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UNLEASH

ELEMENTARY CALCULUS

BEST SELLER

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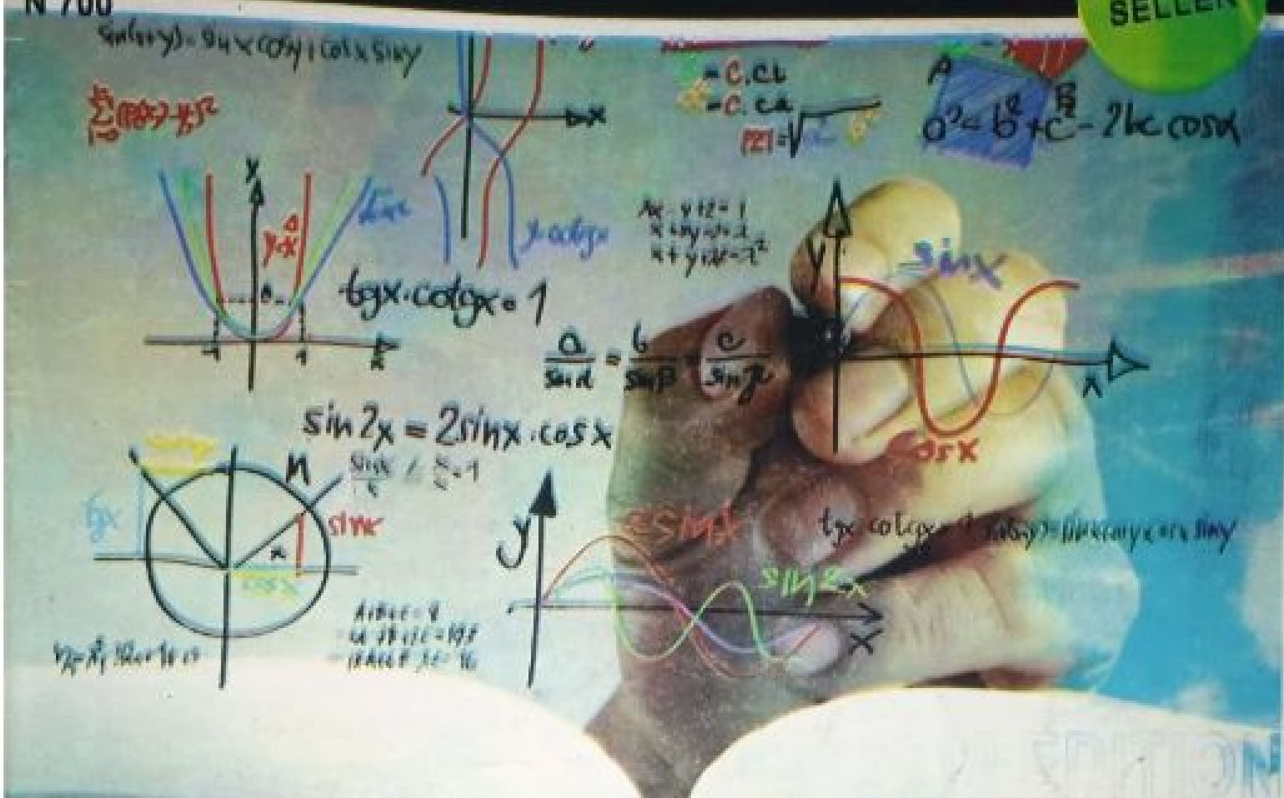

2019 EDITION

MTH 102

COMBINED LECTURE NOTE AND PAST QUESTION ON

2018, 2017, 2016, 2015, 2014, 2013, 2012, 2011, 2010, 2009, 2008 EXAM AND TEST

AN LFX PRODUCTION

BEST SELLER, BEST MATHEMATICS MATERIAL.

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RELATION

A relation is a set of ordered pairs;
For Example: $\{(1,3), (2,5), (4,9)\}$ (x, y)

A relation will be a function if no two different ordered pairs have the same first coordinate; Example.

CHECK 09/10 EXAM Q76 SOLVE IT

1) let $A = \{(3,5), (7,9), (10,12)\}$
A is a function since, no different ordered pair have the same x-value (first coordinate)

2) let $B = \{(2,5), (3,7), (2,9)\}$
B is not a function since. Different ordered pairs have same first co-ordinate "2".

3) let $C = \{(2,4), (1,1), (-2,4)\}$
C is a function, we are only interested in the x-value; No different ordered pairs have the same first co-ordinate

DOMAIN AND RANGE OF OEDERED PAIRS

Domain of a function given as an ordered pairs is the set of the x-values.
While range of same function is the y-values without repetitions.

EXAMPLE 1: 2009/2010 EXAM QUESTION 66

66. Consider the function f defined by $f: \{(-3,0), (-1,4), (0,2), (2,2,)\ (4,-1)\}$, then the domain of f is the set (a) $(-\infty, \infty)$ (b) $[-3,4]$ (c) $\{-3,-1,0,2,4\}$ (d) $\{-3,-1,2,4\}$

SOLUTION

Domain of ordered pairs is the first co-ordinate i.e the x-values of the relation.

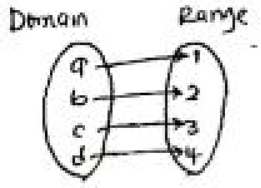
Dom $F = \{-3, -1, 0, 2, 4\}$

MAPPING

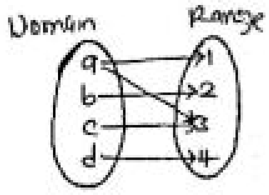
Mapping shows how the elements are paired, its like a flow chart for a function showing the input and the output values.

A mapping diagram consists of two parallel columns. The first column represents the domain of a function F, and the other column for its range. Lines or arrows are drawn from domain to range, to represent the relation between any two elements.

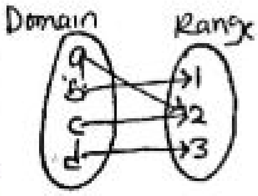
Consider the following types of mapping



A function represented by the mapping in which each element of the range (co-domain) is paired with exactly one element of the domain is called One-to-one mapping



In this mapping, Elements in the domain have mapped more than one element in the range; if one element in the domain is mapped with more than one element in the co-domain, The mapping is called; One-to-many mapping. This mapping is a relation which is not a function

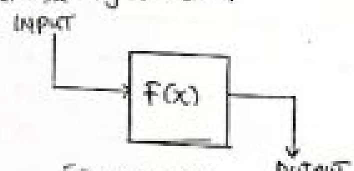


In this mapping, the second element of the range "2" is associated with more than one element in the domain, if element(s) in range have mapped more than one element in the domain, it is called many-to-one mapping

FUNCTION

A function is a rule that assigns to each number $x \in X$ a unique $F(x)$ in Y, denoted by $F: X \to Y$, where X and Y are sets of real number.

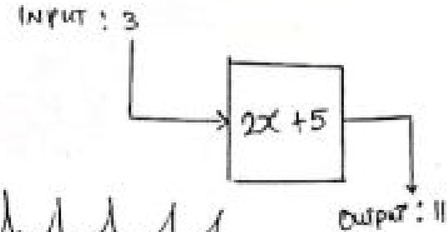
Consider the figure below;



[figure 1.0]

From the above (figure 1.0), the output depends on the rule which the function $f(x)$ will assign. Where the function depends on the input.

Example 1



Note: Functions in FUTO will not be given to you with box, diagram.

Given that $y = f(x)$, x is called the independent variable, while y is dependent variable.

EXAMPLE 2

let f(x) = 2x²+ 3x -1, find f(2)

SOLUTION

F(2) means to put x as 2

$\Rightarrow F(2) = 2(2)^2 + 3(2) - 1$
$= 2(4) + 6 - 1$
$= 8 + 6 - 1$
$= 13$

EXAMPLE 3: 2009/2010 EXAM QUESTION 53

53. Let $f(x) = (x-2)(8-x)$ for $2 \le x \le 8$. Find $f(-1)$ (a) -33 (b) 8 (c) undefined (d) 33

SOLUTION

F(-1) means to put x as -1

But the function was only defined on the interval $2 \le x \le 8$ [meaning numbers between 2 and 8 where 2 and 8 is inclusive]

Hence $F(-1)$ = undefined
= Does not exist (D.N.E)

EXAMPLE 4: 2008/2009 EXAM QUESTION 14

14. If a function is defined by $f(x-3) = 7 + 2x - \left(\frac{1}{3}\right)x^3$. What is the value of $f(0)$? (a) 7 (b) 10 (c) -8 (d) 4

SOLUTION

$F(x-3) = 7 + 2x - \frac{1}{3}x^3$

F(0) means to put x as 0 but if we put x = 0, what we will achieve is F(-3).

We equate, $x - 3 = 0$
$x = 3$

putting x = 3 will yield F(0)

$\Rightarrow F(3-3) = 7 + 2(3) - \frac{1}{3}(3)^3$

$F(0) = 7 + 6 - \frac{27}{3}$
$= 7 + 6 - 9$
$= 4$

EXAMPLE 5: 2009/2010 EXAM QUESTION 56

56. Given that $f(x) = \begin{cases} x-3, x \ge 3 \\ 3-x, x < 3 \end{cases}$ $find\ f(-1)\}$ (a) 4 (b) -4 (c) 0 (d) 8

SOLUTION

The function F(x) is a piecewise defined function with a breaking point 3.

F(-1) means to put x = -1, but we have two branches, we will choose the lower branch where F(x) = 3 - x, because -1 is a number less than 3

$F(-1) = 3 - (-1)$
$= 4$

EXAMPLE 6: 2009/2010 EXAM QUESTION 22

22. Evaluate $f(t,z) = 2e^t + 7e^z + 2z - 1$ where $t = 2, z = 3$ (a) 21 (b) $2e^{et} + 3e^z$ (c) $2e^2 + 7e^3 + 5$ (d) $9e^2 + 5$

SOLUTION

$F(t, z) = 2e^t + 7e^z + 2z - 1$

putting t = 2 and z = 3

$\Rightarrow F(2, 3) = 2e^2 + 7e^3 + 2(3) - 1$
$= 2e^2 + 7e^3 + 5$

EXAMPLE 7: 2010/2011 EXAM QUESTION 62

62. If $f(x) = \int_2^x (y-5)dy$. Find $f(4)$: (a) -4 (b) 4 (c) 2 (d) 20

SOLUTION

F(4) means, putting x = 4.

$F(4) = \int_2^4 (y-5)\,dy$

but $\int_2^4 (y-5)dy$ is a definite integral.

$F(4) = \left.\frac{y^2}{2} - 5y\right|_2^4$

$= \left[\frac{(4)^2}{2} - 5(4)\right] - \left[\frac{(2)^2}{2} - 5(2)\right]$

$= (8 - 20) - (2 - 10)$
$= -12 - (-8)$
$= -4$

For a better understanding to the above solution, check the lecture note on definite integral.

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EXAMPLE 8. 2010/2011 EXAM QUESTION 1

1. If $f(x) = x^3 - 2x^2 - 3x + 8$, find $f(a+2) - f(a)$
(a) $2a^3 + a^2 + 4a - 6$ (b) $2a^3 + 2a^2 - 2a + 6$ (c) $6a^2 + 4a - 6$ (d) $6a^2 - 4a + 6$

SOLUTION

$F(a+2)$ means to put $x = a+2$
$F(a)$ means to put $x = a$

$F(a+2) = (a+2)^3 - 2(a+2)^2 - 3(a+2) + 8$
$= (a^3 + 6a^2 + 12a + 8) - (a^2 + 4a + 4) - (3a + 6) + 8$
$= a^3 + 4a^2 + a + 2$

$F(a) = a^3 - 2a^2 - 3a + 8$

Now, $F(a+2) - F(a) = a^3 + 4a^2 + a + 2 - a^3 + 2a^2 + 3a + 8$
$= 6a^2 + 4a - 6$

EXAMPLE 9: 2011/2012 TEST QUESTION 22

22. If $f(x) = x^2 - 3x - 8$ and $f(k) = 2$ then $k =$
(a) (-2, 5) (b) (2, -5) (c) (-2, -5) (d) (-3, -8)

SOLUTION

$F(x) = x^2 - 3x - 8$
$F(k)$ means putting $x = k$
$\Rightarrow F(k) = k^2 - 3k - 8$
but $F(k) = 2$, we have
$k^2 - 3k - 8 = 2$
$k^2 - 3k - 8 - 2 = 0$
$k^2 - 3k - 10 = 0$
Factorizing
$k^2 - 5k + 2k - 10 = 0$
$k(k-5) + 2(k-5) = 0$
$(k-5)(k+2) = 0$
$k = 5$ or $k = -2$

EXAMPLE 10: 2014/2015 EXAM QUESTION 35

35. If $f(x) = 3^{x-1}$ find $f(x+3) - f(x)$ in terms of $f(x)$ (a) $\frac{15}{2}.f(x)$ (b) $\frac{13}{2}.f(x)$ (c) $26.f(x)$ (d) $13.f(x)$ (e) $15.f(x)$

SOLUTION

$F(x) = 3^{x-1}$
$F(x+3)$ means to put x in $F(x)$ as $x+3$
$F(x+3) = 3^{(x+3)-1} = 3^{x+2} = 3^{(x-1)+3}$
Now, $F(x+3) - F(x) = 3^{(x-1)+3} - 3^{x-1}$
Factorizing 3^{x-1}, we have
$F(x+3) - F(x) = 3^{x-1}(3^3 - 1)$
$= 3^{x-1}(27 - 1)$
$= 26(3^{x-1})$
$= 26F(x)$

EXAMPLE 11: 2008/2009 EXAM QUESTION 17

17. If $Z(t) = 1\sqrt{t-2}$, then find $Z(b+2)$ (a) $1\sqrt{b}$ (b) $\sqrt{b}$ (c) $\frac{1}{b}$ (d) b

SOLUTION

$Z(b+2)$ means to put t as $b+2$

$Z(b+2) = \frac{1}{\sqrt{b+2-2}} = \frac{1}{\sqrt{b}}$

EXAMPLE 12: 2013/2014 EXAM QUESTION 20

20. Given $f(x) = x^2$ and $b = 2$, find the value of $\frac{f(x)-f(b)}{x-b}$ (a) $x+3$ (b) $x-2$ (c) $x-4$ (d) $x+4$ (e) $x+2$

SOLUTION

$F(x) = x^2$. $F(b)$ means $F(2)$
$F(2) = 2^2 = 4$

Therefore, $\frac{F(x) - F(b)}{x-b} = \frac{x^2 - 4}{x-2}$
$= \frac{(x+2)(x-2)}{x-2}$ Difference of two squares
$= x+2$

EXAMPLE 13: 2014/2015 EXAM QUESTION 1

1. Let x be the input and y the output, which of the equations does not represent a function? (a) $y = x + 4$ (b) $y - x^2 = 0$ (c) $y = 4x - 3$ (d) $y^2 - x = 0$ (e) $-3x^2 + y = 4$

SOLUTION

Option A, B, C and E represent a functions because the are/can be expressed explicitly in terms of y.
But option D; $y^2 - x = 0$
$y^2 = x$
$y = \pm\sqrt{x}$
Every function should have a unique image
Hence D not a function.

TYPES OF FUNCTION

GREATEST INTEGER FUNCTION

This is the greatest integer, which is less or equal to x given by $F(x) = [x]$

EXAMPLE 1:

Given that f(x) = (x), find f(2.7)

SOLUTION

2·7 is a number between two integers 2 and 3, but 2 is less than 3
Hence $F(2\cdot7) = [2\cdot7] = 2$

EXAMPLE 2:

Evaluate (-4.8)

SOLUTION

$-4\cdot8$ is a number between two integers -4 and -5, but -5 is less than -4, Hence $[-4\cdot8] = -5$

EXAMPLE 3: 2011/2012 EXAM QUESTION 10

10. The greatest integer function is given by $f(x) = [x]$. Find $[x]$ for $x = -7.5$ (a) 8 (b) -7 (c) -8 (d) 7

SOLUTION

7·5 is a number between two integers 7 and 8 but 7 is less than 8, Hence $[7\cdot5] = 7$

TRANSCEDENTAL FUNCTION

This is a function which cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.
Examples include:
Exponential function, logarithm... Trignometric fxn

EXAMPLE 1: 2009/2010 EXAM QUESTION 29

29. Which of the following is not a transcendental function? (a) $2sinx - x = 0$ (b) $\log x - 1 = 0$ (c) $e^x sinx - \frac{1}{2x} = 0$ (d) $3x^2 + 3x - 3 = 0$

SOLUTION

Option A; $2\sin x - x = 0$ Contains a sinx which is a Trignometric fxn, it is trancendental

Option B; $\log x - 1 = 0$ Contains a logarithm function, Hence it is tranlendental

Option C; $e^x \sin x - \frac{1}{2x} = 0$, This fxn contains exponential and Trignometric fxn, hence it is Tranlendental.

Option D; $3x^2 + 3x - 3 = 0$. This fxn is expressed in terms of finite sequence of the algebraic operation of addition and subtraction
Option D ($3x^2 + 3x - 3 = 0$) is not trancendental

UNLEASH believe you can solve 2010/2011 NO 11

POLYNOMIAL FUNCTION

Given a function $F(x) = a_1x^n + a_2x^{n-1} + \cdots + a_{n-1}x + a_n$
Where a_1, a_2, a_{n-1} and a_n are arbitrary constant

The function is said to be a polynomial in x. Where n is a non-negative integer.

Examples of polynomial;

$f(x) = 2x + 7$; Degree '1' (linear)

$g(x) = 4x^2 + 3x - 8$; Degree '2' (Quadratic)

$h(x) = x^3 + 3x^2 + 5x + 10$; Degree '3' (cubic)

$Q(x) = 5$; Degree '0' (Constant)

Note: A polynomial function is defined at all point in the real line, hence it is continous everywhere.

RATIONAL FUNCTION

Given that $F(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions in x and $q(x) \neq 0$; Example; $F(x) = \frac{2x+1}{3x+5}$

ODD FUNCTION

A function is said to be odd if it satisfies the definition that $F(x) = -F(-x)$ for all x in the domain of F.

EXAMPLE 1:

Sin x is an odd function

SOLUTION

Observe; $\sin(30) = 0.5$

$-\sin(-30) = -(-0.5)$

$-\sin(-30) = 0.5$

Since $F(x) = \sin x$ is equal to $-F(-x) = -\sin(-x)$
Sinx is an odd function

Since F(x) is an odd fxn Check 2014/2015 Exam no 18 -- Answer it?

EXAMPLE 2: 2009/2010 EXAM QUESTION 2

2. Two functions f and g such that $g(f(x)) = x$ and $f(g(y)) = y$ are said to be (a) inverse functions (b) composite functions (c) opposite functions (d) one to one function

SOLUTION

Option A, B and D are even functions

let $F(x) = x^3 + 8x$

$F(-x) = (-x)^3 + 8(-x)$

$F(-x) = -x^3 - 8x$

Now, $-F(x) = -(-x^3 - 8x)$ $= x^3 + 8x$; Since $F(x) = -F(-x)$; option C is odd

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A function is said to be even if it satisfy that;

$f(x) = f(-x)$

EXAMPLE 1:

Cos x = Cos (-x)

SOLUTION

Cos x is even

EXAMPLE 2:

$F(x) = x^2 + 4$

SOLUTION

$F(-x) = (-x)^2 + 4$

$= x^2 + 4$

Since $F(x) = F(-x)$; $F(x)$ is even.

EXAMPLE 3: 2009/2010 EXAM QUESTION 24

24. If a function $f(x)$ is such that $f(-x) = f(x)$, then it is said to be (a) even function of x (b) monotone function x (c) composite function of x (d) continuous function of x

SOLUTION

If $F(-x) = F(x)$, then the function is even.

ABSOLUTE VALUE FUNCTION

This function ensures non-negativity of numbers, it is defined by the piecewise function

$$F(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

denoted by $f(x) = |x|$

Example: $|-3.5| = 3.5$

COMPOSITION OF A FUNCTION

Given two functions $F: A \to B$ and $g: B \to C$, then $A \xrightarrow{f} B \xrightarrow{g} C$ is called the composition of function f and g, which is denoted by $g[F(x)]$

Note: $g \circ F$ means $g \circ F(x) = g[F(x)]$

$F \circ g$ means $F \circ g(x) = F[g(x)]$

EXAMPLE 1: 2008/2009 EXAM QUESTION 13

13. Let f, g be two functions given by $f = x^2 + 3, g(x) = 2x + 1$. What is the value of $(f o g)(2)$? (a) 19 (b) 38 (c) 28 (d) 10

SOLUTION

$F(x) = x^2 + 3$, $g(x) = 2x + 1$

$f \circ g: [g(x)]^2 + 3 \Rightarrow (2x+1)^2 + 3$

Now $(f \circ g)(2)$ means to put x as 2

Hence $(f \circ g)(2) = [2(2)+1]^2 + 3 = 25 + 3 = 28$

EXAMPLE 2: 2009/2010 EXAM QUESTION 49

49. Given that $g(x) = x + 1$ and $f(x) = \sqrt{x - 1}$, what is $f(g(x))$? (a) $x^{\frac{1}{2}}$ (b) $-\sqrt{x}$ (c) $x - 1$ (d) $x + 1$

SOLUTION

$g(x) = x + 1$

$F(x) = \sqrt{x - 1}$

$F[g(x)]$ implies putting x as $g(x)$ in $F(x)$.

$\Rightarrow \sqrt{g(x) - 1}$

$\therefore F[g(x)] = \sqrt{(x+1) - 1}$

$= \sqrt{x + 1 - 1}$

$= \sqrt{x} = x^{1/2}$.

EXAMPLE 3: 2009/2010 EXAM QUESTION 31

31. If $f(x) = x^2 - 1$ and $g(x) = 2x + 3$, then $f(g(x)) = 0$ is (a) 2 or -1 (b) -2 or 1 (c) -2 or -1 (d) 2 or 1

SOLUTION

$F(x) = x^2 - 1$

$g(x) = 2x + 3$

$F[g(x)]$ means to put x as $g(x)$ in $F(x)$

We have: $[g(x)]^2 - 1$

$F[g(x)] = [2x+3]^2 - 1$

$= (4x^2 + 12x + 9) - 1$

$= 4x^2 + 12x + 8$

at $F[g(x)] = 0$

$\Rightarrow 4x^2 + 12x + 8 = 0$

Dividing through by 4, we have

$x^2 + 3x + 2 = 0$

Factorizing,

$(x+1)(x+2) = 0$

$x = -1$ or $x = -2$.

EXAMPLE 4: 2009/2010 EXAM QUESTION 4

4. If $f(x) = x^2 + 3$ and $g(x) = 2x + 1$, find $f(g(x))$ (a) $8x + 4$ (b) $(2x + 1)^2$ (c) $8x^2 + 4x + 4$ (d) $4x^2 + 4x + 4$

SOLUTION

$F(x) = x^2 + 3$

$g(x) = 2x + 1$

F[g(x)] means to put x as g(x) in F(x)

$F[g(x)] = [g(x)]^2 + 3$

$= [2x+1]^2 + 3.$

$= (2x+1)(2x+1) + 3$

$= 4x^2 + 2x + 2x + 1 + 3$

$= 4x^2 + 4x + 4$

EXAMPLE 5: 2010/2011 EXAM QUESTION 47

47. If $g(x) = x^2$ and $f(x) = x^2 - 7$, find $g(f(2))$
(a) 25 (b) -25 (c) 2.5 (d) -5

SOLUTION

$g(x) = x^2,\ F(x) = x^2 - 7$

$F(2) = 2^2 - 7 = -3$

$g[F(2)] = g(-3) = (-3)^2 = 9$

EXAMPLE 6: 2011/2012 EXAM QUESTION 37

37. let f, g be two functions given by $f(x) = 2x$ and $g(x) = 4x^2 + 3$, what is the value of and $(f o g)(2)$ (a) 19 (b) 28 (c) 38 (d) 10

SOLUTION

$F(x) = 2x,\ g(x) = 4x^2 + 3$

$(F \circ g)(2)$ means $F[g(2)]$

$g(2) = 4(2)^2 + 3 = 4(4) + 3 = 19$

$F[g(2)] = F(19) = 2(19) = \underline{38}$

EXAMPLE 7: 2011/2012 TEST QUESTION 2

2. If $f(x) = x + 1$ and $g(x) = x^3 + 2x + 1$ then gof is (a) $x^2 + 2x + 2$ (b) $x^3 + 3x + 1$ (c) $x^3 + 3x^2 + 5x + 4$ (d) $x^3 + 2x^2 + 5x + 4$

SOLUTION

$F(x) = x+1,\ g(x) = x^3 + 2x + 1$

$g \circ F$ means $g[F(x)]$

$g[F(x)]$ means to put x as F(x) in g(x).

$\Rightarrow g[F(x)] = [F(x)]^3 + 2[F(x)] + 1$

$= (x+1)^3 + 2(x+1) + 1$

$= (x+1)(x+1)(x+1) + 2(x+1) + 1$

$= x^3 + x^2 + 2x^2 + 2x + x + 1 + 2x + 2 + 1$

$= x^3 + 3x^3 + 5x + 4$

EXAMPLE 8: 2012/2013 EXAM QUESTION 62

62. Given that $f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$. $g[f(x)]$ is (a) $6x^2 + 3x - 3$ (b) $3x^2 - 6x - 13$ (c) $3x^2 + 6x - 13$ (d) $3x^2 + 6x - 5$

SOLUTION

$f(x) = x^2 + 2x - 3 \quad g(x) = 3x - 4$

g[F(x)] means, putting x as f(x) in g(x)

$\Rightarrow g[F(x)] = 3[F(x)] - 4$ — Now we substitute the given equivalence of f(x)

$= 3[x^2 + 2x - 3] - 4$

$= 3x^2 + 6x - 9 - 4$

$= 3x^2 + 6x - 13$

EXAMPLE 9: 2012/2013 EXAM QUESTION 63

63. Given that $f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$. $f[g(x)]$ is (a) $9x^2 - 24x + 16$ (b) $3x^2 + 6x + 16$ (c) $9x^2 - 18x + 5$ (d) $9x^2 + 18x - 5$

SOLUTION

$F(x) = x^2 + 2x - 3,\ g(x) = 3x - 4$

F[g(x)] means, putting x as g(x) in F(x)

$\Rightarrow F[g(x)] = [g(x)]^2 + 2[g(x)] - 3$ — We substitute the value of g(x)

$= (3x-4)^2 + 2(3x-4) - 3$

$= 9x^2 - 24x + 16 + 6x - 8 - 3$

$= 9x^2 - 18x + 5$

EXAMPLE 10:

Ex. If $f: [R \to \mathfrak{R}$ is given by $f(x) = 3x - 4$. Then, find $(f o f^{-1})(x)$

SOLUTION

$F(x) = 3x - 4,\ F^{-1}(x) = \frac{x+4}{3}$ — Check inverse fxn for better understanding

$F \circ F^{-1}$ means $F[F^{-1}(x)]$

$\Rightarrow$ putting x as $F^{-1}(x)$ in F(x)

We have; $F[F^{-1}(x)] = 3[F^{-1}(x)] - 4$

$= 3\left(\frac{x+4}{3}\right) - 4$

$= (x+4) - 4$

$= x$

EXAMPLE 11: 2012/2013 TEST QUESTION 19

19. If $f(x) = 2x + 3$ and $g(x) = x^2$. Find $g[f(x)]$ (a) $2x + 3$ (b) $2x^2 + 3$ (c) $2x - 3$ (d) $4x^2 + 12x + 9$

SOLUTION

$F(x) = 2x + 3,\ g(x) = x^2$

g[F(x)] means, putting x as f(x) in g(x)

$g[F(x)] = [F(x)]^2$

$= (2x+3)^2$

$= (2x+3)(2x+3)$

$= 4x^2 + 6x + 6x + 9$

$= 4x^2 + 12x + 9$

EXAMPLE 12: 2012/2013 TEST QUESTION 13

13. If $f(x) = x^2 - x$ and $g(x) = f(x+1) - f(-x)$ then $g(x) =$ (a) $3x$ (b) 0 (c) $2x$ (d) 1

SOLUTION

$F(x) = x^2 - x$, $F(-x) = (-x)^2 - (-x) = x^2 + x$

$g(x) = F(x+1) - F(-x)$

$F(x+1)$ mean putting x as $x+1$ in $f(x)$

$F(x+1) = (x+1)^2 - (x+1)$

$= x^2 + 2x + 1 - x - 1$

$= x^2 + x$

$g(x) = F(x+1) - F(-x)$

$= (x^2+x) - (x^2+x) = 0$

EXAMPLE 13: 2012/2013 TEST QUESTION 2

2. Let $f(x) = x^2 + 1$ and $g(x) = x + 1$. Find $(fog)(a^2)$ (a) $(a+2)^3$ (b) $a^4 + a^2 - 2$ (c) $a^4 + 2a^2 + 2$ (d) $a^2 + 2$

SOLUTION

Fog means $F[g(x)]$

$(Fog)(a^2)$ mean $F[g(x)]$ where x becomes a^2.

