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Quantum Compiler Optimizations

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Abstract

A quantum computer consists of a set of quantum bits upon which operations called gates are applied to perform computations. In order to perform quantum algorithms, physicists would like to design arbitrary gates to apply to quantum bits. However, the physical limitations of the quantum computing device restrict the set of gates that physicists are able to apply. Thus, they must compose a sequence of gates from the permitted gate set, which approximates the gate they wish to apply - a process called *quantum compiling*.

Austin Fowler proposes a method [2] that finds optimal gate sequences in exponential time, but which is tractable for common problems. In this paper, I present several optimizations to this algorithm. While my optimizations do not improve its overall exponential behavior, they improve its empirical performance by one to two orders of magnitude.

1 Background

In classical computing, we can generally rely on the correctness of hardware because of the size of the circuit components. For example, if an atom on a hard disk drive changed its spin orientation, or lost an electron, the hard drive's functionality would not be impaired because it takes many thousands of atoms to represent and store a single bit of data. However, in quantum computing, data is stored in quantum bits, which are represented by tiny particles like trapped ions. These qubits are very easy to perturb, potentially corrupting calculations based on them. Thus, we use redundancy, in the form of error-correcting codes, to minimize the impact of individual errors.

The Steane code is one representation of a quantum bit. It uses seven physical qubits to represent one Steane code qubit, and can tolerate an arbitrary error in one of the seven qubits. We can perform any desired operation on a Steane code qubit by applying a combination of H (Hadamard) and T gate operations [4]. T gates are generally complicated to implement in quantum computing hardware, so we seek to use a minimal number. For practical purposes, in addition to H, we can use the Pauli X operator X, the Pauli Z operator Z, the single qubit phase gate S, and its inverse $S^\dagger$ [**?**]. The gates H, X, Z, S, and $S^\dagger$ generate a group under multiplication, called the Clifford group. Thus, any sequence of gates we choose will alternate between a member of the Clifford group and a T gate. A $T^\dagger$ gate is also used in this implementation, bringing the total number of non-identity gates in Fowler's gate set to 25.

A *single-qubit quantum compiler* finds sequences of gates which yield matrices that are "close" to a gate we would like to apply to a quantum bit. Each gate has a corresponding matrix that represents the operation it would perform on a quantum bit. How close one gate is to another is given by the "Fowler" distance:

$$dist(U, U_l) = \sqrt{\frac{2 - \left| tr\left(U \cdot U_l^\dagger\right) \right|}{2}} \tag{1}$$

The longer the gate sequence is, the more closely it can approximate a desired target gate that is not in the universal instruction set. However, a longer gate sequence takes more time to compute on a real quantum computer, increasing the probability of a computation error. An optimal quantum compiler will find gate sequences which:

1. have a minimal Fowler distance from the target gate.

2. have a minimal length.

2 Fowler's Algorithm

Austin Fowler presents an algorithm that iterates over sequences in order from smallest to largest [?]. For each sequence, it multiplies the sequence gates' matrices together to generate a 2×2 unitary matrix representing the complete operation that sequence would perform. The simple brute-force iteration runs in time exponential in sequence length, since all sequences of length n are produced by appending all elements of the universal instruction set to all sequences of length $n - 1$.

To reduce the run time, Fowler's algorithm intelligently skips redundant sequences. The algorithm creates a list of unique sequences for all sequences of length N. Then, for each sequence S of length $N + 1$, it searches for sub-sequences of length N. If a sub-sequence Y is not in the list, then it is not unique. That means it performs the same operation as a sequence V which is in the unique sequence list. Since Y and V are the same, then if you were to replace Y with V in your sequence S, you would get a sequence W that does the same thing S does.

Since the algorithm iterates over all sequences of length N, it will encounter W anyway (or it has already encountered W). Therefore, it should skip sequence S. In fact, it should increment the sub-sequence Y until it is a unique sequence U. Fowler's algorithm contains a tree lookup structure which, for any given sequence, records the next unique sequence U. The algorithm can determine what sequence to skip to by simply accessing this tree. It still requires time exponential in sequence length, but interesting results can now be obtained in mere days using consumer computer hardware.

