Weekly Challenge 12: Cardinality
CS/MATH 113 Discrete Mathematics

q2-team-49

Habib University — Spring 2023

1. Cardinality and Set Operations (0 points)

This ungraded problem provides the background for the next, graded problem for which you will require one or more of the results below. Find and go over the proofs of these results. No submission is needed for this problem.

(a) The union of countably many countable sets is countable.

(b) The superset of an uncountable set is uncountable.

(c) The powerset of a countable set is uncountable.

2. Fibonacci Unchained (10 points)

Consider an infinite matrix, A, in which the entry in the i-th row and j-th column is defined as follows.

$$a[i,j] = a[i,j-1] + a[i,j-2], \quad a[i,1] = i, a[i,2] = i+1, \quad i \geq 1, j \geq 1.$$

Consider a matrix, B, obtained from A as follows.

$$b[i,j] = a[i,i]$$

Argue whether the number of entries in B is countable.

ANSWER: - According to the given formula, the matrix A can be represented as:

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\ 2 & 3 & 5 & 8 & 13 & 21 & 34 & \cdots \\ 3 & 4 & 7 & 11 & 18 & 29 & 47 & \cdots \\ 4 & 5 & 9 & 14 & 23 & 37 & 60 & \cdots \\ 5 & 6 & 11 & 17 & 28 & 45 & 73 & \cdots \\ 6 & 7 & 13 & 20 & 33 & 53 & 86 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Now as the rule is defined above, matrix B consists of diagonal elements of matrix A and the entry to a column is the same for the entire row.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 3 & 3 & 3 & 3 & 3 & \cdots \\ 7 & 7 & 7 & 7 & 7 & \cdots \\ 14 & 14 & 14 & 14 & 14 & \cdots \\ 28 & 28 & 28 & 28 & 28 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

To prove the number of entries or elements in matrix B are countable, we show a bijection between its elements and the elements of the set of positive integers $\mathbb{Z}^+$ (which we already know is countable).

Now, we can access and map each element of matrix B diagonally in such a way that:

the first diagonal consists of 1 which is mapped to 1 in the set $\mathbb{Z}^+$
the second diagonal consists of 3 and 1 in which only 3 is mapped to 2 as 1 has already been mapped.

Continuing in this fashion $f(3) = 7$, $f(4) = 11$, $\cdots$

In this way, every entry in matrix B will be mapped to an element in $\mathbb{Z}^+$ establishing a bijection and proving that the number of entries in matrix B is countable.