MTH 102 THEORY QUESTIONS

GENERAL INSTRUCTION: Answer each Question in the space Provided Below it

1(a). Verify that $\lim_{x\to 4}\frac{1}{2}(3x-1)=\frac{11}{2}$

(b) Evaluate $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$

(a)

$\forall\ \varepsilon>0\ \delta>0\ \exists\ |f(x)-L|<\varepsilon : 0<(x-4)$

$|\frac{1}{2}(3x-1)=\frac{11}{2}|<\varepsilon\ |\frac{3x-12}{2}|<\varepsilon$

Introducing coordinate

$\frac{1}{2}|3(x-4)+12-12|<\varepsilon$

$\frac{1}{2}|3\delta|<\varepsilon$

$\frac{3\delta}{2}<\varepsilon$

$\delta<\frac{2\varepsilon}{3}$,, Choose $\delta\le[\frac{2\varepsilon}{3},1]$

(b)

$\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}=\frac{0}{0}$ (indeterminate)

By Rationalization

$\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-(\sqrt{x+1}+1)}$

$\lim_{x\to 0}\frac{x(\sqrt{x+1}+1}{x+1-1}$

$\lim_{x\to 0}(\sqrt{x+1}+1)$

Taking limit at $x=0$

$\sqrt{0+1}+1=\sqrt{1}+1$

$=\underline{\underline{2}}$

2. Evaluate the integral $\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}=\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{4\sin x\cos x}}$

$\frac{1}{2}\int_0^{\pi/4}\frac{1}{\cos^3 x(\sin x)^{1/2}(\cos x)^{1/2}}dx$

$\frac{1}{2}\int_0^{\pi/4}\frac{1}{\cos^{7/2}x(\sin x)^{1/2}}dx$

$\frac{1}{2}\int\frac{\frac{1}{\cos^4 x}}{\frac{\cos^{7/2}x\sin^{1/2}x}{\cos^4 x}}$

$\frac{1}{2}\int_0^{\pi/4}\frac{\sec^4 x}{\cos^{-1/2}x\sin^{1/2}x}dx$

$\frac{1}{2}\int_0^{\pi/4}\frac{(1+\tan^2 x)\sec^2 x\,dx}{\sqrt{\tan x}}$

$\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{t^2}+\frac{t^2}{\sqrt{t}}\right)dt$

$\frac{1}{2}\int_0^{\pi/4}\left(2t^{1/2}+\frac{2}{5}t^{5/2}\right]_0^{\pi/4}$

$=\underline{\underline{6/5}}$

3. (a) Discuss the continuity of the function below

$$f(x) = \begin{cases} \frac{2-x}{x^2+x-6} & ; \ x \neq 2 \\ -\frac{1}{5} & ; \ x = 2 \end{cases}$$

(b) Determine the points where the function given by $g(t) = \frac{4t+10}{t^2-2t-15}$ is not continuous.

(a) $f(x)$ is well defined at $f(x) = -1/5$

$\lim_{x\to 2} \frac{2-x}{x^2+x-6} = \lim_{x\to 2} \frac{2-x}{(x+3)(x-2)}$

Taking the limit

$\lim_{x\to 2} \frac{-1}{x+3} = \frac{-1}{2+3} = -\frac{1}{5}$

Since $\lim_{x\to 2} f(x) = \frac{-1}{2+3} = f(x) = \frac{-1}{5}$

The function is continuous

(b) $g(t)$ is not continuous at denominator $= 0$

$t^2 - 2t - 15 = 0$

$(t-5)(t+3) = 0$

$t = 5$ or $t = -3$

The point where the function is not continuous at $t = 5$ or $t = -3$

4. (a) Using first principle, find the derivative of the function $f(x) = \sqrt{x}$

(b) Find the slope of the graph of $f(x)$ at the points $(1, 1)$ and $(4, 2)$.

(a) $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ by Rationalization

$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$\frac{dy}{dx} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

(b) Slope $= \frac{\Delta y}{\Delta x}$

$= \frac{\sqrt{2} - \sqrt{1}}{\sqrt{4} - \sqrt{1}}$

$= \frac{\sqrt{2} - 1}{3}$

$= 0.138$

5. (a) Using chain rule, find the first derivative of $y = \tan^2 3x$

(b) Find $\frac{dy}{dx}$ given that $y^3 + y^2 - 5y - x^2 = -4$

(a)

$y = \tan^2 3x$, Let $u = \tan 3x$

So that $y = u^2$ $\frac{dy}{du} = 2u$

$\frac{du}{dx} = 3\sec^2 3x$ (By chain rule)

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 2u \cdot 3\sec^2(3x)$

$\frac{dy}{dx} = 6\tan 3x \sec^2(3x)$

(b)

Differentiating Implicitly

$y^3 + y^2 - 5y - x^2 = -4$

$3y^2 \frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x =$

$(3y^2 + 2y - 5)\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$

6. (a) Determine the slope of the graph $3(x^2 + y^2)^2 = 100xy$ at the point $(3, 1)$

(b) Find the first, second and third derivatives of the function $y = e^{x^4}$

(a)

$3(x^2+y^2)^2 = 100xy$ by Implicit

$3(x^4 + 2x^2y^2 + y^4) = 100xy$

$3x^4 + 6x^2y^2 + 3y^4 = 100xy$

$12x^3 + 12xy^2 + 12x^2y^2 + 12y^3 y' = 100y + 100xy$

$(12x^2y + 12y^3 - 100x)y' = -(12x^2 + 12xy^2 - 100y)$

$y' = -\frac{(12x^3 + 12xy^2 - 100y)}{12x^2y + 12y^2 - 100x} = \frac{1}{3}$

$y'(3,1) = 1/3$

(b)

$y = e^{x^4}$

$y' = 4x^3e^{x^4}$

$y'' = 16x^6e^{x}$

$y''' = 4x^3 16x^6 e^{x^4}$

$y''' = 64x^9e^{x^4}$

7. (a) Find $\frac{dy}{dx}$ if $y = \cos 2t, \quad x = \sin t$

(b) Using quotient rule, find the first derivative of the function $y = \frac{((3x+1)(\cos 2x))}{e^{2x}}$

a) $y = \cos 2t, \; dy/dt = -2\sin 2t, \; x = \sin t; \; dx/dt = \cos t$

$dy/dx = dy/dt \cdot \frac{dt}{dt} = -2\sin 2t \cdot \frac{1}{\cos t} = \frac{-2\sin 2t}{\cos t}$

b) Let $u = (3x+1)(\cos 2x); \; v = e^{2x}$

using $\frac{v\,du - u\,dv}{v^2} = \frac{e^{2x}\left[3\cos 2x - 2(3x+1)(\sin 2x)\right] - (3x+1)(\cos 2x) \cdot 2e^{2x}}{(e^{2x})^2}$

$$\frac{dy}{dx} = \frac{e^{2x}\left[3\cos 2x - 2(3x+1)\sin 2x - (3x+1)(\cos(2x)(2)\right]}{e^{4x}}$$

8. Differentiate the following functions

(a) $y = \ln\left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}$. (b) $y = \left(\frac{\sin 4x}{1+\cos x}\right)^5$

(a)

Let $u = \frac{1-3x^2}{1+3x^2}, \; y = \ln u^{1/2}$

$\frac{du}{dx} = \frac{(1+3x^2)(-6x) - (1-3x^2)(6x)}{(1+3x^2)^2}$

$\frac{dy}{dx} = \frac{-12x}{(1+3x^2)^2}$

$y = \ln u^{1/2}, \; \frac{dy}{du} = \frac{1}{2u}$

$= \frac{dy}{dx} = \left(\frac{1+3x^2}{1-3x^2}\right) \cdot \left(\frac{-6x}{(1+3x^2)^2}\right)$

(b)

Let $u = \frac{\sin 4x}{1+\cos x}; \; y = u^5$

$\frac{du}{dx} = \frac{1+\cos x(4\cos 4x + \sin x \sin 4x}{(1+\cos x)^2}$

$\frac{dy}{du} = 5u^4$

$\frac{dy}{dx} = \frac{5\left[\sin 4x\left[(1+\cos x)(4\cos 4x) + \sin x \sin 4x\right]\right.}{(1+3x^2)^3}$

9. Find the equation of the tangent and the normal lines to the curves

(a) $y = 2x^3 - x^2 + 3x + 1$ at the point $x = 1$

(b) $x^2y + y^3x + 3x - 13 = 0$ at the point $(1, 2)$

(a)

Eqn of tangent: $y - y' = m(x - x_1)$

Eqn of normal $y - y' = \frac{-1}{m}(x - x_1)$

Where $m = \frac{dy}{dx}(x=1) = 6x^2 - 2x + 3$

$m = 6(1)^2 - 2(1) + 3 = 7$

$y' = 2(1)^3 - (1)^2 + 3(1) + 1 = 5$

$y - 5 = 7(x - 1)$

$y - 7x + 2 = 0$ is Eqn of Tangent

$7y + x - 36 = 0$ is Eqn of Normal

(b)

Differentiating Implicitly

$m = \frac{-(2xy + y^3 + 3)}{x^2 + 3y^2x}$

$m = \frac{-2(1)(2) + 2^3 + 3)}{(1)^2 + 3(2)^2(1)} = \frac{-15}{13}$

$y - 2 = \frac{-15}{13}(x - 1)$

$13y + 15x - 41 = 0$ Eqn of Tangent

$15y - 13x - 17 = 0$ Eqn of Normal

10. Find the range of values of x for which each of the following functions is increasing (a) $\frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 1$

(b) $1 + 3x - 2x^2$ (c) $x^3 - 12x$

(10(a))

A Function increases if $\frac{dy}{dx} > 0$

$\frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5} > 0$

$x > -2, 1, 3$ or $(-2, \infty)$

(10(b))

The function Increases at $\frac{dy}{dx} > 0$

$y = 1 + 3x - 2x^2$ $\frac{dy}{dx} = 3 - 4x > 0$

$3 > 4x \Rightarrow x < 3/4$

Range: $(-\infty, 3/4)$

(10c)

$y = x^3 - 12x$

$\frac{dy}{dx} = 3x^2 - 12$

$3x^2 - 12 > 0$

$3x^2 > 12$

$x > \pm 2$

The function Increase at

$x > \pm 2$

Range: $(-2, \infty)$

11. Find the range of values of x for which each of the following functions is decreasing: (a) $\frac{x^2}{2} - 5x + 1$ (b) $x^3 - \frac{5}{2}x^2 - 2x + 1$

12. If the radius of a sphere decreases by 0.1% , find the percentage decrease in the (a) surface area (b) volume

(11)a

$y' < 0$ Implies decreasing function

$y = \frac{x^2}{2} - 5x - 1 \quad y' = x - 5$

$x - 5 < 0; \quad x < 5$

$Rf \{x : x < 5 \ \forall \ x \in \mathbb{R}\}$ function decreases

(b)

$y = x^3 - \frac{5}{2}x^2 - 2x + 1$

$y' = 3x^2 - 5x - 2 < 0$

$(3x+1)(x-2) < 0; \ x < -\frac{1}{3} \text{ or } x < 3$

$Rf : \{x : x < -\frac{1}{3} \ (x < 2\}$ function will decrease

(12a)

Let the radius r $\quad r = 0.1r = 0.7r$

$\%$ decrease $= \frac{4\pi r^2 - 4\pi(0.7\delta)^2}{4\pi r^2(1/100)}$

$\frac{4\pi^2 - (1-0.7)^2}{4\pi r^2} \times 100 = 17\%$

(b)

$\frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi(0.7r)^3 \times 100}{\frac{4}{3}\pi r^3}$

$\%$ decrease $= 27.1\%$

13. (a) If a side of a square increases by 0.5%, find the approximate percentage increase in the area

(b) Find the approximate increase in the surface area of a spherical balloon if its radius increases from 4cm to 4.025cm.