$F[g(x)] = [g(x)]^2 + 1$

$= (x+1)^2 + 1$

$= (x^2 + 2x + 1) + 1$

$= x^2 + 2x + 2$

$F[g(x)](a^2) = a^2 + 2a + 2$

EXAMPLE 14: 2013/2014 EXAM QUESTION 51

51. Let $f(x) = 2x + 1$. Find the function $g(x)$ such that $(fog)(x) = x^3$ (a) $\frac{1}{2}(3^x + 1)$ (b) $\frac{1}{2}(3^x - 1)$ (c) $x^3 + 1$ (d) $x^3 - 1$ (e) $\frac{1}{2}x^3$

SOLUTION

$F(x) = 2x + 1$,

$F[g(x)] = 2[g(x)] + 1$

but $F[g(x)] = x^3$

$\Rightarrow 2[g(x)] + 1 = x^3$

$2[g(x)] = x^3 - 1$

$g(x) = \frac{x^3 - 1}{2} = \frac{1}{2}(x^3 - 1)$

$F[g(x)]$ means to put x as $g(x)$ in $F(x)$

EXAMPLE 15: 2013/2014 EXAM QUESTION 29

29. Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$, find $(fog)(5)$ (a) 29512 (b) 80 (c) -99 (d) 99 (e) 24

SOLUTION

$(Fog)(5)$ means $F[g(5)]$

$g(5) = 1 - 20(5)$

$= 1 - 100 = -99$

$F[g(5)] = F(-99)$

$= 3(-99)^2 - (-99) + 10$

$= 3(9801) + 99 + 10$

$= 29403 + 99 + 10$

$= 29512$

EXAMPLE 16: 2013/2014 EXAM QUESTION 14

14. Let $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, find $(gof)(x)$ (a) $2x^2 - 3$ (b) $4x^2 + 4x - 1$ (c) $2x^3 + x^2 - 4x - 2$ (d) $3x + 2$ (e) $x^2 - 2x - 1$

SOLUTION

gof means $g[F(x)]$

$g[F(x)]$ implies putting x as $F(x)$ in $g(x)$

$\Rightarrow g[F(x)] = [F(x)]^2 - 2$

$= (2x+1)^2 - 2$

$= (4x^2 + 4x + 1) - 2$

$= 4x^2 + 4x - 1$

EXAMPLE 17: 2013/2014 EXAM QUESTION 9

9. If $f(x) = 2x$ and $g(x) = 4x^2 + 3$, find $(fog)(2)$ (a) 38 (b) 64 (c) 16 (d) 32 (e) 67

SOLUTION

$F(x) = 2x$, $g(x) = 4x^2 + 3$

$(Fog)(2)$ means $F[g(2)]$

but $g(2) = 4(2)^2 + 3$

$= 4(4) + 3$

$= 19$

$(Fog)(2) = F[g(2)] = F(19)$

$= 2(19)$

$= 38$

EXAMPLE 18:2014/2015 EXAM QUESTION 41

41. Given $f(x) = 2x - 5$ and $g(x) = 3x + 1$. Find $(gof)(3)$ (a) -2 (b) 2 (c) -3 (d) 3 (e) 4

SOLUTION

$(goF)(3)$ means $g[F(3)]$

Try to Solve Similar question on 2015/2016 Question No 40

$F(3) = 2(3) - 5$

$= 6 - 5 = 1$

$(goF)(3) = g[F(3)]$

$= g(1)$

$= 3(1) + 1$

$= 3 + 1 = 4$

INVERSE FUNCTION

An inverse function is a function that "reverses" another function: If the function f applied to an input x gives an output of y, then applying its inverse function g to y gives the output x, and vice versa. That is if $f(x) = y$, then $g(y) = x$ is the inverse function.
The inverse function of $f(x)$ is denoted by $f^{-1}(x)$.

HOW TO OBTAIN INVERSE

STEP I: let $y = f(x)$
STEP II: Make x the subject formular
STEP III: Interchange y with x and x with $f^{-1}(x)$.

EXAMPLE 1: 2009/2010 EXAM QUESTION 54

54. Let $g(x) = \frac{2x-3}{x+5}, x \in R, x \neq 5$ find $g^{-1}(x)$ (a) $\frac{5x+3}{2-x}$ (b) $\frac{2-x}{5x+3}$ (c) $\frac{-2-x}{5x+3}$ (d) $\frac{5x-3}{2+x}$

SOLUTION

let $y = g(x)$

$\Rightarrow \quad y = \frac{2x-3}{x+5}$

Cross multiplying as to make x subject formula;

$\Rightarrow y(x+5) = 2x-3$

$xy + 5y = 2x-3$

$xy - 2x = -3-5y$

$x(y-2) = -(3+5y)$

$x = \frac{-(3+5y)}{y-2}$

Interchanging y with x and x with $g^{-1}(x)$

$\Rightarrow g^{-1}(x) = \frac{-(3+5x)}{x-2}$

$= \frac{5x+3}{2-x}$

EXAMPLE 2: 2009/2010 EXAM QUESTION 3

3. The inverse function $f(x) = 2x+1$ is (a) $g(y) = \sqrt{y+1}$ (b) $g(y) = \frac{y-1}{2}$ (c) $g(y) = \frac{2x+1}{2}$ (d) $g(y) = \sqrt{x+1}$

SOLUTION

$f(x) = 2x+1$

let $y = f(x)$

$\Rightarrow y = 2x+1$

making x the subject formula

$y-1 = 2x$

$\frac{y-1}{2} = x$

$x = \frac{y-1}{2}$

Since, the options are functions of y then we will not interchange y with x

We have; $g(y) = \frac{y-1}{2}$

EXAMPLE 3: 2009/2010 EXAM QUESTION 2

2. Two functions f and g such that $g(f(x)) = x$ and $f(g(y)) = y$ are said to be (a) inverse functions (b) composite functions (c) opposite functions (d) one to one function

SOLUTION

If $g[f(x)] = x$ and $f[g(y)] = y$

The functions are inverse of each other.

EXAMPLE 4: 2010/2011 EXAM QUESTION 28

28. Find the inverse of the function $g(x) = \frac{3x+2}{x-3}$ (a) $\frac{3y+2}{y-3}$ (b) $\frac{3x+2}{x-3}$ (c) $\frac{3y+2}{x-3}$ (d) $\frac{3x+2}{y-3}$

SOLUTION

let $y = g(x)$

$\Rightarrow y = \frac{3x+2}{x-3}$

making x the subject formula

$y(x-3) = 3x+2$

$xy - 3y = 3x+2.$

$xy - 3x = 2+3y.$

$x(y-3) = 2+3y$

$x = \frac{3y+2}{y-3}$

Interchanging y with x and x with $g^{-1}(x)$

$\Rightarrow g^{-1}(x) = \frac{3x+2}{x-2}$

EXAMPLE 5: 2010/2011 EXAM QUESTION 39

39. Find f^{-1} if $f(x) = \frac{3x-1}{4}$ (a) $\frac{4x-1}{3}$ (b) $\frac{4x+1}{3}$ (c) $\frac{3x-1}{4}$ (d) $\frac{3x+1}{4}$

SOLUTION

let $y = F(x)$

$\Rightarrow y = \frac{3x-1}{4}$

making x the subject, we cross multiply

$4y = 3x-1$

$4y+1 = 3x$

$\frac{4y+1}{3} = x$

$x = \frac{4y+1}{3}$

Interchanging y with x and x with F^{-1}

$\Rightarrow F^{-1} = \frac{4x+1}{3}$

EXAMPLE 6: 2011/2012 EXAM QUESTION 56

56. The inverse of x^2+1 is (a) x^2-1 (b) $\sqrt{x-1}$ (c) $\pm\sqrt{x-1}$ (d) D.N.E

SOLUTION

let $y = F(x)$ and $f(x) = x^2+1$

$\Rightarrow y = x^2+1$

making x the subject formula

$y-1 = x^2$ which means

also that $x^2 = y-1$,

taking square root of both sides,

$x = \pm\sqrt{y-1}$

Interchangin y with x and x with $F^{-1}(x)$

$\Rightarrow F^{-1}(x) = \pm\sqrt{x-1}$

Try to Solve Similar Question on 2015/2016 Q 64

EXAMPLE 7: 2012/2013 TEST QUESTION 29

29. If $f(x) = 7x-5,\ x \in R,\ f(x)^{-1}$ is (a) $\frac{1}{7x-5}$ (b) $\frac{1}{7x+5}$ (c) $\frac{x+5}{7}$ (d) $\frac{x+7}{5}$

SOLUTION

let $y = F(x)$, where $f(x) = 7x-5$

$\Rightarrow y = 7x-5$; making x the subject

$y+5 = 7x$

Dividing both sides by 7

$\frac{y+5}{7} = x$; $x = \frac{y+5}{7}$

Interchanging y with x and x with $F^{-1}(x)$

$\Rightarrow F^{-1}(x) = \frac{x+5}{7}$

EXAMPLE 8: 2013/2014 EXAM QUESTION 21

21. If $f(x) = \frac{x+4}{2x}$, find f^{-1} (a) 4 (b) 3 (c) 2 (d) 1 (e)

SOLUTION

let $y = F(x)$

$\Rightarrow y = \frac{x+4}{2x}$

Now, we cross multiply to make x the subject

$y(2x) = x+4$

$2xy - x = 4$

$x(2y-1) = 4$

$x = \frac{4}{2y-1}$

Interchanging y with x and x with F^{-1}

$\Rightarrow F^{-1} = \frac{4}{2x-1}$

$F^{-1}(1)$ means to put $x=1$

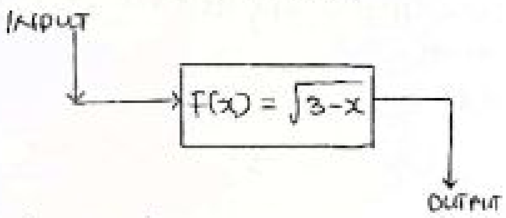
$F^{-1}(1) = \frac{4}{2(1)-1} = \frac{4}{1} = 4$

DOMAIN OF A FUNCTION

Given a function $F: A \rightarrow B$, then the set A which is the points for which the function is defined is called the domain of F, denoted by Dom F.

Hence, the set of all values of x the independent variable for which $y = f(x)$ is defined is known as domain of $f(x)$.

Consider the figure below:



From the above figure, if we input "2" our output will be $\sqrt{1} = 1$ again, if we input 4, our output will be $\sqrt{-1}$ = undefined for all real numbers. Hence, our input "4" is not included in the domain of $F(x)$.

let's consider different types of function and study it's domain. In this Elementary Calculus we shall consider the following functions:

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DOMAIN OF A POLYNOMIAL FUNCTION

The Domain of a polynomial is all real numbers ($\mathbb{R}$) because a polynomial function is defined at all point in the real line.

EXAMPLE 1: 2010/2011 EXAM QUESTION 51

51. If $f(x) = 3x^2 - 5x + 4$, find the domain of $f'(x)$. (a) $[0,8]$ (b) $\Re$ (c) $(-\infty, 0]$ (d) N

SOLUTION

If $F(x) = 3x^2 - 5x + 4$

$F'(x) = 6x - 5$

Dom $F = \mathbb{R}$ (All real numbers)

The Function is a polynomial function $F'(x)$ is the derivative of $F(x)$ which will also yield a polynomial. Hence Dom $F = (-\infty, \infty)$

Check the topic Derivatives to understand how to obtain $F'(x)$ from $F(x)$

DOMAIN OF AN ABSOLUTE VALUE FUNCTION

Given a function $F(x) = |x|$ the Domain of $F(x)$, Dom $F = \mathbb{R}$ All real numbers; i.e $(-\infty, \infty)$

EXAMPLE 1: 2013/2014 EXAM QUESTION 28

28. Find the domain and range of the function $f(z) = |z-6| - 3$ A. R, R^+ B. R; $[-3, \infty)$ C. R; R D. $\left(-\infty, \frac{4}{7}\right); \left[0, \frac{4}{7}\right)$ E. R^+; R

SOLUTION

Domain of $F(z) = |z-6| - 3$ is All real numbers

Dom $F = \mathbb{R}$

$= (-\infty, \infty)$

Absolute value function is defined at all points in the real line

DOMAIN OF A LOGARITHMIC FUNCTION

Given $F(x) = \log x$, then Domain of $F(x)$ Dom $F = \{x : x > 0, x \in \mathbb{R}\}$

EXAMPLE 1: 2013/2014 EXAM QUESTION 25

25. Find $Dom(f)$ if $f(x) = \ln(7x^2 - x^2 - 10)$ A. R B. $x \geq 0$ C. $2 \leq x < 5$ D. $2 \leq x \leq 5$ E. $2 < x < 5$

SOLUTION

$F(x) = \ln(7x - x^2 - 10)$

$F(x)$ will be defined if $(7x - x^2 - 10) > 0$

factorizing the quadratic

$(x-2)(x-5) < 0$

$f(x)$ is a logarithmic function. Hence if $7x - x^2 - 10 > 0$, then it is defined

Truth table

	$x < 2$	$2 < x < 5$	$x > 5$
$x-2$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
Product	$+$	$-$	$+$

The Solution Set less than zero is our Dom $f = 2 < x < 5$

DOMAIN OF TRIGONOMETRY

EXAMPLE 1: 2011/2012 TEST QUESTION 12

12. If $(x) = \sin x$, *then the domain and range is* (a) R,R (b) R^+, R^+ (c) R-{0} (d) R,{-1,1}

SOLUTION

$F(x) = \sin x$

Dom $F = \mathbb{R}$

$= (-\infty, \infty)$

For every real number we input into the sine fxn we must have an answer (output), that is a real number. Since, there is no restriction in the input ∴ Dom $F = \mathbb{R}$

EXAMPLE 2:

Ex. If $f(x) = Tanx$, *find Dom F.*

SOLUTION

$F(x) = \tan x$

$\Rightarrow F(x) = \dfrac{\sin x}{\cos x}$

$F(x)$ is defined if $\cos x \neq 0$

Dom $F = \mathbb{R} - \left\{\left(\frac{2n+1}{2}\right)\pi : n \in \mathbb{Z}\right\}$

Tanx is a function when splited becomes rational. Hence the denominator is not allowed to be zero. Values of x that will make $\cos x = 0$ includes 90, 270, 540 -- $\left(\frac{2n+1}{2}\right)\pi$

Domain of $\cos x$ is same as Domain of $\sin x$ which is $\mathbb{R}$

DOMAIN OF RATIONAL FUNCTION

EXAMPLE 1: 2008/2009 EXAM QUESTION 47

47. Consider the function $y = \frac{x^2-1}{x-1}$, the function is defined for all values of x except (a) $x = 1$ (b) $x = 0$ (c) $x = -1$ (d) $x = 2$

SOLUTION

$y = \dfrac{x^2 - 1}{x - 1}$

Equating denominator to zero,

$\Rightarrow x - 1 = 0$

$x = 1$

Hence at $x = 1$ the function it undefined

This function is a rational function, hence it will be defined, if the denominator is not equal to zero. 'Question' what will make the denominator zero? find it and subtract from real numbers.

Note: Dom $y = \mathbb{R} - \{1\}$

EXAMPLE 2: 2009/2010 EXAM QUESTION 50

50. For what values of x is the function $\frac{2x}{x^2-3x+2}$ defined?
(a) $R-\{1,2\}$ (b) $x>2$ (c) $x<1$ (d) $x>-\infty$

SOLUTION

let $f(x) = \frac{2x}{x^2+3x+2}$

f(x) is defined if

$x^2+3x+2 \neq 0$

Factorising: $(x+1)(x+2)\neq 0$

$x+1\neq 0$ or $x+2\neq 0$

$x\neq 1$ or $x\neq 2$

The function is a rational function and hence, it will be defined if the denominator is not equal to zero.
Dom $f = \mathbb{R}-\{1,2\}$

EXAMPLE 3: 2012/2013 EXAM QUESTION 44

44. The domain of $\frac{x^3+3}{(x-1)(x+5)(x+2)}$ *is A. R B. R −* $\{0,1\}$ *C. R −* $\{0,2\}$ *D. R −* $\{-5,-2,1\}$

SOLUTION

$\frac{x^3+3}{(x-1)(x+5)(x+2)}$ will be defined

if $(x-1)(x+5)(x+2)\neq 0$

$\Rightarrow x-1\neq 0$ or $x+5\neq 0$ or $x+2\neq 0$, we have $x\neq 1$ or $x\neq -5$, $x\neq -$

Dom $F = \mathbb{R}-\{-5,-2,1\}$

Domain of rational function is given by $\mathbb{R}-\{$what will make the denominator zero$\}$
Which is $\mathbb{R}-\{-5,-2,1\}$

EXAMPLE 4: 2012 / 2013 TEST QUESTION 18

18. Which of the following is the domain and range of the function; $f(x)=\frac{2}{5x+6}$

$(a) Dom(f) = R, Ran(f) = R - \{0\}$ $(b) Dom(f) = R - I\{-6\}, Ran(f) = R$ $(c) Dom(f) = R - \{-\frac{6}{5}\}, = Ran(f) = R - \{0\}$

(d) None of the above

SOLUTION

$f(x) = \frac{2}{5x+6}$

f(x) will be defined if $5x+6\neq 0$

$5x \neq -6$

$x \neq -\frac{6}{5}$

Dom $F = \mathbb{R}-\{-\frac{6}{5}\}$

Domain of a fraction is $\mathbb{R}-\{$what will make the under $=0\}$
thus; $\mathbb{R}-\{-\frac{6}{5}\}$

DOMAIN OF A SQUARE ROOT FUNCTION

Given $f(x)=\sqrt{g(x)}$, then Domain of f(x)

Dom $F = \{x: g(x)\geq 0, x\in\mathbb{R}\}$

EXAMPLE 1: [illegible] EXAM QUESTION 33

33. What is the domain and range of the function $y = \sqrt{x+3}$ (a) $x\geq -3$ and $y\geq 0$ (b) $x>3$ and $y<0$ (c) $x\geq 0$ and $y\leq 3$ (d) $x<0$ and $y<3$

SOLUTION

$y=\sqrt{x+3}$

For y to be defined

$x+3\geq 0$

$x\geq -3$

Dom $y = \{x: x\geq -3, x\in\mathbb{R}\}$

y is a square root function, hence it is defined when the content of the square root is greater or equal to zero
i.e $x+3\geq 0$
Dom $y=[-3,\infty)$

EXAMPLE 2: 2013/2014 EXAM QUESTION 10

10. Find the domain and range of the function $y = \sqrt{4-x^2}$ A. (-2,2); R B. [-2,2]; R^+ C. R; R^- D. (2,4) E. [2,4]; R^+

SOLUTION

$y=\sqrt{4-x^2}$

y will be defined if

$4-x^2\geq 0$

$2^2-x^2\geq 0$

$(2+x)(2-x)\geq 0$

Dom $F=[-2,2]$

Truth table

	$x<-2$	$-2\leq x\leq 2$	$x>2$
$2+x$	−	+	+
$2-x$	+	+	−
Product	−	+	−

Dom $F=\{x: -2\leq x\leq 2, x\in\mathbb{R}\}$

DOMAIN OF COMPOSITION FUNCTION

EXAMPLE 1: 2008/2009 EXAM QUESTION 41

41. Determine the domain of $f(x)=\log\sqrt{x^2-1}$ (a) $(-\infty,-1)\cup(1,\infty)$ (b) R (c) $(-\infty,\infty)$ (d) $R-\{1\}$

SOLUTION

$F(x)=\log\sqrt{x^2-1}$

The function will be defined if $x^2-1>0$

$x^2-1^2>0$

$(x+1)(x-1)>0$

$x=-1$ or $x=1$

F(x) is a composition of logarithm and square root function. For square root, dom f is $x^2-1\geq 0$. For logarithm, dom f $\sqrt{x^2-1}>0$, Now we compute the intersection.

Truth Table

	$x<-1$	$-1<x<1$	$x>1$
$x+1$	−	+	+
$x-1$	−	−	+
Product	+	−	+

The Solution Set, which is the Domain is given as;

Dom $F=\{x: x<-1\}\cup\{x: x>1\}$

Dom $F=(-\infty,-1)\cup(1,\infty)$

The answer in the given objective is on interval notation $(-\infty,-1)\cup(1,\infty)$ learn this notation

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EXAMPLE 2: 2009/2010 EXAM QUESTION 55

55. Determine the domain of the function $\frac{x^2}{\sqrt{x^2-4}}$ (a) $R-(-2,2)$ (b) $R-[-2,2]$ (c) $R-\{0\}$ (d) $R-\{2\}$

SOLUTION

let $F(x) = \frac{x^2}{\sqrt{x^2-4}}$

$F(x)$ will be defined if $x^2-4>0$

$x^2-2^2>0$

$(x+2)(x-2)>0$

Truth table

	$x<-2$	$-2<x<2$	$x>2$
[illegible]	−	+	+
[illegible]	−	−	+
Product	+	−	+

This function is a composition of square root and a fraction (rational) function. For rational the denominator is not equal to zero i.e. $\sqrt{x^2-4}\neq 0$. For square root, the discriminant is greater or equal to zero i.e. $x^2-4\geq 0$. Computing the intersection we have: $(-\infty,-2)\cup(2,\infty)$

The solution set is given by: Dom F = $\{x: x<-2\}\cup\{x: x>2\}$

OR Dom F = $\mathbb{R}$ − {where the function is undefined}

Dom F = $\mathbb{R}-\{x: -2\leq x\leq 2\}$

$= \mathbb{R}-[-2,2]$

EXAMPLE 3: 2011/2012 TEST QUESTION 1

1. What is the domain of $\sqrt{\frac{x}{2-x}}$ (a) $[0,2]$ (b) $[R]$ (c) $[1,2]$ (d) $[0,2]$

SOLUTION

$F(x)$ will be defined if $\frac{x}{2-x}\geq 0$, where $x\neq 2$

Solving $\frac{x}{2-x}\geq 0$

$\frac{x(2-x)^2}{2-x}\geq 0(2-x)^2$

$x(2-x)\geq 0$

The function is a composition of square root and rational function. For square root: $\frac{x}{2-x}\geq 0$. For rational fxn; $2-x\neq 0$. Now, we compute their intersection

Truth Table

	$x<0$	$0<x<2$	$x>2$
x	−	+	+
$2-x$	+	+	−
Product	−	+	−

Recall that $x\neq 2$. Hence, we open "2"

Hence Dom f = $\{x: 0\leq x<2, x\in\mathbb{R}\}$

$= [0,2)$

Use the same process and solve 2012/2013 Test Q1

EXAMPLE 4: 2012/2013 EXAM QUESTION 1

1. Dom of $f(x)=\frac{1}{\sqrt{9-x^2}}$ is ______ A. (0,3) B. (-3,3] C. (-3,3) D. (0, ∞)

SOLUTION

$F(x) = \frac{1}{\sqrt{9-x^2}}$

$F(x)$ will be defined if $9-x^2>0$

$3^2-x^2>0$

$(3+x)(3-x)>0$

Truth table

	$x<-3$	$-3<x<3$	$x>3$
$3+x$	−	+	+
$3-x$	+	+	−
Product	−	+	−

The solution set, which is Dom F = $\{x: -3<x<3, x\in\mathbb{R}\}$

$=(-3,3)$

The function is a composition of rational/fraction and square root function. For square root: $9-x^2\geq 0$. For rational: denominator $\neq 0$ => $\sqrt{9-x^2}\neq 0$. Hence the intersection gives $9-x^2>0$. Solving $9-x^2>0$, Dom F = $(-3,3)$

EXAMPLE 5: 2012/2013 TEST QUESTION 4

4. Determine the domain of $f(x)=(1-2x)^{-1}-(3x+x)^{1/2}$ (a) R- $\{\frac{1}{2}\}$ (b) $(-\infty,\frac{1}{2})\cup(\frac{1}{2},3)$ (c) $(-3,\frac{1}{2})\cup(\frac{1}{2},\infty)$ (d) $(-3,\frac{1}{2})\cup(3,\infty)$

SOLUTION

$F(x) = \frac{1}{1-2x} - \sqrt{3+x}$

$F(x)$ will be defined if

$1-2x\neq 0$ and $3+x\geq 0$

$1\neq 2x$ and $x\geq -3$

$x\neq 1/2$ and $x\geq -3$

Dom F = $[-3,\frac{1}{2})\cup(1/2,\infty)$

$F(x)$ is a composition of rational and square root function; For square root; $3+x\geq 0$. For rational; $1-2x\neq 0$. Computing the intersecti... $\{3+x\geq 0\}\cap\{1-2x\neq 0\}$ gives $[-3,\frac{1}{2})\cup(1/2, ...$

EXAMPLE 6: 2014/2015 EXAM QUESTION 40

40. Find the domain and range of the fraction y = $\frac{x}{|x-3|...}$
(a) $\mathcal{R}-\{3\},\mathcal{R}$ (b) $\mathcal{R}-\{2,4\},\mathcal{R}$ (c) $\mathcal{R}-\{2\},\mathcal{R}-$ (d) $\mathcal{R}-\{0\},\mathcal{R}-\{0\}$ (e) $\mathcal{R}-\{2,4\},\mathcal{R}-\{0\}$

SOLUTION

y will be defined if $|x-3|-1\neq 0$

=> $-(x-3)-1\neq 0$

or

$(x-3)-1\neq 0$

$x\neq 4$ or $-x+2\neq 0$

$x\neq 4$ or $x\neq 2$

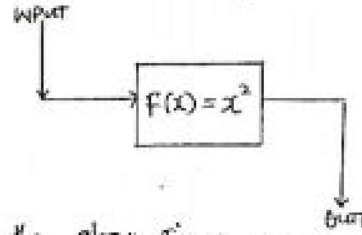
Dom y = $\mathbb{R}-\{2,4\}$

y is a composition of rational number and absolute value function. For absolute value; it is defined at all p... For rational function; $|x-3|-1\neq 0$, therefo... Dom F = $(-\infty,2)\cup(2,4)\cup...$

RANGE OF A FUNCTION

Given a function $F: A \to B$, then the Set B is called the image of the function. Hence, the range of a function is the set all possible value of the dependent variable "y", Given that $y = F(x)$. It is denoted by R_F

Consider the Figure below



From the above figure, if we input "3", our output will be "9".

