Lecture 2

Econ 2001

2015 August 11

Lecture 2 Outline

1. Fields
2. Vector Spaces
3. Real Numbers
4. Sup and Inf, Max and Min
5. Intermediate Value Theorem

Announcements*:*
- Friday's exam will be at 3pm, in WWPH 4716; recitation will be at 1pm. The exam will last an hour.

Number Concept

- "A satisfactory discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept." (Rudin)
- What does this mean? All those concepts use some idea of distance between points.
- Distance is reasonably easy to think about when dealing with points on the real line... but we want a concept that works for abstract sets.
- So, we need a tool that can give meaning to the idea that points are "close" or "far".
- Concepts like close and far imply the existence of some well behaved unit of measure (Rudin's "number concept").
- This unit of measureis the set of real numbers.
- Next, describe the properties one wants from an abstract number system.

Ordered Sets

Definition

Let A be a set. An order on A is a relation denoted by $<$ with the following two properties

- If $x \in A$ and $y \in A$ then one and only one of the following statements is true

$$x < y, \qquad x = y, \qquad x > y$$

- If $x, y, z \in A$

$$x < y \text{ and } y < z \qquad \Rightarrow \qquad x < z$$

- An ordered set is a set on which an order is defined.

Remark

- These definitions make no mention of numbers: very abstract sets can have an order.

Notation

- The notation $x \leq y$ indicates $x < y$ or $x = y$ or both.
  - Therefore $x \leq y$ means $\neg(x > y)$

Fields

- A number concept should allow basic operations like addition and multiplication.
- A binary operation combines two elements of a set to produce another element of that set.
  - A binary operation on X is a function from $X \times X$ to X.
- Spaces on which these operations are possible have special structure, so that such functions are well defined, and have familiar and useful properties.

- A field is a set in which *addition* and *multiplication* satisfy standard properties, like commutative and associative, and where *zero* and *one* can be defined and have the usual meaning.
  - elements of a field are "number-like"
    - $\mathbb{R}$ (the set of all real numbers) is the main example of field.

Fields I

Definition

A field $(F, +, \cdot)$ is a 3-tuple consisting of a set F and two binary operations $+, \cdot : F \times F \to F$ such that the following nine properties hold

1. Associativity of $+$:
$$\forall \alpha, \beta, \gamma \in F,\ (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$
2. Commutativity of $+$:
$$\forall \alpha, \beta \in F,\ \alpha + \beta = \beta + \alpha$$
3. Existence of additive identity:
$$\exists ! 0 \in F \text{ such that } \forall \alpha \in F,\ \alpha + 0 = 0 + \alpha = \alpha$$
4. Existence of additive inverse:
$$\forall \alpha \in F\ \exists ! (-\alpha) \in F \text{ such that } \alpha + (-\alpha) = (-\alpha) + \alpha = 0$$

   Define $\alpha - \beta = \alpha + (-\beta)$

Fields II

Definition

5. Associativity of $\cdot$:

$$\forall \alpha, \beta, \gamma \in F,\ (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

6. Commutativity of $\cdot$:

$$\forall \alpha, \beta \in F,\ \alpha \cdot \beta = \beta \cdot \alpha$$

7. Existence of multiplicative identity:

$$\exists ! 1 \in F \text{ s.t. } 1 \neq 0 \text{ and } \forall \alpha \in F,\ \alpha \cdot 1 = 1 \cdot \alpha = \alpha$$

8. Existence of multiplicative inverse:

$$\forall \alpha \in F \text{ s.t. } \alpha \neq 0\ \exists ! \alpha^{-1} \in F \text{ s.t. } \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$$

   Define $\frac{\alpha}{\beta} = \alpha \beta^{-1}$.

9. Distributivity of multiplication over addition:

$$\forall \alpha, \beta, \gamma \in F,\ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Examples of Fields

Examples: Fields

Using the standard definitions of addition and multiplication:

- $\mathbf{R}$ is a field (verify this).
- $\mathbf{Q}$ is a field (verify this).
- $\mathbf{N}$ is not a field: no additive identity.
- $\mathbf{Z}$ is not a field; no multiplicative inverse for 2.

Ordered Fields

Definition

An ordered field is a field F which is also an ordered set such that

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$.
2. $xy > 0$ if $x, y \in F$, $x > 0$, and $y > 0$.

- The set of real numbers **R** is a field which has an order (less than or eqaul to) and another property.
- The order is defined by the binary relation less than or equal to.

