,j=2} + \underbrace{(2-1)}_{i=2,j=1} + \underbrace{(2-2)}_{i=2,j=2} + \underbrace{(3-1)}_{i=3,j=1} + \underbrace{(3-2)}_{i=3,j=2} = 3$$

Same procedure for the remaining parts

Problem 9.[Chapter 2.4, Question 39,40,41,42] Find the following (Use Table 2, Chapter 2.4)

(a) $\sum_{k=100}^{200} k$
(b) $\sum_{k=99}^{200} k^3$
(c) $\sum_{k=10}^{20} k^2(k-3)$
(d) $\sum_{k=10}^{20}(k-1)(2k^2+1)$

Solution:
(a) $\displaystyle\sum_{k=1}^{200} k - \sum_{k=1}^{99} k = \frac{200(201)}{2} - \frac{99(100)}{2} = 20100 - 4950 = 15150$
(c) $\displaystyle\sum_{k=1}^{20} k^3 - 3k^2 = \sum_{k=1}^{20} k^3 - \sum_{k=1}^{20} 3k^2$
$\displaystyle\sum_{k=1}^{20} k^3 - \sum_{k=1}^{9} k^3 + (-\sum_{k=1}^{20} 3k^2 + \sum_{k=1}^{9} 3k^2) = 34320$

Problem 10. [Chapter 2.4, Question 45] What are the values of the following products ?

(a) $\prod_{i=0}^{10} i$
(b) $\prod_{i=5}^{8} i$
(c) $\prod_{i=1}^{100}(-1)^i$
(d) $\prod_{i=1}^{10} 2$

Solution:
(a) $\displaystyle\prod_{i=0}^{10} i = 0 \times 1 \times 2 \times \cdots \times 10 = 0$. Since we have a 0, the product is 0.

(b) $\displaystyle\prod_{i=5}^{8} i = 5 \times 6 \times 7 \times 8 = 1680$

(c) $\displaystyle\prod_{i=1}^{100}(-1)^i = -1 \times 1 \times -1 \times 1 \times \cdots \times 1$. Every term is either a 1 or -1, so the product is either a 1 or -1. Whenever i is odd, we get a -1, so we need to know the total number of odd is to get the product. Odd numbers will be 1, 3, 5, ..., 99. So we have a total of 50 odd numbers. Then $(-1)^5 0 = 1$. Hence the result of the product is 1.

(d) $\prod_{i=1}^{10} 2 = 2 \times 2 \times 2 \cdots \times 2 = 2^{10} = 1024$