(a)

Let S be one of the sides

$S + 0.5S$ = new side

Area of square $= S^2$

$\%$ Increase in area $=$

$\frac{S^2 + (S + 0.5S)^2 \times 100}{S^2}$

$\frac{S^2 + (1.5S)^2 \times 100}{S^2}$

$\frac{S^2(1 + (1.5)^2 \times 100}{S^2} = 325\%$

(b)

$\Delta r = 0.025 \quad r = 4cm$

Surface area $= \frac{4}{3}\pi r^3$

$\frac{\Delta A}{\Delta r} = 4\pi r^2$

$\Delta A = 4\pi^2 \Delta r$

$\Delta A = 4 \times 3.142 \times (4)^2 \times (0.025)$

$\Delta A = \underline{\underline{5.0272}}$

14. Find the approximate values of the following using differentials

(a) $\sqrt{25.01}$ (b) $\sqrt[3]{125.1}$ (a) $\sqrt{36.01}$

(a)

$F(x) = \sqrt{x}$

$x + \Delta x = 25 + 0.01$

$\Delta x = 0.01$

$F'(x) = \frac{1}{2\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

$\sqrt{25.01} = \sqrt{25} + \frac{1}{10} \times 0.01$

$= 5 + 0.003$

$= 5.003$

(b)

$f(x) = 3\sqrt{x} = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f(x+\Delta x) = f(x) + f'(x)\Delta x$

$\sqrt[3]{125.1} = \sqrt[3]{125} + \frac{1}{3}(125)^{-2/3} \times 0.1$

$= 5.00013$

(c)

$x + \Delta x = 36.0$

$\Delta x = 36.01 - 36$

$\Delta x = 0.01$

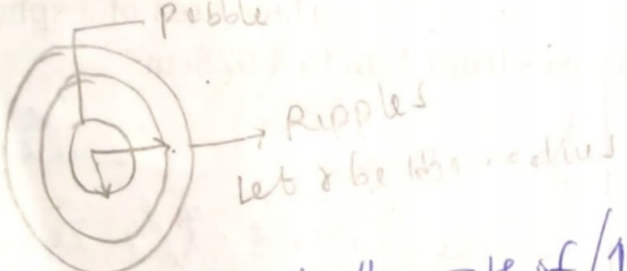
$f(x) = \sqrt{x}$

$\sqrt{36.01} =$

$\sqrt{36} + \frac{1}{2\sqrt{36}} \times 0.01$

$= 6.00083$

15. A pebble is dropped into a calm pound causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4feet, at what rate is the total area A of the disturbed water changing?



Since r is increasing at the rate of 1 foot per second, we have

$\frac{dv}{dt} = 1 ft/sec$

Let $A = \pi r^2$ be the area

$\frac{dA}{dt} = 2\pi r^2 = dr/dt$

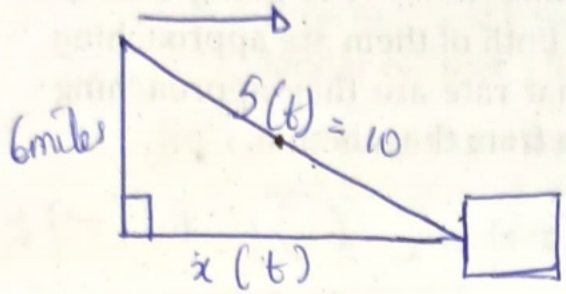
$r = 4ft, \frac{dr}{dt} = 1ft/.$

$\frac{dA}{dt} = 2\pi(4ft)(1ft/s)$

$\frac{dA}{dt} = r(ft)^2/s$

∴ The total area of the disturbed water changes at the rate of $r(ft)^2/s$

16. An airplane is flying on a flight path that will take it directly over a radar tracking statin. The airplane is flying at an altitude of 6 miles and s miles from the station. If s is decreasing at a rate of 400 miles/hour when $s = 10$miles. What is the speed of the plane?



$x^2+6^2 = s^2$

$\frac{d}{dt}(x^2+6^2) = \frac{d}{dt}(s^2)$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{dx}{dt} = \frac{s}{x}\, ds/dt$, but $s = 10$ miles

and $ds/dt = -400$ miles/hour

From $x^2+6^2 = 10^2$, $x = r$

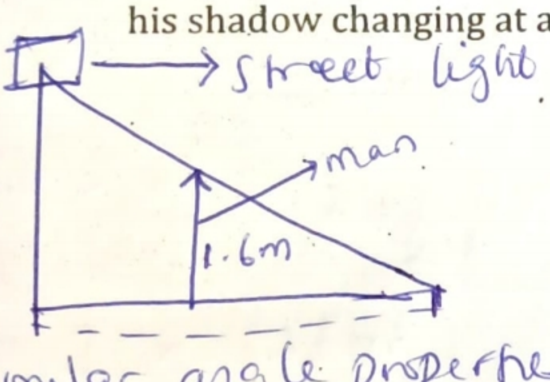
Substituting in $\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$

$\frac{dx}{dt} = \frac{10}{6}(-400)$ mile/hr

$\frac{dx}{dt} = -666.66667$

$\frac{dx}{dt} \approx -666.7$ mile/hr = speed

17. A man 1.6m tall walks away from a street light a speed of 1.5m per second. If the light is 5m above the pavement. How fast is the length of his shadow changing at any time?



By similar angle properties

$\frac{5}{1.6} = \frac{x+y}{y}$ ∴ $5y = 1.6x + 1.6y$

$5y - 1.6y = 1.6x$

$y = \frac{1.6x}{3.4}$

$\frac{dy}{dt}(x+y) = \frac{d}{dt}\left(x+\frac{1.6x}{34}\right)$

$\frac{d}{dt}(x+y) = \frac{d}{dt}\left(x+\frac{1.6x}{3.4}\right)$

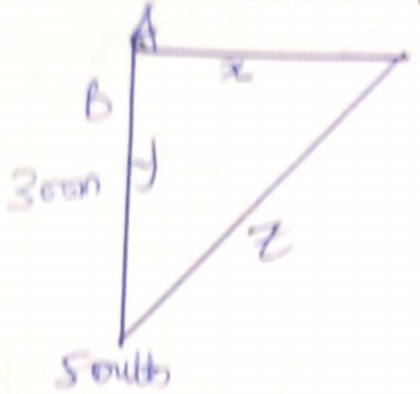
$= \frac{d}{dt}\left(\frac{25}{17}x\right) = \frac{25}{17}\frac{dx}{dt}$

but $\frac{dx}{dt} = 1.5$ m/s

∴ $\frac{25}{17}(1.5\text{ m/s}) = \frac{75}{34}$ m/s

or 2.206m/s is how fast the shadow will change

18. Two cars A and B are going in some directions. A is going east at 60km/h and B is going south at 48km/h. Both of them are approaching towards a junction (intersection) at what rate are they approaching each other when A is 400m and B is 300m from the junction.



From pythagoras theorem

$$z = \sqrt{x^2+y^2}$$

$$z = \sqrt{(400)^2+(300)^2} = 500m$$

$$z^2 = x^2+y^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$$

We know that $\frac{dx}{dt} = 60km/h$

and $\frac{dy}{dt} = 48km/h$

$$\frac{dz}{dt} = \frac{400(-60)+300(-48)}{500}$$

$$\frac{dz}{dt} = -76.8km/h$$

The rate at which the two cars are approaching each other is $-76.8km/hr$

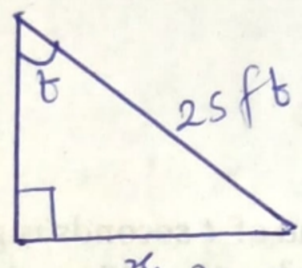
19. A ladder 25feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top moving down the wall when the base of the ladder is 7feet, 15 feet and 24feet from the wall.

(b) Consider the triangle formed by the side of the house, the ladder and the ground, Find the rate at which the area of the triangle is changing when the base of the ladder is 7feet from the wall.

(C) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

(a) y



We know that $x^2+y^2=(25)^2$

$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

$\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}$

But $y=\sqrt{(25)^2-(7)^2}=24$

$\frac{dy}{dt}=-\frac{7ft}{24ft}(2ft/s)$

$\frac{dy}{dt}=-\frac{7}{12}ft/s$

for $x=15ft$ $y=20ft$

$dy/dt=-15/20(2f/s)=\frac{-15}{10}ft/s$

$y=\sqrt{(25)^2-(15)^2}=20$

for $x=24ft$, $y=7ft$

$dy/dx=\frac{-24ft}{7ft}(2ft/s)$

$\frac{dy}{dt}=-6.857ft/s$

(b)

Area $=\frac{1}{2}bh=\frac{1}{2}xy$

$\frac{dA}{dt}=\frac{1}{2}\left[\frac{dx}{dt}y+x\frac{dy}{dt}\right]$

$\frac{dA}{dt}=\frac{1}{2}\left[2ft/s(24)+7\left(\frac{-4}{12}\right)\right]$

$\frac{dA}{dt}=21.958(ft)^2/s$

(c)

$\sin\theta=x/h$ $\frac{dx}{dt}(\sin\theta)=\frac{d}{dt}\left(\frac{x}{h}\right)$

$\cos\theta\frac{d\theta}{dt}=\frac{1}{h}\frac{dx}{dt}$

$\frac{d\theta}{dt}=\frac{1}{y}dx/dt$

$\frac{d\theta}{dt}=1/24(2ft/s)=\frac{1}{12}$ sign units

$d\theta/dt=1/12$ significant units

20. A body is projected vertically upward and the height h meters reached after a time t seconds is given by $h = 196t - 4.9t^2$. Find (a) the timetaken to reach the greatest height (b) the greatest height reached.

(a)

$h = 196t - 4.9t^2$

The time taken to reach the greatest height is at $dy/dt = 0$

$\frac{dh}{dt} = 196 - 9.8t = 0$

$196 - 9.8t = 0$

$t = \frac{196}{9.8} = 20$

$t = 20secs$

(b)

The greatest height reach is

$h = 196(20) - 4.9(20)$

$h = 3920 - 1960 = 1960$

$h = 1960m$

21. The position of two particles P_1 and P_2 at the end of t seconds is given by $s_1 = 3t^3 - 12t^2 + 18t + 5$; $s_2 = -t^3 + 9t^2 - 12t$. When do the two particles have the same velocity.

$S_1 = 3t^3 - 12t^2 + 18t + 5$

$S_2 = -t^3 + 9t^2 - 12t$

The two particles have the velocity if $\frac{dS_1}{dt} = \frac{dS_2}{dt}$

$\frac{dS_1}{dt} = 9t^2 - 24t + 18$

$9t^2 - 24t + 18 = -3t^2 + 18t - 12$

$9t^2 + 3t^2 - 24t - 18t + 18 + 12$

$12t^2 - 42t + 30 = 0$

$t_1 = 5/2, \ t_2 = 1$

is the point where the two

Particles have the same Velocity

22. Determine the critical points of the function $f(x) = x^4 - 6x^2 + 8x + 10$ and state their nature

$f(x) = x^4 - 6x^2 + 8x + 10$

$f'(x) = 4x^3 - 12x + 8$

critical point is at $f'(x) = 0$

$4x^3 - 12x + 8 = 0$

$x_1 = -2, x_2 = 1, x_3 = -4$

$F''(x) = 12x^2 - 12 = x^2 - 1$

$F''(\cancel{x_2}) = F''(-2) = (-2)^2 - 1 = 3 > 0$

$F''(1) = (1)^2 - 1 = 0$

$F''(4) = (4)^2 - 1 = 15 > 0$

$x_1 = -2$ and $x_3 = 4$ are minimum points since

$F''(x_0) > 0$

$x_2 = 1$ is a saddle point

23. Determine the critical points of $f(x) = x^3 - 6x^2 + 9x + 1$ and state their nature

$f(x) = x^3 - 6x^2 + 9x + 1.$

$F'(x) = 3x^2 - 12x + 9$

$F'(x) = 0$ Implies that

$3x^2 - 12x + 9 = 0$

$x_1 = 3 \quad x_2 = 1$

For nature of critical points, we obtain second derivative

$f''(x) = 6x - 12 = x - 2$

$f''(x_1) = 3 - 2 = 1 > 0$

$f''(x^2) = 1 - 2 = -1 < 0$

$x_3 = 3$ is a minimum point

$x_2 = 1$ is a maximum point

24. Find the greatest product of two numbers whose sum is 12.

Let the two numbers be x and y

$x + y = 12$ --- (i) sum

$y = 12 - x$ --- (ii)

Product xy: -- (iii)

$P = x(12-x) = 12 - x^2$

$P' = 12 - 2x$

$P'' = -2 < 0$

$P'' = -2 < 0$ Implies

greatest product/maximum

Let $P' = 0$ $P' = 12 - 2x = 0$

$x = 6$

from $y = 12 - x$

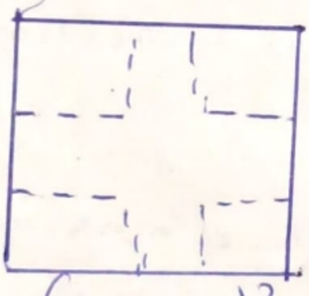
$y = 12 - 6 = 6$

$x = 6$ $y = 6$ are the two numbers

The greatest product $= 6 \times 6$

$= \underline{\underline{36}}$

25. An open tank of square base is to be made from a square sheet of metal of sides 20m by cutting square pieces from each of the corners, lifting the resulting flaps and soldering the edges together. Find the maximum capacity of such tank.



