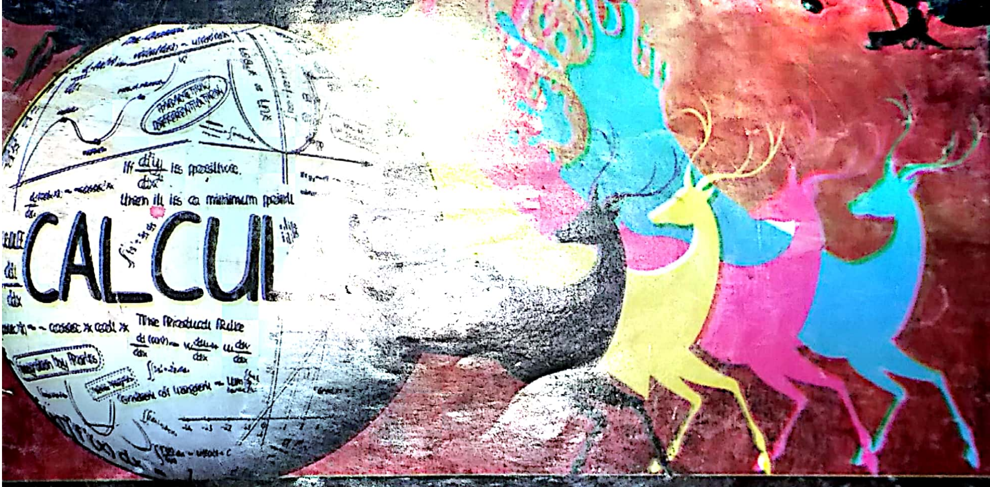
MATHEMATICS

in my BONES

COMPILED BY O'KRI[illegible]O

MTH102



SOLUTION MANUAL

FOR YEAR 1 & PRE-DEGREE; LECTURE NOTE, PAST QUESTIONS AND DETAILED SOLUTIONS

MTH102 TABLE OF CONTENT/COURSE OUTLINE

FUNCTIONS

Definition: Let A and B be two non-empty sets. A function f from A into B denoted by $f: A \to B$ is a rule or relation that assigns a unique element A to an element of B. A is called the domain of f while B is called the co-domain of f.

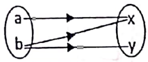
Let f be a function for each x belonging to the domain of f, the corresponding element assigned by f is denoted by $f(x)$ and is called the image of x under f.

The rule is often expressed in the form of an equation $y = f(x)$ with the provision that for any input x there is a unique value for y.

Different outputs are associated with different inputs- the function is said to be single valued because for a given input there is only one output. For instance, the equation: $y = 3x + 5$, expresses the rule 'multiply the value of x by three and add five' and this rule is the function.

On the other hand the equation $y^2 = x$, which is the same as $y = \pm\sqrt{x}$, expresses the rule "take the positive and negative square roots of the value of x". This rule is not a function because to each value of the input $x > 0$ there are two different values of output y.

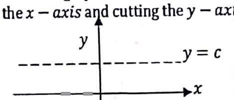
A relation fails to be a function of an element if the domain maps to more than one element in the co-domain



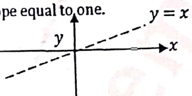
SPECIAL TYPES OF FUNCTIONS

Some functions occur frequently in calculus and they can be recognized by their special names

1. THE CONSTANT FUCTION: This is given as $f(x) = c$ where c is a constant the graph of this function is a straight line parallel to the $x - axis$ and cutting the $y - axis$



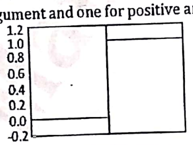
2. THE IDENTITY FUNCTION: This is given as $f(x) = x$ the graph of the identity function is a straight line through the origin with slope equal to one.



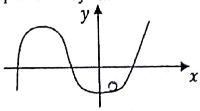
3. THE HEAVISIDE FUNCTION: This is given as

$$f(x) = \begin{cases} 0, x \leq 0 \\ 1, x > 0 \end{cases}$$

It is a discontinuous function whose value is zero for negative argument and one for positive argument.

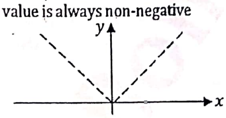


4 THE POLYNOMAL FUNCTION: This is given by $f(x) = ax^n + bx^{n-1} + \cdots + cx + d$ where $a, b, \ldots, c, d$ are constants and n is a non negative integer the degree of the polynomial is n provided $a \neq 0$. If $n = 2$, we have the quadratic function i.e $ax^2 + bx + c$. The graph of the polynomial function depends very much on the degree of the polynomial.



5 THE ABSOLUTE VALUE FUNCTION: It is given as $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

Its value is always non-negative

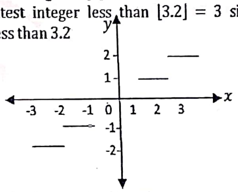


6. THE GREATEST INTEGER FUNCTION

This is given as- $f(x) = \lfloor x \rfloor$ (i.e the greatest integer less or equal to x).

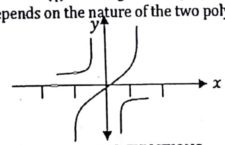
If the number x is an integer use that integer. If the number is not an integer, use the next smaller integer. For instance the greatest integer of $\lfloor 4 \rfloor = 4$ since 4 is an integer.

The greatest integer less than $\lfloor 3.2 \rfloor = 3$ since 3 is the integer less than 3.2



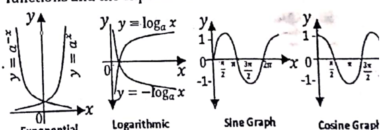
7.THE RATIONAL FUNCTION: This is given as $f(x) = \frac{p(x}{q(x}$

Where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$ for al $x \in \mathbb{R}$ since $dom(q) = \mathbb{R}$. Again the graph of the rationa function depends on the nature of the two polynomials



8. THE TRANSCENDENTAL FUNCTIONS

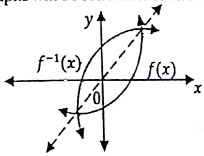
These are functions which are not algebraic i.e functio that cannot be expressed as sums, products or quotients finite polynomials. Common examples are t trigonometric function and their inverses, the logarith functions and the exponential functions.

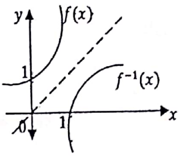


9. INVERSE FUNCTION

The inverse of a function f is denoted by f^{-1}.

n order to find the inverse of a function, we equate the unction to y and proceed to make x the subect of the ormula, in the end we interchange y with x and x with y which thus produces the inverse of the function.
Since functioins and inverse functions contain the same numbers in their ordered pair, just in reverse order, their graphs will be reflection of one another.





Example 1
Find f^{-1} if $f(x) = 6x - 2$
Solution
Given $f(x) = 6x - 2$
Let $y = 6x - 2$.
$y + 2 = 6x$
$x = \frac{y+2}{6} \Rightarrow f^{-1} = \frac{x+2}{6}$

Example 2:
Find the inverse of the function
$f(x) = \frac{3-5xy}{2x}$
Solution
Give $f(x) = \frac{3-5xy}{2x}$
Let $y = \frac{3-5xy}{2x}$
$2xy = 3 - 5xy$
$2xy + 5xy = 3$
$(2x + 5x)y = 3$
$7xy = 3$
$x = \frac{3}{7y}$
$y = \frac{3}{7x}$
That is $f^{-1} = \frac{3}{7x}$

10. EVEN FUNCTION: The function $f(x)$is said to be an even function if $f(-x) = f(x)$
Examples include $f(x) = 1, x^2, x^4 \dots x^{2n}, |x|$ $n = 0,1,2,3,$ $\dots \cos x, \cosh x$ etc

NOTE

- The sum or difference of any number of even functions is also an even function
- The integral of an even function between a symmetrical portion/interval is 2 times(twice) the integral of one half the interval E.g
$$\int_{-6}^{6} f(x)dx = 2\int_{0}^{6} f(x)dx.$$
if $f(x)$is even
- The product of two even functions is also an even function

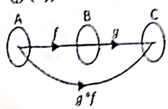
11. ODD FUNCTION: The function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$.
The following are some examples of odd functions $f(x) = x, x^3, x^5, \dots, \sin x, \tan x$ etc.

NOTE

- The sum of any number of odd functions is also an odd function
- The integral of an odd function between a symmetrical portion is always zero(0) i.e
$$\int_{-5}^{5} f(x)dx = 0 \quad if\ f(x) is\ odd$$
- The product of any two (2) odd functions is an even function.
- The sum or difference of an odd function and an even function is neither odd nor even.

COMPOSITION (FUNCTION OF A FUNCTION)

The composition of two function $f(x)$and $g(x)$, written as $f \circ g = f(g(x))$



$f: A \to B$ and $g: B \to C$, then $g°f: A \to C$

Example 1
Let $f: N \to N$ be given by $f(x) = x + 1$ and $g: N \to N$ be given by $g(x) = x^2$. Compute
(i) $f(g(x))$
(ii) $g(f(x))$
Solution:
Given $g(x) = x^2$, $f(x) = x + 1$
(i) $f(g(x)) = x^2 + 1$ and
(ii) $g(f(x)) = g(x+1) = (x+1)^2$
notice that $f(g(x)) \neq g(f(x))$

Example 2
Two functions $f(x)$ and $g(x)$ are defined on the set of real numbers by $f(x) = 3x^2 - 2$ and $g(x) = x + 3$
Find:
(a) $f(-2)$
(b) $g^{-1}(-3/4)$
(c) The value of x for which $f(g(x)) = g(f(x))$
Solution:
Given, $f(x) = 3x^2 - 2$ and $g(x) = x + 3$
(a) $f(-2) = 3(-2)^2 - 2$
$3(4) - 2 = 12 - 2 = 10$
(b) let y be the image of x under g, then $y = g(x) = x + 3$
$$y = x + 3$$
$$x = y - 3$$
$$\therefore g^{-1}(x) = x - 3$$
$$g^{-1}\left(-\frac{3}{4}\right) = -\frac{3}{4} - 3 = -\frac{15}{4}$$
(c) $f(g(x)) = f(x+3)$
$= 3(x+3)^2 - 2$

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$3(x^2 + 6x + 9) - 2$
$= 3x^2 + 18x + 27 - 2$
$= 3x^2 + 18x + 25$
Again, $g(f(x)) = g(3x^2 - 2)$
$3x^2 - 2 + 3$
$= 3x^2 + 1$
If $f(g(x)) = g(f(x))$, then
$3x^2 + 18x + 25 = 3x^2 + 1$
$18x + 25 = 1$
$18x = 1 - 25$
$18x = -24$

$$x = -\frac{24}{18}$$
$$x = -\frac{4}{3}$$

Example 3
Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x - 5 & ; \quad x > 2 \\ x^2 - 2|x| & ; \quad x \le 2 \end{cases}$$
$$g(x) = 3x + 1$$

find the values of:
(i) $f(-2)$ (ii) $g(-3)$ (iii) $(g \circ f)(2)$ (iv) $(g \circ f)(3)$

Solution
(i) $f(-2)$
since $x = -2$, $x \le 2$
thus $f(x) = x^2 - 2|x|$
$f(-2) = (-2)^2 - 2|2| = 4 - 4 = 0$
(ii) $g(x) = 3x + 1$
$g(-3) = 3(-3) + 1$
$= -9 + 1 = -8$
(iii) $g \circ f\ (2)$
since $x = 2$ we use $f(x) = x^2 - 2|x|$
$g[x^2 - 2|x|] = 3[x^2 - 2|x|] + 1$
$= 3x^2 - 6|x| + 1$
$(g \circ f)(2) = 3(2)^2 - 6|2| + 1$
$3(4) - 12 + 1 = 12 + 1 = 1$
(iv) $(g \circ f)3$
Since $x = 3$ and $x > 2$ we use $f(x) = 2x - 5$
$g \circ f = g[2x - 5] = 3[2x - 5] + 1$
$= 6x - 15 + 1$
$= 6x - 14$
$(g \circ f)3 = 6(3) - 14$
$18 - 14 = 4$

Example 4
If $f(x) = 2^x$ show that
$f(x+3) - f(x-1) = \frac{15}{2} f(x)$

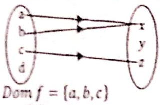
Solution
$f(x) = 2^x$; $f(x+3) = 2^{x+3}$; $f(x-1) = 2^{x-1}$
thus, $f(x+3) - f(x-1) = 2^{x+3} - 2^{x-1}$
$= 2^x . 2^3 - 2^x 2^{-1}$
$= 8(2^x) - \frac{1}{2}(2^x)$
$(8 - \frac{1}{2})(2^x)$
$(\frac{16-1}{2})(2^x)$
$\frac{15}{2}(2^x) = \frac{15}{2} f(x)$

Example 5
Given that $f(x-3) = x^2 + 2x - 3$, find the value of $f(2)$

Solution:
$$f(x-3) = x^2 + 2x - 3$$
To find $f(2)$, we equate $x - 3$ and 2
$$x - 3 = 2 \Rightarrow x = 5$$
$$\therefore f(2) = f(5) = 5^2 + 2(5) - 3$$
$$= 25 + 10 - 3 = 32$$
$$\therefore f(2) = 32$$

DOMAIN, CO-DOMAIN & RANGE

The domain of a function is the set of points where function is said to be defined, it is denoted by $Dom\ f$.
The range of a function is the set of all possible values of th function, it is denoted by R_f
Given that $f: A \to B$ is a function, A is called the domain of this is the subset of $\mathbb{R}$ for which the function is defined. B the co-domain of f, for the duration of this course, w assume the co-domain to be $\mathbb{R}$. The range is the set of a images of A under f i.e
$ran(f) = \{y \in B : y = f(x) for\ some\ x \in A\}$



$Dom\ f = \{a, b, c\}$
$ran(f) = \{x, z\}$
$co - domain = \{x, y, z\}$

Example 1: Find the domain of the following:
i) $f(x) = \sqrt{3 - x}$
ii) $f(x) = \sqrt{1 + 5x}$
iii) $f(x) = \frac{1}{\sqrt{3-x}}$
iv) $f(x) = \ln(\sqrt{1 + 5x})$
v) $f(x) = \frac{1}{\ln(3x+9)}$
vi) $f(x) = \frac{\sqrt{2-x}}{x^2 - 1}$

Solution:
i. The function is defined if $3 - x \ge 0$ since squar root holds for only non-negative numbers,
$$3 \ge x \ or\ x \le 3$$
$$Dom(f) = (-\infty, 3]$$
ii. The function is defined if $1 + 5x \ge 0$, sinc square root holds for only non-negativ numbers,
$$5x \ge -1 \ or\ x \ge -\frac{1}{5}$$
$$Dom(f) = [-\frac{1}{5}, \infty)$$
iii. The function is defined if $3 - x > 0$, since squar root holds for only non-negative numbers an also we avoid zero being in the denominator,
$$3 > x \Rightarrow x < 3$$
$$Dom(f) = (-\infty, 3)$$
iv. The function is defined if $1 + 5x > 0$, since l holds for only non-negative numbers,
$$5x > -1 \ or\ x > -\frac{1}{5}$$

$$Dom(f) = \left(-\frac{1}{5}, \infty\right)$$

v. The function is defined if

$$\ln(3x+9) \neq 0 \quad \& \quad 3x+9 > 0$$

since ln holds for only non-negative numbers and also we avoid zero being in the denominator

$$3x+9 \neq e^0 \ \& \ 3x > -9$$

$$3x+9 \neq 1 \ \& \ \frac{3x}{3} > -\frac{9}{3}$$

$$x \neq -\frac{8}{3} \ \& \ x > -3$$

$$or \ \left(-3, -\frac{8}{3}\right) \cup \left(-\frac{8}{3}, \infty\right)$$

vi. The function is defined if

$$2 - x > 0 \ \& \ x^2 - 1 \neq 0$$

since square root holds for only non-negative numbers and also we avoid zero being in the denominator

$$2 > x \ \& \ x^2 \neq 1$$

$$x < 2 \ \& \ x \neq \pm 1$$

$$or \ (-\infty, -1) \cup (-1, 1) \cup (1, 2)$$

N.B. The range of a function is the domain of the inverse function.

Example 2. Find the range of the following functions:

i. $f(x) = x^2 + 3$

ii. $f(x) = \sqrt{1+3x}$

iii. $f(x) = \frac{1}{x-3}$

Solution:

i. Since, $x^2 \geq 0 \Rightarrow x^2 + 3 \geq 3 \Rightarrow f(x) \geq 3$

Thus, range $= [3, \infty)$

$$Or \ y = [f(y)]^2 + 3$$

$$[f(y)]^2 = y - 3$$

$$f(y) = \sqrt{y-3}$$

$$Range = Dom\, f(y) = [3, \infty)$$

ii. $f(x) = \sqrt{1+3x}$

$$y = \sqrt{1+3f(y)} \Rightarrow y^2 = 1 + 3f(y)$$

$$f(y) = \frac{y^2-1}{3}$$

$$range = Dom\, f(y) = [0, \infty)$$

iii. $f(x) = \frac{1}{x-3}$

$$y = \frac{1}{f(y)-3} \Rightarrow yf(y) - 3y = 1$$

$$\Rightarrow yf(y) = 1 + 3y$$

$$\Rightarrow f(y) = \frac{1+3y}{y}$$

$$Range = Dom\, f(y) = \mathbb{R}\backslash\{0\} \ or \ (-\infty, 0) \cup (0, \infty)$$

Example 3: Determine the domain and range of the following function f: $\mathbb{R} \to \mathbb{R}$

(i) $f(x) = \frac{1}{\sqrt{49-x^2}}$ (ii) $f(x) = \frac{x}{\sqrt{x^2-9}}$

(iii) $f(x) = \frac{x^2}{\sqrt{x^2-9}}$ (iv) $f(x) = \frac{2x}{(x-2)(x+1)}$

(v) $f(x) = \sin 5x$

(vi) $f(x) = \cos x$

(vii) $f(x) = \log(x - 5)$

(viii) $f(x) = x^2 + 2$

(ix) $f(x) = \frac{1}{x+2}$ (x) $f(x) = \frac{2x^2 - \log(x+5)}{\sqrt{16-x}}$

Solution

(i) $f(x) = \frac{1}{\sqrt{49-x^2}}$

function is defined if and only if

$49 - x^2 > 0$

$7^2 - x^2 > 0$

$(7-x)(7+x) > 0$

Using truth table to get the true solution

	$x < -7$	$-7 < x < 7$	$x > 7$
$(7-x)$	+	+	−
$(7+x)$	−	+	+
$(7-x)(7+x)$	−	+	−

Solution is : $\{x : -7 < x < 7\}$

∴ Domain of f = Dom f = $\{x : -7 < x < 7\}$

or $(-7, 7)$

Range of f

For all values of x in the $f(x)$ produces positive numbers apart from zero , $R_f = \mathbb{R}^+ / \{0\}$ or $(0, \infty)$

(ii) $f(x) = \frac{x^2}{\sqrt{x^2-9}}$

The function is defined if

$$x^2 - 9 > 0$$

$$x^2 - 3^2 > 0$$

$$(x-3)(x+3) > 0$$

Using truth table to get the true solution

	$x < -3$	$-3 < x < 3$	$x > 3$
$x-3$	−	−	+
$x+3$	−	+	+
$(x-3)(x+3)$	+	−	+

The solution = $\{x : x < -3 \cup x > 3\}$

Thus, Domain of f

$= \{x : x < -3 \cup x > 3\} \ or \ (\infty, -3) \cup (3, \infty)$

Range of f => for all values of x in the domain, $f(x)$ produces only non-negative numbers, thus

$R_f = (0, \infty)$

(iii) $f(x) = \frac{x}{\sqrt{x^2-9}}$

The domain is the same as above

$= \{x : x < -3 \cup x > 3\} \ or \ (\infty, -3) \cup (3, \infty)$

Range of f => for all values of x in the domain, $f(x)$ produces negative numbers, zero and positive numbers thus

$R_f = \mathbb{R}$

(iv) $f(x) = \frac{2x}{(x-2)(x+1)}$

the function is defined

if $(x-2)(x+1) \neq 0$

$$x \neq 2, -1$$

Hence, Domain of f = Dom f = $\mathbb{R} - \{-1, 2\}$

Range of f =

$f(x)$ can produce negative numbers, zero or positive numbers. Thus

$$R_f = \mathbb{R}$$

(v) $f(x) = \sin 5x$

Domain = Dom f = $\mathbb{R}$

Range = $R_f = [-1, 1]$

(vi) $f(x) = \cos x$

Domain = Dom f = $\mathbb{R}$

Range = $R_f = [-1, 1]$

NOTE: The range of $\cos x$ and $\sin x$ is $[-1,\ 1]$

(vii) $f(x) = \log(x - 5)$

the function is defined if $x - 5 > 0$

$$x > 5$$

Dom f = $\{x : x > 5\}$ or $(5, \infty)$

Since $f(x)$ can be negative, zero or positive Range of f $= R_f = \mathbb{R}$

(viii) $f(x) = x^2 + 2$

the function is defined for all real values of x,

thus, Dom f = $\mathbb{R}$

$R_f = [2, \infty)$

(ix) $f(x) = \frac{1}{x+2}$

the function is defined if $x + 2 \neq 0$

$$x \neq -2$$

Hence, dom f = $\mathbb{R}$ - {-2}

Also, no matter how large or small x becomes, $f(x)$ will never be equal to zero

$\therefore R_f = \mathbb{R} - \{0\}$

(x) $f(x) = \frac{2x^2 - \log(x+5)}{\sqrt{16-x}}$

the function is defined if $16 - x > 0$ and $x + 5 > 0$

$\Rightarrow x < 16$ and $x > -5$

$$\Rightarrow (-\infty, 16) \cap (-5, \infty)$$

Dom f = $(-5, 16)$

Since $f(x)$ can produce positive, zero or negative values, $R_f = \mathbb{R}$

Example 4: Determine the domain and range of the following function

(i) $f(t) = \sqrt{4 - 7t}$ (ii) $f(x) = |x - 4|$ (iii) $f(x) = |x - 3| - 5$

Solution:

(i) $f(t) = \sqrt{4 - 7t}$

$f(t)$ is defined if $4 - 7t \geq 0 \Rightarrow 4 \geq 7t \Rightarrow \frac{4}{7} \geq t$ or $t \leq \frac{4}{7}$

Domain = $(-\infty, \frac{4}{7}]$

The minimum value is gotten if $t = \frac{4}{7}$ i.e 0

The maximum value is gotten if $t = 0$ i.e 2

Thus, Range = $[0, 2]$

(ii) $f(x) = |x - 4|$

$Dom\ (f) = \mathbb{R}$, the function is defined for all values of x

$Ran\ (f) = [0, \infty)$, the smallest value we can get from absolute values is 0 and it keeps increasing down to ∞.

(iii) $f(x) = |x - 3| - 5$

$Dom\ (f) = \mathbb{R}$, the function is defined for all values of x

$Ran\ (f) = [-5, \infty)$, the smallest value we can get from the function is $0 - 5 = -5$ and it keeps increasing down to ∞.

Example 5: Find the domain of $f(x) = \sqrt{\frac{x}{2-x}}$

Solution:

The function is defined if $\frac{x}{2-x} \geq 0 \cup x \neq 2$

Solving the inequality $\frac{x}{2-x} \geq 0 \Rightarrow \frac{x}{2-x} \times (2-x)^2 \geq 0$

$$x(2 - x) \geq 0$$

Using truth table to get the true solution

	$x \leq 0$	$0 \leq x \leq 2$	$x \geq 2$
x	−	+	+
$(2 - x)$	+	+	−
$(7 - x)(7 + x)$	−	+	−

The solution = $\{x : 0 \leq x \leq 2\}$

Domain = $0 \leq x \leq 2 \cup x \neq 2$ or $0 \leq x < 2$ or $[0, 2)$

DOMAIN AND RANGE OF SOME TYPES OF FUNCTIONS

1. CONSTANT FUNCTION:

$$f(x) = k$$

$$Dom\ (f) = \mathbb{R},\ Ran\ (f) = k$$

2. LINEAR FUNCTIONS:

$$f(x) = ax + b;\ a, b \in \mathbb{R}, a \neq 0$$

$$Dom\ (f) = \mathbb{R},\ Ran\ (f) = \mathbb{R}$$

3. QUADRATIC FUNCTIONS:

$$f(x) = ax^2 + bx + c;\ a, b, c \in \mathbb{R}, a \neq 0$$

$$Dom\ (f) = \mathbb{R}$$

$$Ran(f) = \left[-\frac{b}{2a}, -\infty\right) \text{ if } a > 0$$

$$\text{or } \left(-\infty, -\frac{b}{2a}\right] \text{ if } a < 0$$

4. EXPONENTIAL FUNCTIONS:

$$f(x) = e^x$$

$$Dom\ (f) = \mathbb{R},\ Ran\ (f) = (0, \infty)$$

5. LOGARITHMIC FUNCTIONS:

$$f(x) = \log_a x$$

$$Dom\ (f) = (0, \infty),\ Ran\ (f) = \mathbb{R}$$

6. TRIGONOMETRIC FUNCTIONS:

$$f(x) = \sin x \text{ or } \cos x$$

$$Dom\ (f) = (0, \infty),\ Ran\ (f) = [-1, 1]$$

7. ABSOLUTE VALUE FUNCTIONS:

$$f(x) = |x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

$$Dom\ (f) = \mathbb{R},\ Ran\ (f) = [0, \infty)$$

LIMITS

The limit of the function $f(x)$ as x tends to x_0, written as $\lim_{x \to x_0} f(x) = L$ is defined thus;

If given $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ such that if $|f(x) - L| < \varepsilon$ whenever $|x - x_0| < \delta$.

This means that the limit as x tends to x_0 is L. In other words, we seek a δ that depends on ε such that the above is satisfied.

Example1:

Prove that

$$\lim_{x \to 3} 2x + 6 = 12$$

Solution

$f(x) = 2x + 6, \quad L = 12, \quad x_0 = 3$

Let $\varepsilon > 0$, then

$|F(x) - L| < \varepsilon$, implies

$\Rightarrow |2x + 6 - 12| < \varepsilon \Rightarrow |2x - 6| < \varepsilon$

$\Rightarrow |2(x - 3)| < \varepsilon \Rightarrow 2|x - 3| < \varepsilon$

But for $|x - x_0| = |x - 3| < \delta$

$\therefore$ we have $2\delta < \varepsilon$

$\delta < \frac{\varepsilon}{2}$

So choose $\delta = \frac{\varepsilon}{2}$.

Which means that given any $\varepsilon > 0$, there exist $\delta(\varepsilon) = \frac{\varepsilon}{2}$ such that $|2x + 6 - 12| < \varepsilon$ whenever $|x - 3| < \frac{\varepsilon}{2}$.

Therefore, $\lim_{x\to 3} 2x + 6 = 12$

Example 2:

Prove that

$\lim_{x\to -1} x^2 + 3x - 4 = -6$

$f(x) = x^2 + 3x - 4, \quad L = -6, x_0 = -1$

$|f(x) - L| < \varepsilon$, implies

$|x^2 + 3x - 4 - (-6)|$

$= |x^2 + 3x - 4 + 6|$

$= |x^2 + 3x + 2| < \varepsilon$

Next, we replace x with $x + 1$ and cancel out what makes the outcome different from the original inequality.

$|(x + 1)^2 + 3(x + 1) - 2x - 2|$

$= |(x + 1)^2 + 3(x + 1) - 2(x + 1)|$

$= |(x + 1)^2 + (x + 1)|$

$\leq |x + 1|^2 + |x + 1| < \varepsilon$

We restrict x to $0 < |x + 1| < \delta$

And thus we have

$$\delta^2 + \delta < \varepsilon$$

Since δ is a very small positive number, we expect $\delta^2 \ll \delta$

Hence, replacing δ^2 by δ, we have

$$\delta + \delta < \varepsilon$$

$2\delta < \varepsilon \Rightarrow \delta < \varepsilon/2$

Hence, choose $\delta(\varepsilon) = \frac{\varepsilon}{2}$.

Example 3:

Find $\delta(\varepsilon) > 0$ such that

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = 6$$

Solution:

$$f(x) = \frac{x^2 - 9}{x - 3}; \; L = 6; \; x_0 = 3$$

$$\left|\frac{x^2 - 9}{x - 3} - 6\right| < \varepsilon$$

$$\left|\frac{x^2 - 9 - 6x + 18}{x - 3}\right| < \varepsilon$$

$$\left|\frac{x^2 - 6x + 9}{x - 3}\right| < \varepsilon$$

Let $x - 3 = k$

$$\left|\frac{x^2 - 6x + 9}{k}\right| < \varepsilon$$

$$\frac{1}{k}|x^2 - 6x + 9| < \varepsilon$$

Introducing the co-ordinates.

$$\frac{1}{k}|[(x - 3)^2 + 6x - 9] - [6(x - 3) + 18] + 9| < \varepsilon$$

$$\frac{1}{k}|(x - 3)^2 + 6x - 9 - 6x + 18 - 18 + 9| < \varepsilon$$

$$\frac{1}{k}|(x - 3)^2| < \varepsilon$$

$$\frac{1}{k}\delta^2 < \varepsilon$$

$$\delta^2 < k\varepsilon$$

But, $\delta^2 \ll \delta$,

$$\delta < k\varepsilon$$

Choose, $\delta(\varepsilon) = k\varepsilon$ such that

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = 6$$

LIMIT THEOREMS

If the limits of the function $f(x)$ and $g(x)$ exist at any point $x = x_0$ then:

a. $\lim_{x\to x_0}[f(x) \pm g(x) \pm h(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) \pm \lim_{x\to x_0} h(x)$

b. $\lim_{x\to x_0} f(x).g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$

c. $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ provided $\lim_{x\to x_0} g(x) \neq 0$

d. $\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)}$ whenever n is any positive integer.

e. $\lim_{x\to x_0} c\, f(x) = c \lim_{x\to x_0} f(x)$

f. $\lim_{x\to x_0} (g \circ f)(x) = g \circ \left(\lim_{x\to x_0} f(x)\right)$

EVALUATION OF LIMIT

There are various ways by which we can evaluate the limit of a function, we will consider the following approach:

1. SUBSTITUTION

In many cases it is possible to evaluate the limits by substituting the point into the function. In applying this method it is imperative for us to be careful, since the existence of the function does not guarantee the existence of the limit of the function at the same point. Substitution fail if it leads to an indeterminate form.

Example1: Evaluate

$$\lim_{x\to 1} \sqrt{x^2 + 2}$$

Solution

$$\lim_{x\to 1} \sqrt{x^2 + 2}$$

$$= \sqrt{(-1)^2 + 2} = \sqrt{1 + 2} = \sqrt{3}$$

$$\therefore \lim_{x\to 1} \sqrt{x^2 + 2} = \sqrt{3}$$

Example 2: Evaluate the limit

$\lim_{x} \tan^2 x$

Solution:

$\lim_{x\to\frac{\pi}{4}} \tan^2 x = \lim_{x\to\frac{\pi}{4}} (\tan x)^2$

$= (\lim_{x\to\frac{\pi}{4}} \tan x)^2 = 1^2 = 1$

2. SIMPLICATION

In many cases direct substitution may lead to an indeterminate form. Whenever this occures, it is therefore imperative to simplify the expression before substitution.

We may require the method of factorization and rationalization

Examples

Evaluate the following limits at the given points

(i) $\frac{x^2+2x+1}{x+1}$; $x = -1$

(ii) $\frac{9-x^2}{3-\sqrt{x+6}}$; $x = 3$

(iii) $\frac{x^2-\sqrt{x}}{x-1}$; $x = 1$

Solution

(i) $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$

$= \frac{(-1)^2+2(-1)+1}{-1+1} = \frac{1-2+1}{0} = \frac{0}{0}$

Which is indeterminate, now we factorize

$\lim_{x\to -1} \frac{x^2+2x+1}{x+1} = \lim_{x\to -1} \frac{(x+1)(x+1)}{x+1}$

$x \to -1$

$= \lim_{x\to -1} (x+1) = -1+1 = 0$

(ii) $\lim_{x\to 3} \frac{9-x^2}{3-\sqrt{x+6}}$

$\frac{9-3^2}{3-\sqrt{3+6}} = \frac{9-9}{3-\sqrt{9}} = \frac{9-9}{3-3} = \frac{0}{0}$

Which is indeterinate rationalizing

$$\frac{9-x^2}{3-\sqrt{x+6}} \times \frac{3+\sqrt{x+6}}{3+\sqrt{x+6}}$$

$$= \frac{(9-x^2)(3+\sqrt{x+6})}{3^2-(\sqrt{x+6})2}$$

$$\frac{(9-x^2)(3+\sqrt{x+6})}{9-(x+6)}$$

$$\frac{(9-x^2)(3+\sqrt{x+6})}{9-x-6}$$

$$\frac{(9-x^2)(3+\sqrt{x+6})}{3-x}$$

$$\frac{(3-x)(3+x)(3+\sqrt{x+6})}{3-x}$$

$\lim_{x\to 3} (3+x)(3+\sqrt{x+6})$

$(3+3)(3+\sqrt{3+6})$

$(3+\sqrt{9}) = 6(3+3)$

$(6) = 36$

(iii) $\lim_{x\to 1} (\frac{x^2-\sqrt{x}}{x-1}) = \frac{1^2-\sqrt{1}}{1-1}$

$\frac{1-1}{1-1} = \frac{0}{0}$

Which is indeterminate next, we rationalise with the term bearing the surdic sign

$$\frac{x^2-\sqrt{x}}{x-1} \times \frac{x^2+\sqrt{x}}{x^2+\sqrt{x}}$$

$$\frac{(x^2)^2-(\sqrt{x})^2}{(x-1)(x^2+\sqrt{x})}$$

$$\frac{x^4-x}{(x-1)(x^2+\sqrt{x})}$$

$$= \frac{(x-1)(x^3+x^2+x)}{(x-1)(x^2+\sqrt{x})}$$

$$\lim_{x\to 1} \frac{x^3+x^2+x}{x^2+\sqrt{x}}$$

$$= \frac{1^3+1^2+1}{1^2+\sqrt{1}} = \frac{1+1+1}{1+1} = \frac{3}{2}$$

3. INDETERMINATE FORMS AND L'HOPITAL'S RULE

L'hopital's rule named after the 17th century French Mathematician Guillaume de l'Hopital states that for functions $f(x)$ and $g(x)$ which are differentiable and $g(x) \neq 0$ at point $x = x_0$

where

$$\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{0}{0}$$

Then,

$$\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \lim_{x\to x_0} \left[\frac{f'(x)}{g'(x)}\right]$$

If $\lim_{x\to x_0} \left[\frac{f'(x)}{g'(x)}\right]$ exists

Example 1

Evaluate the folowing limits at the given points

(i) $\frac{1-\tan x}{\sin x - \cos x}$; $x = \pi/4$

(ii) $\frac{\tan x - x}{x^3}$; $x = 0$

(iii) $\frac{\sin 2x}{\sin 3x}$; $x = 0$

Solution

i.

$$\lim_{x\to \pi/4} \left[\frac{1-\tan x}{\sin x - \cos x}\right]$$

$\frac{1-\tan \pi/4}{\sin \pi/4 - \cos \pi/4} = \frac{1-\tan 45}{\sin 45 - \cos 45}$

$= \frac{1-1}{1/\sqrt{2} - 1/\sqrt{2}} = \frac{0}{0}$

Which is indeterminate

Applying l'Hopital's rule

$$\frac{\frac{d}{dx}(1-\tan x)}{\frac{d}{dx}(\sin x - \cos x)} = \frac{-\sec^2 x}{\cos x + \sin x}$$

$$= \frac{\frac{-1}{(\cos\frac{\pi}{4})^2}}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \frac{\frac{-1}{(\frac{1}{\sqrt{2}})^2}}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}$$

$-\frac{\frac{1}{\frac{1}{2}}}{\frac{2}{\sqrt{2}}} = -2 \times \frac{\sqrt{2}}{2} = -\sqrt{2}$

ii.

$\lim_{x\to 0} (\frac{\tan x - x}{x^3})$

$\frac{\tan 0 - 0}{0^3} = \frac{0-0}{0} = \frac{0}{0}$

Which is indeterminate
Applying l'Hopital's rule

$$\frac{\frac{d}{dx}(tanx - x)}{\frac{d}{dx}(x^3)}$$

$$\lim_{x \to 0} \frac{sec^2 x - 1}{3x^2}$$

$$\frac{(\sec 0)^2 - 1}{3\,(0)^2} = \frac{1-1}{0} = \frac{0}{0}$$

Which is indeterminate
Applying l'Hopitals rule again

$$\frac{\frac{d}{dx}(tan^2 x)}{\frac{d}{dx}(3x^2)} = \frac{2tanx\, sec^2x}{6x}$$

$$\lim_{x \to 0} \frac{2\,tanx\, sec^2x}{6x} = \frac{2\,tan0\, sec^2 0}{6(0)} = \frac{0}{0}$$

Which is indeterminate

$$\frac{2\tan x\, sec^2x}{6x} = \frac{2tanx\,(1 - tan^2x)}{6x}$$

$$\frac{2[tanx - tan^3x]}{6x}$$

$$\frac{\frac{1}{3}\frac{d}{dx}[tanx - tan^3x]}{\frac{d}{dx}(x)}$$

$$\frac{\frac{1}{3}sec^2x - 3tan^2x\,sec^2x}{1}$$

$$\lim_{x \to 0} [\tfrac{1}{3}(sec^2x - 3tan^2x\,sec^2x)]$$

$$\frac{1}{3}[(\sec 0)^2 - 3(tan0)^2\,(\sec 0)^2]$$

$$= \tfrac{1}{3}[(\sec 0)^2 - 3(\tan 0)^2\,(sec\,0)^2]$$

$$= \tfrac{1}{3}[1 - 0] = 1/3$$

iii.