Again, if we input "-2", our output will be "4" a positive number.

Conclusion: From the function $F(x) = x^2$ in the above figure, every output must be positive. Hense the range of $F(x) = x^2$ is all positive real numbers, denoted by $R_F = \mathbb{R}^+ + \{0\}$

$R_F = [0, \infty)$

The difference between this [2,5] and (2, 5) will be discussed in the LFK CLASS

EXAMPLE 1: 2010/2011 EXAM QUESTION 57

57. If $f = \{(x, y): -3 \le x \le 2, y = x^2\}$, find the range of f. (a) $[-1, 4]$ (b) $(0, 9)$ (c) $[0, 9]$ (d) $[0, 9)$

SOLUTION

$y = x^2$ For $-3 \le x \le 2$

minimum Value of $x = -3$

maximum Value of $x = 2$

If we input "-3" the output of $y = x^2$ will be "9"

again, if we input "2" the output of $y = x^2$ will be "4"

If we input "0" in $y = x^2$, the output will be "0".

Note: x is defined only on the interval of numbers between -3 and 2, where -3 and 2 is inclusive. Now, we check out for the minimum output from the our defined interval.

Hence, the range is $R_F = [0, 9]$

Range is the "[minimum, maximum]" output.

EXAMPLE 2: 2011/2012 EXAM QUESTION 33

33. What is the domain and range of the function $y = \sqrt{x+3}$ (a) $x \ge -3$ and $y \ge 0$ (b) $x > 3$ and $y < 0$ (c) $x \ge 0$ and $y \le 3$ (d) $x < 0$ and $y < 3$

SOLUTION

Range of y (R_y) is given by

$R_y = \{x: x \geq 0, x \in \mathbb{R}\}$

Range of every Square root function must have "0" as minimum output.

EXAMPLE 3: 2011/2012 TEST QUESTION 7

7. The range of the function $y = \sqrt{4 - x^2}$ is (a) $-2 \le y \le 2$ (b) $-\infty < y < \infty$ (c) $0 \le y \le 2$ (d) $0 < y < 2$

SOLUTION

$y = \sqrt{4 - x^2}$

$y = \sqrt{2^2 - x^2}$

Range of y. Given by

$R_y = [0, 2]$

$= \{x: 0 \le x \le 2, x \in \mathbb{R}\}$

The minimum output of every Square root function is zero... But for $\sqrt{4-x^2}$, we cannot have an output above "2" due to the restriction of our Domain.

EXAMPLE 4: 2011/2012 TEST QUESTION 12

12. If $(x) = \sin x$, *then the domain and range is* (a) R,R (b) R^+, R^+ (c) R-{0} (d) R,{-1,1}

SOLUTION

$F(x) = \sin x$

The range of $f(x) = \sin x$, given by $R_F = \{x: -1 \le x \le 1, x \in \mathbb{R}\}$

Observe that if you punch Sine or Cosine of any number your answer (output) will not be above "1" or below "-1".

Hence $R_F = [-1, 1]$

EXAMPLE 5: 2012/2013 TEST QUESTION 18

18. Which of the following is the domain and range of the function; $f(x) = \frac{2}{5x+6}$
(a) $Dom(f) = R, Ran(f) = R - \{0\}$ (b) $Dom(f) = R - \{-6\}, Ran(f) = R$ (c) $Dom(f) = R - \{-\frac{6}{5}\}, = Ran(f) = R - \{0\}$
(d) None of the above

SOLUTION

$f(x) = \frac{2}{5x+6}$

Let $y = f(x)$

$\Rightarrow y = \frac{2}{5x+6}$

making x the Subject

$y(5x+6) = 2$

$5xy + 6y = 2$

$5xy = 2 - 6y$

$x = \frac{2-6y}{5y}$

Dom x = Range y = $\mathbb{R} - \{0\}$

The range of a Simple linear rational function is given by $\mathbb{R}$ - {Horizontal Asymptote} which gives $\mathbb{R} - \{0\}$. Check Asymptote as to understand H.A

EXAMPLE 6: 2013/2014 EXAM QUESTION 28

28. Find the domain and range of the function $f(z) = |z-6| - 3$ A. R. R^+ B. R; $[-3, \infty)$ C. R; R $(-\infty, \frac{4}{7})$; $[0, \frac{4}{7})$ E. R^+; R

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SOLUTION

Range of $F(z)$

$R_F = [-3, \infty)$

Range of Absolute value function $|z-6|$ is $\mathbb{R}^+$, but in this case; we have $|z-6|-3$. The minimum output belong -3. Hence Range f is given by $\{z : z \geq -3, z \in \mathbb{R}\}$

EXAMPLE 7: 2009/2010 EXAM QUESTION 61

61. Determine the range of the function $f(x) = \frac{3x}{x+4}$ (a) $R-\{3\}$ (b) R (c) $R-\{4\}$ (d) $R-\{0\}$

SOLUTION

$F(x)$ is a linear rational function

let $y = F(x)$

$\Rightarrow y = \frac{3x}{x+4}$

making x the subject

$y(x+4) = 3x$

$xy + 4y = 3x$

$xy - 3x = -4y$

$x(y-3) = -4y$

$x = \frac{-4y}{y-3}$

$x = \frac{4y}{3-y}$

Given a linear rational function, your range is given by $\mathbb{R} - \{H \cdot A\}$ where $H \cdot A$ means Horizontal Asymptote.

Dom x = Range y

Therefore, Range $F(x) = \mathbb{R} - \{3\}$

LIMIT OF FUNCTION

The limit of a function $F(x)$ is the behaviour of that function near a particular input (x_0).

Function may not be defined at some point but tend to have a value within the neigbourhood of same point.

For instance;

let $F(x) = \frac{x^2-4}{x-2}$, Evaluate $F(2)$

$F(2)$ means to put x as 2 in $F(x)$

$F(2) = \frac{2^2-4}{2-2} = \frac{4-4}{2-2} = \frac{0}{0}$

Which gives a meaningless value - But we can evaluate $F(x)$ using values in the neigbourhood of 2.. known as the limit as x tends to 2.

Consider the table below.

x-value	Value of F(x)
1.7	3.7
1.8	3.8
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

From this table, observe that we are putting x as numbers close to "2" but less than 2, it is known as limit as x tends to "2" from the left.. denoted by $\lim_{x \to 2^-} F(x)$

From the above table

$\lim_{x \to 2^-} \frac{x^2-4}{x-2} = 4$

x-values	values of f(x)
2.2	4.2
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.00001	4.00001

From this table, observe that we are putting x as numbers close to "2" but bigger than "2"; it is known as the right hand neighbourhood. Denoted by $\lim_{x \to 2^+} F(x)$

From the table above

$\lim_{x \to 2^+} \frac{x^2-4}{x-2} = 4$

Since $\lim_{x \to 2^-} F(x) = \lim_{x \to 2^+} F(x)$

The limit exist and it is equal to 4.

Alternatively, we can evaluate the limit using an algebraic process known as factorization. Thus

$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{x^2-2^2}{x-2}$

$\lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2} x+2$

$= 2+2 = 4$

Alternatively Again, we can apply the rule of G.F.A De L'hopital. Who said that we should differentiate the function individually whenever we have $\frac{0}{0}$

$\lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{4-4}{2-2}$

$= \frac{0}{0}$

Applying L'hopital's rule

$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} 2x$

$= 2(2)$

$= 4$

LIMIT OF PIECEWISE

Piecewise function is defined on branches/interval. For Example; Consider the piecewise function and investigate if the limit exist.

$$F(x) = \begin{cases} 3x+2, & x < 2 \\ 3x-2, & x \geq 2 \end{cases}$$

Evaluating the left Hand limit,

$\lim_{x \to 2^-} 3x+2 = 3(2)+2$

$= 8$

Left Hand L is the branch where $x < 2$

Evaluating the Right hand limit

$\lim_{x \to 2^+} 3x-2 = 3(2)-2$

$= 4$

Since the LHL $\neq$ RHL, that is

$\lim_{x \to 2^-} F(x) \neq \lim_{x \to 2^+} F(x)$

The limit as $x \to 2$ does not exist (DNE)

INDETERMINATE FORMS

Sometimes, when evaluating the limit of a function at some point, we obtain meaningless form/value we cannot determine; it includes the following;

$\frac{0}{0}$, 1^{∞}, ∞^{∞}, 0^0, ∞^0, $\frac{\infty}{\infty}$, $\infty - \infty$, e.t.c

L'Hopital's rule can only be used to evaluate the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

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STANDARD LIMITS

1) $\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad (n>0)$

2) $\lim_{x\to 0} \frac{1}{x^n} = \infty \quad (n>0)$

3) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

L' HOPITAL'S THEOREM

Check 2010/2011 Exam Q 66 Answer (?

Let f and g be two functions which are differentiable at f(x) and g(x), again let x_0 be any point such that;

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$

SQUEEZE THEOREM

Suppose that $g(x) \le f(x) \le p(x)$ holds for all x in some open interval contain x_0 except possibly at $x = x_0$ itself and suppose also that:

$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} p(x) = L$

Then $\lim_{x\to x_0} f(x) = L$ also.

EXAMPLE 1: 2012/2013 EXAM QUESTION 38

38. Evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ A. 0 B. 1 C.2 D. ONE

SOLUTION

$\lim_{x\to 0} x^2 \sin\frac{1}{x}$

From Trignometry, we know that the range of sine function is $[-1, -1]$

Hence; $-1 \le \sin(\frac{1}{x}) \le 1$

multiplying through by x^2

$-x^2 \le x^2\sin(\frac{1}{x}) \le x^2$

From squeeze theorem

if $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = L$

Then $\lim_{x\to 0} x^2 \sin\frac{1}{x} = L$

$\lim_{x\to 0} -x^2 = -0^2 = 0$

$\lim_{x\to 0} x^2 = 0^2 = 0$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2$

then $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$

LIMITS AT INFINITY

The limit of a function f(x) as x tends to infinity may or may not exis

EXAMPLE 1: 2009/2010 EXAM QUESTION 51

51. The limit of $x^{\frac{1}{x}}$ as $x \to \infty$ is (a) 0 (b) e (c) ln x (d) 1

SOLUTION

$\lim_{x\to\infty} x^{1/x} = \infty^{1/\infty} = \infty^0$ [Indeterminate form]

Let $y = x^{1/x}$, Then taking natural log of both sides,

$\ln y = \ln x^{1/x}$

$\Rightarrow \ln y = \frac{1}{x}\ln x$, Therefore

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\ln\infty}{\infty} = \frac{\infty}{\infty}$ L' Hopital's Indeterminate form

Applying L' Hopital's rule,

$\Rightarrow \lim_{x\to\infty} \left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}\right] = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$

Recall that $\lim_{x\to\infty} \ln y = 0$, Then taking exponential of both sides, $y = e^0$; $y = 1$

EXAMPLE 2: 2010/2011 EXAM QUESTION 26

26. Evaluate $\lim_{x\to\infty} \frac{1+5x}{6+2x-x^2}$ (a) $\frac{5}{5}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

SOLUTION

Evaluate $\lim_{x\to\infty} \frac{1+5x}{6+2x-x^2}$

Solution

$\lim_{x\to\infty} \frac{1+5x}{6+2x-x^2} = \frac{1+5(\infty)}{6+2(\infty)-(\infty)^2}$

$= \frac{\infty}{\infty}$

Applying L' Hopital's rule

$\lim_{x\to\infty} \left[\frac{\frac{d}{dx}(1+5x)}{\frac{d}{dx}(6+2x-x^2)}\right] = \lim_{x\to\infty} \frac{5}{2-2x} = \frac{5}{2-\infty}$

$= \frac{5}{-\infty} = 0$

EXAMPLE 3: 2011/2012 EXAM QUESTION 1

1. $\lim_{x\to\infty} \frac{2x^2-3x+1}{3x^2+5x+1}$ (a) -1 (b) $\frac{2}{3}$ (c) $-\frac{3}{5}$ (d) ∞

SOLUTION

$\lim_{x\to\infty} \frac{2x^2-3x+1}{3x^2+5x+1} = \frac{2(\infty)^2-3(\infty)+1}{3(\infty^2)+5(\infty)+1} = \frac{\infty}{\infty}$

Dividing through by the highest power of x (x^2):

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}}{\frac{3x^2}{x^2} + \frac{5x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{1}{x^2}}{3 + \frac{5}{x} + \frac{1}{x^2}}$$

Putting x as ∞, we have

$$\lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{1}{x^2}}{3 + \frac{5}{x} + \frac{1}{x^2}} = \frac{2 - \frac{3}{\infty} + \frac{1}{\infty^2}}{3 + \frac{5}{\infty} + \frac{1}{\infty^2}}$$

Recall that a number divided by infinity (∞) is zero (0).

$$= \frac{2 - 0 + 0}{3 + 0 + 0} = \frac{2}{3}$$

Check similar question and solve 11/12 Q1

EXAMPLE 4: 2014/2015 EXAM QUESTION 45

45. Find $\lim_{x\to\infty}(\sqrt{x-4} - \sqrt{x})$ (a) 0 (b) 4 (c) -4 (d) 1 (e) -1

SOLUTION

$$\lim_{x\to\infty} \sqrt{x+4} - \sqrt{x} = \sqrt{\infty+4} - \sqrt{\infty}$$
$$= \infty - \infty \text{ [Indeterminate Form]}$$

We can't apply L'Hopital for this form, hence we rationalize, we have

$$\lim_{x\to\infty} \frac{\sqrt{x+4} - \sqrt{x}}{1} \times \frac{\sqrt{x+4} + \sqrt{x}}{\sqrt{x+4} + \sqrt{x}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{(x+4) - (x)}{\sqrt{x+4} + \sqrt{x}} = \lim_{x\to\infty} \frac{4}{\sqrt{x+4}+\sqrt{x}} = \frac{4}{\infty+\infty} = 0$$

EXAMPLE 5:

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \left(1 + \frac{1}{\infty}\right)^\infty = (1+0)^\infty$$
$$= 1^\infty \text{ (Indeterminate Form)}$$

Let $y = \left(1 + \frac{1}{x}\right)^x$

Taking natural logarithm of both sides,

$$\ln y = \ln\left(1 + \frac{1}{x}\right)^x$$

$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{(1/x)} \ln\left(1 + \frac{1}{x}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x}$$

Taking the limit again,

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \frac{\ln\left(1 + \frac{1}{\infty}\right)}{1/\infty} = \frac{\ln 1}{0} = \frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \left\{ \frac{\frac{d}{dx}\left[\ln\left(1 + \frac{1}{x}\right)\right]}{\frac{d}{dx}(1/x)} \right\}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \left(\frac{-x^{-2}}{1 + x^{-1}}\right) \cdot \left(\frac{1}{-x^{-2}}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{1 + x^{-1}} = \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)$$

Taking the limit again,

$$\lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right) = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = \frac{1}{1} = 1$$

If $\lim_{x\to\infty} \ln y = 1$, then taking e of both sides,

$$e^{\ln\left(\lim_{x\to\infty} y\right)} = e^1$$

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$$

SHORT-CUT

$$\lim_{x\to\infty}\left(1 + \frac{\boxed{A}}{\boxed{B}x}\right)^x = e^{A/B}$$

Example;

$$\lim_{x\to\infty}\left(1 + \frac{1}{2x}\right)^x = e^{1/2}$$

Where $A = 1$, $B = 2$

PROVES OF LIMIT

Let $F(x)$ be defined on a domain D. Then the limit of $F(x)$ as it gets closer to $x_0 \in D$ is denoted as;

$$\lim_{x\to x_0} F(x) = L$$

Thus it is defined for every number $\varepsilon > 0$, there exist a corresponding number $\delta > 0$ such that

$$|F(x) - L| < \varepsilon \text{ whenever } 0 < |x - x_0| < \delta$$

Where we choose $\delta \le \frac{\varepsilon}{K}$

EXAMPLE 1: 2011/2012 EXAM QUESTION 40

40. Verifying the $\lim_{x\to 2}(4x+3)11$ gives (a) $\delta = \epsilon$ (b) $\delta = \frac{\epsilon}{4}$ (c) $\delta = \frac{\epsilon}{2}$ (d) $\delta = \frac{2\epsilon}{11}$

SOLUTION

$$\lim_{x\to 2} 4x + 3 = 11$$

$x_0 = 2$, $F(x) = 4x+3$, $L = 11$

Using $|F(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$

$\Rightarrow |(4x+3)-11| < \varepsilon$

Transformin the content of the absolute value in terms of δ.

$|4(x-2)+8+3-11| < \varepsilon$

$|4\cdot(x-2)| < \delta$ Hence we choose $\delta \le \frac{\varepsilon}{4}$

$|4\,\delta| < \varepsilon$

EXAMPLE 2: 2011/2012 TEST QUESTION 3

3 Verifying that $\lim_{x\to 4} \frac{3x+1}{2} = \frac{11}{2}$ gives (a) $\delta = \frac{\varepsilon}{6}$ (b) $\delta = \frac{\varepsilon}{3}$ (c) $\delta = \frac{2\varepsilon}{3}$ (d) $\delta = \frac{\varepsilon}{2}$

SOLUTION

$\lim_{x\to 4} \frac{3x+1}{2} = \frac{11}{2}$ Can be reduced to

$\lim_{x\to 4} (3x+1) = 11$

Using the Formular

$|F(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$

Where $x_0 = 4$, $F(x) = 3x+1$, $L = 11$

Substituting,

$|(3x+1) - 11| < \varepsilon$ whenever $0 < |x-4| < \delta$

Now, we transfor in terms of δ

$|3(x-4)+12+1-11| < \varepsilon$

$|3(x-4)| < \delta$

$|3\delta| < \delta$

Hence we choose $\delta \le \frac{\varepsilon}{3}$

EXAMPLE 3: 2013/2014 EXAM QUESTION 34

34. $\lim_{x\to 3} 2x = 6$, then with $\in =$ 0.0001 *what is the corresponding δ that satifies this limit?* A. 0.5 B. 0.05 C. 0.005 D. 0.0005 E. 0.00005

SOLUTION

$\lim_{x\to 3} 2x = 6$ For $\varepsilon = 0.0001$

Using Epsilon-Delta definition of Limits,

$|F(x) - L| < \varepsilon$, whenever $0 < |x - x_0| < \delta$

where $F(x) = 2x$, $x_0 = 3$, $L = 6$

$|2x - 6| < \varepsilon$, whenever $0 < |x-3| < \delta$

$|2(x-3)+6-6| < \varepsilon$

$|2(x-3)| < \varepsilon$

$|2\,\delta| < \varepsilon$, Hence we choose

$\delta \le \frac{\varepsilon}{2}$, at $\varepsilon = 0.0001$

$\delta \le \frac{0.0001}{2} \le 0.00005$

$\frac{1}{10}$ *what is the corresponding δ that satifies this limit?* A. $\frac{1}{30}$ B. 10 C. $\frac{1}{20}$ D. $\frac{1}{5}$ E. $\frac{1}{40}$

SOLUTION

$\lim_{x\to 6} 3x - 4 = 14$ For $\varepsilon = \frac{1}{10}$

using $|F(x) - L| < \varepsilon$, whenever $0 < |x - x_0| < \delta$

where $F(x) = 3x-4$, $x_0 = 6$, $L = 14$

$\Rightarrow |(3x-4) - 14| < \varepsilon$, whenever $0 < |x-6| < \delta$

Transforming the content of the absolute value expression in terms of δ,

$|3(x-6)+18-4-14| < \varepsilon$

$|3(x-6)| < \varepsilon$

$|3\delta| < \varepsilon$, Hence we choose

$\delta \le \frac{\varepsilon}{3}$, For $\varepsilon = \frac{1}{10}$

$\delta \le \frac{\frac{1}{10}}{\frac{3}{1}} = \frac{1}{10} \times \frac{1}{3}$

$\delta \le \frac{1}{30}$

LIMIT AT REAL - VALUE (X_0)

If a limit of a function tends to a real value, Then it will exist, if the left hand limit is equal to the right hand limit, However we use some other process to evaluate limit at finite point on the real line which are as follows;

- Factorization
- Rationalization
- L'Hopital
- E.T.C

Consider the following examples

EXAMPLE 1: 2008/2009 EXAM QUESTION 44

44. Find the $\lim_{x\to 0} \frac{1-\cos mx}{x^2}$, where m is an arbitrary non-zero constant (a) $\frac{m^2}{m}$ (b) 0 (c) undefined (d) $\frac{1}{2}$

SOLUTION

$\lim_{x\to 0} \frac{1-\cos mx}{x^2} = \frac{1-\cos(0)}{0^2} = \frac{1-1}{0} = \frac{0}{0}$

Applying L' hopital's rule

$\lim_{x\to 0} \frac{-[-m\sin(mx)]}{2x} = \frac{m\sin(0)}{2(0)} = \frac{0}{0}$

Applying L' hopital's rule again

$\lim_{x\to 0} \frac{m^2\cos mx}{2} = \frac{m^2\cos(0)}{2} = \frac{m^2}{2}$

EXAMPLE 2: 2008/2009 EXAM QUESTION 20

20. $\lim_{x\to 0}\frac{\sin 2x}{x}$ (a) 1 (b) 2 (c) sin2 (d) 0

SOLUTION

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{\sin 2(0)}{0} = \frac{0}{0}$$

Applying L'hopital's rule

$$\lim_{x\to 0}\frac{2\cos 2x}{1} = 2\cos(0) = 2(1) = 2$$

EXAMPLE 3: 2008/2009 EXAM QUESTION 19

19. Find $\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4}$ (a) $\frac{1}{4}$ (b) 0 (c) $\frac{1}{2}$ (d) 4

SOLUTION

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4} = \frac{(2)^2-3(2)+2}{(2)^2-4} = \frac{4-6+2}{4-4} = \frac{0}{0}$$

Applyin L'hopital's rule, we have

$$\lim_{x\to 2}\frac{2x-3+0}{2x-0} = \frac{2(2)-3}{2(2)} = \frac{1}{4}$$

EXAMPLE 4: 2009/2010 EXAM QUESTION 73

73. The limit of $\frac{x^2+x+2}{x^2-2x-3}$ as $x \to 3^+$ is (a) ∞ (b) $-\infty$ (c) 0 (d) none of the above

SOLUTION

$$\lim_{x\to 3^+}\frac{x^2+x+2}{x^2-2x-3}$$

$\lim x \to 3^+$ mean to approach 3 from the right i.e R.H.L

taking the limits

$$\lim_{x\to 3^+}\frac{(3)^2+3+2}{(3)^2-2(3)-3} = \frac{9+3+2}{9-6-3} = \frac{14}{0} = \infty$$

Since, we approached "3" from right

$$\lim_{x\to 3^+}\frac{x^2+x+2}{x^2-2x-3} = +\infty = \infty$$

Note: assuming its from the left, our L would be $-\infty$

EXAMPLE 5: 2009/2010 EXAM QUESTION 67

Solve $lim\left(\frac{\tan x - x}{x^3}\right)$ as x tends to 0 (a) $\frac{1}{3}$ (b) $\frac{2}{15}$ (c) 0 (d) 3

SOLUTION

maclaurin series of $\tan x$ is given

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

Subtituting the value of $\tan x$ as far as x^3

$$\Rightarrow \lim_{x\to 0}\frac{x+\frac{x^3}{3}-x}{x^3} = \lim_{x\to 0}\frac{1}{3}\cdot\frac{x^3}{x^3} = \frac{1}{3}$$

Check Taylors and maclaurin series @ application of Derivatives

EXAMPLE 6: 2009/2010 EXAM QUESTION 58

58. Find $lim\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)$ as x tends to 0 (a) 1 (b) 0 (c) -1 (d) - 2

SOLUTION

$$\lim_{x\to 0}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) = \frac{1+\frac{1}{0}}{1-\frac{1}{0}} = \frac{1+\infty}{1-\infty} = \frac{\infty}{-\infty}$$

(Indeterminate form)

Applying L'Hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(1-\frac{1}{x}\right)}\right] = \lim_{x\to 0}\left[\frac{-\frac{1}{x^2}}{\frac{1}{x^2}}\right]$$

$$= \lim_{x\to 0} -\frac{1}{x^2}\cdot\frac{x^2}{1} = \lim_{x\to 0} -\frac{1}{1} = -1$$

EXAMPLE 7: 2009/2010 EXAM QUESTION 34

34. Evaluate $\left(\frac{\sqrt{(4+x)}-2}{x}\right)$ as x tends to 0 is (a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{2}{3}$ (d) 2

SOLUTION

$$\lim_{x\to 0}\left(\frac{\sqrt{(4+x)}-2}{x}\right) = \frac{\sqrt{4+0}-2}{0} = \frac{2-2}{0} = \frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to 0}\frac{\frac{d}{dx}\left[\sqrt{(4+x)}-2\right]}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{1}{2\sqrt{4+x}} = \frac{1}{2\sqrt{4+0}} = \frac{1}{2(2)} = \frac{1}{4}$$

Note: If $y = \sqrt{f(x)}$ Then $\frac{dy}{dx} = \frac{f'(x)}{2\sqrt{f(x)}}$

EXAMPLE 8 : 2009/2010 EXAM QUESTION 33

33. Find $\left(\frac{1-\cos 2x}{x^2}\right)$ as x tends to 0 is (a) -2 (b) 2 (c) 0 (d) 1

SOLUTION

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \frac{1-\cos 2(0)}{(0)^2} = \frac{1-1}{0} = \frac{0}{0}$$

(Indeterminate form)

Applying L'hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(1-\cos 2x)}{\frac{d}{dx}(x^2)}\right] = \lim_{x\to 0}\frac{-(-2\sin 2x)}{2x}$$

$$= \lim_{x\to 0} \frac{2\sin 2x}{2x} = \frac{2\sin 2(0)}{2(0)} = \frac{0}{0}$$

Applying L' Hopital's rule again, we have

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(2\sin 2x)}{\frac{d}{dx}(2x)}\right] = \lim_{x\to 0}\frac{2(2)\cos 2x}{2}$$

$$= \lim_{x\to 0}\frac{4\cos 2x}{2} = \frac{4\cos 2(0)}{2} = \frac{4(1)}{2} = 2$$

EXAMPLE 9: 2009/2010 EXAM QUESTION 32

32. The limit of $\frac{3x^3+2x^2-4}{6x^3-x+3}$ as x tends to 0 is (a) 2 (b) $-\frac{1}{2}$ (c) -2 (d) $\frac{1}{2}$

SOLUTION

$$\lim_{x\to 0}\frac{3x^3+2x^2-4}{6x^3-x+3} = \frac{3(0)^3+2(0)^2-4}{6(0)^3-(0)+3}$$

$$= \frac{3(0)+2(0)-4}{6(0)-0+3} = \frac{-4}{3}$$

EXAMPLE 10: 2009/2010 EXAM QUESTION 28

28. Evaluate the limit $\left(\frac{x^2+2x-15}{x^2-9}\right)$ as $x \to 3$ (a) 1 (b) 8 (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

SOLUTION

$$\lim_{x\to 3}\frac{x^2+2x-15}{x^2-9} = \frac{(3)^2+2(3)-15}{(3)^2-9} = \frac{9+6-15}{9-9} = \frac{0}{0}$$ (Indeterminate form)

