Project 5

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Abstract—Some dynamic model's are greatly simplified by implementention of Langrangian equations. The use of a Langrangian overhauls Newton style internal mechanics and focuses on

1 Introduction: Inverted Pendulum Problem

An inverted pendulum is visualized in the figure below. With a swinging pendulum bob of mass m_2, a sliding base of mass m_1, and a pendulum of length L.

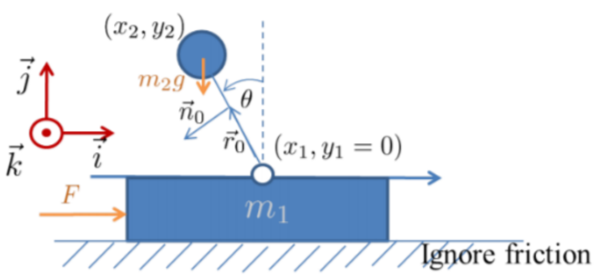

Fig. 1. Inverted Pendulum

Variables x_1 and θ will be used to describe the system. Describing the bob's coordinates in terms of those variables is the following:

$$\begin{aligned} x_2 &= x_1 - L\sin(\theta) \\ y_2 &= y_1 + L\cos(\theta) = L\cos(\theta) \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{x_2} &= \dot{x_1} - L\cos(\theta)\dot{\theta} \\ \dot{y_2} &= \dot{y_1} - L\sin(\theta)\dot{\theta} = -L\sin(\theta)\dot{\theta} \end{aligned} \tag{2}$$

Having described the bob and mass in terms of (x_1, θ) the system needs to be solved for $\ddot{x}$, and $\ddot{\theta}$. Normally, these functions would be solved for from several force summation equations, but in this system, we solve using a Langrangian to simplify solutions.

2 Langrangian Solution

A Lagrangian is ideal for systems of conservative forces as it is dependent on energy. The general form, where KE is Kinetic Energy and PE is Potential Energy, is the following:

$$\begin{aligned} \mathcal{L} &= KE - PE \\ \mathcal{L} &= \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) - m_2 g y_2 \end{aligned} \tag{3}$$

The Euler-Lagrangian Equation is the following, and will be used to solve the system:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_1}}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = F \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{5}$$

Use eq. (1) and (2) to substitute for $\dot{x_2}$, $\dot{y_2}$, into the Langrangian equation (3) and then differentiate i.e. solve the Euler-Lagrangian Equation:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2((\dot{x_1} - L\cos(\theta)\dot{\theta})^2 \\ &+(\dot{y_1} - L\sin(\theta)\dot{\theta})^2) - m_2 g(-L\sin(\theta)\dot{\theta}) \end{aligned} \tag{6}$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}((\dot{x_1}^2 - L^2\dot{\theta}^2 \\ &-2L\dot{x_1}\cos(\theta)\dot{\theta}) - m_2 g(-L\sin(\theta)\dot{\theta}) \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x_1}} &= m_1\dot{x_1} + \frac{1}{2}(2\dot{x_1} + 2L\theta\cos(\theta)) \\ &= \dot{x_1}(m_1 + m_2) + \dot{\theta}m_2 L\cos(\theta) \end{aligned} \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_1}}\right) = \ddot{x_1}(m_1+m_2) + m_2L^2(\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)) \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \tag{10}$$

Putting the two terms of the Euler-Lagrangian eq (4) together, we derive that:

$$\ddot{x_1}(m_1+m_2) + m_2L(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) = F \tag{11}$$

Then the second Euler-Lagrangian equation (5) is solved:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2}m_2(2L^2\dot{\theta} - 2L\cos\theta\dot{x_1}) \tag{12}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2}m_2(2L^2\ddot{\theta} - 2L\ddot{x_1}\cos\theta + 2L\sin\theta\dot{x_1}\dot{\theta}) \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2}m_2(2L\sin\theta\dot{x_1}\dot{\theta} + 2gL\sin\theta) \tag{14}$$

Subtracting the two terms for the second Euler-Lagrangian equation (6) gives:

$$0 = m_2L^2\ddot{\theta} - m_2L\cos\theta\ddot{x} - m_2gL\sin\theta \tag{15}$$

The project manual describes these solutions equations in the form of:

$$[M]\begin{bmatrix}\ddot{x_1}\\ \ddot{\theta}\end{bmatrix} = B(\theta, \dot{\theta}, x_1.\dot{x_1}, F)$$

Where M is a 2x2 matrix. We take the solved Euler-Lagrangian solutions (11) and (15) and put them in a form that can be converted easily to this matrix algebra.

$$(m_1+m_2)\ddot{x_1} - m_2L^2\cos\theta\ddot{\theta} = F - m_2L^2\dot{\theta}^2\sin\theta \tag{11}$$

$$-m_2L^2\cos\theta\ddot{x_1} + m_2L^2\ddot{\theta} = m_2gL\sin\theta \tag{15}$$

Our solutions for $\ddot{x}$ and $\ddot{\theta}$ from a Lagrangian approached are then expressed in the matrix form:

$$\begin{bmatrix} m_1+m_2 & -m_2L^2\cos\theta \\ -m_2L^2\cos\theta & m_2L^2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} F - m_2L\dot{\theta}^2\sin\theta \\ m_2gL\sin\theta\end{bmatrix} \tag{16}$$

These accleration equations can then solve for position using Matlab integration like ODE45().

Kyle Jeffrey is a Senior Robotics Engineering Student at the University of California Santa Cruz. He is the Secretary of the Engineering Fraternity Tau Beta Pi and the lead Hardware Engineer at the on campus startup Yektasonics.