Modelling of a 5-Bar-Linkage Manipulator With One Flexible Link

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Abstract In this paper, the model of a 5-bar-linkage robot where the top link is flexible is examined. The modelling process is described and applied to the simpler problem of a single link beam. The model of the 5-bar-linkage robot with the top link flexible is derived and then simplified using various assumptions which are discussed. It is shown that under these assumptions, it may be possible to control two joints using a typical rigid body controller while using the third joint to control the vibrations. This greatly simplifies the control problem.

Introduction

In recent years, there has been a growing interest in the design and control of lightweight robots. Several researchers [1,2,4,7,10] have studied the modelling and control of a single link flexible beam. The robot configuration examined in this paper is the 5-bar-linkage configuration [8,9] (See Figure 1). Here, the 4th link in the robot is assumed to be flexible. The link has a cross-section where the height is much greater than the width. Thus, the beam vibrates primarily in the direction perpendicular to the plane defined by the links of the robot. This prevents the beam from flexing vertically when picking up a load and this is ideal for pick and place operations (See Figure 2). The 5-bar-linkage configuration is used since, in the rigid link case, the inertia matrix of the robot can be designed to be diagonal with two of the diagonal elements also invariant. It is therefore natural to use this configuration when extending to the case where one of the links is flexible.

In this paper, the modelling process is first described. The modelling method is then applied to the single link beam experiment. Finally, the modelling method is used to find the dynamic equations of a 5-bar-linkage robot with the last link flexible.

Rigid 5-Bar-Linkage Manipulator

The rigid 5-bar-linkage manipulator has been extensively studied [8,9] (See Figure 3). It is assumed that all the links lie in a plane and are symmetric in three perpendicular directions. The length of link i is l_i while the distance to the center of gravity of link i is given by l_{ci}. The manipulator is constructed such that the quadrilateral formed by the links is a parallelogram.

Using the method outlined in [11], the nonzero elements of the inertia matrix M are;

$$(M)_{11}=(I_{22}^1C_{q_2}^2+I_{33}^1S_{q_2}^2+m_1l_{c1}^2C_{q_2}^2)+(I_{22}^2S_{q_3}^2+I_{33}^2C_{q_3}^2+m_2l_{c2}^2C_{q_3}^2) \quad (1)$$
$$+(I_{22}^3C_{q_2}^2+I_{33}^3S_{q_2}^2+m_3(l_2C_{q_3}+l_{c1}C_{q_2})^2)+(I_{22}^4S_{q_3}^2+I_{33}^4C_{q_3}^2$$
$$+m_4(l_1C_{q_2}-l_{c4}C_{q_3})^2)$$
$$(M)_{22}=I_{11}^1+m_1l_{c1}^2+I_{11}^3+m_3l_{c1}^2+m_4l_1^2$$
$$(M)_{33}=I_{11}^2+m_2l_{c2}^2+m_3l_2^2+I_{11}^4+m_4l_{c4}^2$$
$$(M)_{23}=(M)_{32}=(m_3l_2l_{c1}-m_4l_{c4}l_1)C_{q_2-q_3}$$

where C_a is $\cos(a)$, S_a is $\sin(a)$, m_i is the mass of link i, and I^i is the inertia tensor of the ith link about a frame fixed symmetrically to the center of gravity of the ith link. The most important advantage of this configuration is that, if $m_3l_2l_{c3}=m_4l_1l_{c4}$, then the inertia matrix becomes decoupled. As well, two of the diagonal elements of the inertia matrix are invariant.

Manipulators With One Flexible Link

The previous section described the inertia matrix of the rigid body 5-bar-linkage manipulator. Once the inertia matrix of a rigid body robot is found, then the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=\tau_j\,,\ j=1,..n \quad (2)$$

where

$L=K-V$=Lagrangian, K=Kinetic Energy, V=Potential Energy

q_j=generalized coordinates, τ_j=generalized forces

can be used. The generalized coordinates here are the joint angles q_1, q_2 and q_3. The generalized forces are the torques τ_1, τ_2 and τ_3 exerted at the three motors. Once the potential energy V is found in terms of the generalized coordinates, the dynamic equations are easily found.