Axioms for $\mathbb{R}$

The real numbers

- **R** is a field (with the "usual" $+$ and $\cdot$, where additive identity defines 0 and multiplicative identity defines 1) that also satisfies the following axioms.

1. **Order Axiom:** There is a complete ordering $\leq$. That is $\leq$ is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) binary relation such that

$$\forall \alpha, \beta \in \mathbf{R} \text{ either } \alpha \leq \beta \text{ or } \beta \leq \alpha$$

   Note, the order is compatible with $+$ and $\cdot$, i.e.

$$\forall \alpha, \beta, \gamma \in \mathbf{R} \begin{cases} \alpha \leq \beta & \Rightarrow \quad \alpha + \gamma \leq \beta + \gamma \\ \alpha \leq \beta, 0 \leq \gamma & \Rightarrow \quad \alpha\gamma \leq \beta\gamma \end{cases}$$

   (notation: $\alpha \geq \beta$ means $\beta \leq \alpha$, and $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$).

2. **Completeness Axiom:** Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$ satisfy

$$\ell \leq h \;\; \forall \ell \in L, \text{ and } \forall h \in H$$

   Then

$$\exists \alpha \in \mathbf{R} \text{ such that } \ell \leq \alpha \leq h \;\; \forall \ell \in L, \; \forall h \in H$$

Real Numbers and Rational Numbers

- Rational numbers fail completeness.
- In other words, the set of rationals has "holes".
  - Let A be all positive rationals p such that $p^2 < 2$ and B consist of all positive rationals p such that $p^2 > 2$. A contains no largest number and B contains no smallest (this was a question in Problem Set 1).
  - Yet, there is no rational q such that $q^2 = 2$ (proved yesterday in class).
- This is true even though given any two rationals $m < n$ there aways exist another rational q such that $m < q < n$ ($q = \frac{m+n}{2}$).
- The set of real number does not have the same holes.
  - For every real $x > 0$ and every integer $n > 0$ there is a unique positive real y such that $y^n = x$ (this defines $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$).
- The reals fill the holes in the rationals (the rational are dense in the reals).

Upper and Lower Bounds

- Since **R** is an ordered field, given one of its subsets we can talk about the real numbers that are larger or lower than any element of that subset.

Definition

Given a set $X \subseteq \mathbf{R}$,

- u is an upper bound for X if

$$u \geq x \qquad \forall x \in X$$

- ℓ is a lower bound for X if

$$\ell \leq x \qquad \forall x \in X;$$

- X is bounded above if there exists an upper bound for X.
- X is bounded below if there exists a lower bound for X.

Sups and Infs

- One can define the smallest among the upper bounds and the largest among the lower bounds.

Definition

Suppose $X \subseteq \mathbf{R}$ is bounded above. The supremum of X, written $\sup X$, is the smallest upper bound for X; that is, $\sup X$ satisfies

1. $\sup X \geq x \; \forall x \in X$ — $\sup X$ is an upper bound
2. $\forall y < \sup X$, $\exists x \in X$ such that $x > y$ — there is no smaller upper bound

Definition

Suppose $X \subseteq \mathbf{R}$ is bounded below. The infimum of X, written $\inf X$, is the greatest lower bound for X; that is, $\inf X$ satisfies

1. $\inf X \leq x \; \forall x \in X$ — $\inf X$ is a lower bound
2. $\forall y > \inf X$, $\exists x \in X$ such that $x < y$ — there is no greater lower bound

Sups and Infs

NOTATION

- If X is not bounded above, write $\sup X = \infty$.
- If X is not bounded below, write $\inf X = -\infty$.
- Convention: $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

Remark

- $\sup X$ and $\inf X$ need not be elements of X.

Examples

- $\sup(0,1) = 1 \notin (0,1)$
- $\inf(0,1) = 0 \notin (0,1)$.
- How about $(0,1]$?

The Supremum Property

The Supremum Property

- Every nonempty set of real numbers that is bounded above has a supremum which is a real number.
- Every nonempty set of real numbers that is bounded below has an infimum which is a real number.

Theorem

The Supremum Property and the Completeness Axiom are equivalent.

- This is an if and only if statement.
- Proof in the next two slides.

Proof of the Supremum Property I

Proof.

$\boxed{\text{Assume the Completeness Axiom}}$ and show that $\sup X$ and $\inf X$ exist and are a real numbers.