$V(x) = (20-2x)^2 \cdot x$

$V(x) = 400x - 80x^2 + 4x^2$

Solving for critical values

$x_1 = 0$ $x_2 = 10$, $x_3 = 10$

Now, we obtain the first derivative of $V(x)$

$V(x) = 400x - 80x^2 + 4x^3$

$\frac{dv}{dx} = 400 - 160x + 12x^2 = 0$

$x_1 = 10$; $x_2 = 10/3$

$V(10) = 400 - 80(10)^2 + 4(10)^3$

$V(10) = 0$

$V(10/3) = 400(10/3) - 80(10/3)^2 + 4(10/3)^3 = 592.6$

$V(10/3) = 592.6$

So the minimum capacity is $592.6m^3$

26. The total cost, y naira, of manufactturing x units of an article is given by the relation $y = \frac{5}{4}x^2 + \frac{20}{x}$.

Find (i) the number of units of the articles for which the cost of manufacture is least (ii) the corresponding least cost.

Solution

$$y = \frac{5}{4}x^2 + \frac{20}{x}$$

$$\frac{dy}{dx} = \frac{5}{2}x - \frac{20}{x^2}$$

$$\frac{dy}{dx} = \frac{5}{2}x^3 - 20 = 0$$

$$x_1 = 2, \quad x_2 = -1 + \sqrt{3}i$$

$$x_3 = -1 - \sqrt{3i}$$

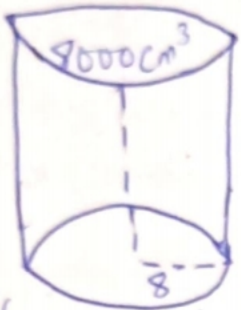
i) The number of units of the articles is $x = 2$ since it cant be Imaginary

ii) Corresponding least cost is

$$y = 5/4(2)^2 + \frac{20}{2}$$

$$y = 5 + 10 = 15 \text{ units}$$

27. A cylindrical tank closed at both ends with circular base is to hold $8000 cm^3$ of liquid. Find the minimum amount of metal sheet required in the construction of the vessel.



$V = \pi r^2 h$, $A = 2\pi(r^2 + rh)$

$h = \frac{V}{\pi r^2}$

$A = 2\pi\left(2r + \frac{(v)(-1)}{\pi r^2}\right)$

let $A' = 0$

$2r = V/\pi r^2$. $V = 2\pi r^3$

Since $V = 8000 cm^3$

$2\pi r^3 = 8000$

$r^3 = \frac{8000}{2\pi}$ $r = \left(\frac{8000}{2\pi}\right)$

$r = 10.84 cm$

from $h = \frac{V}{\pi r^2}$

$h = \frac{8000}{\pi(10.84)^2} = 21.668 m$

The minimum sheet required in the construction is $21.7 m$

28. (a) Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \frac{0}{0}$

Where is Indeterminate

using L'Hopital's rule

$\lim_{x\to 1^+} \frac{d/dx(x-1-\ln x)}{d/dx(x-1)(\ln x)}$

$\lim_{x\to 1^+}\left(\frac{1 - 1/x}{1 - 1/x \ln x}\right) = \frac{0}{0}$

Using L'Hopital rule

$\lim_{x\to 1^+}\left(\frac{1}{2+\ln x}\right) = \frac{1}{2}$

(b) Evaluate $\lim_{x\to 0}\frac{e^{2x}-1}{x}$

(b)

$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \frac{0}{0}$ (Indeterminate)

By direction

Applying L'Hopital rule

$\lim_{x\to 0} = \frac{d/dx(e^{2x}-1)}{d/dx(x)}$

$\lim_{x\to 0}\frac{2e^{2x}}{1} = \underline{\underline{2}}$

29. Find the Taylor and Maclaurin series for the function $f(x) = \cos x$ at $c = \frac{\pi}{6}$ and $c = 0$ respectively.

$f(x) = f(x_0) + \frac{f'(x_0)x}{1!} + \frac{f^n(x_0)x^n}{n!}$

$f(x) = \cos x;\ f(\pi/6) = \cos(\pi/6) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x;\ f''(\pi/6) = -1/2$

$f''(x) = -\cos x;\ f''(\pi/6) = -\sqrt{3}/2$

$f'''(x) = \sin x\ \ f'''(\pi/6) = 1/2$

$\therefore f(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}x - \frac{\sqrt{3}}{2}\frac{x^2}{2!} + \frac{1}{2}\frac{x3}{3!} + \ldots$

At $c = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}\ \forall n = 0, 1, 2 \ldots$$

When $n = 0$ we have

$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4} - \frac{x^6}{6!} + \ldots$

30. (a) Find the horizontal asymptotes of the function $f(x) = \frac{3x^2}{x^2+1}$

(b) Determine all the vertical asymptotes of the graph of the function $f(x) = \frac{x^2+1}{x^2-1}$

(a)

$f(x) = \frac{3x^2}{x^2+1}$ for the horizontal asymptote, we take

$\lim_{x\to\infty} \frac{3x^2}{x^2+1}$

$\lim_{x\to\infty} \frac{\frac{3x^2}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \frac{3}{1 + \frac{1}{\infty^2}} = 3.$

$\therefore$ Horizontal asymptote at $y = 3$

(b)

$f(x) = \frac{x^2+1}{x^2-1}$, the vertical asymptote is at denominator equals to zero

$x^2 - 1 = 0\ \ (x+1)(x-1) = 0$

$x = 1$ or $x = -1$

Vertical asymptote is at $x = -1$ or $x = 1$

31. The angle θ, swept by a particle moving round a circle at time t is given by $\theta = t^2 + 3t - 5$. Find the angular velocity and acceleration at $t = 5$seconds. Solution

$\theta = t^2 + 3t - 5$

Angular velocity occurs at $\frac{d\theta}{dt}$

$\frac{d\theta}{dt} = 2t + 3$

$\frac{d\theta}{dt}(t=5) = 2(5) + 3 = 13m/s$

Angular acceleration occurs at $\frac{d^2\theta}{dt^2}$

$\frac{d^2\theta}{dt^2} = \frac{d}{dt}(2t+3) = \underline{\underline{2m/s^2}}$

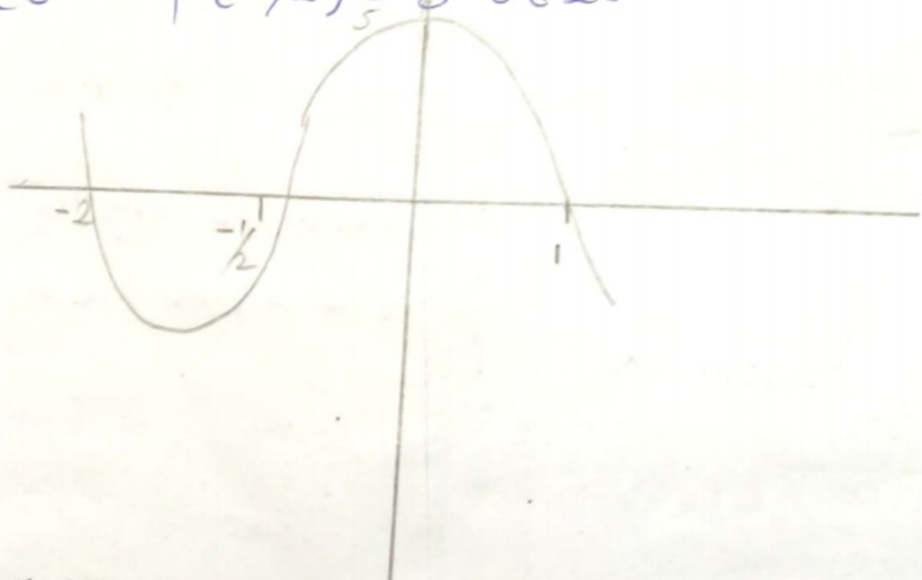
32. Using curve sketching techniques, sketch the graph of $y = x^4 + 2x^3 - 3x^2 - 4x + 4$

$\frac{dy}{dx} = 4x^3 + 6x^2 - 6x - 4 = 0;\ x_1 = -2,\ x_2 = 1,\ x_3 = -\frac{1}{2}$

$d^2y/dx^2 = 12x^2 + 12x - 6;\ f''(-2) = 12(-2)^2 + 12(-2) - 6 > 0$

$f''(1) = 12(1)^2 + 12(1) - 6 > 0$ $f''(-\frac{1}{2}) = 12(-\frac{1}{2})^2 + 12(-\frac{1}{2}) - 6$

$f''(-\frac{1}{2}) < 0$ $f(-\frac{1}{2}) = 5.0625$



33. Integrate the following with respect to x

(a) $\int (x^3-5)3x^2 dx$ (b) $\int \frac{x^3}{(3x^4-5)^6} dx$

(a)

$\int (x^3-5)3x^2\, dx$

$= \int 3x^5 dx - \int 15x^2 dx$

$= \frac{1}{2}x^6 - 5x^3 + C$

(b)

$\int \frac{x^3}{(3x^4-5)^6} dx$

By u-substitution

Let $u = (3x^4-5)$

$dy/dx = 12x^3$; $dx = \frac{du}{12x^3}$

$\int \frac{x^3}{(3x^4-5)^6} dx = \int \frac{x^3}{u^6} \cdot \frac{du}{12x^3}$

$= \frac{1}{12}\int \frac{1}{u^6} du = \frac{1}{12}\int u^{-6} du$

$\frac{1}{12} \frac{u^{-5}}{-5} = \frac{-1}{60} u^{-5} + C$

$= \frac{-1}{60}(3x^4-5)^{-5} + C$

34. Find the integral of the following

(a) $\int \sin^3 x \cos^4 x\ dx$ (b) $\int \frac{\cos^3 x}{\sqrt[3]{\sin x}} dx$

(a)

$\int \sin^3 x \cos^4 x\, dx$

$\int \sin^2 x (\sin x) \cos^4 x\, dx$

$\int (1-\cos^2 x)\ \cos^4 x \sin x\, dx$

let $u = \cos^2 x$, $du = -2\cos x \sin x$

$\int (1-u^2)u^2(-du) = \int (u^2-1)u^4 du$

$\int (u^6-u^4) du = \frac{u^7}{7} - \frac{u^5}{5} + C$

$\frac{\cos 7x}{7} - \frac{\cos 5x}{5} + C$

(b)

Let $u = \sin x$ $dx = du/\cos x$

$\int \frac{\cos 3x}{\sqrt[3]{\sin x}} dx = \int \frac{\cos^3 x}{3\sqrt{u}} \cdot \frac{du}{\cos x}$

$\int \frac{\cos^2 x}{3\sqrt{u}} du = \int \frac{1-\sin^2 x}{3\sqrt{u}} du$

$\int \frac{1-u^2}{3\sqrt{u}} du = \int u^{-1/3} - \int u^{5/3} du$

$\frac{3}{2}(1^{2/3} - 3/8(1^{8/3} + C$

$\frac{3}{2}(\sin x)^{2/3} - 3/8(\sin x)^{2/3} + C$

35. Integrate the following (a) $\int \frac{(x^3+x^2+4x)}{x^2+x-2}\, dx$ (b) $\int \frac{2x^3-x^2-x}{2x-3}\, dx$

(a)