Applying L' Hopital's rule

$$\lim_{x\to 3}\left[\frac{\frac{d}{dx}(x^2+2x-15)}{\frac{d}{dx}(x^2-9)}\right] = \lim_{x\to 3}\frac{2x+2}{2x}$$

Taking the limit, we have

$$\lim_{x\to 3}\frac{2x+2}{2x} = \frac{2(3)+2}{2(3)} = \frac{6+2}{6} = \frac{8}{6} = \frac{4}{3}$$

EXAMPLE 11: 2009/2010 EXAM QUESTION 26

26. Find the behaviour of $\frac{1}{x}$ as $x \to 0$ from the left hand (a) 8 (b) ∞ (c) $-\infty$ (d) -8

SOLUTION

$$\lim_{x\to 1}\frac{4x^4+3x^2-1}{x^3+7} = \frac{4(1)^4+3(1)^2-1}{(1)^3+7} = \frac{4+3-1}{1+7}$$

$$= \frac{6}{8} = \frac{3}{4}$$

EXAMPLE 12: 2010/2011 EXAM QUESTION 65

65. Evaluate $\lim_{x\to 0}\frac{x}{\sqrt{x+9}-3}$ (a) ∞ (b) 3 (c) 0 (d) 6

SOLUTION

$$\lim_{x\to 0}\frac{x}{\sqrt{x+9}-3} = \frac{0}{\sqrt{0+9}-3} = \frac{0}{\sqrt{9}-3} = \frac{0}{3-3} = \frac{0}{0}$$ (Indeterminate form)

Applying L' Hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(\sqrt{x+9}-3)}\right] = \lim_{x\to 0}\frac{1}{\frac{1}{2\sqrt{x+9}}}$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{1}\times\frac{2\sqrt{x+9}}{1} = \lim_{x\to 0} 2\sqrt{x+9}$$

Taking the limit

$$\lim_{x\to 0} 2\sqrt{x+9} = 2\sqrt{0+9} = 2\sqrt{9} = 2(3)$$
$$= 6$$

EXAMPLE 13: 2010/2011 EXAM QUESTION 59

59. Evaluate the following $\lim_{h\to\pi}\frac{2(-3+h)^2-18}{h}$ (a) 12 (b)... (c) 6 (d) -18

SOLUTION

$$\lim_{h\to\pi}\frac{2(-3+h)^2-18}{h} = \frac{2(-3+\pi)^2-18}{\pi}$$

$$= \frac{2(\pi-3)^2-18}{\pi} = \frac{2(\pi^2-6\pi+9)-18}{\pi}$$

$$= \frac{2\pi^2-12\pi+18-18}{\pi} = \frac{2\pi(\pi-6)}{\pi} = 2(\pi-6)$$

EXAMPLE 14: 2010/2011 EXAM QUESTION 58

58. Evaluate the following $\lim_{x\to\pi}\frac{\cos x}{x}$ (a) $\frac{1}{\pi}$ (b) 0 (c) $\frac{\pi}{2}$ (d)...

SOLUTION

$$\lim_{x\to\pi}\frac{\cos x}{x} = \frac{\cos\pi}{\pi}$$ Note: $\pi = 180°$

$$= \frac{\cos 180}{\pi} = \frac{-1}{\pi}$$

EXAMPLE 15: 2010/2011 EXAM QUESTION 48

48. Evaluate $\lim_{x\to 1}\left\{\frac{x^3+x^2-x-1}{x^2+2x-3}\right\}$ (a) 4 (b) -4 (c) 1 (d) -1

SOLUTION

$$\lim_{x\to 1}\frac{x^3+x^2-x-1}{x^2+2x-3} = \frac{(1)^3+(1)^2-(1)-1}{(1)^2+2(1)-3}$$

$$= \frac{1+1-1-1}{1+2-3} = \frac{0}{0}$$ [Indeterminate form]

Applying L' Hopital's rule:

$$\lim_{x\to 1}\left[\frac{\frac{d}{dx}(x^3+x^2-x-1)}{\frac{d}{dx}(x^2+2x-3)}\right] = \lim_{x\to 1}\frac{3x^2+2x-1}{2x+2}$$

Taking the limit again,

$$\lim_{x\to 1}\frac{3x^2+2x-1}{2x+2} = \frac{3(1)^2+2(1)-1}{2(1)+2} = \frac{4}{4}$$

EXAMPLE 16: 2010/2011 EXAM QUESTION 32

32. Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ (a) does not exist (b) 0 (c) 1 (d)...

SOLUTION

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \frac{1}{\sin 0} - \frac{1}{0} = \frac{1}{0} - \frac{1}{0} = \infty-\infty$$ (Indeterminate form)

We cannot apply L' Hopital's rule

Hence, we have

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \frac{0-\sin(0)}{0(\sin(0))} = \frac{0-0}{0\cdot 0} = \frac{0}{0}$$

Applying L'hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(x-\sin x)}{\frac{d}{dx}(x\sin x)}\right] = \lim_{x\to 0} \frac{1-\cos x}{x\cos x+\sin x}$$

Taking the limit again;

$$\lim_{x\to 0} \frac{1-\cos x}{x\cos x+\sin x} = \frac{1-\cos 0}{(0)\cos(0)+\sin(0)} = \frac{1-1}{0+0} = \frac{0}{0}$$

Applying L' Hopital's rule again,

$$\lim_{x\to 0} \frac{1-\cos x}{x\cos x+\sin x} = \lim_{x\to 0} \frac{\sin x}{-x\sin x+\cos x+\cos x}$$

Taking the limit again,

$$\lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{\sin 0}{2\cos 0 - 0\sin(0)} = \frac{0}{2(1)} = 0$$

EXAMPLE 17: 2010/2011 EXAM QUESTION 29

29. What is the value of $\lim_{x\to 1} \frac{x^2-1}{x-1}$ (a) 1 (b) 0 (c) 2 (d) -2

SOLUTION

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{(1)^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 1}\left[\frac{\frac{d}{dx}(x^2-1)}{\frac{d}{dx}(x-1)}\right] = \lim_{x\to 1} \frac{2x}{1} = \frac{2(1)}{1} = 2$$

EXAMPLE 18: 2010/2011 EXAM QUESTION 27

27. Evaluate $\lim_{x\to 0} \frac{2x^2-1}{1+6x}$ (a) 1 (b) -2 (c) 2 (d) -1

SOLUTION

$$\lim_{x\to 0} \frac{2x^2-1}{1+6x} = \frac{2(0)-1}{1+6(0)} = \frac{0-1}{1+0} = -1$$

EXAMPLE 19: 2010/2011 EXAM QUESTION 22

22. What is the limit of $\frac{x-4}{x^2-x-12}$ as x approaches 4? (a) $\frac{1}{7}$ (b) $\frac{1}{3}$ (c) $\frac{3}{7}$ (d) $\frac{7}{3}$

SOLUTION

$$\lim_{x\to 4} \frac{x-4}{x^2-x-12} = \frac{4-4}{(4)^2-(4)-12} = \frac{0}{0}$$ (indeterminate form)

Applying L'Hopital's rule

$$\lim_{x\to 4}\left[\frac{\frac{d}{dx}(x-4)}{\frac{d}{dx}(x^2-x-12)}\right] = \lim_{x\to 4} \frac{1}{2x-1} = \frac{1}{2(4)-1} = \frac{1}{7}$$

EXAMPLE 20: 2010/2011 EXAM QUESTION 6

6. Evaluate $\lim_{x\to 1} \frac{x^4-1}{x-1}$ (a) 0 (b) 2 (c) 4 (d) ∞

SOLUTION

$$\lim_{x\to 1} \frac{x^4-1}{x-1} = \frac{(1)^4-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$$ [Indeterminate Form]

Applying L' Hopital's rule

$$\lim_{x\to 1}\left[\frac{\frac{d}{dx}(x^4-1)}{\frac{d}{dx}(x-1)}\right] = \lim_{x\to 1} \frac{4x^3}{1} = 4(1)^3 = 4$$

EXAMPLE 21: 2010/2011 EXAM QUESTION 13

13. Find $\lim_{x\to 0^-} \frac{1}{x^2}$ (a) 0 (b) $+\infty$ (c) $-\infty$ (d) 1

SOLUTION

$$\lim_{x\to 0^-} \frac{1}{x^2} = \frac{1}{0^2} = \infty$$

Observe the table below

x-values	$\frac{1}{x^2}$
-0.1	100
-0.01	10,000
-0.001	1,000,000

From the table as x moves closer to 0 from the left (negative) the function $\frac{1}{x^2}$ increase so large to infinity

EXAMPLE 22: 2011/2012 EXAM QUESTION 55

55. Evaluate $\lim_{x\to 1} \frac{x^2-\sqrt{x}}{x-1}$ (a) $\frac{3}{2}$ (b) $\frac{2}{5}$ (c) $\frac{1}{3}$ (d) 1

SOLUTION

$$\lim_{x\to 1} \frac{x^2-\sqrt{x}}{x-1} = \frac{(1)^2-\sqrt{1}}{(1)-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 1}\left[\frac{\frac{d}{dx}(x^2-\sqrt{x})}{\frac{d}{dx}(x-1)}\right] = \lim_{x\to 1}\left(\frac{2x-\frac{1}{2\sqrt{x}}}{1}\right)$$

Taking the limit,

$$\lim_{x\to 1}\left(2x-\frac{1}{2\sqrt{x}}\right) = 2(1)-\frac{1}{2(1)}$$

$$= 2-\tfrac{1}{2}$$

$$= 3/2$$

EXAMPLE 23: 2011/2012 EXAM QUESTION 50

50. Evaluate $\lim_{x\to 0} \frac{e^{bx}-1}{x^2}$ (a) $\frac{b}{2}$ (b) $\frac{1}{2}$ (c) $\frac{b^2}{2}$ (d) D.N.E

SOLUTION

$$\lim_{x\to 0} \frac{e^{bx}-1}{x^2} = \frac{e^{b(0)}-1}{0^2} = \frac{e^0-1}{0} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(e^{bx}-1)}{\frac{d}{dx}(x^2)}\right] = \lim_{x\to 0} \frac{be^{bx}}{2x}$$

Taking the limit, we have

$$\lim_{x\to 0} \frac{be^{bx}}{2x} = \frac{be^{b(0)}}{2(0)} = \frac{be^0}{0} = \frac{b(1)}{0}$$

$= \frac{b}{0} = \infty$

Since the limit is not a finite number,
Then, it Does not exist (D.N.E)

EXAMPLE 24: 2011/2012 EXAM QUESTION 34

34. Find $\lim_{x\to 0} e^{\sin x}$ (a) 0 (b) -1 (c) e^{π} (d) 1

SOLUTION

$$\lim_{x\to 0} e^{\sin x} = e^{\sin(0)}$$
$$= e^{0}$$
$$= 1$$

EXAMPLE 25: 2011/2012 EXAM QUESTION 2

2. Evaluate $\lim_{x\to 3} \frac{x^4-81}{x-3}$ (a) 27 (b) 108 (c) 0 (d) 32

SOLUTION

$$\lim_{x\to 3} \frac{x^4-81}{x-3} = \frac{(3)^4-81}{3-3} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 3} \left[\frac{\frac{d}{dx}(x^4-81)}{\frac{d}{dx}(x-3)}\right] = \lim_{x\to 3} \frac{4x^3}{1}$$

Taking the Limit again,

$$\lim_{x\to 3} 4x^3 = 4(3)^3$$
$$= 108$$

EXAMPLE 26: 2011/2012 TEST QUESTION 23

23. $\lim_{x\to 1} \sqrt{x-1}$ is (a) 0 (b) -1 (c) 1 (d) D.N.E

SOLUTION

$$\lim_{x\to 1} \sqrt{x-1} \neq 0$$

$$\lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0$$

$$\lim_{x\to 1^-} \sqrt{x-1} = \sqrt{-\#} = \text{Undefined}$$

Since $\lim_{x\to 1^+} \sqrt{x-1} \neq \lim_{x\to 1^-} \sqrt{x-1}$

The $\lim_{x\to 1} \sqrt{x-1}$ Does not exist (D.N.E)

Note: Because of it a Square root Function and we Cannot evaluate the Left hand limit of x, Hence the limit D.N.E

EXAMPLE 27: 2011/2012 TEST QUESTION 8

8. Find $\lim_{x\to 1}(\sqrt{x+1} - \sqrt{2x+18})$ (a) does not exist (D.N.E) (b) $\sqrt{2} - 2\sqrt{2}$ (c) 0 (d) $3\sqrt{2}$

SOLUTION

$$\lim_{x\to 1} \sqrt{x+1} - \sqrt{2x+18} = \sqrt{1+1} - \sqrt{2(1)+18}$$
$$= \sqrt{2} - \sqrt{20}$$
$$= \sqrt{2} - 2\sqrt{5}$$

EXAMPLE ... QUESTION 4

4. Find the limit of $\frac{9-x^2}{3-\sqrt{x+5}}$ at $x = 3$ (a) 0 (b) 36 (c) ... ∞

SOLUTION

$$\lim_{x\to 3} \frac{9-x^2}{3-\sqrt{x+6}} = \frac{9-(3)^2}{3-\sqrt{(3)+6}} = \frac{9-9}{3-3}$$

Applying L' Hopital's rule,

$$\lim_{x\to 3} \left[\frac{\frac{d}{dx}(9-x^2)}{\frac{d}{dx}(3-\sqrt{x+6})}\right] = \lim_{x\to 3} \frac{-2x}{-\frac{1}{2\sqrt{x+6}}}$$

Taking the Limit,

$$\lim_{x\to 3} \left(\frac{2x}{1} \times \frac{2\sqrt{x+6}}{1}\right) = \lim_{x\to 3} (2\sqrt{x+6})(2x)$$

$$\lim_{x\to 3} (2\sqrt{x+6})(2x)$$ Taking the limit again,

$$= [2\sqrt{(3)+6}][2(3)]$$
$$= 2(3)(6)$$
$$= 36.$$

EXAMPLE 29: 2012/2013 EXAM QUESTION 39

39. Evaluate $\lim_{x\to 0} \frac{e^x-1}{x}$ A. 0 B. ∞ C. 1 D. 10

SOLUTION

$$\lim_{x\to 0} \frac{e^x-1}{x} = \frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0}$$

Applying L' Hopital's theorem

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1$$

EXAMPLE 30: 2012/20213 EXAM QUESTION 17

17. Evaluate $\lim_{x\to 0}(\sec^3 2x)^{\cot^2 3x}$ $A. e^{\frac{1}{2}}$ $B. \frac{2}{3}$ $C. e^{\frac{2}{3}}$ D.0

SOLUTION

$$\lim_{x\to 0} (\sec^3 2x)^{\cot^2 3x} = \left[\frac{1}{[\cos 2(0)]^3}\right]^{\frac{1}{(\tan 0)^2}} = \left(\frac{1}{0}\right)^0$$

$$= \infty^0 \text{ (Indeterminate form)}$$

let $y = (\sec^3 2x)^{\cot^2 x}$

Taking natural logarithm of both sides

$$\ln y = \cot^2(3x)\ln(\sec^3 2x)$$

but $\cot^2 3x = \frac{1}{\tan^2 3x}$

and $\ln(\sec^3 2x) = 3\ln(\sec 2x)$

$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{3\ln(\sec 2x)}{\tan^2 3x} = \frac{0}{0}$$

Applying L' Hopital's theorem

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \left[\frac{\frac{d}{dx}(3\ln(\sec(2x)))}{\frac{d}{dx}(\tan^2 3x)}\right]$$

$$= \lim_{x\to 0}\left(\frac{\tan 2x}{\tan 3x \sec^2 3x}\right)$$

$$= \lim_{x\to 0}\frac{\tan 2x}{\tan 3x} \cdot \lim_{x\to 0}\frac{1}{\sec^2 3x}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0}\frac{\tan 2x}{\tan 3x} = \frac{0}{0}$$

Applying L' Hopital's rule again,

$$\lim_{x\to 0}\frac{2\sec^2 2x}{3\sec^2 3x} = \lim_{x\to 0}\frac{2}{3} = \frac{2}{3}$$

Recall that

$$\lim_{x\to 0} \ln y = \frac{2}{3}$$

Taking e of both sides

$$e^{\ln\left(\lim_{x\to 0} y\right)} = e^{2/3}$$

$$\lim_{x\to 0}(\sec^2 2x)^{\cot^2 3x} = e^{2/3}$$

EXAMPLE 31: 2012/2013 EXAM QUESTION 12

12. Evaluate $\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}$ A. 3 B. 4 C. 5 D. 6

SOLUTION

$$\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}} = \frac{4-(2)^2}{3-\sqrt{(2)^2+5}} = \frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to 2}\left[\frac{\frac{d}{dx}(4-x^2)}{\frac{d}{dx}(3-\sqrt{x^2+5})}\right] = \lim_{x\to 2}\left(\frac{-2x}{-\frac{2x}{2\sqrt{x^2+5}}}\right)$$

Taking the limit,

$$\lim_{x\to 2}\left(\frac{-2x}{1}\times\frac{2\sqrt{x^2+5}}{-2x}\right) = 2\sqrt{(2)^2+5}$$

$$= 2\sqrt{9}$$

$$= 6$$

While evaluating limit involving square root, you have to know how to differentiate square root functions

E.g If $f(x) = \sqrt{g(x)}$

SHORTCUT

$$f'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$$

For instance

If $y = \sqrt{2x^3+5x}$

$$\frac{dy}{dx} = \frac{6x^2+5}{2\sqrt{2x^3+5x}}$$

EXAMPLE 32: 2012/2013 EXAM QUESTION 3

3. Evaluate $\lim_{x\to 0}\frac{\sqrt{x-2}-\sqrt{2}}{x}$ A. Undefined B. $\frac{0}{0}$ C. $\frac{\sqrt{2}}{4}$ D. $\frac{1}{\sqrt{2}}$

SOLUTION

$$\lim_{x\to 0^+}\frac{\sqrt{x-2}-\sqrt{2}}{x} = \text{undefined.}$$

Observe the term $\sqrt{x-2}$, we can't evaluate the right hand limit.

EXAMPLE 33:2012/2013 TEST QUESTION 20

20. Solve $\lim_{x\to 0}\left[\frac{2}{x}+\frac{1}{1-\cos x}\right]$ (a) 1 (b) 3 (c) 0 (d) 2

SOLUTION

$$\lim_{x\to 0}\left[\frac{2}{x}+\frac{1}{1-\cos x}\right] = \frac{2}{0}+\frac{1}{1-\cos 0}$$

$$= \frac{2}{0}+\frac{1}{1-1}$$

$$= \infty + \infty$$

$$= \infty$$

The limit does not exist. (DNE)

EXAMPLE 34: 2012/2013 TEST QUESTION 11

11. $\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}$ is (a) 2 (b) 3 (c) -2 (d) -4

SOLUTION

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x} = \frac{(\sin 0)^2}{1-\cos 0} = \frac{0^2}{1-1} = \frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(\sin^2 x)}{\frac{d}{dx}(1-\cos x)}\right] = \lim_{x\to 0}\frac{2\sin x\cos x}{-(-\sin x)}$$

$$\lim_{x\to 0} 2\cos x = 2\cos 0$$

$$= 2$$

EXAMPLE 35: 2013/2014 EXAM QUESTION 54

54. Evaluate $\lim_{x\to 0}\frac{|x|}{x}$ A. 1 B. −1 C. 0 D. e E. undefined

SOLUTION

$$\lim_{x\to 0}\frac{|x|}{x} = \frac{0}{0}\ \text{[Indeterminate Form]}$$

From definition

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0\end{cases}$$

Evaluating the LHL of "0"

$$\lim_{x\to 0^-}\frac{-x}{x} = -1$$

Left hand limit (LHL) is the branch where $x < 0$.... "$-x$"

Evaluating the right hand limit

$$\lim_{x\to 0^+} \frac{x}{x} = 1$$

Since $\lim_{x\to 0^-} \frac{|x|}{x} \neq \lim_{x\to 0^+} \frac{|x|}{x}$

The $\lim_{x\to 0} \frac{|x|}{x}$ Does not exist (DNE)

EXAMPLE 36: 2013/2014 EXAM QUESTION 30

30. Evaluate $\lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t}$ A. $\frac{5}{8}$ B. $\frac{8}{5}$ C. $-\frac{5}{8}$ D. $-\frac{8}{5}$ E. 0

SOLUTION

$$\lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t} = \frac{4-\sqrt{3(4)+4}}{4-4} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{t\to 4} \frac{1-\frac{3}{2\sqrt{3t+4}}}{-1}$$

$$\lim_{t\to 4} -\left(1-\frac{3}{2\sqrt{3t+4}}\right) = -\left(1-\frac{3}{2\sqrt{3(4)+4}}\right)$$

$$= -\left(1-\frac{3}{8}\right)$$

$$= -5/8$$

EXAMPLE 37: 2014/2015 EXAM QUESTION 17

17. Evaluate $\lim_{x\to 0} \frac{e^{3x}-1}{x}$ (a) 1 (b) 0 (c) -1 (d) 3 (e) -2

SOLUTION

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = \frac{e^{3(0)}-1}{0} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 0} \frac{3e^{3x}}{1} = 3e^{3(0)} = 3e^0 = 3$$

EXAMPLE 38: 2014/2015 EXAM QUESTION 34

34. Evaluate $\lim_{x\to 0} \left\{\frac{\cosh x - e^x}{x}\right\}$ (a) 0 (b) 1 (c) -1 (d) 0.5 (e) 2

SOLUTION

$$\lim_{x\to 0} \frac{\cosh x - e^x}{x} = \frac{\cosh(0)-e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$$

Applying L' Hopital's rule

$$\lim_{x\to 0} \left[\frac{\frac{d}{dx}(\cosh x - e^x)}{\frac{d}{dx}(x)}\right] = \lim_{x\to 0} \frac{\sinh x - e^x}{1}$$

$$= \frac{\sinh(0)-e^0}{1} = \frac{0-1}{1} = -1$$

EXAMPLE 39: 2014/2015 EXAM QUESTION

32. Evaluate $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$ (a) 0 (b) -1 (c) 0.5 (e) 0.25

SOLUTION

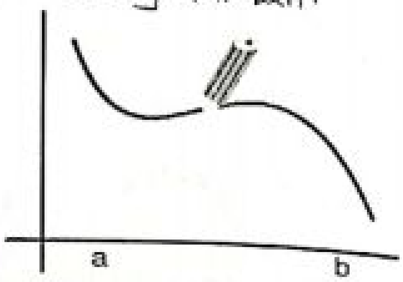
$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \frac{1-\cos 0}{0+0^2} = \frac{1-1}{0+0} =$$

Applying L' Hopital's rule

$$\lim_{x\to 0} \frac{0+\sin x}{1+2x} = \frac{\sin(0)}{1+2(0)} = \frac{0}{1+0} =$$

CONTINUITY

Continuity simply means drawing without lifting the pen.



DEFINITION/ CONDITION FOR CONTINUITY

A function $F(x)$ is said to be continuous at a point $x = x_0$, if and only if it satisfy the following conditions.

i) The function must be defined i.e $F(x_0)$ must exist.

ii) The limit must exist. i.e $\lim_{x\to x_0} F(x)$ must exist

iii) The function at $x=x_0$ must equal the limit as x tends to x_0 of $f(x)$. i.e $F(x_0) = \lim_{x\to x_0} F(x)$

Note: polynomial functions are continuous at all points in the real line. i.e $(-\infty, \infty)$

EXAMPLE 1:

Ex. Show that $f(x) = \begin{cases} \frac{e^x - e^{-x}}{4\sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$ is continuous at $x = 0$

SOLUTION

$F(0)$ exist: $F(0) = 1/2$

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{e^x - e^{-x}}{4\sin x} = \frac{1-1}{0} = \frac{0}{0}$$

Applying L'Hopital's rule,

$$\lim_{x\to 0}\left[\frac{\frac{d}{dx}(e^{x}-e^{-x})}{\frac{d}{dx}(4\sin x)}\right]=\lim_{x\to 0}\frac{e^{x}+e^{-x}}{4\cos x}$$

$$=\frac{e^{0}+e^{-0}}{4\cos 0}=\frac{1+1}{4}=\frac{2}{4}=\frac{1}{2}$$

Since $F(0)=\lim_{x\to 0}F(x)=\frac{1}{2}$

The function is continous at $x=0$

EXAMPLE 2: 2009/2010 EXAM QUESTION 74

74. The function $f(x)=log\sqrt{x^2-1}$ is not continuous at one of the following points of x? (a) 2 (b) 1 (c) 3 (d) -2

SOLUTION

For $F(x)$ to be continous, then $F(x)$ must be defined. i.e $F(x)$ must exist.

But at $x=1$

$$F(1)=\log\sqrt{(1)^2-1}$$
$$=\log\sqrt{0}$$
$$=\log 0$$
$$=\text{undefined}$$

Hence at $x=1$ $F(x)$ D.N.E

$F(x)$ is not continous at $x=1$

EXAMPLE 3: 2011/2012 EXAM QUESTION 35

35. Determine where the function is not continuous $g(t)=\frac{4t+10}{t^2-2t-15}$ (a) 3 or 5 (b) -3 or 5 (c) -3 or -5 (d) 3 or -5

SOLUTION

$g(t)$ is a rational function, and will be discontinoys if the denominator is equal to zero. thus

For $g(t)=\frac{4t+10}{t^2-2t-15}$ to be discontinous, then

$t^2-2t-15=0$; $(t-5)(t+3)=0$

$t=5$ or $t=-3$

EXAMPLE 4: 2011/2012 TEST QUESTION 14

14. $f(x)=\begin{cases}\frac{x^2-4}{x-2} & if\ x\neq \\ k & if\ x=2\end{cases}$

2 *then* $f(x)$ *continuous at* $x=2$ *if* k *is* (a) 2 (b) 4 (c) 6 (d) 8

SOLUTION

$$F(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ k, & x=2\end{cases}$$

$F(2)$ exist; $F(2)=k$

$$\lim_{x\to 2}F(x)=\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{2^2-4}{2-2}=\frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to 2}\left[\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)}\right]=\lim_{x\to 2}\frac{2x}{1}=\frac{2(2)}{1}$$
$$=4$$

Since the function is continous at $x=2$

$\Rightarrow F(2)=\lim_{x\to 2}F(x)$

$\Rightarrow k=4$

CONTINUITY OF A PIECEWISE

A piecewise function is defined on intervals, Hence, We are investigating the interval of continuity of a function.