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$$

$$\lim_{x \to 0} \frac{\sin 2(0)}{\sin 3\,(0)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

Which is indeterminate
Applying l'hopitals rule

$$\frac{\frac{d}{dx}(\sin 2x)}{\frac{d}{dx}(\sin 3x)} = \frac{2\cos 2x}{3\cos 3x}$$

$$\lim_{x \to 0} \frac{2\cos 2x}{3\cos 3x}$$

$$\frac{2\,cos2\,(0)}{3\cos 3(0)} = \frac{2\cos 0}{3\cos 0}$$

$$= \frac{2\,(1)}{3\,(1)} = \frac{2}{3}$$

Example 2

Evaluate $\lim_{x \to 0} \frac{tanx}{x}$

Solution

From basic trigonometry

$$\tan x = \frac{sinx}{cosx}$$

Thus, $\lim_{x \to 0} \frac{tanx}{x} = \lim_{x \to 0} \frac{\frac{sinx}{cosx}}{x}$

$= \lim_{x \to 0} \frac{sinx}{x\,cosx} = \lim_{x \to 0} \frac{1}{cosx} \lim_{x \to 0} \frac{sinx}{x}$

NOTE: The limiting value of $\frac{sinx}{x}$ as $x \to 0$ equals 1

$\therefore \lim_{x \to 0} \frac{1}{cosx} \cdot \lim_{x \to 0} \frac{sinx}{x}$

$\frac{1}{cos} \cdot 1 = \frac{1}{1} \cdot 1 = 1$

4. LIMITS TO INFINITY

To evaluate the limit of a rational function (i.e a function the form $\frac{f(x)}{g(x)}$, $g(x) \neq 0$) as x tends to infinity, we divide ea term in that function by the highest power of x occuring that function, and further substitute infinity.

Example 1

Evaluate the following limits

(i) $\frac{2x^2 - 3}{x^2 - 5x - 6}$; $x = \infty$

(ii) $\frac{3x^3 + 2x^2 + 7x + 9}{5x^2 + 17}$; $x = \infty$

(iii) $\frac{3^x - 3^{-x}}{3^x + 3^{-x}}$; $x = \infty$

Solution:

(i) $\lim_{x \to \infty} \frac{2x^2 - 3}{x^2 - 5x - 6}$

$$\lim_{x \to \infty} [\frac{\frac{2x^2}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{5x}{x^2} - \frac{6}{x^2}}]$$

$$\lim_{x \to \infty} [\frac{2 - \frac{3}{x^2}}{1 - \frac{5}{x} - \frac{6}{x^2}}]$$

$$\frac{2 - \frac{3}{\infty^2}}{1 - \frac{5}{\infty} - \frac{6}{\infty^2}} = \frac{2}{1} = 2$$

(ii)

$$\lim_{x \to \infty} [\frac{3x^3 + 2x^2 + 7x + 9}{5x^2 + 17}]$$

The highest power of x here is x^3

$$\lim_{x \to \infty} [\frac{\frac{3x^3}{x^3} + \frac{2x^2}{x^3} + \frac{7x}{x^3} + \frac{9}{x^3}}{\frac{5x^2}{x^3} + \frac{17}{x^3}}]$$

$$\lim_{x \to \infty} [\frac{3 + \frac{2}{x} + \frac{7}{x^2} + 9/x^3}{\frac{5}{x} + \frac{17}{x^3}}]$$

$$\frac{3 + \frac{2}{\infty} + \frac{7}{\infty^2} + \frac{9}{\infty^3}}{\frac{5}{\infty} + \frac{17}{\infty^3}}$$

$$\frac{3 + 0 + 0 + 0}{0 + 0} = \frac{3}{0} = \infty$$

Hence the limit does not exist

NOTE: If the degree of the numerator is greater than the degree of the denominator, then the limit does not exist.

(iii)

$$\lim_{x \to \infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

We divide both denominator by 3^x

$$\lim_{x \to \infty} \frac{\frac{3^x}{3^x} - \frac{3^{-x}}{3^x}}{\frac{3^x}{3^x} + \frac{3^{-x}}{3^x}}$$

$$\lim_{x \to \infty} \frac{1 - \frac{1}{3^{2x}}}{1 + \frac{1}{3^{2x}}} = \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}}$$

$$\frac{1 - 0}{1 + 0} = \frac{1}{1} = 1$$

Example 2: Evaluate the limit

$$\lim_{x \to \infty} \left(1 + \frac{1}{2x}\right)^x$$

Solution:

$$\lim_{x \to \infty} \left(1 + \frac{1}{2x}\right)^x$$

$$let\ y = \left(1 + \frac{1}{2x}\right)^x$$

$$\ln y = \ln\left(1 + \frac{1}{2x}\right)^x \Rightarrow \ln y = x \ln\left(1 + \frac{1}{2x}\right)$$

$$\ln y = \frac{\ln\left(1 + \frac{1}{2x}\right)}{1/x} \Rightarrow \lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{2x}\right)}{1/x}$$

$$= \frac{\ln\left(1 + \frac{1}{2\infty}\right)}{1/\infty} = \frac{\ln(1+0)}{0} = \frac{0}{0}$$

Which is indeterminate, applying L'Hopitals rule,

$$\lim_{x \to \infty} \ln y = \frac{-\frac{1}{2x^2}\left(\frac{1}{1 + 1/2x}\right)}{-1/x^2} = \lim_{x \to \infty} \frac{1}{2}\left(\frac{1}{1 + 1/2x}\right)$$

$$\lim_{x \to \infty} \ln y = \frac{1}{2}\left(\frac{1}{1 + 1/2\infty}\right) = \frac{1}{2}\left(\frac{1}{1+0}\right) = \frac{1}{2}$$

$$\lim_{x \to \infty} \ln y = \frac{1}{2}$$

since ln is a continuous function, we have that,

$$\lim_{x \to \infty} \ln y = \ln \lim_{x \to \infty} y$$

$$thus,\ \ \ln \lim_{x \to \infty} y = \frac{1}{2}$$

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(1 + \frac{1}{2x}\right)^x = e^{\frac{1}{2}}$$

5. THE SQUEEZE (SANDWICH OR PINCHING) THEOREM

Thistheorem states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \qquad \text{then} \lim_{x \to a} g(x) = L$$

Example

Evaluate (i) $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$ (ii) $\lim_{x \to 0} \sqrt{x} e^{\cos(\pi/x)}$

Solution

(i) Note that we cannot use

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x^2 . \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ because $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

However, since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ we have that,

$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$.

But $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} -x^2 = 0$ and so we obtain

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the squeeze theorem.

(ii) since $-1 \leq \cos(\pi/x) \leq 1$;

$$e^{-1} \leq e^{\cos(\pi/x)} \leq e^1$$

Multiplying through by $\sqrt{x}$ we have;

$$\sqrt{x} e^{-1} \leq \sqrt{x} e^{\cos(\pi/x)} \leq \sqrt{x} e^1.$$

But $\lim_{x \to 0} \sqrt{x} e^{-1} = \lim_{x \to 0} \sqrt{x} e^1 = 0$; so by the Squeeze theorem we have

$$\lim_{x \to 0^+} \sqrt{x} e^{\cos(\pi/x)} = 0$$

CONTINUITY

A function $f(x)$ is said to be continuous at a point x_0 if it satisfies the following conditions:

(i) $f(x_0)$ exists

(ii) $\lim_{x \to x_0} f(x)$ exists, and

(iii) $f(x_0) = \lim_{x \to x_0} f(x_0)$

NOTE: If any of the conditions (i), (ii) and (iii) fails then the function is not continuous or the function is discontinuous.

Examples

Investigate the contnuity of the following functions:

(i) $f(x) = \begin{cases} \frac{x^2 - 4x + 3}{x^2 - 9} & x \neq 3 \\ 1/3 & ; x = 3 \end{cases}$

(ii) $f(x) = \begin{cases} \frac{4 - x}{\sqrt{x} - 2} & ; x \neq 4 \\ 1/4 & ; x = 4 \end{cases}$

(iii) $f(x) = \begin{cases} \frac{\sqrt{x+2} - \sqrt{2}}{4 \sin x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

(iv) $f(x) = \begin{cases} \frac{e^x - e^{-x}}{4 \sin x} & x \neq 0 \\ 1/2 & ; x = 0 \end{cases}$

Solution

(i)

$$\begin{cases} \frac{x^2 - 4x + 3}{x^2 - 9} & x \neq 3 \\ 1/3 & ; x = 3 \end{cases} \quad at\ x = 3$$

From definition,

$$f(3) = {}^1/_3$$

Next $\lim\limits_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9}$

$$\frac{3^2 - 4(3) + 3}{3^2 - 9}$$
$$\frac{9 - 12 + 3}{9 - 9} = \frac{0}{0}$$

Which is indeterminate factorizing

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 1)(x - 3)}{(x - 3)(x + 3)}$$

$$\lim_{x \to 3} \frac{x - 1}{x + 3} = \frac{3 - 1}{3 + 3} = \frac{2}{6} = \frac{1}{3}$$

Thus $\quad f(3) = \lim\limits_{x \to 3} f(x) = \frac{1}{3}$

$f(x)$ is continuous at $x = 3$

(ii)

$$\begin{cases} \frac{4 - x}{\sqrt{x} - 2} & ; x \neq 4 \\ & at\ x = 4 \\ {}^1/_4 & ; x = 4 \end{cases}$$

By definition

$$f(4) = {}^1/_4$$

$$\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2} = \frac{4 - 4}{\sqrt{4} - 2} = \frac{4 - 4}{2 - 2} = \frac{0}{0}$$

Which is indeterminate, rationalizing, we have

$$\lim_{x \to 4} \left[\frac{4 - x}{\sqrt{x} - 2} \times \frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right]$$

$$\lim_{x \to 4} \left[\frac{(4 - x)(\sqrt{x} + 2)}{x - 4}\right]$$

$$\lim_{x \to 4} \left[\frac{(4 - x)(\sqrt{x} + 2)}{x - 4}\right]$$

$$\lim_{x \to 4} \left[\frac{-(x - 4)(\sqrt{x} + 2)}{x - 4}\right]$$

$$\lim_{x \to 4} [-(\sqrt{x} + 2)]$$

$$-(\sqrt{4} + 2)$$

$$-(2 + 2) = -4$$

$$f(4) \neq \lim_{x \to 4} f(x)$$

Thus, $f(x)$ is not continuous at $x = 4$.

(iii)

$$f(x) = \begin{cases} \frac{\sqrt{x + 2} - \sqrt{2}}{4 \sin x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

By difinition

$$f(0) = \frac{1}{4}\sqrt{2} = \frac{\sqrt{2}}{4}$$

Next, $\lim\limits_{x \to 0} \left[\frac{\sqrt{x + 2} - \sqrt{2}}{4 \sin x}\right]$

$$\frac{\sqrt{0 + 2} - \sqrt{2}}{4 \sin 0} = \frac{\sqrt{2} - \sqrt{2}}{4\ (0)}$$

$$= \frac{0}{0}$$

Which is indeterminate

Applying the rule of l'Hopital

$$\frac{\frac{d}{dx}(\sqrt{x + 2} - \sqrt{2})}{\frac{d}{dx}(4 \sin x)} = \frac{{}^1/_2\ (x + 2)^{-1/2}}{4 \cos x}$$

$$= \frac{\frac{1}{2} \frac{1}{(x + 2)^{1/2}}}{4 \cos x} = \frac{1}{8(\sqrt{x + 2}) \cos x}$$

$$\lim_{x \to 0} \left[\frac{1}{8(\sqrt{x + 2}) \cos x}\right]$$

$$= \frac{1}{8(\sqrt{0 + 2}) \cos 0}$$

$$= \frac{1}{8(\sqrt{2}) \cdot 1} = \frac{1}{8\sqrt{2}}$$

$$\frac{1}{8\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{8 \times 2} = \frac{\sqrt{2}}{16}$$

But $f(0) \neq \lim\limits_{x \to 0} f(x)$

Thus, $f(x)$ is not continuous at $x = 0$

(iv)

$$f(x) = \begin{cases} \frac{e^x - e^{-x}}{4 \sin x} & ; x \neq 0 \\ & at\ x = 0 \\ {}^1/_2 & ; x = 0 \end{cases}$$

By definition

$$f(0) = {}^1/_2$$

Next,

$$\lim_{x \to 0} \left[\frac{e^x - e^{-x}}{4 \sin x}\right]$$

$$\frac{e^0 - e^0}{4 \sin 0} = \frac{1 - 1}{4(0)} = \frac{0}{0}$$

Which is indeterminate, applying l'Hopital's rule

$$\frac{\frac{d}{dx}[e^x - e^{-x}]}{\frac{d}{dx}[4 \sin x]}$$

$$= \frac{e^x + e^{-x}}{4 \cos x}$$

$$\lim_{x \to 0} \left[\frac{e^x + e^{-x}}{4 \cos x}\right]$$

$$\frac{e^0 + e^0}{4 \cos 0} = \frac{1 + 1}{4(1)}$$

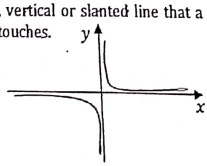
$$= \frac{2}{4} = \frac{1}{2}$$

Since $f(0) = \lim\limits_{x \to 0} f(x) = {}^1/_2$

$f(x)$ is therefore continuous at $x = 0$.

ASYMPTOTE

An asymptote is a value that you get closer and closer to, but never quite reach. In Mathematics an asymptote is a horizontal, vertical or slanted line that a graph approaches but never touches.



A line $x = a$ is said to be a vertical asymptote of the curve $y = f(x)$ if any of the following for statements is true.

(i) $\lim_{x \to a^-} f(x) = \infty$

(ii) $\lim_{x \to a^-} f(x) = -\infty$

(iii) $\lim_{x \to a^+} f(x) = \infty$

(iv) $\lim_{x \to a^+} f(x) = -\infty$

Alternatively: suppose $f(x) = \frac{p(x)}{q(x)}$ where $p(x), q(x)$ are functions, vertical asymptote occurs at $x = a$ if $g(a) = 0$.

A line $x = b$ is a horizontal asymptote of the curve $y = f(x)$ if either

(i) $\lim_{x \to \infty} f(x) = \text{b}$ or

(ii) $\lim_{x \to -\infty} f(x) = \text{b}$

Example 1:

Determine the vertical and horizontal asymptotes of the graph of the following

(i) $y^2 = \frac{x}{x-1}$

(ii) $f(x) = \frac{x^2+1}{x^2-1}$

Solution

(i) $y^2 = \frac{x}{x-1}$

$$y = f(x) = \pm\sqrt{\frac{x}{x-1}} = \pm\frac{\sqrt{x}}{\sqrt{x-1}}$$

We equate the denominator of $f(x)$ to zero

$$\sqrt{x-1} = 0$$
$$x - 1 = 0$$
$$x = 1$$

Thus, the vertical asymptote is $x = 1$

For horizontal asymptote

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \pm\sqrt{\frac{x}{x-1}}$$
$$= \pm \lim_{x \to \infty} \sqrt{\frac{x/x}{x/x - 1/x}}$$
$$= \pm \lim_{x \to \infty} \sqrt{\frac{1}{1 - \frac{1}{x}}}$$
$$= \pm\sqrt{\frac{1}{1 - \frac{1}{\infty}}} = \pm\sqrt{\frac{1}{1-0}}$$
$$= \pm\sqrt{1/1} = \pm 1$$

Thus $y = 1$ and $y = -1$ are the horizontal asyptotes

(ii) $f(x) = \frac{x^2+1}{x^2-1}$

We seek a number $x = a$ for which the function gives ∞ as on the RHS, i.e. equating the denominator to zero

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm\sqrt{1}$$
$$x = \pm 1$$

The verical asymptotes are

$$x = +1 \text{ and } x = -1$$

For horizontal asymptotes,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2+1}{x^2-1}$$
$$\lim_{x \to \infty} \frac{x^2/x^2 + 1/x^2}{x^2/x^2 - 1/x^2}$$
$$\lim_{x \to \infty} \frac{1 + 1/x^2}{1 - 1/x^2}$$
$$\frac{1 + 1/\infty^2}{1 - 1/\infty^2} = \frac{1+0}{1-0}$$
$$= 1$$

$\therefore$ horizontal asymptote, $y = 1$

Example 2

Determine all vertical asymptotes of the graph of the following:

(i) $f(x) = \cot x$

(ii) $f(x) = \frac{1}{\sqrt{49 - x^2}}$

Solution

(i) $f(x) = \cot x$

$$f(x) = \frac{1}{\tan x} = \frac{\frac{1}{\sin x}}{\cos x}$$

$$f(x) = \frac{\cos x}{\sin x}$$

To get the vertical asymtotes equate $\sin x = 0$

$$x = \sin^{-1}(0) + 2n\pi$$

For $\text{n} = 0, 1, 2, \ldots$

$\therefore$ the vertical asymptotes are $x = 0, 2\pi, 4\pi, \ldots$

(ii) $y = \frac{1}{\sqrt{49 - x^2}}$

The vertical asymptotes are gotten when

$$\sqrt{49 - x^2} = 0$$
$$49 - x^2 = 0$$
$$x^2 = 49$$
$$x = \pm\sqrt{49}$$
$$x = \pm 7$$

$\therefore$ the vertical asymptotes are $x = 7$ and $x = -7$

THE DERIVATIVE

The derivative of a function of a real variable measures the sensitivity to change of a quantity (dependent variable) which is determined by another quantity (the independent variable).

Derivatives are fundamental tool of calculus. For example the derivative of the position of a moving object with respect to time is the objects velocity. This measures how quick the position of the object changes when time is advanced.

The derivative of a function of a single variable at a chosen input value, when it exists, is the slope of the tangent line to the graph of the function at that point.
The tangent line is the best linear approximation of the function near that input value. For this reason, the derivative is often decribed as the "instantaneous rate of change" i.e the ratio of the instantaneous change in the dependent variable to that of the independent variable.
The process of finding a derivative is called differentiation. The reverse process is called anti-differentiation. The fundamental theorem of calculus states that anti-differentiation is the same as integration.
Differentiation and intergration constitute the two fundamental operations in single variable calculus.

DERIVATIVE AT A POINT

Let $y = f(x)$ be a real valued function of a real variable and let x be in the domain of f, then $f(x)$ is said to be differentiable at x if:
$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists.
If the limit exists, the result is called the derivative of $y = f(x)$ at x. It is denoted by $\frac{dy}{dx}$ (read as dee y dee x) or $f'(x)$ (read as f prime of x). If $f'(x)$ exists, we can call it

(i) The derivative of $f(x)$ at x
(ii) The instantaneous rate of change of $f(x)$ at x.
(iii) The slope of the graph of $f(x)$ at any x
(iv) The differential coefficient of $f(x)$.

DIFFERENTIATION FROM THE FIRST PRINCIPLE

To differentiate a function $f(x)$ from the first principle, we make use of the fact that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Or $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Example1

Differentiate $f(x) = x^2$, using first principle

Solution

$$f(x) = x^2$$

$f(x+h) = (x+h)^2$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$\frac{dy}{dx} = \lim_{h \to 0} (2x+h)$

$\frac{dy}{dx} = 2x + 0$

$\frac{dy}{dx} = 2x$

Example2

Use the defintion of the derivative to find the first derivative of $y = \sqrt{x}$

Solution

$f(x) = \sqrt{x}, \quad f(x+h) = \sqrt{x+h}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rationalizing, we have

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{1}{\sqrt{x+h}+\sqrt{x}}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Example3

Differentiate $f(x) = \cos x$, using first principle

Solution

Given $f(x) = \cos x$

$f(x+h) = \cos(x+h)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x+h)-\cos x}{h}$$

From basic trigonometry

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{-\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h/2}$$

$$\frac{dy}{dx} = \lim_{h \to 0} -\sin\left(\frac{2x+h}{2}\right) \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$\frac{dy}{dx} = \lim_{h \to 0} -\sin\left(\frac{2x+h}{2}\right) 1$$

$$\frac{dy}{dx} = -\sin\left(\frac{2x+0}{2}\right)$$

$$\frac{dy}{dx} = -\sin x$$

<u>Example 4</u>: using first principle,

Find $f'(x)$ if $f(x) = \sin x$,

<u>Solution</u>

$f(x) = \sin x$, $f(x+h) = \sin(x+h)$

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

From basic trigonometry,

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) 1$$

$f'(x) = \cos\frac{2x}{2} = \cos x$

$\therefore f'(x) = \cos x$

STANDARD DERIVATIVES

Here is a list of basic derivatives

s/n	$f(x)$	$f'(x)$ or $\frac{dy}{dx}$
1	x^n	nx^{n-1}
2	$\ln x$	$1/x$
3	e^x	e^x
4	e^{kx}	ke^{kx}
5	a^x	$a^x \ln a$
6	$\cos x$	$-\sin x$
7	$\sin x$	$\cos x$
8	$\tan x$	$\sec^2 x$
9	$\cosh x$	$\sinh x$
10	$\sinh x$	$\cosh x$
11	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
12	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
13	$\tan^{-1} x$	$\frac{1}{1+x^2}$
14	$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
15	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
16 .	$\tanh^{-1} x$	$\frac{1}{1-x^2}$

FORMULAE FOR DIFFERENTIATION

Here we develop rules, methods and theorems that will hel us find the derivatives of various types of functions whic are combinations of those treated previously.

THEOREM 2.1

If c is a constant and if $f(x)$ is a differentiable function, then

$$\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}\,f(x)$$

Example 1:

Find the derivative of $f(x) = 5\tan x$.

Solution

$$\frac{d}{dx}(5\tan x) = 5\,\frac{d}{dx}\tan x$$

$= 5\sec^2 x$

Example 2

Find $\frac{dy}{dx}$ if $y = 7e^{2x}$

Solution

$$\frac{d}{dx}[7e^{2x}] = 7\,\frac{d}{dx}\,[e^{2x}]$$

$= 7\,[2e^{2x}] = 14e^{2x}$

THEOREM 2.2 (derivative of the sum (or difference))

The derivative of the sum (or difference) of two differentiable functions is equal to the sum (or difference) of their derivatives. i.e if f and g are differentiable functions of x, then

$$\frac{d}{dx}(f \pm g) = \frac{d}{dx}\,f(x) \pm \frac{d}{dx}\,g(x)$$

Example 3

Use the differentiation formular to find the derivative of the function $y = 5x^3 - 10x^2 + 5x + 6$

Solution

$$y = 5x^3 - 10x^2 + 5x + 6$$

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^3) - 10\,\frac{d}{dx}(x^2) + 5\frac{d}{dx}(x) + \frac{d}{dx}\,(6)$$

$$\frac{dy}{dx} = 5(3x^{3-1}) - 10\,(2x^{2-1}) + 5(x^{1-1}) + 0$$

$$\frac{dy}{dx} = 15x^2 - 20x + 5x^0$$
$$\frac{dy}{dx} = 15x^2 - 20x + 5$$

THEOREM 2.3 (PRODUCT RULE)

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

i.e if u and v are differentiable functions of the same variable, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 1

Find $\frac{dy}{dx}$ if $y = (x^2 + 1)(2x - 3)$

Solution

$$y = (x^2 + 1)(2x - 3)$$

Using product rule

Let $u = x^2 + 1$, $\frac{du}{dx} = 2x$

And $v = 2x - 3$, $\frac{dv}{dx} = 2$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x^2 + 1)(2) + (2x - 3)2x$$
$$\frac{dy}{dx} = 2x^2 + 2 + 4x^2 - 6x$$
$$\frac{dy}{dx} = 6x^2 - 6x + 2$$

Or $\frac{dy}{dx} = 2(3x^2 - 3x + 1)$

Example 2

Given that $f(x) = (x^2 + 3)\sin 5x$

Find $f'(x)$.

Solution

$f(x) = (x^2 + 3)\sin 5x$

Set $u = x^2 + 3$ and $v = \sin 5x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 5\cos 5x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = (x^2 + 3)(5\cos 5x) + (\sin 5x)(2x)$$
$$\frac{dy}{dx} = 5(x^2 + 3)\cos 5x + 2x\sin 5x$$

THEOREM 2.4 (QUOTIENT RULE)

The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

i.e if u and v are differentiable functions of the same variable and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1

Find the derivative of

$$f(x) = \frac{x}{x + \cos x}$$

Solution

$$f(x) = \frac{x}{x + \cos x}$$

Set $u = x$, $v = x + \cos x$

$\frac{du}{dx} = 1$, $\frac{dv}{dx} = 1 - \sin x$

Using quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{(x + \cos x)(1) - x(1 - \sin x)}{(x + \cos x)^2}$$
$$\frac{dy}{dx} = \frac{x + \cos x - x + \sin x}{(x + \cos x)^2}$$
$$\frac{dy}{dx} = \frac{\cos x + \sin x}{(x + \cos x)^2}$$

Example 2

Find the differential coefficient of

$$y = \frac{1 + \tan x}{1 - \tan x}$$

Solution

$$y = \frac{1 + \tan x}{1 - \tan x}$$

Using quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

But, $u = 1 + \tan x$, $v = 1 - \tan x$

$$\frac{du}{dx} = \sec^2 x \quad , \quad \frac{dv}{dx} = -\sec^2 x$$

$$\frac{dy}{dx} = \frac{(1 - \tan x)\sec^2 x - (1 + \tan x)(-\sec^2 x)}{(1 - \tan x)^2}$$
$$\frac{dy}{dx} = \frac{\sec^2 x - \tan x\sec^2 x + \sec^2 x + \tan x\sec^2 x}{(1 - \tan x)^2}$$
$$\frac{dy}{dx} = \frac{2\sec^2 x}{(1 - \tan x)^2}$$

Example 3

If $y = \operatorname{cosec} x$ find $\frac{dy}{dx}$

Solution

$$y = \operatorname{cosec} x = \frac{1}{\sin x}$$

Set $u = 1$, $v = \sin x$

$$\frac{du}{dx} = 0 \quad , \quad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{\sin x(0) - 1(\cos x)}{(\sin x)^2}$$
$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x}$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$
$$\frac{dy}{dx} = -\cot x \operatorname{cosec} x$$

THEOREM 2.5 (CHAIN RULE)

If f and g are differentiable functions and $y = f(u)$ when $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

NOTE: The method of differentiating function of a function is also called Chain rule.

Example 1

Find the deriavtive of $f(x) = (7x^4 - 5x + 3)^{-5}$

Solution

$$f(x) = (7x^4 - 5x + 3)^{-5}$$

Let $u = 7x^4 - 5x + 3$

So that $y = u^{-5}$

$$\frac{dy}{du} = -5u^{-5-1}$$
$$\Rightarrow \frac{dy}{du} = -5u^{-6}$$
$$\frac{du}{dx} = 4(7x^{4-1}) - 5x^{1-1} + 0$$
$$\frac{du}{dx} = 28x^3 - 5x^0$$
$$\Rightarrow \frac{du}{dx} = 28x^3 - 5$$

$\therefore$ By Chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -5u^{-6} \times (28x^3 - 5)$$
$$\frac{dy}{dx} - 5(28x^3 - 5)u^{-6}$$
$$\frac{dy}{dx} = \frac{-5(28x^3 - 5)}{u^6}$$

But $u = 7x^4 - 5x + 3$

$$\therefore \frac{dy}{dx} = \frac{-5(28x^3 - 5)}{(7x^4 - 5x + 3)^6}$$

Example 2

Find the differential coefficient of $y = \sin^2 2x$

Solution

$$y = \sin^2 2x$$

Or $y = [\sin 2x]^2$

Set $u = \sin 2x$, so that $y = u^2$

$\frac{dy}{du} = 2u$; $\frac{du}{dx} = 2\cos 2x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 2u \times 2\cos 2x$$
$$\frac{dy}{dx} = 4u\cos 2x$$

But $u = \sin 2x$

$$\therefore \frac{dy}{dx} = 4\sin 2x \cos 2x$$

Example 3

Find $\frac{dy}{dx}$ if

$$y = \sqrt{\frac{x-1}{x+1}}$$

Solution:

$$y = \sqrt{\frac{x-1}{x+1}} = \left(\frac{x-1}{x+1}\right)^{1/2}$$

Let $u = \frac{x-1}{x+1}$, so that

$$y = u^{1/2}$$
$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}$$

Using quotient rule to find du/dx

$$\frac{du}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$
$$\frac{du}{dx} = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$
$$\frac{du}{dx} = \frac{x+1 - x+1}{(x+1)^2}$$
$$\frac{du}{dx} = \frac{2}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times \frac{2}{(x+1)^2}$$
$$\frac{dy}{dx} = u^{-\frac{1}{2}} \times \frac{1}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{u^{1/2}(x+1)^2}$$

But $u = \frac{x-1}{x-1}$

$$\frac{dy}{dx} = \frac{1}{\left[\frac{(x-1)}{x+1}\right]^{1/2}(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{\frac{(x-1)^{1/2}}{(x+1)^{1/2}}(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{(x-1)^{1/2}(x+1)^{\frac{3}{2}}}$$

PARAMETRIC EQUATIONS

Suppose $x = x(t)$ and $y = y(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Note: $\frac{dt}{dx} = \frac{1}{dx/dt}$

The second derivative of equations defined parametrically obtained from the formula:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

Example 1

If $x = e^t \sin t$, $y = e^t \cos t$,

Find $\frac{dy}{dx}$

Solution

$x = e^t \sin t$,

By produt rule

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t$$

$$y = e^t \cos t$$

By product rule,

$$\frac{dy}{dt} = e^t \cos t - e^t \sin t$$

But $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ or $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$

$$\frac{dy}{dx} = (e^t \cos t - e^t \sin t) \times \frac{1}{e^t \cos t + e^t \sin t}$$

$$\frac{dy}{dx} = \frac{e^t \cos t - e^t \sin t}{e^t \cos t + e^t \sin t}$$

$$\frac{dy}{dx} = \frac{e^t [\cos t - \sin t]}{e^t [\cos t + \sin t]}$$

$$\frac{dy}{dx} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

Example2

If $y = a(1 - t\sin t)$ and $x = a(t - t\cos t)$ where a is a constant, find $\frac{dy}{dx}$

Solution

$$y = a(1 - t\sin t)$$

$$\frac{dy}{dt} = a[0 - (\sin t + t\cos t)]$$

$$\frac{dy}{dt} = a[0 - \sin t - t\cos t]$$

$$\frac{dy}{dt} = -a(\sin t + t\cos t)$$

$$x = a(t - t\cos t)$$

$$\frac{dx}{dt} = [1 - (\cos t - t\sin t)]$$

$$\frac{dx}{dt} = a(1 - \cos t + t\sin t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$\frac{dy}{dx} = -a(\sin t + t\cos t) \times \frac{1}{a(1 - \cos t + t\sin t)}$$

$$\frac{dy}{dx} = \frac{-a(\sin t + t\cos t)}{a(1 - \cos t + t\sin t)}$$

$$\frac{dy}{dx} = -\left(\frac{\sin t + t\cos t}{1 - \cos t + t\sin t}\right)$$

Example 3

If $x = \frac{2}{1+t^2}$, $y = \frac{2}{t(1+t^2)}$

Find $\frac{dy}{dx}$

Solution

$$x = \frac{2}{1+t^2} = 2(1+t^2)^{-1}$$

By chain rule

$$\frac{dx}{dt} = -2(1+t^2)^{-1-1}(2t)$$

$$\frac{dx}{dt} = -4t(1+t^2)^{-2} = \frac{-4t}{(1+t^2)^2}$$

$$y = \frac{2}{t(1+t^2)} = \frac{2}{t+t^3} = 2(t+t^3)^{-1}$$

$$\frac{dy}{dt} = -2(t+t^3)^{-1-1}(1+3t^2)$$

$$\frac{dy}{dt} = \frac{-2(1+3t^2)}{(t+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$\frac{dy}{dx} = \frac{-2(1+3t^2)}{(t+t^3)^2} \times \frac{1}{\frac{-4t}{(1+t^2)^2}}$$

$$\frac{dy}{dx} = \frac{-2(1+3t^2)}{(t+t^3)^2} \times \frac{(1+t^2)^2}{-4t}$$

$$\frac{dy}{dx} = \frac{(1+3t^2)(1+t^2)^2}{2t(t+t^3)^2}$$

$$\frac{dy}{dx} = \frac{(1+3t^2)(1+t^2)^2}{2t[t(1+t^2)]^2}$$

$$\frac{dy}{dx} = \frac{(1+3t^2)(1+t^2)^2}{2t \cdot t^2(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{1+3t^2}{2t^3}$$

Example 4

If $x = 3e^{-t}$ $y = \frac{1}{2}e^t$;   Find $\frac{dy}{dx}$

Solution.

$$x = 3e^{-t}; \quad \frac{dx}{dt} = -3e^{-t}$$

$$y = \frac{1}{2}e^t, \quad \frac{dy}{dt} = \frac{1}{2}e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$\frac{dy}{dx} = \frac{1}{2}e^t \times \frac{1}{-3e^{-t}}$$

$$\frac{dy}{dx} = \frac{1}{2}e^t \times \frac{e^t}{-3}$$

$$\frac{dy}{dx} = -\frac{1}{6}e^{2t}$$

Example 5: The equations of a curve is defined parametrically by $y = 1 + \cos t$ and $x = \sin t$. Find

Solution:

$$y = 1 + \cos t, \quad x = \sin t$$

$$\frac{dy}{dt} = -\sin t; \quad \frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$\frac{dy}{dx} = -\sin t \times \frac{1}{\cos t} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}(-\tan t)\left(\frac{1}{\cos t}\right)$$

$$= (-\sec^2 t)(\sec t)$$

$$\frac{d^2y}{dx^2} = -\sec^3 t$$

IMPLICIT FUNCTION

If $y = x^2 + 5x - 3$, y is completely defined in terms of x and y is called an explicit function of x.

When the relationship between x and y is more involved, it may not be posible to separate y completely on the left hand side, e,g $xy + \cos y = 3$ in such a cases, y is called an implicit function of x

NOTE: Here we attach the differentiatial coefficient ${dy}/{dx}$ whenever we differentiate a term containing y

Example1:

If $x^2 - 3xy + 4y = 0$ defines y as an implicit function of x. Find $\frac{dy}{dx}$

Solution:

$$x^2 - 3xy + 4y = 0$$

Differentiating both sides with respect to x, we differentiate the xy term using product rule

$$2x - 3\left[y + x\frac{dy}{dx}\right] + 4\frac{dy}{dx} = 0$$

$$2x - 3y - 3x\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$$

$$(4 - 3x)\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{4 - 3x}$$

Example 2

If $x^2 + 2xy + 3y^2 = 4$, find $\frac{dy}{dx}$

Solution

Given $x^2 + 2xy + 3y^2 = 4$

Differentiate both sides w.r.t. x

$$2x + 2\left[x\frac{dy}{dx} + y(1)\right] + 6y\frac{dy}{dx} = 0$$

$$2x + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 6y\frac{dy}{dx} = -2x - 2y$$

$$(2x + 6y)\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 6y}$$

$$\frac{dy}{dx} = \frac{-2(x + y)}{2(x + 3y)}$$

$$\frac{dy}{dx} = -\frac{(x + y)}{(x + 3y)}$$

Example 3

If $x^3 + y^3 + 3xy^2 = 7$ find $\frac{dy}{dx}$

Solution

$$x^3 + y^3 + 3xy^2 = 7$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left[x\left(2y\frac{dy}{dx}\right) + (1)y^2\right] = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 = 0$$

$$3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} = -3x^2 - 3y^2$$

$$(3y^2 + 6xy)\frac{dy}{dx} = -3x^2 - 3y^2$$

$$\frac{dy}{dx} = \frac{-3x^2 - 3y^2}{3y^2 + 6xy}$$

$$\frac{dy}{dx} = \frac{-3(x^2 + y^2)}{3(y^2 + 2xy)}$$

$$\frac{dy}{dx} = -\frac{(x^2 + y^2)}{(y^2 + 2xy)}$$

HIGHER ORDER DERIVATIVES

Since the derivative of a function is another function, it is possible to differentiate the derived function again to produce another function. The process can be continued for as long as the resulting function if differentiable. If $y = f(x)$, differentiating once we have $\frac{dy}{dx} = f'(x)$,

Differentiating again, we have, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = [f'(x)]'$

The second derivative of y can also be expressed as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ (read as dee square y, dee x squared)

We can continue the process to get

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4}$$

In general, the nth derivative of y is

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^n(x)$$

Example 1

If $y = \frac{x}{e^x}$, show that

$$\frac{d^2y}{dx^2} = (x - 2)e^{-x}$$

Solution

Given, $y = \frac{x}{e^x}$,set , $u = x$, $v = e^x$

$$\frac{du}{dx} = 1, \quad \frac{dv}{dx} = e^x$$

Using quotient rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{e^x(1) - x(e^x)}{(e^x)^2}$$

$$\frac{dy}{dx} = \frac{e^x - xe^x}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{(1 - x)e^x}{e^{2x}}$$

Again set $u = (1 - x)e^x$

And $v = e^{2x}$, so that

$$\frac{du}{dx} = (1 - x)e^x - e^x$$

$$\frac{dv}{dx} = 2e^{2x}$$

By quotient rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(1-x)e^x}{e^{2x}}\right]$$

$$= \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

$$= \frac{e^{2x}[(1-x)e^x - e^x] - (1-x)e^x[2e^{2x}]}{[e^{2x}]^2}$$

$$= \frac{(1-x)e^{3x} - e^{3x} - 2(1-x)e^{3x}}{e^{4x}}$$

$$\frac{d^2y}{dx^2} = \frac{[(1-x) - 1 - 2(1-x)]e^{3x}}{e^{4x}}$$

$$\frac{d^2y}{dx^2} = \frac{[1 - x - 1 - 2 + 2x]e^{3x}}{e^{4x}}$$

$$\frac{d^2y}{dx^2} = (x-2)e^{3x-4x}$$

$\frac{d^2y}{dx^2} = (x-2)e^{-x}$ proved

Example 2

If $y = e^{-ax}$ where a is a constant, show that

$y'' + 2ay' + a^2y = 0$

Solution

$$y = e^{-ax}, \quad y' = -ae^{-ax}$$

$$y'' = \frac{d^2y}{dx^2} = a^2e^{-ax}$$

$\therefore y'' + 2ay' + a^2y$

$= a^2e^{-ax} + 2a(-ae^{-ax}) + a^2e^{-ax}$

$= a^2e^{-ax} + a^2e^{-ax} - 2a^2e^{-ax}$

$= 2a^2e^{-ax} - 2a^2e^{-ax} = 0$ proved

Example 3

If $x = 2\cosh t$, $y = 2\sinh t$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solution

$$x = 2\cosh t \quad , \quad y = 2\sinh t$$

$$\frac{dx}{dt} 2\sinh y, \frac{dy}{dt} = 2\cosh t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$

$$\frac{dy}{dx} = 2\cosh t \times \frac{1}{2\sinh t}$$

$$\frac{dy}{dx} = \frac{2\cosh t}{2\sinh t}$$

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t}$$

$$\frac{dy}{dx} = \coth t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\coth t) = \frac{d}{dx}\left(\frac{\cosh t}{\sin h\, t}\right)$$

$$= \frac{d}{dt}\left(\frac{\cos h\, t}{\sin h\, t}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{\cos h\, t}{\sin h\, t}\right) \times \frac{1}{\frac{dx}{dt}}$$

$$\left[\frac{\sin h\, t\,(\sin h\, t) - \cosh t\,(\cosh t)}{(\sin h\, t)^2}\right] \times \frac{1}{2\sin h\, t}$$

$$= \left[\frac{\sin h^2 t - \cos h^2 t}{\sin h^2 t}\right] \times \frac{1}{2\sinh t}$$

$$= \frac{-(\cosh^2 t - \sinh^2 t)}{\sinh^2 t} \cdot \frac{1}{2\sinh t}$$

$$\frac{-(\cosh^2 t - \sinh^2 t)}{2\sinh^3 t}$$

$$= \frac{-1}{2\sinh^3 t}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{2}\text{cosech}^3 t$$

Example 4

If $x = \frac{1 + \ln t}{t^2}$, $y = \frac{3 + 2\ln t}{t}$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solution

$$x = \frac{1 + \ln t}{t^2}$$

By quotient rule

$$\frac{dx}{dt} = \frac{t^2\left(\frac{1}{t}\right) - (1 + \ln t)(2t)}{(t^2)^2}$$

$$\frac{dx}{dt} = \frac{t - 2t - 2t\ln t}{t^4}$$

$$\frac{dx}{dt} = \frac{-t - 2t\ln t}{t^4}$$

$$\frac{dx}{dt} \frac{-t(1 + 2\ln t)}{t^4}$$

$$\frac{dx}{dt} = \frac{-(1 + 2\ln t)}{t^3}$$

$$y = \frac{3 + 2\ln t}{t}$$

By quotient rule,

$$\frac{dy}{dt} = \frac{t\left(0 + \frac{2}{t}\right) - (3 + 2\ln t)(1)}{t^2}$$

$$\frac{dy}{dt} = \frac{2 - 3 - 2\ln t}{t^2}$$

$$\frac{dy}{dt} = \frac{-1 - 2\ln t}{t^2}$$

$$\frac{dy}{dt} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{-(1 + 2\ln t)}{t^2} \times \frac{1}{\frac{-(1 + 2\ln}{t^3}}$$

$$\frac{dy}{dx} = \frac{(1 + 2\ln t)}{t^2} \times \frac{t^3}{-(1 + 2\ln}$$

$$\frac{dy}{dx} = -\frac{(1 + 2\ln t)}{t^2} \times \frac{t^3}{-(1 + 2\ln}$$

$$\frac{dy}{dx} = t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(t) = \frac{d}{dt}(t)\frac{dt}{dx}$$

$$= \frac{d}{dt}(t) \cdot \frac{1}{dx/dt}$$

$1 . \frac{1}{\frac{-(1+2\ln t)}{t^3}}$

$$\frac{d^2y}{dx^2} = \frac{-t^3}{(1+2\ln t)}$$

PPLICATIONS OF DIFFERENTIATION

this section, we will consider some applications of fferentiation.