By long division

$\int x\,dx + \int \frac{4}{x+2}\,dx + \int \frac{2}{x+1}\,dx$

$\frac{x^2}{2} + 4\ln(x+2) + 2\ln(x-1) + C \in \mathbb{R}$

(b)

Using long division

$\int x^2\,dx + \int x\,dx + \int 1\,dx$

$+ \int \frac{3}{2x-3}\,dx$

$\frac{x^3}{3} + \frac{x^2}{2} + x + \frac{3}{2}$

$\ln|2x-3| + C \;\forall c \in \mathbb{R}$

36. Using the method of integration by parts, integrate the following

(a) $\int x^2 \sin x\, dx$ (b) $\int \sin^{-1} x\, dx$

(a)

$\int u\,dv = uv - \int v\,du$

let $u = x^2$ $dv = \sin x$

$du = 2x$, $v = -\cos x$

Substituting in formula

$x^2(-\cos x) - \int -\cos x\,(2x)\,dx$

$-x^2\cos x + 2\int x\cos x\,dx$

Integrating

let $u = x$ $du = dx$

$dv = \cos x$ $V = \sin x$

$I_2 = -x^2\cos x + 2x\sin x + 2\cos x$

$+ C \quad C \in \mathbb{R}$

(b)

By Integration by part

$I = uv - \int v\,du$

Let $u = \sin^{-1}(x)$ $du = \frac{-1}{\sqrt{1-x}}$

$dv = 1$, $V = x$

$I = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

By u-substitution

Let $u = 1-x^2$ $du = -2x$

$x\sin^{-1}(x) + \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$

$x\sin^{-1}(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$x\sin^{-1}(x) + \frac{1}{2}\frac{u^{1/2}}{1/2} + C$

$x\sin^{-1}(x) + \sqrt{1-x^2} + C$

37. Evaluate the following (a) $\int \frac{1}{x^2+4x+2}dx$ (b) $\int \frac{1}{\sqrt{9-x^2}}dx$

(a)

$\int \frac{1}{x^2+4x+2}dx$ Can be written as

$\int \frac{1}{(x+2)^2-2}dx - \int \frac{1}{(x+2)^2-(\sqrt{2})^2}dx$

is now in the form;

$\int \frac{1}{Z^2-A^2}dZ = \frac{1}{2A}\ln\left|\frac{Z-A}{Z+A}\right| + C$

$\therefore \int \frac{1}{(x+2)^2-(\sqrt{2})^2}dx = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}\right|$

$+ C \in IR$

(b)

Using the standard Integral property

$\int \frac{1}{\sqrt{9-x^2}}dx = \sin^{-1}(x/3)$

$+ C \in IR$

$I = \sin^{-1}\left(\frac{x}{3}\right) + C$

38. (a). Find a reduction formular for $\int x^n e^x dx$

(b) Using the reduction formular obtained find $\int x^2 e^x dx$

a) $\int x^n e^x dx = x^n e^x - n\int e^x x^{n-1} dx = x^n e^x I - n I n-1$

b) $\int x^2 e^x dx$ Implies that $n=2$

$In = x^n e^x - n In-1$; $I_2 = x_2 e^x - 2I_1$

$I_1 = x^1 e^x - 1 \cdot I_0$

$I_0 = \int x^0 e^x dx = \int 1 e^x dx = \int e^x dx = e^x$

So, combining I_2 . I_1 and I_0, We have

$I_2 = e^x[x^2-2x+2] + C \;\forall \in R$

39. (a). Find a reduction formula for $\int x^n \cos x \, dx$

(b) Using the reduction formular obtained find $\int x^2 \cos x \, dx$

a) $\int x^n \cos x \, dx = I_n = x^n \sin x - n \int x^{n-1} \sin x \, dx$

$= x^n \sin x - n \int x^{n-1} \sin x \, dx$

$I_n = x^n \sin x + n x^{n-1} \cos (x - n(n-1)) \int x^{n-2} \cos x \, dx$

b) $\int x^2 \cos x \, dx$ for $n = 2$

$I_2 = x^2 \sin x + 2x \cos x - 2(2-1) \int x^0 \cos x \, dx$

$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + c \quad \forall c \in \mathbb{R}$

40. Evaluate the following integrals

(a) $\int_0^2 \frac{dx}{\sqrt{x^2+2x+3}}$ (b) $\int_0^{\frac{\pi}{4}} (\tan^2 x + 3) \, dx$

a): $\int_0^2 \frac{dx}{\sqrt{x^2+2x+3}} = \int_0^2 \frac{1}{\sqrt{(x+1)^2+2}} dx = \int_0^2 \frac{1}{\sqrt{t^2+2}} dt$

$= \ln(t + \sqrt{t^2 + (\sqrt{2})^2}) = \ln(x+1 \sqrt{(x+1)^2 + (\sqrt{2})^2} = 0.838$

b): $\int_0^{\pi/4} (\tan^2 x + 3) \, dx = \int_0^{\pi/4} (1 + \sec^2 x + 3 \, dx$

$[x + \tan x + 3x]_0^{\pi/4} = [4x + \tan x]_0^{\pi/4}$

$4[\pi/4 + \tan(\pi/4)] = \pi + 1$

41. •Find the area of the region between the graph of $y = x^2 + 4x + 3$ and $y = x^2$ between the intervals $x = 1$and $x = 3$.

Solution

To find the area between the two region, We evaluate the y-values ie $y = x^2+4x+3 = x^2$ and Integrate;

$= x^2+4x+3 = x^2; \; 4x+3 = y$

Area $\int_1^3 (4x+3)\,dx = \left[\frac{4x^2}{2} + 3x\right]_1^3 = 27-5$

Area = 22 sq units

42. Find the area enclosed between the curve $y = x^3$ and the straight line $y = x$

Solution

We Equate the y-values and Integrate $y = x^3$ and $y = x$

$y = x^3 - x$

Area $= \int (x)^3 - x)\,dx = \left(\frac{x^4}{4} - \frac{x^2}{2} + c\right)$ sq units

43. Find the area bounded by the curve $y = x^2 + 3$, the x- axis and the ordinates $x = 0$, $x = 5$

Solution

$y = x^2 + 3$

$$\text{Area} = \int_0^5 (x^2+3)\,dx = \left[\frac{x^3}{3} + 3x\right]_0^5$$

$$\text{Area} = \frac{(5)^3}{3} + 3(5) - 0 = \frac{125}{3} + 5$$

$$= 56.67 \text{ sq units}$$

44. Evaluate the integrals (a) $\int_1^e x \ln x \, dx$ (b) $\int_0^{\pi/2} (\sin x - \cos x) dx$

USING LIATE TRICK;

Signs	Differentiation	Integration
+	$\ln x$	x
−	$1/x$	$\frac{x^2}{2}$
+		

$\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx$

$\left[\frac{x^2}{2} \ln x - \frac{1}{4} x^2\right]_1^e = \frac{-2e^2+1}{4}$

(b)

$\int_0^{\pi/2} (\sin x - \cos x) dx$

$\int_0^{\pi/2} \sin x \, dx - \int_0^{\pi/2} \cos x \, dx$

$-\cos x \Big|_0^{\pi/2} - \sin x \Big|_0^{\pi/2}$

$-\cos(\pi/2) + \cos(0) - \sin(\pi/2) - \sin(0)$

$1 - 1 = 0$

$\therefore \int_0^{\pi/2} [\sin x - \cos x) dx = 0$

45. Find the integral $\int \frac{dx}{5+4\cos x}\, dx$

Solution

By u-substitution trick

let $t = \tan\frac{x}{2}$, we have

$5+4\cos x = 5+4\left(\frac{1-t^2}{1+t^2}\right) = 5+4\left(\frac{1-t^2}{1+t^2}\right)$

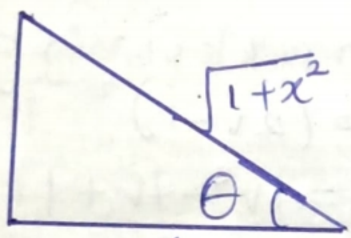
Using Lcm

$= \frac{5+5t^2+4-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2} = 2\int\frac{dt}{9+t^2}$

$= \frac{2}{3}\tan^{-1}(t/3)+C$ Replacing x;

$= \frac{2}{3}\tan^{-1}\left[\frac{\tan x/3}{3}\right] + C \in \mathbb{R}$

46. Using integration by trigonometric substitution integrate $\int \frac{dx}{x^2\sqrt{1+x^2}}$



Let $\sin\theta = \sqrt{1+x^2}$

$x^2 = \sin^2\theta - 1$; $x = \sqrt{\sin^2\theta - 1}$

$\theta = \sin^{-1}\left(\sqrt{1+x^2}\right)$

$\frac{dt}{dx} = \frac{x}{\sqrt{1+x^2}}$; $dx = \frac{\sqrt{1+x^2}}{x}$

$\int\frac{dx}{x^2\sqrt{1+x^2}} = \int\frac{1}{(\sin^2\theta-1)} \times \frac{\sqrt{1+x^2}}{x}$

$\int\frac{1}{(\sin^2\theta-1)\sin\theta} \times \frac{\sqrt{1+x^2}}{x}$

$\int\frac{1}{x^3}dx = \frac{x^{-2}}{-2} + C$

47. Find c in the interval $[-2, 1]$ which satisfies the Mean Value Theorem for $f(x) = x^3 - 2x^2 + x + 1$ Solution

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$f(x) = x^3 - 2x^2 + x + 1$; $f'(x) = 3x^2 - 4x + 1$

$f'(c) = 3c^2 - 4c + 1$

$f(a) = (-2)^3 - 2(-2)^2 + 1 = -17$

$f(b) = (1)^3 - 2(1)^2 + 1 + 1 = 1$

$$\frac{1 - (-17)}{1 - (-2)} = \frac{18}{3} = 6$$

$f'(c) = 3c^2 + 4c + 1 = 16$; $c_1 = 2.12$, $c_2 = -0.786$

$\therefore c_2 = -0.786$ satisfies mean value theorem ie $-0.786 \in I$

48. If $f(x) = 3x + 1$, $g(x) = 2x - 1$ and $h(x) = x^2$. Find (i) $f \circ g$ (ii) $g \circ f$ (iii) $h \circ f$ (iv) $h \circ g$ (v) $f \circ h$ (vi) $f \circ g \circ h$

(i)

$f(x) = 3x + 1$

$g(x) = 2x - 1$

$f \circ g = 3(2x - 1) + 1$

$f \circ g = 6x - 2$

(ii)

$g \circ f = 2(3x + 1) - 1$

$g \circ f = 6x + 2 - 1$

$g \circ f = 6x + 1$

(iii)

$h \circ f = (3x + 1)^2$

$h \circ f = 9x^2 + 6x + 1$

(iv)

$h \circ g(v)$; first we obtain $g(v) = 2v - 1$

$h \circ g(v) = (2v - 1)^2$

$h \circ g(v) = 4v^2 - 4v + 1$

(v)

$f \circ h = 3(x^2) + 1$

$f \circ h = 3x^2 + 1$

(vi)

$f \circ g \circ h = 6(x^2) - 2$

$f \circ g \circ h = 6x^2 - 2$

49. Find the inverse functions $(f^{-1}(x))$ of the functions below:

(a) $f(x) = x^3 + 3$ (b) $f(x) = \frac{x+1}{x-2}$; $x \neq 2$

(a)

$f(x) = x^3 + 3$

Equate $f(x)$ to y

$y = x^3 + 3$.

make x the subject formula

$y - 3 = x^3$

$(y-3)^{1/3} = x$

$\therefore f^{-1}(x) = (x-3)^{1/3}$

(b)

$y = \frac{x+1}{x-2}$

Cross multiply.