As I will demonstrate later, Fowler's tree data structure requires memory that scales exponentially with the sequence length. Thus, the algorithm consists of two stages:

1. During the first stage, it builds the structure until it stores all unique sequences up to length W, where $W = 15$ for most of the experiments.

2. After the structure is built, it enters the second stage, where it generates sequences – and uses the structure to skip them – but it doesn't add them to the structure.

This dramatic change in behavior between stages explains some interesting features in the following graphs. Also, it means I can only infer behavior on longer sequences from behavior in the second stage, which explains my focus on data produced during that stage.

3 Experimental Goal

In order to empirically measure the impact of my optimizations, I need a consistent experimental goal to test on every version of the algorithm. For this research, I chose to approximate the $\frac{\pi}{6}$ gate $\exp(i\frac{\pi}{12}\sigma_z)$ to 10^{-7} accuracy. Since this approximation is currently very time-consuming, I can use it to empirically evaluate the impact of my enhancements.

4 Existing Performance

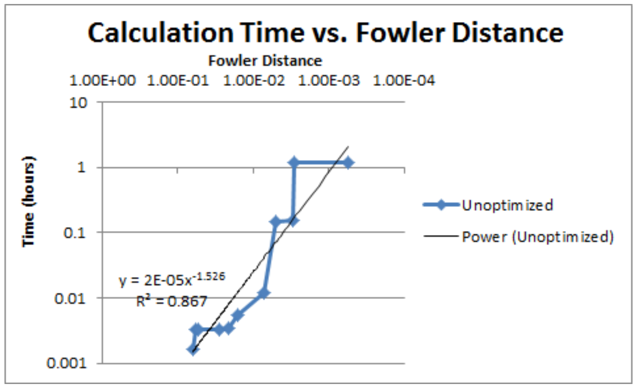
In order to better understand the performance characteristics of the Fowler algorithm, I modified its C source code to obtain performance-related statistics. In this section, I present the data I gathered, along with some explanations for unusual data features and speculations on how the statistics should change for a meaningful performance improvement. Most of these benchmarks ran on an Amazon Elastic Compute Cloud Medium computer, which contains a 2-2.4 GHz processor and 3.75 GB of memory.

4.1 Code Profiling

I ran a profiler (`gprof`) to determine where the performance bottlenecks are. Initially, I thought that memory accesses would dominate the program's runtime, because of the size of the data structures involved. However, the program spends 92.49% of its time inside mathematical functions, meaning that calculation is the dominant operation. In the first stage of the algorithm, the program spends 98.5% of its time checking for unique matrices. In the second stage, it spends most of its time multiplying gate matrices together to calculate the matrix for a given sequence.

4.2 Calculation Time vs. Fowler Distance

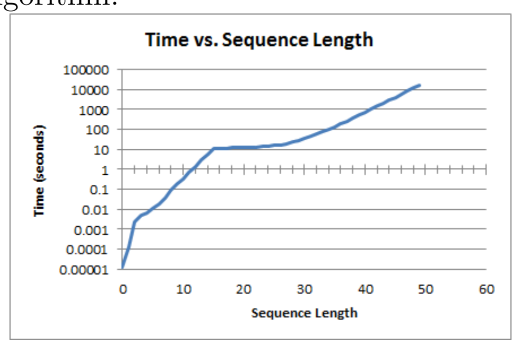
The figure below indicates how much time will be required to obtain a given Fowler distance using Fowler's original source code. For the purposes of this paper, the Fowler source code is "unoptimized", as it does not contain my optimizations and is a baseline for comparison. This graph is perhaps the most important graph of them all, since we often want gates with a certain specific precision.

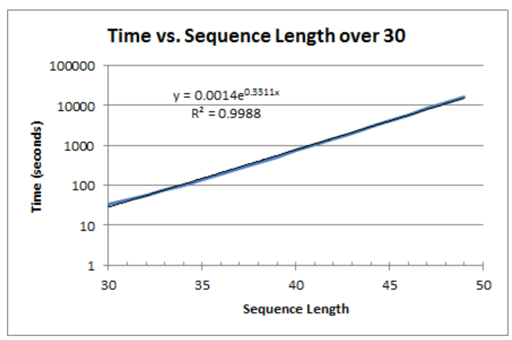


Since there aren't very many unique distances, there are not enough data points to establish a clear trend. A power function appears to fit the data somewhat closely, though. This power function predicts that the unoptimized version of the program will take about 110 years to approximate the gate to a distance closer than 10^{-7}! This massive exponential expansion explains why Fowler's original paper had no gates with a precision better than 10^{-4}, since it would take at least a day for the $\frac{\pi}{6}$ gate to compile to even that precision!