EXAMPLE 1: 2014/2015 EXAM QUESTION 3

3 At what points, if any, is the following functions not continuous?

$g(x)=\begin{cases}x-3; & x\le -1\\ x^2+1; & -1<x\le 2\\ x^3+4; & x>2\end{cases}$ (a) -3 (b) none (c) 1, -2 (d) -1, 3 (e) -1,2

SOLUTION

$g(-1)=(-1)-3=-4$

g(-1) means to put x=-1 at the branch where $x\le -1$

$$\lim_{x\to -1^-}x-3=(-1)-3=-4$$

$$\lim_{x\to -1^+}=(-1)^2+1=2$$

Since $\lim_{x\to -1^-}g(x)\neq\lim_{x\to -1^+}g(x)$

Then $g(x)$ is not continous at $x=-1$

again, $g(2)=(2)^2+1=5$

Reflect on limit of a piecewise function as to recall when a limit exist

$$\lim_{x\to 2^-}x^2+1=(2)^2+1=5$$

$$\lim_{x\to 2^+}x^3+4=(2)^3+4=12$$

Since $\lim_{x\to 2^-}g(x)\neq\lim_{x\to 2^+}g(x)$

The function is not continous at $x=2$

$g(x)$ is not continous at $x=-1$ and 2

EXAMPLE 2: 2008/2009 EXAM QUESTION 21

21. The value of k given that $f(x)=$
$\begin{cases}x+2 & if\ x\le 1\\ 5+kx^2 & if\ x>1\end{cases}$ *continous at* $x=1$

(a) $\frac{2}{3}$ (b) 4 (c) $\pm\frac{5}{2}$ (d) $-\frac{1}{4}$

SOLUTION

$$F(x) = \begin{cases} x+2 \ ; x \le 1 \\ 5+kx^2 ; x > 1 \end{cases}$$

$$F(1) = 1+2 = 3$$

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x+2 = 1+2 = 3$$

$$\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} 5+kx^2 = 5+k(1)^2$$

$$= 5+k$$

Since the function F(x) is continuous

$\Rightarrow 5+k = 3$ — Equating Left and Righ hand Limit

$K = 3-5$

$K = -2$

FUTO MOST FAVORITE QUESTION ON CONTINUITY EVERY YEAR IS THE CONDITIONS FOR CONTINUITY

ASYMPTOTES

This is a line Tangent to a Curve at infinity. The Curve touches the asymptotes at two Coincident points at infinity.

Asymptote is used in Curve/grave sketching.

We have two asymptotes in a function

1) VERTICAL ASYMPTOTES

A line $x = x_0$ is said to be a vertical asymptote, to a Curve $y = F(x)$ if

$$\lim_{x \to x_0^-} F(x) = \pm\infty \quad \text{or} \quad \lim_{x \to x_0^+} F(x) = \pm\infty$$

2) HORIZONTAL ASYMPTOTES

A line $x = x_0$ is a horizontal asymptote of the Curve $y = F(x)$, IF

$$\lim_{x \to \infty} F(x) = x_0 \quad \text{or} \quad \lim_{x \to -\infty} F(x) = x_0$$

EXAMPLE 1: 2011/2012 TEST QUESTION 24

24. The vertical and horizontal asymptotes of the graph of th function $f(x) = \frac{4x^2}{x^2-9}$ are respectively; (a) 3,4 (b) 4,3 (c 3, ∞ (d) ∞,3

SOLUTION

$$F(x) = \frac{4x^2}{x^2-9}$$

For vertical asymptotes:

While evaluating Vertical Asymptotes we equate denominator to Zero

$$x^2 - 9 = 0$$

$$x^2 - 3^2 = 0$$

$$(x+3)(x-3) = 0$$

$$x = -3 \text{ or } x = 3$$

EXAMPLE 2: 2013/2014 EXAM QUESTION 55

55. Determine all the vertical asymptotes of the graph y = $\frac{x^x+2x-8}{x^2-4}$ A. $x = 2$ B. $x = -2$ C. $x = 0$ D. $x = \pm 2$ E. x = 4

SOLUTION

$$y = \frac{x^2+2x-8}{x^2-4}$$

Factorising,

$$y = \frac{(x+4)(x-2)}{(x+4)(x-4)}$$

$$y = \frac{x-2}{x-4} \ ; \text{ Equating denominator to zero}$$

$$x-4 = 0 \ ; \ x = 4$$

We know that while evaluating Vertical asymptotes we equate denominator to Zero; but we first factorise as to reduce the fraction

EXAMPLE 3: 2013/2014 EXAM QUESTION 37

37. Find the vertical asymptote of the function $y = \frac{x-1}{x^2-1}$ A. $x = -1$ B. $x = 1$ C. $x = -\frac{1}{2}$ D. $x = 0$

SOLUTION

$$y = \frac{x-1}{x^2-1}$$

Factorizing;

$$y = \frac{x-1}{x^2-1^2} \ ; \ y = \frac{x-1}{(x+1)(x-1)}$$

Simplifying;

$$y = \frac{1}{x+1}$$

Equating denominator to zero

$$x+1 = 0 \ ; \ x = -1$$

EXAMPLE 4: 2013/2014 EXAM QUESTION 15

15. Let the asymptote to the curve $x^3(x^2+2) = y^2(x-4)$ parallel to the y – axis. A. x=4 B. y=4 C. x=1 D. y=2 E. x=3

SOLUTION

$$x^3(x^2+2) = y^2(x-4)$$

Making y the subject,

$$y = \pm\sqrt{\frac{x^3(x^2+2)}{x-4}}$$

Note: The asymptote parallel to y-axis is the vertical

For vertical asymptote, we equate denominator to zero $x = 4$

EXAMPLE 5: 2015/2016 EXAM QUESTION 4

4. Determine all the horizontal asymptotes of the function $f(x) = \frac{2x-1}{x+1}$ (a) 2 (b) -2 (c) 0 (d) 10 (e) 4

SOLUTION

let $y = f(x)$, where $f(x) = \frac{2x-1}{x+1}$

$\Rightarrow y = \frac{2x-1}{x+1}$

making 'x' the subject

$(x+1)y = 2x-1$

$xy + y = 2x - 1$

$xy - 2x = -1 - y$

$x(y-2) = -(1+y)$

$x = \frac{1+y}{2-y}$

Equating denominator to zero

$2 - y = 0$; $y = 2$

For Horizontal Asymptote; we make the independent variable "x" the subject formula, Then, we equate denominator to zero. Alternatively, if we take the limit of the function f(x) as $x \to \infty$ Then the limit is our horizontal asymptote ~ UNLEASH

Alternatively:

$\lim_{x \to \infty} f(x)$ is Horizontal Asymptote

$\Rightarrow \lim_{x \to \infty} \frac{2x-1}{x+1} = \frac{2(\infty)-1}{\infty+1} = \frac{\infty}{\infty}$

Applying L' hopital's rule

$\lim_{x \to \infty} \frac{2}{1} = 2$; Hence H.A is at $y = 2$

DIFFERENTIATION

The derivative of a function $y = f(x)$ is defined as the rate of change of y, when the change in x is very small.

The process of finding derivatives is called differentiation.

DIFFERENTIABILITY OF A FUNCTION

A function $y = f(x)$ is said to be differentiable at x, if

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ exist.}$$

If this limit exist, the result is called the derivative of $y = f(x)$ at x which is denoted by $\frac{dy}{dx}$ or $f'(x)$

CONCEPT OF CONTINUITY AND DIFFERNTIABILITY

Continuity and differentiability are intimately related. The continuity of a function is necessary but not sufficient condition for its differentiability. Hence for a function to be differentiable at a particular point, it must first satisfy the condition of continuity at that point, in other words all differentiable functions are continuous, but not all continuous function are differentiable.

DIFFERENTIATION FROM FIRST PRINCIPLE

Given that $y = f(x)$, then a slight increament in x (Δx) will cause a slight increament in y (Δy). Mathematically,

$y + \Delta y = f(x + \Delta x)$

$\Delta y = f(x + \Delta x) - y$

but $y = f(x)$

$\Rightarrow \Delta y = f(x + \Delta x) - f(x)$ --- (i)

This equation is known as Rate of Change in y.

Dividing both sides of eqn (i) by Δx

$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$ --- (ii)

This equation is known as Average rate of change of y with respect to x.

Taking the limit as $\Delta x \to 0$ of $\frac{\Delta y}{\Delta x}$

$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

This equation is known as Instantaneous rate of change or Derivative of y w.r.t x. Hence the formular for first priciple is given by $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

EXAMPLE 1:

Ex. Given $f(x) = x^2 + 2x + 4$. find $f'(x)$ from first principle

SOLUTION

$f(x) = x^2 + 2x + 4$

$f(x + \Delta x) = (x + \Delta x)^2 + 2(x + \Delta x) + 4$

$= x^2 + 2x\Delta x + \Delta x^2 + 2x + 2\Delta x + 4$

using $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, we have;

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 2x + 2\Delta x + 4 - (x^2 + 2x + 4)}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 + 2\Delta x}{\Delta x}$$

Now, we factor out Δx

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x + 2)}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} 2x + \Delta x + 2$$

Taking the limit

$$\lim_{\Delta x \to 0} 2x + \Delta x + 2 = 2x + 0 + 2$$
$$= 2x + 2$$

EXAMPLE 2:

Ex. If $y = lnx$, find $\frac{dy}{dx}$ from first principle

SOLUTION

let $y = f(x)$

$\Rightarrow f(x) = \ln x = \log_e x$

$F(x+\Delta x) = \ln(x+\Delta x) = \log_e(x+\Delta x)$

using $f'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - f(x)}{\Delta x}$

$$\Rightarrow F'(x) = \lim_{\Delta x \to 0} \frac{\log_e(x+\Delta x) - \log_e x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\log_e\left(\frac{x+\Delta x}{x}\right)\right]$$

$$= \lim_{\Delta x \to 0} \frac{x}{\Delta x} \cdot \frac{1}{x}\left[\log_e\left(1 + \frac{\Delta x}{x}\right)\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{x}\log_e\left(1 + \frac{1}{\frac{x}{\Delta x}}\right)^{\frac{x}{\Delta x}}$$

let $n = \frac{x}{\Delta x}$, observe that

if $\Delta x \to 0$, then $n \to \infty$, Hence

$$f'(x) = \lim_{n \to \infty} \frac{1}{x}\log_e\left(1 + \frac{1}{n}\right)^n$$

but $\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

$$\Rightarrow f'(x) = \frac{1}{x}\log_e e = \frac{1}{x}$$

EXAMPLE 3: 2010/2011 EXAM QUESTION 67

67. What is the value of Δy *if* $y = x^2 - 3x + 5$ and when $x = 1, \Delta x = 0.02$ (a) -0.0196 (b) 0.0196 (c) 0.196 (d) -0.196

SOLUTION

$y = x^2 - 3x + 5$

$y + \Delta y = (x+\Delta x)^2 - 3(x+\Delta x) + 5$

$\Delta y = (x+\Delta x)^2 - 3(x+\Delta x) + 5 - y$

but $y = x^2 - 3x + 5$

$\Delta y = x^2 + 2x\Delta x + \Delta x^2 - 3x - 3\Delta x + 5 - x^2 + 3x - 5$

$\Delta y = 2x\Delta x + \Delta x^2 - 3\Delta x$

but when $x = 1$; $\Delta x = 0.02$

$\Delta y = 2(1)(0.02) + (0.02)^2 - 3(0.02)$

$= 0.04 + 0.0004 - 0.06$

$= -0.0196$

EXAMPLE 4: 2012/2013 EXAM QUESTION 4

4. Given that: $f(x) = (x+1)^3$

Find the average rate of change of the function as x changes from 1-5 A. 31 B. 52 C. 62 D. 208

SOLUTION

Average rate of change is given by $\frac{\Delta y}{\Delta x} = \frac{F(x+\Delta x) - f(x)}{\Delta x}$

where $f(x) = (x+1)^3$

$F(x+\Delta x) = (x+\Delta x+1)^3$

where $\Delta x = 5 - 1 = 4$, $x = 1$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{(x+4+1)^3 - (x+1)^3}{4}$$

$$= \frac{(1+4+1)^3 - (1+1)^3}{4}$$

$$= \frac{216 - 8}{4} = 52$$

RULES OF DIFFERENTIATION

Given that $y = F(x) = ax^n$, then

$$\frac{dy}{dx} = f'(x) = nax^{n-1}$$

Where "a" is a constant.

EXAMPLE 1:

Ex. If $f(x) = 2x^5 + 3x^4 - 12x^3$. Find $f'(x)$

SOLUTION

$f(x) = 2x^5 + 3x^4 - 12x^3$

$F'(x) = (5\times 2)x^{5-1} + (3\times 4)x^{4-1} - (12\times 3)x^{3-1}$

$= 10x^4 + 12x^3 - 36x^2$

To Differentiate, you multiply the power of "x" by its coefficient and subtract '1' from the used power.

Thus if $y = 2x^3$ (multiply 3 by 2) (subtract 1)

$\frac{dy}{dx} = 6x^2$ (multiply)

g'(... g(x ... Now, ... g'(... LOG ... This ... deriv ... EXAM ... 36. If y ... (a) − ... Tak ... Diff ... mu ... EXAM ... 32. D

(0.02)

4

on as x

$$g'(x) = \frac{1}{\left(\frac{a-x}{a+x}\right)}\left[\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}\right]$$

$$= \frac{a+x}{a-x}\left[\frac{-a-x-a+x}{(a+x)^2}\right]$$

$$= \frac{a+x}{a-x}\left[\frac{-2a}{(a+x)^2}\right]$$

$$g'(x) = \frac{-2a}{(a-x)(a+x)}$$

Now, $g'(0)$ means to put $x=0$

$$g'(0) = \frac{-2a}{(a-0)(a+0)} = \frac{-2a}{a^2} = -2a^{-1}$$

LOGARITHMIC DIFFERENTIATION

This Technique allows us to find derivative of some complex functions

=1

EXAMPLE 1: 2008/2009 EXAM QUESTION 36

36. If $y = 10^{-3x}$, then the first differential coefficient is (a) $-3(10^{-2x})$ (b) $3ln10$ (c) 3 (d) $-3(10^{-3x}ln10)$

SOLUTION

$y = 10^{-3x}$

Taking $\log_e$ of both sides

$\log_e y = \log_e 10^{-3x}$ $\boxed{\log_e x = \ln x}$

ON

zen

$\ln y = -3x\ln 10$

Differentiating implicitly

!-1

$$\frac{1}{y}\frac{dy}{dx} = -3\ln 10$$

multiplying both sides by y

$$\frac{dy}{dx} = y(-3\ln 10)$$

but $y = 10^{-3x}$

$$\Rightarrow \frac{dy}{dx} = 10^{-3x}(-3\ln 10) = -3(10^{-3x}\ln 10)$$

e^{3-1}

EXAMPLE 2: 2008/2009 EXAM QUESTION 32

32. Differentiate $y^x = x$ w.r.t. x (a) $\frac{y}{x}\left(\frac{1}{x} - lny\right)$ (b) $\frac{x}{y}\left(\frac{1}{x} - lny\right)$ (c) $\frac{y}{x}\left(\frac{1}{lny} - x\right)$ (d) lny

SOLUTION

$y^x = x$

Taking $\ln$ of both sides,

$\ln y^x = \ln x$

$x\ln y = \ln x$

Differentiating implicitly

$$x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \ln y = \frac{1}{x}$$

$$\frac{x}{y}\frac{dy}{dx} = \frac{1}{x} - \ln y$$

multiplying both sides by $\frac{y}{x}$ as to make $\frac{dy}{dx}$ the subject

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{1}{x} - \ln y\right)$$

EXAMPLE 3: 2008/2009 EXAM QUESTION 31

31. Differentiate $y = \log_2 a^x$ (a) $a^x lna$ (b) $2^x ln2$ (c) $\frac{a^x ln}{ln2}$ (d) $\frac{lna}{ln2}$

SOLUTION

$y = \log_2 a^x$

Converting to indicial form

$2^y = a^x$

Taking $\ln$ of both sides

$\ln 2^y = \ln a^x$

$y\ln 2 = x\ln a$

$\ln 2 . \frac{dy}{dx} = \ln a$

Dividing both sides by $\ln 2$

$$\frac{dy}{dx} = \frac{\ln a}{\ln 2}$$

EXAMPLE 4: 2008/2009 EXAM QUESTION 4

4. Differentiate $y = x^4 e^{3x}\tan x$. (a) $\frac{4}{x} + 3 + \frac{sin^2x}{tanx}$ (b) $\frac{4}{x} - 3 + \frac{sin^2x}{tanx}$ (c) $x^4e^{3x}\tan x\left[\frac{4}{x} + 3 + \frac{sin^2x}{tanx}\right]$ (d) $x^4e^3\tan x\left[\frac{4}{x} + 3 + \frac{sin^2x}{tanx}\right]$

SOLUTION

$y = x^4 e^{3x}\tan x$

Taking $\ln$ of both sides

$\ln y = \ln(x^4 . e^{3x}\tan x)$

$\ln y = \ln x^4 + \ln e^{3x} + \ln\tan x$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \frac{4x^3}{x^4} + \frac{3e^{3x}}{e^{3x}} + \frac{\sec^2 x}{\tan x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}$$

multiplying both sides by y

$$\frac{dy}{dx} = y\left(\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right)$$

but $y = x^4 e^{3x}\tan x$

$$\Rightarrow \frac{dy}{dx} = x^4 e^{3x}\tan x\left(\frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}\right)$$

EXAMPLE 5: 2009/2010 EXAM QUESTION 8

8. If $y = \frac{x^2 sinx}{cos2x}$ find $\frac{dy}{dx}$ (a) $\left[\frac{2}{x} + cotx + 2tan2x\right]\frac{x^2 sinx}{cos2x}$ (b) $\frac{2}{1+x^2}$ (c) $x^4e^{3x}tanx\left[\frac{4}{x} + 3 + \frac{sec^2x}{tanx}\right]$ (d) $\frac{1+x}{...}$

$y = \frac{x^2 \sin x}{\cos 2x}$

Taking ln of both sides,

$\ln y = \ln\left(\frac{x^2 \sin x}{\cos 2x}\right)$

$\ln y = \ln x^2 + \ln \sin x - \ln \cos 2x$

Differentiating implicitly,

$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2} + \frac{\cos x}{\sin x} - \frac{(-2\sin 2x)}{\cos 2x}$

Note: If $y = \ln[f(x)]$
Then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$\frac{\cos x}{\sin x} = \frac{1}{\tan x}$
and $\frac{1}{\tan x} = \cot x$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \cot x + 2\tan 2x$

multiplying both sides by y

$\frac{dy}{dx} = y\left[\frac{2}{x} + \cot x + 2\tan 2x\right]$

but $y = \frac{x^2 \sin x}{\cos 2x}$

$\frac{dy}{dx} = \frac{x^2 \sin x}{\cos 2x}\left[\frac{2}{x} + \cot x + 2\tan 2x\right]$

EXAMPLE 6: 2013/2014 EXAM QUESTION 42

42. Find $\frac{dy}{dx}$ $if\ y = x^x, x > 0$ $A.\ x^x \ln x$ $B.\ x(x^{x-1})$ $C.\ x^x(1-\ln x)$ $D.\ x^x(1+\ln x)$ $E.\ x(1-\ln x)$

SOLUTION

$y = x^x$

taking ln of both sides

$\ln y = \ln x^x$

$\ln y = x\ln x$

Differentiating implicitly

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x\,(1)$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = y(1+\ln x)$

$\frac{dy}{dx} = x^x(1+\ln x)$

EXAMPLE 7: 2013/2014 EXAM QUESTION 31

31. If $y = a^x$, where $a \in R$, find $\frac{dy}{dx}$ $A.\ \frac{a^x}{\ln a}$ $B.\ x\ln a$ $C.\ xa^{x-1}$ $D.\ a^x \ln a$ E. $a\ln x$

SOLUTION

$y = a^x$

Taking ln of both sides

$\ln y = \ln a^x$

$\ln y = x\ln a$

Differentiating implicitly

$\frac{1}{y}\frac{dy}{dx} = \ln a$

multiplying both sides by y.

$\frac{dy}{dx} = y\ln a$

but $y = a^x$

$\Rightarrow \frac{dy}{dx} = a^x \ln a$

HIGHER ORDER DIFFERENTIATION

If $y = f(x)$, Then differentiating once, We have

$$\frac{dy}{dx} = f'(x)$$

If we differentiate twice, we have

$$\frac{d^2y}{dx^2} = f''(x)$$

If we differentiate again, we have the 3rd derivative as

$$\frac{d^3y}{dx^3} = f'''(x)$$

In general, The n^{th} order derivative of y with respect to x is;

$$\frac{d^ny}{dx^n} = f^n(x)$$

EXAMPLE 1: 2011/2012 EXAM QUESTION 39

39. Given that $f(x) = x^4 + 5x^3 - 3x^2 - 6x + 8$. Find $f''(x)$ at $x = 1$ (a) 31 (b) 36 (c) 54 (d) 24

SOLUTION

$F(x) = x^4 + 5x^3 - 3x^2 - 6x + 8$

$F'(x) = 4x^3 + 15x^2 - 6x - 6$

$F''(x) = 12x^2 + 30x - 6$

$F''(1) = 12(1)^2 + 30(1) - 6 = 36$

EXAMPLE 2: 2011/2012 EXAM QUESTION 27

27. Evaluate $\frac{d^3y}{dy^3}$ for $y = x^3e^x$ at $x = 0$ (a) 1 (b) 0 (c) 6 (d) 3

SOLUTION

$y = x^3e^x$

$\frac{dy}{dx} = x^3e^x + 3x^2e^x$

$\frac{d^2y}{dx^2} = (x^3e^x + 3x^2e^x) + 3x^2e^x + 6xe^x$

$\frac{d^2y}{dx^2} = x^3e^x + 6x^2e^x + 6xe^x$

$\frac{d^3y}{dx^3} = x^3e^x + 3x^2e^x + 6x^2e^x + 12xe^x + 6xe^x + 6e^x$

08 ... at $\frac{d^3y}{dx^3} =$ ...

EXAMPLE ...

66. Find $\frac{d^3y}{dx^3}$ ... $-27\cos$ ...

using ...

EXAMPL...

4. Find y if ... $x\cos x$ ... $2\cos x$ ...

EXAMP...

44. If y = ... (c) a^2y ...

Now, ...

at $x=0$

$$\frac{d^3y}{dx^3} = 0e^0 3(0)^2e^0 + 6(0)^2e^0 + 12e^0 + 6(0)e^0 + 6e^0$$

$$= 0+0+0+0+0+6(1)$$

$$= 6$$

EXAMPLE 3: 2013/2014 EXAM QUESTION 66

66. Find $\frac{d^3y}{dx^3}$ if $y = \cos 3x$ A. $-27\sin 3x$ B. $27\sin 3x$ C. $-27\cos 3x$ D. $20\cos 3x$ E. $-27\cos x$

SOLUTION

$y = \cos 3x$

let $u = 3x$, $\frac{du}{dx} = 3$

using $\Rightarrow y = \cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \times 3$$

$$= 3\cos 3x$$

Differentiating again in same process

$$\frac{d^2y}{dx^2} = (-3)(3)\sin 3x$$

$$= -9\sin 3x$$

Differentiating again,

$$\frac{d^3y}{dx^3} = (-9)(3)\cos 3x$$

$$\frac{d^3y}{dx^3} = -27\cos 3x$$

EXAMPLE 4: 2013/2014 EXAM QUESTION 4

4. Find y if $y = x\sin x$ A. $2\cos x - x\sin x$ B. $\sin x + x\cos x$ C. $\sin x - x\cos x$ D. $x\sin x - 2\cos x$ E. $x\sin x - \cos x$

SOLUTION

$y = x\sin x$

using product rule

$$\frac{dy}{dx} = x\cos x + \sin x$$

using product rule; for $x\cos x$

$$\frac{d^2y}{dx^2} = -x\sin x + \cos x + \cos x$$

$$= -x\sin x + 2\cos x$$

$$= 2\cos x - x\sin x$$

EXAMPLE 5: 2014/2015 EXAM QUESTION 44

44. If $y = e^{-ax}$, find $(y'' + 3ay')$ in terms of y (a) y^2 (b) $(ay)^2$ (c) a^2y (d) $-a^2y$ (e) ay^2

SOLUTION

$y = e^{-ax}$, $y' = -ae^{-ax}$,

$y'' = a^2e^{-ax}$

Now, $y'' + 2ay' = a^2e^{-ax} + 2a(-ae^{-ax})$

$$= a^2e^{-ax} - 2a^2e^{-ax}$$

$$= (a^2 - 2a^2)e^{-ax}$$

$$= -a^2e^{-ax} = -a^2y$$

INVERSE TRIGONOMETRY FUNCTION

EXAMPLE 1:

Ex. Given that $y = \sin^{-1}x$, then what is $\frac{dy}{dx}$?

SOLUTION

If $y = \sin^{-1}x$, Then $x = \sin y$

$$\frac{dx}{dy} = \cos y \; ; \; \frac{dy}{dx} = \frac{1}{\cos y}$$

From elementary Trignometry:

$$\cos^2y + \sin^2y = 1$$

$$\cos^2y = 1 - \sin^2y$$

$$\cos y = \sqrt{1-\sin^2y}$$

Substituting the value of $\cos y$;

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2y}}$$

but $x = \sin y$

$\Rightarrow x^2 = \sin^2y$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

PARAMETRIC DIFFERENTIATION

Given that $y = f(t)$ and $x = f(t)$

Then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

EXAMPLE 1: 2009/2010 EXAM QUESTION 36

68. What is $\frac{dy}{dx}$ in terms of parameter t when $y = t^2$ and $x = t^3$ (a) $\frac{2}{3t}$ (b) $\frac{2t}{3}$ (c) $\frac{3t}{2}$ (d) $\frac{3}{2t}$

SOLUTION

$$y = t^2 \; ; \; \frac{dy}{dt} = 2t$$

$$x = t^3 \; ; \; \frac{dx}{dt} = 3t^2$$

using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Where $\frac{dt}{dx} = \frac{1}{3t^2}$

$$\Rightarrow \frac{dy}{dx} = (2t)\left(\frac{1}{3t^2}\right)$$

$$\frac{dy}{dx} = \frac{2}{3t}$$

EXAMPLE 2: 2009/2010 EXAM QUESTION 75

75. $y = \cos 2t$ and $x = \sin t$, what is y'' (a) -4 (b) $4\tan t$ (c) $-4\cos 2t$ (d) $\sec^2 2t$

SOLUTION

$$y = \cos 2t \; ; \; \frac{dy}{dt} = -2\sin 2t$$

$x = \sin t$; $\frac{dx}{dt} = \cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$; where $\frac{dt}{dx} = \frac{1}{\cos t}$

$\frac{dy}{dx} = (-2\sin 2t) \cdot \left(\frac{1}{\cos t}\right)$

$\frac{dy}{dx} = -2\frac{\sin 2t}{\cos t}$

But From Elementary Trignometry

$\sin(A+B) = \sin A\cos B + \sin B\cos A$

Hence $\sin(2t) = \sin(t+t) = \sin t\cos t + \sin t\cos t$

$\Rightarrow \sin 2t = 2\sin t\cos t$

$\frac{dy}{dx} = \frac{-2 \cdot 2\sin t\cos t}{\cos t} = -4\sin t$

$\frac{d^2y}{dx^2} = \left[\frac{d(-4\sin t)}{dt}\right]\left(\frac{dt}{dx}\right) = -4\cos t \cdot \frac{1}{\cos t}$

$= -4$

EXAMPLE 3: 2009/2010 EXAM QUESTION 64

64. If $y = t^2$ and $x = \frac{1}{t}$, what is $\frac{dy}{dx}$ (a) $-2t^3$ (b) $2t^3$ (c) $-\frac{2}{t}$ (d) $\frac{2}{t}$

SOLUTION

$y = t^2$, $\frac{dy}{dt} = 2t$

$x = \frac{1}{t}$; $x = t^{-1}$; $\frac{dx}{dt} = -t^{-2}$

$\frac{dt}{dx} = -t^2$

$\frac{dy}{dx} = (2t)(-t^2)$

$= -2t^3$

EXAMPLE 4: 2012/2013 EXAM QUESTION 36

36. If $y = \sin 2t$ and $x = \cos 2t$, find $\frac{dy}{dx}$ A. $-\tan 2t$ B. $\cot 2t$ C. $-\cot 2t$ D. $-\tan 2t$

SOLUTION

$y = \sin 2t$; $\frac{dy}{dt} = 2\cos 2t$

$x = \cos 2t$; $\frac{dx}{dt} = -2\sin 2t$

but $\frac{dt}{dx} = \frac{1}{-2\sin 2t}$

using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2\cos 2t}{-2\sin 2t} = -\cot 2t$

EXAMPLE 5: 2012/2013 TEST QUESTION 10

10. Find y' if $x = 2\cos t + \cos 2t$ and $y = 2\sin t - \sin 2t$ for $t = \frac{\pi}{2}$ (a) 1 (b) -1 (c) 2 (d) 10

SOLUTION

$x = 2\cos t + \cos 2t$

$\frac{dx}{dt} = -2\sin t - 2\sin 2t$

$\frac{dt}{dx} = \frac{1}{-2(\sin t + \sin 2t)}$

$y = 2\sin t - \sin 2t$

$\frac{dy}{dt} = 2\cos t - 2\cos 2t$

$\frac{dy}{dx} = \frac{2\cos t - 2\cos 2t}{-2(\sin t + \sin 2t)}$

$\frac{dy}{dx} = \frac{2(\cos t - \cos 2t)}{-2(\sin t + \sin 2t)}$

$\frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin t + \sin 2t}$ at $t = \frac{\pi}{2}$

$= \frac{\cos 90 - \cos 180}{\sin 90 + \sin 180} = \frac{0-(-1)}{1+0} = \frac{1}{1} = 1$

EXAMPLE 6: 2014/2015 EXAM QUESTION 36

36. Find $\frac{dy}{dx}$ in terms of y if $x = 3e^{-t}$ and $y = \frac{1}{2}e^{-t}$ (a) ... (b) $\frac{2}{3}y$ (c) $\frac{2}{3}y^2$ (d) $\frac{-2}{3}y^2$ (e) $\frac{1}{2}y^2$

SOLUTION

$x = 3e^{-t}$; $\frac{dx}{dt} = -3e^{-t}$

$\frac{dt}{dx} = -3e^{t}$

Again ; $y = \frac{1}{2}e^{-t}$; $\frac{dy}{dt} = -\frac{1}{2}e^{-t}$

using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{1}{2}e^{-t}(-\frac{1}{3}e^{t})$

$\Rightarrow \left(\frac{-1}{2} \cdot \frac{-1}{3}\right)e^{-t+t} = \frac{1}{6}e^0 = \frac{1}{6}$

EXAMPLE 7: 2014/2015 EXAM QUESTION 14

14. Find $\frac{dy}{dx}$ if $x = 2 + \sin^2 t$ and $y = 5 - \sec t$ (a) $\tan t$ (b) $\frac{1}{2}\sec^3 t$ (c) $\cot^2 t$ (d) $\sin^2 t$ (e) $\sec^2 t$

SOLUTION

$x = 2 + \sin^2 t$, $y = 5 - \sec t$

using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$; where

$\frac{dy}{dt} = \frac{d}{dt}\left(5 - \frac{1}{\cos t}\right) = (\cos t)^{-2}\sin t$

$\frac{dy}{dt} = \frac{-\sin t}{\cos^2 t}$

$\frac{dx}{dt} = 2\sin t\cos t$; $\frac{dt}{dx} = \frac{1}{2\sin t\cos t}$

$$\frac{dy}{dt} = \frac{-\sin t}{\cos^2 t} \times \frac{1}{2\sin t\cos t}$$

$$= -\frac{1}{2\cos^3 t} = -\frac{1}{2}\cdot\frac{1}{\cos^3 t}$$

$$= -\frac{1}{2}\sec^2 t$$

IMPLICIT DIFFERENTIATION

An expression is said to be explicit if $y = F(x)$, but if an equation is given by;

$y = F(x,y)$, Then it is implicit.