RATE OF CHANGE

order to solve problems on rate of change, we employ the lea of chain rule.

xample 1

the position of a particle at a time tsec along a straight line given by $s = \left(\frac{4t^3}{3} + t + 5\right)m$

ind the velocity and acceleration of $t = 2$sec. when is the article at rest?

olution

$$s = \frac{4t^3}{3} + t + 5$$

'elocity $= \frac{ds}{dt} = 4t^2 + 1$

'elocity at 2sec $= \left[\frac{ds}{dt}\right]_{t=2}$

$= 4(2)^2 + 1 = 17ms^{-1}$

Acceleration $= \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$

$= \frac{d}{dt}(4t^2 + 1) = 8t$

Acceleration at 2sec $= \left[\frac{d^2s}{dt^2}\right]_{t=2}$

$= 8(2) = 16ms^{-2}$

The particle is at rest when $\frac{ds}{dt} = 0$

$\therefore 4t^2 + 1 = 0$

$4t^2 = -1 \Rightarrow t^2 = \frac{-1}{4} \Rightarrow t = \pm\sqrt{-\frac{1}{4}}$

Which is imaginary therefore the partcle can never be at rest.

Example 2:

The radius of a cylindrical object is decreasing at the rate of $0.5\ m/s$. find the rate of change of the volume given that its radius is 11m and height 13m.

Solution:

Volume of Cylinder, $v = \pi r^2 h$

$$\frac{dv}{dr} = 2\pi rh$$

Rate of change of radius,

$$\frac{dr}{dt} = 0.5\,{}^{m}/_{s}$$

r= 11 m, h = 13

The rate of change of the volume

$$\frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt}$$

$$\frac{dv}{dt} = 2\pi rh \times 0.5$$

$$\frac{dv}{dt} = 2 \times \pi \times 11 \times 13 \times 0.5$$

$$\frac{dv}{dt} = 14\pi m^3/s$$

Example 3

The positions of two part'cles P_1 and P_2 at the end of t sec are given by

$$P_1 = 3t^3 - 12t^2 + 18t + 5,$$

$P_2 = -t^3 + 9t^2 - 12t$, when do the two particles have the same velocity?

Solution

The particles have the same velocity when

$$\frac{dP_1}{dt} = \frac{dP_2}{dt}.$$

$$\frac{dP_1}{dt} = 9t^2 - 24t + 18$$

$$\frac{dP_2}{dt} = -3t^2 + 18t - 12$$

$9t^2 - 24t + 18 = -3t^2 + 18t - 12$

$9t^2 + 3t^2 - 24t - 18t + 18 + 12 = 0$

$12t^2 - 42t + 30 = 0$

Divide through by 6

$2t^2 - 7t + 5 = 0$

$2t^2 - 2t - 5t + 5 = 0$

$2t(t - 1) - 5(t - 1) = 0$

$(t - 1)(2t - 5) = 0$

$$t = 1 \text{ or } {}^5/_2$$

Thus, the particles have the same velocity at

$$t = \frac{5}{2}\text{seconds and } t = 1 \text{ second}$$

2. APPROXIMATIONS

Example 1

The outside diameter of a thin sherical shell is 12cm think, use differentials to approximate the volume of the interior of the shell.

Solution

$$v = \frac{4}{3}\pi r^3, \quad r = \frac{d}{2} = \frac{12}{2} = 6cm$$

$$v + \Delta v = \frac{4}{3}\pi(r + \Delta r)^3$$

$$\Delta v = \frac{4}{3}\pi(r + \Delta r)^3 - v$$

$$\Delta v = \frac{4}{3}\pi(r + \Delta r)^3 - \frac{4}{3}\pi r^3$$

$$\Delta v = \frac{4}{3}\pi[(r + \Delta r)^3 - r^3]$$

But $\Delta v = \frac{4}{3}\pi[(6 - 0.3)^3 - 6^3]$

$$\Delta v = \frac{4}{3}\pi[(5.7 - 6^3]$$

$$\Delta v = \frac{4}{3}\pi[(185.193 - 216]$$

$$\Delta v = -41.076\pi$$

$\therefore$ Approximate volume $= v + \Delta v$

$= \frac{4}{3}\pi 6^3 - 41.076\pi$

$= 288\pi - 41.076\pi$

$= 246.924\pi\ cm^3$

Example 2:

If $F = {}^{m}/_{r^2}$ is given relation between F and r (m is constant), find the approximate change of F when r is reduced from 2 to 1.98

Solution

$F = \frac{m}{r^2}$

$$F + \Delta F = \frac{m}{(r+\Delta r)^2}$$

$$\Delta F = \frac{m}{(r+\Delta r)^2} - F$$

$$\Delta F = \frac{m}{(r+\Delta r)^2} - \frac{m}{r^2}$$

$$\Delta F = m\left[\frac{1}{(r+\Delta r)^2} - \frac{1}{r^2}\right]$$

$$r = 2, \Delta r = -(2 - 1.98) = -0.02$$

$$\therefore \Delta F = m\left[\frac{1}{(2-0.02)^2} - \frac{1}{2^2}\right]$$

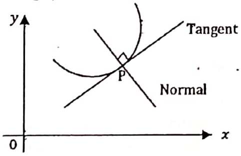
$= m\,(0.2551 - 0.25)$

$\Delta F = 0.0051m$ (m is a constant)

3. EQUATION OF TANGENT AND NORMAL

A tangent to a curve , $y = f(x)$ is a straight line which touches the curve, i.e it cuts the curve in two coincident points.

The normal to a curve, $y = f(x)$ at a point P parpendicular to the tangent at the curve at P



we note the following

- The equation of a line passing through (x_1, y_1) with slope m is given as: $y - y_1 = m(x - x_1)$
- If two line are parallel with slopes m_1 and m_2, then their slopes are equal, i.e m_1 and m_2
- If two line are perpendicular with slope $m_1 m_2 = -1$

Example 1

Find the equation of the tangent and normal line to the graph of functions given below at the points

(i) $x^2 + y^2 = 16$; $(2,3)$

(ii) $x^2 + xy + y^2 = 0$; $(11,1)$

(iii) $x = t + \frac{1}{t}$, $y = t + 1$; $t = 3$

Solution

Note: we need to first determine the slope of the tangent i.e $\frac{dy}{dx}$

(i) $x^2 + y^2 = 16$; $(2,3)$

Differentiating w.r.t x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Slope of tangent at (2, 3),

$$m_1 = \frac{dy}{dx}\Big|_{(2,3)} = \frac{-2}{3}$$

Equation of tangent

$$y - y_1 = m_1(x - x_1)$$

$$y_1 = 3\ ,\quad x_1 = 2\ , m_1 = \frac{-2}{3}$$

$(y - 3) = \frac{-2}{3}(x - 2)$

$3y - 9 = -2x + 4$

$3y = -2x + 4 + 9$

$$3y = -2x + 13$$

$$y = \frac{-2}{3}x + \frac{13}{3}$$

Slope of normal is gotten if $m_1 m_2 = -1$

$$\frac{-2}{3}m_2 = -1$$

$$m_2 = -1 \times \frac{-3}{2}$$

$$m_2 = \frac{3}{2}$$

Equation of normal

$$y - y_1 = m_2(x - x_1)$$

$$y_1 = 3,\quad x_1 = 2\ ,\ m = \frac{3}{2}$$

$$(y - 3) = \frac{3}{2}(x - 2)$$

$$2y - 6 = 3x - 6$$

$$2y = 3x - 6 + 6$$

$$2y = 3x$$

$$y = \frac{3}{2}x$$

(ii) $x^2 + xy + y^2 = 0$; $(11,1)$

Differentiating, we have

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$$

$$m_1 = \frac{dy}{dx}\Big|_{(11,1)} = \frac{-[2(11)+}{11 + 2(1}$$

$$m_1 = \frac{-[22 + 1]}{11 + 2};\ m_1 = \frac{-2}{1}$$

Equation of tangent,

$$y - y_1 = m_1(x - x_1)$$

$$y - 1 = \frac{-23}{13}(x - 11)$$

$$13(y - 1) = -23(x - 11$$

$$13y - 13 = -23x + 253$$

$$13y = -23x + 253 + 13$$
$$13y = -23x + 266$$
$$y = \frac{-23}{13}x + \frac{266}{13}$$

Slope of normal, $m_1m_2 = -1$

$$\frac{-23}{13}m_2 = -1$$
$$m_2 = -1 \times \left(\frac{-13}{23}\right) \Rightarrow m_2 = \frac{13}{23}$$

Equation of normal

$$y - y_1 = m_2(x - x_1)$$
$$y - 1 = \frac{13}{23}(x - 11)$$
$$23(y-1) = 13(x - 11)$$
$$23y - 23 = 13x - 143$$
$$23y = 13x\ 143 + 23$$
$$23y = 13x - 120$$
$$y = \frac{13}{23}x - \frac{120}{23}$$

(iii) $x = t + \frac{1}{t} = t + t^{-1}$

$$\frac{dx}{dt} = 1 - t^{-2} = 1 - \frac{1}{t^2}$$
$$\frac{dx}{dt} = \frac{t^2 - 1}{t^2}$$
$$y = t + 1$$
$$\frac{dy}{dt} = 1$$

$\therefore$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt}$$
$$\frac{dy}{dx} = 1 \times \frac{1}{\frac{t^2-1}{t^2}}$$
$$\frac{dy}{dx} = 1 \times \frac{t^2}{t^2-1} = \frac{t^2}{t^2-1}$$
$$m_1 = \frac{dy}{dx}\Big|_{t=3} = \frac{3^2}{3^2-1} = \frac{9}{8}$$
$$x(t) = t + \frac{1}{t} \quad , \quad y(t) = t + 1$$
$$x(3) = 3 + \frac{1}{3} = \frac{10}{3}$$
$$y(3) = 3 + 1 = 4$$
$$(x_1, y_1) = \left(\frac{10}{3}, 4\right)$$

Equation of tangent

$$(y - y_1) = m_2(x - x_1)$$
$$(y - 4) = \frac{9}{8}\left(x - \frac{10}{3}\right)$$
$$8(y-4) = 9\left(x - \frac{10}{3}\right)$$
$$8y = 9x - 30 + 32$$
$$8y = 9x + 2$$
$$y = \frac{9}{8}x + \frac{2}{8}$$
$$y = \frac{9}{8}x + \frac{1}{4}$$

Equation of nomal

$$m_1m_2 = -1$$
$$\frac{9}{8}m_2 = -1$$
$$m_2 = -1 \times \frac{8}{9}$$
$$m_2 = \frac{-8}{9}$$
$$(y - y_1) = m_2(x - x_1)$$
$$(y - 4) = \frac{-8}{9}\left(x - \frac{10}{3}\right)$$
$$y - 4 = \frac{-8}{9}x + \frac{80}{27}$$
$$y = \frac{-8}{9}x + \frac{80}{27} + 4$$
$$y = \frac{-8}{9}x + \frac{188}{27}$$

4. MAXIMUM AND MINIMUM POINTS

The function, $y = f(x)$at the point x_0 is said to have a turning/critical/stationary point $[x_0, f(x_0)]$ if the first derivative equals zero i.e.

$$f'(x_0) = 0.$$

Maximum point is the point (x_0, y_0) for which $f''(x) < 0$
Minimum point is the point (x_0, y_0) for which $f''(x) > 0$
saddle point is the point (x_0, y_0) for which $f''(x) = 0$

Second derivative test
If $f(x)$ and $f'(x)$ are differentiable then 3 cases arise, if

(i) $f''(x) > 0$, then $[x_0, f(x_0)]$ is a minimum point.
(ii) $f''(x) < 0$, then$[x_0, f(x_0)]$ is a maximum point
(iii) $f''(x) = 0$, *then* $[x_0, f(x_0)]$ is a point of inflexion or saddle point.

Example 1
Find the nature of the turning points of the function

$$y = x^3 - 6x^2 + 9x + 16$$

Solution

$$y = x^3 - 6x^2 + 9x + 16; \quad \frac{dy}{dx} = 3x^2 - 12x + 9$$

For turning points $\frac{dy}{dx} = 0$ *i.e* $3x^2 - 12x + 9 = 0$

$$or\ x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0; \ x = 1,3$$

Substitute for x in the given equation

$$y(x) = x^3 - 6x^2 + 9x + 16$$
$$y(1) = (1)^3 - 6(1)^2 + 9(1) + 16 = 20$$
$$y(3) = (3)^3 - 6(3)^2 + 9(3) + 16 = 16$$

The turning points are $(1, 20)$ *and* $(3, 16)$
Now to get the nature of the turning point,we get the secon derivative,

$$f''(x) = 6x - 12$$

At $(1,20)$; $f''(1) = 6(1) - 12 = -6 < 0$

Since $f''(x) < 0$, $(1, 20)$ is a maximum point and its maximum value is 20.

At $(3,16)$; $f''(3) = 6(3) - 12 = 6 > 0$

Since $f''(x) > 0$, $(3,16)$ is a minimum point and its minimum value is 16.

Example 2

Determine the nature of the graph $f(x) = x^4 - 4x^3 + 12$

Solution

$$f(x) = x^4 - 4x^3 + 12$$
$$f'(x) = 4x^3 - 12x^2$$

The turning point is at $f'(x) = 0$

$$i.e\ \ 4x^3 - 12x^2 = 0$$
$$or\ \ x^3 - 3x^2 = 0$$
$$x^2(x-3) = 0$$
$$\therefore x = 0\ or\ 3$$

Substitute $x = 0$ or 3 in the given equation

$$f(x) = x^4 - 4x^3 + 12$$
$$f(0) = 0^4 - 4(0)^3 + 12 = 12 > 0$$
$$f(3) = 3^4 - 4(3)^3 + 12 = -15 < 0$$

The turning points are $(0,12)$ and $(3,-15)$

Next, to get the nature of the turning points, we get the second derivative, $f''(x) = 12x^2 - 24x$

At $(0,12)$; $f''(0) = 12(0)^2 - 24(0) = 0$

Since $f''(0) = 0$, $(0,12)$ is a saddle point or point of inflexion.

At $(3,-15)$; $f''(3) = 12(3)^2 - 24(3) = 36 > 0$

Since $f''(3) > 0$, $(3,-15)$ is a minimum point and the minimum value is -15.

5. ROLLE'S THEOREM

This is a special case of the mean value theorem (to be explained later) as proposed in 1690 by the French Mathematician Michel Rolle (1652 - 1719).

It states that "let $a < b$, If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) and $f(a) = f(b)$, then there is a point c in (a,b) with $f'(c) = 0$".

Example 1

Let $f(x) = 4x^2 - 20x + 29$. Show that $f(x)$ satisfies the hypotheses of Rolle's theorem in the interval [1, 4], hence find all real numbers c in the open interval (1, 4) such that $f'(c) = 0$

Solution

$$f(x) = 4x^2 - 20x + 29$$
$$a = 1, \quad b = 4$$
$$f(a) = f(1) = 4(1)^2 - 20(1) + 29 = 13$$
$$f(b) = f(4) = 4(4)^2 - 20(4) + 29 = 13$$

$\therefore\ f(a) = f(b)$

Thus $f(x)$ satisfies the hypothesis of Rolle's theorem

There is a real number $c \in (1,4)$ such that $f'(c) = 0$

Recall, $f(x) = 4x^2 - 20x + 29$

$$f'(x) = 8x - 20$$
$$f'(c) = 8c - 20$$
$$8c - 20 = 0$$
$$8c = 20$$
$$c = \frac{20}{8}$$
$$c = \frac{5}{2} = 2.5$$

$\therefore c \in (1,4)$

Example 2

Show that the function $f(x) = (x-a)(x-b)$, $x \in (a,b)$ satisfies the condition of Rolle's theorem. hence find all real numbers c in the open interval (a,b).

Solution

$$f(x) = (x-a)(x-b)$$
$$f(a) = (a-a)(a-b) = 0$$
$$f(b) = (b-a)(b-b) = 0$$

$\therefore f(a) = f(b)$

Next, we seek a point $c \in (a,b)$, such that $f'(c) = 0$

By product rule

$$f'(x) = (x-a)(1) + (x-b)(1)$$
$$f'(x) = (x-a) + (x-b)$$
$$f'(c) = (c-a) + (c-b)$$
$$f'(c) = 2c - (a+b)$$
$$2c - (a+b) = 0$$
$$2c = a + b$$

$c = \frac{a+b}{2} \in (a,b)$

6. MEAN VALUE THEOREM

Let $a < b$. if f is continuous on $[a,b]$ and differentiable on (a,b) then there is a point c in (a,b) with

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Example 1

Find the point in the given interval that satisfies the hypothesis of the Mean Value theorem.

(i) $f(x) = 1 - x^2$; $[0,3]$

(ii) $f(x) = x^2 + 2x^2 - x$; $[-1,2]$

Solution

(i) $f(x) = 1 - x^2$; $[0,3]$

$$f(a) = f(0) = 1 - 0^2 = 1$$
$$f(b) = f(3) = 1 - 3^2 = 1 - 9 = -8$$
$$f'(x) = -2x$$
$$f'(c) = -2c$$

But $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$-2c = \frac{-8-1}{3-0}$$
$$-2c = \frac{-9}{3}$$
$$-2c = -3$$
$$c = \frac{-3}{-2}$$
$$c = \frac{3}{2} = 1.5 \ \in (0,3)$$

(ii) $f(x) = x^3 + 2x^2 - x$; $[-1,2]$

$$f(a) = f(-1) = (-1)^3 + 2(-1)^2 - (-1)$$
$$= -1 + 2(1) + 1 = 2$$

$$f(b) = f(2) = 2^3 + 2(2)^2 - 2$$
$$= 8+8-2 = 14$$
$$f'(x) = 3x^2 + 4x - 1$$
$$f'(c) = 3c^2 + 4c - 1$$

But $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$3c^2 + 4c - 1 = \frac{14-2}{2-(-1)}$$
$$3c^2 + 4c - 1 = \frac{12}{3}$$
$$3c^2 + 4c - 1 = 4$$
$$3c^2 + 4c - 1 - 4 = 0$$
$$3c^2 + 4c - 5 = 0$$

Using the quadratic formular, we get

$$c = \frac{-4 \pm \sqrt{16-4(3)(-5)}}{6}$$
$$c = \frac{-4 \pm \sqrt{76}}{6}$$
$$c = \frac{-4+\sqrt{76}}{6} \quad \text{OR} \quad \frac{-4-\sqrt{76}}{6}$$

$$c = 0.7863 \quad \text{OR} -2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means we will exclude the second one [since it isn't in the interval (-1, 2)] thus, the number we seek in this problem is

$$c = 0.7863$$

Example2

Suppose we know that $f(x)$ is continuous and differentiable on [6, 15]. Let's also suppose we know that $f(6) = -2$ and that we know that $f'(x) \leq 10$. What is the largest possible value for $f(15)$?

Solution

Lets start with the conclusion of the Mean Value Theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(15) - f(6)}{15 - 6}$$

$$f(15) - f(6) = f'(c)[15 - 6]$$
$$f(15) - (-2) = 9f'(c)$$
$$f(15) = -2 + 9f'(c)$$
$$f(15) = -2 + 9f'(c)$$

Now, we know that $f'(x) \leq 10$ so in particular we know that $f'(c) \leq 10$

This gives the following $f(15) = -2 + 9f'(c)$

$$f(15) \leq -2 + 9(10)$$
$$f(15) \leq 88$$

All we did was replace $f'(c)$ with its largest possible value. This means that the largest possible value for $f(15)$ is 88.

7. TAYLOR AND MACLAURIN'S SERIES

The series,

$$\sum_{n=0}^{\infty} \frac{f^n(x_0)(x - x_0)^n}{n!}$$

is called the TAYLOR SERIES for the function f

About the point $x = x_0$.

A function f that possesses a Taylor series expansion about $x = x_0$, with a radius of convergence $\rho > 0$ is said to be analytic at $x = x_0$. If it possesses a Taylor series expansion it must be differentiable.

The Maclaurin series is a special case of Taylor's series about $x_0 = 0$

Thus, the Maclaurin's series expansion of $f(x)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)\, x^n}{n!}$$

Example 1:

Determine the Taylor series about the point $x = 0$ for the function $f(x) = \cos x$

Solution

$$f(x) = \cos x \ , \quad x_0 = 0$$
$$f(x) = \cos x \ , f'(0) = \cos 0 = 1$$
$$f'(x) = -\sin x \ , \ f'(0) = -\sin 0 = 0$$
$$f''(x) = -\cos x, \ f''(0) = -\cos 0 = -1$$
$$f'''(x) = \sin x \ , f'''(0) = \sin 0 = 0$$
$$f^{iv}(x) = \cos x \ , f^{iv}(0) = \cos 0 = 1$$

To determine the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)(x-x_0)^n}{n!}$$
$$= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + .. + \frac{f^n(0)x^n}{n!}$$
$$\therefore \cos x = \frac{1}{0!} + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots.$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 2: Determine whether the function $f(x) = \ln$ possesses a Taylor series about $(a)\ x = 1 (b) x = 0$

Solution

(a) Given , $f(x) = \ln n$, $x_0 = 1$

$$f(x) = \ln x, \quad f(1) = \ln 1 = 0$$
$$f'(x) = \frac{1}{x}, \quad f(1) = {}^1/_1 = 1$$
$$f''(x) = \frac{-1}{x^2}, f''(1) = \frac{-1}{1^2} = -1$$
$$f'''(x) = \frac{2}{x^3}, f'''(1) = \frac{2}{1^3} = 2$$
$$f^{(iv)}(x) = \frac{-6}{x^4}, f^{(iv)}(1) = \frac{-6}{1^4} = -6$$

To determine the Tayor series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)(x - x_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^n(1)(x-1)^n}{n!}$$

$$= \frac{f(1)}{0!} + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(iv)}(1)}{4!}(x-1)^4 + \cdots$$

$$= 0 + \frac{(x-1)}{1!} - \frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

$\therefore$ the Taylor series of $\ln x$ about $x = 1$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

(b) $f(x) = \ln x \;\; , x_o = 0$

$f(x) = \ln x$, $f(0) = \ln 0 =$ undefined

$f'(x) = \frac{1}{x}$, $f'(o) = \frac{1}{0} =$ undefined

$\therefore$ the function $f(x) = \ln x$ does not possess the Taylor series expansion about $x_0 = 0$

8. RELATED RATES

if $y = y(x)$ is such that

$$y = y(t), \;\; x = x(t)$$

Then applying the chain rule we obtain,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This equation relates the time rate of change of y to that of x at any position, x.

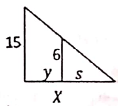
Example 1

A man 6ft tall walks at a rate of 5ft/sec away from a light that is 15ft above the ground. When he is 10ft away from the base of the light,

(a) At what rate is the tip of his shadow moving?

(b) At what rate is the length of his shadow changing?

Solution



(a) Given that $\frac{dy}{dt} = 5ft/sec$, we seek $\frac{dX}{dt}$

Using similar triangles

$$\frac{15}{X} = \frac{6}{X-y}$$
$$15X - 15y = 6X$$
$$9X = 15y$$
$$X = \frac{5}{3}y$$
$$\frac{dX}{dt} = \frac{5}{3}\frac{dy}{dt}$$
$$\frac{dX}{dt} = \frac{5}{3} \times 5ft/sec$$
$$\frac{dX}{dt} = \frac{25}{3}ft/sec$$

(b) Given that $\frac{dy}{dt} = 5ft/sec$, we seek $\frac{ds}{dt}$

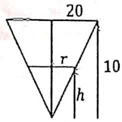
Using similar triangles

$$\frac{6}{s} = \frac{15}{y+s}$$
$$6y + 6s = 15s$$
$$9s = 6y$$
$$s = \frac{2}{3}y$$
$$\frac{ds}{dt} = \frac{2}{3}\frac{dy}{dt}$$
$$\frac{ds}{dt} = \frac{2}{3} \times 5ft/sec$$
$$\frac{ds}{dt} = \frac{10}{3}ft/sec$$

Example 2

A cone-shaped funnel is emptying water at a rate $5cm^3/hr$. The base (positioned at the top) radius is 20 and the height is 10cm. At what rate is the depth of t water in the funnel changing when the depth of the water 6cm.

Solution



$$given, h = 6cm, \;\; \frac{dV}{dt} = 5cm^3/hr$$

By similar triangles,

$$\frac{20}{10} = \frac{r}{h}$$
$$10r = 20h \;\; \Rightarrow r = 2h$$

We know that the general equation for the volume of a co is $V = \frac{1}{3}\pi r^2 h \qquad (*)$

Substituting $r = 2h$ in $(*)$

$$V = \frac{1}{3}\pi(2h)^2 h$$
$$V = \frac{4}{3}\pi h^3$$

Differentiate with respect to time, solve for $\frac{dh}{dt}$, and plug values,

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \Rightarrow \frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{4\pi h^2}\frac{dV}{dt}$$
$$\frac{dh}{dt} = \frac{1}{4\pi(6)^2} \times 5cm^3/hr$$
$$\frac{dh}{dt} \approx 0.0111cm/hr$$

INTEGRATION

Integration is the inverse of differentiation, and its principal concern is to find the value of a function of a variable when its differential coefficient is known i.e, give that

$$\frac{dy}{dx} = 3x + 5$$

We can conveniently find y.

TERMS ASSOCIATED WITH INTEGRATION

INTEGRAL: If the function $f(x)$ is such that when differentiated yeilds the given function $\psi(x)$; then $f(x)$ is said to be the integral of $\psi(x)$ with respect to x.

Thus, if $\frac{d}{dx}[f(x)] = \psi(x)$ then

$$\int \frac{d}{dx}[f(x)]dx = \int \psi(x)dx$$

$\therefore \int \psi(x)dx = f(x) + c$

Where c is any constant called constant of integration.

INTEGRATION: The operation of obtaining the intergral is called intergration, and this operation is denoted by.

$$\int ...dx$$

Note: The symbol $\int$ alone is meaningless without dx which denote what variable we are integrating with respect to.

The symbol "$\int \psi(x)dx$" is read "integral of $\psi(x)$ with respect to x"

INTEGRAND:

The function $\psi(x)$ upon which integration is to be effected is called the integrand

Notice that:

- $f(x) = \int \psi(x)dx$

$f(x)$ is called the indefinite integral of $\psi(x)$

- In the equation $f(x) = \int_a^b \psi(x)dx$

$f(x)$ is called the definite integral of $\psi(x)$.

STANDARD INTERGRALS

S/N	$f(x)$	$\int f(x)dx$
1	x^n	$\frac{x^{n+1}}{n+1}+c$
2	1	$x + c$
3	a	$ax + c$
4	e^x	$e^x + c$
5	a^x	$\frac{a^x}{\ln a}+c$
6	$\frac{1}{x}$	$\ln x + c$
7	$\sin x$	$-\cos x + c$
8	$\cos x$	$\sin x + c$
9	$\sec^2 x$	$\tan x + c$
10	$\operatorname{cosec}^2 x$	$-\cot x + c$
11	$\tan x$	$-\ln \cos x + c$
12	$\cot x$	$\ln \sin x + c$
13	$\frac{f'(x)}{f(x)}$	$\ln f(x) + c$
14	$\sec x$	$\ln(\sec x + \tan x) + c$
15	$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) + c$
16	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
17	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
18	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) + c$
19	$\frac{1}{1-x^2}$	$\tanh^{-1} x + c$
20	$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) + c$

INTERGRATION AS AN INVERSE OF DIFFERENTIATION

Example 1:

Integrate x^3 with respect to x.

Solution:

Now since $\frac{d}{dx}(x^4) = 4x^3$

Integrating both side with respect to x, we have:

$\int \frac{d}{dx}(x^4)dx = \int 4x^3\,dx$ or

$4\int x^3 dx = \int \frac{d}{dx}(x^4)dx$

$4\int x^3\,dx = x^4$

Dividing each side by 4 gives:

$$\int x^3 dx = \frac{x^4}{4} + C$$

In General,

$$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$$

$\therefore \int \frac{d}{dx}(x^{n+1})dx = (n+1)\int x^n\,dx$

Thus, $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ Provided $n \neq -1$

n in x^n may be an integer or a rational number

Thus, $\int x^{1/3}dx = \frac{x^{1/3+1}}{1/3+1} + C$

$$= \frac{x^{4/3}}{4/3} + C$$

$$= \frac{3}{4}x^{4/3} + C$$

RULES FOR INTERGRATION

1. Let $f(x) = \psi'(x)$ so that $\psi(x) = \int f(x)dx$; suppose K is a constant, then:

$$\int Kf(x)dx = K\int f(x)\,dx = K\psi(x)$$

It follows that a constant factor may be brought before the sign of integral

2. Let $\psi(x)$ and $f(x)$ be intergrable functions of $f(x)$, then;

$$\int[\psi(x) \pm f(x)]dx = \int \psi(x)dx \pm \int f(x)dx$$

Thus the integral of a sum or difference of a finite number of functions is the sum or difference of their separate integrals.

Example 1: Solve

(i) $\int 2x\,dx$

(ii) $\int(x^2 + x)dx$

Solution

(i) $\int 2x\,dx = 2\int x dx$

$$= 2\frac{x^2}{2} + C$$

$$= x^2 + C$$

(ii) $\int(x^2 + x)dx$

$$= \int x^2 dx + \int x dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + C$$

3. If $f(x) = \psi(x)$, then

$$\int f(x+k)dx = \psi(x+k)$$

Where k is any constant, i.e. the addition of a constant to the variable makes no difference to the form of the integral E.g

If $\int x^2 = \frac{x^3}{3} + C$, then:

$$\int (x+2)^2 dx = \frac{(x+2)^3}{3} + C$$

WARNING: rule 3 applies only when x is replaced by $(x+h)$ and does not cover the case when the expression in the bracket is not linear

Thus $\int(x^2 + 2)^2 \neq \frac{(x^2+2)^3}{3} + C$

Example 2: Evaluate

(i) $\int \frac{x^2-x+1}{x^3}\,dx$

(ii) $\int(x^2 + 1)^2 dx$

(iii) $\int(\sqrt{x} - \frac{1}{x})^2 dx$

Solution

(i) $\int \frac{x^2-x+1}{x^3}\,dx$

$$= \int\left(\frac{x^2}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}\right)dx$$

$$= \int\left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)dx$$

$$= \int\frac{1}{x}\,dx - \int\frac{1}{x^2}\,dx + \int\frac{1}{x^3}\,dx$$

$$= \int\frac{1}{x}\,dx - \int x^{-2}dx + \int x^{-3}dx$$

$$= \ln x + \frac{x^{-2+1}}{-2+1} + \frac{x^{-3+1}}{-3+1} + C$$

$$= \ln x - \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C$$

$$= \ln x + \frac{1}{x} - \frac{1}{2x^2} + C$$

(ii) $\int(x^2 + 1)^2 dx$

$$= \int(x^2 + 1)(x^2 + 1)dx$$

$$= \int(x^4 + 2x^2 + 1)dx$$

$$= \int x^4 dx + 2\int x^2 dx + \int 1dx$$

$$= \frac{x^5}{5} + \frac{2x^3}{3} + x + C$$

(iii) $\int(\sqrt{x} - \frac{1}{x})^2 dx$

$$= \int\left(\sqrt{x} - \frac{1}{x}\right)\left(\sqrt{x} - \frac{1}{x}\right)dx$$

$$\int\left(x - 2x^{-12} + \frac{1}{x^2}\right)dx$$

$$= \int x dx - 2\int x^{-12}dx + \int dx$$

$$= \frac{x^2}{2} - \frac{2x^{1/2}}{12} + \frac{x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} - 4x^{12} - \frac{1}{x} + C$$

$$\text{OR } \frac{x^2}{2} - 4\sqrt{x} - \frac{1}{x} + C$$

DEFINITE INTEGRAL

If $f(x) = \int \psi(x)dx$, then

$\int_a^b \psi(x)dx = f(b) - f(a)$ is the definite integral over the range of values of x from a to b; a and b are the lower and upper limits of integration, and also called range. It follows

$$\int_a^b \psi(x)dx = [f(x)]_a^b$$

$$= f(b) - f(a) = -f(a) + f(b)$$

$$= -[f(a) - f(b)]$$

$$= -\int_b^a \psi(x)dx$$

Thus, the integral is changed in sign if the limits are interchanged.

Example 1

Evaluate $\int_1^2 (x^2 - 2)dx$

Solution

$$\int_1^2 (x^2 - 2)dx = [\frac{x^3}{3} - 2x]_1^2$$

$$= \left[\frac{2^3}{3} - 2(2)\right] - \left[\frac{1^3}{3} - 2(1)\right]$$

$$= \left(\frac{8}{3} - 4\right) - \left(\frac{1}{3} - 2\right)$$

$$=^{-4}/_3 - (^{-5}/_3)$$
$$= -^{4}/_3 + ^{5}/_3$$
$$= ^{1}/_3$$

METHOD OF INTEGRATION

1. INTEGRATION BY ALGEBRAIC SUBSTITUTION.

An integral may be reduced to a standard integral by making an algebraic substitution and this is done by changing the independent variable, say x to another one, u, where the correspondence between x and u are know. A clue of knowing when to apply this is if given $\int f(x)g(x)dx$ and $\frac{du}{dx} = kf$; then substitution can be applied.