4.3 Time vs. Sequence Length

This metric is related to the above metric because longer sequences tend to have better precision. However, the relationship between time and sequence length is much clearer, as can be witnessed by the much smoother curve. While this graph may not have as much practical significance, it is much easier to relate this graph to the underlying implementation of the algorithm.

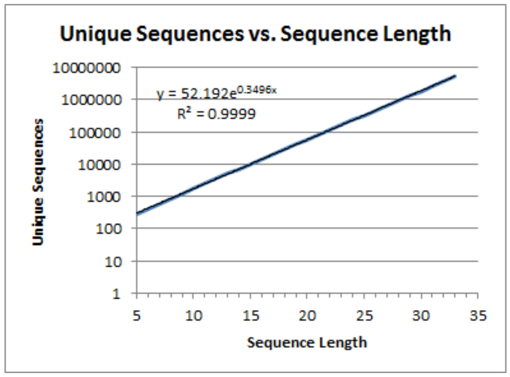




From sequence length 0 to 2, the line has a steep slope. This feature probably exists because the processor cache has not warmed up yet. Between 2 and 15, every sequence generated by the algorithm is checked against a list of unique sequences, to see if it's unique. This check only occurs up to a certain sequence length: 15 in this case. After that, the algorithm speeds up very rapidly until it reaches about a sequence length of 30. Then, the graph becomes a clean exponential curve.

To improve performance, I will effectively need to shift this curve down, producing longer sequences in less time.

4.4 Unique Sequences Per Sequence Length



This metric provides insight into the algorithm's storage requirements. It is clear that Fowler's optimizations have not altered the fundamental exponential nature of the problem. For sequences longer than about 3, the number of unique sequences grows exponentially with the sequence length. Since I am more worried about time rather than space, I will not mind if this curve shifts up. However, I do need to make sure that my optimizations do not consume too much memory.

5 Ways to Improve Performance

To optimize the performance, I need to:

1. Speed up calculations such as matrix multiplication.
2. Reduce the number of calculations required for a given gate sequence length.

There are quite a few possible approaches to approaches 1 and 2. Some of these approaches were taken this quarter, yet others will be left for future work.

5.1 "Meet in the Middle" Bidirectional Search

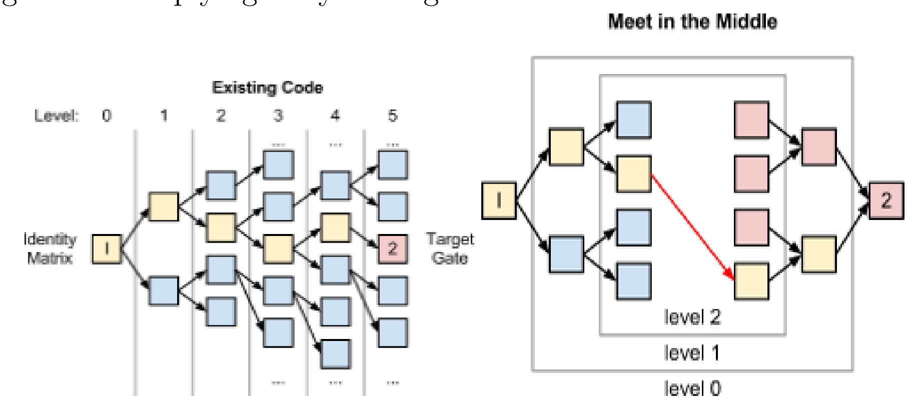
A traditional "uni-directional" search seeks a path from a start state to a goal state by starting from the start state and exploring all possible paths. A bidirectional search starts searching from the goal state as well. Thus, the search paths will "meet in the middle": each search only has to take $\frac{N}{2}$ steps to meet the other search. Thus, instead of taking $O\left(a^N\right)$ time, the algorithm only takes $O\left(a^{\frac{N}{2}}\right)$ time. One will need some data structure to store the paths, but inserting into this data structure does not require exponential time. Thus, for a given amount of time, the algorithm could compute gate sequences that are twice as long. This approach is the most promising, and it was implemented in software.