EXAMPLES OF IMPLICIT DIFFERENTIATION

1] $x^2 + y^2 = 4$ [ii] $x^2 + 2xy + x^3 = 10$

3] $x\sin y + 4x^2y = e^x$ [4] $\sin xy + x^2 = y^2$

SOLUTION TO IMPLICIT DIFFERENTIATION

Given $y = f(x,y)$, We differentiate with respect to x and attach dx, we also attach dy when we differentiate w.r.t. y.

Then we divide all through by dx and make the corresponding $\frac{dy}{dx}$ the Subject Formular.

EXAMPLE 1:

Ex. Find y' if $y^2 + x^2 = 4$

SOLUTION

$$y^2 + x^2 = 4$$

Differentiating implicitly

$$2y\,dy + 2x\,dx = 0$$

Dividing through by dx

$$2y\frac{dy}{dx} + 2x\frac{dx}{dx} = \frac{0}{dx}$$

$$2y\frac{dy}{dx} + 2x = 0$$

making $\frac{dy}{dx}$ Subject formular

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

EXAMPLE 2: 2008/2009 EXAM QUESTION 46

46. Find y' given that $xy + x - 2y - 1 = 0$ (a) $y' = \frac{1}{(2-x)}$ (b) $y' = \frac{(1+x)}{(2-x)}$ (c) $y' = 2 - x$ (d) $y' = \frac{(1+y)}{(2-x)}$

SOLUTION

$$xy + x - 2y - 1 = 0$$

$$x\frac{dy}{dx} + y + 1 - 2\frac{dy}{dx} = 0$$

Alternatively, you will attach $\frac{dy}{dx}$ when you differentiate w.r.t. y and nothing when you differentiate w.r.t. x.

Factorising $\frac{dy}{dx}$ and making it subject

$$x\frac{dy}{dx} - 2\frac{dy}{dx} = -1 - y$$

$$\frac{dy}{dx}(x-2) = -(1+y)$$

Dividing both sides by $x-2$

$$\frac{dy}{dx} = \frac{-(1+y)}{x-2}$$

multiplying Negative by the Numerator and denominator of the R.H.S

$$\frac{dy}{dx} = \frac{1+y}{-x+2}\ ;\ \frac{dy}{dx} = \frac{1+y}{2-x}$$

EXAMPLE 3: 2009/2010 EXAM QUESTION 65

65. What is, $\frac{dy}{dx}$, if $x^2y^2 - x - y = 0$? (a) $\frac{1-2xy^2}{2x^2y-1}$ (b) $\frac{1+2xy^2}{2x^2y-1}$ (c) $\frac{2x^2y-1}{1-2xy^2}$ (d) $\frac{2x^2y+1}{1-2xy^2}$

SOLUTION

$$x^2y^2 - x - y = 0$$

Differentiating implicitly

$$(x^2)\left(2y\frac{dy}{dx}\right) + y^2(2x) - 1 - \frac{dy}{dx} = 0$$

$$2x^2y\frac{dy}{dx} + 2xy^2 - 1 - \frac{dy}{dx} = 0$$

$$2x^2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2xy^2$$

$$\frac{dy}{dx}(2x^2y - 1) = 1 - 2xy^2$$

$$\frac{dy}{dx} = \frac{1 - 2xy^2}{2x^2y - 1}$$

EXAMPLE 4: 2009/2010 EXAM QUESTION 36

36. Given that $U = f(x,y) = x\sin y + y\sin x$, if $x = \log t$, $y = e^t$, find $\frac{dU}{dt}$ (a) $(\sin y + y\cos x)\left(\frac{1}{t}\right) + (x\cos y + \sin x)e^t$ (b) $(\sin x + x\cos x)\left(\frac{1}{t}\right) + (x\cos y + \sin x)e^t$ (c) $(\sin y + x\cos x)\left(\frac{1}{t}\right) + (x\cos x + \sin x)e^t$ (d) $(\sin y + y\cos x)e^t + (x\cos y + \sin x)\left(\frac{1}{t}\right)$

SOLUTION

using $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Given that $x = \log_e t$, Then $\frac{dx}{dt} = \frac{1}{t}$

Again if $y = e^t$, Then $\frac{dy}{dt} = e^t$

But $\frac{\partial u}{\partial x} = u_x = \frac{\partial(x\sin y + y\sin x)}{\partial x}$

$= \sin y + y\cos x$

Again; $\frac{\partial u}{\partial y} = u_y = \frac{\partial(x\sin y + y\sin x)}{\partial y}$

$= x\cos y + \sin x$

Hence: $\frac{dy}{dt} = \frac{1}{t}(\sin y + y\cos x) + e^t(x\cos y + \sin x)$

EXAMPLE 5: 2009/2010 EXAM QUESTION 35

35. Given that $f(x,y) = x^2 + 3xy^2 - 5$, find $\frac{dy}{dx}$ at $x = y = 1$ (a) $\frac{5}{6}$ (b) $\frac{2}{6}$ (c) $-\frac{5}{6}$ (d) $\frac{6}{5}$

SOLUTION

Let $F(x,y) = x^2 + 3xy^2 - 5 = 0$

Differentiating implicitly

$2x + 3x(2y\frac{dy}{dx}) + 3(y^2) = 0$

$2x + 6xy\frac{dy}{dx} + 3y^2 = 0$

$6xy\frac{dy}{dx} = -2x - 3y^2$

$\frac{dy}{dx} = \frac{-(2x + 3y^2)}{6xy}$

at $x = 1,\ y = 1$

$\frac{dy}{dx} = \frac{-[2(1) + 3(1)^2]}{6(1)(1)}$

$= -\frac{5}{6}$

EXAMPLE 6: 2010/2011 EXAM QUESTION 55

55. If $x^2 + y^2 - 2x - 6y + 5 = 0$ find $\frac{dy}{dx}$ at (3,2) (a) -2 (b) 2 (c) 3 (d) -3

SOLUTION

$x^2 + y^2 - 2x - 6y + 5 = 0$

Differentiating implicitly,

$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$

Factorizing $\frac{dy}{dx}$;

$2y\frac{dy}{dx} - 6\frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx}(2y - 6) = 2 - 2x$

Dividing both sides by $2y - 6$

$\frac{dy}{dx} = \frac{2 - 2x}{2y - 6}$

at (3,2); $x = 3$ and $y = 2$

$\Rightarrow \frac{dy}{dx} = \frac{2 - 2(3)}{2(2) - 6} = \frac{2 - 6}{4 - 6}$

$= \frac{-4}{-2} = 2$

EXAMPLE 7: 2010/2011 EXAM QUESTION 19

19. Given the function $2y^2x - x = 6 + y$, what is $\frac{dy}{dx}$ (a) $\frac{1-2y^2}{4xy-1}$ (b) $\frac{4xy-1}{1-2y^2}$ (c) $\frac{4xy+1}{1-2y^2}$ (d) $\frac{4xy-1}{1+2y^2}$

SOLUTION

$2y^2x - x = 6 + y$

$2y^2x - x - y - 6 = 0$

Differentiating implicitly

$2y^2 + 4y\frac{dy}{dx}(x) - 1 - \frac{dy}{dx} = 0$

$4xy\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y^2$

$\frac{dy}{dx}(4xy - 1) = 1 - 2y^2$

$\frac{dy}{dx} = \frac{1 - 2y^2}{4xy - 1}$

EXAMPLE 8: 2011/2012 EXAM QUESTION 54

54. If $xy + \sin(xy) = 3$, find $\frac{dy}{dx}$ at $\left(1, \frac{\pi}{3}\right)$ (a) 2π (b) 3π (c) -2π (d) $-\pi$

SOLUTION

$xy + \sin xy = 3$

Using partial Differentiation;

$\frac{dy}{dx} = -\frac{f_x}{f_y}$

Where $f_x = y + y\cos xy$

$f_y = x + x\cos xy$

$\frac{dy}{dx} = -\frac{(y + y\cos xy)}{x + x\cos xy}$

$= \frac{-y(1 + \cos xy)}{x(1 + \cos xy)}$

$\frac{dy}{dx} = -\frac{y}{x}$; at $(1, \frac{\pi}{3})$

$x = 1,\ y = \pi/3$

$\frac{dy}{dx} = -\frac{\frac{\pi}{3}}{1} = -\frac{\pi}{3}$

EXAMPLE 9: 2011/2012 EXAM QUESTION 9

9. Find $\frac{dy}{dx}$ at $(x,y) = (-1,1)$ if $x^3 + y^3 - x^2y + 1 = 0$ (a) $\frac{7}{2}$ (b) $-\frac{5}{2}$ (c) $-\frac{9}{2}$ (d) none of the above

SOLUTION

$x^3 + y^3 - x^2y + 1 = 0$

Differentiating implicitly;

$3x^2 + 3y^2\frac{dy}{dx} - \left(x^2\frac{dy}{dx} + 2xy\right) = 0$

$3x^2 + 3y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy = 0$

factorising $\frac{dy}{dx}$;

$3y^2\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - 3x^2$

$\frac{dy}{dx}(3y^2 - x^2) = 2xy - 3x^2$

Dividing both sides by $3y^2 - x^2$

$\frac{dy}{dx} = \frac{2xy - 3x^2}{3y^2 - x^2}$

at $(-1,1)$; $x = -1$ and $y = 1$

$\frac{dy}{dx} = \frac{2(-1)(1) - 3(-1)^2}{3(1)^2 - (-1)^2} = \frac{-2-3}{3-1}$

$= \frac{-5}{2}$

EXAMPLE 10: 2011/2012 TEST QUESTION 9

9. If $4x^2 - 8xy + 4y^3 = 0$ then $\frac{dy}{dx}$ at $(2,3)$ is (a) $\frac{2}{27}$ (b) $\frac{4}{23}$ (c) $\frac{2}{23}$ (d) $\frac{4}{27}$

SOLUTION

$4x^2 - 8xy + 4y^3 = 0$

Differentiating implicitly;

$8x - 8x\frac{dy}{dx} - 8y + 12y^2\frac{dy}{dx} = 0$

$12y^2\frac{dy}{dx} - 8x\frac{dy}{dx} = 8y - 8x$

$\frac{dy}{dx}(12y^2 - 8x) = 8y - 8x$

Dividing both sides by $(12y^2 - 8x)$

$\frac{dy}{dx} = \frac{8y - 8x}{12y^2 - 8x}$; at $(2,3)$

$\frac{dy}{dx} = \frac{8(3) - 8(2)}{12(3)^2 - 8(2)} = \frac{24-16}{108-16} = \frac{4}{41}$

EXAMPLE 11: 2012/2013 EXAM QUESTION 37

37. If $xy + \sin(xy) = 5$, find $\frac{dy}{dx}$ at $(1,1)$ A. 1 B. -2 C. -1 D. 2

SOLUTION

$xy + \sin(xy) = 5$

$\frac{dy}{dx} = -\frac{y}{x}$ Check Solution on Example 8

at $(1,1)$; $x=1$ and $y=1$

$\frac{dy}{dx} = \frac{-1}{1} = -1$ 2011/2012 Exam Question 14

EXAMPLE 12: 2012/2013 EXAM QUESTION 45

45. Find y' for which $x^2 + y^2 - 2x - 6y + 7 = 0$ A. $\frac{1-x}{y-3}$ B. $\frac{y-3}{1-x}$ C. $\frac{x-1}{y-3}$ D. $\frac{1-x}{3-y}$

SOLUTION

$x^2 + y^2 - 2x - 6y + 7 = 0$

Differentiating implicitly;

$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} - 6\frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx}(2y - 6) = 2 - 2x$

$\frac{dy}{dx} = \frac{2-2x}{2y-6} = \frac{2(1-x)}{2(y-3)} = \frac{1-x}{y-3}$

EXAMPLE 12: 2012/2013 TEST QUESTION 23

23. Given the implicit function $xy + \sin y - 13 = 0$, what is its derivative? (a) $y(x - \cos\ y)^{-1}$ (b) $-y(x + \cos\ y)^{-1}$ (c) $x(y - \cos\ y)^{-1}$ (d) $x(y + \cos x)^{-1}$

SOLUTION

$xy + \sin y - 13 = 0$

Differentiating implicitly,

$x\frac{dy}{dx} + y + \frac{dy}{dx}\cos y = 0$

$x\frac{dy}{dx} + \frac{dy}{dx}\cos y = -y$

$\frac{dy}{dx}(x + \cos y) = -y$

$\frac{dy}{dx} = \frac{-y}{x + \cos y}$

$= -y(x + \cos y)^{-1}$

EXAMPLE 13: 2012/2013 TEST QUESTION 15

15. Determine the gradient of the equation: $xy^2 + 2x + y - 2 = 0$

(a) $y' = \frac{2-y^2}{1+2xy}$ (b) $y' = \frac{y^2+2}{1-2xy}$ (c) $y' = -\frac{(y^2+2)}{1+2xy}$ (d) $y' = \frac{y^2-y^2}{1+2xy}$

SOLUTION

$xy^2 + 2x + y - 2 = 0$

$\underbrace{x \cdot 2y\frac{dy}{dx} + (1)\cdot y^2}_{\text{Product rul}} + 2 + 1\cdot\frac{dy}{dx} - 0 = 0$

$\boxed{x\cdot 2y\frac{dy}{dx} + 1\cdot y^2} + 2 + 1\cdot\frac{dy}{dx} - 0 = 0$

Product rule using implicit Differentiation

$2xy\frac{dy}{dx} + y^2 + 2 + \frac{dy}{dx} = 0$

$2xy\frac{dy}{dx} + \frac{dy}{dx} = -2 - y^2$

$\frac{dy}{dx} = \frac{-(2+y^2)}{2xy+1} = \frac{-(y^2+2)}{1+2xy}$

EXAMPLE 14: 2013/2014 EXAM QUESTION 56

56. Find $\frac{d^2y}{dx^2}$ at $(1,1)$ if $x^2+y^2=25$
A. -25 B. 25 C. 5 D. -5 E. 1

SOLUTION

$x^2 + y^2 = 25$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

let $y = \frac{dy}{dx}$

$\Rightarrow y' = -\frac{x}{y}$

Cross multiplying

$y\cdot y' = -x$

using product rule for the L.H.S

$y\cdot y'' + y'\cdot y' = -1$

$y\cdot y'' + (y')^2 = -1$

making y'' the Subject Formula

$y'' = \frac{-1-(y')^2}{y}$; but $y' = \frac{-x}{y}$

$\Rightarrow y'' = \frac{-1-\left(\frac{x}{y}\right)^2}{y} = \frac{-1-\frac{x^2}{y^2}}{y} = \frac{-y^2-x^2}{y^3}$

at $(1,1)$; $y'' = -2$

EXAMPLE 15: 2014/2015 EXAM QUESTION 43

43. If $x^2 - 3xy + 4y = 0$, find y' at $(1, 1)$ (a) 0 (b) 1 (c) -1 (d) 4 (e) 2

SOLUTION

$x^2 - 3xy + 4y = 0$

Differentiating implicitly

$2x - 3x\frac{dy}{dx} - 3y + 4\frac{dy}{dx} = 0$

$4\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$

$\frac{dy}{dx}(4-3x) = 3y - 2x$

$\frac{dy}{dx} = \frac{3y-2x}{4-3x}$

at $(1,1)$; $x=1$ and $y=1$

$\frac{dy}{dx} = \frac{3(1)-2(1)}{4-3(1)} = \frac{3-2}{4-3}$

$= \frac{1}{1} = 1$

EXAMPLE 16: 2014/2015 EXAM QUESTION 6

6. Find the $\frac{dy}{dx}$ of the equation $x^2 + xy + \cos y = 8y$ (a) $\frac{-(2x+y)}{x-\sin y-8}$ (b) $\frac{y-x}{x^2-\sin y-8}$ (c) $\frac{y-x}{x^2-\cos y-8}$ (d) $\frac{-2x-y}{x-\cos y-8}$ (e) $\frac{2x-y}{x-\cos y-8}$

SOLUTION

$x^2 + xy + \cos y = 8y$

Differentiating implicitly

$2x + x\frac{dy}{dx} + y - \frac{dy}{dx}\sin y = 8\frac{dy}{dx}$

Collecting term on $\frac{dy}{dx}$;

$x\frac{dy}{dx} - \frac{dy}{dx}\sin y - 8\frac{dy}{dx} = -y - 2x$

$\frac{dy}{dx}(x - \sin y - 8) = -(y+2x)$

$\frac{dy}{dx} = \frac{-(2x+y)}{x-\sin y-8}$

APPLICATION OF DIFFERENTIATION

Differentiation plays a major role in elementary calculus, for the interest of MTH 102, we shall study the following Applications.

i) Tangent and Normal
ii) Approximations
iii) Related rates
iv) Extremum
v) Kinematics
vi) Rolle's theorem
vii) mean value theorem
viii) Taylor's/ maclurin Series

TANGENT AND NORMAL

A. Tangent to a Curve $y = F(x)$ is a straight line which cuts the curve at a point (x_0, y_0).

The slope of a Tangent to a Curve at any point $P(x_0, y_0)$ is given by $\frac{dy}{dx} = f'(x) = M$

Hence, the equation of Tangent line to a curve is given by;

$$\boxed{\frac{y-y_0}{x-x_0} = M} \quad \text{or} \quad y - y_0 = M(x-x_0)$$

Normal is a line perpendicular to the tangent of a curve at any point $P(x_0, y_0)$

it is given by;

$$\boxed{\frac{y-y_0}{x-x_0} = -\frac{1}{M}}$$

EXAMPLE 1: 2008/2009 EXAM QUESTION 48

48. Find the equation of the tangent line to the $y = \frac{1}{\sqrt{x-1}}$ when $x = 2$. (a) $y - 1 = \frac{-1}{2}(x-2)$ (b) $y - 1 = \frac{1}{2}(x-2)$ (c) $y + 1 = -\frac{-1}{2}(x-2)$ (d) $y + 1 = \frac{1}{2}(x-2)$

SOLUTION

$y = \frac{1}{\sqrt{x-1}} = \frac{1}{(x-1)^{1/2}} = (x-1)^{-1/2}$

When $x = 2$

$\Rightarrow y = (2-1)^{-1/2} = (1)^{-1/2} = 1$

Hence, point of evaluation is $(2,1)$

Where $\frac{dy}{dx} = -\frac{1}{2}(x-1)^{-3/2}$

at $x = 2$

$\frac{dy}{dx} = -\frac{1}{2}(2-1)^{-3/2} = -\frac{1}{2}(1)^{-3/2}$

$= -\frac{1}{2}$

Equation of Tangent is given by;

$\frac{y-y_0}{x-x_0} = M$

Where $x_0 = 2$, $y_0 = 1$, $M = \frac{dy}{dx}\Big|_{x=2} = \frac{-1}{2}$

$\Rightarrow \frac{y-1}{x-2} = -\frac{1}{2}$; $y - 1 = -\frac{1}{2}(x-2)$

EXAMPLE 2: 2008/2009 EXAM QUESTION 34

34. Find the equation of normal line to the curve; $y = 3x^2 + 2x + 4$ at $x = 1$ (a) $8y - x - 71 = 0$ (b) $8y + x - 71 = 0$ (c) $8y - x + 71 = 0$ (d) $8y + x + 71 = 0$

SOLUTION

$y = 3x^2 + 2x + 4$

at $x = 1$; $y = 3(1) + 2(1) + 4$

$y = 3 + 2 + 4$

When $x = 1$, $y = 9$

Hence; Evaluation point $(x_0, y_0) = (1, 9)$

$\frac{dy}{dx} = 6x + 2$; $\frac{dy}{dx}\Big|_{x=1} = 6(1) + 2 = 8$

$\Rightarrow M = 8$

Equation of Normal is given by

$\frac{y-y_0}{x-x_0} = -\frac{1}{M}$

$\Rightarrow \frac{y-9}{x-1} = -\frac{1}{8}$

$8(y-9) = -1(x-1)$

$8y - 72 = -x + 1$

$8y + x - 73 = 0$

EXAMPLE 3: 2012/2013 EXAM QUESTION 22

22. Find the equation of the tangent to the curve $x^2y + y^3x + 3x - 13 = 0$ at $(1,2)$ A. 15y+13x-41=0 B. 13y+15x-41=0 C. 5y+3x-4=0 D. 3y+5x-4=0

SOLUTION

$x^2y + y^3x + 3x - 13 = 0$ at $(1,2)$

$\frac{dy}{dx} = -\left(\frac{2xy + y^3 + 3}{x^2 + 3y^2x}\right)$

at $(1,2)$

$\frac{dy}{dx} = -\left[\frac{2(1)(2) + (2)^3 + 3}{(1)^2 + 3(2)^2(1)}\right]$

$= -\left(\frac{4 + 8 + 3}{1 + 12}\right)$

$= -\frac{15}{13}$

Using. $\frac{y-y_0}{x-x_0} = M$

Where $y_0 = 2$, $x_0 = 1$, $M = -\frac{15}{13}$

$\frac{y-2}{x-1} = \frac{-15}{13}$

$13(y-2) = -15(x-1)$

$13y - 26 = -15x + 15$

$13y + 15x - 41 = 0$

EXAMPLE 4: 2012/2013 EXAM QUESTION 48

48. The function of the tangent to curve $y = 3x^2 - 5x$ at $(1, -2)$ is given by A. $x - 1$ B. $x + 5$ C. $x - 3$ D. $x +$ [illegible]

SOLUTION

$y = 3x^2 - 5x$

$\frac{dy}{dx} = 6x - 5$

at $x = 1$; $\frac{dy}{dx} = 6(1) - 5 = 1$

Equation of tangent is given by; $\frac{y-y_0}{x-x_0} = M$

Where $y_0 = -2, x_0 = 1, m = 1$

$$\Rightarrow \frac{y-(-2)}{x-1} = 1$$

$$\Rightarrow \frac{y+2}{x-1} = 1$$

Cross multiplying

$$y + 2 = x - 1$$
$$y = x - 3$$

EXAMPLE 5: 2011/2012 TEST QUESTION 16

16. The equation of the tangent to the curve $x^2 + 3xy + y^2 = 5$ at $(1,1)$ is (a) $y = x$ (b) $x + y = 1$ (c) $x + y = 5$ (d) $x + y = 2$

SOLUTION

$$x^2 + 3xy + y^2 = 5$$

$$\frac{dy}{dx} = -\left(\frac{2x + 3y}{3x + 2y}\right)$$

at $(1,1)$:

$$\frac{dy}{dx} = -\frac{5}{5} = -1$$

using equation of tangent given by

$$\frac{y - y_0}{x - x_0} = M$$

where $y_0 = 1$, $x_0 = 1$, $M = -1$

$$\frac{y-1}{x-1} = -1$$

Cross multiplying;

$$(y-1) = -1(x-1)$$
$$y - 1 = -x + 1$$
$$x + y = 2$$

EXAMPLE 6: 2011/2012 EXAM QUESTION 3

3. The equation of tangent line to the curve $y = 3x^2 - 4x + 1$ when $x = 2$ is (a) $y = 8x - 11$ (b) $y = 8x + 11$ (c) $y = \frac{1}{8}x - 11$ (d) $y = 2x + 13$