Example 1

Evaluate $I = \int x(x^2 - 2)^4 dx$

Solution

$I = \int x(x^2 - 2)^4 dx$

Let $u = x^2 - 2$

$$\frac{du}{dx} = 2x, \quad dx = \frac{du}{2x}$$

Substituting in $I = \int x(x^2 - 2)^4 dx$

$$\int x u^4 \frac{du}{2x} = \int u^4 \frac{du}{2}$$

$$\frac{1}{2}\int u^4\, du = \frac{1}{2}\left(\frac{u^5}{5}\right) + C$$

$$= \frac{u^5}{10} + C$$

But $u = x^2 - 2$

$$\therefore I = \frac{(x^2-2)^5}{10} + C$$

Example 2

Solve $\int \frac{x^3}{\sqrt{x^4+2}}\, dx$

Solution

$$\int \frac{x^3}{\sqrt{x^4 + 2}}\, dx$$

Let $u = x^4 + 2$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

Substituting in the given problem , we have

$$\int \frac{x^3}{\sqrt{u}} \cdot \frac{du}{4x^3}$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{4}\int \frac{1}{u^{1/2}}\, du$$

$$= \frac{1}{4}\int u^{-1/2}\, du$$

$$= \frac{1}{4}\, \frac{u^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{1}{4}\frac{u^{1/2}}{1/2} + C$$

$$= \frac{1}{4} \times 2u^{1/2} + C$$

$$= \frac{1}{2}u^{1/2} + C$$

$$= \frac{1}{2}\sqrt{u} + C$$

But $u = x^4 + 2$

$$\therefore \int \frac{x^3}{x^4+2}\, dx = \frac{1}{2}\sqrt{x^4+2} + C$$

Example3

Evaluate $\int \frac{\sin x}{1+\cos x}\, dx$

Solution

$$\int \frac{\sin x}{1 + \cos x}\, dx$$

Let $u = 1 + \cos x$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$

Substituting in the given problem

$$\int \frac{\sin x}{u} \frac{du}{-\sin x}$$

$$-\int \frac{1}{u}\, du$$

$$-\ln u + C$$

But $u = 1 + \cos x$

$$\int \frac{\sin x}{1 + \cos x}\, dx = -\ln(1 + \cos x) +$$

Example 4

Evaluate $\int \frac{dx}{e^x + e^{-x}}$

Solution

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{e^x + e^{-x}}\, dx$$

$$but \quad \frac{1}{e^x + e^{-x}} = 1 \div \left(e^x + \frac{1}{e^x}\right)$$

$$= 1 \div \left(\frac{e^{2x}+1}{e^x}\right) = 1 \times \left(\frac{e^x}{e^{2x}+1}\right)$$

$$= \frac{e^x}{e^{2x}+1}$$

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x}+1}\, dx$$

Let $u = e^x$, so that

$$\frac{du}{dx} = e^x$$

$$dx = \frac{du}{e^x}$$

$\Rightarrow dx = \frac{du}{u}$

Substituting in $\int \frac{e^x}{e^{2x}+1}\, dx$

We have,

$$\int \frac{u}{u^2+1}\frac{du}{u}$$

$$\int \frac{1}{u^2+1}\, du$$

$$= \tan^{-1}u + C$$

But $u = e^x$

$$= \tan^{-1}(e^x) + C$$

$$\int \frac{dx}{e^x + e^{-x}} = \tan^{-1}(e^x) + C$$

2. INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Here, we will make use of the fact that integration is the converse of differentiation to integrate certain trigonometric function.

Example 1

Evaluate the following

(i) $\int \sin ax\, dx$

(ii) $\int \cos ax\, dx$

(iii) $\int \sec^2 ax\, dx$

(iv) $\int \csc^2 ax\, dx$

Solution

(i) $\int \sin ax\, dx$

Recall $\frac{d}{dx}(\cos ax) = -a\sin ax$

$$\int \frac{d}{dx}(\cos ax)dx = -a\int \sin ax dx$$

$$\cos ax = -a\int \sin ax\, dx$$

$$\therefore \int \sin ax\, dx = -\frac{1}{a}\cos ax$$

(ii) Likewise

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + c$$

(iii) $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + c$

(iv) $\int \csc^2 ax\, dx = \frac{1}{a}\cot ax + C$

Example2

Find $\int (3\cos x + 2x^2)\, dx$

Solution

$$\int (3\cos x + 2x^2)\, dx$$

$$= 3\int \cos x\, dx + 2\int x^2\, dx$$

$$= 3\sin x + \frac{2}{3}x^3 + C$$

TO INTEGRATE CERTAIN TRIGONOMTERIC FUNCTIONS, the following familiar identities are employed:

(i) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

(ii) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Thus, $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

AND

$$\int \cos^2 x dx = \int \frac{1}{2}(1 + \cos 2x)dx$$

$$= \int \frac{1}{2}dx + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

Example 3

Find $\int \cos^2 2x dx$

Solution:

Since, $\cos^2 x = \frac{1}{2}\ (1 + \cos 2x)$

Then

$$\cos^2 2x = \frac{1}{2}\ (1 + \cos 4x)$$

Therefore,

$$\int \cos^2 2x\, dx = \int \frac{1}{2}(1 + \cos 4x)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4x\, dx$$

$$= \frac{x}{2} + \frac{1}{8}\sin 4x + C$$

Example 4

Evaluate $\int \tan^2 3x\, dx$

Solution

Since $1 + \tan^2 x = \sec^2 x$

$$\Rightarrow \tan^2 x = \sec^2 x - 1$$

Therefore,

$$\tan^2 3x = \sec^2 3x - 1$$

Thus

$$\int \tan^2 3x\, dx = \int (\sec^2 3x - 1)dx$$

$$= \int \sec^2 3x\, dx - \int dx$$

$$= \frac{1}{3}\tan 3x - x + C$$

Integration of powers of sines and cosines: $\sin^m \times \cos^n x\, d$ where m or n is odd integer (not both)

Example 5:

Find $\int \cos^3 x \sin^2 x\, dx$

Solution

Recall $\sin^2 x + \cos^2 x = 1$

$\therefore \cos^2 x = 1 - \sin^2 x$

And

$$\cos^3 x = \cos x\,(1 - \sin^2 x)$$

$$\therefore \int \cos^3 x \sin^2 x\, dx = \int \cos x(1 - \sin^2 x)\sin^2 x dx$$

$$= \int (\sin^2 x - \sin^4 x)\cos x dx$$

$$\Rightarrow dx = \frac{du}{\cos x}$$

$$\int (\sin^2 x - \sin^4 x)\cos x \cdot \frac{du}{\cos x}$$

$$= \int (u^2 - u^4)\,du$$

$$= \int (u^2 - u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Example 6

Evaluate $\int \sin^3 x\, dx$

Solution

$$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$$

$$= \int (1 - \cos^2 x)\sin x\, dx$$

$$= \int \sin x\, dx - \int \cos^2 x \sin x\, dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + c$$

Example 7

Evaluate $\int \cos^3 x\, dx$

Solution

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$$

$$= \int (1 - \sin^2 x)\cos x\, dx$$

$$= \int \cos x\, dx - \int \sin^2 x \cos x\, dx$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Further examples involving even powers and odd powers

Example 1

Evaluate $\int \sec^2 x \tan^2 x\, dx$

Solution

Let $u = \tan x$,

Then $\frac{du}{dx} = \sec^{2x}$, $dx = \frac{du}{\sec^2 x}$

$$\therefore \int \sec^2 x \tan x\, dx = \int \sec^2 x \cdot u^2 \frac{du}{\sec^2 x}$$

$$= \int u^2 du = \frac{u^3}{3} + C$$

$$= \frac{\tan^3 x}{3} + C$$

Example 2

Evaluate $\int \cos^2 x \sin^4 x\, dx$

Solution

Recall $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

And $\sin^2 x = \frac{1}{2}(1 + \cos 2x)$

And $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\int \cos^2 x \sin^4 x dx = \int \frac{1}{2}(1 + \cos 2x)[\frac{1}{2}(1 - \cos 2x)]^2$$

$$= \int \frac{1}{2}(1 + \cos 2x)\frac{1}{4}(1 - \cos 2x)^2 dx$$

$$= \int \frac{1}{8}(1 + \cos 2x)(1 - \cos 2x)(1 - \cos 2x)dx$$

$$= \frac{1}{8}\int (1 - \cos^2 2x)(1 - \cos 2x)dx$$

$$= \frac{1}{8}\int (\sin^2 2x)(1 - \cos 2x)dx$$

$$= \frac{1}{8}\int \sin^2 2x - \sin^2 2x \cos 2x$$

$$= \frac{1}{8}\int \left[\frac{1}{2}(1 - \cos 4x) - \sin^2 2x \cos 2x\right] dx$$

$$\frac{1}{8}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4x - \sin^2 2x \cos 2x\right) dx$$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{2}\frac{\sin 4x}{4} - \frac{1}{2}\frac{\sin^3 2x}{3}\right] + C$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

Example 3:

Evaluate $\int \sin^3 x \cos^5 x\, dx$

Solution

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{du}{\cos x}$$

$$\int \sin^3 x \cos^5 x\, dx = \int u^3 \cos^5 x \frac{du}{\cos x}$$

$$= \int u^3 \cos^4 x\, du$$

$$= \int u^3 [\cos^2 x]^2 du$$

$$\int u^3 [1 - \sin^2 x]^2 du$$

$$\int u^3 [1 - u^2]^2 du$$

$$= \int u^3 [1 - 2u^2 + u^4] du$$

$$= \int [u^3 - 2u^5 + u^7] du$$

$$= \int u^3 du - 2\int u^5 du + \int u^7 du$$

$$\frac{u^4}{4} - 2\frac{u^6}{3} + \frac{u^8}{8} + C$$

$$\frac{u^4}{4} - \frac{u^6}{3} + \frac{u^8}{8} + C$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C$$

NOTE:

(i) $\int f(x) f'(x) dx = \frac{[f(x)]^2}{2} = C$

(ii) $\int \frac{f'(x)}{f(x)} = \ln f(x) + C$

Example 1

Evaluate $\int \frac{\cos x}{1+\sin x}\, dx = \int \frac{d(1+\sin x)}{1+\sin x}$

Solution

$$\int \frac{\cos x}{1+\sin x}\, dx = \int \frac{d(1+\sin x)}{1+\sin x}$$
$$= \ln(1+\sin x) + C$$

Example 2

Evaluate $\int \frac{4x^2}{x^3-7}\, dx$

solution

$$\int \frac{4x^2}{x^3-7} = \frac{4}{3}\int \frac{3x^2}{x^3-7}$$
$$= \frac{4}{3}\ln(x^3-7) + C$$

Example 3

Evaluate $\int \frac{\ln x}{x}\, dx$

Solution

$\int \frac{\ln x}{x}\, dx = \int \ln x . \frac{1}{x}\, dx$

$$\int \ln x\, d(\ln x) = \frac{(\ln x)^2}{2} + C$$

Example 4

Find $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\, dx$

Solution

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}} = \int \sin^{-1}x . \frac{1}{\sqrt{1-x^2}}\, dx$$
$$\int \sin^{-1}x\; d(\sin^{-1}x)$$
$$= \frac{(\sin^{-1}x)^2}{2} + C$$

Integration of products of sines and cosines: the following formula are important

1. $Sin\,A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
2. $Cos\,A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
3. $Cos\,A \sin B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
4. $Sin\,A \sin B = \frac{-1}{2}[\cos(A+B) - \cos(A-B)]$

Example 1

Evaluate

$$\int \sin 4x \cos 2x\, dx$$

Solution

$$\int \sin 4x \cos 2x\, dx$$
$$= \int \frac{1}{2}[\sin(4x+2x) + \sin(4x-2x)]dx$$
$$= \frac{1}{2}\int (\sin 6x + \sin 2x)dx$$
$$= \frac{1}{2}\left[\frac{-1}{6}\cos 6x - \frac{1}{2}\cos 2x\right] + C$$
$$= \frac{-1}{12}\cos 6x - \frac{-1}{4}\cos 2x + C$$
$$= \frac{-1}{12}(\cos 6x + 3\cos 2x) + C$$

3. TRIGONOMETRIC SUBSTITUTIONS

The expression $\sqrt{a^2 - x^2}$ may be reduced to a rational form by changing the variable to say θ,

When $x = a\sin\theta$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1-\sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= a\cos\theta$$

Likewise for

$\sqrt{a^2+x^2}$; set $x = a\tan\theta$

For $\sqrt{x^2-a^2}$; set $x = a\sec\theta$

Example 1 .

Find $\int \frac{1}{\sqrt{9-x^2}}\, dx$

Solution

Let $x = 3\sin\theta$

$$\frac{dx}{d\theta} = 3\cos\theta$$
$$dx = 3\cos\theta\, d\theta$$

Thus,

$$\int \frac{1}{\sqrt{9-x^2}}\, dx = \int \frac{3\cos\theta\, d\theta}{\sqrt{9-(3\sin\theta)^2}}$$
$$= \int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$
$$= \int \frac{3\cos\theta\, d\theta}{\sqrt{9(1-\sin^2\theta)}}$$
$$= \int \frac{3\cos\theta\, d\theta}{\sqrt{9\cos^2\theta}}$$
$$\int \frac{3\cos\theta}{3\cos\theta}d\theta = \int d\theta = \theta + c$$

But $x = 3\sin\theta \Rightarrow \theta = \sin^{-1}(x/3)$

$\therefore \int \frac{1}{\sqrt{9-x^2}}\, dx = \sin^{-1}(x/3) + c$

Example 2

Find $\int \frac{1}{a^2+x^2} dx$

Solution

Let $x = a\tan\theta$

$$\frac{dx}{d\theta} = a\sec^2\theta$$
$$dx = a\sec^2\theta\, d\theta$$

$$\therefore \int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta}$$
$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$
$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\sin^2\theta}$$
$$= \int \frac{1}{a}\, d\theta$$
$$= \frac{1}{a}\int d\theta$$
$$= \frac{\theta}{a} + C$$

But $x = a\tan\theta$

$\therefore \theta = \tan^{-1}(x/a)$

$\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$

Example 3

Find $\int_0^1 \frac{dx}{9[\frac{25}{9}+x^2]}$

solution

$$\int_0^1 \frac{dx}{9[\frac{25}{9}+x^2]}$$

$$= \int_0^1 \frac{dx}{9\left[\left(\frac{5}{3}\right)^2+x^2\right]}$$

$$= \frac{1}{9}\int_0^1 \frac{dx}{\left[\left(\frac{5}{3}\right)^2+x^2\right]}$$

Using,

$$\int_0^1 \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$$

$$\frac{1}{9}\int \frac{dx}{(\frac{5}{3})^2+x^2} = \frac{1}{9}\ [\frac{1}{5/3}\tan^{-1}\left(\frac{x}{5/3}\right)]_0^1$$

$$= \frac{1}{9}\ [\frac{3}{5}\tan^{-1}(\frac{3x}{5})]_0^1$$

$$[\frac{1}{15}\tan^{-1}(\frac{3x}{5})]_0^1$$

$$\frac{1}{15}\ [\tan^{-1}\frac{3(1)}{5} - \tan^{-1}\frac{3(0)}{5}]$$

$$\frac{1}{15}\ [\tan^{-1}\frac{3}{5} - \tan^{-1} 0]$$

$= 0.036$

4. EXPONENTIAL INTEGRALS

Since,

$$\frac{d}{dx}(e^x) = e^x$$

$$\int e^x dx = e^x + C$$

In General,

$$\frac{d}{dx}e^{f(x)}dx = \frac{e^{f(x)}}{f'(x)} + C$$

Where $f'(x)$ is the derivative of $f(x)$.

Example 1

Evaluate $\int x^2e^{x^3}dx$

Solution

$$\frac{d}{dx}\left(e^{x^3}\right) = 3x^2e^{x^3}$$

$$d\left(e^{x^3}\right) = 3x^2e^{x^3}\,dx$$

$$\int d\left(e^{x^3}\right) = 3\int x^2e^{x^3}dx$$

$$e^{x^3} = 3\int x^2e^{x^3}\,dx$$

$\therefore \int x^2e^{x^3}dx = \frac{1}{3}e^{x^3} + C$

Example 2

Evaluate $\int 3^{2x}dx$

Solution

Let $y = 3^{2x}$

Take the natural logarithm (ln) of both sides

$\ln y = \ln 3^{2x}$

$\ln y = 2x\ln 3$

differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = 2\ln 3$$

$$\frac{1}{y}\frac{dy}{dx} = 2\ln 3^2$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 9$$

$$\frac{dy}{dx} = y\ln 9$$

$$\frac{dy}{dx} = 3^{2x}\ln 9$$

$$dy = 3^{2x}\ln 9\,dx$$

$$\int dy = \int 3^{2x}\ln 9\,dx$$

$$y = \int 3^{2x}\ln 9\,dx$$

That is,

$$3^{2x} = \int 3^{2x}\ln 9\,dx$$

$$3^{2x} = \ln 9\int 3^{2x}\,dx$$

$$\int 3^{2x}\,dx = \frac{3^{2x}}{\ln 9} + C$$

5. INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS

Certain rational functions can only be integrable when they have resolved into partial fraction, others can be integrable after reducing by long division.

Example 1:

Evaluate $\int \frac{x+1}{x-1}\,dx$

Solution

N:B: If the degree of the numerator is higher than or equal to that of the denominator, it can be resolved thus

$$\frac{x+1}{x-1} = \frac{(x-1)+1+1}{x-1} = 1 + \frac{2}{x-1}$$

$$\therefore \int \frac{x+1}{x-1}\,dx = \int\left(1+\frac{2}{x-1}\right)dx$$

$$= x + 2\ln(x-1) + C$$

Example 2

Evaluate $\int \frac{x^3}{x-1}\,dx$

Solution

By long division we have that

$$\frac{x^3}{x-1} = x^2 + x + 1 + \frac{1}{x-1}$$

$$\int \frac{x^3}{x-1}dx = \int (x^2 + x + 1)dx + \int \frac{1}{x-1}dx$$

$= \frac{x^3}{3} + \frac{x^2}{2} + x + \ln(x-1) + C$

Example 3

Evaluate $\int \frac{11x+2}{6x^2-x-1}dx$

Solution

By partial fractions

$$\frac{11x+2}{6x^2-x-1} = \frac{11x+2}{(3x+1)(2x-1)} = \frac{A}{3x+1} + \frac{B}{2x-1}$$

Multiplying through by $(3x+1)(2x-1)$

$$11x + 2 = A(2x-1) + B(3x+1)$$

Set $x = -\frac{1}{3}$

$$11\left(-\frac{1}{3}\right) + 2 = A[2\left(-\frac{1}{3}\right) - 1]$$

$$-\frac{11}{3} + 2 = A[-\frac{2}{3} - 1]$$

$$\frac{-11+6}{3} = A[\frac{-2-3}{3}]$$

$$\frac{-5}{3} = \frac{-5}{3}A \Rightarrow A = 1$$

Set $x = \frac{1}{2}$

$$11(1/2) + 2 = B[3(1/2) + 1]$$

$$\frac{11}{2} + 2 = B[3/2 + 1]$$

$$\frac{11+4}{2} = B[\frac{3+2}{2}]$$

$\frac{15}{2} = \frac{5}{2}B \Rightarrow B = 3$

$\therefore \frac{11x+2}{6x^2-x-1} = \frac{1}{3x+1} + \frac{3}{2x-1}$

$$\int \frac{11x+2}{6x^2-x-1}dx = \int \frac{1}{3x+1}dx + \int \frac{3}{2x-1}$$

$$= \frac{1}{3}\ln(3x+1) + \frac{3}{2}\ln(2x-1) + C$$

Example 4

Evaluate $\int \frac{x+3}{(x+2)^2}dx$

Solution

By partial fractions,

$$\frac{x+3}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Multiply through by $(x+2)^2$

$$x + 3 = A(x+2) + B$$

Comparing coefficients of x

$A = 1$

Comparing constants

$2A + B = 3$

$2(1) + B = 3 \Rightarrow 2 + B = 3$

$B = 3 - 2 \Rightarrow B = 1$

$\therefore \frac{x+3}{(x+2)^2} = \frac{1}{x+2} + \frac{1}{(x+2)^2}$

$$\int \frac{x+3}{x+2}dx = \int \frac{1}{x+2}dx + \int \frac{1}{(x+2)^2}$$

$$\int \frac{1}{x+2}dx + \int \frac{1}{(x+2)^2}$$

$\ln(x+2) + \frac{(x+2)^{-2+1}}{-2+1} + C$

$\ln(x+2) + \frac{(x-2)^{-1}}{-1} + C$

$\ln(x+2) - \frac{1}{(x-2)} + C$

Example 5

Evaluate $\int \frac{3x^2-2x+5}{(x-1)(x^2+5)}$

Solution

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+5}$$

Multiply through by $(x-1)(x^2+5)$

$$3x^2 - 2x + 5 = A(x^2+5) + (Bx+C)(x-1)$$

$$3x^2 - 2x + 5 = A(x^2+5) + Bx^2 - Bx + Cx - C$$

$$3x^2 - 2x + 5 = (A+B)x^2 + (c-B)x + 5A - C$$

Comparing coefficients of x

$$A + B = 3 \quad \text{—} (i)$$

$$C - B = -2 \quad \text{—} (ii)$$

$$5A - C = 5 \quad \text{—} (iii)$$

$$(i) + (ii)$$

$$A + C = 1 \quad \text{—} (iv)$$

Solving (iii) and (iv) simultaneously

$$5A - C = 5 \quad \text{—} (iii)$$

$$A + C = 1 \quad \text{—} (iv)$$

$$(iii) + (iv)$$

$$6A = 6 \Rightarrow A = 1$$

$$Put\ A = 1\ in\ (iv)$$

$$1 + C = 1 \Rightarrow C = 0$$

Put $C = 0$ into (ii)

$$0 - B = -2$$

$$\Rightarrow B = 2$$

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)} = \frac{1}{x-1} + \frac{2x}{x^2+5}$$

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)}dx = \int \frac{1}{x-1}dx + \int \frac{2x}{x^2+5}dx$$

$$= \ln(x-1) + \ln(x^2+5) + C$$

6. INTEGRATION BY PARTS

Recall from product rule of differentiation that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x, we have

$$\int \frac{d}{dx}(uv)dx = \int \left(\frac{dv}{dx}\right)dx + \int \left(v\frac{du}{dx}\right)dx$$

$$\int d(uv) = \int u\,dv + \int v\,du$$

$uv = \int u\,dv + \int v\,du$

$uv - \int v\,du = \int u\,dv$

$\therefore \int u\,dv = uv - \int v\,du$

This technique for integrating a product of two functions is called integration by parts.

NOTE:

(i) Your choice of u must be easily differentiable,

(ii) Your choice of dv is such that it shall be easily integrable.

Example 1

Evaluate $\int x^2 e^x \, dx$

Solution

$$\int x^2 e^x \, dx$$

Let $U = x^2$, $dv = e^x dx$

$$\frac{du}{dx} = 2x \quad , \quad \int dv = \int e^x dx$$

$du = 2x\,dx$, $v = e^x$

Applying, $\int u dv \quad = uv - \int v\, du$

$$\int x^2 e^x dx = x^2 e^x - \int 2x\, e^x dx$$

$$= x^2 e^x - 2\int x e^x \, dx + 6 - *$$

Integrating xe^x by part again we have

$U = x$, $dv = e^x$

$du = dx$, $V = e^x$

$$\int xe^x = xe^x - \int e^x dx$$

$$\int xe^x = xe^x - e^x$$

Substitute in *

$$\int x^2 e^x dx = x^2 e^x - 2[xe^x - e^x]$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= e^x (x^2 - 2x + 2) + C$$

ALTERNATIVELY

Using tabular form we differentiate u till it gets to zero we integrate dv till it completes the table, next we alternate with plus and minus sign and sum up products

$$\int x^2 e^x dx$$

u		dv
x^2	+	e^x
$2x$	−	e^x
2	+	e^x
0		e^x

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + c$$

$$\int x^2 e^x dx = e^x (x^2 - 2x + 2) + C$$

Example 2

Evaluate $\int \ln x \, dx$

Solution

$$\int \ln z \, dz$$

Let $U = \ln z$, $dv = dz$

$$\frac{du}{dz} = \frac{1}{z} \quad , \quad \int dv = \int dz$$

$du = \frac{1}{z} dz$, $v = z$

$$\int u\, dv = uv - \int v\, du$$

$$\int \ln z\, dz = z\ln z - \int z \frac{1}{z}$$

$$\int \ln z\, dz = \ln z - \int dz + C$$

$$\int \ln z\, dz = z\ln z - z + C$$

Example 3

Evaluate $\int \sqrt{z} \ln z \, dz$

Solution

$\int \sqrt{z} \ln z\, dz = \int z^{1/2} \ln z\, dz$

Let $u = \ln z$, $dv = z^{\frac{1}{2}} dz$

$$\frac{du}{dz} = \frac{1}{z} \quad , \int dv = \int z^{1/2} dz$$

$du = \frac{1}{z} dz$, $v = \frac{z^{1/2+1}}{1/2+1}$

$v = \frac{z^{3/2}}{3/2}$ or $v = \frac{2}{3} z^{3/2} \frac{1}{z} dz + C$

Using, $\int u dv \quad = uv - \int v du$

$$\int \sqrt{z} \ln z = \frac{2}{3} z^{3/2} \ln z - \int \frac{2}{3} z^{3/2} \frac{1}{z} \, dz + C$$

$= \frac{2}{3} z^{3/2} \ln z - \frac{2}{3} \int z^{1/2} dz + C$

$= \frac{2}{3} z^{3/2} \ln z - \frac{2}{3} \frac{z^{1/2+1}}{1/2+1} + C$

$= \frac{2}{3} z^{3/2} \ln z - \frac{2}{3} \left(\frac{z^{3/2}}{3/2} \right) + C$

$= \frac{2}{3} z^{3/2} \ln z - \frac{2}{3} \left(\frac{2}{3} z^{3/2} \right) + C$

$= \frac{2}{3} z^{3/2} \ln z - \frac{4}{9} z^{3/2} + C$

SOME APPLICATIONS OF INTEGRATION.

Integration can be used to find the equation of a curve when the gradient function is given , and the arbitrary constant that ensues can be evaluated by using the initial conditions given.

MOTION OF PARTICLES

Example 1

A particle starts with initial speed $25\ ms^{-1}$. its acceleration at any time t is $(12 - 2t)\ ms^{-2}$ find the speed at the end of 6 seconds.

Solution

The acceleration is given by $\frac{dv}{dt}$

$\therefore \frac{dv}{dt} \quad = 12 - 2t$

$$dv = (12 - 2t) dt$$

$$\int dv = \int (12 - 2t) dt$$

$$v = 12t - t^2 + C$$

When $t = 0$, $v = 25\,ms^{-1}$, we get

$$25 = 12(0) - 0^2 + C$$
$$C = 25$$
$$\therefore v = 12t - t^2 + 25$$

Thus the speed when $t = 6$ is:

$v = 12(6) - 6^2 + 25$

$v = 61 ms^{-1}$

AREA UNDER A CURVE

Suppose that $y = f(x)$ is the equation of a curve, and area required to find the area enclosed by the curve on the = $x - axis$ with co-ordinates $x = a$ is $x = b$;

This area

$A = \int_a^b f(x)dx$.

Example 1

Calculate the area between the $x - axis$ and the curve $y = 2x - x^2$

Solution

The curve crosses the axis where $y = 0$

i.e. $2x - x^2 = 0$

$$x(2 - x) = 0$$
$$x = 0 \text{ or } 2$$

Thus we have that

$$\int_0^2 y\,dx = \int_0^2 (2x - x^2)dx$$

$= [\frac{2x^2}{2} - \frac{x^3}{3}]_0^2$

$= [x^2 - \frac{x^3}{3}]_0^2$

$$\left[2^2 - \frac{2^3}{3}\right] - [0 - 0]$$
$$= \left(4 - \frac{8}{3}\right) - 0$$

$= {}^4/_3$ square units.

Example 2

A curve passes through the point (2, 3) and has the gradient function $3x - 2$. Find the equation of the curve.

Solution

The gradient function is given by $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = 3x - 2$

$dy = (3x - 2)dx$

$$\int dy = \int (3x - 2)dx$$
$$y = \frac{3x^2}{2} - 2x + C - *$$

Put $x =$, $y = 3$ in *

We have,

$3 = \frac{3(2)^2}{2} - 2(2) + C$

$3 = 6 - 4 + C$

$3 = 2 + C$

$C = 1$

Put $C = 1$ into *

$\therefore y = \frac{3x^2}{2} - 2x + 1$

Example 3

Find the area enclosed between the curve $y = x - 3$

Solution

To get the limits $x = a$ & $x = b$ we equate both values of y

$3x - x^2 = x - 3$

$x^2 + x - 3x - 3 = 0$

$x^2 + x - 2x - 3 = 0$

$(x + 1)(x + 3) = 0$

$$x = -1 \text{ or } x = 3$$

Thus the required area is

$= \int_{-1}^3 [(3x - x^2) - (x - 3)]dx$

$$= \int_{-1}^3 (-x^2 + 2x + 3)dx$$
$$= [\frac{-x^3}{3} + x^2 + 3x]_{-1}^3$$

$= \left[\frac{-(3)^3}{3} + 3^2 + 3(3)\right] - [-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)]$

$$\left[\frac{-27}{3} + 9 + 9\right] - \left[\frac{1}{3} + 1 - 3\right]$$

$= 102/3$ square units.

VOLUMES OF REVOLUTION

If the graph $y = f(x)$ between $x = a$ and $x = b$ right angles, (i.e. 2π radians) about the $x - axis$, then the volume of the solid generated is:

$V = \pi\int_a^b y^2\,dx$

If the graph between $y = c$ and $y = d$ is rotated through four right - angles about the $y - axis$, then the volume of the solid generated is:

$V = \pi\int_c^d x^2\,dy$.

Examples 1

The finite area enclosed by the curve $y = x^2 + 2$ and the line $y = 4x + 2$ is rotated completely about the $x - axis$. Calculate the volume of the solid generated.

Solution

To get the limits, we equate the y values

$$x^2 + 2 = 4x + 2$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$
$$x = 0 \text{ or } x = 4$$

The volume of the solid generated

= (volume of the solid generated by the line) - (volume generated by the curve)

$= \int_0^4 \pi(4x + 2)^2dx - \int_0^4 \pi(x^2 - 2)^2dx$

$= \pi\int_0^4 [(4x + 2)^2 - (x^2 - 2)^2]dx$

$= \pi\int_0^4 [(16x^2 + 16x + 4) - (x^4 - 4x^2 + 4]dx$

$= \pi\int_0^4 (-x^4 + 20x^2 + 16x)dx$

$$=\pi\left[\frac{-x^5}{5}+\frac{20x^3}{3}+\frac{16x^2}{2}\right]_0^4=\pi\left[\frac{-x^5}{5}+\frac{20}{3}x^3+8x^2\right]_0^4$$

$$=\pi\left\{\left(\frac{-(4)^5}{5}+\frac{20}{3}(4)^3+8(4)^2\right)-(0)\right\}$$

$$=\pi\left(\frac{-1024}{5}+\frac{1280}{3}+128\right)$$

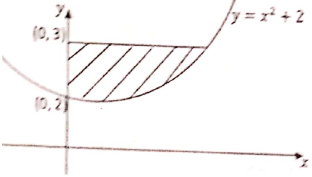
$$=\pi\left(\frac{-3072+6400+1920}{15}\right)=\pi\left(\frac{5248}{15}\right)$$

$$=\frac{5248}{15}\pi \text{ cubic units}$$

Example 2:

The portion of the curve $y=x^2+2$ between the points $(0,2)$ and $(0,3)$ is rotated through 2π radians about the y - axis to form the surface bowl. Find the volume of the bowl.

Solution



$V=\int_2^3 \pi x^2\,dx$

But, $y=x^2+2$

$x^2=y-2$

$V=\int_2^3 \pi(y-2)dy=\pi\int_2^3(y-2)dy$

$=\pi\left[\frac{y^2}{2}-2y\right]_2^3=\pi\left\{\left(\frac{3^2}{2}-2(3)\right)-\left(\frac{2^2}{2}-2(2)\right)\right\}$

$\pi\{(^9/_2-6)-(^4/_2-4)\}=\pi(^{-3}/_2+2)=\frac{\pi}{2}$ cubic units.

EXERCISE:

1. Determine the domain and range of the functions given by:
   (i) $y=\sqrt{4-x^2}$ (iii) $f(x)=-x^2+2x+3$
   (ii) $f(x)=|x-3|-10$
2. If $g(x)=\frac{1-x}{1+x}$; $x\in\mathbb{R}$, the set of real numbers, find the value of x for which $g'(x)$ does not exist
3. Prove the limit
   $$\lim_{x\to3}2x^2-x+7=22$$
4. Evaluate the following limits at the given points
   (i) $\frac{4-x}{\sqrt{x}-2}$; $x=4$ (ii) $\frac{x^3-27}{x-3}$; $x=3$
5. If $f(x)=2^x$, show that
   $$\frac{f(x+3)}{f(x-1)}=f(4)$$
6. Find the derivative of $3x^2-5x+6$ using first principle.
7. Find the differential coefficient of
   (i) $(x^2-6)(x^2+3)$
   (ii) $x^3+x^2y=5$
   (iii) $(1-3x^2)^5$
   (iv) $\sin^2(2x+1)$
   (v) $\sqrt{\frac{x}{2+x}}$
8. The positions of two particles p_1 and p_2 at the end of t sec. are given by $p_1=3t^3-12t^2+18t+5$, $p_2=-t^3+9t^2-12t$ when do the two particles have the same velocity?
9. If the gradient of the curve $y=2kx^2+x+1$ at $x=1$ is 9. Find k.
10. Verify that the function $f(x)=\sin x+\cos x$ satisfies the hypothesis of Rolle's theorem in the interval $[0,2\pi]$. Hence find the real number c in the open interval $(0,2\pi)$ such that $f'(c)=0$
11. Find the point in the interval $[1,4]$ of the function $f(x)=\sqrt{x}$ that satisfies the hypothesis of the mean value theorem.
12. Determine the Taylor series about the point $x=1$ for the function $f(x)=\cos x$
13. Evaluate the following integrals.
    (i) $\int\frac{x^2-2x+1}{x^3}dx$ (ii) $\int\cos 5x\sin 2x\,dx$
    (iii) $\int\frac{\sec^2 x}{1+\tan x}dx$ (iv) $\int_0^{\frac{\pi}{2}}(\cos x-\sin 2x)dx$

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI
SCHOOL OF PHYSICAL SCIENCES
DEPARTMENT OF MATHEMATICS
ELEMENTARY MATHEMATICS I (MTH 102) RAIN SEMESTER TEST: 2016/2017

TIME ALLOWED: 10 MINS DATE: 25/10/2017

INSTRUCTIONS: ATTEMPT ALL QUESTIONS. ANSWER THE QUESTIONS IN THE SPACES PROVIDED IN THE QUESTION PAPER

NAME____ REG. NUMBER____ DEPT____

1. Given $f(x)=-x^2+6x-11$, find $f(t-3)$. $-t^2+12t-38$
2. Find the domain of function, $|x-6|-3$ R
3. Find the range of function, $|x-6|-3$ $[-3,\infty)$
4. Find the domain of function, $g(t)=100$ R
5. Find the range of function, $p(t)=12$ 12
6. If $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)$, then the normal limit exists. TRUE OR FALSE? True
7. Given the functions $f(x)$ and $g(x)$, suppose we have $\lim_{x\to c}f(x)=\infty$ and $\lim_{x\to c}g(x)=L$, for some real numbers, c and L, then $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right)=\infty$. TRUE OR FALSE? False
8. Using the two functions in question 7, if $L>0$, then $\lim_{x\to c}[f(x)g(x)]=\infty$. TRUE OR FALSE? True
9. If r is a positive rational number and c is any real number, find $\lim_{x\to\infty}\left(\frac{c}{x^r}\right)$ 0
10. Evaluate the following limit $\lim_{x\to[illegible]}([illegible])$ 0

HEAD OF DEPARTMENT
MATHEMATICS
DATE 20/2/2018

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

SCHOOL OF PHYSICAL SCIENCES DEPARTMENT OF MATHEMATICS

2017/2018 MTH 102 TEST TIME: 1 Hr DATE: 02/08/2018 Instruction: Answer all questions.

NAME............................ REG. NO............................ DEPT............

1. Evaluate $\lim_{x\to 0}\frac{x}{3-\sqrt{9+x}}$ ....
2. Obtain the inverse $f^{-1}(x)$ of $f(x)=\frac{5x+3}{4x-7}$ ....
3. Determine the domain and range of the function $g(x)=\frac{\log(2x-1)}{x+1}$ ....
4. Find the value(s) of t if $f(x)=x^2+5x+3$ and $f(t)=-3$ ....
5. Find $y'(2,0)$ if $5x^2-x^3\sin y+5xy=10$ ....
6. Evaluate $\lim_{x\to\pi}\frac{\sin^2 x}{1+\cos x}$ ....
7. Find $(g\circ f)(x)$ if $f(x)=x^2-1$ and $g(x)=\sqrt{x+1}$ ....
8. Evaluate $f'(x)$ if $f(x)=\sqrt{2x^2+3x}$. Hence, find $f'(1)$ ....
9. Find the domain and range of $g(x)=(x^2-2x-3)^{-1}$ ....
10. For what value(s) of k is $f(x)$ continuous if $f(x)=\begin{cases}k, & x\le 1\\ 2x+4, & x>1\end{cases}$ ....