5.2 Optimized Unique Matrix Lookup

The algorithm checks to see if a matrix is unique by calculating the distance between it and all other matrices. Since 98.5% of the application's run time is spent in this function, optimizing it could yield significant improvements in performance in the first stage. However, in the second stage, no more unique matrix checks are performed; therefore, no time will be spent in this function. Unless the first stage lasts a long time, it may not be worth the implementation trouble. This optimization was easy to implement since the C++ standard template library provides a red-black binary search tree.

6 Bidirectional Search

Searching for the correct gate is like searching through nodes in a tree: for a given sequence of gates, the computer must choose which gate to add to the sequence to come closer to the target gate. In the diagrams below, the arrows represent a choice of gate, and the boxes represent matrices. When an arrow is drawn from some box A to a box B, box B is the matrix resulting from multiplying A by some gate matrix.

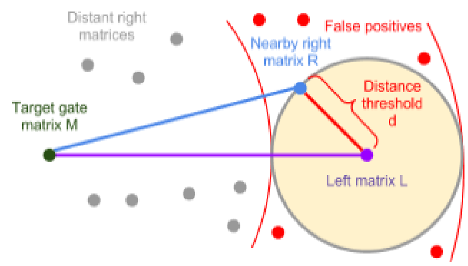


In the example shown in these figures, the existing code must go through five levels of searching in order to reach the target gate. At each new level, the algorithm considers adding all of the available gates to *each* sequence generated by the previous level. Thus, each step multiplies the number of matrices to consider by 25. So, for a sequence of length N, there will be 25^N operations. The "meet in the middle" figure reveals that starting the search from the start and the goal results in the computer exploring fewer levels. Each side would only have to explore half as many levels since the searches meet in the middle. Instead of 25^N operations, the computer can ideally perform $2 \cdot 25^{N/2}$ operations using the MITM (meet in the middle) algorithm.

6.1 The Search Index

The critical component of the MITM algorithm is the structure that allows the paths to connect. This structure effectively creates the red arrow in the MITM figure above, matching up left matrices with right matrices. It must be designed carefully to ensure optimal performance of the algorithm. For a given left matrix, it should find a minimal number of right matrices which are close to the left matrix. Thus, the data structure needs a way to parameterize all of the matrices stored in it, using parameters that are related to the Fowler distance between two matrices.

The simplest approach is to choose some reference matrix M, and store the right matrices in a tree map, using their distances from M as keys. Then, to find right matrices that are "close" to a left matrix L, the algorithm simply measures the distance from L to M, and performs a range query for all right matrices that have about the same distance to M. This trick works because the Fowler distance measure obeys the triangle inequality: if two matrices L and R are within some distance d of each other, then the difference in their distances to some other matrix M will not be greater than d. In the figure below, this fact is true for all matrices inside the circle.



For my implementation, I use the target gate as the reference matrix, and I choose d to be 10^{-10} less than the smallest distance found so far. Since the left matrix must check its distance from the target gate anyway, we can re-use the distance calculation without having to cache it. Note that it is possible for two matrices to be far away from each other while still having the same distance to M. Thus, the range query may return false positives, which are shown between the red lines in the figure. The triangle inequality property simply guarantees that the range query will not leave out potential candidates.

6.2 Building the Structure

For each sequence S the algorithm generates, a corresponding matrix M is generated. M represents the transformation that S would perform on a quantum bit. The algorithm usually assumes that S is a prefix of the solution, meaning that other gates will be added to the end of S to reach the target gate G. However it's also possible to consider S as a suffix, in which gates are added onto the beginning of S. In this case, S would work backwards from G, attempting to come close to the identity matrix, rather than the other way around. If the computer knows M, it can work backwards by multiplying the inverse of M with G to get a matrix M2. Then, prefix sequences can see if S is their suffix by comparing their matrices to M2. If a prefix matrix is close to M2, then it would be close to G if it were multiplied by M.