$yx - 2y = x + 1$

Collect like terms

$yx - x = 1 + 2y$

$x(y-1) = 1 + 2y$

$x = \frac{1+2y}{y-1}$; $y \neq 1$

$\therefore f^{-1}(x) = \frac{1+2x}{x-1}$, $x \neq 1$

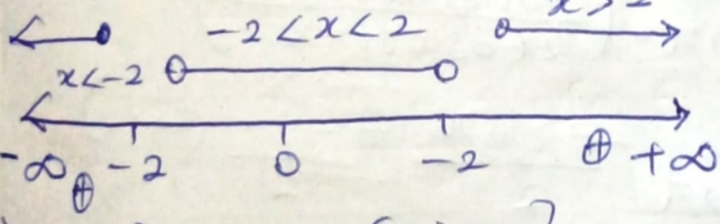
50. Determine the domain and range of the function $f(x): \mathbb{R} \rightarrow \mathbb{R}$ such that (i) $f(x) = \frac{x^2}{\sqrt{x^2-4}}$ (ii) $f(x) = \frac{2x}{(x-2)(x+1)}$

(i)

$f(x) = \frac{x^2}{\sqrt{x^2-4}}$ is defined

if $x^2 - 4 > 0$

$(x+2)(x-2) > 0$



$D_f = [x : x < -2 \cup x > 2]$

$R_f = \mathbb{R}^+$ or $[0, \infty]$

ie Range $= [0, \infty]$

(ii)

$f(x) = \frac{2x}{(x-2)(x+1)}$ is defined

If denominator $\neq 0$

$= (x-2)(x+1) \neq 0$

$= x \neq 2$ or $x \neq -1$

Domain $= [\mathbb{R} - [-1, 2]$

Range $= \mathbb{R}$ or $(-\infty, \infty)$

51. Evaluate the following limits:

(a) $\lim_{x\to 0}(\sin x)^{\tan x}$ (b) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

(a)

$\lim_{x\to 0}(\sin x)^{\tan x} = 0$

Let $y = \lim_{x\to 0}(\sin x)^{\tan x}$

$\ln y = \lim_{x\to 0}\frac{\sin x}{1/\tan x}$

$\ln y = \sin x \cdot \frac{\sin x}{\cos x}$

$\ln y = \sin(0)\frac{/\sin(0)}{\cos(0)} = 0$

$y = e^0 = 1$

(b)

Using L'hopital rule

$\lim_{x\to 0}\frac{d/dx(\sin x - x)}{d/dx(x\sin x)}$; by LCm

$\lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} = \frac{0}{0}$

Using L'hopital rule

$\lim_{x\to 0}\frac{-\sin x}{\cos x + \cos x - x\sin x}$

$= \frac{-\sin(0)}{2\cos(0) - 0\sin(0)} = \frac{0}{2}$

$= 0$

52. Evaluate the following limit

(a) $\lim_{x\to 3}\frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$ (b) $\lim_{x\to\pi}\frac{1+\cos^3 x}{1-\cos^2 x}$

(a)

By L'Hopital's rule

$\lim_{x\to 3}\frac{d/dx(x^3-7x^2+15x-9)}{d/dx(x^4-5x^3+27x-27)} = \frac{0}{0}$

Indeterminate

Applying L'Hopital's again

$\lim_{x\to 3}\frac{6x-14}{12x^2-30x} = \frac{2}{7}$

(b)

$\lim_{x\to\pi}\frac{1+\cos^3 x}{1-\cos^2 x} = \frac{0}{0}$

By L'Hopital's rule

$\lim_{x\to\pi}\frac{d/dx(1+\cos^3 x)}{d/dx(1-\cos^2 x)}$

$\lim_{x\to\pi}\left(\frac{-3\cos^2 x\sin x}{2\cos x\sin x}\right)$

$\lim_{x\to\pi}\frac{-3\cos x}{2} = \frac{-3}{2}(-1)$

$= \frac{3}{2}$

53. Find the slope of the tangent line for the curve $xy^3 - yx^3 = 6$, at the point $(1, -1)$

Solution

$xy^3 - yx^3 = 6$ at $(1, -1)$

By Implicit differentiation

$3xy^2\, dy/dx + y^3 - x^3\, dy/dx - 3yx^2 = 0$

$3xy^2\, dy/dx - x^3\, dy/dx = 3yx^2 - y^3$

$(3xy^2 - x^3)\, dy/dx = 3yx^2 - y^3$.

$dy/dx = \dfrac{3yx^2 - y^3}{3xy^2 - x^3}$

$$\frac{dy}{dx}(1,-1) = \frac{3(-1)(1)^2 - (-1)^3}{3(1)(-1)^2 - (1)^3} = \underline{\underline{-1}}$$

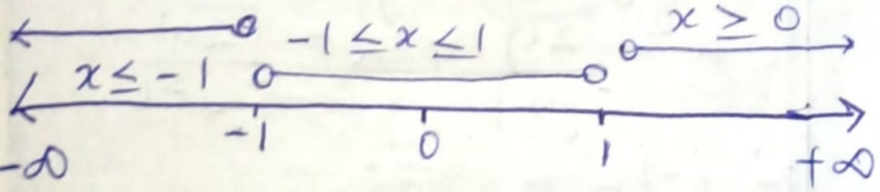
54. Find the domain of the and range of the following functions (a) $f(x) = \sqrt{1 - x^2}$ (b) $f(x) = \frac{x-4}{(x^2-2x-15)}$ (c) $f(x) = \ln(x - 8)$.

(a)

$f(x) = \sqrt{1-x^2}$ is defined where

$1 - x^2 \geq 0$

$(1+x)(1-x) \geq 0$



(b)

For $\dfrac{x-4}{x^2-2x-15}$ is defined if

$x^2 - 2x - 15 \neq 0$

$(x-5)(x+3) \neq 0$

$x \neq 5$ or $x \neq -3$

Domain $= IR - [-3, 5]$

Range $= IR$ or $(-\infty, \infty)$

(c)

$f(x) = \ln(x-8)$ is well defined if $x - 8 > 0$

$= x > 8$

$Df = [x : x > 8, \forall x \in IR]$

$Rf = IR$ or $(-\infty, \infty)$

55. If $f(x) = x^2 - 1, \quad -2 \leq x < 4$ and $g(x) = \frac{2}{x-3}, \quad 0 < x \leq 5$. Find the common domain of $h(x) = f(x) + g(x)$.

Solution

$f(x) = x^2 - 1, g(x) = \frac{2}{x-3}$

$f(x) + g(x) = h(x)$

$h(x) = \frac{x^2-1}{1} + \frac{2}{x-3}$ by L.C.M

$h(x) = \frac{(x-3)(x^2-1)+2}{x-3} = \frac{x^3 - x^2 - x + 5}{x-3}$

$Df = [R - \{3]$ or $(-\infty, 3) \cup (3, \infty)$

56. Find the inverse $f^{-1}(x)$ of the following functions (a) $f(x) = 3x - 7$ (b) $f(x) = \frac{x^2}{(x^2-2)}$

(a)

$f(x) = 3x - 7$

Equate the function to y

$y = 3x - 7$

Make x subject formula

$y + 7 = 3x$

$\frac{y+9}{3} = x$

Change the position of x with y, $\frac{x+7}{3} = y$

$\therefore f^{-1}(x) = \frac{x+7}{3}$ //

(b)

$f(x) = x^2/x^2 - 2$

Let $y = \frac{x^2}{x^2-2}$; Cross multiply

$x^2y - 2y = x^2$

$x^2y - x^2 = 2y$

$x^2(y-1) = 2y$

$x^2 = \frac{2y}{y-1}$; $x = \sqrt{\frac{2y}{y-1}}$

$\therefore f^{-1}(x) = \sqrt{\frac{2x}{x-1}}$

$f^{-1}(x) = \sqrt{\frac{2x}{x-1}}, \quad x \neq 1$

57. (a) Find the even and odd part of the function $f(x) = 3x^2 - 2x + 1$

(b) Determine whether the function $f(x) = sin^4 x tan x$ is even or odd.

(a)

$f(x) = 3x^2 - 2x + 1$

for even function $f(-x) = f(x)$

for odd function, $f(-x) = -f(x)$

The part $3x^2 + 1$ is even

Since $f(x) = 3(-x)^2 + 1$

$f(x) = 3x^2 + 1 = f(x)$

The part $-2x$ is odd

Since $f(-x) = -2(-x) = 2x$

$f(-x) \neq -f(x)$

$3x^2 + 1$ is even function part

$-2x$ is odd function part

(b)

$f(x) = \sin^4 x \tan x$

The product of two odd function is also an odd function

Hence $f(x) = \sin^4 x \tan x$ is an odd function

58. (a) Let $h(x) = x + 4$ and $g(x) = 2x^2$. Find $g(h(x))$ (b) Given $h(x) = 5x - 3$and $f(x) = x^2 + 3$. Find $h(f(x))$ (c) Given $f(x) = x^2 - 4x + 3$, $g(x) = 3x^2 - 2x$ and $h(x) = 3x^2 - 9x + 1$, Find $f(x) + g(x) + h(x)$

(a)

$h(x) = x + 4, \ g(x) = 2x^2$

$g(h(x)) = 2[x + 4]^2$

$g(h(x)) = 2(x^2 + 8x + 16)$

$g(h(x)) = 2x^2 + 16x + 32$

or

To simplest term

$g(h(x)) = x^2 + 8x + 16$

(b)

$h(x) = 5x - 3$

$f(x) = x^2 + 3$

$h(f(x) = 5x^2 + 12$

(c)

$f(x) = x^2 - 4x + 3$

$g(x) = 3x^2 - 2x$

$h(x) = 3x^2 - 9x + 1$

$f(x) + g(x) + h(x)$

$$
\begin{array}{r}
+ x^2 - 4x + 9 \\
3x^2 - 2x + 0 \\
3x^2 - 9x + 1 \\
\hline
7x^2 - 15x + 4
\end{array}
$$

59a) Given $f(x) = \frac{x^2-6x+7}{x+5}$. Find the oblique (slant) asymptote.

(b) Given $f(x) = \frac{4x^2-5x}{x^2-2x+1}$. Find the horizontal asymptote.

(c) Given $f(x) = \frac{x+6}{x^2-5x}$. Find the vertical asymptote.

(a)

$$x+5 \overline{)\, x^2-6x+7 \,}$$ quotient $x-11$

$-(x^2-5x+0)$

$0-11x+7$

$-(-11x-55)$

62

Slant oblique Asymptote

$y = x-11$

(b)

Horizontal Asymptote

$$\lim_{x\to\infty} \frac{\frac{4x^2}{x^2} - \frac{5x}{x^2}}{x^2/x^2 - 2x/x^2 - 1/x^2}$$

$$\lim_{x\to\infty} \frac{4-0}{1-0} = 4$$

Horizontal Asymptote at $y = 4$

(c)

$f(x) = \frac{x+6}{x^2-5x}$

Vertical asymptote at $x^2-5x=0$

$x(x-5)=0$

$x=0$ or $x=5$

Vertical Asymptote $= \underline{0, 5}$

60. Investigate the continuity of the following functions

(a) $f(x) = \begin{cases} 2x^2 - x, & x < 2 \\ 3x, & x \geq 2 \end{cases}$ (b) $f(x) = \begin{cases} \frac{4-x}{\sqrt{x}-2}, & x \neq 4 \\ \frac{1}{4}, & x = 4 \end{cases}$

(a)

Check if limit is equal to $f(x)$

$\therefore \lim_{x\to 2} 2x^2 - x = 2(2)^2 - 2$

$= 6$

$F(x) = 3x$; $F(2) = 6$

The function is continuous

since $\lim_{x\to 2} f(x) = F(x) = 6$

(b)

The function is well defined at $x = 4$

Next, we take limit

$\lim_{x\to 4} \frac{4-x}{\sqrt{x}-2} \times \frac{\sqrt{x}+2}{\sqrt{x}+2}$

$\lim_{x\to 4} \sqrt{x}+2 = \sqrt{4}+2$

$= 4$

Since $\lim_{x\to 4} f(x) \neq f(4)$, the function is not continuous at $x = 4$

61 Evaluate the first derivatives of the following functions (a) $y = x^2 cosx$. (b) $y = (3x+5)^4$ (c)Given $y = e^{-4x}\cos(3x)$ (d)$y = sinh^{-1}(tanx)$.