SOLUTION

$$y = 3x^2 - 4x + 1 \quad \text{at } x = 2$$

$$y = 3(2)^2 - 4(2) + 1 \; ; \; y = 5$$

$$\frac{dy}{dx} = 6x - 4$$

at $x = 2$

$$\left.\frac{dy}{dx}\right|_{x=2} = 6(2) - 4 = 12 - 4 = 8$$

using

$$\frac{y - y_0}{x - x_0} = M$$

Where $y_0 = 5$, $x_0 = 2$, $m = 8$

$$\Rightarrow \frac{y-5}{x-2} = 8$$

Cross multiplying

$$y - 5 = 8(x-2)$$
$$y - 5 = 8x - 16$$
$$y = 8x - 11$$

EXAMPLE 7: 2011/2012 EXAM QUESTION 57

57. What is the normal to the curve $y = x^2$ at the point $(3,4)$ (a) $\frac{-1}{6}$ (b) 6 (c) -6 (d) $\frac{1}{6}$

SOLUTION

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

at $(3,4)$: $x = 3$, $y = 4$

Hence $\left.\frac{dy}{dx}\right|_{x=3} = 2(3) = 6$

using

$$\frac{y - y_0}{x - x_0} = -\frac{1}{m}$$

where $y_0 = 4$, $x_0 = 3$, $m = 6$

$$\Rightarrow \frac{y-4}{x-3} = -\frac{1}{6}$$

Cross multiplying:

$$6(y-4) = -(x-3)$$
$$6y - 24 = -x + 3$$
$$6y + x - 27 = 0$$

The Gradient of Normal of $y = \frac{-1}{6}x + \frac{27}{6}$ is $\frac{-1}{6}$

EXAMPLE 8: 2010/2011 EXAM QUESTION 30

30. Find the equation of the normal at the point $(1,4)$ of the curve $y = x + \frac{3}{x}$ (a) $y = 1 - \frac{3}{x^2}$ (b) $2y = x + 7$ (c) $2x + y = 6$ (d) $2x - y = 6$

SOLUTION

$$y = x + \frac{3}{x} \; ; \; y = x + 3x^{-1}$$

$$\frac{dy}{dx} = 1 - 3x^{-2}$$

at $(1,4)$: $x = 1$

$$\frac{dy}{dx} = 1 - 3(1)^{-2}$$
$$= 1 - 3 = -2$$

Using

$$\frac{y - y_0}{x - x_0} = M$$

$$\Rightarrow \frac{y-4}{x-1} = -2$$

Cross multiplying

$$-2(x-1) = y - 4$$
$$-2x + 2 = y - 4$$
$$2x + y = 6$$

EXAMPLE 9: 2010/2011 EXAM QUESTION 23

23. What is the equation of the tangent of the curve $y = x^2 - 4x + 5$ at $(1,2)$? (a) $y + 2x - 4 = 0$ (b) $y - 2x + 4 = 0$ (c) $y + 2x - 4 = 0$ (d) $y + 2x + 4 = 0$

$y = x^2 - 4x + 5$ at $(1,2)$

$\frac{dy}{dx} = 2x - 4$

at $(1,2)$: $x = 1$

$\frac{dy}{dx}\Big|_{x=1} 2(1) - 4 = -2$

using $\frac{y - y_0}{x - x_0} = M$

where $y_0 = 2,\ x_0 = 1,\ M = -2$

$\Rightarrow \frac{y-2}{x-1} = \frac{-2}{1}$

$y - 2 = -2(x-1)$

$y - 2 = -2x + 2$

$y + 2x = 4$

The equation of Normal is given by

$\frac{y-2}{x-1} = \frac{1}{2}$; $2(y-2) = x - 1$

$2y - 4 = x - 1$; $2y - x = 3$

APPROXIMATIONS

From First Principle, the rate of change of $y = F(x)$, with respect to x is given by:

$$\frac{\Delta y}{\Delta x} = \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$\Rightarrow \frac{dy}{dx} = F'(x) \approx \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

The approximation will be more exact if the limit of Δx is very small i.e "if Δx tends close to zero"

EXAMPLE 1:

Ex. Without using tables or calculators find the square root of 17

SOLUTION

let $F(x) = \sqrt{x}$; $F'(x) = \frac{1}{2\sqrt{x}}$

using $F'(x) \approx \frac{F(x+\Delta x) - F(x)}{\Delta x}$

making $F(x+\Delta x)$ the subject, we have

$F(x+\Delta x) \approx F'(x)\Delta x + F(x)$

Again, let $\Delta x = 1$ and $x = 16$

$F(16+1) \approx F'(16)(1) + F(16)$; where $F'(16) = \frac{1}{8}$

and $F(16) = 4$; substituting

$F(17) = \sqrt{17} \approx \frac{1}{8} + 4 \approx 4.125$

RELATED RATES / RATE OF CHANGE

Most physical quantities have dependent function. If $y = y(x)$ is such that $y = y(t)$ and $x = x(t)$. Then applying Chain rule:

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

EXAMPLE 1: 2008/2009 EXAM QUESTION 18

18. It is estimated that months from now, the population of a certain community will be $p(t) = t^2 + 20t + 8{,}000$. At what rate will the population be changing with respect to time 15 months from now? (a) 51 (b) 52 (c) 20 (d) 50

SOLUTION

$P(t) = (t^2 + 20t + 8{,}000)$ months

$P'(t) = 2t + 20$

at $t = 15$ months

$P'(t) = 2(15) + 20$

$= 30 + 20$

$= 50$

Hence the rate is 50.

EXAMPLE 2: 2010/2011 EXAM QUESTION 34

34. From a spherical balloon of radius r, gas escapes at the rate of 500cm³/sec. Find how its radius is changing (a) $4\pi r^2$ (b) $8\pi r$ (c) $-\frac{2}{r}$ (d) $-\frac{125}{\pi r^2}$

SOLUTION

The volume of a sphere V is given by

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt}$ is the changing of r with respect to t

The gas is excaping, thus the rate of volume escaping with time will be negative

$\frac{dv}{dt} = -500cm^3$

using chain rule,

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dv}{dt}}{\frac{dv}{dr}} = \frac{-500}{\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)}$$

$$= \frac{-500}{4\pi r^2} = \frac{-125}{\pi r^2}$$

EXAMPLE 3: 2013/2014 EXAM QUESTION 38

38. It is estimated that months from now, the population of a c[...] community will be $p(t) = t^2 + 20t + 8000$, at what rate [...] the population be changing 20 months from now? A.51 B. [...] 60 D. 50 E. 64

SOLUTION

$P(t) = t^2 + 20t + 8000$

$P'(t) = 2t + 20$

at $t = 20$

The rate $P'(t)$ is given by

$P'(20) = 2(20) + 20$

$= 40 + 20$

$= 60$

EXAMPLE 4: 2013/2014 EXAM QUESTION 57

57. Air is being pumped into a spherical balloon at a rate of 4.5 cubic inches per minute. Find the rate of change of the radius when the radius is 2 inches. A. 9 inches per minute B. 0.9 inch per minute C. 0.09 inch per minute D. 0.009 inch per minute E. 1 inch per minute.

SOLUTION

Volume of Sphere $= \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$, $\frac{dv}{dt} = 4.5$, $r = 2$

Using chain rule

$\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$\Rightarrow 4.5 = (4\pi r^2)(\frac{dr}{dt})$

$\frac{dr}{dt} = \frac{4.5}{4\pi r^2}$, where $r = 2$

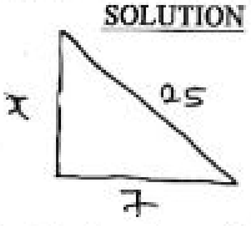
$= \frac{4.5}{4\pi (2)^2} = \frac{4.5}{16(3.142)}$

$= 0.09$ inch per minute

EXAMPLE 2: 2013/2014 EXAM QUESTION 58

58 A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2ft/sec, how fast is the top moving down the wall when the base of the ladder is 7 feet from the wall? A. $\frac{12}{7}ft/sec$ B. $\frac{7}{12}ft/sec$ C. $-\frac{12}{7}ft/sec$ D. $\frac{7}{12}ft/sec$ E. 10 ft/sec

SOLUTION



let x be the hieght of the wall. using pythagoras rule

$x^2 = 25^2 - 7^2$

$= 625 - 49$

$= 579$

$x = \sqrt{579}$

$= 24$

The rate at which the top is moving is $\frac{dx}{dt} = \frac{7}{24} \times 2 = \frac{7}{12}$

EXAMPLE 3: 2012/2013 TEST QUESTION 7

7. A circle with radius 5cm has its radius increasing at the rate 0.2cm/s. what will be the corresponding increase in the area π (b) 2π (c) 4π (d)5π

SOLUTION

The area (A) of a circle is given by; $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r\, dr$

where $r = 5cm$, dr (increase in radius) $= 0.2cm/s$

$dA = 2\pi(5)(0.2)$

$= 10\pi(2/10)$

$= 2\pi$

EXAMPLE 4: 2012/2013 EXAM QUESTION 33

33. Find the rate of change of the volume of a sphere with respect to its radius r when r=1 A. 4π B.8π C. 12π D. 24π

SOLUTION

Volume of a Sphere (v) is given by; $V = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = \frac{12\pi r^2}{3} = 4\pi r^2$

at $r = 1$

$\frac{dv}{dr} = 4\pi(1)^2$

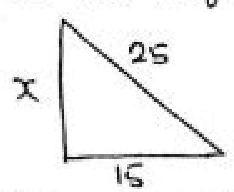
$= 4\pi$

EXAMPLE 5: 2011/2012 EXAM QUESTION 47

47. The bottom of a 25ft ladder on a wall is being pulled at ... how fast is it sliding down the wall when the bottom is ... from the wall? (a) $-\frac{9}{4}ft/sec$ (b) $\frac{8}{5}ft/sec$ (c) $-\frac{8}{5}ft/sec$ (d) $\frac{9}{4}ft/sec$

SOLUTION

let x be the hieght of the wall



Using pythagoras rule

$x^2 = 25^2 - 15^2$

$x^2 = 625 - 225$

$x = \sqrt{400} = 20$

The rate at which the top is moving is $\frac{dx}{dt} = \frac{15}{20} \times 3 = \frac{9}{4}$ ft/sec

EXTREMUM

Extremum, in Calculus is any point at which the value of a function is largest (maximum) or smallest (minimum).

CRITICAL POINT

Given a function $y = f(x)$, then a point x_0 is said to be a turning point, critical or stationary point if $f'(x_0) = 0$ i.e, if $\frac{dy}{dx} = 0$

Critical point is also known as Turning point

NATURE OF TURNING POINTS

The nature of a turning point could either be minimum or maximum otherwise inconclusive. i.e Saddle point.

The nature of turning/critical point is determined by a second derivative test.

SECOND DERIVATIVE TEST

This test is used to determine whether a turning point is a maximum or minimum point.

let $y = f(x)$ with $x = x_0$ as a turning point. Then the test says

- If $\left.\frac{d^2f}{dx^2}\right|_{x=x_0} < 0$, Then $x = x_0$ is a maximum point
- If $\left.\frac{d^2f}{dx^2}\right|_{x=x_0} > 0$, Then $x = x_0$ is a minimum point
- If $\left.\frac{d^2f}{dx^2}\right|_{x=x_0} = 0$, Then $x = x_0$ is inconclusive, in otherwords Saddle point.

EXAMPLE 1: 2008/2009 EXAM QUESTION 37

37. Obtain the maximum point of the function $y = x^3 - 6x + 9x + 1$ (a) (3,1) (b) (1,5) (c) (3,5) (d) (1,3)

SOLUTION

$y = x^3 - 6x^2 + 9x + 1$

At turning point $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

Dividing through by 3

$x^2 - 4x + 3 = 0$

Factorizing

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$ (turning point)

Using second derivative test

$\frac{d^2y}{dx^2} = 6x - 12$

at $x = 1$

$\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0$ (Negative)

at $x = 3$

$\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0$ (positive)

For second derivative test when-ever the second derivative is positive, then the function has a maximum point.

When $x = 3$; $y = (3)^3 - 6(3)^2 + 9(3) + 1$

maximum point is given by $(3, 1)$

EXAMPLE 2: 2009/2010 EXAM QUESTION 15

15. Find the maxima of the function $2x^3 + 3x^2 - 36x + 10$ (a) -34 (b) 91 (c) -3 (d) 2

SOLUTION

If $y = 2x^3 + 3x^2 - 36x + 10$

$\frac{dy}{dx} = 6x^2 + 6x - 36$

At critical points ; $\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

Dividing through by 6

$\Rightarrow x^2 + x - 6 = 0$

Factorizing,

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

Using second derivative test

$\frac{d^2y}{dx^2} = 12x + 6$

at $x = -3$; $12(-3) + 6 = -30 < 0$ (Negative)

The point $x = -3$ is maximum

at $x = 2$; $12(2) + 6 = 30 > 0$ (positive)

The point $x = 2$ is minimum

The maxima is the maximum value, which is the y-value when $x = -3$ is substituted in the function $y = 2x^3 + 3x^2 - 36x + 10$

$\boxed{\text{maxima} = 91}$

EXAMPLE 3: 2010/2011 EXAM QUESTION 54

54. Consider the function $y = x^3 - 2x^2 + x + 4$. If $x = \frac{1}{3}$ is a turning point, what is the nature? (a) minimum (b) maximum (c) inflexion (d) saddle

SOLUTION

$y = x^3 - 2x^2 + x + 4$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

at turning point; $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 4x + 1 = 0$

$(x-1)(3x-1) = 0$

$x = \frac{1}{3}$ is a turning point.

Using second derivative test;

$\frac{d^2y}{dx^2} = 6x - 4$

at $x = \frac{1}{3}$

$\frac{d^2y}{dx^2} = 6(\frac{1}{3}) - 4 = -2 < 0$ (Negative)

Hence $x = \frac{1}{3}$ is a maximum point.

EXAMPLE 4: 2010/2011 EXAM QUESTION 70

70. Find two numbers whose sum is 10 and whose product is as large as possible (a) (5, 10) (b) (10, 5) (c) (5, 5) (d) (15, 5)

SOLUTION

let the two numbers be x and y and let P be the product

$\Rightarrow x + y = 10 \quad \ldots (i)$

$P = xy \quad \ldots (ii)$

From eqn (i) $\quad y = 10 - x$

$\Rightarrow P = x(10 - x)$

$P = 10x - x^2$

$\frac{dP}{dx} = 10 - 2x$

at turning point $\frac{dP}{dx} = 0$

$\Rightarrow 10 - 2x = 0$

$10 = 2x$; $x = \frac{10}{2}$

$x = 5$, Hence $y = 10 - 5$

$x = 5$ and $y = 5$

The two numbers are 5, 5

EXAMPLE 5: 2012/2013 EXAM QUESTION 43

43. At what value of x is the function $y = x^2 - 2x - 3$ *minimum* A. 1 B. -1 C. -4 D. 4

SOLUTION

$y = x^2 - 2x - 3$

$\frac{dy}{dx} = 2x - 2$

at turning point $\frac{dy}{dx} = 0$

$\Rightarrow 2x - 2 = 0$; $x = \frac{2}{2} = 1$

The value of x, for which the Function ...

EXAMPLE 6: 2012/2013 EXAM QUESTION

29. Determine the maximum value of $y = 3x^2 - x^3$ A.0 B... D.6

SOLUTION

$y = 3x^2 - x^3$; $\frac{dy}{dx} = 6x - 3x^2$

At turning point; $\frac{dy}{dx} = 0$

$\Rightarrow 6x - 3x^2 = 0$; $3x(2 - x) = 0$; $x = 0$ or $x = 2$

using second derivative test

$\frac{d^2y}{dx^2} = 6 - 6x$; at $x = 2$

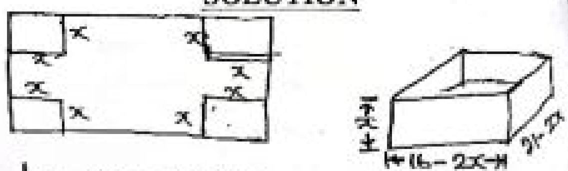
$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 6 - 6(2) = -6 < 0$ (Negative)

maximum value; is y-value when $x = 2$; $y = 4$

EXAMPLE 7: 2012/2013 EXAM QUESTION 30

30. An open box is to be made from a rectangular sheet 16 in by ... in by cutting out a square from each corner and bending up the sides. What should be the length of a side of the square for the box's volume to be a maximum? (a) 3in (b) 2in (c) $\frac{28}{3}$ in (d) $\frac{3}{2}$ in

SOLUTION



let V = volume

$\Rightarrow V = x(16 - 2x)(21 - 2x)$

$V = (16x - 2x^2)(21 - 2x)$

$V = 336x - 32x^2 - 42x^2 + 4x^3$

$V = 4x^3 - 74x^2 + 336x$

$\frac{dV}{dx} = 12x^2 - 148x + 336$

at critical points; $\frac{dV}{dx} = 0$

$\Rightarrow 12x^2 - 148x + 336 = 0$

$(3x - 28)(x - 3) = 0$

$x = \frac{28}{3}$ or $x = 3$

using second derivative test

$\frac{d^2y}{dx^2} = 24x - 148$

at $x = 3$

$\frac{d^2y}{dx^2} = 24(3) - 148 = -76 < 0$ (Negative)

Hence; $x = 3$inch for volume to be maximum.

Max volume $= 3 \times 10 \times 15 = 450$ inch^3

EXAMPLE 8: 2011/2012 TEST QUESTION 6

6. What is the nature of -2 as a critical point of $f(x) = x^4 - 6x^2 + 8x + 10$ (a) ... (d) ...

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$f(x) = x^4 - 6x^2 + 8x + 10$
$f'(x) = 4x^3 - 12x + 8$
$f''(x) = 12x^2 - 12$
using Second derivative test
$f''(-2) = 12(-2)^2 - 12$
$= 12(4) - 12 = 36 > 0$
Since the result is positive
$x = -2$ is minimum

EXAMPLE 9: 2011/2012 EXAM QUESTION 52

52. Find the maximum value of $y = x^3 - 9x^2 + 15x$ (a) 3 (b) 2 (c) 1 (d) 7

SOLUTION

$y = x^3 - 9x^2 + 15x$

$\frac{dy}{dx} = 3x^2 - 18x + 15$

at turning points; $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 18x + 15 = 0$
Dividing through by 3
$x^2 - 6x + 5 = 0$
factorising; $(x-1)(x-5) = 0$
$x = 1$ or $x = 5$
using Second derivative test to find the maximum point;

$\frac{d^2y}{dx^2} = 6x - 18$

at $x = 1$; $6(1) - 18 = -12 < 0$ (negative)
Hence $x = 1$ is a maximum point.
But maximum value is the Corresponding y-value; therefor $y = (1)^3 - 9(1)^2 + 15(1)$
maximum value is $y = 7$

EXAMPLE 10: 2011/2012 EXAM QUESTION 65

65. At what point on the curve $y = x^3 - 2x^2 + x + 4$ is y a minimum (a) (1, 4) (b) (-1, 4) (c) (2, 6) (d) $\left[\frac{1}{3}, \frac{11^2}{27}\right]$

SOLUTION

$y = x^3 - 2x^2 + x + 4$

$\frac{dy}{dx}$ $3x^2 - 4x + 1$

Evaluating the Critical points, we equate $\frac{dy}{dx}$ to zero

$\Rightarrow 3x^2 - 4x + 1 = 0$
factorising;
$(x-1)(3x-1) = 0$
$x = 1$ or $x = 1/3$ is the Critical points

Using Second derivative test,

$\frac{d^2y}{dx^2} = 6x - 4$

at $x = 1$

$\frac{d^2y}{dx^2} = 6(1) - 4 = 2 > 0$ [positive]

The function is minimum at $x = 1$
When $x = 1$, $y = (1)^3 - 2(1)^2 + (1) + 4$
The function is maximum at $(1, 4)$

INCREASING FUNCTION

A function $y = f(x)$ is said to be increasing, if the gradient is greater than zero. i.e if $\frac{dy}{dx} > 0$ or $f'(x) > 0$

DECREASING FUNCTION

A function $y = f(x)$ is said to be decreasing, if the gradient is less than zero. i.e if $\frac{dy}{dx} < 0$ or $f'(x) < 0$

STATIONARY FUNCTION

A function $y = f(x)$ is stationary if the gradient is equal to zero i.e $\frac{dy}{dx} = 0$

TEST ON CONCAVITY

let $f(x)$ be a function whose Second derivative exist on an open interval I

- If $f''(x) > 0$ for all x in I, then the graph of $f(x)$ is Concave upwards on I
- If $f''(x) < 0$ for all x in I, then the graph of $f(x)$ is concave downwards on I
- If $f''(x) = 0$; Then it is inflection point

EXAMPLE 1: 2008/2009 EXAM QUESTION 42

42. Identify the interval of increase of the function $f(x) = x^4 + 8x^3 + 18x^2 - 8$ (a) $x < -3$ (b) $-3 < x < 0$ (c) $x > 0$ (d) $x < 0$

SOLUTION

$f(x) = x^4 + 8x^3 + 18x^2 - 8$

$f'(x) = 4x^3 + 24x^2 + 36x$

for a function to be increasing

$f'(x) > 0$

$\Rightarrow 4x^3 + 24x^2 + 36x > 0$

Dividing through by 4

$x^3 + 6x^2 + 9x > 0$

$x(x^2 + 6x + 9) > 0$

$x(x+3)(x+3) > 0$

$x(x+3)^2 > 0$

Truth Table

	$x < -3$	$-3 < x < 0$	$x > 0$
x	$-$	$-$	$+$
$(x+3)^2$	$+$	$+$	$+$
product	$-$	$-$	$+$

The solution set is $\{x : x > 0\}$

Hence the interval of increase is $x > 0$

EXAMPLE 2: 2008/2009 EXAM QUESTION 33

33. Determine the stationary points on the curve $y = x^3 - 3x^2 - 9x$. (a) $(1, -5)$ and $(3, 27)$ (b) $(1, -5)$ and $(3, -27)$ (c) $(1, 5)$ and $(-3, 27)$ (d) $(1, -5)$ and $(-3, -7)$

SOLUTION

$y = x^3 - 3x^2 - 9x$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

at Stationary point; $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 6x - 9 = 0$

Dividing through by 3

$x^2 - 2x - 3 = 0$

Factorising

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

When $x = 3$; $y = (3)^3 - 3(3)^2 - 9(3)$
$= -27$

When $x = -1$; $y = (-1)^3 - 3(-1)^2 - 9(-1)$
$y = 5$

Hence the stationary points is given by $(-1, 5)$ and $(3, -27)$

EXAMPLE 3: 2009/2010 EXAM QUESTION 63

63. What is the point of inflexion of the function $y = x^3 - 2x^2 + x + 3$ (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 3 (d) $\frac{1}{3}$

SOLUTION

$y = x^3 - 2x^2 + x + 3$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

$\frac{d^2y}{dx^2} = 6x - 4$

at point of inflexion; $\frac{d^2y}{dx^2} = 0$

$\Rightarrow 6x - 4 = 0$

$6x = 4$; $x = \frac{4}{6}$

$x = \frac{2}{3}$

EXAMPLE 4: 2009/2010 EXAM QUESTION 13

13. Given the function $y = x^3 - 3x^2 + 3x$, find the point of inflexion (a) $x = 2, y = 2$ (b) $x = 1, y = 1$ (c) $x = 1, y = 2$ (d) $x = 2, y = 1$

SOLUTION

$y = x^3 - 3x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 6x + 3$

$\frac{d^2y}{dx^2} = 6x - 6$

The point of inflexion is given when $\frac{d^2y}{dx^2} = 0$

$\Rightarrow 6x - 6 = 0$

$6x = 6$; $x = \frac{6}{6}$

$x = 1$

When $x = 1$, $y = (1)^3 - 3(1)^2 + 3(1)$

Hence, when $x = 1$, $y = 1$

EXAMPLE 5: 2009/2010 EXAM QUESTION 12

12. Which of the following explains points of inflexion (a) $\frac{dy}{dx} = 0$, however, this is not a necessary condition (b) $\frac{d^2y}{dx^2} = 0$, this is a necessary condition (c) $\frac{d^3y}{dx^3} \neq 0$, it may be positive or negative but it is a necessary condition (d) all of the above

SOLUTION

The necessary condition for inflexion is that $\frac{d^2y}{dx^2}$ (second derivative) must equal to zero.

EXAMPLE 6: 2011/2012 EXAM QUESTION 60

60. For what value of x is the function $1 + 3x - 2x^2$ increasing? (a) $x < \frac{3}{4}$ (b) $x < \frac{-3}{4}$ (c) $x >$ (d) $x > \frac{-3}{4}$

SOLUTION

let $y = 1 + 3x - 2x^2$

$\frac{dy}{dx} = 3 - 4x$

The function will be increasing

$\frac{dy}{dx} > 0$

$$3-4x>0$$
$$\Rightarrow 3>4x$$
$$\frac{3}{4}>x$$
Hence $x<3/4$

KINEMATICS

If an object is moving along a straight line; its distance (s) from a fixed point on the line is a function of time (t).
i.e "S" depends on "t", denoted by S(t). If S(t) is differentiable at $t=t_0$, then the derivative is given by $\boxed{\frac{d[S(t)]}{dt}=S'(t)}$ = Velocity at t_0.

Again, if V(t) is differentiable at $t=t_0$; Then $\boxed{S''(t)=V'(t)}$ = acceleration at t_0.

EXAMPLE 1: 2009/2010 EXAM QUESTION 38

38. A body moves in a straight line so that its distance S-meter from a fixed point 0 on the line after t seconds is given by $S=t^2-6t+10$. Find t when the body comes to rest (a) 4 seconds (b) 3 seconds (c) 2 seconds (d) 5 seconds

SOLUTION

$$S=t^2-6t+10$$
$$V=\frac{ds}{dt}=2t-6$$
The object will be at rest when the velocity is zero
$$\Rightarrow V=2t-6=0$$
$$2t-6=0$$
$$2t=6$$
$$t=6/2$$
$$t=3\text{ seconds}$$
The object will be at rest @ t = 3secs

SOLUTION II

$$S=t^2-6t+10$$
$$V=S'=2t-6$$
$$a=V'=S''=2$$
Hence at any time the acceleration has a constant value of $2m/s^2$

EXAMPLE 2: 2011/2012 TEST QUESTION 10

10. If a particle's motion is given by $s(t)=(2t^2+5t-3)m$. and t change from 2 sec s to 5 sec s, then it's average velocity is (a) 4m/s (b) 3cm/s (c) 5m/s (d) 2m/s.