Therefore, the middle structure simply needs to store as many matrices N as possible, with pointers to their corresponding sequences. It stores a list of binary search trees by sequence length, so that all short sequences can be examined before long sequences.

The middle structure only has so much room to store entries, though. Since the number of unique sequences scales exponentially with the sequence length, the structure can store entries up to some length L before running out of memory. Thus, the MITM algorithm does not always cut the number of search levels in half; instead, it subtracts L from the number of search levels required to find a solution. This approach replaces the $O(25^N)$ cost of exploring sequences of length N with a $O(25^{N-L})$ cost, since a well-optimized middle structure should not have an exponential lookup time.

6.3 Performing the Search

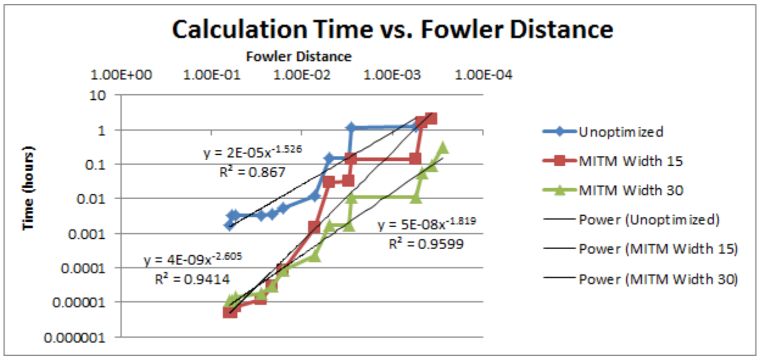
Whenever the algorithm finds a new unique sequence P, it checks the middle structure to see if one of the suffixes S can connect it to the target gate G. Since suffixes are searched by ascending length, the first result should be of optimal length. The search function is given a distance parameter that indicates the maximum tolerable Fowler distance for the match; all matrices that are farther away are skipped. If a result is found, the search function also returns the distance D from P's matrix to S's matrix, so that the distance threshold can be reduced to $D - \epsilon$ (some small value). That way, future searches will only return more precise matches.

One problem that I noted after obtaining my results is that the real sequence may not be of optimal length. The Clifford group contains elements that are composed of multiple real gates, but each Clifford group element is considered to be one gate in this algorithm. Since every sequence alternates between Clifford group elements and T gates, the number of real gates in the sequence of length n returned by the algorithm is about $n/2 + 3(n/2)$. However, the resulting sequence will still have an optimal real length: the Clifford group elements are ordered such that the ones comprised of multiple real gates are visited later by the algorithm, meaning they are added to the structure at a later time. Thus, if the structure uses a stable sort, these longer sequences will be considered later. I am not entirely certain that my structure does so, however, which would be a good topic for future research.

Another potential problem is that a very good suffix may be skipped because a "sufficient" suffix was encountered first. For speed, the MITM algorithm returns the first suffix that is within the desired distance threshold. Technically, if this event occurs, the improved suffix would be discovered at the next search level, so this problem should not impact correctness. However, that means the best result might not be returned as early as possible. One sufficient correction would be to continue the search; it won't impact performance because new sequences are rarely found. This fix could be implemented in future work.