Now suppose the last link is flexible. This means the vibrations can be described by the beam equation

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$$EI\frac{\partial^4 w}{\partial x^4}=-\rho\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where $w(x,t)$ is the deflection of the flexible link at time t and at a distance x from a predefined origin. The equation is subject to four boundary conditions. This is a distributed system and, as such, the simplified version of the Euler-Lagrange equation as presented above cannot be used. However, as done in [1,3,4,7,10], modal expansions can be used and the variable $w(x,t)$ expressed as

$$w(x,t)=\sum_{i=1}^{\infty}\phi_i(x)q_i(t) \tag{4}$$

where the $\phi_i(x)$ are the eigenfunctions of the beam equation subject to the appropriate boundary conditions. Since the coefficients $q_i(t)$ decrease as i increases, it is reasonable to truncate this series after a finite number, say n. Thus, the vibrations are given approximately by

$$w(x,t)=\sum_{i=1}^{n}\phi_i(x)q_i(t). \tag{5}$$

It is reasonable to use as generalized coordinates for this distributed system the variables $q_1, q_2 \ldots q_n$ since the eigenfunctions are orthogonal to each other and these variables describe the position of this system. This allows the use of the Euler-Lagrange equations above.

Thus, when dealing with rigid body manipulators which have one flexible link, the generalized coordinates are the joint angles of the motors and the time-varying coefficients of the eigenfunctions of the vibration modes. The Euler-Lagrange equation can then be used to derive the equations of motion. This will now be done for the single link flexible beam.

Modelling of a Single Link Flexible Robot

In this section, a state space equation is derived for a flexible link driven at one end by a motor. The beam has a cross-section where the height of the beam is much greater than the width, thus constraining the beam to vibrate primarily in the horizontal direction. All deflections of the beam are assumed to be small and shear deformation and rotary inertia effects are ignored. The beam is fixed to the motor hub at one end and is free at the other.

A torque T is applied at the hub of the motor (see Figure 4). The beam has a moment of inertia I_b, a linear mass density ρ, and a length L. The angular rotation of the beam is denoted as $\theta(t)$. Points on the beam have their position fixed by the variable x which is the distance of that point from the hub. The elastic deformation at x is given as $w(x,t)$ while $y(x,t)=w(x,t)+\theta(t)x$ is the total movement of that point. One can describe the deflection $w(x,t)$ as the infinite sum

$$w(x,t)=\sum_{i=1}^{\infty}q_i(t)\phi_i(x) \tag{6}$$

where $\phi_i(x)$ are the normalized clamped-free eigenfunctions [6].

A frame $x'-y'$ is attached rigidly to the point where the beam is attached to the motor hub (see Figure 5). In other words, the frame $x'-y'$ rotates such that the slope of the beam at $x=0$ is zero. The position of a point on the beam at a distance x from the hub, $P(x)$, is given by

$$P(x)=R_{z,\theta}\begin{bmatrix}x\\w\end{bmatrix}=\begin{bmatrix}C_\theta x-S_\theta w\\S_\theta x+C_\theta w\end{bmatrix}. \tag{7}$$

Now, the kinetic energy K and the potential energy V are given by

$$K=\frac{1}{2}I_h\dot{\theta}^2+\frac{1}{2}\int_0^L\dot{P}^t\dot{P}dm \tag{8}$$

$$V=\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2dx$$

where I_h is the hub inertia of the motor. The Lagrangian $L=K-V$ can be found as

$$L=\frac{1}{2}(I_h+I_b)\dot{\theta}^2+\frac{1}{2}\sum_{i=1}^{\infty}\dot{q}_i^2+\dot{\theta}\sum_{j=1}^{\infty}\dot{q}_j\int_0^L\phi_j x dm-\frac{1}{2}\sum_{k=1}^{\infty}q_k^2\omega_k^2. \tag{9}$$

where ω_k are the resonance frequencies of the clamped-free eigenfunctions.