(a)

$y = x^2 \cos x$, using product rule

Let $u = x^2$, $du = 2x$; $v = \cos x$

$dv = -\sin x$

$\frac{dy}{dx} = \frac{u\,dv}{dx} + \frac{v\,du}{dx}$

$dy/dx = -x^2 \sin x + 2x \cos 2x$

(b)

$y = (3x+5)^4$

Using chain Rule

Let $u = 3x+5$ $du = 3$

$y = u^4$; $dy/du = 5u^4$

$dy/dx = 15(3x+5)^4$

(c)

$y = e^{-4x}\cos(3x)$

$dy/dx = -3e^{-4x}\sin(3x) - 4e^{-4x}\cos(3x)$

(d)

$y = \sinh^{-1}(\tan x)$

Let $u = \tan x$; $du = \sec^2 x$

$y = \sinh^{-1}(u)$; $dy/du = \frac{1}{\sqrt{1+x^2}}$

$\frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+x^2}}$

62. Evaluate the following integrals (a)$\int \frac{cosx}{1-sinx} dx$ (b)$\int x^2 lnx\, dx$ (c) $\int \frac{x+1}{x^2-3x+2} dx$

(a)

$\int \frac{\cos x}{1-\sin x} dx$

Let $u = 1-\sin x$

$du = -\cos x\, dx = -\frac{du}{\cos x}$

$\int \frac{\cos x}{1-\sin x} dx = -\int \frac{\cos x}{u} \cdot \frac{du}{\cos x}$

$-\int \frac{1}{u} du = -\ln u + C$

$\therefore -\ln|1-\sin x| + C$

$\forall C \in \mathbb{R}$

(b)

$\int x^2 \ln x\, dx$

$I = uv - \int v\, du$

$u = \ln x$, $du = 1/x$

$dv = x^2$, $v = x^3/3$

$I = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$

$I = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$

$I = \frac{x^3}{3}\left[\ln x - \frac{1}{3}\right] + C$

$\forall C \in \mathbb{R}$

(c)

Resolve into partial fraction, we have

$\frac{x+1}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$

Set $x = 1, 2$,

$A = \frac{1}{3}$ and $B = \frac{-1}{2}$

$\frac{1}{3}\int \frac{1}{x-2} dx - \frac{1}{2}\int \frac{1}{x-1} dx$

$\frac{1}{3}\ln|x-2| - \frac{1}{2}\ln|x-1| + C$

$\forall C \in \mathbb{R}$

63. Differentiate from first principle the following functions (a) $y = \sin 3x$ (b) $y = (2x^2+2x)(2x+2)$ (c) $y = \frac{1}{x^3}+x$ (d) $y = \frac{3}{1-x}$

(a)

$y = \sin 3x$

$y'' = 3\cos 3x$

(b)

$y = (2x^2+2x)(2x+2)$

Let $u = 2x^2+2x$, $v = 2x+2$

$du = u\,dv + v\,du$

$\frac{dy}{dx} = 2(4x+2) = 8x+4$

(c)

$y = \frac{1}{x^3} + x$

$dy/dx = \frac{-3}{x^4} + 1$

(d)

$y = \frac{3}{1-x}$

$\frac{dy}{dx} = \frac{(1-x)(0)(-3)(1-x)}{(1-x)^2}$

$\frac{dy}{dx} = \frac{-3(1-x)}{(1-x)^2}$

64. Find the asymptotes of the following curves (a) $x^2(x^2+2) = y^3(x+5)$ (b) $x^3 - xy^2 + 4y^2 - 5 = 0$ (c) $y = \frac{2x-16}{2x+3}$

(a)

$x^2(x+2) = y^3(x+5)$

$y^3 = \frac{x^2(x+2)}{x+5}$

$y = \left(\frac{x^2(x+5)}{x+5}\right)^{1/3}$

The function V.A at

$x + 5 = 0$

$\therefore$ V.A is at $x = -5$

(b)

Make y subject formula

$y^2(4-x^2) = 5-x^3$

$y^2 = \frac{5-x^3}{4-x^2}$

$y = \left(\frac{5-x^3}{4-x^2}\right)^{1/2}$

V.A $= 4-x^2 = 0$

V.A is at $x = \pm 2$

It has no horizontal asymptote

The slant asymptote is

$$\begin{array}{r|l} & x \\ \hline 4-x^2 & 5-x^3 \\ & -(4x-x^3) \\ \hline & 4x \end{array}$$

slant $= y = x$

(c)

$y = \frac{2x-16}{2x+3}$ V.A is at $2x+3=0$

V.A $= -3/2$

$\lim_{x\to\infty} \frac{2x-6}{2x+3} = 1$

H.A $= 1$

It has no slant Asymptote

65. (a) Find the gradient of the tangent to the curve $y=\frac{x^2}{x^2+1}$ at the point with x=1 (b)Find the equation of the tangent and normal to the curve $y=3x^2-5x$ at the point (1, -2) (c) Find the equation of the tangent to the curve $y=2x^2-x+3$ which is parallel to the line y=3x-2

(a)

$y=\frac{x^2}{x^2+1}$ at $x=1$

Gradient $= m = dy/dx$

$dy/dx = \frac{(x^2+1)(x) - x^2(2x)}{(x^2+1)^2}$

Gradient $= 1/2$

(b)

$y = 3x^2 - 5x$; $(1, -2)$

$dy/dx = 6x - 5$, $6(1) - 5 = 1$

$m = 1$

Eqn of Tangent is

$y - y_0 = m(x - x_0)$

$y + 2 = 1(x - 1)$

$y - x + 3 = 0$

Eqn of Normal line

$y - y_0 = \frac{-1}{m}(x - x_0)$

$y + 2 = -1(x - 1)$

$y + 2 = -x + 1$

$y + x + 1 = 0$ is the eqn of Normal line

(c)

$y = 2x^2 - x + 3$

Parallel to $y = 3x - 2$

Slope $= \frac{-3}{-1} = 3$

$\frac{dy}{dx} = 4x - 1 = 3$

$4x - 1 - 3 = 0$

$4x - 4$, $x = 1$

To get y, are substituted $x = 1$ in $y = 2x^2 - x + 3$

$\therefore y = 2(1)^2 - 1 + 3 = 4$

Eqn of Tangent is

$y - 4 = 3(x - 1)$

$y - 3x - 1 = 0$

$y - 3x - 1 = 0$ is the Eqn of Tangent

66. Sketch the following curves (a) $y=\frac{2x+1}{x-1}$ (b) $y=\frac{x^2-12x+27}{x^2-4x+15}$ (c) $y=\frac{x^2}{x^2-1}$

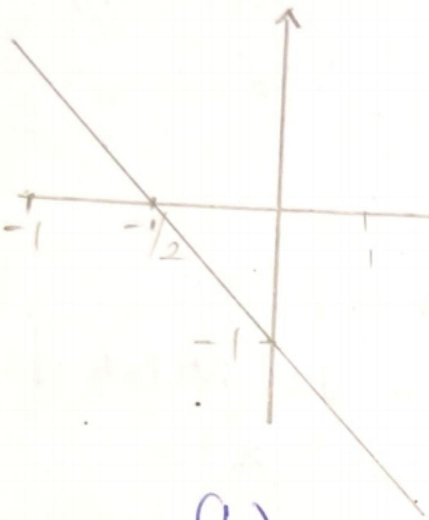
(d) $y=\sqrt{25-x^2}$ (e) $y=x+\frac{1}{x}$ (f) $y=\frac{x}{x^2+1}$ (g) $y=\frac{(x+2)(x-3)}{x+1}$

(a)

$y = \frac{2x+1}{x-1}$

When $x=0$, $y=-1$

When $y=0$, $x=-\frac{1}{2}$

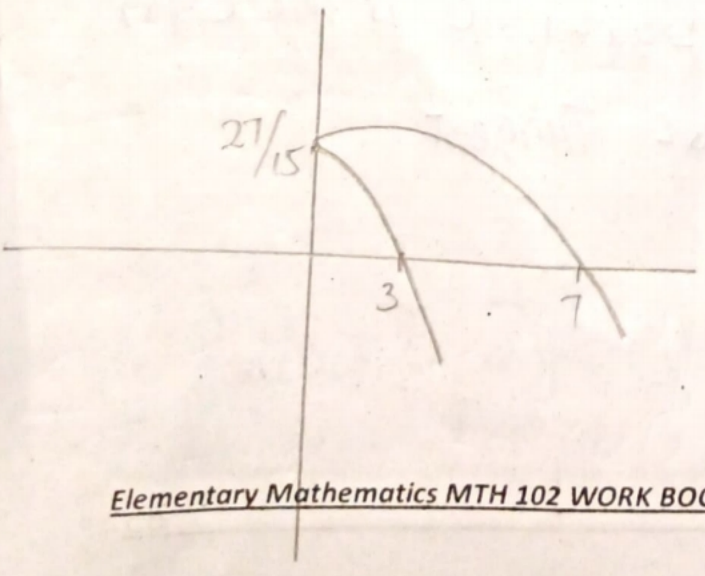


(b)

$y = \frac{x^2-12x+27}{x^2-4x+15}$

When $x=0$, $y=27/15$

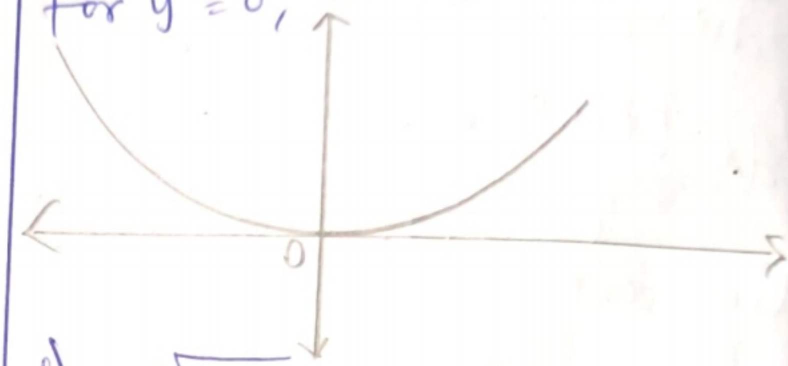
When $y=0$, $x=3, 9$



(c)

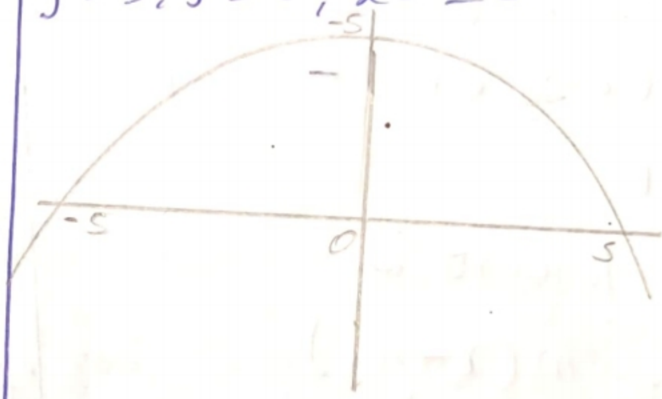
$y = \frac{x^2}{x^2-1}$ for $x=0$, $y=0$

for $y=0$, $x=0$



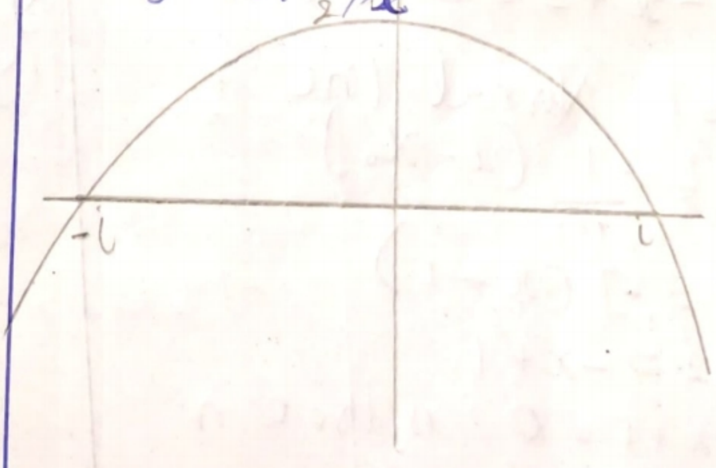
d) $y=\sqrt{25-x^2}$, $x=0$

$y=5$, $y=0$, $x=\pm 5$



(e)