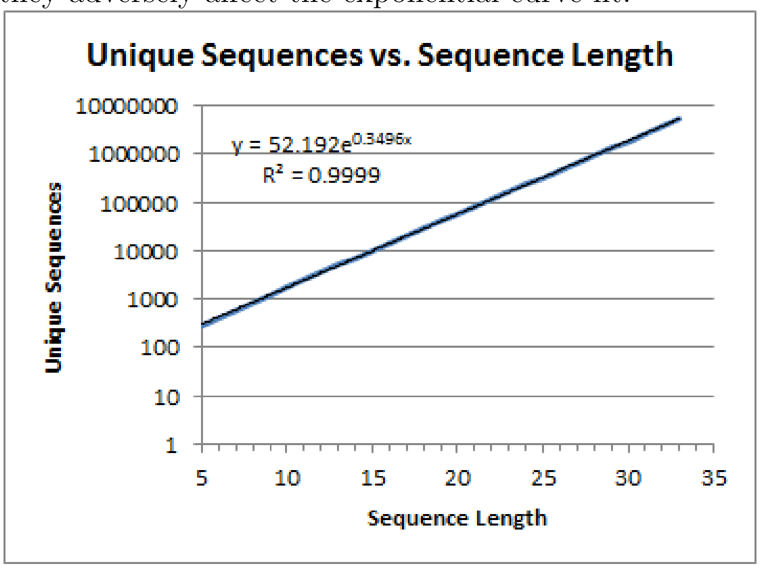
6.4 Results

As the graph below shows, the "meet in the middle" (MITM) optimization improved performance by an order of magnitude. Instead of taking about one hour to calculate a gate sequence that is within 10^{-3} of the target gate, it takes about ten minutes. The Unoptimized and MITM Width 15 lines both used a "width" of 15, meaning that the middle structure and Fowler's data structures stored sequences of length 15. The actual improvement appears to depend on the width of the middle structure: when sequences of length 30 are stored in it, the time is cut by two orders of magnitude instead of one. *Note: "unoptimized" refers to Fowler's existing algorithm without the MITM optimization, not to a simple brute-force enumeration.*

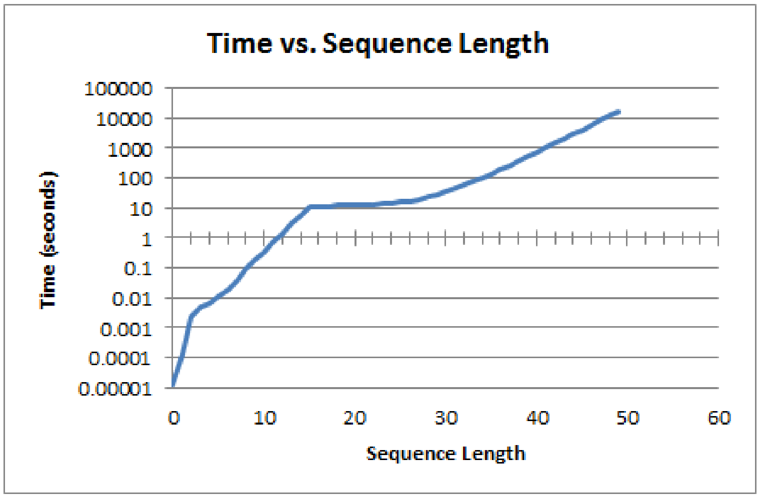


Fowler's unoptimized algorithm also improves performance when the width is increased, because his data structures can cache more data. Thus, it makes sense that increasing the width to 30 from 15 results in a larger improvement than just turning on the MITM optimization.

The memory requirements are much clearer as well: the number of unique sequences increases exponentially with the sequence length. I omitted data for sequences of length less than five because they adversely affect the exponential curve fit.



Finally, I noticed that the number of sequences per unit of time was much larger in the optimized versions than in the unoptimized versions, confirming my hypothesis. It clearly makes sense to keep expanding the middle structure if possible: beyond sequences of length 30, the MITM implementation with width 15 slows down relative to the implementation with width 30. However, in the long run, the MITM optimization does not change the base of the exponential that governs the algorithm run time: notice that all of the lines are roughly parallel towards the right side of the graph.



I managed to approximate the $\frac{\pi}{6}$ gate to 6.8×10^{-5} precision in about 3 hours and 5 minutes. The result is 72 gates long:

$$HTHT(HS)THTHTHT(HS)THT(HS)T(HS)$$
$$T(HS)T(HS)THTHT(HS)THTHT(HXZ)$$
$$THTHTHTHTHT(HS)THTHT(HS)THT(HS)$$
$$THTHTHTHT(HS)THT(HXS)T^{\dagger}$$

7 Change of Basis

Since the Fowler distance is phase independent, we can adjust gates to remove their global phase. Thus, it is possible to represent a quantum gate in $SU(2)$ by using just four real numbers. In the equation below, σ_x, σ_y, and σ_z are the Pauli basis matrices. Since they are multiplied by i, the basis is called the *modified Pauli basis*.