By definition, the q_j's can be considered the generalized coordinates of the system. Thus, one can apply the Euler-Lagrange equations. The equations of motion are therefore

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}-\frac{\partial L}{\partial\theta}=T, \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \ , \ i=1,2\cdots \tag{10}$$

It is easily shown that

$$\ddot{\theta}=\frac{T+\sum_{j=1}^{\infty}q_j\omega_j^2\int_0^L\phi_j x dm}{I_h} \tag{11}$$

$$\ddot{q}_i=-\frac{T}{I_h}\int_0^L\phi_i x dm-q_i\omega_i^2\left(1+\frac{(\int_0^L\phi_i x dm)^2}{I_h}\right)-\sum_{j\neq i}^{\infty}\frac{q_j\omega_j^2\int_0^L\phi_j x dm\int_0^L\phi_i x dm}{I_h}$$

Here, Bessel's equality, $I_b=\int_0^L x^2dm=\sum_{j=1}^{\infty}(\int_0^L\phi_j x dm)^2$, is used.

Thus, the state space equations (written only to the sixth order for simplicity) are

$$\dot{x}=Ax+Bu \quad , \quad y=Cx \tag{13}$$

where

$$x=\begin{bmatrix}\theta\\\dot{\theta}\\q_1\\\dot{q}_1\\q_2\\\dot{q}_2\\q_3\\\dot{q}_3\end{bmatrix} \quad B=\begin{bmatrix}0\\\frac{1}{I_h}\\0\\-\frac{\gamma_1}{I_h}\\0\\-\frac{\gamma_2}{I_h}\\0\\-\frac{\gamma_3}{I_h}\end{bmatrix} \quad C=[\,1\;0\;\phi_1(L)\;0\;\phi_2(L)\;0\;\phi_3(L)\;0\,]$$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1^2\frac{\gamma_1}{I_h} & 0 & \omega_2^2\frac{\gamma_2}{I_h} & 0 & \omega_3^2\frac{\gamma_3}{I_h} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1^2(1+\gamma_1^2/I_h) & 0 & -\omega_2^2\frac{\gamma_2\gamma_1}{I_h} & 0 & -\omega_3^2\frac{\gamma_1\gamma_3}{I_h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_1^2\frac{\gamma_1\gamma_2}{I_h} & 0 & -\omega_2^2(1+\gamma_2^2/I_h) & 0 & -\omega_3^2\frac{\gamma_3\gamma_2}{I_h} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2\frac{\gamma_1\gamma_3}{I_h} & 0 & -\omega_2^2\frac{\gamma_2\gamma_3}{I_h} & 0 & -\omega_3^2(1+\gamma_3^2/I_h) & 0 \end{bmatrix}$$

$$\gamma_i=\int_0^L \phi_i x dm$$

It can be experimentally shown that, for large I_h, the poles of the transfer function converge to the resonance frequencies of the clamped-free eigenfunctions while, for small I_h, they approach the resonance frequencies of the pinned-free eigenfunctions. The resonance frequencies of the model for various hub inertias are calculated and plotted (see Figure 6). Also included in Figure 6 are three experimental results obtained from the experimental setup at the University of Waterloo. There is good agreement in these preliminary results and with the model from Hastings and Book [3].

Model of a 5-Bar-Linkage Manipulator with One Flexible Link

The model of the 5-bar-linkage manipulator with the 4th link flexible is described in this section (See Figure 7). Because links 1,2 and 3 are rigid, we can assume that only the part of link 4 labelled as h_4 in Figure 7 is flexible. From the boundary conditions, one can see that the vibrations of the beam satisfy the clamped-free boundary conditions. Thus, if $w(a,t)$ is the deflection of the elastic portion of link 4 where a is the distance from the point l in Figure 7, then we let

$$w(a,t)=\sum_{i=1}^{\infty} q_i(t)\phi_i(a) \tag{14}$$

where $\phi_i(a)$ are the clamped-free eigenfunctions. The clamped-free eigenfunctions are chosen because the deflection is described relative to a frame attached to the rigid body and situated where the flexible link would be if it were rigid (see Figure 8). There cannot be a nonzero slope at the origin O.