$y = x + \frac{1}{x}$



67. Evaluate the following limits (a) $\lim_{x\to 0}\frac{\cosh-e^x}{x}$ (a)$\lim_{x\to 0}\frac{x-\sin x}{x^2}$ (c)$\lim_{x\to 1}\frac{x^3+x^2-x-1}{x^2+2x-3}$ (d)$\lim_{x\to\infty}\frac{2x^2+x-1}{3x^2-x+7}$ (e) $\lim_{x\to-\infty}\frac{5x^2-x+2}{3x^3-4}$

(a)

$\lim_{x\to 0}\frac{\cosh x-e^x}{x} = \frac{0}{0}$

L'Hopital's rule method

$\lim_{x\to 0}\frac{\sinh x-e^x}{1} = -1$

b) $\lim_{x\to 0}\frac{x-\sin x}{x^2} = \frac{0}{0}$ using L'Hopital rule

$\lim_{x\to 0}\frac{1-\cos x}{2x} = \frac{0}{0}$, Again L'Hopital rule

$\lim_{x\to 0}\frac{\sin x}{2} = \underline{\underline{0}}$

c) $\lim_{x\to 1}\frac{x^3+x^2-x-1}{x^2+2x-3} = \frac{0}{0}$

By L'Hopital rule

$\lim_{x\to 1}\frac{3x^2+2x-1}{2x+2} = \underline{\underline{1}}$

d) $\lim_{x\to\infty}\frac{2x^2+x-1}{3x^2-x+7} = \frac{2}{3}$

e) $\lim_{x\to\infty}\frac{5/x-\frac{1}{x^2}+\frac{2}{x^3}}{3-4/x^3}$

$=\frac{0}{3} = \underline{\underline{0}}$

68. (a) Find the first and second derivatives of the following function, $x + y + \sin y = 3$ (b) Find an expression for $\frac{dy}{dx}$ for the cycloid given by $x = a(t + \sin t), y = a(1 - \cos t)$

(a)

$x+y+\sin y = 3$

By Implicit differentiation

$1+dy/dx + \cos y\, dy/dx = 0$

$(1+\cos y)dy/dx = -1$

$dy/dx = \frac{-1}{(1+\cos y)}$

$\frac{d^2y}{dx^2} = \frac{(1+\cos y)\cos+1(-\sin y)\,dy/dx}{(1+\cos y)^2}$

$\frac{d^2y}{dx^2} = \frac{\sin y}{(1+\cos y)^2}\cdot\frac{1}{(1+\cos y)}$

$\frac{d^2y}{dx^2} = \frac{\sin y}{(1+\cos y)^3}$

(b)

$x = a(t+\sin t)$

$y = a(1-\cos t)$

$dy/dt = a\sin t$

$dx/dt = a(1+\cos t)$

$dy/dx = dy/dt\cdot\frac{dt}{dx}$

$\frac{dy}{dx} = a\sin t\cdot\frac{1}{a(1+\cos t)}$

$\frac{dy}{dx} = \frac{\sin t}{1+\cos t}$

69. (a) Find $\frac{dy}{dx}$ of the function $y=(\frac{x-1}{x+1})^2$ (b) The equations of a curve in parametric form are x= 4cosθ-4sinθ-1, find $\frac{dy}{dx}$ at the point where $\theta=\frac{\pi}{2}$

(a)

$y=\left(\frac{x-1}{x+1}\right)^2$, By chain Rule

let $u=\frac{x-1}{x+1}$; $y=u^2$

$dy/du = 2u$, $\frac{du}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$\frac{du}{dx} = \frac{2}{(x+1)^2}$

$\frac{dy}{dx} = 2u \cdot \frac{2}{(x+1)^2} = \frac{4(x-1)}{(x+1)^3}$

(b)

$x=4\cos\theta-4\sin\theta-1$

let $x=4\cos\theta$, $y=4\sin\theta$

$dx/d\theta = -4\sin\theta$

$dy/d\theta = 4\cos\theta$

$dy/dx = dy/d\theta \cdot d\theta/dx$

$\frac{dy}{dx} = 4\cos\theta \cdot \frac{1}{-4\sin\theta}$

$\frac{dy}{dx} = \frac{-4\cos\theta}{4\sin\theta} = \frac{-\cos\theta}{\sin\theta}$

at $\theta=\frac{\pi}{2}$; $\frac{-\cos(90)}{\sin 90} = \frac{0}{1}$

$= \underline{0}$

70. (a) If $y=sin^2(x^2+1), find$ $\frac{dy}{dx}$ (b) Find $\frac{dy}{dx}$ if $5x^2y^3-3xy^{-2}=4xy^2-5x^{-1}y^4+x^4$

(a)

$y=\sin^2(x^2+1)$

let $u=x^2+1$

$\frac{du}{dx} = 2x$

$y=\sin^2 u$

$\frac{dy}{du} = 2\sin u\cos u$

$\frac{dy}{dx} = \frac{dy}{du} = \frac{du}{dx}$

$\frac{dy}{dx} = 2\sin u\cos u \cdot 2u$

$= 4x\sin(x^2+1)\cos(x^2+1)$

(b)

Differentiating Impuicitly

$10xy^3+15x^2y^2 dy/dx-3y^{-2}+2y^{-3}dy/dx$
$=4y^2+8xy\frac{dy}{dx}+5x^{-2}y^4-20x^{-1}y^3 dy/dx+4x^3$

Factor out y'

$\frac{dy}{dx} = \frac{4y^2+5x^{-2}y^4+4x^3-10xy^3+1y}{15x^2y^2+2y^{-3}-8xy+20x^{-1}y^3}$

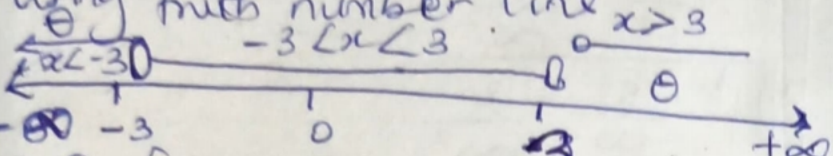
71. Determine the domain and range of the following functions $F: R \to R$ such that (i) $\frac{1}{\sqrt{9-x^2}}$ (ii) $\frac{2x}{(x-2)(x+1)}$ (iii) $\sqrt{\frac{x}{2-x}}$ (iv) $\frac{x^2-1}{\sqrt{4x^3}}$

(i)

$f(x) = \frac{1}{\sqrt{9-x^2}}$ is defined at

$9-x^2 > 0$; $(3+x)(3-x) > 0$

using truth number line



$Df = \{-3<x<3; x \in \mathbb{R}\}$

$Rf = (0, \infty)$

(ii)

$f(x) = \frac{2x}{(x-2)(x+1)}$ is defined

if $(x+1)(x-2) \neq 0$

$Df = \mathbb{R} - [1, 2]$

$Rf = \mathbb{R}$

(iii)

$f(x) = \sqrt{x/2-x}$ is defined

if $2-x>0$ $2>x$; $x<2$

$Df = (-\infty, 2)$

$Rf = \mathbb{R}$ or $[0, \infty)$

(iv)

$f(x) = \frac{x^2-1}{\sqrt{4x^3}}$ is defined

if $4x^3 > 0 \Rightarrow x > 0$

$Df = \{x \in \mathbb{R} : x > 0\}0$

$Df = (0, \infty)$

$Rf = [0, \infty]$ or $\mathbb{R}^+$

72. Verify that lim $\frac{3x-5}{4x^2+9} = \frac{1}{25}$

$\forall \varepsilon > 0$ and $f > 0$

$\exists |f(x) - L| < \varepsilon$; $0 < |x - x_0| < f$

$\left| \frac{3x-5}{4x^2+9} - \frac{1}{25} \right| < \varepsilon$

By L.C.M

$\left| \frac{75x - 4x^2 - 134}{25(4x^2+9)} \right| < \varepsilon$

Let $k = 4x^2+9$ and Introduce Coordinate

$\frac{1}{25k} |91f - 4f_2| < \varepsilon$

$\frac{1}{25k} |87f| < \varepsilon$

$87f < 25k \cdot \varepsilon$

$f < \frac{25k \cdot \varepsilon}{87}$

73. Evaluate the following at the given points,

(i) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ (ii) $\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}$ (iii) $\lim_{x\to 1}\frac{x^5-3x^4+5x-3}{4x^5+2x^3-5x^2-1}$

(iv) $\lim_{x\to\infty}\frac{2x^2+3x+1}{3x^2-5x+2}$ (v) $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}$ (vi) $\lim_{x\to\infty}\frac{3-e^x}{x^2}$

(i)

$\lim_{x\to 1}\frac{\sqrt{x-1}}{x-1} = \frac{0}{0}$.

Using L'Hopital rule

$\lim_{x\to 1}\frac{d}{2\sqrt{x}} = \frac{1}{2}$

(ii)

$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2} = \frac{0}{0}$

By factorization

$\lim_{x\to 1}\frac{(x+1)(x-1)}{(x-2)(x-1)}$

$\lim_{x\to 1}\frac{x+1}{(x-2)} = \frac{2}{1-2} = -2$

(iii)

$\lim_{x\to 1}\frac{x^5-3x^4+5x-3}{4x^5+2x^3-5x^2-1} = \frac{0}{0}$

Using L'Hopital rule

$\lim_{x\to 1}\frac{5x^4-12x^3+5}{20x^4+6x^2-5} = \frac{-2}{21}$

(iv)

$\lim_{x\to\infty}\frac{2+3/x+1/x^2}{3-5/x+2/x^2}$

$\frac{2+\frac{2}{\infty}+\frac{1}{\infty^2}}{3-\frac{5}{\infty}+\frac{2}{\infty^2}} = \frac{2}{3}$.

(v)

$\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \frac{0}{0}$

By L'Hopital rule

$\lim_{x\to 0}\frac{1}{2\sqrt{x+2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ //

(vi)

$\lim_{x\to\infty}\frac{3-e^x}{x^2} = \frac{\infty}{\infty}$

By L'Hopital rule

$\lim_{x\to\infty}\frac{-e^x}{2x} = \frac{\infty}{\infty}$

L'Hopital's rule again

$\lim_{x\to\infty}\frac{-e^x}{2} = \frac{-\infty}{2} = -\infty$

74. Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = \begin{cases} 2x - 5 & x > 2 \\ x^2 - 2|x| & x \leq 2 \end{cases}$

$$g(x) = 3x + 1$$

Find the values of (i) $f(-2)$ (ii) $g(-3)$ (iii) $(gof)(2)$

(i).