- Let $X \subseteq \mathbf{R}$ be a nonempty set that is bounded above. Let U be the set of all upper bounds for X.
- Since X is bounded above, $U \neq \emptyset$. If $x \in X$ and $u \in U$, $x \leq u$ since u is an upper bound for X.
- So
$$x \leq u \ \forall x \in X, u \in U$$
- By the Completeness Axiom,
$$\exists \alpha \in \mathbf{R} \text{ such that } x \leq \alpha \leq u \quad \forall x \in X, u \in U$$
α is an upper bound for X, and it is less than or equal than every other upper bound for X, so it is the least upper bound for X, so $\sup X = \alpha \in \mathbf{R}$.
- The case in which X is bounded below is similar. (show it)
- Thus, the Supremum Property holds. $\square$

Proof of the Supremum Property II

Proof.

$\boxed{\text{Assume the Supremum Property}}$ and show that the completeness axiom holds.

- Suppose $L, H \subseteq \mathbf{R}$, $L \neq \emptyset \neq H$, and

$$\ell \leq h \; \forall \ell \in L, h \in H$$

- Since $L \neq \emptyset$ and L is bounded above (by any element of H), $\sup L$ exists and it is a real number, so let $\alpha = \sup L$.
- By the definition of supremum, α is an upper bound for L, so

$$\ell \leq \alpha \; \forall \ell \in L$$

- Suppose $h \in H$. Then h is an upper bound for L, so by the definition of supremum, $\alpha \leq h$.
- Therefore, we have shown that

$$\ell \leq \alpha \leq h \; \forall \ell \in L, \; \forall h \in H$$

so the Completeness Axiom holds. □

The Archimedean Property

The Supremum Property: *Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.*

- The supremum property is useful to prove other properties of real numbers

The Archimedean Property

$$\forall x, y \in \mathbf{R},\ y > 0,\ \exists n \in \mathbf{N} \text{ such that } x < ny = \underbrace{(y + \cdots + y)}_{n \text{ times}}$$

Proof.

Problem Set 2.
This is a proof by contradiction, using the Supremum Property. $\square$

Maximum and Minimum

Definition

Given a set $X \subseteq \mathbf{R}$, we say a is a maximum for X (denoted $\max X$) if

$$a \in X \qquad \text{and} \qquad a \geq x \ \forall x \in X$$

and we say b is a minimum for X if

$$b \in X \qquad \text{and} \qquad b \leq x \ \forall x \in X$$

- Maximum and minimum do not always exist even if the set is bounded, but the sup and the inf do always exist if the set is bounded.
- If sup and inf are also elements of the set, then they coincide with max and min.

Simple Result

Theorem

Given a set $X \subseteq \mathbf{R}$, if $\max X$ exists it is equal to $\sup X$.

Proof.

Let $a \equiv \max X$. We need to show that a equals $\sup X$. To do this we must show: that (1) a is an upper bound, and (2) for every other upper bound a', we have $a' \geq a$.

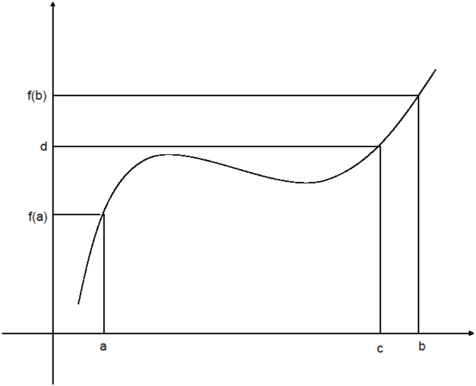
1. is easy: $\forall y < \sup X$, $\exists x \in X$ such that $x > y$
   - a is an upper bound on X since $a \geq x \ \forall \, x \in X$ by definition of $\max X$.
2. is also easy: let $a' \neq a$ be an upper bound $(a' \geq x \ \forall \, x \in X)$ and suppose $a' < a$.
   - Since $a \in X$ (because $a = \max X$), this yields an immediate contradiction (a' cannot be an upper bound). $\square$

- Prove that if $\min X$ exists it is equal to $\inf X$ as exercise.

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Let $a, b \subset \mathbf{R}$. Suppose $f : [a, b] \to \mathbf{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.



- Proof: construct $B = \{x \in [a, b] : f(x) < d\}$ and then show $c = \sup B$.

Proof of the Intermediate Value Theorem I

$\boxed{f : [a, b] \to \mathbf{R} \text{ continuous with } f(a) < d < f(b)} \Rightarrow \boxed{\exists c \in (a, b) \text{ s.t } f(c) = d}$

- This proof uses the Supremum Property.

Proof.