Now, the Euler-Lagrange equations are used to derive the dynamic equations. The first three generalized coordinates are considered to be the q_1, q_2 and q_3 of the rigid body robot. For notational convenience, the generalized coordinates corresponding to the time-varying coefficients of $w(a,t)$ are labelled q_4, q_5, ... Thus

$$w(a,t)=\sum_{i=4}^{\infty} q_i(t)\phi_i(a). \tag{15}$$

where $\phi_4(a)$ is the first mode eigenfunction, $\phi_5(a)$ is the second and so on. The generalized forces for q_1, q_2 and q_3 are the torques exerted at the motors, T_1, T_2 and T_3. The generalized forces for $q_4, q_5 \ldots$ are zero.

For links 1,2 and 3 (see Figure 7), the inertia matrix stays the same since they only depend on $\dot{q}_1,\dot{q}_2$ and $\dot{q}_3$. For link 4, the flexibility must be considered. It is assumed that the portion of link 4 up to point l (see Figure 7) acts rigidly due to the structure of the rigid links. Thus, only the elastic portion labelled as h_4 in Figure 7 needs to be considered. (The rigid portion is considered separately.)

Let us examine the frame obtained after the homogeneous transformations

$$T=Rot_{z,q_1}Rot_{z,f}. \tag{16}$$

where $f=q_3-\pi$. This is the frame at $o-x'y'z'$ (see Figure 9). For this new frame, the vibrations $w(a,t)$ are exclusively in the x'-direction. The location of a point da on the elastic link with respect to this new frame is

$$\begin{bmatrix} w(a,t) \\ -l_1C_{q_2-q_3}+a \\ -l_1S_{q_2-q_3} \end{bmatrix} \tag{17}$$

The location of the point da with respect to the base frame, P_a, is

$$P_a=Rot_{z,q_1}Rot_{z,f}\begin{bmatrix} w \\ y \\ z \end{bmatrix} = P_e+P_r \tag{18}$$

where P_e is the position due to the elasticity of the beam and P_r is the rigid body position of the link. The kinetic energy K of the elastic portion of link 4 is given by

$$K=\frac{1}{2}\rho\int_0^{h_4}\dot{P}_a^t\dot{P}_a da =\frac{1}{2}\rho\int_0^{h_4}\dot{P}_e^t\dot{P}_e da+\rho\int_0^{h_4}\dot{P}_e^t\dot{P}_r da+\frac{1}{2}\rho\int_0^{h_4}\dot{P}_r^t\dot{P}_r da. \tag{19}$$

The last term is the rigid body kinetic energy term and can be added to the other rigid body inertias to form the inertia matrix of the total rigid 5-bar-linkage robot. Thus, denoting the inertia matrix of the rigid 5-bar-linkage robot by M_r, the total kinetic energy, K_t, of the robot is

$$K_t=\frac{1}{2}\dot{q}M_r\dot{q}+\frac{1}{2}\rho\int_0^{h_4}\dot{P}_e^t\dot{P}_e da+\rho\int_0^{h_4}\dot{P}_e^t\dot{P}_r da+I_h^1\dot{q}_1^2+I_h^2\dot{q}_2^2+I_h^3\dot{q}_3^2 \tag{20}$$

where $q=[q_1,q_2,q_3,q_4,\ldots]^t$ and I_h^i is the hub inertia of the ith motor. Thus,

$$K_t=\frac{1}{2}\dot{q}^t M_t\dot{q} \tag{21}$$

where

$$M_t=\begin{bmatrix} M_{11}+\sum_{i=4}^{\infty}\rho q_i^2+I_h^1 & & \mathit{Symmetric} & & \\ -l_1 S_{q_2}\sum_{i=4}^{\infty}\rho q_i\int_0^{h_4}\phi_i da & M_{22}+I_h^2 & & & \\ S_{q_3}\sum_{i=4}^{\infty}\rho q_i\int_0^{h_4}\phi_i a da & & M_{33}^2+I_h^3 & & \\ \rho\int_0^{h_4}\phi_4(-l_1C_{q_2}+aC_{q_3})da & & & \rho & \\ \rho\int_0^{h_4}\phi_5(-l_1C_{q_2}+aC_{q_3})da & & & & \rho \\ \cdot & & & & \\ \cdot & & & & \end{bmatrix}$$

where M_{11}, M_{22}, M_{33} are defined in (1).