$$\begin{aligned} A &= a_0 \cdot I + a_1 \cdot \sigma_x + a_2 \cdot \sigma_y + a_3 \cdot \sigma_z & (2)\\ &= a_0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + a_3 \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & (3)\\ &= \begin{pmatrix} a_0 + a_3 \cdot i & a_2 + a_1 \cdot i \\ -a_2 + a_1 \cdot i & a_0 - a_3 \cdot i \end{pmatrix} & (4) \end{aligned}$$

One advantage of this new basis is that the trace distance between two gates A and B is just double the dot product of their vectors a and b:

$$\begin{aligned} tr\left(A \cdot B^{\dagger}\right) &= tr\left(\begin{pmatrix} a_0 + a_3 \cdot i & a_2 + a_1 \cdot i \\ -a_2 + a_1 \cdot i & a_0 - a_3 \cdot i \end{pmatrix} \cdot \begin{pmatrix} b_0 + b_3 \cdot i & b_2 + b_1 \cdot i \\ -b_2 + b_1 \cdot i & b_0 - b_3 \cdot i \end{pmatrix}^{\dagger}\right) & (5)\\ &= tr\left(\begin{pmatrix} a_0 + a_3 \cdot i & a_2 + a_1 \cdot i \\ -a_2 + a_1 \cdot i & a_0 - a_3 \cdot i \end{pmatrix} \cdot \begin{pmatrix} b_0 - b_3 \cdot i & -b_2 - b_1 \cdot i \\ b_2 - b_1 \cdot i & b_0 + b_3 \cdot i \end{pmatrix}\right) & (6) \end{aligned}$$

$$= tr\left(\left(\begin{array}{cc} (a_0+a_3i)\cdot(b_0-b_3i)+ \\ (a_2+a_1i)\cdot(b_2-b_1i) & \cdots \\ \cdots & (-a_2+a_1i)\cdot(-b_2-b_1i)+ \\ & (a_0-a_3i)\cdot(b_0+b_3i)\end{array}\right)\right) \quad (7)$$

$$= \begin{array}{c}(a_0+a_3i)\cdot(b_0-b_3i)+(a_2+a_1i)\cdot(b_2-b_1i)+ \\ (-a_2+a_1i)\cdot(-b_2-b_1i)+(a_0-a_3i)\cdot(b_0+b_3i)\end{array} \quad (8)$$

$$= \begin{array}{c}(a_0b_0+a_3b_3-a_0b_3i+b_0a_3i)+(a_2b_2+a_1b_1-a_2b_1i+a_1b_2i) \\ (a_2b_2+a_1b_1+a_2b_1i-a_1b_2i)+(a_0b_0+a_3b_3+a_0b_3i-a_3b_0i)\end{array} \quad (9)$$

$$= 2\left(a_0b_0+a_1b_1+a_2b_2+a_3b_3\right)=2\cdot a\cdot b \quad (10)$$

This result is important because multiplication is an expensive operation in computer calculation, relative to addition. Traditionally, calculating the trace distance between two 2×2 matrices A and B requires one to obtain the diagonal elements of the product AB, which requires 4 complex number multiplications. Since every complex number multiplication requires 4 real-number multiplications, 16 real multiplications must be performed in total. If the matrices are in the modified Pauli basis, on the other hand, only four real multiplications are required.

The other advantage is that multiplying two gates requires only 16 real multiplications. A traditional 2×2 matrix multiplication, on the other hand, requires 8 complex number multiplications, or 32 real multiplications.

The final advantage is storage size: this new basis can be stored in half the space that a full 2×2 matrix would require.

The advantages of this basis are outlined in this table:

Task	**Regular Matrices**	**Pauli Basis**	**Improvement**
Find trace distance	16 real multiplies	4 real multiplies	4x speedup
Multiply matrices	32 multiplies	16 multiplies	2x speedup
Store a matrix	8 real numbers	4 real numbers	1/2 storage

8 Future Work

8.1 Using multidimensional spatial indices for the bidirectional search middle structure

The bidirectional search index only uses one parameter to index the right matrices. For the reference matrices M which I chose, many matrices had similar Fowler distances to M. Thus, while the algorithm was able to avoid iterating over some right matrices, it still had to iterate over many matrices that were not close to a given left matrix. In fact, only .0003% of the matrices returned by the index were actual matches.