$f(x) = x^2 - 2|x|$

$f(-2) = (-2)^2 - 2|-2|$

$f(-2) = 4 - 2|2| = 4 - 4 = 0$

(ii)

$g(x) = 3x + 1$

$g(-3) = 3(-3) + 1$

$g(-3) = -9 + 1 = -8$

(iii)

$(g \circ f)(2)$

Recall that $g \circ f$

$3(x^2 - 2|x|) + 1$

$3x^2 - 6|x| + 1$

$(g \circ f)(2) = 3(2)^2 - 6(2) + 1$

$= 12 - 12 + 1 = \underline{\underline{11}}$

75. If $f(t) = 3^t$, show that $f(t + 3) + f(t - 1) = 27f(t)$

Solution

$f(t) = 3^t$

$f(t+3) + f(t-1) = 3^{t+3} + 3^{t-1}$

Obeying the law of Indices

$3^{t+3} + 3^{t-1} = 3^t \cdot 3^3 + 3^t \cdot 3^{-1}$

Factor out 3^t

$3^t[3^3 + 1/3] = 3^t \cdot [27 + 1/3] = 3^{-1}[22/3]$

$= 3^t[27 \cdot 33] = 27[3^{-1}] = 27f(t)$

$= f(t+3) + f(t-1) = 27f(t)$

Shown

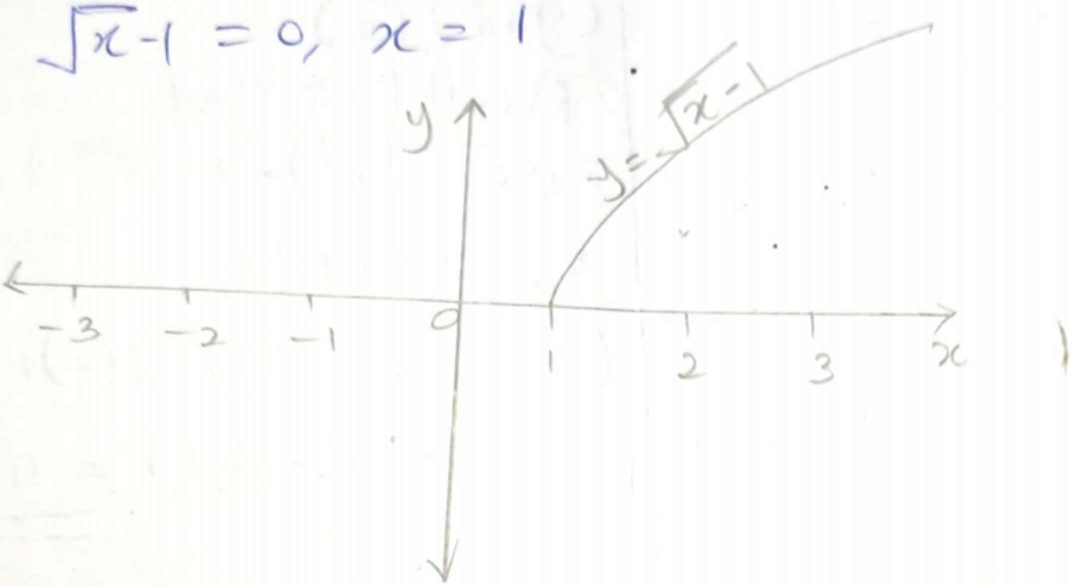
76. Sketch the graph of the following

(i) $y = \sqrt{x-1}$: $x \geq 1$ (ii) $f(x) = 3 - x^2$: $-4 \leq x \leq 4$

(i.)

$y = \sqrt{x-1}$, when $x = 1$ $y = 0$

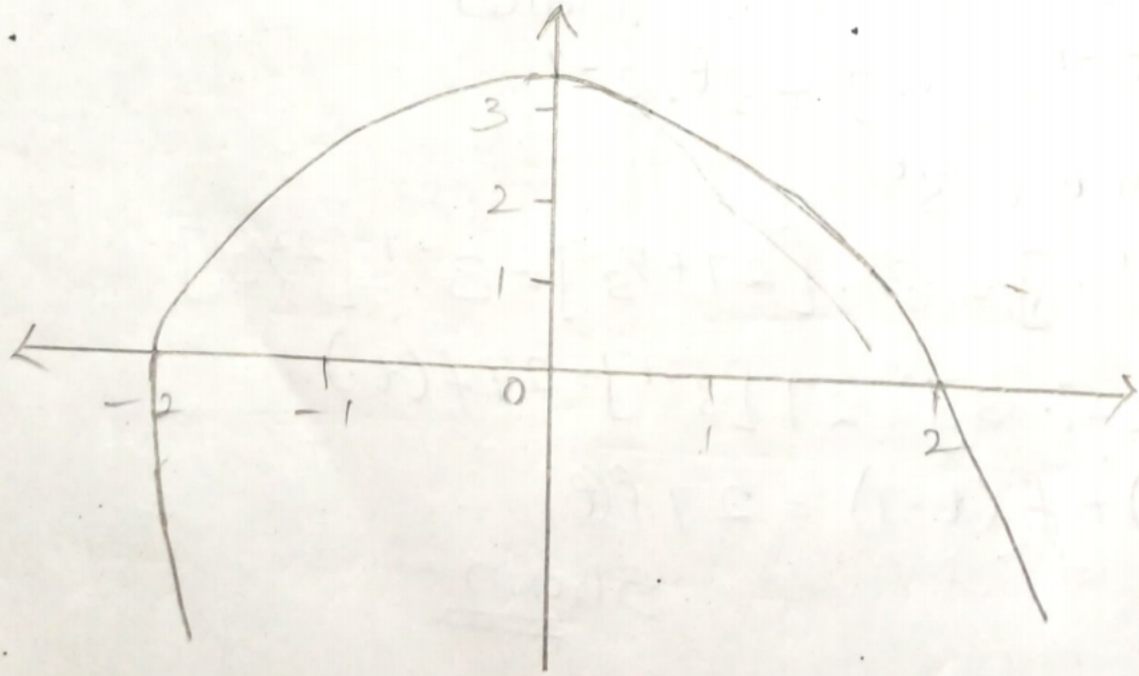
$\sqrt{x-1} = 0$, $x = 1$



(ii.)

$y = 3 - x^2$, $-4 \leq x \leq 4$; when $x = 0$ $y = 3$

But $3 - x^2 = 0$ $x = -2$ or 2 are the root



77. Determine the vertical asymptotes of the graph of the following functions:

(i) $f(x) = \frac{5+2x^2}{2-x-x^2}$ (ii) $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$

(iii) $f(x) = \cot x$ (iv) $f(x) = \frac{x-2}{x^2-4}$

(i)

$f(x) = \frac{5+2x^2}{2-x-x^2}$

Vertical asymptote is at $2-x-x^2=0$

$(2+x)(1-x)=0$

$x=-2$ or $x=1$ is the V.A.

(ii)

$f(x) = \frac{x^2-25}{x^3-6x^2+5x}$

V.A is at $x^3-6x^2+5x=0$

$x=0,5,1$ are the vertical Asymptote

(iii)

$f(x) = \cot x = \frac{\cos x}{\sin x}$

Vertical asymptote is at

$\sin x = 0 = x = n\pi$

$n = 0, 1, 2 \ldots$

(iv)

$f(x) = \frac{x-2}{x^2-4}$

factorizing

$f(x) = \frac{x-2}{(x+2)(x-2)}$

V.A is at $x=-2$

78. Determine the equation of the tangent and normal to the graph $x(xy + y^3 + 3) - 13 = 0$ at the point (1,2)

Solution

$x^2y + xy^3 + 3x - 13 = 0$; $2xy + x^2\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} + 3 = 0$

$(x^2+3xy^2)dy/dx = -(2xy+y^3+3)$

$\frac{dy}{dx} = -\frac{(2xy)+y^3+3}{x^2+3xy^2}$

$\frac{dy}{dx}$ at $x=1$ $y=2$, $\frac{dy}{dx} = \frac{(2(1)(2)+(2)^3+3)}{(1)^2+3(1)(2)^2}$

$\frac{dy}{dx}(1,2) = \frac{-15}{13}$

79. Determine the first, second and third derivatives of the following function.

(i) (i) $2x^{13} - 3x^8 + x^7 - 5x^3$ (ii) $\sin 2x^2$ (iii) e^{x^3}

$y = 2x^{13} - 3x^8 + x^7 - 5x^3$

$y' = 26x^{12} - 24x^7 + 7x^6 - 15x^2$

$y'' = 312x^{11} - 168x^6 + 42x^5 - 30x$

$y''' = 3432x^{10} - 1008x^5 + 210x^4 - 30$

(iii)

$y = e^{x^3}$

$y' = 3x^2 e^{x^3}$

$y'' = 9x^4 e^{x^3}$

$y''' = 27x^6 e^{x^3}$

(ii)

$y = \sin(2x^2)$

$y' = 4x \cos 2x^2$

$y'' = -16x^2 \sin 2x^2$

$y''' = -64x^3 \cos 2x^2$

80. By first principle, find the derivative of $y = \sqrt{t}$ at the point (1,1).

Discuss the behaviour of the function at point (0,0)

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(t+h) - f(-1)}{h} = \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h}$

By Rahonalizahon

$\frac{dy}{dt} = \lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \times \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}}$

$\frac{dy}{dt} = \lim_{h \to 0} \frac{\cancel{t}\,\cancel{+}\,h - t}{h(\sqrt{t+h}) + t)} = \lim_{h \to 0} \frac{h}{h(\sqrt{t+h} + \sqrt{t})}$

$\frac{dy}{dt} = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{\sqrt{t} + 0 + \sqrt{t}} = \frac{1}{2\sqrt{t}}$

$\frac{dy}{dt}(1,1) = \frac{1}{2}$, $dy/dt\,(0,0) = \frac{1}{2\sqrt{0}} = \frac{1}{0}$ is undefined at (0,0)

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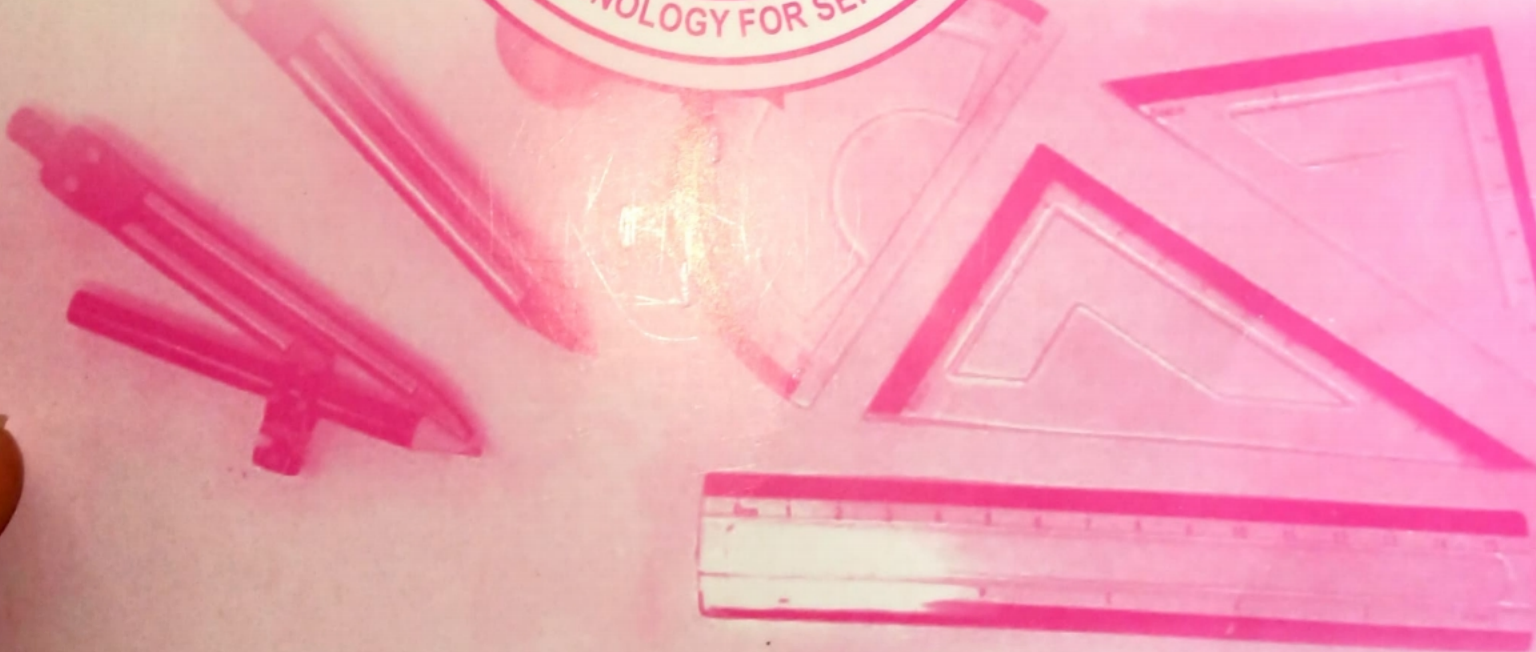

ELEMENTARY MATHEMATICS II
(MTH 102)