SOLUTION TO MTH102 TEST 2017/2018

(1) $\lim_{x\to 0}\frac{x}{3-\sqrt{9+x}}=\frac{0}{3-3}=\frac{0}{0}$ indeterminate

rationalizing

$\frac{x}{3-\sqrt{9+x}}\times\frac{3+\sqrt{9+x}}{3+\sqrt{9+x}}=\frac{x(3+\sqrt{9+x})}{9-(9+x)}=-(3+\sqrt{9+x})$

$\lim_{x\to 0}-(3+\sqrt{9+x})=-6$

(2) $f(x)=\frac{5x+3}{4x-7}$; let $y=\frac{5x+3}{4x-7}$

$4xy-7y=5x+3$

$4xy-5x=7y+3$

$x(4y-5)=7y+3$

$x=\frac{7y+3}{4y-5}$

$f^{-1}(x)=\frac{7x+3}{4x-5}$

(3) $g(x)=\frac{\log(2x-1)}{x+1}$

The function is defined for $2x-1>0 \cup x\neq -1$

$2x>1$

$x>1/2$

$Dom(g)=(1/2,\infty)$

Range $=\mathbb{R}$

(4) $f(x)=x^2+5x+3$

$f(t)=t^2+5t+3=-3$

$t^2+5t+3+3=0 \Rightarrow t^2+5t+6=0$

$(t+2)(t+3)=0$

$t=-2$ or -3

(5) $5x^2-x^3\sin y+5xy=10$

$10x-3x^2\sin y-x^3\cos y\frac{dy}{dx}+5y+5x\frac{dy}{dx}=0$

$(5x-x^3\cos y)\frac{dy}{dx}=3x^2\sin y-10x-5y$

$y'=\frac{dy}{dx}=\frac{3x^2\sin y-10x-5y}{5x-x^3\cos y}$

$y'(2,0)=\frac{3(2)^2\sin 0-10(2)-5(0)}{5(2)-2^3\cos 0}=-10$

(6) $\lim_{x\to\pi}\frac{\sin^2x}{1+\cos x}=\frac{(\sin\pi)^2}{1+\cos\pi}=\frac{0^2}{1-1}=\frac{0}{0}$ indeterminate

Using L'Hopital's rule

$\lim_{x\to\pi}\frac{2\sin x\cos x}{-\sin x}=\frac{2\sin\pi\cos\pi}{-\sin\pi}=\frac{0}{0}$ indeterminate

Using L'Hopital rule again

$\lim_{x\to\pi}\frac{-2\sin^2x+2\cos^2x}{-\cos x}=\frac{-2\sin^2\pi+2\cos^2\pi}{-\cos\pi}=\frac{0+2(-1)^2}{-(-1)}=2$

(7) $f(x)=x^2-1$; $g(x)=\sqrt{x+1}$

$f\circ g(x)=f[g(x)]=(\sqrt{x+1})^2-1=x+1-1=x$

(8) $f(x)=\sqrt{2x^2+3x}$; let $u=2x^2+3x$, $y=u^{1/2}$

$\frac{dy}{du}=\frac{1}{2}u^{-1/2}=\frac{1}{2\sqrt{u}}$ $\quad\frac{du}{dx}=4x+3$

$f'=\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2\sqrt{u}}\times 4x+3=\frac{4x+3}{2\sqrt{2x^2+3x}}$

$f'(1)=\frac{4(1)+3}{2\sqrt{2(1)^2+3(1)}}=\frac{7}{2\sqrt{5}}$ or $\frac{7}{2\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{7\sqrt{5}}{10}$

(9) $g(x)=(x^2-2x-3)^{-1}=\frac{1}{x^2-2x-3}=\frac{1}{(x+1)(x-3)}$

$Dom(g)=\mathbb{R}\setminus\{-1,3\}$

$Ran(g)=\mathbb{R}\setminus\{0\}$

(10) $\lim_{x\to 1^-}(k)=k$, $\lim_{x\to 1^+}(2x+4)=2(1)+4=6$

The function is continuous if $\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)$

$\therefore k=\underline{\underline{6}}$

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

SCHOOL OF PHYSICAL SCIENCES

DEPARTMENT OF MATHEMATICS

ELEMENTARY MATHEMATICS II (MTH 102) RAIN SEMESTER TEST SESSION: 2015/2016

ALLOWED: 45 MINS DATE: 24/08/2016

IUCTIONS: ATTEMPT ALL QUESTIONS. EACH QUESTION CARRIES EQUAL MARKS.

1a) Find the domain and range of the following functions: (a) $g(t) = \sqrt{4-7t}$

(b). $f(t) = |z-6| - 3$ (c) $g(x) = 8$ (d) $f(x) = \sqrt{(4-x^2)}$

2. Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$, find each of the following: (a) $(f\circ g)(5)$ (b) $(f\circ g)(x)$

(c) $(g\circ f)(x)$ (d) $(g\circ g)(x)$.

3. Evaluate the following: (a). $\lim_{x\to -1} \frac{x^2+6x+5}{3x^2-6x-9}$ (b) $\lim_{x\to\infty} \frac{x^4-2}{7x^4+3x+4}$ (c) $\lim_{x\to -2} 3x^2 + 5x - 9$

4. Obtain the first derivative, $\frac{dy}{dx}$, of the following: (a) $xy + x - 2y - 1 = 0$

(b) $y = \sin(2x^2 + 4x)$ (c) $x^3 - xy + y^3 = 1$ (d) $y = (x^2+2)(x-2)$

5. What are the points of the curve $y = 4x^3 - 15x^2 - 18x + 79$ where the gradient is zero?

MTH 102 TEST SOLUTION FOR 2015/2016 SESSION

1 (a) $g(t) = \sqrt{4-7t}$

$g(t)$ is defined if

$4 - 7t \geq 0$

$4 \geq 7t \Rightarrow \frac{4}{7} \geq t$

or $t \leq \frac{4}{7}$

Dom $g = \{t : t \leq \frac{4}{7}\}$ or $(-\infty, \frac{4}{7}]$

Range $= [0, 2]$

(b) $f(t) = |z - 6| - 3$

Dom $f = \mathbb{R}$

Range $= [-3, \infty)$

(c) $g(x) = 8$

Dom $g = \mathbb{R}$

Range $= 8$

(d) $f(x) = \sqrt{4 - x^2}$

$f(x)$ is defined if $4 - x^2 \geq 0$

$(2-x)(2+x) \geq 0$

The turning values are $x = -2, 2$

Using the truth table

	$x \leq -2$	$-2 \leq x \leq 2$	$x \geq 2$
$(2-x)$	+	+	−
$(2+x)$	−	+	+
$(2-x)(2+x)$	−	+	−

Solution is $= \{x : -2 \leq x \leq 2\}$

Dom $f = [-2, 2]$

For range $4 - x^2 \geq 0 \Rightarrow 4 \geq x^2$

or $x^2 \leq 4$ or $0 \leq x^2 \leq 4$

or $0 \leq x \leq 2$

Range $= [0, 2]$

2 $f(x) = 3x^2 - x + 10$

$g(x) = 1 - 20x$

(a) $f\circ g(5) = f[g(5)] = f[1 - 20(5)] = f(-99)$

$f(-99) = 3(-99)^2 - (-99) + 10 = 29512$

(b) $f\circ g(x) = f[g(x)] = f(1 - 20x)$

$= 3(1-20x)^2 - (1-20x) + 10$

$= 3(1 - 40x + 400x^2) - 1 + 20x + 10$

$= 3 - 120x + 1200x^2 - 1 + 20x + 10$

$= 1200x^2 - 100x + 12$

(c) $(g\circ f)(x) = g[f(x)] = g[3x^2 - x + 1$

$= 1 - 20(3x^2 - x + 10) = 1 - 60x^2 + 20x$

$= -60x^2 + 20x - 199$

(d) $(g\circ g)(x) = g[g(x)] = g[1 - 20x]$

$= 1 - 20(1 - 20x) = 1 - 20 + 400x$

3 (a) $\lim_{x\to -1} \frac{x^2+6x+5}{3x^2-6x-9} = \frac{(-1)^2+6(-1)+5}{3(-1)^2-6(-1)-9}$

$= \frac{1-6+5}{3+6-9} = \frac{0}{0}$

Which is indeterminate

Factorizing, we have: $\lim_{x\to -1} \frac{(x+1)(x+5)}{(x+1)(x-3)}$

$= \lim_{x\to -1} \frac{x+5}{x-3} = \frac{-1+5}{-1-3} = \frac{4}{-4} = -1$

(b) $\lim_{x\to\infty} \frac{x^4-2}{7x^4+3x+4} = \lim_{x\to\infty} \frac{\frac{x^4}{x^4} - \frac{2}{x^4}}{\frac{7x^4}{x^4} - \frac{3x}{x^4} + \frac{4}{x^4}}$

$= \lim_{x\to\infty} \frac{1 - \frac{2}{x^4}}{7 - \frac{3}{x^3} + \frac{4}{x^4}} = \frac{1 - \frac{2}{\infty^4}}{7 - \frac{3}{\infty^3} + \frac{4}{\infty^4}}$

$= \frac{1-0}{7-0+0} = 1/7$

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(c) $\lim_{x \to -2} 3x^2 + 5x - 9$

$3(-2)^2 + 5(-2) - 9$

$3(4) + 5(-2) - 9$

$12 - 10 - 9 = -7$

4\. a) $xy + x - 2y - 1 = 0$

$y + x\frac{dy}{dx} + 1 - 2\frac{dy}{dx} = 0$

$(x-2)\frac{dy}{dx} = -y - 1$

$\frac{dy}{dx} = \frac{-y-1}{x-2}$

(b) $y = \sin(2x^2 + 4x)$

let $u = 2x^2 + 4x$; $y = \sin u$

$\frac{du}{dx} = 4x + 4$; $\frac{dy}{du} = \cos u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times (4x+4)$

$= (4x+4)\cos u$

$= 4(x+1)\cos(2x^2+4x)$

(c) $x^3 - xy + y^3 = 1$

$3x^2 - x\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$

$(3y^2 - x)\frac{dy}{dx} = y - 3x^2$

$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$

(d) $y = (x^2+2)(x-2)$

$u = x^2 + 2$; $v = x - 2$

$\frac{du}{dx} = 2x$; $\frac{dv}{dx} = 1$

By Product rule

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$(x^2+2)(1) + (x-2)(2x)$

$x^2 + 2 + 2x^2 - 4x$

$= 3x^2 - 4x + 2$

5\. $y = 4x^3 - 15x^2 - 18x + 79$

$\frac{dy}{dx} = 12x^2 - 30x - 18$

at $\frac{dy}{dx} = 0$

$12x^2 - 30x - 18 = 0$

$2x^2 - 5x - 3 = 0$

$2x^2 - 6x + x - 3 = 0$

$2x(x-3) + 1(x-3) = 0$

$(x-3)(2x+1) = 0$

$x = 3$ or $-\frac{1}{2}$

at $x = 3$

$y = 4(3)^3 - 15(3)^2 - 18(3) + 79$

$y = 4(27) - 15(9) - 18(3) + 79$

$y = -2$

$(3, -2)$

at $x = -\frac{1}{2}$

$y = 4(-\frac{1}{2})^3 - 15(-\frac{1}{2})^2 - 18(-\frac{1}{2}) + 79$

$y = -\frac{4}{8} - \frac{15}{4} + \frac{18}{2} + 79$

$y = -\frac{1}{2} - \frac{15}{4} + 9 + 79$

$y = \frac{-2 - 15 + 36 + 316}{4}$

$y = \frac{335}{4}$

$\therefore (-\frac{1}{2}, \frac{335}{4})$

Thus the points on the curve where the gradient is zero are $(3,-2)$ and $(-\frac{1}{2}, \frac{335}{4})$.

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI
DEPARTMENT OF MATHEMATICS
MTH102 TEST 2013/2014 SESSION TIME: 1 HOUR
INSTRUCTION: ANSWER ALL QUESTIONS

1(a) Find the domain and range of the following functions (i) $f(x)=\sqrt{4-x^2}$ (ii) $f(x)=\dfrac{x^2}{\sqrt{x^2-16}}$

(b) Evaluate the following limits (i) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x^2}$ (ii) $\lim\limits_{t\to 4}\dfrac{t-\sqrt{3t+4}}{4-t}$

2(a) Find $\dfrac{dy}{dx}$ if (i) $x^3y^3+x=y$ (ii) $4x^2-8xy+4y^3=0$

(b) Given $f(x)=3x^2-x+10$ and $g(x)=1-20x$, find (i) $(f\circ g)(5)$ (ii) $(g\circ f)(2)$

3(a) Evaluate the following integrals (i) $\int_{1}^{6}|x-3|\,dx$ (ii) $\int_{1}^{e}x\ln x\,dx$

(b) Determine the critical points of the following functions (i) $f(x)=x^3+3x^2-9x$ (ii) $f(x)=12x^4-28x^3+21x^2-6x$

MTH 102 TEST SOLUTION FOR 2013/2014 SESSION

1a(i) $f(x)=\sqrt{4-x^2}$

$f(x)$ is defined if $4-x^2\geq 0$

$\Rightarrow (2-x)(2+x)\geq 0$

The turning values are $x=2,-2$

Using the truth table

	$x<-2$	$-2\leq x\leq 2$	$x\geq 2$
$2-x$	+	+	−
$2+x$	−	+	+
$(2-x)(2+x)$	−	+	−

Solution is $=\{x: -2\leq x\leq 2\}$

Dom $f=[-2,2]$

For the range, $4-x^2\geq 0$

$4\geq x^2$

$x^2\leq 4$ or $0\leq x^2\leq 4$

$0\leq x\leq 2$

Range $=[0,2]$

(ii) $f(x)=\dfrac{x^2}{\sqrt{x^2-16}}$

$f(x)$ is defined if $x^2-16>0$

or $(x-4)(x+4)>0$

The turning values are $x=4,-4$

Using the truth table,

	$x<-4$	$-4<x<4$	$x>4$
$(x-4)$	−	−	+
$(x+4)$	−	+	+
$(x-4)(x+4)$	+	−	+

Solution is $=\{x: x<-4 \cup x>4\}$

Dom $f=(-\infty,-4)\cup(4,\infty)$

Range $=[0,\infty)$

1b)(i) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x^2}=\dfrac{\sqrt{0^2+16}-4}{0^2}=\dfrac{0}{0}$

Which is indeterminate,

Simplifying, we have

$$\frac{\sqrt{x^2+16}-4}{x^2}\times\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}=\frac{x^2+16-4^2}{x^2(\sqrt{x^2+16}+4)}$$

$$=\frac{x^2}{x^2(\sqrt{x^2+16}+4)}=\frac{1}{\sqrt{x^2+16}+4}$$

Taking the limit

$$\lim_{x\to 0}\frac{1}{\sqrt{x^2+16}+4}=\frac{1}{\sqrt{16}+4}=\frac{1}{4+4}=\frac{1}{8}$$

(ii) $\lim\limits_{t\to 4}\dfrac{t-\sqrt{3t+4}}{4-t}=\dfrac{4-\sqrt{3(4)+4}}{4-4}=\dfrac{0}{0}$

Which is indeterminate,

By L'Hopital's rule

$$\lim_{t\to 4}\frac{1-\frac{3}{2}(3t+4)^{-\frac{1}{2}}}{-1}=\lim_{t\to 4}-\left(1-\frac{3}{2\sqrt{3t+4}}\right)$$

$$=\lim_{t\to 4}\left[\frac{3}{2\sqrt{3t+4}}-1\right]=\frac{3}{2\sqrt{3(4)+4}}-1$$

$$=\frac{3}{2(4)}-1=\frac{3}{8}-1=-\frac{5}{8}$$

2a)(i) $x^3y^3+x=y$

$$3x^2y^3+3x^3y^2\frac{dy}{dx}+1=\frac{dy}{dx}$$

$$3x^3y^2\frac{dy}{dx}-\frac{dy}{dx}=-3x^2y^3-1$$

$$(3x^3y^2-1)\frac{dy}{dx}=-(3x^2y^3+1)$$

$$\frac{dy}{dx}=\frac{-(3x^2y^3+1)}{3x^3y^2-1}$$

(ii) $4x^2-8xy+4y^3=0$

$$8x-8\left[x\frac{dy}{dx}+(1)y\right]+(3)4y^2\frac{dy}{dx}=0$$

$$8x-8x\frac{dy}{dx}-8y+12y^2\frac{dy}{dx}=0$$

$$12y^2\frac{dy}{dx}-8x\frac{dy}{dx}=8y-8x$$

$$(12y^2-8x)\frac{dy}{dx}=8y-8x$$

$$\frac{dy}{dx}=\frac{8y-8x}{12y^2-8x}=\frac{4(2y-2x)}{4(3y^2-2x)}=\frac{2y-2x}{3y^2-2x}$$

2b) $f(x) = 3x^2 - x + 10$, $g(x) = 1 - 20x$

(i) $(f \circ g)(x) = f[g(x)] = 3(1-20x)^2 - (1-20x) + 10$

$(f \circ g)(5) = 3[1 - 20(5)]^2 - (1 - 20(5)) + 10$

$= 3[1-100]^2 - [1-100] + 10$

$= 3(-99)^2 - (-99) + 10$

$= 29403 + 99 + 10 = 29512$

(ii) $(g \circ f)(x) = g[f(x)] = 1 - 20(3x^2 - x + 10)$

$= 1 - 60x^2 + 20x - 200$

$(g \circ f)(2) = 1 - 60(2)^2 + 20(2) - 200$

$= 1 - 60(4) + 40 - 200$

$= 1 - 240 + 40 - 200 = -399$

3(a) (i) $\int_1^6 |x-3|\,dx$

$x - 3 = 0$

$\therefore x = 3$ (mid-point)

$\int_1^3 -(x-3)dx + \int_3^6 (x-3)dx$

$\int_1^3 (3-x)dx + \int_3^6 (x-3)dx$

$\left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^6$

$\left(3(3) - \frac{3^2}{2}\right) - \left(3(1) - \frac{1^2}{2}\right) + \left(\frac{6^2}{2} - 3(6)\right) - \left(\frac{3^2}{2} - 3(3)\right)$

$\left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right) + (18 - 18) - \left(\frac{9}{2} - 9\right)$

$9 - \frac{9}{2} - 3 + \frac{1}{2} - \frac{9}{2} + 9$

$15 + \frac{-9+1-9}{2} = 15 - \frac{17}{2}$

$= \frac{30-17}{2} = \frac{13}{2}$

(ii) $\int_1^e x \ln x\,dx$

let $u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

$\int u\,dv = uv - \int v\,du$

$\left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x}dx$

$\left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\int_1^e x\,dx$

$\left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$

$\frac{1}{2}[x^2 \ln x]_1^e - \frac{1}{4}[x^2]_1^e$

$\frac{1}{2}[e^2 \ln e - 1^2 \ln 1] - \frac{1}{4}[e^2 - 1^2]$

$\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}e^2 + \frac{1}{4}$

$\therefore \int_1^e x \ln x\,dx = \frac{1}{4}(e^2 + 1)$

3b (i) $f(x) = x^3 + 3x^2 - 9x$

$f'(x) = 3x^2 + 6x - 9$

at $f'(x) = 0 \Rightarrow 3x^2 + 6x - 9 = 0$

or $x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$x = 1$ or -3

$f(1) = 1^3 + 3(1)^2 - 9(1) = -5$

$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) = 27$

Thus the critical points are $(1, -5)$ and $(-3, 27)$

(ii) $f(x) = 12x^4 - 28x^3 + 21x^2 - 6x$

$f'(x) = 48x^3 - 84x^2 + 42x - 6$

at $f'(x) = 0 \Rightarrow 48x^3 - 84x^2 + 42x - 6 = 0$

or $8x^3 - 14x^2 + 7x - 1 = 0$

$x = 1$ gives 0 on RHS, thus $x - 1$ is a factor

By Long division of Polynomials

$$\begin{array}{r|l} & 8x^2 - 6x + 1 \\ \hline x - 1 & 8x^3 - 14x^2 + 7x - 1 \\ & 8x^3 - 8x^2 \\ \hline & -6x^2 + 7x \\ & -6x^2 + 6x \\ \hline & x - 1 \\ & x - 1 \\ \hline & 0 \end{array}$$

Thus we have

$8x^3 - 14x^2 + 7x - 1 = (x-1)(8x^2 - 6x + 1) = 0$

$\Rightarrow (x-1)[8x^2 - 2x - 4x + 1] = 0$

$\Rightarrow (x-1)[2x(4x-1) - 1(4x-1)] = 0$

$\Rightarrow (x-1)(2x-1)(4x-1) = 0$

$\Rightarrow x = 1$ or $\frac{1}{2}$ or $\frac{1}{4}$

$f(1) = 12(1)^4 - 28(1)^3 + 21(1)^2 - 6(1) = -1$,

$f(1/2) = 12\left(\frac{1}{2}\right)^4 - 28\left(\frac{1}{2}\right)^3 + 21\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right)$

$= \frac{3}{4} - \frac{7}{2} + \frac{21}{4} - 3 = \frac{3 - 14 + 21 - 12}{4} =$

$= \frac{-2}{4} = -\frac{1}{2}$

$f\left(\frac{1}{4}\right) = 12\left(\frac{1}{4}\right)^4 - 28\left(\frac{1}{4}\right)^3 + 21\left(\frac{1}{4}\right)^2 - 6\left(\frac{1}{4}\right)$

FEDERAL UNIVERSITY OF [illegible]

SCHOOL OF PHYSICAL SCIENCES [illegible]

2017/2018 RAIN SEMESTER EXAMINATION

MTH 102: ELEMENTARY MATHEMATICS II DATE: [illegible] TIME: 2 HOURS

Instructions: Answer all questions. NOTA denotes "None of the Above", DNE denotes "Does not Exist", AOTA denotes "All of the Above"

HEAD OF DEPARTMENT [illegible] DATE

[illegible]valuate the limit $\lim_{x\to\infty}$ [illegible] (a) 0 (b) ∞ (c) 35 (d) $\frac{1}{2}$ (e) $\frac{1}{3}$

[illegible]nd the equation of the normal to the curve $y = x^3 - 4x^2 + 5x + 7$ at the point (2, 1) (a) $y = x + 1$ (b) $y = 2x$ (c) $y = -x + 3$ (d) $y = x - 1$ (e) $y = 5x + 7$

[illegible] particle moves along a straight line so that the distance travelled after time t secs is given by $S(t) = 2t^3 - 6t^2 + 4t - 5$. Find the acceleration of the body at time $t = 2$ secs (a) $20ms^{-2}$ (b) $-6ms^{-2}$ (c) $-50ms^{-2}$ (d) $75ms^{-2}$ (e) $-20ms^{-2}$

Given $f(x) = 5x - 2$ and $g(x) = \frac{x+2}{5}$, find respectively $(f \circ g)(x)$ and $(g \circ f)(x)$. (a) $-x, x$ (b) $x, -x$ (c) x, x (d) $x, 2x$ (e) NOTA

Obtain the minimum point of the function $y = x^3 - 6x^2 + 9x + 10$ (a) (3, 10) (b) (3, −17) (c) (3, 5) (d) (1, 3) (e) (1, 14)

Find the vertical asymptote(s) of $f(x) =$ [illegible] (a) -5 (b) 2 (c) -5, 5 (d) 0 (e) 3

Obtain the domain of the function $f(x) =$ [illegible] (a) $(-\infty, -9) \cup (-9, \infty)$ (b) $(-\infty, -2)$ (c) $(-\infty, 9)$ (d) $(-\infty, -9) \cup (9, \infty)$ (e) $(2, \infty)$

Find $\lim$ [illegible] (a) 0 (b) ∞ (c) $\frac{1}{\pi}$ (d) $\frac{\pi}{6}$ (e) $\frac{1}{\pi}$

For what value of k is the function $f(x) = \begin{cases} x + 5, & x \leq 1 \\ 5 + kx^2, & x > 1 \end{cases}$ continuous at $x = 1$.

(a) 0 (b) -1 (c) 1 (d) -2 (e) 2.

Find the rate of change of the volume of a spherical balloon if the radius is increasing at the rate of $0.5cms^{-1}$ given that the radius is given by $r = 5cm$. (a) $6\pi cm^3s^{-1}$ (b) $50\pi cm^3s^{-1}$ (c) $90\pi cm^3s^{-1}$ (d) $25\pi cm^3s^{-1}$ (e) $100\pi cm^3s^{-1}$

Evaluate $\int x\ln x \, dx$ (a) $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ (b) $\frac{x^2}{4}\ln x - \frac{x^2}{12} + c$ (c) $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ (d) $x^2\ln x + c$

[illegible]. Evaluate $\lim_{x\to\infty}(1 + \frac{1}{3x})^{2x}$ (a) e^2 (b) e (c) $e^{\frac{2}{3}}$ (d) $e^{\frac{1}{3}}$ (e) $e^{\frac{3}{2}}$.

[illegible]. Find $\frac{dy}{dx}$ if $y(t) = 1 + \sec 2t$ and $x(t) = \tan 2t$. (a) $\tan 2t$ (b) $\sec 2t$ (c) $\sin 2t$ (d) $-\tan 2t$ (e) $\csc 2t$

[illegible]4. Find y' and y'' at (1, 1) of $x^2y + xy^2 = 2$. (a) $-1, 0$ (b) 1, 1 (c) $-1, -1$ (d) 1, 0 (e) $-1, 1$

[illegible]5. Evaluate $\lim_{x\to 1} x^{\frac{1}{x-1}}$ (a) $\frac{1}{e}$ (b) e (c) e^2 (d) e^{-2} (e) NOTA

[illegible]6. Evaluate $\lim_{x\to 2}$ [illegible] (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

[illegible]7. Determine the domain of $f(x) = \sqrt{\frac{x}{2-x}}$ (a) $\mathbb{R} \setminus \{2\}$ (b) [0, 2) (c) [illegible] (e) NOTA

[illegible]. Find the range of $g(x) = (2x^2 - 3x)^{-\frac{1}{2}}$ (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{0\}$ (c) $[0, \infty)$ (d) [illegible]

[illegible]. Find the domain of $f(x) =$ [illegible] (a) $\mathbb{R} \setminus \{\pm 1\}$ (b) $(1, \infty)$ (c) $\mathbb{R}$ (d) $\mathbb{R} \setminus \{1\}$ (e) [illegible]

[illegible]. Evaluate $\lim_{x\to\infty}(1 + \frac{1}{2x})^x$ (a) 0 (b) $\frac{1}{e}$ (c) e (d) $\sqrt{e}$ (e) NOTA

[illegible]. Find the domain of $h(x) = (2x^2 - 3x)^{-\frac{1}{2}}$. (a) $\mathbb{R}$ (b) $(-\infty, 0) \cup (\frac{3}{2}, \infty)$ (c) $(-\infty, 0)$ (d) $[0, \infty)$ (e) $(-\infty, 0]$

[illegible]. Evaluate $\int x^2e^{x^3-3}dx$. (a) $\frac{1}{3}e^{x^3-3}$ (b) e^{x^3-3} (c) $3e^{x^3-3}$ (d) $3xe^{x^3-3}$ (e) NOTA

[illegible]. Which of these best describes the continuity of $f(x)$ in $[a, b]$? (a) $f(x)$ is continuous at each point of (a, b) (b) $\lim_{x\to a} f(x) = f(a)$ (c) $\lim_{x\to b} f(x) = f(b)$ (d) $f(x)$ exists (e) NOTA

[illegible]. Evaluate $\int_1^2 \frac{3}{2}\sqrt{x}dx$ (a) $\sqrt{2}$ (b) 2 (c) $\sqrt{8}$ (d) $\sqrt{3}$ (e) [illegible]

[illegible]. The minimum point of $z = x^3 - 2x^2 + x + 4$ is? (a) (1, 4) (b) [illegible] (c) (3, 8) (d) (2, 12) (e) (3, 27)

[illegible]. Evaluate $\lim_{x\to 0^+} x\cot x$ (a) 0 (b) 1 (c) ∞ (d) -1 (e) $\frac{1}{2}$

[illegible]. If $f(x) = k$, then $Dom(f)$ and $Rng(f)$ are (a) $\mathbb{R}, k$ (b) $\mathbb{R}, \{k\}$ (c) $k, \mathbb{R}$ (d) $\{k\}, \mathbb{R}$ (e) NOTA

[illegible]. If $x^2 + y^2 - 2x - 6y + 10 = 0$, find $y'(3, 2)$ (a) 5 (b) -5 (c) 4 (d) 2 (e) -4

[illegible]. Evaluate $\int \sinh x \cosh x dx$ (a) $\frac{\sinh^2 x}{2} + c$ (b) $\frac{\cosh^2 x}{2} + c$ (c) $\tanh x + c$ (d) $\sinh x \tanh x + c$

[illegible]. Evaluate $\lim_{x\to 0^+}$ [illegible] (a) 0 (b) 1 (c) ∞ (d) -1 (e) 2

[illegible]. Which of these is false? (a) Not all relations are functions (b) Asymptote is the tangent to a curve at infinity (c) The range of every function is $\mathbb{R}$ (d) A bijection a mapping. (e) All linear functions have inverse.

If $f(x) =$ [illegible], then $f(x)$ is continuous at? (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{-5\}$ (c) [illegible] (d) [illegible] (e) [illegible]

If $f(x) = 2x$ and $g(x) = 4x^2 + 3$, then $g \circ f$ is? (a) $8x^2 + 6$ (b) $8x^2 + 3$ (c) $16x^4 + 3$ (d) $16x^2 + 3$ (e) $16x^2 + 6$

If $f(x) =$ [illegible], find $Dom(f)$. (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{-\frac{1}{2}\}$ (c) $\mathbb{R} \setminus \{\frac{1}{2}\}$ (d) $\mathbb{R}_+$ (e) [illegible]

Evaluate $\lim$ [illegible] (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1 (e) -1

Evaluate $\int$ [illegible] dx (a) [illegible] (b) [illegible] (c) [illegible] (d) [illegible] (e) NOTA

Determine the critical point(s) of the following function $f(x) = 12x^4 - 28x^3 + 21x^2 - 5x$ (a) 0, −1, 1 (b) 0, 1, $\frac{1}{2}$ (c) 0, −1, 2 (d) 1, $\frac{1}{2}$, $\frac{3}{4}$ (e) 0, 1, 2

Evaluate $\lim$ [illegible] (a) -1 (b) 1 (c) undefined (d) 0 (e) NOTA

18. Find y' if $y = \sin(x + y)$. (a) [illegible] (b) [illegible] (c) [illegible] (d) DNE (e) NOTA

19. Given $y =$ [illegible], find $y'(0)$. (a) 6 (b) -6 (c) 0 (d) -1 (e) DNE

20. Evaluate $\int$ [illegible] dx (a) 4 (b) 8 (c) 16 (d) 32 (e) 0

21. Given $a \neq 0$. Determine $\int$ [illegible] (a) [illegible] (b) [illegible] (c) [illegible] (d) [illegible] (e) NOTA

22. A body moves along a straight line according to the law [illegible]. Determine its acceleration after 2 secs. (a) [illegible] (b) [illegible] (c) [illegible] (d) [illegible] (e) [illegible]

23. Determine the relative extreme of $g(x) = (x - 2)^3$ (a) -1 (b) 0 (c) 1 (d) 2 (e) 3

24. Which of the following is true about $f(x) = \frac{1}{x}$. (a) $\lim f(x) = 0$ (b) $f(x)$ is even (c) $f(x)$ is discontinuous at $x = 0$ (d) $\lim f(x) = 1$ (e) NOTA

25. Evaluate $\int_0^1 x^2(x^3 + 1)^2 dx$ (a) 3 (b) 6 (c) 9 (d) 18 (e) 27

26. Determine $\lim$ [illegible] (a) Undefined (b) 1 (c) 0 (d) -1 (e) $\frac{1}{2}$

27. Find the area of the region bounded by $f(x) = |x - 2|$ and x-axis for $-1 \leq x \leq 1$ (a) 9 (b) 4 (c) 6 (d) 3 (e) NOTA

28. Evaluate $\int$ [illegible] dx (a) $\frac{1}{2}$ (b) [illegible] (c) [illegible] (d) [illegible] (e) 2

29. Find $k \in [0, 2]$ that satisfies the Mean Value Theorem for $f(x) = x^3 + 2x^2 + 1$ (a) $\frac{1}{3}$ (b) [illegible] (c) [illegible] (d) [illegible] (e) 2

30. All except one are odd functions (a) $3x^3 + 4\cos x$ (b) $\tan x$ (c) $\sin x$ (d) $x^3 - 4x$ (e) [illegible]

31. Obtain $k(x)$ for which $\lim 2x^2 + x - k = -1$ (a) [illegible] (b) [illegible] (c) [illegible] (d) [illegible] (e) [illegible]

32. Find the horizontal asymptote(s) of $f(x) =$ [illegible] (a) 2 (b) −3 (c) 2 (d) 1 (e) [illegible]

33. Solve $\lim$ [illegible] (a) 3 (b) 2 (c) 1 (d) 0 (e) DNE

[illegible] Evaluate $\int$ [illegible]

[illegible] Find $\frac{dy}{dx}$ if $y = e^{\sin x}$ [illegible]

[illegible] $f^{-1}(x)$ of $f(x) = x^2 - 7$ (a) $\sqrt{x - 7}$ (b) [illegible] (c) [illegible]

58. The total cost of manufacturing x units of an article is given by [illegible]. Find the number of units of the article for which the cost of manufacturing is least. (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

59. Find $\frac{d^3y}{dx^3}$ if $y = \cos 3x$ (a) $-27\sin 3x$ (b) $27\sin 3x$ (c) $-27\cos 3x$ (d) $27\cos 3x$

60. Find $dom(f)$ if $f(x) = \ln(7x - x^2 - 10)$ (a) $\mathbb{R}$ (b) [2, 5] (c) (2, 5] (d) [2, 5) (e) (2, 5)

61. Evaluate $\int$ [illegible] dx (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

62. If a function is continuous at some point, then it is differentiable at that point. (a) Contrary (b) True and False (c) False (d) True (e) Neither True nor False

63. If $f(x) = 3^{x-1}$. Find $f(x + 3) - f(x)$ in terms of $f(x)$. (a) [illegible] (b) [illegible] (c) $26f(x)$ (d) $13f(x)$ (e) [illegible]

64. Solve $\lim$ [illegible] (a) 1.5 (b) 1 (c) 2.5 (d) 1.3 (e) 3.5

65. Find $\frac{dy}{dx}$ given that $y =$ [illegible] and $x =$ [illegible] (a) [illegible] (b) [illegible] (c) [illegible] (d) $(1 + t)^2$ (e) NOTA

66. Obtain the Maclaurin's series of $f(x) = e^{-x}$. (a) $\sum \frac{x^n}{n!}$ (b) [illegible] (c) $\sum -\frac{x^n}{n!}$ (d) [illegible] (e) [illegible]

67. Solve $\int$ [illegible] dx. (a) ln2 (b) 2 (c) -2 (d) 0 (e) 1

68. Find $g'(2)$ if $g(x) =$ [illegible] (a) [illegible] (b) [illegible] (c) [illegible] (d) [illegible] (e) [illegible]

69. Pick the odd one (a) $f(x) = |x|$ is an even function (b) $g(x) = e^x$ is a positive function (c) At local minimum $f'(x)$ changes from sign from +ve to −ve as it passes (d) Every linear function is injective (e) Every function is a relation.