The modified Pauli basis offers an excellent way to parameterize the right matrices in a spatial index:

1. Its compact representation requires less space than a full matrix would. In fact, since one can derive one component from any other three components, only three components are strictly required. Space is not the only advantage; certain spatial indices, such as k-d trees, perform better with low-dimensional data. Hardware implementations of the algorithm also benefit from simpler calculation circuitry.

2. Since the trace distance is just the dot product of a left matrix vector a with a right matrix vector b, all right matrices b that are close to some left matrix satisfy this equation:

$$-D \leq a \cdot b \leq D \tag{11}$$

   where D is some constant related to the maximum trace distance between the two gates. Geometrically, this means all of the close right matrices are between two parallel hyperplanes. The process of finding points between the hyperplanes should be straightforward to optimize. Many spatial indices group points into bounding volumes like boxes or spheres; checking to see if these volumes are between the parallel hyperplanes is a simple process.

Libraries such as FLANN [3] provide a wide variety of spatial indices to use.

8.2 Map-Reduce Parallelism

The Fowler algorithm can be broken down into a cycle for each sequence length. Each cycle is essentially a map-reduce job. During the map phase, we assign one gate to each computer, and that computer will consider all sequences of length n which start with that gate. Once all computers have finished the cycle, the reduce phase will merge the data structures for unique matrices, as well as the discovered gate sequences.

There are several advantages to map-reduce parallelism: Unique sequence data structures can be shared with all the units between cycles. Thus, all units can benefit from each unit's work in each subsequent calculation cycle. If you keep track of the data structure contents after the final stage, you can restart the algorithm from this final stage. No specialized hardware (such as a FPGA) is required. Anyone with access to Amazon's Elastic MapReduce service, or a Hadoop cluster, can use a map-reduce algorithm.

Map-reduce parallelism will probably divide the algorithm's run-time for a given sequence length by the number of computers involved. Thus, if there are 25 computers (for 25 gates), then the algorithm ought to run up to 25 times faster. However, since all of the computers must merge their data after each cycle, the faster computers must wait for the slower ones. Due to the complexity of the map-reduce setup, this method was not implemented this quarter. However, Amazon provides a map-reduce framework that should be straightforward to use and scale, should someone decide to adapt the program.

9 Related Work

A variation of the MITM algorithm was independently invented by researchers at the Institute for Quantum Computing at the University of Waterloo [1]. This group also seeks to find quantum circuits of optimal length implementing a given quantum gate. There are a few key differences between their research and the work presented here:

1. Their work applies the algorithm to multiple-qubit gates, and does not combine it with Fowler's algorithm.

2. They focus on finding *exact* matches, rather than approximate ones.

Their future work may benefit from the approximate matching technique discussed in this paper, as well as the brief discussion of using spatial indices and a change of basis to accelerate matching. My research will benefit from their more rigorous treatment of the algorithm, as well as its extension to multiple qubits.

10 Summary

I considered a variety of optimizations to Fowler's quantum compiler algorithm. Then, I implemented the "meet in the middle" algorithm in software, as well as a change of basis technique, and I presented the results here. While the algorithm certainly provides a dramatic performance boost, it also requires a lot of memory to maintain the middle index structure I introduced. Future work involves using map-reduce parallelism and better spatial indices to improve performance.

11 Acknowledgements

I performed most of this research independently, but received significant guidance and assistance from the following individuals and organizations. Without their involvement, this research project would not have happened!

1. **Paul Pham** – the UW graduate student who suggested the research topic for this project, and who provided essential quantum computing context and advice. I had weekly meetings with him, and I worked with him on his pulse sequence board two years ago. He is working on his own quantum compiler based on the Solovay-Kitaev Theorem.

2. **Austin Fowler** – a Research Fellow in Quantum Computer Science at the University of Melbourne. He wrote the original paper describing the sequence-skipping optimization, upon which my research is based. He also supplied the C source code to his algorithm, so that I could test my optimizations.

3. **Aram Harrow** – my faculty advisor, who came up with smart suggestions for error accumulation analysis and calculation optimization. He also indirectly proposed the MITM algorithm at the beginning of this research project.

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