To find the dynamic equations of the robot, the potential energy, P, of the robot is also needed. Recall that the dimensions of the beam were chosen so that there is no vertical bending but only horizontal bending. The variable q_1 merely rotates the plane containing the robot. Thus, the potential energy does not depend on q_1, but only on q_2 and q_3. Also, since there is only horizontal bending, the elastic deflections do not change the effective height of the last link. Thus, the deflection of the elastic link does not add or subtract from the gravitational potential energy P_h. It only contributes an elastic potential energy P_e.

Thus,

$$P=P_e+P_h \tag{22}$$

where

$$P_e=\frac{EI}{2}\int_0^{h_4}\left(\frac{\partial^2 w}{\partial a^2}\right)^2 da. \tag{23}$$

$$P_h=m_1gl_{c1}S_{q_2}+m_2gl_{c2}S_{q_3}+m_3g(l_2S_{q_3}+l_{c3}S_{q_2})+m_4g(l_1S_{q_2}-l_{c4}S_{q_3}).$$

Now, the equations of motion can be derived using the Euler-Lagrange equations. The generalized forces are the torques T_1, T_2, and T_3 for the first three generalized coordinates respectively and are zero for the other generalized coordinates.

Simplification of the Dynamic Equations

The previous development leads to complicated expressions. In this section, one method of simplifying the dynamic equations is presented. Several terms may be neglected in the inertia matrix, M_t. First, consider the (1,1) element.

$$(M_t)_{11}=M_{11}(q_2,q_3)+\rho\int_0^{h_4} w^2(a,t)da+I_h^1 \tag{25}$$

Assuming the deflections w are small, the second term may be dropped. Now, consider the (1,2) and (1,3) elements.

$$(M_t)_{12}=-l_1S_{q_2}\rho\int_0^{h_4} w(a,t)da \tag{26}$$

$$(M_t)_{13}=S_{q_3}\rho\int_0^{h_4} w(a,t)a da \tag{27}$$

Since the deflection w is small, these elements are small compared to the $(M_t)_{11}$, $(M_t)_{22}$ and $(M_t)_{33}$ terms. However, if q_2 and q_3 are both equal to $\pm\frac{\pi}{2}$, then the $(M_t)_{j1}$ terms for $j\geq 4$ are zero and the $(M_t)_{12}$ and $(M_t)_{13}$ terms cannot be neglected. The reason the $(M_t)_{j1}$ terms ($j\geq 4$) disappear is that these angles correspond to the configuration with the robot pointing straight up or straight down. In this configuration, rotations of the motor q_1 cannot affect the vibrations and thus the coupling terms are zero. To get around this problem, let us assume that the robot is always at least δ degrees from a straight up or straight down configuration. With a suitable choice of δ, the $(M_t)_{12}$ and $(M_t)_{13}$ terms can be neglected. After dropping these terms, it is possible to again solve for the dynamic equations. The new inertia matrix becomes

$$\begin{bmatrix} M_{11}(q_2,q_3)+I_h^1 & & & & \\ & M_{22}+I_h^2 & & & \\ & & M_{33}+I_h^3 & & \\ \rho\int_0^{h_4}\phi_4(-l_1C_{q_2}+aC_{q_3})da & & & \rho & \\ \rho\int_0^{h_4}\phi_5(-l_1C_{q_2}+aC_{q_3})da & & & & \rho \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix} \tag{28}$$

where M_{11}, M_{22}, and M_{33} are defined in (1). After calculating the coriolis and potential energy terms, the dynamic equations are

$$1.\ (M_{11}(q_2,q_3)+I_h^1)\ddot{q}_1+\sum_{i=4}^{\infty}\rho\int_0^{h_4}\phi_i(-l_1C_{q_2}+aC_{q_3})da\ddot{q}_i+\dot{q}_2\dot{q}_1\xi_1(q_2,q_3) \tag{29}$$

$$+\dot{q}_2\sum_{i=4}^{\infty}\dot{q}_i\rho\int_0^{h_4}\phi_i l_1 da S_{q_2}-\dot{q}_3\sum_{j=4}^{\infty}\dot{q}_j\rho\int_0^{h_4}\phi_j a da S_{q_3}+\dot{q}_3\dot{q}_1\xi_2(q_2,q_3)=T_1$$