70. Solve $\lim$ [illegible] (a) -2 (b) undefined (c) 1 (d) -1 (e) NOTA

HEAD OF DEPARTMENT

MATHEMATICS

DATE

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MTH 102 EXAM SOLUTION 2017/2018

1) $\lim_{x\to\infty} \dfrac{35x^6+3x^2+8x}{45x^6-2x^2+3x}$

Dividing by the highest power of x

$$\lim_{x\to\infty} \frac{\frac{35x^6}{x^6}+\frac{3x^2}{x^6}+\frac{8x}{x^6}}{\frac{45x^6}{x^6}-\frac{2x^2}{x^6}+\frac{3x}{x^6}} = \lim_{x\to\infty} \frac{35+\frac{3}{x^4}+\frac{8}{x^5}}{45-\frac{2}{x^4}+\frac{3}{x^5}}$$

$$\therefore \frac{35+\frac{3}{\infty^4}+\frac{8}{\infty^5}}{45-\frac{2}{\infty^4}+\frac{3}{\infty^5}} = \frac{35+0+0}{45-0+0} = \frac{35}{45} = \frac{7}{9} \quad (d)$$

2) $y = x^3-4x^2+5x+7$

$\frac{dy}{dx} = 3x^2-8x+5 \quad (\text{slope} = m_1)$

at $(2,1) \Rightarrow m_1 = 3(2)^2-8(2)+5 = 1$

For equation of normal, we need m_2

but $m_1m_2 = -1 \Rightarrow (1)m_2 = -1 \Rightarrow m_2 = -1$

Eqn of Normal, $m_2 = \frac{y-y_1}{x-x_1} \Rightarrow -1 = \frac{y-1}{x-2}$

$\Rightarrow y-1 = -x+2 \Rightarrow y = -x+2+1$

$\therefore y = -x+3 \quad (c)$

3) $S(t) = 3t^3-8t^2+4t-5$

$v = \frac{ds}{dt} = 9t^2-16t+4$

$a = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(9t^2-16t+4)$

$a = 18t-16 \quad$ at $t = 2$secs

$a = 18(2)-16 = 20ms^{-2} \quad (a)$

4) $f(x) = 5x-2, \quad g(x) = \frac{x+2}{5}$

$(f\circ g)(x) = f[g(x)] = f\left[\frac{x+2}{5}\right] = 5\left(\frac{x+2}{5}\right)-2$

$= x+2-2 = x$

$(g\circ f)(x) = g[f(x)] = g(5x-2) = \frac{(5x-2)+2}{5}$

$= \frac{5x}{5} = x$

Ans $= x, x \quad (c)$

5) $y = x^3-6x^2+9x+10$ —— *

$\frac{dy}{dx} = 3x^2-12x+9$

The critical point is at $\frac{dy}{dx} = 0$

$3x^2-12x+9 = 0 \Rightarrow x^2-4x+3 = 0$

Solving, we get $x = 3$ or 1

Substituting in the given equation *

$y = 3^3-6(3)^2+9(3)+10 = 10$

$y = 1^3-6(1)^2+9(1)+10 = 14$

Thus, the critical points are $(3,10)$ and $(1,14)$

To determine the nature of the critical points, we obtain the second derivative

$\frac{d^2y}{dx^2} = 6x-12$

at $(3,10)$, $6(3)-12 = 6 > 0$ (minimum pt)

$\therefore (3,10)$ is a minimum point. (a)

6) $f(x) = \frac{x^2+2x-35}{x^2-25} = \frac{(x+7)(x-5)}{(x-5)(x+5)} = \frac{x+7}{x+5}$

Vertical asymtote $\Rightarrow x+5 = 0 \Rightarrow x = -5 \quad (a)$

7) $f(x) = \frac{2x+7}{\sqrt{x^2-81}}$

The function is defined if $x^2-81 > 0$

$\Rightarrow (x-9)(x+9) > 0$

Using the truth table the solution is

$\{x<-9\} \cup \{x>9\}$

Thus, domain $= (-\infty,-9) \cup (9,\infty) \quad (d)$

8) $\lim_{x\to 0} \frac{\sin mx}{m\sin nx} = \frac{\sin 0}{m \sin 0} = \frac{0}{0}$ (indeterminate)

Using L'Hopital's rule,

$\lim_{x\to 0} \frac{m\cos mx}{mn\cos nx} = \frac{m\cos 0}{mn\cos 0} = \frac{m}{mn} = \frac{1}{n} \quad (c)$

9) $f(x) = \begin{cases} x+5, & x\le 1 \\ 5+kx^2, & x>1 \end{cases}$

Left hand limit $\lim_{x\to 1^-}(x+5) = 6$

Right hand limit $\lim_{x\to 1^+}(5+kx^2) = 5+k$

The function is continuous if the left hand limit is equal to the right hand limit

$6 = 5+k \Rightarrow k = 6-5$

$k = 1 \quad (c)$

10) $V = \frac{4}{3}\pi r^3, \quad \frac{dr}{dt} = 0.5cms^{-1}, \quad r = 5cm$

$\frac{dv}{dr} = 4\pi r^2$, but $\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$\frac{dv}{dt} = 4\pi r^2 \times (0.5) = 2\pi r^2 = 2\pi(5)^2$

$= 50\pi cm^3s^{-1} \quad (b)$

(11) $\int x\ln x\,dx$

let $u=\ln x$, $dv=x$

$\frac{du}{dx}=\frac{1}{x}$, $v=\frac{x^2}{2}$

Using integration by part

$\int udv = uv - \int vdu$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ (C)

(12) $\lim_{x\to\infty}\left(1+\frac{1}{9x}\right)^{2x}$

let $y=\left(1+\frac{1}{9x}\right)^{2x}$

Take the ln of both sides

$\ln y = \ln\left(1+\frac{1}{9x}\right)^{2x}$

$\ln y = 2x\ln\left(1+\frac{1}{9x}\right)$ or $\ln y = \frac{2\ln\left(1+\frac{1}{9x}\right)}{1/x}$

Taking the limit, we get indeterminate

Hence By L'Hopitals rule

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{2\cdot\frac{-1}{9x^2}\left(\frac{1}{1+1/9x}\right)}{-1/x^2} = \frac{2}{9}$

$\ln \lim_{x\to\infty} y = \frac{2}{9}$; Take e of both sides

$\lim_{x\to\infty} y = e^{2/9}$ (d)

(13) $y(t)=1+\sec 2t$ and $x(t)=\tan 2t$

$\frac{dy}{dt}=2\sec 2t\tan 2t$; $\frac{dx}{dt}=2\sec 2t$

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{dx/dt}$ (Parametric differentiatio)

$\frac{dy}{dx}=2\sec 2t\tan 2t\times\frac{1}{2\sec 2t}=\tan 2t$ (a)

(14) $x^3y+xy^3=2 \Rightarrow f(x,y)=x^3y+xy^3-2$

$f_x=3x^2y+y^3$, $f_y=x^3+3xy^2$

$y'=\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-(3x^2y+y^3)}{x^3+3xy^2}$ at $(1,1)$

$\frac{-[3(1)^2(1)+1^3]}{1^3+3(1)(1)^2}=\frac{-4}{4}=-1$

$y''=\frac{d^2y}{dx^2}=\frac{-f_x}{f_y}=\frac{-(6xy)}{6xy}=1$ (e)

(15) $\lim_{x\to1} x^{\frac{1}{x-1}}$

Let $y=x^{\frac{1}{x-1}}$

Take ln of both sides

$\ln y=\ln x^{\frac{1}{x-1}}$

$\ln y=\frac{1}{x-1}\ln x$

$\ln y=\frac{\ln x}{x-1}$

If we put the limit (1) we get Indeterminate

Using L'Hopital's rule

$\lim_{x\to1}\ln y=\lim_{x\to1}\frac{1/x}{1}$

$\lim_{x\to1}\ln y=1$

$\ln\lim_{x\to1} y=1$

Take e of both sides

$\lim_{x\to1} y=e^1=e$ (b)

(16) $\lim_{x\to2}\frac{4-x^2}{3-\sqrt{x^2+5}}=\frac{0}{0}$

Indeterminate. Rationalizing, we have

$\frac{4-x^2}{3-\sqrt{x^2+5}}\times\frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}=\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}$

$=\frac{(4-x^2)(3+\sqrt{x^2+5})}{(4-x^2)}=3+\sqrt{x^2+5}$

$\lim_{x\to2}\left[3+\sqrt{x^2+5}\right]=\underline{\underline{6}}$ (d)

(17) $f(x)=\sqrt{\frac{x}{2-x}}$

The Function is defined if

$\frac{x}{2-x}\geq0$, and $x\neq2$

$\frac{x}{2-x}\cdot(2-x)^2\geq0\cdot(2-x)^2$

$x(2-x)\geq0$

The turning values are $x=0,2$

Using the truth table,

	$x\leq0$	$0\leq x\leq2$	$x\geq2$
x	$-$	$+$	$+$
$(2-x)$	$+$	$+$	$-$
	$-$	$+$	$-$

Solution $=0\leq x\leq2$

But the function is defined for $x\neq2$

Thus, domain $=[0,2)$ (b)

18) $y = \sin(x+y)$

By implicit differentiation

$$\frac{dy}{dx} = \cos(x+y) + \cos(x+y)\cdot\frac{dy}{dx}$$

$$[1-\cos(x+y)]\frac{dy}{dx} = \cos(x+y)$$

$$\frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)} \qquad (a)$$

19) Given $y = \frac{x}{\sqrt{x-1}}$

let $u = x$, $v = \sqrt{x-1}$, $\frac{du}{dx} = 1$, $\frac{dv}{dx} = \frac{1}{2\sqrt{x-1}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{(Quotient Rule)}$$

$$= \frac{\sqrt{x-1}\,(1) - x\left(\frac{1}{2\sqrt{x-1}}\right)}{(\sqrt{x-1})^2} = \frac{\frac{2(x-1)-x}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{x-1}{2\sqrt{x-1}}\cdot\frac{1}{x-1} = \frac{1}{2\sqrt{x-1}} \text{ or } \frac{(x-1)^{-1/2}}{2}$$

$$\frac{d^2y}{dx^2} = y'' = \frac{-\frac{1}{2}(x-1)^{-3/2}}{2} = -\frac{1}{4}(x-1)^{-3/2} = \frac{1}{4}$$

(e)

20) $\int_2^6 |6-2x|\,dx$

$6 - 2x = 0 \Rightarrow x = 3$

Splitting the integrals

$$\int_2^3 (6-2x)dx + \int_3^6 -(6-2x)dx$$

$$\int_2^3 (6-2x)dx + \int_3^6 (2x-6)dx$$

$$[6x - x^2]_2^3 + [x^2 - 6x]_3^6$$

$$\{[6(3)-3^2] - [6(2)-2^2]\} + \{[6^2-6(6)] - [3^2-6(3)]\}$$

$$[9-8] + [0-(-9)] = 1+9 = 10 \qquad (c)$$

21) $\int \frac{dx}{x^2-a^2} = \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}dx$

$$\frac{1}{2a}\int\left[\frac{x+a}{(x-a)(x+a)} - \frac{x-a}{(x-a)(x+a)}\right]dx = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c \qquad (a)$$

22) $s(t) = \frac{t^3-4t}{2}$, $v(t) = \frac{ds}{dt} = \frac{3t^2-4}{2} = \frac{3t^2}{2} - 2$

$a(t) = \frac{6t}{2} = 3t \Rightarrow$ at $t = 2sec$; $a = 6ms^{-2}$ (e)

23) $g(x) = (x-2)^{2/3}$; $g'(x) = (x-2)^{-1/3}$

$$g'(x) = \frac{1}{(x-2)^{1/3}}$$

If we equate $g(x) = 0$, there's no value of x for which $g'(x) = 0$, hence no correct option

24) $f(x) = \frac{|x|}{x}$ NOTA (None of the Above)

(e)

25) $\int_0^1 x^2(x^3+2)^2 dx$ —— *

let $u = x^3+2$; $\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$

Substitute in the given equation*

$$\int_0^1 x^2 u^2 \frac{du}{3x^2} = \frac{1}{3}\int_0^1 u^2 du = \left[\frac{1}{3}\cdot\frac{u^3}{3}\right]_0^1$$

$$= \left[\frac{(x^3+2)^3}{9}\right]_0^1 = \frac{(1^3+2)^3}{9} - \frac{(0^3+2)^3}{9} = \frac{27}{9} - \frac{8}{9} = \frac{19}{9}$$

26) $\lim_{x\to 0}\frac{1-\cos x}{x} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$,

Indeterminate.

Using L'Hopital's rule

$$\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{\sin x}{1} = \frac{\sin 0}{1} = 0 \qquad (c)$$

27) $\int_{-1}^5 |x-2|dx = \int_{-1}^2 -(x-2)dx + \int_2^5 (x-2)dx$

$$= \int_{-1}^2 (2-x)dx + \int_2^5 (x-2)dx$$

$$= \left[2x - \frac{x^2}{2}\right]_{-1}^2 + \left[\frac{x^2}{2} - 2x\right]_2^5$$

$$= \left\{\left[2(2) - \frac{2^2}{2}\right] - \left[2(-1) - \frac{(-1)^2}{2}\right]\right\} + \left\{\left[\frac{5^2}{2} - 2(5)\right] - \left[\frac{2^2}{2} - 2(2)\right]\right\}$$

$$= \left\{[4-2] - \left[-2-\frac{1}{2}\right]\right\} + \left\{\left[\frac{25}{2}-10\right] - [2-4]\right\}$$

$$= \left\{2 + \frac{5}{2}\right\} + \left\{\frac{5}{2} + 2\right\} = \frac{10}{2} + 4 = 5+4 = 9$$

(a)

28) $\int_e^{e^2}\frac{\ln x}{x}dx = \int_e^{e^2}\ln x\cdot\frac{1}{x}dx$

Which is of the form $\int_e^{e^2} f(x)f'(x)dx$

Hence the solution is $\left[\frac{(\ln x)^2}{2}\right]_e^{e^2}$

$$\frac{(\ln e^2)^2}{2} - \frac{(\ln e)^2}{2} = \frac{\ln e^4}{2} - \frac{\ln e^2}{2}$$

$$= \frac{4\ln e}{2} - \frac{2\ln e}{2} = 2 - 1 = 1$$

Direct punching (calculator) gives $1\cdot 5 = \frac{3}{2}$ (b)

Closest option.

(29) $f(x) = x^3 + 2x^2 + 1$

$f(b) = f(3) = 46$

$f(a) = f(0) = 1$

$f'(x) = 3x^2 + 4x \Rightarrow f'(K) = 3K^2 + 4K;\ K \in [0,3]$

By the Mean Value theorem

$$f'(K) = \frac{f(b) - f(a)}{b - a}$$

$$3K^2 + 4K = \frac{46 - 1}{3 - 0} \Rightarrow 3K^2 + 4K = 15$$

$$3K^2 + 4K - 15 = 0$$

$$3K^2 + 9K - 5K - 15 = 0$$

$$3K(K+3) - 5(K+3) = 0 \Rightarrow (K+3)(3K-5) = 0$$

$$\Rightarrow K = -3,\ 5/3$$

But $K = -3 \notin [0,3]$, Hence $K = 5/3$ (c)

(30) All are odd except $3x^2 + 4\cos x$ (a)

(31) $f(x) = 2x^2 + x - 4$ $\quad L = -1$

$|f(x) - L| < \varepsilon \Rightarrow$

$|2x^2 + x - 4 - (-1)| < \varepsilon \Rightarrow |2x^2 + x - 3| < \varepsilon$

Introducing the co-ordinates, we have

$|2(x-1)^2 + (x-1) + 4x - 4| < \varepsilon$

$|2(x-1)^2 + (x-1) + 4(x-1)| < \varepsilon$

$\leq 2|x-1|^2 + |x-1| + 4|x-1| < \varepsilon$

$2\delta^2 + \delta + 4\delta < \varepsilon$

$2\delta^2 + 5\delta < \varepsilon$

But $\delta^2 \leq \delta$

$2\delta + 5\delta < \varepsilon \Rightarrow 7\delta < \varepsilon \Rightarrow \delta < \frac{\varepsilon}{7}$

So, we choose $\delta = \frac{\varepsilon}{7}$ for which $\lim_{x \to 1}(2x^2 + x - 4) = -1$ (e)

(32) $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4}$

Taking Limit to infinity gives the horizontal asymptote

$$\lim_{x \to \infty} \frac{x^2 + 2x - 3}{x^2 - 4} = 1 \qquad (d)$$

(33) $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1} = \dfrac{1^3 - 1}{1 - 1} = \dfrac{0}{0}$ Indeterminate

$\lim_{x \to 1} \dfrac{(x-1)(x^2 + x + 1)}{(x-1)}$ (By long division)

$\lim_{x \to 1}(x^2 + x + 1) = 1^2 + 1 + 1 = 3$ (a)

OR By L'Hopital's $\lim_{x \to 1} \dfrac{3x^2}{1} = 3(1)^2 = 3$

(34) $\int \dfrac{dx}{\sqrt{x^2 - 9}}$ ——— *

Using trigonometric substitution

let $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta$

Substituting in *

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{(3\sec\theta)^2 - 9}} = \int \frac{3\sec\theta\tan\theta}{\sqrt{9(\sec^2\theta - 1)}}\, d\theta$$

But $\sec^2\theta - 1 = \tan^2\theta$

$$\frac{3\sec\theta\tan\theta}{\sqrt{9\tan^2\theta}} = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C_1$$

replace θ terms with x terms;

$\sec\theta = \frac{x}{3}$ and $\sec^2\theta = \left(\frac{x}{3}\right)^2 \Rightarrow \sec^2\theta = \frac{x^2}{9}$

$\tan^2\theta + 1 = \frac{x^2}{9} \Rightarrow \tan^2\theta = \frac{x^2}{9} - 1 \Rightarrow \tan^2\theta = \frac{x^2 - 9}{9}$

$\tan\theta = \sqrt{\frac{x^2 - 9}{9}} \Rightarrow \tan\theta = \frac{\sqrt{x^2 - 9}}{3}$

$$\therefore \int \frac{dx}{\sqrt{x^2 - 9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right| + C_1$$

$$= \ln(1/3)(x + \sqrt{x^2 - 9}) + C_1 = \ln(1/3) + \ln(x + \sqrt{x^2 - 9}) + C_1$$

$$= \ln(x + \sqrt{x^2 - 9}) + C$$

(35) $g(x) = (2x^2 - 3x)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2x^2 - 3x}}$

Since the numerator is 1, and the denominator is a square root function which is non-negative, hence the range $= (0, \infty)$ (c)

(36) $f(x) = \dfrac{\sin(x^2 - 1)}{x^2 + 1}$

Domain $= \mathbb{R}$

(37) $\lim_{x \to \infty}\left(1 + \frac{1}{2x}\right)^x = e^{1/2} = \sqrt{e}$ (d)

(See lecture note for solution)

(38) $h(x) = (2x^2 - 3x)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2x^2 - 3x}}$

The function is defined for

$2x^2 - 3x > 0$

$x(2x - 3) > 0$

Turning values are $x = 0,\ 3/2$

Using the truth table

	$x<0$	$0<x<3/2$	$x>3/2$
x	$-ve$	$+ve$	$+ve$
$2x-3$	$-ve$	$-ve$	$+ve$
	$+$	$-$	$+$

solution is $\{x : x < 0 \cup x > 3/2\}$

Domain: $(-\infty, 0) \cup (3/2, \infty)$ (b)

(39) $\int x^2 e^{x^3 - 3}\, dx$

Let $u = x^3 - 3$, $\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$

Substituting, $\int x^2 e^u \frac{du}{3x^2} = \frac{1}{3}\int e^u\, du$

$= \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3 - 3} + c$ (a)

(40) All of the Above (AOTA) (e)

(41) $\int_0^2 \frac{3}{2}\sqrt{x}\,dx = \frac{3}{2}\int_0^2 x^{1/2}\,dx = \left[\frac{3}{2}\frac{x^{3/2}}{3/2}\right]_0^2$

$\left[\frac{3}{2}\cdot\frac{2}{3}x^{3/2}\right]_0^2 = \left[x^{3/2}\right]_0^2 = 2^{3/2} - 0^{3/2} = 2^{3/2}$

$= \sqrt{2^3} = \sqrt{8}$ (c)

(42) $z = x^3 - 2x^2 + x + 4$

$\frac{dz}{dx} = 3x^2 - 4x + 1$

The critical point is at $\frac{dz}{dx} = 0$

$3x^2 - 4x + 1 = 0$

$3x^2 - 3x - x + 1 = 0$

$3x(x-1) - 1(x-1) = 0$

$(x-1)(3x-1) = 0$

$x = 1, \frac{1}{3}$

Substitute in the original equation

$z(1) = 1^3 - 2(1)^2 + 1 + 4 = 4$

$z(\frac{1}{3}) = \frac{1}{3}^3 - 2(\frac{1}{3})^2 + \frac{1}{3} + 4 = \frac{112}{27}$

$(1,4)$ and $(\frac{1}{3}, \frac{112}{27})$ are the critical points, we take second derivative To verify the nature, we take second derivative

$\frac{d^2z}{dx^2} = 6x - 4$

at $(1,4)$ $\frac{d^2z}{dx^2} = 6(1) - 4 = 2 > 0$

Thus $(1,4)$ is a minimum point (a)

(43) $\lim_{x\to 0^+} x\cot x = \lim_{x\to 0^+} x\cdot\frac{\cos x}{\sin x} = \frac{0\cos 0}{\sin 0} = \frac{0}{0}$

Which is indeterminate,

Applying L'Hopital's rule,

$\lim_{x\to 0^+} \frac{-x\sin x + \cos x}{\cos x} = \frac{-0\sin 0 + \cos 0}{\cos 0}$

$= \frac{1}{1} = 1$ (b)

(44) $f(x) = K$

$Dom(f) = \mathbb{R}$

$Rng(f) = K$ (a)

(45) $x^2 + y^2 - 2x - 6y + 10 = 0$

This is an implicit function

$2x + 2yy' - 2 - 6y' = 0$

$(2y-6)y' = 2 - 2x$

$y' = \frac{2-2x}{2y-6} \Rightarrow y' = \frac{1-x}{y-3}$

$y'(3,2) = \frac{1-3}{2-3} = \frac{-2}{-1} = 2$ (d)

(46) $\int \sinh x \cosh x \, dx$ — *

This is of the form $\int f(x) f'(x)\,dx$

∴ Solution is $\frac{\sinh^2 x}{2} + c$ (a)

Alternatively: let $u = \sinh x$

$\frac{du}{dx} = \cosh x \Rightarrow dx = \frac{du}{\cosh x}$

Substitute in *

$\int u\cdot\cosh x\cdot\frac{du}{\cosh x} = \int u\,du = \frac{u^2}{2} + c$

$= \frac{\sinh^2 x}{2} + c.$

(47) $\lim_{x\to 0^+}\frac{\sin x}{x^2} = \frac{\sin 0}{0^2} = \frac{0}{0}$ Indeterminate

Using L'Hopital's rule

$\lim_{x\to 0^+}\frac{\cos x}{2x} = \frac{\cos 0}{2(0)} = \frac{1}{0}$ undefined

Using L'Hopital's rule again

$\lim_{x\to 0^+}\frac{\sin x}{2} = \frac{\sin 0}{2} = \frac{0}{2} = 0$

(48) (C)

(49) $f(x) = \frac{\sqrt{x^2-9}}{x+5}$

$f(x)$ is continuous at

$(-\infty, -5) \cup (-5, -3] \cup [3, \infty)$

(50) $f(x) = 2x$ $g(x) = 4x^2 + 3$

$g\circ f = g[f(x)] = g(2x) = 4(2x)^2 + 3$

$= 4(4x^2) + 3 = 16x^2 + 3$ (d)

(51) $f(x) = \frac{x+3}{2x+1}$

$f(x)$ is defined for $\mathbb{R}\setminus\{2x+1=0\}$

$\mathbb{R}\setminus\{x = -\frac{1}{2}\}$

$Dom(f) = \mathbb{R}\setminus\{-\frac{1}{2}\}$

(52) $\lim_{x\to 0}\frac{\cosh x - e^x}{x} = \frac{\cosh 0 - e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$

Which is indeterminate, By L'Hopitals rule

$\lim_{x\to 0}\frac{\sinh x - e^x}{1} = \frac{\sinh 0 - e^0}{1} = \frac{0-1}{1} = -1$ (e)

(53) $\int_2^3 \frac{3x^2-2x+5}{(x-1)(x^2+5)}dx = \int_2^3 \frac{3x^2-2x+5}{x^3-x^2+5x-5}dx$

$= \ln(x^3 - x^2 + 5x - 5)\Big|_2^3$

$= \ln(3^3 - 3^2 + 5(3) - 5) - \ln(2^3 - 2^2 + 5(2) - 5)$

$= \ln(27 - 9 + 15 - 5) - \ln(8 - 4 + 10 - 5)$

$= \ln(28) - \ln(9) = \ln(\frac{28}{9})$ (a)

(54) $f(x) = 12x^4 - 28x^3 + 21x^2 - 6x$

$f'(x) = \frac{dy}{dx} = 48x^3 - 84x^2 + 42x - 6$

Critical point is at $\frac{dy}{dx} = 0$

$48x^3 - 84x^2 + 42x - 6 = 0$

$8x^3 - 14x^2 + 7x - 1 = 0$

$(x-1)(2x-1)(4x-1) = 0$

$x = 1, \frac{1}{2}, \frac{1}{4}$ (d)

(55) $\lim_{x\to a} \frac{|x-a|}{x-a} = \lim_{x\to a} \frac{\pm(x-a)}{x-a} = \pm 1$.

Since the limit is not unique, it is undefined (c)

(56) $y = e^{\sin x}$

let $= \;, y = e^u$

$\frac{du}{dx} = \cos x \qquad \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times \cos x = \cos x e^u = \cos x e^{\sin x}$ (b)

(57) $f(x) = x^2+2 \Rightarrow y = x^2+2$

$x^2 = y-2 \Rightarrow x = \pm\sqrt{y-2}$

$f^{-1}(x) = \pm\sqrt{x-2}$

The inverse of a function is suppose to be unique. So the inverse does not exist (e)

(58) $y = \frac{5}{4}x^2 + \frac{20}{x}$

$y' = \frac{5}{2}x - \frac{20}{x^2}$

For least manufacturing cost $y' \leq 0$

$\frac{5}{2}x - \frac{20}{x^2} \leq 0$

$\frac{5}{2}x^3 - 20 \leq 0 \Rightarrow \frac{5}{2}x^3 \leq 20$

$x^3 \leq 20 \cdot \frac{2}{5} \Rightarrow x^3 \leq 8 \Rightarrow x \leq 2$ (b)

(59) $y = \cos 3x$

$\frac{dy}{dx} = -3\sin 3x \quad , \quad \frac{d^2y}{dx^2} = -9\cos 3x$

$\frac{d^3y}{dx^3} = -9(3)(-\sin 3x) = 27\sin 3x$ (b)

(60) Find dom(f) if $f(x) = \ln(7x - x^2 - 10)$

$f(x)$ is defined for $7x - x^2 - 10 > 0$

or $-x^2 + 7x - 10 > 0 \Rightarrow x^2 - 7x + 10 < 0$

$\rightarrow x^2 - 2x - 5x + 10 < 0$

$x(x-2) - 5(x-2) < 0$

$(x-2)(x-5) < 0$

Turning values are 2, 5, Using truth table

	$x<2$	$2<x<5$	$x>5$
$x-2$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
	$+$	$-$	$+$

Solution is $\{x: 2<x<5\}$

Dom(f) = $(2, 5)$

(61) $\int_0^{\pi/2} \cos(-4x)dx = \left[-\frac{\sin 4x}{4}\right]_0^{\pi/2} = -\frac{1}{4}[\sin 4(\pi/2) - \sin 0]$

$= -\frac{1}{4}[\sin 2\pi - \sin 0] = -\frac{1}{4}[0-0] = 0$ (a)

(62) True (d)

(63) $f(x) = 3^{x-1}$, $f(x+3) = 3^{x+3-1} = 3^{x+2}$

$f(x+3) - f(x) = 3^{x+2} - 3^{x-1} = 3^x \cdot 3^2 - 3^x \cdot 3^{-1}$

$= 3^x(3^2 - 3^{-1}) = (9 - \frac{1}{3})3^x = \frac{26}{3}3^x = 26(3^{x-1}) = 26f(x)$

(C)

(64) $\lim_{x\to 1} \frac{x^2 - \sqrt{x}}{x-1} = \frac{1^2 - \sqrt{1}}{1-1} = \frac{0}{0}$ indeterminat[e]

using L'Hopitals rule

$\lim_{x\to 1} \frac{2x - \frac{1}{2}x^{-1/2}}{1} = \frac{2(1) - \frac{1}{2}(1)^{-1/2}}{1} = \frac{2-\frac{1}{2}}{1}$

or 1.5 (d)

(65) $y = \frac{3+2t}{1+t}$; $x = \frac{2-3t}{1+t}$

$\frac{dy}{dt} = \frac{(1+t)(2) - (3+2t)\cdot 1}{(1+t)^2}$; $\frac{dx}{dt} = \frac{(1+t)(-3) - (2-3t)}{(1+t)^2}$

$\frac{dy}{dt} = \frac{2+2t-3-2t}{(1+t)^2} = \frac{-1}{(1+t)^2}$; $\frac{dx}{dt} = \frac{-3-3t-2+3t}{(1+t)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{dx/dt} = \frac{-1}{(1+t)^2} \times \frac{1}{\frac{-5}{(1+t)^2}} = \frac{-1}{(1+t)^2} \times \frac{(1+t)^2}{-5}$

$= \frac{1}{5}$ (a)

(66) $f(x) = e^{-x}$, $x_0 = 0$.

$f(x) = e^{-x}$; $f(0) = e^0 = 1$

$f'(x) = -e^{-x}$; $f'(0) = -e^0 = -1$

$f''(x) = e^{-x}$; $f''(0) = e^0 = 1$

$f'''(x) = -e^{-x}$; $f'''(0) = -e^0 = -1$

$f^{(iv)}(x) = e^{-x}$; $f^{(iv)}(0) = e^0 = 1$

Maclaurin series is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$

$= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(iv)}(0)x^4}{4!} + \cdots$

$= 1 + (-1)x + \frac{(1)x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{(1)x^4}{4!} + \cdots$

$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (b)

(67) $\int_0^{\pi/2} \frac{\sin 2x}{1+\cos^2 x}dx = \int \frac{2\sin x \cos x}{1+\cos^2 x}dx = \int \frac{2\cos x}{1+\cos^2 x}\sin x\, dx$

let $u = \cos x$, $\frac{du}{dx} = -\sin x \Rightarrow dx = \frac{du}{-\sin x}$

Substitute in $\int \frac{2u}{1+u^2} \sin x \cdot \frac{du}{-\sin x} = -\int \frac{2u}{1+u^2}du = -\ln|1+$

$= \left[-\ln|1+\cos^2 x|\right]_0^{\pi/2} = -\ln(1+\cos^2 \pi/2) + \ln(1+\cos 0)$

$= -\ln(1) + \ln 2 = \ln 2$ (a)

(68) $y(x) = \frac{x^2+1}{x^2-1}$; Using quotient rule

$y' = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}, \frac{-4x}{(x^2-1)}$

$y'(2) = \frac{-4(2)}{(2^2-1)^2} = \frac{-8}{3^2} = \frac{-8}{9}$ (d)

(69) (C)

(70) $\lim_{x\to 0}\left(\operatorname{cosec} x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$ = Und[efi]ned

$\lim_{x\to 0}\left(\frac{x - \sin x}{x \sin x}\right) = \frac{0 - \sin 0}{0 \sin 0} = \frac{0}{0}$ indeterminate

using L'Hopitals $\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x + x\cos x}\right) = \frac{1-\cos 0}{\sin 0 - 0\cos 0} = \frac{0}{0}$

L'Hopitals again $\lim_{x\to 0}\left(\frac{\sin x}{\cos x + \cos x - x\sin x}\right) = \frac{\sin 0}{2\cos 0 - 0\sin 0} = \frac{0}{2} =$

FEDERAL UNIVERSITY OF TECHNOLOGY, OW.
SCHOOL OF PHYSICAL SCIENCES
DEPARTMENT OF MATHEMATICS
ELEMENTARY MATHEMATICS II (MTH 102) RAIN
SEMESTER EXAM: 2016/2017
TIME ALLOWED: 2Hrs. 30 Mins.
DATE: 13/11/2017

INSTRUCTIONS: ATTEMPT ALL QUESTIONS.

1. Find respectively the Domain and Range of the function $f(x) = \sqrt{9-x^2}$ (A) $(-3,3)$; $(0,3)$ (B) $[-3,3]$; $(0,3)$ (C) $[-3,3]$; $[0,3]$ (D) $(0,3)$; $(-3,3)$; (E) None of the above

2. Determine all the vertical asymptotes of the graph $f(x) = \frac{x^2-x-6}{x^2-9}$ (A) -3 (B) 2 (C) -3, 3 (D) 0 (E)3

3. Determine all the horizontal asymptotes of the function $f(x) = \frac{5x^2+7x-1}{x^2-25}$ (A) 2 (B) -2 (C) -5,5 (D)25 (E) 5

4. Evaluate $\int x\ln x\, dx$. (A) $\frac{x^2}{2}\ln x - \frac{x^4}{12} + c$ (B) $\frac{x^3}{3}\ln x - \frac{x^5}{15} + c$ (C) $\frac{x^2}{2}\ln x + c$ (D) $\ln x + c$ (E) None of the above.

5. Evaluate $\int \cos^2 x\, dx$ (A) $\frac{x}{2} - \frac{\sin 2x}{4} + c$ (B) $\frac{x}{2} + \frac{\sin 2x}{2} + c$ (C) $\frac{x}{2} + \frac{\sin 2x}{4} + c$ (D) $\frac{x}{2} + \frac{\cos 2x}{4} + c$ (E) None of the above

5. The radius of a sphere is increasing at the rate of 0.25cm/sec when r=7cm, find the rate at which the volume is increasing.
(A) $44cm^2/sec$ (B) $45cm^2/sec$ (C) $160cm^2/sec$ (D) $156cm^2/sec$ (E) $154cm^2/sec$

6. Find the slope of the tangent line to the circle $x^2 + y^2=25$ at the point (3,4).
(A) 3/4 (B) -3/4 (C) 1/4 (D) -5/4 (E) 1/2

7. Let f be defined by $x^3 + 2x^2 + 1$. Using the mean value theorem find any real number c in the interval [0, 3].
(A) -3 (B) 5/3 (C) 2 (D) 4/3 (E) None of the above.

8. If $x^2 - 2y^2 = 4$. Find $\frac{dy}{dx}$ at the point (1, 3).
(A) 3 (B) 0 (C) -3 (D) -1/3 (E) None of the above.

9. Evaluate $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$. (A) e^3 (B) e (C) 1 (D) e^3 (E) None of the above.

10. Obtain the value of δ if $\lim_{x\to 2}(3x + 7) = 1$.
(A) $\frac{\varepsilon}{2}$ (B) ε (C) $\frac{\varepsilon}{3}$ (D) $\frac{\varepsilon}{6}$ (E) None of the above.

11. A particle moves along a line so that at time t secs its position is $s(t) = t^3 - 6t^2 + 7t - 2$ metres. Find the time t_0 at which acceleration equals zero. (A) 4 secs (B) 3 secs (C) 2 secs (D) 12 secs (E) 0.5 secs.

12. Obtain the maximum point of the function $y = x(x-1)^2$. (A) (1, 0) (B) (1/3, 4/27) (C) (1, 1/3) (D) (1/3, 0) (E) None of the above.

13. A function is given by $f(t) = \sin \pi t$. Find the instantaneous rate of change at $t = {}^1/_4$. (A) $\frac{\pi}{2}$ (B) π (C) $\frac{\pi\sqrt{3}}{2}$ (D) $\frac{\pi\sqrt{2}}{2}$ (E) None of the above.

14. Find all the vertical asymptotes for the graph of $f(x) = \frac{x^4}{4-x^2}$. (A) 0, 2 (B) 0, 4 (C) -2, 2 (D) 0, -2 (E) 0, -4

15. All of these except one can be expressed as $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. (A)The derivative of f(x) at x_0. (B) The average rate of change of f(x) at x_0. (C) The slope of the graph of f(x) at x_0 (D) The differential coefficient of f(x). (E) None of the above.

16. A Giraffe 6ft tall walks at the rate of 4.4ft/sec. At what rate is the length of its shadow increasing?
(A) $\frac{66}{35}ft/sec$ (B) $\frac{36}{7}ft/sec$ (C) $\frac{22}{5}ft/sec$ (D) 26.4ft/sec (E) $\frac{3}{2.2}ft/sec$.

17. Find f^{-1} if $f(x) = 4x + 3$. (A) $\frac{x-3}{4}$ (B) $\frac{1}{4x+3}$ (C) $\frac{1}{4x}$ (D) $\frac{4}{4x+3}$ (E) $\frac{4x+3}{4}$

18. Find the horizontal asymptote of the graph $f(x) = \frac{(3x-2)(2x+3)}{(2x+1)(x+2)}$. (A) 3/2 (B) 3 (C) 0 (D) 9/2 (E) 2/3

19. Obtain the integral $\int x^2(x^3+7)^4 dx$. (A) $\frac{(x^3+7)^5}{15} + K$ (B) $\frac{(x^5+7x^2)^5}{5} + K$ (C) $\frac{x^2(x^3+7)^5}{15} + K$ (D) $\frac{x^3(x^3+7)^5}{8} + K$ (E) $\frac{(x^5+7x^2)^5}{4} + K$.

20. Two functions f(x) and g(x) are defined on the set of real numbers by $f(x) = 2x + 3$ and $g(x) = x^2 + 2x - 7$. Find f(g(2)). (A) 2 (B) 3 (C) 5 (D) 6 (E) 25.

21. Determine respectively the Domain and Range of the function, $(x) = \frac{1}{\sqrt{9-x^2}}$. (A) (-3, 3) ; R^+ (B) [-3, 3] ; R^+ (C) (-3, 3) ; R (D) [-3, 3] ; R (E) R ; R

22. Find the inverse of the function $f(x) = \frac{1}{2}x - 3$.
(A) 2(x +1) (B) 2(x + 2) (C) 2(x + 3) (D) 2(x + 6) (E) 2x + 3

23. Find δ > 0 such that |f(x) – L| < 0.01 whenever 0 < |x - 2| < δ for the limit, $\lim_{x\to 2}(3x + 2) = 8$.
(A) 0.033 (B) 0.33 (C) 0.00033 (D) 0.0033 (E) 0.000033.

24. Evaluate the limit, $\lim_{x\to\infty}\frac{2x^2-3x+1}{3x^2+5x+1}$. (A) 3/2 (B) 2/3 (C) 1/2 (D) 2/5 (E) 5/2

25. Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point (3, 1). (A) 9/13 (B) -9/13 (C) -13/9 (D) 13/9 (E) 2/13.

26. Determine all vertical asymptotes of the graph c $f(x) = \frac{x^2+2x-8}{x^2-4}$. (A) 2, -2 (B) 2 (C) -2 (D) 0 (E) 2, 2

27. Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
(A) $y = 3x - \sqrt{2}$ (B) $y = 3x + \sqrt{2}$ (C) $\sqrt{2}\,x - 3$ (D) $\sqrt{2}\,x + 3$ (E) $-3x - \sqrt{2}$

28. Find $\frac{dy}{dx}$ if $y = \cos 2t$, $x = \sin t$. (A) 4sint (B) -4sint (C) 4cost (D) -4cost (E) 4sin2t

29. Find the greatest product of two numbers wh sum is 12. (A) 36 (B) 64 (C) 2 (D) 6 (E) 2

30. Determine the minimum points of the functio $f(x) = x^4 - 6x^2 + 8x + 10$. (A) 0 (B) 1, 1, -2 (D) 1, -2 (E) -2

31. Evaluate $\int \frac{7+x-2x^2}{2-x} dx$.
(A) $x^2+3x-\ln(2-x)+C$ (B) $x^3+2x-\ln(2+x)+C$ (C) $\frac{x^2}{2}-x+\ln(1+x)+C$
(D) $\ln(1+x)+C$ (E) $\ln(1-x)+C$

32. A body is projected vertically upward and the height h metres after a time t secs is given by $h = 196t - 4.9t^2$. Find the time taken to reach the greatest height. (A) 2 (B) 12 (C) 24 (D) 20 (E) 36

33. Given $f(z) = -z^2+6z-11$, find $f(4z-1)$.
(A) $16z^2+32z-18$ (B) $-16z^2-32z-18$
(C) $-16z^2+32z-18$ (D) $-16z^2+32z+18$
(E) None of the above.

34. Find the domain of the function, $y(z) = \sqrt{4-7z}$.
(A) $(-\infty,\infty)$ (B) $\left(-\infty,\frac{4}{7}\right]$ (B) $(0,\infty)$ (C) $\left(-\infty,\frac{4}{7}\right)$
(D) $[0,\infty)$ (E) None of the above.

35. Find the range of the function, $y(z) = \sqrt{4-7z}$.
(A) $(-\infty,\infty)$ (B) $\left(-\infty,\frac{4}{7}\right]$ (B) $(0,\infty)$ (C) $\left(-\infty,\frac{4}{7}\right)$
(D) $[0,\infty)$ (E) None of the above.

36. Find the Domain of the function,
$g(t) = |t-3| - 10$ (A) $(-10,\infty)$ (B) $(-10,\infty]$ (B) $(0,\infty)$ (C) $(-\infty,\infty)$ (D) $[-10,\infty)$ (E) None of the above.

37. Find the Range of the function,
$g(t) = |t-3| - 10$ (A) $(-10,\infty)$ (B) $(-10,\infty]$ (B) $(0,\infty)$ (C) $(-\infty,\infty)$ (D) $[-10,\infty)$ (E) None of the above.

38. Given the functions $f(x)$ and $g(x)$, suppose we have $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$, for some real numbers, c and L, find $\lim_{x\to c}[f(x) \pm g(x)]$. (A) 0 (B) ∞ (C) $-\infty$ (D) $(-\infty,\infty)$ (E) None of the above.

39. Using the two functions in question 38, if L<0, find $\lim_{x\to c}[f(x)g(x)]$ (A) 0 (B) ∞ (C) $-\infty$ (D) $(-\infty,\infty)$ (E) None of the above.