- Let
$$B = \{x \in [a, b] : f(x) < d\}$$
- $a \in B$, so $B \neq \emptyset$; and $B \subseteq [a, b]$, so B is bounded above
- By the Supremum Property, $\sup B$ exists and is a real number, so let $c = \sup B$.
- Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$.
- Assume $f(c) \not< d$ and $f(c) \not> d$.
  - Since $f(c) \not< d$ and $f(c) \not> d$, we must have $f(c) = d$ (by the order axiom).
  - Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$.
- This finishes the proof... but we still have to show that $f(c) \not< d$ and $f(c) \not> d$.

$\square$

Proof of the Intermediate Value Theorem II

Proof.

Need to show that $f(c) \not< d$. Suppose not: $f(c) < d$.

- Since $f(b) > d$, we have $c \neq b$, and therefore $c < b$.
- Let $\varepsilon = \frac{d - f(c)}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$
\begin{array}{rcrcl}
|x - c| < \delta & \Rightarrow & |f(x) - f(c)| & < & \varepsilon \\
 & & f(x) & < & f(c) + \varepsilon \\
 & & & = & f(c) + \frac{d - f(c)}{2} \\
 & & & = & \frac{f(c) + d}{2} \\
 & & & < & \frac{d + d}{2} \\
 & & & = & d
\end{array}
$$

- hence $f(x) < d$, and therefore $(c, c + \delta) \subseteq B$ (remember $B = \{x \in [a, b] : f(x) < d\}$),
- this implies $c \neq \sup B$, a contradiction; therefore we have $f(c) \not< d$. $\square$

Proof of the Intermediate Value Theorem III

Proof.

Need to show that $f(c) \not> d$. Suppose not: $f(c) > d$.

- Since $f(a) < d$, $a \neq c$, so $c > a$.
- Let $\varepsilon = \frac{f(c)-d}{2} > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$
\begin{array}{rcl}
|x-c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| & < & \varepsilon \\
f(x) & > & f(c) - \varepsilon \\
& = & f(c) - \frac{f(c)-d}{2} \\
& = & \frac{f(c)+d}{2} \\
& > & \frac{d+d}{2} \\
& = & d
\end{array}
$$

  so $(c-\delta, c+\delta) \cap B = \emptyset$ (remember $B = \{x \in [a,b] : f(x) < d\}$).
- Thus, either there exists $x \in B$ with $x \geq c + \delta$ (in which case c is not an upper bound for B) or $c - \delta$ is an upper bound for B (in which case c is not the least upper bound for B);
- in either case, $c \neq \sup B$, a contradiction; thus $f(c) \not> d$. $\square$

Implications of the Intermediate Value Theorem

- The intermediate value theorem can be used in all sorts of ways

Corollary

There exists $x \in \mathbf{R}$ *such that* $x^2 = 2$*.*

Proof.

Let $f(x) = x^2$, for $x \in [0, 2]$.

- f is continuous (we proved this yesterday).
- $f(0) = 0 < 2$ and $f(2) = 4 > 2$,
- so by the Intermediate Value Theorem, there exists $c \in (0, 2)$ such that $f(c) = 2$, i.e. such that $c^2 = 2$.

$\square$

- In Problem Set, you have to show (wihout cheating) that for every real number $x > 0$ and every integer $n > 0$ there is a unique positive real number y such that $y^n = x$.
    - This defines $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Vector Space

- Now that we have defined a good number concept, we define useful properties of the set that collects the objects of interest.
- This set should have (at least) two minimal properties:
  - one should be able to add two of its elements, and
  - one should be able to scale up and down any of its elements.
    - These properties are useful to give meaning to the idea that elements of a set are large or small, and close or far.

- A vector (or linear) space is a set with linear structure and where some basic operations are possible.
  - The basic operations are *addition* and *multiplication by a scalar* (an element of a field), and one wants them to have the usual properies.
  - Linear structure means that the sum of any two objects is also an element of the set (*closure under addition*), and that a scalar multiple of any object is also in the set (*closure under multiplication by a scalar*).
  - The set should contain a "zero" element, and multiplying any element by the scalar one should not change it.
- Elements of a vector space are "vector-like".
  - $\mathbb{R}^n$ (the set of n-dimensional real numbers) is the main example of vector space.