$$2.\ (M_{22}+I_h^2)\ddot{q}_2-\frac{1}{2}\xi_1(q_2,q_3)\dot{q}_1^2-\dot{q}_1\sum_{j=4}^{\infty}\dot{q}_j\rho l_1 S_{q_2}\int_0^{h_4}\phi_j da \tag{30}$$

$$+(m_1gl_{c1}+m_3gl_{c3}+m_4gl_1)C_{q_2}=T_2$$

$$3.(M_{33}+I_h^3)\ddot{q}_3-\frac{1}{2}\xi_2(q_2,q_3)\dot{q}_1^2+\dot{q}_1\sum_{j=4}^{\infty}\dot{q}_j\rho S_{q_3}\int_0^{h_4}\phi_j a da \tag{31}$$

$$+(m_2gl_{c2}+m_3gl_2-m_4gl_{c4})C_{q_3}=T_3$$

$$4.\int_0^{h_4}\phi_j(-l_1C_{q_2}+aC_{q_3})da\ddot{q}_1+\ddot{q}_j+l_1S_{q_2}\int_0^{h_4}\phi_j da\dot{q}_1\dot{q}_2 \tag{32}$$

$$-S_{q_3}\int_0^{h_4}\phi_j a da\dot{q}_3\dot{q}_1+q_j\omega_j^2=0,\ \ j=4,5\ \ldots$$

with

$$\xi_1(q_2,q_3)=2C_{q_2}S_{q_2}(-I_{22}^1+I_{33}^1+m_1l_{c1}^2-I_{22}^3+I_{33}^3)-2m_3l_{c1}S_{q_2}(l_2C_{q_3}+l_{c1}C_{q_2})$$
$$-2m_4l_1S_{q_2}(l_1C_{q_2}-l_{c4}C_{q_3})$$
$$\xi_2(q_2,q_3)=2S_{q_3}C_{q_3}(I_{22}^2-I_{33}^2-m_2l_{c2}^2+I_{22}^4-I_{33}^4)-2m_3l_2S_{q_3}(l_2C_{q_3}+l_{c1}C_{q_2})$$
$$+2m_4l_{c4}S_{q_3}(l_1C_{q_2}-l_{c4}C_{q_3})$$

Now, in (30) and (31), the coriolis terms can be written as

$$-\dot{q}_1\rho l_1S_{q_2}\int_0^{h_4}\dot{w}(a,t)da \tag{33}$$

and

$$\dot{q}_1\rho S_{q_3}\int_0^{h_4}\dot{w}(a,t)ada \tag{34}$$

If it can be shown that these terms are small compared to the other terms, then they can be neglected. In that case, (30) and (31) become the equations that one would obtain assuming a rigid robot. In other words, this implies that one could use a typical rigid body controller on joints 2 and 3, and design a controller for joint 1 to damp out the vibrations of the last link.

Future Work

The model in (29-32) is much simpler than the model obtained using (21) and (22). However, more work must still be done on this model. It is important to see how the neglected terms will affect the stability of the system after a controller is designed based on the simpler model. As well, some practical dimensions will be assigned to the robot, and the dynamic equations will be calculated to see if the above assumptions really are valid in practice. Further simplifications of the model will also be studied in the future. The design of a controller will be studied and a prototype of this robot will be built.

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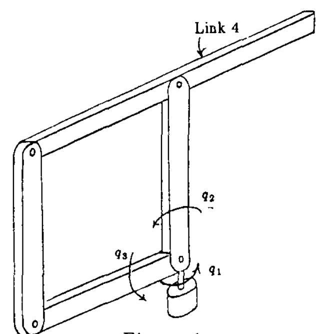