40. Evaluate $\lim_{x\to\infty}(2x^4 - x^2 - 8x)$. (A) 0 (B) ∞ (C) $-\infty$ (D) $(-\infty,\infty)$ (E) 2

41. Evaluate the following limit, $\lim_{x\to-\infty}\left(e^{x^4-5x^2+1}\right)$
(A) 0 (B) ∞ (C) $-\infty$ (D) $(-\infty,\infty)$ (E) Undefined.

42. Evaluate $\lim_{t\to-\infty}\left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$. (A) 0 (B) ∞ (C) $-\infty$ (D) $(-\infty,\infty)$ (E) Undefined

43. Which of the following is not a transcendental function. (A) $\cos 2x$ (B) e^{2x} (C) $\ln x$ (D) $\ln(x^2+2)$ (E) None of the above.

44. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. (A) e (B) 0 (C) 1 (D) e^x (E) None of the above.

45. Evaluate $\lim_{x\to0}\frac{(2-2\cos x)}{x}$. (A) -1 (B) 0 (C) 1 (D) 2 (E) Undefined.

46. Given that $y = \frac{1}{\sqrt{(x-1)}}$, find $\frac{dy}{dx}$. (A) $\frac{1}{2}(x-1)^{\frac{3}{2}}$
(B) $-\frac{1}{2}(x-1)^{\frac{3}{2}}$ (C) $\frac{1}{2}(x-1)^{-\frac{3}{2}}$
(D) $-\frac{1}{2}(x-1)^{-\frac{3}{2}}$ (E) $-\frac{1}{2}(x-1)^{-1}$

47. Evaluate $\int k\,dx$, where k is a constant. (A) $0+C$ (B) $k+C$ (C) $kx+C$ (D) $1+C$ (E) $x+C$.

48. Evaluate $\int \frac{2x+3}{x^2+3x+6} dx$. (A) $\ln(2x+3)+C$
(B) $-\ln(2x+3)+C$ (C) $\ln(x^2+3x+6)+C$
(D) $-\ln(x^2+3x+6)+C$ (E) $-\ln C(x^2+3x+6)$

49. Evaluate $\int xe^{x^2+3}dx$. (A) $\frac{xe^{x^2+3}}{x^2+3}+C$
(B) $\frac{e^{x^2+3}}{5}+C$ (C) $\frac{e^{x^2+3}}{2}+C$ (D) $\frac{xe^{x^2+3}}{2}+C$
(E) $\frac{xe^{x^2+3}}{5}+C$

50. Given $f(x) = \log_e x$. Evaluate $\int f(x)dx$.
(A) $\frac{1}{x}+C$ (B) e (C) $\ln x - x + C$ (D) $x\ln x - x + C$
(E) $\ln x + x + C$.

MTH 102 EXAM SOLUTION 2016/2017

1. Given $f(x) = \sqrt{9-x^2}$

$f(x)$ is defined if $9-x^2 \geq 0$

$(3-x)(3+x) \geq 0$, turning values $-3, 3$

Using the truth table

	$x \leq -3$	$-3 \leq x \leq 3$	$x \geq 3$
$3-x$	+	+	−
$3+x$	−	+	+
$(3-x)(3+x)$	−	+	−

The solution is $= \{x : -3 \leq x \leq 3\}$ or $[-3, 3]$

Range $= [0, 3]$ (C)

2. $f(x) = \frac{x^2-x-6}{x^2-9} = \frac{(x-3)(x+2)}{(x-3)(x+3)} = \frac{x+2}{x+3}$

To determine the vertical asymptote

$x+3 = 0$

$\Rightarrow x = -3$. (A)

3. $f(x) = \frac{5x^2+7x-1}{x^2-25}$

To determine the horizontal asymptote we take limit of the function to infinity

$\lim_{x\to\infty} \frac{5x^2+7x-1}{x^2-25} = \lim_{x\to\infty} \frac{\frac{5x^2}{x^2}+\frac{7x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{25}{x^2}}$

$\lim_{x\to\infty} \frac{5+\frac{7}{x}-\frac{1}{x^2}}{1-\frac{25}{x^2}} = \frac{5+\frac{7}{\infty}-\frac{1}{\infty}}{1-\frac{25}{\infty}} = 5$ (E)

4. $I = \int x \ln x \, dx$

Using integration by Part

$\int u\,dv = uv - \int v\,du$ .... *

let $u = \ln x$; $dv = x$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x}$, $V = \frac{x^2}{2}$

Substitute in *

$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{dx}{x} = \frac{x^2}{2}\ln x - \int \frac{x}{2}dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c \Rightarrow \frac{x^2}{2}\left[\ln x - \frac{1}{2}\right] + c$ (E)

5a. $\int \cos^2 x \, dx$

Using the trigonometric identity

$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{1}{2}(\cos 2x + 1)$

$\int \cos^2 x\,dx = \frac{1}{2}\int(\cos 2x+1)\,dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right] + c = \frac{x}{2} + \frac{\sin 2x}{4} + c$ (C)

5b. The volume of a sphere is given by $\frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

The rate of change of radius $\frac{dr}{dt} = 0.25$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.25$

$= 4\pi(7)^2 \times 0.25 = 154$ (E)

6. $x^2+y^2 = 25$

$f(x,y) \Rightarrow x^2+y^2-25 = 0$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x \Rightarrow y\frac{dy}{dx} = -x$

$\frac{dy}{dx} = \frac{-x}{y}$ at $(3,4) \Rightarrow \frac{dy}{dx} = \frac{-3}{4}$ (B)

7. $f(x) = x^3+2x^2+1$

$f(0) = 0^3+2(0)^2+1 = 1$

$f(3) = 3^3+2(3)^2+1 = 46$

$f'(x) = 3x^2+4x$

$f'(c) = 3c^2+4c$

but $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{f(3)-f(0)}{3-0}$

$3c^2+4c = \frac{46-1}{3}$

$3c^2+4c = \frac{45}{3}$

$3c^2+4c = 15$

$3c^2+4c-15 = 0$

$c = 5/3, -3 \therefore c = 5/3 \in [0,3]$ (B)

8. $x^2-2y^2 = 4$

Differentiating implicitly

$2x - 4y\frac{dy}{dx} = 0$

$2x = 4y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{4y}$

$\frac{dy}{dx} = \frac{x}{2y}$ at $(1,3)$, $\frac{dy}{dx} = \frac{1}{6}$ (E)

9. let $y = \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$

Taking the Natural log of both sides

$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x\right]$

By the law of limit, we have

$\ln y = \lim_{x\to\infty}\left[\ln\left(1+\frac{3}{x}\right)^x\right]$

$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{3}{x}\right)\right]$

$$\ln y = \lim_{x\to\infty}\left[\frac{\ln(1+3/x)}{1/x}\right]$$

$$\ln y = \frac{\ln(1+3/\infty)}{1/\infty} = \frac{\ln(1+0)}{0} = \frac{0}{0}$$

Which is indeterminate, Using L'Hopital's rule,

$$\ln y = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}[\ln(1+3/x)]}{\frac{d}{dx}[1/x]}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{d}{dx}[\ln(1+3x^{-1})]}{\frac{d}{dx}[x^{-1}]}\right\}$$

$$\ln y = \lim_{x\to\infty}\left\{\frac{\frac{1}{1+3x^{-1}}(-3x^{-2})}{-x^{-2}}\right\} = \lim_{x\to\infty}\left\{\frac{\frac{1}{(1+\frac{3}{x})}\left(\frac{-3}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right\}$$

$$\ln y = \lim_{x\to\infty}\left\{\frac{-3}{x^2(1+\frac{3}{x})}\cdot\left(\frac{-x^2}{1}\right)\right\}$$

$$\ln y = \lim_{x\to\infty}\left\{\frac{3}{(1+3/x)}\right\}$$

$$\ln y = \left\{\frac{3}{1+3/\infty}\right\} = \frac{3}{1+0}$$

$$\ln y = \frac{3}{1}$$

$$\ln y = 3$$

Take e of both sides

$y = e^3$ (A)

(10) $\lim_{x\to -2}(3x+7) = 1$

$f(x) = 3x+7$, $L = 1$, $x_0 = -2$

$|f(x) - L| < \varepsilon$

$\Rightarrow |3x+7-1| < \varepsilon$

$|3x+6| < \varepsilon$

$|3(x+2)| < \varepsilon$

$3|x+2| < \varepsilon$

whenever $|x-x_0| = |x-(-2)| = |x+2| < \delta$

$\therefore 3|x+2| < \varepsilon \Rightarrow 3\delta < \varepsilon$

$\delta < \frac{\varepsilon}{3}$ (C)

(11) $S(t) = t^3 - 6t^2 + 7t - 2$

$v = \frac{ds}{dt} = 3t^2 - 12t + 7$

$a = \frac{dv}{dt} = 6t - 12$

at $a = 0$

$6t_0 - 12 = 0$

$6t_0 = 12$

$t_0 = \frac{12}{6} \Rightarrow t_0 = 2\text{ secs}$ (C)

(12) $y = x(x-1)^2$

$y = x[(x-1)(x-1)] = x[x^2 - 2x + 1]$

$y = x^3 - 2x^2 + x$ — $*$

$\frac{dy}{dx} = 3x^2 - 4x + 1 = 0$

$\Rightarrow (x-1)(3x-1) = 0 \Rightarrow x = 1$ or $\frac{1}{3}$

$\frac{d^2y}{dx^2} = 6x - 4$

at $x = 1/3$, we have $6(\frac{1}{3}) - 4 = -2$

$-2 < 0$ (maximum

Substitute in $*$ for $x = \frac{1}{3}$

$y = (\frac{1}{3})^3 - 2(\frac{1}{3})^2 + \frac{1}{3} = 4/27$

The maximum point (x, y) is given by $(1/3, 4/27)$

(13) $f(t) = \sin\pi t$

$\frac{df}{dt} = \pi\cos\pi t$ at $t = 1/4$

$\left.\frac{df}{dt}\right|_{t=1/4} = \pi\cos\pi(\frac{1}{4}) = \pi\cos\frac{\pi}{4}$

$= \pi\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{2}$ (D)

(14) $f(x) = \frac{x^2}{4-x^2}$

For vertical Asymptotes, $4 - x^2 = 0$

$\Rightarrow (2-x)(2+x) = 0 \Rightarrow x = 2$ or -2

or $x = [-2, 2]$ (C)

(15) E

(16) If 4.4ft tall : 1 ft/sec

6 ft tall : x

cross multiplying, $x = \frac{6ft \times 1ft/s}{4.4ft}$

$= \frac{6}{4.4}$ ft/sec $= \frac{3}{2.2}$ ft/sec (E)

(17) $f(x) = 4x + 3$

let $f(x) = y$

$y = 4x + 3$

$y - 3 = 4x$

$4x = y - 3$

$x = \frac{y-3}{4}$

$f^{-1}(x) = \frac{x-3}{4}$ (A)

(18) $f(x) = \frac{(3x-2)(2x+3)}{(2x+1)(x+2)}$

$f(x) = \frac{6x^2 + 9x - 4x - 6}{2x^2 + 4x + x + 2}$

$$f(x) = \frac{6x^2+5x-6}{2x^2+5x+2}$$

For horizontal asymptote, we take limit as x tends to ∞.

$$\Rightarrow \lim_{x\to\infty} \frac{6x^2+5x-6}{2x^2+5x+2} = \lim_{x\to\infty}\left[\frac{\frac{6x^2}{x^2}+\frac{5x}{x^2}-\frac{6}{x^2}}{\frac{2x^2}{x^2}+\frac{5x}{x^2}+\frac{2}{x^2}}\right]$$

$$\lim_{x\to\infty}\left[\frac{6+5/x-6/x^2}{2+5/x+2/x^2}\right] = \frac{6+5/\infty-6/\infty^2}{2+5/\infty+2/\infty^2}$$

$$= \frac{6+0+0}{2+0+0} = \frac{6}{2} = \underline{\underline{3}} \quad (B)$$

(19) $\int x^2(x^3+7)^4\,dx$ — $*$

let $u = x^3+7 \Rightarrow \frac{du}{dx} = 3x^2$

$\Rightarrow dx = \frac{du}{3x^2}$

Substitute in $*$

$$\int x^2 u^4 \cdot \frac{du}{3x^2} = \frac{1}{3}\int u^4\,du$$

$$\frac{1}{3}\left[\frac{u^5}{5}\right] + K = \frac{u^5}{15} + K$$

but $u = x^3+7$

$$\therefore \frac{(x^3+7)^5}{15} + K \quad (A)$$

(20) $f(x) = 2x+3$, $g(x) = x^2+2x-7$

$g(2) = 2^2+2(2)-7 = 1$

$f[g(2)] = f(1) = 2(1)+3 = 5 \quad (C)$

(21) $f(x) = \frac{1}{\sqrt{9-x^2}}$

The function is defined if

$9-x^2 > 0$

$(3-x)(3+x) > 0$

The turning values are $-3, 3$

using the truth table

	$x<-3$	$-3<x<3$	$x>3$
$3-x$	$+$	$+$	$-$
$3+x$	$-$	$+$	$+$
$(3-x)(3+x)$	$-$	$+$	$-$

The solution is $= \{x : -3<x<3\}$

Domain $= (-3, 3)$

Range $= \mathbb{R}^+ \quad (A)$

(22) $f(x) = \frac{1}{2}x - 3$, let $f(x) = y$

$y = \frac{1}{2}x - 3 \Rightarrow y+3 = \frac{1}{2}x$

$2(y+3) = x$ or $x = 2(y+3)$

$f^{-1}(x) = 2(x+3) \quad (C)$

(23) $\lim_{x\to 2}(3x+2) = 8$

$f(x) = 3x+2$, $L = 8$, $x_0 = 2$

$|f(x) - L| < \varepsilon$

$|3x+2-8| < 0.01$

$|3x-6| < 0.01$

$|3(x-2)| < 0.01$

$3|x-2| < 0.01$

Whenever $0 < |x-2| < \delta$

$3\delta < 0.01$

$\delta < \frac{0.01}{3}$

$\delta < 0.0033 \quad (D)$

(24) $\lim_{x\to\infty} \frac{2x^2-3x+1}{3x^2+5x+1}$

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{5x}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2-3/x+1/x^2}{3+5/x+1/x^2}$$

$$= \frac{2-3/\infty+1/\infty^2}{3+5/\infty+1/\infty^2} = \frac{2}{3} \quad (B)$$

(25) $3(x^2+y^2)^2 = 100xy$

$3[x^4+2x^2y^2+y^4] = 100xy$

$3x^4+6x^2y^2+3y^4-100xy = 0$

Differentiating implicitly, we have

$12x^3+6\left[2x^2y\frac{dy}{dx}+2xy^2\right]+12y^3\frac{dy}{dx}-100\left[x\frac{dy}{dx}+y\right]=0$

$\frac{dy}{dx}\left[12x^2y+12y^3-100x\right] = 100y-12xy^2-12x^3$

$$\frac{dy}{dx} = \frac{100y-12xy^2-12x^3}{12x^2y+12y^3-100x} \quad \text{at } (3,1)$$

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{100(1)-12(3)(1)^2-12(3)^3}{12(3)^2(1)+12(1)^3-100(3)} = \frac{13}{9} \quad (D)$$

(26) $f(x) = \frac{x^2+2x-8}{x^2-4} = \frac{(x+4)(x-2)}{(x-2)(x+2)}$

$= \frac{x+4}{x+2}$

The vertical asymptote is at

$x+2 = 0 \Rightarrow x = -2 \quad (C)$

(27) $x^2(x^2+y^2) = y^2$

$x^4+x^2y^2-y^2 = 0$

Differentiating implicitly

$4x^3+2xy^2+2x^2y\frac{dy}{dx}-2y\frac{dy}{dx} = 0$

$[2x^2y-2y]\frac{dy}{dx} = -4x^3-2xy^2$

$$\frac{dy}{dx} = -\frac{4x^3-2xy^2}{2x^2y-2y} = \frac{-2[2x^3+xy^2]}{2[x^2y-y]}$$

$$\frac{dy}{dx} = \frac{-(2x^3+xy^2)}{x^2y-y} \quad \text{at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$= \frac{-\left[2\left(\frac{\sqrt{2}}{2}\right)^3+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^2\right]}{\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)} = \frac{-\left[2\left(\frac{2\sqrt{2}}{8}\right)+\left(\frac{2\sqrt{2}}{8}\right)\right]}{\frac{2\sqrt{2}}{8}-\frac{\sqrt{2}}{2}}$$

$$= \frac{-\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{4}\right]}{\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{2}} = \frac{-\left[\frac{3\sqrt{2}}{4}\right]}{-\frac{\sqrt{2}}{4}} = \frac{3\sqrt{2}}{4}\times\frac{4}{\sqrt{2}} = 3$$

$m = 3$

Equation of tangent is given by

$y - y_0 = m(x - x_0)$

$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$

$y - \frac{\sqrt{2}}{2} = 3x - \frac{3\sqrt{2}}{2}$

multiplying through by 2

$2y - \sqrt{2} = 6x - 3\sqrt{2}$

$2y = 6x - 3\sqrt{2} + \sqrt{2}$

$2y = 6x - 2\sqrt{2}$

$y = 3x - \sqrt{2}$ (A)

(28) $y = \cos 2t$; $x = \sin t$

$\frac{dy}{dt} = -2\sin 2t$, $\frac{dx}{dt} = \cos t$

but $\frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\cos t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -2\sin 2t \times \frac{1}{\cos t}$

$= -\frac{2\sin 2t}{\cos t}$

but $\sin 2t = \sin(t+t) = \sin t\cos t + \sin t\cos t$

$\therefore \sin 2t = 2\sin t\cos t$

Thus, $\frac{dy}{dx} = \frac{-2\sin 2t}{\cos t} = \frac{-2(2\sin t\cos t)}{\cos t}$

$= -4\sin t$ (B)

(29)

$1 \times 11 = 11$

$2 \times 10 = 20$

$3 \times 9 = 27$

$4 \times 8 = 32$

$5 \times 7 = 35$

$6 \times 6 = 36$

The greatest product is 36 (A)

(30) $f(x) = x^4 - 6x^2 + 8x + 10$

$\frac{df}{dx} = 4x^3 - 12x + 8$, at $\frac{df}{dx} = 0$

$4x^3 - 12x + 8 = 0$

$x^3 - 3x + 2 = 0$

$x = -2, 1, 1$

$\frac{d^2f}{dx^2} = 12x^2 - 12$ at $x = -2$

$12(-2)^2 - 12 = 36 > 0$ minimum

(E)

(31) $\int \frac{7 + x - 2x^2}{2 - x}dx = \int \frac{-2x^2 + x + 7}{-x + 2}dx$

By long division

$$\begin{array}{r} 2x+3 \\ -x+2 \overline{\big) -2x^2+x+7} \\ \underline{-2x^2+4x} \\ -3x+7 \\ \underline{-3x+6} \\ 1 \end{array}$$

$\therefore \int \frac{7+x-2x^2}{2-x}dx = \int\left(2x+3+\frac{1}{2-x}\right)dx$

$= \frac{2x^2}{2} + 3x - \ln(2-x) + C$

$= x^2 + 3x - \ln(2-x) + C$ (A)

(32) $h = 196t - 4.9t^2$

$\frac{dh}{dt} = 196 - 9.8t = 0$

$196 = 9.8t$

$t = \frac{196}{9.8} = 20$ sec (D)

(33) $f(z) = -z^2 + 6z - 11$

$f(4z-1) = -(4z-1)^2 + 6(4z-1) - 11$

$= -[(4z-1)(4z-1)] + 6(4z-1) - 11$

$= -[16z^2 - 4z - 4z + 1] + 24z - 6 - 11$

$= -16z^2 + 8z - 1 + 24z - 6 - 11$

$= -16z^2 + 32z - 18$ (C)

(34) $y(z) = \sqrt{4 - 7z}$

The function is defined if

$4 - 7z \geq 0$

$4 \geq 7z \Rightarrow \frac{4}{7} \geq z$

or $z \leq \frac{4}{7} \Rightarrow (-\infty, \frac{4}{7}]$ (B)

(35) $y(z) = \sqrt{4 - 7z}$

The minimum number a square root gives is zero (0). in this case set $z = \frac{4}{7}$ we get 0.

The maximum number here is when $z = 0$ we get $y = \sqrt{4} = 2$

Thus the range $= [0, 2]$ (E)

(36) $g(t) = |t - 3| - 10$

g is defined for all real numbers,

Domain $= (-\infty, \infty)$ (C)

(37) $g(t) = |t - 3| - 10$

The absolute value function gives a minimum of zero (0) value, thus the minimum value that can be obtained from $g(t)$ is $0 - 10 = -10$

the maximum value is ∞.

Thus range $= [-10, \infty)$ (D)

(38) Given $\lim_{x \to c} f(x) = \infty$ and

$\lim_{x \to c} g(x) = L$

Then $\lim_{x \to c}[f(x) \pm g(x)] = \infty + L = \infty$

(B)

(39) If $L < 0$

$\lim_{x \to c}[f(x)\, g(x)] = -\infty$ (C)

(40) $\lim_{x\to\infty}(2x^4 - x^2 - 8x)$
$= 2\infty^4 - \infty^2 - 8\infty = \infty$ (B)

(41) $\lim_{x\to-\infty}(e^{t^4-5t^2+1}) = e^{(-\infty)^4 - 5(-\infty)^2+1}$
$e^{\infty - 5\infty + 1} = e^{-\infty} = 0$ (A)

(42) $\lim_{x\to-\infty}\left(\frac{1}{3}t^5 + 2t^3 - t^2 + 8\right)$
$= \frac{1}{3}(-\infty)^5 + 2(-\infty)^3 - (-\infty)^2 + 8$
$= \frac{-\infty}{3} - 2\infty - \infty + 8 = -\infty$ (C)

(43) (E)

(44) (A) is the correct option i.e solution contained in MTH102 exam 2015/2016 Number 60.

(45) $\lim_{z\to 0}\frac{(2 - 2\cos z)}{z} = \frac{2 - 2\cos 0}{0}$
$= \frac{2-2}{0} = \frac{0}{0}$ indeterminate
Applying L'Hopital's rule
$\lim_{z\to 0}\frac{2\sin z}{1} = \frac{2\sin 0}{1} = \frac{2(0)}{1} = \frac{0}{1}$
$= 0$ (B)

(46) $y = \frac{1}{\sqrt{x-1}}$
$y = (x-1)^{-1/2}$
let $u = x-1$, $y = u^{-1/2}$
$\frac{du}{dx} = 1$, $\frac{dy}{du} = -\frac{1}{2}u^{-3/2}$
$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = -\frac{1}{2}u^{-3/2}\times 1$
$= -\frac{1}{2}u^{-3/2} = \frac{-1}{2u^{3/2}}$
but $u = x-1$
$\therefore \frac{dy}{dx} = \frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}(x-1)^{-3/2}$ (D)

(47) $I = \int K\,dx = K\int dx$
$= Kx + C$ (C)

(48) $I = \int\frac{2x+3}{x^2+3x+6}\,dx$ — *
let $u = x^2+3x+6$, $\frac{du}{dx} = 2x+3$
$\Rightarrow dx = \frac{du}{2x+3}$
substitute in *
$I = \int\frac{2x+3}{u}\cdot\frac{du}{2x+3}$

$I = \int\frac{1}{u}\,du = \ln u + c$
but $u = x^2+3x+6$
$I = \ln(x^2+3x+6) + c$

(49) $I = \int x e^{x^2+3}\,dx$ — *
let $u = x^2+3$
$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$
Substitute in *
$I = \int x e^u \frac{du}{2x} = \frac{1}{2}\int e^u du$
$= \frac{1}{2}e^u + c$
but $u = x^2+3$
$\therefore I = \frac{1}{2}e^{x^2+3} + c$ (C)

(50) $f(x) = \log_e x = \ln x$
$I = \int \ln x\,dx$
let $u = \ln x$ $dv = dx$
$\frac{du}{dx} = \frac{1}{x}$ $v = x$
$du = \frac{dx}{x}$
Using integration by Part
$\int u\,dv = uv - \int v\,du$
$\int \ln x\,dx = x\ln x - \int x\frac{dx}{x}$
$= x\ln x - \int dx$
$I = x\ln x - x + c$ (D)

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI
SCHOOL OF PHYSICAL SCIENCES
DEPARTMENT OF MATHEMATICS
ELEMENTARY MATHEMATICS II (MTH 102)
RAIN SEMESTER EXAM. SESSION: 2015/2016
TIME ALLOWED: 3 HOURS DATE: 20/09/2016
INSTRUCTIONS: ATTEMPT ALL QUESTIONS.

1. Find the domain and range of the function $f(x) = \frac{x^2}{\sqrt{x^2-4}}$ (a) $(-\infty, -2) \cup (2, \infty)$; $\mathbb{R}^+$ (b) $(-\infty, -2)$; $\mathbb{R}^+$ (c) $(2, \infty)$; $\mathbb{R}^-$ (d) None of the above (e) $x > 2$; $\mathbb{R}^+$

2. Evaluate $\lim_{x\to 0} \frac{|x|}{x}$ (a) -1 (b) 1 (c) Undefined (d) 0 (e) None of the above

3. Determine all the vertical asymptotes of the graph $f(x) = \frac{x^2+2x-8}{x^2-4}$ (a) -2 (b) 2 (c) -2, 2 (d) 0 (e)3

4. Determine all the horizontal asymptotes of the function $f(x) = \frac{2x-1}{x+1}$ (a) 2 (b) -2 (c) 0 (d) 10 (e) 4

5. A pebble is dropped into a calm pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of $1ft/sec$. When the radius is $4ft$, at what rate is the total area A of the disturbed water changing? (a)$80\pi ft^2/sec$ (b)$800\pi ft^2/sec$ (c) $0.8\pi ft^2/sec$ (d) $8\pi ft^2/sec$

6. For what range of values of x is the function $y = \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{5}{2}x^2 - 6x - 7$ increasing? (a) $\{x : x > 2\}$ (b) $\{x : x < 2\}$ (c) $\{x : x < -3\}$ (d)$\{x : -3 < x < -1\} \cup \{x : x > 2\}$

7. Evaluate $\lim_{x\to 0} (sec^3 2x)^{cot^2 3x}$ (a) 2/4 (b) $e^{2/4}$ (c) 2/3 (d) $e^{2/3}$

8. Evaluate $\int_1^e x\ln x\, dx$ (a) ¾ (b) $\frac{1}{4}e^2$ (c) $\frac{1}{4}(e^2+1)$ (d) $\frac{1}{4}(e^2-1)$

9. Evaluate $\int_{-1}^{2} |x^3 - x|\, dx$ (a) ½ (b) 3/2 (c) 7/2 (d) 11/4

10. Evaluate $\int \frac{dx}{\sqrt{49-x^2}}$ (a) $\sin\left(\frac{x}{7}\right) + c$ (b) $\sin\left(\frac{7}{x}\right) + c$ (c) $sin^{-1}\left(\frac{7}{x}\right) + c$ (d) $sin^{-1}\left(\frac{x}{7}\right) + c$

11. Evaluate $\int \frac{dx}{9-8x-x^2}$ (a) $\frac{1}{10}\ln\left(\frac{1-x}{1+x}\right) + c$ (b) $\frac{1}{10}\ln\left(\frac{9-x}{9+x}\right) + c$ (c) $\frac{1}{10}\ln\left(\frac{9+x}{1-x}\right) + c$ (d) $\frac{1}{10}\ln\left(\frac{9-x}{1+x}\right) + c$

12. Evaluate $\int \cos^2 x\, dx$ (a) $\frac{x}{2} - \frac{sin2x}{4} + c$ (b) $\frac{x}{2} + \frac{sin2x}{2} + c$ (c) $\frac{x}{2} + \frac{sin2x}{4} + c$ (d) $\frac{x}{2} + \frac{cos2x}{4} + c$

13. Evaluate $\int \frac{x-1}{(x+1)(x^2+1)} dx$ (a) $\frac{\sqrt{x^2+1}}{x+1} + c$ (b) $\frac{x+1}{\sqrt{x^2+1}} + c$ (c) $ln\left(\frac{\sqrt{x^2+1}}{x+1}\right) + c$ (d) $\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right) + c$

14. Find c in [1, 4] which satisfies the Mean Value Theorem for $f(x) = 5 - \left(\frac{4}{x}\right)$. (a) ±2 (b) 2 (c) -2 (d) 0

15. A body is projected vertically upward and the height h $meters$ reached after a time $t sec$. is given by $h = 196t - 4.9t^2$. Find the time taken to reach the greatest height and the greatest height reached. (a) 20s, 90m (b) 20s, 1960m (c) 20s, 60m (d) 2s, 6m

16. Find the area of the region enclosed by $y = 3$ and $y = 4x - x^2$ (a) 4sq unit (b) 4/3 sq unit (c) 3 sq unit (d) ¾ sq unit

17. Evaluate $\int sin^3 x\, dx$. $(A). \frac{cos^3 x}{3} - \cos x + c$ (B). $\frac{cos^4 x}{3} + c$ (C). $\frac{cos^3 x}{3} - \sin x + c$ (D). $\frac{sin^3 x}{3} - \cos x + c$ E. $\frac{sin^4 x}{4} + c$.

18. Evaluate the following integral $\int \ln x\, dx$. (A). $x\ln x + c$ B. $x\ln x - x + c$ C. $x\ln x + e^x + c$ D. $x\ln x - x^4 + c$ E. $\frac{\ln x^2}{2} + c$

19. Find the domain and range of the function $f(x) = \frac{5x}{\sqrt{16-x^2}}$. A. $[-4, 4], [0, \infty)$ B. R, R C. $[0, 4], R$ D. $[-4, 4], [0, 4]$ E. $[-4, 4], R$

20. Find the equation of the tangent to the curve $y = x^3 - 4x^2 + 5x + 7$ at the point (2,1).
A. $y = x + 1$ B. $y = 2x$ C. $y = x + 3$
D. $y = x - 1$ E. $y = 5x + 7$

21. Evaluate the limit $\lim_{x\to 0} \frac{1-cos2x}{3x^2}$. A. $\frac{5}{2}$ B. $\frac{5}{7}$ C. 0 D. $\frac{2}{3}$ E. -2

22. Find the rate of change of the volume of a spherical balloon if the radius is increasing at the rate of $0.5cms^{-1}$ given that the radius is given by $r = 5cm$. A. $50\pi cm^3 s^{-1}$ B. $20\pi cm^3 s^{-1}$ C. $90\pi cm^3 s^{-1}$ D. $25\pi cm^3 s^{-1}$ E. $100\pi cm^3 s^{-1}$

23. Evaluate the limit $\lim_{x\to\infty} \left(1 + \frac{1}{2x}\right)^x$. A. e^2 B. e C. $e^{1/3}$ D. $e^{1/5}$ E. $e^{1/2}$

24. A particle moves along a straight line so that the distance travelled after time t seconds is given by $(t) = t^3 - 2t^2 + 4t - 5$. Find the velocity of the body at time $t = 5$ sec. A. $20ms^{-1}$ B. $59ms^{-1}$ C. $50ms^{-1}$ D. $75ms^{-1}$ E. $29ms^{-1}$

25. Find the area of the region bounded by the curve $y = x^3$, the x-axis and the ordinates $x = 1$

and $x = 3$. A. 10 sq. units B. 20 sq. units C. 30 sq. units D. 40 sq. units E. 50 sq. units

26. Find the gradient of the curve $x^2 + 5y^3 + 6xy^2 = 10$ at the point (1,1).
A. $\frac{4}{3}$ B. $\frac{4}{27}$ C. $-\frac{8}{27}$ D. $-\frac{4}{11}$ E. $-\frac{5}{21}$

27. Find the mean value m of the function $f(x) = x^2$ between the limits $x = 2$ and $x = 4$. A. 28/3 B. 3/28 C. 64/3 D. 8/3 E. 3/64

28. Find the surface area of a cone formed by rotating about the x – axis the line $y = 2x$ between $x = 0$ and $x = h$. A. $2\sqrt{5}\pi h^2$ B. $25\sqrt{\pi}h^2$ C) $2\sqrt{5}\pi h^2$ D. $5\sqrt{2}\pi h^2$ E. $2h^2\sqrt{2}\pi$

29. Find the area between the curves $y = \sec^2 x$ and $y = \sin x$ from 0 to $\frac{\pi}{4}$. A. $\frac{\sqrt{2}}{2}$ B. $\frac{\sqrt{3}}{3}$ C. $2\sqrt{2}$ D. $3\sqrt{3}$ E. $\sqrt{2}$

30. Evaluate $\int \frac{dx}{x^2+4x+7}$ A. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+c$ B. $\frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+c$ C. $\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+c$ D. $\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+c$ E. $3\tan^{-1}\left(\frac{x+2}{3}\right)+c$

31. Evaluate $\int \sin^3 x dx$ A. $-\sin x + \frac{\sin^2 x}{2} + c$ B. $\cos x + \frac{\cos^2 x}{2} + c$ C. $-\cos x + \frac{\cos^3 x}{3} + c$ D. $-\cos x - \frac{\cos^2 x}{2} + c$ E. $\sec x - \cos^2 x + c$

32. Find the equation of the normal to the curve $y = 3x^2 - 5x$ at the point $(1,-2)$ A. $y = x - 3$ B. $y = -(x+1)$ C. $y = 2x + 3$ D. $y = x + 3$ E. $y = x + 1$

33. Find y' if $2y = x^2 + \sin y$. A. $2x + \cos y$ B. $2x + \tan y$ C. $\frac{2x}{2-\cos y}$ D. $2x - \cos y$ E. $\frac{2x}{2+\cos y}$

34. Find y' given that $y = (\cos x)^{-\frac{1}{5}}$. A. $\frac{1}{5}\sin x(\cos x)^{-\frac{6}{5}}$ B. $\sin x \cos x^{-\frac{2}{5}}$ C. $-\frac{1}{5}\sin x(\cos x)^{-\frac{2}{3}}$ D. $\frac{6}{5}\cos x(\sin x)^{-\frac{6}{5}}$ E. $\frac{1}{5}\sin x \cos^{-1} x$

35. Evaluate $\lim_{x \to 0}\left\{\frac{\tan x - x}{x^3}\right\}$ A. 0 B. 1 C. $\frac{1}{3}$ D. ∞ E. $\frac{1}{2}$

36. If $y = 3\sin\theta - \sin^3\theta$ and $x = \cos^3\theta$, find y''.
A) $-(3\cos^2\theta\sin^3\theta)^{-1}$ (B $-(3\sin^2\theta\cos^3\theta)^{-1}$ C) $(\cos^2\theta\sin^2\theta)^{-1}$ D $(\cos^3\theta\sin^3\theta)$ E) $(\cos^3\theta)^{-1}\cos 3\theta$

37. Find respectively the domain and range of the following function, $f(t) = |z - 6| - 3$ A. $(-\infty,\infty), [-3,\infty)$ B. R, R C. $(-\infty,\infty), [-3,0)$ D. $(-\infty,\infty), [-3)$ E. None of the above

38 .Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$, find respectively the following (f o g)(x) and (g o f)(x). A. – x, x B. x, x C. –x, -x D. x, 2x E. None of the above.