Vector Spaces I

Definition

A vector space V is a 4-tuple $(V, F, +, \cdot)$ where V is a set of elements, called *vectors*, F is a field, a binary operation $+ : V \times V \to V$ called vector addition, and an operation $\cdot : F \times V \to V$ called scalar multiplication, satisfying

1. Closure under addition
$$\forall \mathbf{x}, \mathbf{y} \in V \qquad (\mathbf{x} + \mathbf{y}) \in V$$
2. Associativity of $+$:
$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V \qquad (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$
3. Commutativity of $+$:
$$\forall \mathbf{x}, \mathbf{y} \in V \qquad \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$
4. Existence of vector additive identity:
$$\exists! \mathbf{0} \in V \quad \text{such that} \quad \forall \mathbf{x} \in V,\ \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

Vector Spaces II

Definition

5. Existence of vector additive inverse:

$$\forall \mathbf{x} \in V \ \exists! \mathbf{y} \in V \quad \text{such that} \quad \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = \mathbf{0}$$

   We define $(-\mathbf{x})$ to be this unique $\mathbf{y}$ (and let $\mathbf{x} - \mathbf{y}$ be $\mathbf{x} + (-\mathbf{y})$).

6. Distributivity of scalar multiplication over vector addition:

$$\forall \alpha \in F, \ \forall \mathbf{x}, \mathbf{y} \in V \qquad \alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$$

7. Distributivity of scalar multiplication over scalar addition:

$$\forall \alpha, \beta \in F, \ \forall \mathbf{x} \in V \qquad (\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$$

8. Associativity of $\cdot$:

$$\forall \alpha, \beta \in F, \ \forall \mathbf{x} \in V \qquad (\alpha \cdot \beta) \cdot \mathbf{x} = \alpha \cdot (\beta \cdot \mathbf{x})$$

9. Multiplicative identity:

$$\forall \mathbf{x} \in V \qquad 1 \cdot \mathbf{x} = \mathbf{x}$$

Subspaces

In this class, vector spaces are always over real numbers.

Definition

A vector space V over $\mathbb{R}$ is $(V, \mathbb{R}, +, \cdot)$ where V is a set of vectors, $+ : V \times V \to V$ is vector addition, $\cdot : \mathbb{R} \times V \to V$ is scalar multiplication, s.t.

1. Closure under addition: the sum of two vectors is a vector.
2. Associativity of $+$.
3. Commutativity of $+$.
4. There exists a unique vector we call **zero**.
5. There exists a unique 'negative' of any vector.
6. Distributivity of scalar multiplication over vector addition (the scalar in $\mathbb{R}$).
7. Distributivity of scalar multiplication over scalar addition (the scalar in $\mathbb{R}$).
8. Associativity of $\cdot$ (the scalar in $\mathbb{R}$).
9. The real number 1 multiplied by a vector yields that same vector (uses $\mathbb{R}$).

Definition

A subset of a vector space that is itself a vector space is called a subspace.

Euclidean Spaces

Definition

For each positive integer n the corresponding Euclidean space, denoted $\mathbf{R}^n$, is the set of all ordered n-tuples

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$

where each of the $x_1, ..., x_n$ are real numbers (sometimes called coordinates).

- The elements of $\mathbf{R}^n$ are vectors (or points).

Notation

- If $\mathbf{x}$ and $\mathbf{y}$ are vectors and if α is a real number:

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \\ \alpha\mathbf{x} &= (\alpha x_1, \alpha x_2, ..., \alpha x_n) \end{aligned}$$

- The zero element of $\mathbf{R}^n$ is the vector

$$\mathbf{0} = (0, ..., 0)$$

Examples of Vector Spaces

Euclidean Spaces are Vector Spaces

R and $\mathbf{R}^n$ are vector spaces (verify this).

The Set of All Real Valued Functions is a Vector Space

Let X be a non empty set; the set of all functions $f : X \to \mathbf{R}$ is a vector space.

- vector addition works since we defined sums of functions as: $(f + g)(x) = f(x) + g(x)$
- scalar multiplication also works since we defined as: $(\alpha f)(x) = \alpha(f(x))$
- vector additive identity: 0 is the function which is identically zero ($f(x) = 0$ for any x).
- vector additive inverse: $(-f)(x) = -(f(x))$

NOTE

- If we take $X = \{1, 2, ..., n\}$, this is the set of all vectors in $\mathbf{R}^n$.
- If we take $X = \{1, 2, ...\}$, this is the set of all infinite real sequences.

Tomorrow

- Introduce the idea of distance between two elements of a set; introduce the idea of length of a vector, and look at special vector spaces where length is a well defined concept; focus on sets where distance coincides with the length of the difference; sequences and limits are defined using thse concepts.

1. Metric and Metric Spaces
2. Norm and Normed Spaces
3. Sequences and Subsequences
4. Convergence
5. Monotone and Bounded Sequences