Figure 1

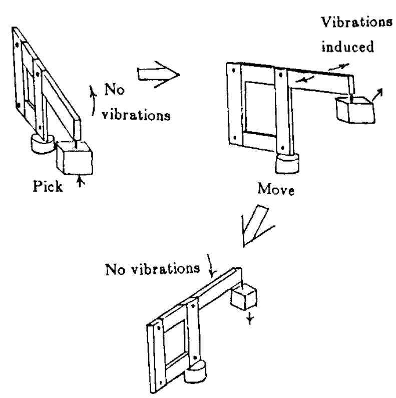

Figure 2

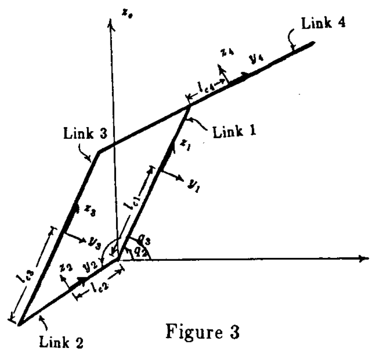

Figure 3

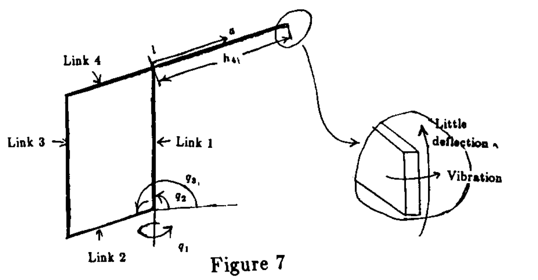

Figure 7

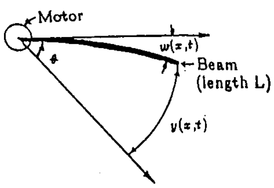

Figure 4

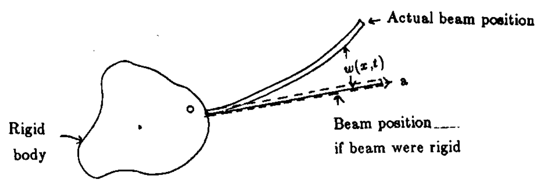

Figure 8

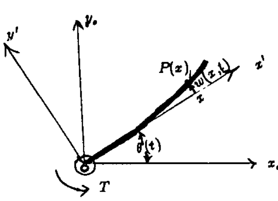

Figure 5

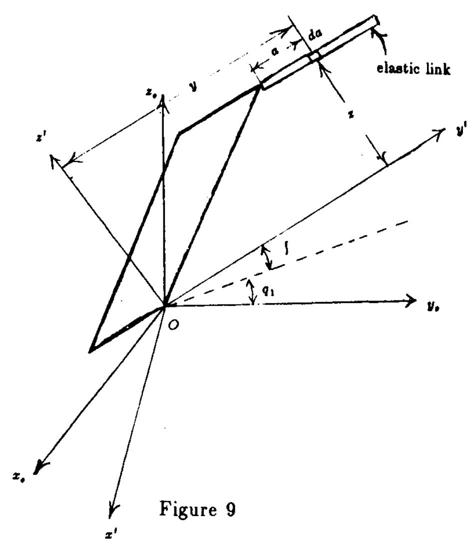

Figure 9

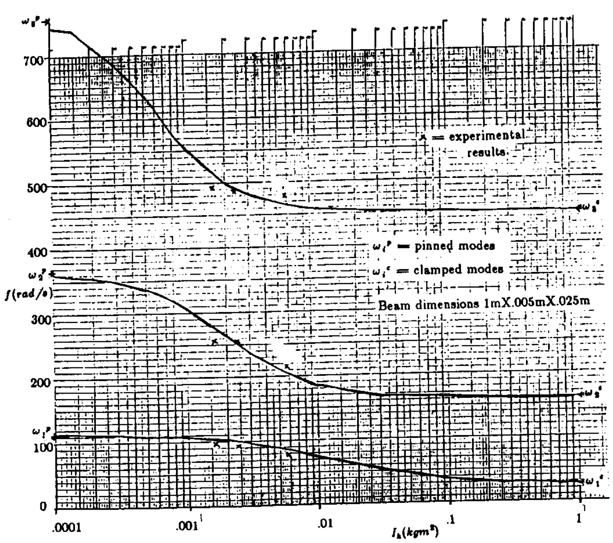

Figure 6