39 Find the domain of the following function, $h(x) = \frac{x}{\sqrt{x^2-9}}$. A. x<-3 & x > 3 B. x > 3 C. x < -3 D. x > 0 E. Undefined

40 Given two functions, $f(x) = x^2+3$ and $g(x) = 2x+1$. Find the value of (fog)(2).
A)19 B)38 C)28 D)10 E)18

41. For what value of k is the function $f(x) = \begin{cases} x+2, & x \le 1 \\ 5+kx^2, & x > 1 \end{cases}$ continuous at x=1?
A) $^2/_3$ B) -2 C)3 D) $^5/_2$ E) $^{-1}/_4$

42. Equations of a curve are defined parametrically by $y(t) = 1 + \cos t$ and $x(t) = \sin t$ .Find $\frac{d^2y}{dx^2}$
A)cos t B) $-\sec^3$t C)sin t D)-sin t E)cosec t

43. Find the $\lim_{x\to 0}\frac{1-\cos mx}{x^2}$ where m is an arbitrary non-zero constant
A) $\frac{m^2}{2}$ B)0 C) $\frac{1}{2}$ D) $\frac{-1}{2}$ E)undefined

44. Obtain y' given that xy+x-2y-1=0
A) $\frac{1}{2-x}$ B) $\frac{1+x}{2-x}$ C)2-x D) $\frac{1+y}{2-x}$ E) $\frac{x}{2x-3}$

45. Obtain the maximum point of the function $y = x^3 - 6x^2 + 9x + 1$. (A) (3,1) B) (1,5) C) (3,5) D) (1,3) E) (1,4)

46. Find the equation of normal line to the curve $y = 3x^2 + 2x + 4$ at $x = 1$.
(A) $8y - x - 71 = 0$ B) $8y + x - 71 = 0$ C) $8y - x + 69 = 0$ (D) $8y + x - 73 = 0$ (E) $8y + x - 70 = 0$

47. What is the domain of the function $y = \sqrt{16 - x^2}$
A)[0,4] B)[-4,0] C) $(-\infty, 4)$ D)[-4,4] E)$[4,\infty)$

48. Evaluate $\lim_{x\to\infty}\frac{1+5x}{6+2x-x^2}$ A)0 B)1 C)1/6 D)∞ E)1/2

49. Obtain $\frac{dy}{dx}$ for $y = e^{\sin x}$ (A) $e^{\cos x}$ (B) $\cos x e^{\sin x}$ C) $\sin x e^{\sin x}$ D) $\cos x e^{\cos x}$ E) $e^{\sin x}$

50. Find the critical points of inflexion on the curve $y = x^2(3x^2 - 10x - 12)$.
A) $^1/_3$, 2 B) 4,-2 C) 0,1 D) 4,6 E) 2,-1

51. The radius of a sphere is increasing at the rate of 0.1cm/sec when r=5cm,find the rates at which the surface Area and the Volume are increasing.
A) $4\pi cm^2/sec$ B) $0.4\pi cm^2/sec$ C) $16\pi cm^2/sec$ D) $0.16\pi cm^2/sec$ E) $\pi cm^2/sec$

52. Find the area of the region bounded by the curve $y = \ln x$, and the lines $x = 1$ and $x = e$
A)2 B)1 C)0 D)0.5 E)2.5

53. A man 6ft tall is walking away from a lamp post 15ft high at the rate of 6ft/sec.At what rate is the end of his shadow moving away from the lamp post? (A)5ft/sec B)0.5ft/sec C)10ft/sec D)12ft/sec E)20ft/sec

54. Obtain the integral $\int_{-1}^{5}|x-2|dx$ A)15 B)12 C)6 D)9 E)3

55. Determine all horizontal asymptotes of the graph of $f(x) = \frac{x^2}{x^2+x+1}$ A)1/2 B)1/3 C)1 D)1/4 E)-1

56. Find the slope of the tangent line to the circle $x^2+y^2=25$ at the point (3,4).
A)3/4 B)-3/4 C)1/4 D)-1/4 E)1/2

57. Evaluate the value of $\delta(\varepsilon)$ for which $\lim_{x\to 2} 3x^2 - 2 = 10$. (a) $\frac{\varepsilon}{2}$, (b) $\frac{\varepsilon}{24}$ (c) $\frac{\varepsilon}{3}$ (d) $\frac{\varepsilon}{15}$ (e) $\frac{\varepsilon}{18}$

58. Which of the following is not an even function. (a) $y = |x|$ (b) $y = (x+2)^2$ (c) $y = \cos x$ (d) $y = \cosh x$ (e) $y = x^2 + 2$

59. Find the critical points of the function $y(x) = x^3 - x$ (a) 0 (b) $\frac{\sqrt{3}}{3}$ (c) $-\frac{\sqrt{3}}{3}$ (d) $\pm\frac{\sqrt{3}}{3}$ (e) -1

60. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ (a) 0 (b) e (c) -1 (d) 1 (e)2

61. find y'' of $y(x) = 2\cosh(3x) + 5\sinh(3x)$ in terms of y. (a) 6y (b) 7y (c) 8y (d) 9y (e) 10y

62. Evaluate $y'(0)$ of $y(x) = \frac{\cos x}{x+1}$ (a) -1 (b) 1 (c) 0 (d) 2 (e)None of the above

63. Find $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ (a) 0 (b) $\frac{a}{b}$ (c) $\frac{b}{a}$ (d) $\frac{a}{b}$

64. Find the inverse of $f(x) = x^2 + 2$ (a) $\sqrt{x-2}$ (b) $\sqrt{2-x}$ (c) $\pm\sqrt{x-2}$ (d) $\sqrt{x+2}$ (e) None of the above

65. Find the domain of the function $f(x) = 4 - \frac{1}{x^2-4}$
(a) $\mathbb{R}$ (b) $\mathbb{R}\setminus\{2\}$ (c) $\mathbb{R}\setminus\{\pm 2\}$ (d) $\mathbb{R}\setminus\{0\}$ (e) $\mathbb{R}\setminus\{1\}$

66. Find the range of the function $f(x) = 4 - \frac{1}{x^2-4}$
(a) $\mathbb{R}\setminus\{4\}$ (b) $\mathbb{R}$ (c) $\mathbb{R}\setminus\{3\}$ (d) $\mathbb{R}\setminus\{0\}$ (e) $\mathbb{R}\setminus\{1\}$

67. What are the points of the curve $y = 4x^3 - 15x^2 - 18x + 79$ where the gradient zero? (a)1, 3 (b)1/2, 3 (c) -1/2 ,-3 (d) -1/2, 3 (e) 0, 1

68. Find the domain and range of the function $g(x) = c$, where c is a constant (A) $\mathbb{R}$; {0} (B) $\mathbb{R}$; {c} (C) $\mathbb{R}\setminus\{0\}$; $\mathbb{R}$ (d) $\mathbb{R}^+$; {1, 2, 3} (e) $\mathbb{R}\setminus\{c\}$; {0}

69. Obtain the first derivative $\frac{dy}{dx}$ of $xy + x - 2y - 1 = 0$ (a) $-\frac{y-1}{x-2}$ (b) $-\frac{(y+1)}{x-2}$ (c) $\frac{y-1}{x-2}$ (d) $-\frac{y-1}{x+2}$ (e) $-\frac{y+1}{x+2}$

70. Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$. Find $(f\circ g)(5)$ (a) 20000 (b) 2220 (c) 21245 (d) 2229 (e) 29314

MTH 102 EXAM SOLUTION 2015/2016

(1) $f(x) = \dfrac{x^2}{\sqrt{x^2-4}}$

The function is defined if:

$x^2 - 4 > 0 \implies (x-2)(x+2) > 0$

The turning values are $x = -2, 2$

Using the truth table,

	$x<-2$	$-2<x<2$	$x>2$
$x-2$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
$(x-2)(x+2)$	$+$	$-$	$+$

Solution is $= \{x : x<-2 \cup x>2\}$

Dom $f = (-\infty, -2) \cup (2, \infty)$

Range $= \mathbb{R}^+$ (A)

(2) $\lim_{x\to 0} \dfrac{|x|}{x}$

If x is negative, $\lim_{x\to 0} \dfrac{-x}{x} = -1$

If x is positive, $\lim_{x\to 0} \dfrac{x}{x} = 1$

Since the limit is not unique, it does not exist (E)

(3) $f(x) = \dfrac{x^2+2x-8}{x^2-4} = \dfrac{(x-2)(x+4)}{(x-2)(x+2)}$

$f(x) = \dfrac{x+4}{x+2}$

For vertical asymptote, we equate the denominator to zero (B)

$x+2 = 0 \implies x = -2$

(4) $f(x) = \dfrac{2x-1}{x+1}$

For horizontal asymptote, we take limit as $x \to \infty$

$\lim_{x\to\infty} \dfrac{2x-1}{x+1} = \lim_{x\to\infty} \dfrac{\frac{2x}{x} - \frac{1}{x}}{\frac{x}{x} + \frac{1}{x}} = \lim_{x\to\infty} \dfrac{2-\frac{1}{x}}{1+\frac{1}{x}}$

$= \dfrac{2-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \dfrac{2-0}{1+0} = \dfrac{2}{1} = 2$ (A)

(5) The variable r and A are related by $A = \pi r^2$.

The rate of change of the radius, r is

$\dfrac{dr}{dt} = 1$ ft/sec

radius, $r = 4$ ft

$\dfrac{dA}{dt} = \dfrac{d}{dt}(\pi r^2) = 2\pi r \dfrac{dr}{dt}$

$\dfrac{dA}{dt} = 2\pi(4)(1)$ ft^2/sec $= 8\pi$ ft^2/sec (D)

(6) Given, $y = \dfrac{x^4}{4} + \dfrac{2}{3}x^3 - \dfrac{5}{2}x^2 - 6x - 7$

The function y will increase for the values of x for which $\dfrac{dy}{dx} > 0$

But $\dfrac{dy}{dx} = x^3 + 2x^2 - 5x - 6$

$\dfrac{dy}{dx} > 0 \implies x^3 + 2x^2 - 5x - 6 > 0$

Solving we get

$(x-2)(x+1)(x+3) > 0$

The turning values gives, $x = 2, -1, -3$

Using the truth table

	$x<-3$	$-3<x<-1$	$-1<x<2$	$x>2$
$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
$x+3$	$-$	$+$	$+$	$+$
$(x-2)(x+1)(x+3)$	$-$	$+$	$-$	$+$

Solution is $= \{x : -3<x<-1\} \cup \{x : x>2\}$ (D)

(7) $\lim_{x\to 0} (\sec^3 2x)^{\cot^2 3x}$

This is of the indeterminate form

Now, let $y = (\sec^3 2x)^{\cot^2 3x}$

Taking the log of both sides

$\ln y = \ln(\sec^3 2x)^{\cot^2 3x}$

$\ln y = \cot^2 3x \ln(\sec^3 2x) = \dfrac{1}{\tan^2 3x} \cdot \ln(\sec 2x)^3$

$\implies \ln y = \dfrac{3\ln\sec 2x}{\tan^2 3x}$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \dfrac{3\ln\sec 2x}{\tan^2 3x}$

$= \lim_{x\to 0} \dfrac{6\tan 2x}{6\tan^2 3x \sec^2 3x}$ [Applying L'Hopitals rule]

$= \lim_{x\to 0} \dfrac{\tan 2x}{\tan 3x}$ [Applying L'Hopitals rule]

$= \lim_{x\to 0} \dfrac{2\sec^2 2x}{3\sec^2 3x}$ [By L'Hopitals rule]

Hence $\lim_{x\to 0} \ln y = \dfrac{2}{3}$

Since the function is continuous,

$\lim_{x\to 0} \ln y = \ln \lim_{x\to 0} y$

So, $\ln \lim_{x\to 0} y = \dfrac{2}{3}$

Take e of both sides

$\lim_{x\to 0} y = \lim_{x\to 0} (\sec^3 2x)^{\cot^2 3x} = e^{2/3}$ (D)

(8) $\int_0^e x \ln x \, dx$

let $u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

$\int u\,dv = uv - \int v\,du$

$= \left[\dfrac{x^2}{2}\ln x\right]_0^e - \int_0^e \dfrac{x^2}{2} \cdot \dfrac{1}{x}dx = \left[\dfrac{x^2}{2}\ln x\right]_0^e - \dfrac{1}{2}\int_0^e x\,dx$

$= \left[\dfrac{x^2}{2}\ln x\right]_0^e - \left[\dfrac{1}{2}\cdot\dfrac{x^2}{2}\right]_0^e = \left[\dfrac{x^2}{2}\ln x\right]_0^e - \left[\dfrac{x^2}{4}\right]_0^e$

$= \left[\dfrac{e^2}{2}\ln e - \dfrac{0^2}{2}\ln 0\right] - \left[\dfrac{e^2}{4} - \dfrac{0^2}{4}\right] = \left[\dfrac{e^2}{2} - 0\right] - \left[\dfrac{e^2}{4} - 0\right]$

$= \dfrac{e^2}{2} - \dfrac{e^2}{4} = \dfrac{2e^2 - e^2}{4} = \dfrac{e^2}{4} = \dfrac{1}{4}e^2$ (B)

⑨ $\int_{-1}^{2}|x^3-x|\,dx$

First, we remove the absolute value sign

$$|x^3-x| = \begin{cases} x^3-x & \text{if } x^3-x \geq 0 \\ -(x^3-x) & \text{if } x^3-x<0 \end{cases}$$

Solving the inequalities $x^3-x\geq 0$ and $x^3-x<0$ respectively we get $-1\leq x\leq 0 \cup x\geq 1$ and $0<x<1$

Thus, we have

$$|x^3-x| = \begin{cases} x^3-x & \text{if } -1\leq x\leq 0 \text{ or } x\geq 1 \\ -(x^3-x) & \text{if } 0<x<1 \end{cases}$$

Therefore,

$$\int_{-1}^{2}|x^3-x|dx = \int_{-1}^{0}(x^3-x)dx - \int_{0}^{1}(x^3-x)dx + \int_{1}^{2}(x^3-x)dx$$

$$= \frac{1}{4}+\frac{1}{4}+\frac{9}{4} = \frac{11}{4} \quad \text{(D)}$$

⑩ $\int\frac{dx}{\sqrt{49-x^2}} = \int\frac{dx}{\sqrt{7^2-x^2}} = \sin^{-1}\left(\frac{x}{7}\right)+c \quad \text{(D)}$

⑪ $\int\frac{dx}{9-8x-x^2} = \int\frac{1}{9-8x-x^2}dx$

By Partial Fraction

$$\frac{1}{9-8x-x^2} = \frac{A}{1-x}+\frac{B}{9+x}$$

$1 = A(9+x)+B(1-x)$

~~set x=1~~ $\Rightarrow 1=10A \Rightarrow A=\frac{1}{10}$

~~set x=-9~~ $\Rightarrow 1=10B \Rightarrow B=\frac{1}{10}$

$$\therefore \int\frac{dx}{9-8x-x^2} = \frac{1}{10}\int\left(\frac{1}{1-x}+\frac{1}{9+x}\right)dx$$

$$= \frac{1}{10}\left[-\ln(1-x)+\ln(9+x)\right]+C$$

$$= \frac{1}{10}\left[\ln(9+x)-\ln(1-x)\right]+C$$

$$= \frac{1}{10}\ln\left(\frac{9+x}{1-x}\right)+C \quad \text{(C)}$$

⑫ $\int\cos^2x\,dx = \int\left(\frac{1+\cos2x}{2}\right)dx$

$$= \frac{1}{2}\int(1+\cos2x)dx = \frac{1}{2}\left(x+\frac{\sin2x}{2}\right)+C$$

$$= \frac{x}{2}+\frac{\sin2x}{4}+C \quad \text{(C)}$$

⑬ $\int\frac{x-1}{(x+1)(x^2+1)}dx$

By Partial Fraction

$$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

Solving, we get $A=-1,\ B=1,\ C=0$

$$\therefore \int\frac{x-1}{(x+1)(x^2+1)}dx = \int\left(\frac{-1}{x+1}+\frac{x}{x^2+1}\right)dx$$

$$= -\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{2x}{x^2+1}dx$$

$$= -\ln(x+1)+\frac{1}{2}\ln(x^2+1)+C$$

$$= \ln(x^2+1)^{\frac{1}{2}}-\ln(x+1)+C$$

$$= \ln\left(\frac{\sqrt{x^2+1}}{x+1}\right)+C \quad \text{(C)}$$

⑭ $f(x) = 5-\left(\frac{4}{x}\right)$

$f(a) = f(1) = 5-\frac{4}{1} = 1$

$f(b) = f(4) = 5-\frac{4}{4} = 4$

$f'(x) = \frac{4}{x^2}, \quad f'(c) = \frac{4}{c^2}$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \frac{4}{c^2} = \frac{4-1}{4-1}$$

$$\frac{4}{c^2} = \frac{3}{3} \Rightarrow \frac{4}{c^2}=1 \rightarrow c^2=4$$

$c=\pm2$ (A)

⑮ Given $h = 196t-4.9t^2$

The greatest height is reached when $\frac{dh}{dt}=0$

$$\frac{dh}{dt} = 196-9.8t = 0$$

$$\Rightarrow 9.8t = 196 \Rightarrow t = \frac{196}{9.8} = 20s$$

The greatest height thus reached is

$h(20) = 196(20)-4.9(20)^2 = 1960m$ (B)

⑯ The two curves cut each other at $4x-x^2=3$ or $x^2-4x+3=0$

$\Rightarrow x=1,3$

$\therefore$ The area of the region enclosed by the curve is given by

$$\int_1^3(4x-x^2-3)dx = \left[2x^2-\frac{x^3}{3}-3x\right]_1^3$$

$$= \left[2(3)^2-\frac{3^3}{3}-3(3)\right]-\left[2(1)^2-\frac{(1)^3}{3}-3(1)\right]$$

$$= (18-9-9)-\left(2-\frac{1}{3}-3\right) = -\left(-\frac{4}{3}\right) = \frac{4}{3} \text{ unit}$$

(B)

⑰ $\int\sin^3x\,dx = \int\sin^2x\cdot\sin x\,dx$

$$= \int(1-\cos^2x)\sin x\,dx = \int(\sin x-\cos^2x\sin x)dx$$

$$= -\cos x+\frac{\cos^3x}{3}+C$$

or $\frac{\cos^3x}{3}-\cos x+C \quad$ (A)

⑱ $\int\ln x\,dx$

Let $u=\ln x, \quad dv=dx$

$du=\frac{1}{x}dx \quad v=x$

$\int u\,dv = uv-\int v\,du$

$\int\ln x\,dx = x\ln x-\int x\cdot\frac{1}{x}dx$

$\int\ln x\,dx = x\ln x-\int dx+C$

$= x\ln x-x+C \quad$ (B)

⑲ $f(x) = \frac{5x}{\sqrt{16-x^2}}$

$f(x)$ is defined if $16-x^2>0$

$\Rightarrow (4-x)(4+x)>0$

The turning values are $x=-4,4$

Using the truth table

	$x<-4$	$-4<x<4$	$x>4$
$4-x$	+	+	−
$4+x$	−	+	+
$(4-x)(4+x)$	−	+	−

solution is = $\{x: -4<x<4\}$

$\therefore$ Dom $f = (-4, 4)$
Range $= \mathbb{R}$ No correct option

(20) $y = x^3 - 4x^2 + 5x + 7$

$\frac{dy}{dx} = 3x^2 - 8x + 5$

$\frac{dy}{dx}$ at $(2, 1) \Rightarrow 3(2)^2 - 8(2) + 5 = 1$

Equation of tangent, $(y - y_1) = m_1(x - x_1)$

$(y - 1) = 1(x - 2) \Rightarrow y - 1 = x - 2$

$y = x - 2 + 1 \Rightarrow y = x - 1$ (D)

(21) $\lim_{x \to 0} \frac{1 - \cos 2x}{3x^2} = \frac{1 - \cos 2(0)}{3(0)^2} = \frac{1-1}{0} = \frac{0}{0}$

Which is indeterminate

Thus, by L'Hopital's rule

$\lim_{x \to 0} \frac{2\sin 2x}{6x} = \frac{0}{0}$

By L'Hopital's rule again.

$\lim_{x \to 0} \frac{4\cos 2x}{6} = \frac{4\cos 2(0)}{6} = \frac{4}{6} = \frac{2}{3}$ (D)

(22) $V = \frac{4}{3}\pi r^3$; $\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.5$

$= 4\pi(5)^2 \times 0.5 = 50\pi\, cm^3 s^{-1}$ (A)

(23) $\lim_{x \to \infty} \left(1 + \frac{1}{2x}\right)^x$

Let $y = \left(1 + \frac{1}{2x}\right)^x$

$\ln y = x \ln\left(1 + \frac{1}{2x}\right)$

$\ln y = \frac{\ln\left(1 + \frac{1}{2x}\right)}{1/x}$

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \left[\frac{\ln\left(1 + \frac{1}{2x}\right)}{1/x}\right] = \frac{\ln\left(1 + \frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{0}{0}$

Applying L'Hopitals rule

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \left[\frac{-\frac{1}{2x^2}\left(\frac{1}{1 + \frac{1}{2x}}\right)}{-1/x^2}\right]$

$= \lim_{x \to \infty} \frac{1}{2}\left(\frac{1}{1 + \frac{1}{2x}}\right) = \frac{1}{2}$

$\lim_{x \to \infty} \ln y = \frac{1}{2}$

But $\lim_{x \to \infty} \ln y = \ln \lim_{x \to \infty} y$ [Since ln is a continuous function]

$\ln \lim_{x \to \infty} y = \frac{1}{2}$

$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(1 + \frac{1}{2x}\right)^x = e^{1/2}$ (E)

(24) $s(t) = t^3 - 2t^2 + 4t - 5$

$V = \frac{ds}{dt} = 3t^2 - 4t + 4$

$V(5) = 3(5)^2 - 4(5) + 4 = 75 - 20 + 4 = 59 ms^{-1}$ (B)

(25) $\int_1^3 x^3 dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4}$

$= \frac{80}{4} = 20$ sq. unit (B)

(26) $x^2 + 5y^3 + 6xy^2 = 10$

$2x + 15y\frac{dy}{dx} + 6y^2 + 12xy\frac{dy}{dx} = 0$

$(15y + 12xy)\frac{dy}{dx} = -(2x + 6y^2)$

$\frac{dy}{dx} = \frac{-(2x + 6y^2)}{15y + 12xy}$

at $(1, 1)$, $\frac{dy}{dx} = \frac{-[2(1) + 6(1)^2]}{15(1) + 12(1)(1)} = -\frac{[2}{15}$

$= -\frac{8}{27}$ (C)

(27) $f(x) = x^2$

$f(2) = 2^2 = 4$

$f(4) = 4^2 = 16$

$f'(x) = 2x \Rightarrow f'(c) = 2c$

$f'(c) = \frac{f(4) - f(2)}{4 - 2}$

$2c = \frac{16 - 4}{4 - 2} \Rightarrow 2c = \frac{12}{2}$

$2c = 6 \Rightarrow c = 3$

(28) $y = 2x$; $\frac{dy}{dx} = 2$, $a = 0$, $b = h$

Area of rotation about the x-a
is given as ; $S_A = \int_a^b 2\pi y\, ds$

Where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \Rightarrow ds = \sqrt{1+2}$

$ds = \sqrt{5}\, dx$

Hence, $S_A = \int_0^h 2\pi(2x)\sqrt{5}\, dx$

$= 2\pi\sqrt{5}\left[x^2\right]_0^h = 2\sqrt{5}\pi h^2$ (C

(29) $\int_0^{\pi/4} (\sec^2 x - \sin x)\, dx$

$\left[\tan x + \cos x\right]_0^{\pi/4}$

$= \left[\tan\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - \left[\tan 0 + \cos 0\right]$

$= \left[1 + \frac{\sqrt{2}}{2}\right] - \left[0 + 1\right] = \frac{\sqrt{2}}{2}$ (A

(30) $\int \frac{dx}{x^2 + 4x + 7} = \int \frac{dx}{(x+2)^2 + 3}$

$= \int \frac{dx}{(x+2)^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) +$

$= \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C$ (B)

(31) (C) Refer to No 17 for solutio

(32) $y = 3x^2 - 5x$; $\frac{dy}{dx} = 6x - 5$

at $(1, -2)$, $\frac{dy}{dx} = 6(1) - 5 = 1 = m_1$

$m_2 = -\frac{1}{m_1} = -\frac{1}{1} = -1$

Equation of normal; $(y - y_1) = m_2(x -$

$[y - (-2)] = -1(x - 1) \Rightarrow (y + 2) = -(x -$

$y + 2 = -x + 1 \Rightarrow y = -x + 1 - 2$

$y = -x - 1$

$y = -(x + 1)$ (B)

(33) $2y = x^2 + \sin y$

$2\frac{dy}{dx} = 2x + \frac{dy}{dx}\cos y$

$2\frac{dy}{dx} - \frac{dy}{dx}\cos y = 2x$

$(2-\cos y)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2-\cos y}$ (C)

(34) $y = (\cos x)^{-\frac{1}{5}}$

Let $u = \cos x$; $y = u^{-\frac{1}{5}}$

$\frac{du}{dx} = -\sin x$; $\frac{dy}{du} = -\frac{1}{5}u^{-\frac{6}{5}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{5}u^{-\frac{6}{5}} \times (-\sin x)$

$= \frac{1}{5}\sin x \cdot u^{-\frac{6}{5}} = \frac{1}{5}\sin x(\cos x)^{-\frac{6}{5}}$ (A)

(35) $\lim_{x\to 0}\left[\frac{\tan x - x}{x^3}\right] = \frac{0}{0}$

By L'Hopitals rule, $\lim_{x\to 0}\left[\frac{\sec^2 x - 1}{3x^2}\right] = \frac{0}{0}$

By L'Hopitals rule, $\lim_{x\to 0}\left[\frac{2\tan x\sec^2 x}{6x}\right] = \frac{0}{0}$

By L'Hopitals rule, $\lim_{x\to 0}\left\{\frac{2(\sec^4 x + 2\tan^2 x\sec^2 x)}{6}\right\}$

$= \frac{2}{6} = \frac{1}{3}$ (C)

(36) $y = 3\sin\theta - \sin^3\theta$; $x = \cos^3\theta$

$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$; $\frac{dx}{d\theta} = 3\cos^2\theta\sin\theta$

$y' = \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}}$

$y' = \frac{3\cos\theta - 3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}$

$= \frac{3\cos\theta[1-\sin^2\theta]}{3\cos\theta[-\cos\theta\sin\theta]} = \frac{1-\sin^2\theta}{-\cos\theta\sin\theta}$

$= \frac{\cos^2\theta}{-\cos\theta\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$

$y'' = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}(-\cot\theta)\cdot\frac{1}{-3\cos^2\theta\sin\theta}$

$= \operatorname{cosec}^2\theta \cdot \frac{1}{-3\cos^2\theta\sin\theta} = \frac{1}{\sin^2\theta}\cdot\frac{1}{-3\cos^2\theta\sin\theta}$

$y'' = \frac{1}{-3\cos^2\theta\sin^3\theta} = (-3\cos^2\theta\sin^3\theta)^{-1}$

(37) $f(t) = |t-6| - 3$

Domain $= \mathbb{R}$ or $(-\infty, \infty)$

Range $= [-3, \infty)$ (A)

(38) $f(x) = 3x - 2$; $g(x) = \frac{1}{3}x + \frac{2}{3}$

$(f\circ g)(x) = f[g(x)] = 3[\frac{1}{3}x + \frac{2}{3}] - 2 = x + 2 - 2 = x$

$(g\circ f)(x) = g[f(x)] = \frac{1}{3}[3x-2] + \frac{2}{3} = x - \frac{2}{3} + \frac{2}{3} = x$

Ans = x, x (B)

(39) $h(x) = \frac{x}{\sqrt{x^2-9}}$

$h(x)$ is defined if $x^2 - 9 > 0$

$(x-3)(x+3) > 0$

The turning values are $x = -3, 3$

using the truth table

	$x<-3$	$-3<x<3$	$x>3$
$x-3$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
$(x-3)(x+3)$	$+$	$-$	$+$

Domain $= \{x : x<-3 \cup x>3\}$

or $(-\infty, -3) \cup (3, \infty)$ (A)

(40) $f(x) = x^2 + 3$; $g(x) = 2x + 1$

$(f\circ g)(x) = (2x+1)^2 + 3$

$(f\circ g)(2) = [2(2)+1]^2 + 3 = 5^2 + 3 = 28$ (C)

(41) If $f(x)$ is continuous at $x = 1$

then, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+2) = 3$ —— *

Similarly

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 5 + Kx^2 = 5 + K$ —— **

Equating * & **

$3 = 5 + K \Rightarrow K = -2$ (B)

(42) $y(t) = 1 + \cos t$ $x(t) = \sin t$

$\frac{dy}{dt} = -\sin t$ $\frac{dx}{dt} = \cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = -\sin t \times \frac{1}{\cos t} = -\tan t$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}(-\tan t)\cdot\frac{1}{\cos t}$

$= -\sec^2 t \cdot \frac{1}{\cos t} = -\sec^2 t \cdot \sec t$

$= -\sec^3 t$ (B)

(43) $\lim_{x\to 0}\frac{1-\cos mx}{x^2} = \frac{1-\cos 0}{0} = \frac{0}{0}$

By L'Hopitals rule $\lim_{x\to 0}\frac{m\sin mx}{2x} = \frac{0}{0}$

By L'Hopitals rule $\lim_{x\to 0}\frac{m^2\cos mx}{2} = \frac{m^2\cos 0}{2}$

$= \frac{m^2}{2}$ (A)

(44) $xy + x - 2y - 1 = 0$

$xy' + y + 1 - 2y' = 0$

$-2y' + xy' = -1 - y$

$-(2-x)y' = -(1+y)$

$y' = \frac{-(1+y)}{-(2-x)} = \frac{1+y}{2-x}$ (D)

(45) $y = x^3 - 6x^2 + 9x + 1$

$y' = 3x^2 - 12x + 9$

Turning point is at $y' = 0$

$3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1$ or 3

$y(1) = 1^3 - 6(1)^2 + 9(1) + 1 = 5$

$y(3) = 3^3 - 6(3)^2 + 9(3) + 1 = -1$

$\therefore$ Turning points are $(1, 5)$ and $(3, -1)$

2nd derivative, $y'' = 6x - 12$

$y''(1) = 6(1) - 12 = -6 < 0$

Thus, maximum point $= (1, 5)$ (B)

(46) $y = 3x^2 + 2x + 4$

$y(1) = 3(1)^2 + 2(1) + 4 = 9$

$y' = 6x + 2$

$m_1 = y'(1) = 6(1) + 2 = 8$

$m_2 = -\frac{1}{m_1} = -\frac{1}{8}$

Equation of normal line;

$y - 9 = -\frac{1}{8}(x - 1)$

$8y - 72 = -x + 1 \Rightarrow 8y + x - 72 - 1 = 0$

$\Rightarrow 8y + x - 73 = 0$ (D)

(47) $y = \sqrt{16 - x^2}$

y is defined if $16 - x^2 \geq 0$

$(4 - x)(4 + x) \geq 0$

Turning values are $x = -4, 4$

Using the truth table

	$x \leq -4$	$-4 \leq x \leq 4$	$x \geq 4$
$4 - x$	+	+	−
$4 + x$	−	+	+
$(4-x)(4+x)$	−	+	−

Solution is $= -4 \leq x \leq 4$

Domain $= [-4, 4]$ (D)

(48) $\lim_{x \to \infty} \frac{1 + 5x}{6 + 2x - x^2} = \lim_{x \to \infty} \frac{\frac{1}{x^2} + \frac{5x}{x^2}}{\frac{6}{x^2} + \frac{2x}{x^2} - \frac{x^2}{x^2}}$

$= \lim_{x \to \infty} \frac{\frac{1}{x^2} + \frac{5}{x}}{\frac{6}{x^2} + \frac{2}{x} - 1} = \frac{\frac{1}{\infty^2} + \frac{5}{\infty}}{\frac{6}{\infty^2} + \frac{2}{\infty} - 1} = \frac{0 + 0}{0 + 0 - 1} = \frac{0}{-1}$

$= 0$ (A)

(49) $y = e^{\sin x}$

Let $u = \sin x$; $y = e^u$

$\frac{du}{dx} = \cos x$; $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times \cos x = \cos x \, e^u$

but $u = \sin x$

$\therefore \frac{dy}{dx} = \cos x \, e^{\sin x}$ (B)

(50) $y = x^2(3x^2 - 10x - 12)$

$y = 3x^4 - 10x^3 - 12x^2$

$\frac{dy}{dx} = 12x^3 - 30x^2 - 24x$

The critical point is at $\frac{dy}{dx} = 0$

$12x^3 - 30x^2 - 24x = 0$

$2x^3 - 5x^2 - 4x = 0$

$x(2x^2 - 5x - 4) = 0$

$x = 0, 3.137, -0.637$ No correct option

(51) $\frac{dr}{dt} = 0.1\,cm/sec$; $r = 5cm$

$A = 4\pi r^2$; $\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times 0.1\,cm/sec$

$= 8\pi \times 5cm \times 0.1cm/sec = 4\pi cm^2/sec$

$V = \frac{4}{3}\pi r^3$; $\frac{dv}{dr} = 4\pi r^2$; $\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

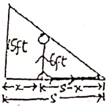
$\Rightarrow \frac{dv}{dt} = 4\pi r^2(0.1) = 4\pi(5)^2(0.1) = 10cm^3/sec$

(52) $\int_1^e \ln x = [x \ln x]_1^e - [x]_1^e$

$= [e \ln e - 1 \ln 1] - [e - 1] = [e - 0] - [e \ldots$

$= e - e + 1 = 1$ (B)

(53)



By similar triangles, we

$\frac{s}{15} = \frac{s - x}{6}$

$6s = 15s - 15x$

or $9s = 15x$

$s = \frac{15}{9}x \Rightarrow \frac{ds}{dt} = \frac{15}{9} \ldots$

$\frac{ds}{dt} = \frac{15}{9} \times 6ft/sec$

$\frac{ds}{dt} = 10ft/sec$ (…

Where:

$\frac{dx}{dt}$ = The rate at which the man is walking away from the Lamp post

$\frac{ds}{dt}$ = rate at which the end of his shadow is moving away from the post.

(54) $\int_{-1}^{5} |x - 2| dx$

Midpoint $x - 2 = 0 \Rightarrow x = 2$

$\therefore \int_{-1}^{5} |x - 2| dx = \int_{-1}^{2} -(x - 2)dx + \int_{2}^{5} (x - 2)dx$

$= \int_{-1}^{2} (2 - x)dx + \int_{2}^{5} (x - 2)dx = \left[2x - \frac{x^2}{2}\right]_{-1}^{2} + \left[\frac{x^2}{2} - 2x\right]_{2}^{5}$

$= \left\{\left[2(2) - \frac{(2)^2}{2}\right] - \left[2(-1) - \frac{(-1)^2}{2}\right]\right\} + \left\{\left[\frac{5^2}{2} - 2(5)\right] - \left[\frac{2^2}{2} - 2(2)\right]\right\}$

$= \left\{[4 - 2] - \left[-2 - \frac{1}{2}\right]\right\} + \left\{\left[\frac{25}{2} - 10\right] - [2 - 4]\right\}$

$= \left[2 - \left(-\frac{5}{2}\right)\right] + \left[\frac{5}{2} - (-2)\right] = 2 + \frac{5}{2} + \frac{5}{2} + 2$

$= 4 + \frac{10}{2} = 4 + 5 = 9$ (D)

(55) $f(x) = \frac{x^2}{x^2 + x + 1}$

$\lim_{x \to \infty} \frac{x^2}{x^2 + x + 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1 + \frac{1}{\infty} + \frac{1}{\infty^2}} = \frac{1}{1 + 0 + 0}$

$= 1$ (C)

(56) $x^2 + y^2 = 25$

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}$ (B)

(57) $f(x) = 3x^2 - 2$; $L = 10$, $x_0 = 2$

$|f(x) - L| < \varepsilon \Rightarrow |3x^2 - 2 - 10| < \varepsilon$

$\Rightarrow |3x^2 - 12| < \varepsilon$, introducing the co-ordinate

$|3(x - 2)^2 + 12x - 24| < \varepsilon$

$|3(x - 2)^2 + 12(x - 2)| < \varepsilon$

$\leq 3|x - 2|^2 + 12|x - 2| < \varepsilon$

but $0 < |x - 2| < \delta$

$\therefore$ we have, $3\delta^2 + 12\delta < \varepsilon$

but $\delta^2 \leq \delta$

$\therefore 3\delta + 12\delta < \varepsilon \Rightarrow 15\delta < \varepsilon \Rightarrow \delta < \frac{\varepsilon}{15}$ (d)

58) By inspection, (B) is the correct option because

$(x+2)^2 \neq (-x+2)^2$

$x^2+4x+4 \neq x^2-4x+4$

59) $y(x) = x^3 - x$, $y'(x) = 3x^2 - 1$

$y'(x) = 0 \Rightarrow 3x^2 - 1 = 0$

$3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\sqrt{\frac{1}{3}}$

$x = \pm\frac{1}{\sqrt{3}} \Rightarrow x = \pm\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{3} = \pm\frac{\sqrt{3}}{3}$ (D)

60) Let $y = \left(1+\frac{1}{x}\right)^x$

$\ln y = x\ln\left(1+\frac{1}{x}\right) \Rightarrow \ln y = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right] = \frac{0}{0}$

By L'Hopital's rule, we have

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty}\left[\frac{\frac{1}{(1+\frac{1}{x})} \times -\frac{1}{x^2}}{-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{1}{1+\frac{1}{x}}\right]$

$\lim_{x\to\infty} \ln y = 1$

But $\lim_{x\to\infty} \ln y = \ln \lim_{x\to\infty} y$ since log is continuous

$\ln \lim_{x\to\infty} y = 1$

$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$ (B)

61) $y(x) = 2\cosh(3x) + 5\sinh(3x)$

$y'(x) = 6\sinh(3x) + 15\cosh(3x)$

$y''(x) = 18\cosh(3x) + 45\sinh(3x)$

$y''(x) = 9\left[2\cosh(3x) + 5\sinh(3x)\right]$

$y''(x) = 9y$ (D)

62) $y(x) = \frac{\cos x}{x+1}$

$y = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x+1)(-\sin x) - \cos x(1)}{(x+1)^2}$

$y'(0) = \frac{(0+1)(-\sin 0) - \cos 0}{(0+1)^2} = \frac{0-1}{1^2} = \frac{-1}{1} = -1$ (A)

63) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$

By L'Hopital's rule $\lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a\cos 0}{b\cos 0} = \frac{a}{b}$ (C)

64) $f(x) = x^2 + 2 \Rightarrow y = x^2 + 2$

$x^2 = y - 2 \Rightarrow x = \pm\sqrt{y-2}$

$f^{-1}(x) = \pm\sqrt{x-2}$

Since the inverse is not unique, it does not exist. (E)

65) $f(x) = 4 - \frac{1}{x^2-4}$

Defined if $x^2 - 4 \neq 0 \Rightarrow x^2 \neq 4$

$\Rightarrow x \neq \pm 2$

$\text{Dom} f = \mathbb{R}\backslash\{\pm 2\}$ (C)

66) $f(x) = 4 - \frac{1}{x^2-4}$ or $y = 4 - \frac{1}{x^2-4}$

$\frac{1}{x^2-4} = 4 - y \Rightarrow x^2 - 4 = \frac{1}{4-y}$

$\Rightarrow x^2 = 4 + \frac{1}{4-y} \Rightarrow x = \pm\sqrt{4 + \frac{1}{4-y}}$

$\text{Range} = \mathbb{R}\backslash\{4\}$ (A)

67) $y = 4x^3 - 15x^2 - 18x + 79$

$\frac{dy}{dx} = 12x^2 - 30x - 18$

$\frac{dy}{dx} = 0 \Rightarrow 12x^2 - 30x - 18 = 0$

$\Rightarrow 2x^2 - 5x - 3 = 0$

Solving, we get $x = -\frac{1}{2}$ or 3 (D)

68) $f(x) = c$

Domain $= \mathbb{R}$

Range $= c$ (B)

69) $xy + x - 2y - 1 = 0$ (Refer to No 49)

$x\frac{dy}{dx} + y + 1 - 2\frac{dy}{dx} = 0$

$x\frac{dy}{dx} - 2\frac{dy}{dx} = -y - 1$

$(x-2)\frac{dy}{dx} = -(y+1)$

$\frac{dy}{dx} = \frac{-(y+1)}{x-2}$, (B)

70) $f(x) = 3x^2 - x + 10$

$g(x) = 1 - 20x$

$(f \circ g)(x) = f[g(x)] = 3(1-20x)^2 - (1-20x) + 10$

$(f \circ g)(5) = 3(1-20(5))^2 - (1-20(5)) + 10$

$= 3(1-100)^2 - (1-100) + 10$

$= 3(-99)^2 - (-99) + 10$

$= 29403 + 99 + 10 = 29512$