SOLUTION

$$S(t)=(2t^2+5t-3)m$$
$$V=\frac{ds}{dt}=4t+5$$
V_1 at $t_1=2$secs
$$\Rightarrow V_1=4(2)+5=13m/s$$
V_2 at $t_2=5$secs
$$\Rightarrow V_2=4(5)+5=25m/s$$
Average Velocity (V) is given by
$$\frac{\Delta v}{\Delta t}=\frac{V_2-V_1}{T_2-T_1}$$
$$\frac{\Delta v}{\Delta t}=\frac{25-13}{5-2}=\frac{12}{3}$$
$$=4m/s$$

EXAMPLE 3: 2013/2014 EXAM QUESTION 12

12. The motion of a particle is given by $s=3t^4-18t^2+7t$ when is the velocity of the particle constant? A. t=1 B. t=-1 C. t=2 D. t=-2 E. t $=\pm1$

SOLUTION

$$S=3t^4-18t^2+7t$$
$$V=\frac{ds}{dt}=12t^3-36t+7$$
$$a=\frac{d^2s}{dt^2}=36t^2-36$$
At Constant velocity; acceleration = 0
$$\Rightarrow 36t^2-36=0$$
$$t^2=\frac{36}{36};\ t^2=1$$
$t=\pm1$ time cannot be negative

Hence; t = 1secs

ROLLE'S THEOREM

If F is a function which is continuous everywhere on a close interval [a, b] gain, if F(a) = F(b); Then there exist one point c in the open interval (a, b) such that F'(c) = 0

EXAMPLE 1: 2013/2014 EXAM QUESTION 7

7. If $f(x)=4x^2-20x+29$, using Rolle's theorem find the critical number when $f(x)$ is defined in [1, 4] A. 13 B. $\frac{5}{2}$ C. $\frac{3}{2}$ D. 2 E. $\frac{1}{2}$

SOLUTION

$$F(x)=4x^2-20x+29$$
at [1, 4]; a = 1, b = 4
$$F(a)=F(1)=4(1)^2-20(1)+29$$
$$=13$$

$f(b) = f(4) = 4(4)^2 - 20(4) + 29$
$= 13$

Since $f(a) = f(b)$, There exist a point (1, 4) Such that $f'(c) = 0$

but $f'(x) = 8x - 20$

$f'(c) = 8c - 20$

From the above theorem

$f'(c) = 0$

$\Rightarrow 8c - 20 = 0$

$8c = 20$

$c = \frac{20}{8} = \frac{5}{2}$

EXAMPLE 2: 2013/2014 EXAM QUESTION 40

40. Find a point $c \in (-2,2)$ *such that* $f(x) = x^2$ satisfies the conditions of Rolle's theorem in the interval [-2,2] A. 0 B. 1 C. 2 D. 3 E. 4

SOLUTION

$f(x) = x^2$

$f(a) = f(-2) = (-2)^2 = 4$

$f(b) = f(2) = (2)^2 = 4$

Since $f(a) = f(b)$, we evaluate $f'(c)$; but

$f'(x) = 2x$

$\Rightarrow f'(c) = 2c$

By Rolle's theorem

$2c = 0$; $c = \frac{0}{2} = 0$

$c = 0$

MEAN VALUE THEOREM

let F be a Continuous Function everywhere on a Closed Interval [a, b] and it's differentiable at each point on the open interval (a, b), then their exist at least one interior point c. Such that;

$$\boxed{F'(c) = \frac{F(b) - F(a)}{b - a}}$$

EXAMPLE 1: 2014/2015 EXAM QUESTION 23

23. If the function $f(x) = x^3 + 3x^2 - 9x + 12$ satisfy the conditions of the mean value theorem on the interval (-2, 0) find the constant c prescribed by the theorem (a) 0.25 (b) 2.75 (c) 3.0 (d) -0.25 (e) 1.5

SOLUTION

$f(x) = x^3 + 3x^2 - 9x + 12$

at interval [-2, 0]

$\Rightarrow a = -2, b = 0$

$f(-2) = (-2)^3 + 3(-2)^2 - 9(-2) + 12 = 34$

$f(0) = (0)^3 + 3(0)^2 - 9(0) + 12 = 12$

$f'(x) = 3x^2 + 6x - 9$

$f'(c) = 3c^2 + 6c - 9$

Applying Mean Value Theorem

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$3c^2 + 6c - 9 = \frac{12 - 34}{0 - (-2)}$

$3c^2 + 6c - 9 = -11$

$3c^2 + 6c + 2 = 0$

$c = -0.42$ or -1.57

TAYLOR'S / MACLURIN SERIES

If $f(x)$ together with its derivative is continous on the interval (a, b) and $f^n(x)$ exist everywhere on the interval except the end point, then there exist at least one value $x = x_0$ Such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)(x - x_0)^n}{n!}$$

Expanding the above Taylor's Series from definition;

$$f(x) = f(x_0) + \frac{f'(x_0)(x - x_0)}{1!} + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \cdots + \frac{f^n(x_0)(x - x_0)^n}{n!}$$

EXAMPLE 1: 2009/2010 EXAM QUESTION 80

80. Find the Taylor's series of the function $f(x) = \frac{1}{1-x}$ about $x = 0$ (a) $\sum_{n=1}^{\infty}(-1)^{n+1}x^n$ (b) $\sum_{n=1}\frac{x^n}{n}$ (c) $\sum_{n=0}^{\infty}x^n$ (d) $\sum_{n=0}^{\infty}\frac{x^n}{n+1}$

SOLUTION

Given a function $f(x) = \frac{1}{1-x}$

$\Rightarrow f(x) = (1-x)^{-1}$

$f'(x) = (1-x)^{-2}$

$f''(x) = 2(1-x)^{-3}$

$f'''(x) = 6(1-x)^{-4}$

From the question, it is expanded about $x = 0$

$f(0) = (1-0)^{-1} = 1$

$f'(0) = -(1-0)^{-2} = 1$

$f''(0) = 2(1-0)^{-3} = 2$
$f'''(0) = -6(1-0)^{-4} = 6$

using $f(x) = \sum_{n=0}^{\infty} f^n(x_0)(x-x_0)^n$; where $x_0 = 0$

$$\Rightarrow \frac{1}{1-x} = 1 + \frac{1(x-0)}{1!} + \frac{2(x-0)^2}{2!} + \frac{(6)(x-0)^3}{3!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots x^n$$

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

EXAMPLE 2: 2011/2012 EXAM QUESTION 68

68. If $f(x) = sinx$, then series $f(x) = ?$ (a) $x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$ (b) $x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (c) $x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ (d) $\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

SOLUTION

$f(x) = \sin x$ $\quad f(0) = \sin 0 = 0$
$f'(x) = \cos x$ $\quad f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$ $\quad f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$ $\quad f'''(0) = -\cos 0 = -1$

using $f(x) = f(0) + \frac{f'(0)(x-0)}{1!} + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \cdots$

$$\Rightarrow \sin x = 0 + \frac{1(x)}{1!} + \frac{0(x)^2}{2!} + \frac{(-1)(x)^3}{3!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

TAYLOR'S SERIES OF SOME FUNCTIONS

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

SIGMA NOTATION

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n, \quad \text{where } x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}, \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

EXAMPLE 1: 2012/2013 TEST QUESTION 5

5. Find the Maclaurin's series of $f(x) = e^{-3x}$ (a) $\sum_{n=0}^{\infty} \frac{(3x)^{n+1}}{(n+1)!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{n+1}}{(n+1)!}$ (c) $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$ (d $\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{n+1}}{(n+1)!}$

SOLUTION

$$F(x) = e^{-3x}$$

The Sigma notation of

$$e^y = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot y^n$$

Now, let $y = -3x$

$$\Rightarrow e^{-3x} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (-3x)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (3x)^n$$

SA: Alternatively you can generate the solutions using normal definition of derivatives

INTEGRATION

This is simply the reverse of differentiation. The integration of a function with respect to x is denoted by:

$$\int F(x)\,dx = f(x) + c$$

* The symbol $\int$ is an integral sign
* The function F(x) is the integral
* dx is the variable of integration.
* f(x) is the integrand
* C is the arbitrary constant of integration

WE HAVE TWO TYPES OF INTEGRATION

[i] Indefinite integration
[ii] Definite integration

Given $\int f(x)dx$; Then it is an indefinite integration because there is no boundary condition (No limit). But; if given $\int_a^b f(x)dx$; Then it is a ~~definite~~ integration. Because it has boundary condition (from a to b)

INTEGRATION TECHNIQUES

This is the distinct method of evaluating the integration of different functions and its combinations.

1. INTEGRATION OF POLYNOMIAL FUNCTION

Given that $\frac{dy}{dx} = ax^n$

Then $y = \int ax^n = \frac{ax^{n+1}}{n+1}$, where $n \neq -1$

Consider; $y = 4x^3 + 3x^2 + 5$

$$\frac{dy}{dx} = 12x^2 + 6x$$

But if we integrate $12x^2+6x$, we will obtain the value of y + constant. the integration is denoted by;

$$y = \int 12x^2 + 6x$$
$$= \frac{12x^{2+1}}{3} + \frac{6x^{1+1}}{2}$$
$$y = 4x^3 + 3x^2 + c$$

Note: integration of simple polynomial function is just adding one(1) to the existing power and dividing by the resultant.

Though, we can find the value of the constant "c" when given additional information. for instance:

Find the value of y;

For $\frac{dy}{dx} = 12x^2+6x$, Given that when $x=1, y=12$

with this information, we substitute $x=1, y=12$ then we find c, thus $12 = 4(1)^3 + 3(1)^2 + c$

$c=5$; $y = 4x^3+3x^2+5$

EXAMPLE 1: 2009/2010 EXAM QUESTION 17

17. Evaluate $\int \frac{x^4+1}{x^2} dx$ (a) $\frac{x^3}{x}$ (b) $\frac{x^3}{3} - \frac{1}{x} + C$ (c) $sec^{-1}x$ (d) $\int \frac{x^5}{5} dx$

SOLUTION

$$\int \frac{x^4+1}{x^2} dx$$
$$\int \frac{x^4}{x^2} dx + \int \frac{1}{x^2} dx$$
$$\int x^2 dx + \int x^{-2} dx$$
$$= \frac{x^3}{3} - \frac{x^{-1}}{1}$$
$$= \frac{x^3}{3} - \frac{1}{x} + C$$

EXAMPLE 2: 2009/2010 EXAM QUESTION 70

70. Given that $\frac{d^2y}{dx^2} = 6x - 4$, the value of $\frac{dy}{dx}$ given that $\frac{dy}{dx} = 3$ when $x = 0$ is (a) $3x^2 - 4x + 3$ (b) $3x^2 - 4x + C$ (c) $3x^2 + 4x + 3$ (d) $6x - 4 + C$

SOLUTION

$$\frac{d^2y}{dx^2} = 6x-4$$
$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = 6x-4$$
$$d\left(\frac{dy}{dx}\right) = (6x-4)dx$$

Integrating both sides,

$$\int d\left(\frac{dy}{dx}\right) = \int (6x-4)dx$$
$$\frac{dy}{dx} = \frac{6x^2}{2} - \frac{4x}{1} + C$$
$$\frac{dy}{dx} = 3x^2 - 4x + C$$

But, when $\frac{dy}{dx} = 3$, $x=0$

$\Rightarrow 3 = 3(0)^2 - 4(0) + C$; $C = 3$

We have that;

$$\frac{dy}{dx} = 3x^2 - 4x + 3$$

EXAMPLE 3: 2010/2011 EXAM QUESTION 43

43. Find $\int \sqrt{x}(x^2-1)dx$ (a) $\frac{2x^{\frac{7}{2}}}{7} + \frac{2x^{\frac{7}{2}}}{3} + C$ (b) $\frac{2x^{\frac{7}{2}}}{7} + C$ (c) $-\frac{2x^{\frac{3}{2}}}{3} + C$ (d) $\frac{2x^{\frac{7}{2}}}{7} - \frac{2x^{\frac{3}{2}}}{3} + C$

SOLUTION

$$\int \sqrt{x}(x^2-1)dx$$

Expanding, we have

$$\int (x^2 \cdot \sqrt{x} - \sqrt{x})dx$$
$$\int (x^{2+\frac{1}{2}} - x^{\frac{1}{2}})dx$$
$$\int (x^{5/2} - x^{1/2})dx$$
$$= \frac{x^{5/2+1}}{7/2} - \frac{x^{1/2+1}}{3/2} + C$$
$$= \frac{2x^{7/2}}{7} - \frac{2x^{3/2}}{3} + C$$

EXAMPLE 4: 2010/2011 EXAM QUESTION 31

31. Find $\int (x^2-2x)dx$ that takes the value 4 at $x=1$ (a) $\frac{x^3}{3} - x^2$ (b) $\frac{x^3}{3} - x^2 + C$ (c) $\frac{x^3}{3} - x^2 + \frac{14}{3}$ (d) $\frac{x^3}{3} - x^2 - \frac{14}{3}$

SOLUTION

Let $y = \int (x^2-2x)dx$

Integrating each terms of the terms, we have.

$$y = \frac{x^3}{3} - \frac{2x^2}{2} + C$$
$$y = \frac{x^3}{3} - x^2 + C$$

TION 70
iven that
) $3x^2$ –

3

ON 43
) $\frac{2x^{\frac{7}{2}}}{7} + C$ (c)

ON 31
$x = 1$ (a)
$\frac{x^3}{3} - x^2$ –

rms,

EXAMPLE 2:

$$\int \frac{e^x}{x+1}\,dx \equiv \int u\,dv = uv - \int v\,du$$

where $u = \frac{1}{x+1} = (x+1)^{-1}$

$$du = -(x+1)^{-2} = -\frac{1}{(x+1)^2}$$

$$dv = e^x \text{, then } v = \int e^x = e^x$$

$$\int \frac{e^x}{x+1}\,dx = \frac{1}{x+1}e^x - \int e^x\left[-\frac{1}{(x+1)^2}\right]$$

$$= \frac{e^x}{x+1} + \int \frac{e^x}{(x+1)^2}$$

Substituting in eqn (*) above

$$\int \frac{xe^x}{(x+1)^2}\,dx = \frac{e^x}{x+1} + \int \frac{e^x}{(x+1)^2}\,dx - \int \frac{e^x}{(x+1)^2}$$

$$= \frac{e^x}{x+1} + C$$

EXAMPLE 3: 2009/2010 EXAM QUESTION 42

42. Evaluate $\int \frac{x+6}{x^2-4}\,dx$ (a) $ln\left[\frac{(x-2)^2}{x+2}\right]$ (b) $ln\left[\frac{(x-2)^2}{(x+2)^2}\right]$ (c) $ln\left[\frac{(x-2)^2}{x+2}\right] + C$ (d) $\ln(x-3)^2 - \ln(x+2) + C$

SOLUTION

$$\int \frac{x+6}{x^2-4}\,dx$$

Resolving $\frac{x+6}{x^2-4}$ into partial fraction

$$\frac{x+6}{x^2-4} = \frac{x+6}{x^2-2^2} = \frac{x+6}{(x+2)(x-2)}$$

$$\frac{x+6}{(x+2)(x-2)} \equiv \frac{A}{x+2} + \frac{B}{x-2}$$

multiplying both sides by $(x+2)(x-2)$

$$x+6 \equiv A(x-2) + B(x+2)$$

Put $x = 2$

$$\Rightarrow 8 = 4B;\ B = 8/4 = 2$$

Put $x = -2$

$$\Rightarrow 4 = -4A;\ A = \frac{-4}{4} = 1$$

We have: $\frac{x+6}{(x+2)(x-2)} = \frac{2}{x-2} - \frac{1}{x+2}$

$$\int \frac{x+6}{x^2-4}\,dx = \int \frac{2}{x-2}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 2\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 2\ln(x-2) - \ln(x+2) + C$$

$$= \ln\left[\frac{(x-2)^2}{(x+2)}\right] + C$$

EXAMPLE 4: 2008/2009 EXAM QUESTION 38

38. Evaluate the indefinite integral $\int \frac{6x+1}{x^2+2x}\,dx$ (a) $\frac{1}{2x} - \frac{13}{2(x+2)} + C$ (b) $\frac{x^2}{2} + 3x + lnx + C$ (c $\frac{1}{2}lnx - \frac{13}{2}\ln(x+2) + C$ (d) $\ln(x+3)^{\frac{1}{2}} + \frac{x^2}{2} + C$

SOLUTION

$$\int \frac{6x+1}{x^2+2x}\,dx$$

We will resolve $\frac{6x+1}{x^2+2x}$ into Partial fraction

$$\frac{6x+1}{x^2+2x} = \frac{6x+1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$$

$$\Rightarrow 6x+1 \equiv A(x+2) + Bx$$

Using Cover-up method

Put $x = 0$, Then

$$1 = 2A;\ A = 1/2$$

Again; Put $x = -2$; Then

$$-11 = -2B;\ B = \frac{11}{2}$$

$$\frac{6x+1}{x^2+2x} = \frac{1}{2x} + \frac{11}{2(x+2)}$$

Therefore

$$\int \frac{6x+1}{x^2+2x}\,dx = \int \frac{1}{2x}\,dx + \int \frac{11}{2(x+2)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{11}{2}\int \frac{1}{x+2}\,dx = \frac{1}{2}\ln x + \frac{11}{2}(x+2) + C$$

EXAMPLE 5: 2011/2012 EXAM QUESTION 23

23. Evaluate $\int (x+1)/(x^2+5x+6)\,dx$ (a) $-\ln(x+2) + 2\ln(x+3) + c$ (b) $\ln(x+2) + 2\ln(x+3) + c$ (c) $\ln(x+2) + 2\ln(x+3)$ (d) $\ln(x+2) - 2\ln(x+3) + c$

SOLUTION

$$\int \frac{x+1}{x^2+5x+6}\,dx$$

Resolving $\frac{x+1}{x^2+5x+6}$ into partial fraction,

$$\frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} \equiv \frac{A}{x+2} + \frac{B}{x+3}$$

multiplying through by $(x+2)(x+3)$

$$\Rightarrow x+1 \equiv A(x+3) + B(x+2)$$

Using Cover-up method

let $x = -3$

$$\Rightarrow -2 = -B;\ B = 2$$

let $x = -2$

$$\Rightarrow -1 = A;\ A = -1$$

$$\frac{x+1}{(x+2)(x+3)} = \frac{2}{x+3} - \frac{1}{x+2}$$

$$\int \frac{x+1}{x^2+5x+6}\,dx = \int \frac{2}{x+3}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 2\int \frac{1}{x+3}\,dx - \int \frac{1}{x+2}\,dx$$

$$= 2\ln(x+3) - \ln(x+2) + C$$

$$= -\ln(x+2) + 2\ln(x+3) + C$$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

In this technique, one need a huge knowledge of Trigonometric Identities to enable suitable Substitution.

For instance; Given $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

SOLUTION

let $x = a\sin\theta$

$$\frac{dx}{d\theta} = a\cos\theta;\ dx = a\cos\theta\,d\theta$$

Where $\theta = \sin^{-1}(x/a)$

Substituting into $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

$\Rightarrow \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta d\theta$

$= \int \frac{a\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}}$

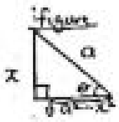
Note $1-\sin^2\theta$ is equal to $\cos^2\theta$

$= \int \frac{a\cos\theta d\theta}{a\cos\theta} = \int d\theta = \theta + c$

Recall that $\theta = \sin^{-1}(x/a)$

$\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(x/a)$

figure



Note: We choosed $x = a\sin\theta$ because of the term $\sqrt{a^2-x^2}$. Observe that $\sin\theta = x/a$ in the figure

EXAMPLE 2: 2012/2013 EXAM QUESTION 46

46. Evaluate $\int \tan x \sec^2 x\, dx$ A. $\frac{\sec^4 x}{2} + c$ B. $\frac{\tan^2 x}{2}$ C. $\frac{\sec^4 x}{2}$ D. $\frac{\tan^{24} x}{2} + c$

SOLUTION

$\int \tan x \sec^2 x\, dx$

let $u = \tan x$; $\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$; Substituting

$\int \tan x \sec^2 x dx = \int u \cdot \sec^2 x \cdot \frac{du}{\sec^2 x}$

$= \int u\, du$

$= \frac{u^2}{2} + c$

$= \frac{\tan^2 x}{2} + c$

EXAMPLE 3: 2008/2009 EXAM QUESTION 7

7. Evaluate $\int \sin^3 x dx$ (a) $-\sin x + \frac{\sin^3 x}{3} + C$. (b) $-\cos x + \frac{\cos^3 x}{3} + C$ (c) $\sin x + \frac{\cos^3 x}{3} + C$ (d) $\cos x + \frac{\sin^3 x}{3} + C$

SOLUTION

$\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$

From Trignometry,

$\sin^2 x + \cos^2 x = 1$; $\sin^2 x = 1 - \cos^2 x$

$\Rightarrow \int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx$

$= \int (\sin x - \cos^2 x \sin x) dx$

$= \int \sin x\, dx - \int \cos^2 x \sin x\, dx$

let $I = \int \cos^2 x \sin x\, dx$

again, let $u = \cos x$; $\frac{du}{dx} = -\sin x$; $dx = \frac{du}{-\sin x}$

$\Rightarrow I = \int u^2 \sin x \cdot \frac{du}{-\sin x} = -\int u^2 du$

$= -\frac{u^3}{3} + c$

Recall that $u = \cos x$

$\Rightarrow I = -\frac{\cos^3 x}{3} + c$

therefore $\int \sin^3 x dx = \int \sin x + I$

$= -\cos x + \frac{\cos^3 x}{3} + c$

EXAMPLE 4: 2010/2011 EXAM QUESTION 68

68. What is the integral of $\cos^2 x$ with respect to x. (a) $\frac{1}{4}(2x + \sin 2x) + C$ (b) $\frac{1}{4}(x + \sin 2x) + C$ (c) $1 + \frac{1}{2}\sin 2x + C$ (d) $\frac{1}{4}(x - \sin 2x) + C$

SOLUTION

$\int \cos^2 x dx$

From trignometry

$\cos^2 x = \frac{1}{2}[\cos 2x + 1]$

$\int \cos^2 x\, dx = \int \frac{1}{2}[\cos 2x + 1]\, dx$

$= \frac{1}{2}\int \cos 2x\, dx + \frac{1}{2}\int 1\, dx$

$= \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + \frac{1}{2}(x) + c$

$= \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + \frac{1}{2}x + c$

$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$

EXAMPLE 5: 2010/2011 EXAM QUESTION 60

60. Evaluate $\int \cos x \sin^5 x\, dx$ (a) $\cos x^5 + C$ (b) $\frac{\cos^6 x}{6} + C$ (c) $\frac{\sin^6 x}{6} + C$ (d) $\sin^6 x \cos x + C$

SOLUTION

$\int \cos x \sin^5 x\, dx$

let $u = \sin x$; $\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$\Rightarrow \int \cos x \sin^5 x\, dx = \int \cos x \cdot u^5 \cdot \frac{du}{\cos x}$

$= \int u^5 du = u^6 + c$

putting $u = \sin x$

$\Rightarrow \int \cos x \sin^5 x\, dx = \sin^6 x + c$

EXAMPLE 6: 2014/2015 EXAM QUESTION 21

21. Evaluate $\int \sin^2 x \cos^3 x\, dx$ (a) $\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$ (b) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$ (c) $\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$ (d) $\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$ (e) none of the above

SOLUTION

$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx$

From trignometry, $\cos^2 x = 1 - \sin^2 x$

$\int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x(1 - \sin^2 x)\cos x\, dx$

let $u = \sin x$, $\frac{du}{dx} = \cos x$; $dx = \frac{du}{\cos x}$

$$\Rightarrow \int u^2(1-u^2)\cos\theta \frac{du}{\cos\theta}$$

$$= \int u^2(1-u^2)\, du$$

$$= \int u^2 - u^4\, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

EXAMPLE 7: 2011/2012 EXAM QUESTION 53

53. Evaluate $\int e^{\sin x}\cos x\, dx$ (a) $e^{\sin x}+c$ (b) $e^{\cos x}+c$ (c) $e^{\tan x}+c$ (d) $e^{\sec x}+c$

SOLUTION

$$\int e^{\sin x}\cos x\, dx$$

let $u = \sin x$; $\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$; Substituting

$$\int e^{\sin x}\cos x\, dx = \int e^{u}\cdot\cos x\cdot\frac{du}{\cos x}$$

$$= \int e^{u}\, du = e^{u} + C$$

$$= e^{\sin x} + C$$

INTEGRATION OF ABSOLUTE VALUE AND PIECEWISE FUNCTION

The definition of an absolute value as a piecewise function is given by;

$$F(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Given $\int_a^b |x|dx$, Then we split the integral into piecewise defined function, we now take the integral from lower limit to a transition point and add it to the integral, from the transition point to the upper limit.

EXAMPLE 1: 2011/2012 EXAM QUESTION 41

41. Evaluate $\int_1^5 |x-2|dx$ (a) 4 (b) 2 (c) 9 (d) 3

SOLUTION

$$\int_1^5 |x-2|\, dx$$

To get the transition point, we equate the content of the absolute value to zero

$x-2=0$; $x=2$; $|x-2| = \begin{cases} (x-2); & x \geq 2 \\ -(x-2); & x<2 \end{cases}$

$$\int_1^5 |x-2|\, dx = \int_1^2 -(x-2)\,dx + \int_2^5 (x-2)\,dx$$

$$= \int_1^2 (-x+2)\,dx + \int_2^5 (x-2)\,dx$$

$$= \left[-\frac{x^2}{2}+2x\right]_1^2 + \left[\frac{x^2}{2}-2x\right]_2^5$$

$$= \frac{1}{2} + \frac{9}{2} = 5$$

EXAMPLE 2: 2013/2014 EXAM QUESTION 24

24. Evaluate $\int_0^6 g(x)dx$ if $g(x) = \begin{cases} x^2, & x \leq 2 \\ 3x-2, & x \geq 2 \end{cases}$ A. 128 B. $\frac{34}{3}$ C. $\frac{128}{3}$ D. $\frac{34}{9}$ E. $\frac{128}{5}$

SOLUTION

$\int_0^6 g(x)\,dx$ where $g(x) = \begin{cases} x^2 ; & x \leq 2 \\ 3x-2 ; & x > 2 \end{cases}$

The transition point is at $x=2$ (breaking point)

$$\int_0^6 g(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 3x-2$$

$$= \frac{x^3}{3}\Big|_0^2 + \frac{3x^2}{2} - 2x\Big|_2^6$$

$$= \left(\frac{8}{3}-0\right) + \left[(54-12)-(6-4)\right]$$

$$= \frac{8}{3} + (42-2) = \frac{8}{3} + 40$$

$$= \frac{8+120}{3} = \frac{128}{3}$$

EXAMPLE 3: 2013/2014 EXAM QUESTION 65

65. Evaluate $\int_{-5}^{3}|x+1|dx$ A. 2 B. 4 C. 8 D. 12 E. 16

SOLUTION

$\int_{-5}^{3}|x+1|\,dx$; $x+1=0$; $x=-1$

Equating the content of the absolute value to zero as to evaluate transition point

$$|x+1| = \begin{cases} (x+1); & x \geq -1 \\ -(x+1); & x < -1 \end{cases}$$

$$\int_{-5}^{3}|x+1|\,dx = \int_{-5}^{-1} -(x+1)\,dx + \int_{-1}^{3}(x+1)\,dx$$

$$= 8 + 8 = 16$$

EXAMPLE 4: 2013/2014 EXAM QUESTION 48

48. Evaluate $\int_1^6 |x-3|dx$ A. $\frac{1}{2}$ B. 1 C. 6 D. $\frac{13}{2}$ E. $\frac{7}{3}$

SOLUTION

$\int_1^6 |x-3|\,dx$; $x-3=0$; $x=3$

$$|x-3| = \begin{cases} (x-3) ; & x \geq 3 \\ -(x-3) ; & x < 3 \end{cases}$$

$$\int_1^6 |x-3|\,dx = \int_1^3 -(x-3)\,dx + \int_3^6 (x-3)\,dx$$

$$= \left[-\frac{x^2}{2}-3x\right]_1^3 + \left[\frac{x^2}{2}-3x\right]_3^6$$

$$= 2 + \frac{9}{2} = 13/2$$

INTEGRATION OF PRODUCT OF SINES AND COSINES

Given the form $\int \cos A \sin B \, dx$
Where A and B are both functions of x.
Then we use a suitable substitution from the following identities.

1) $\sin(A+B) = \sin A\cos B + \sin B\cos A$
2) $\sin(A-B) = \sin A\cos B - \sin B\cos A$
3) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

Observe that; $\sin(A+B) + \sin(A-B)$ will give
$\sin A\cos B + \sin B\cos A + \sin B\cos A - \sin A\cos B$

$\Rightarrow \sin(A+B)+\sin(A-B) = 2\sin A\cos B$; Hence

$$\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$
$$= \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Similarly,

$$\sin B\cos A = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$
$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A\sin B = \frac{1}{2}[\cos(A+B) - \cos(A-B)]$$

EXAMPLE 1: 2009/2010 EXAM QUESTION

69. What is the integral of $\sin 3x\cos x$ with respect to x?
(a) $-\frac{1}{2}\left(\frac{1}{2}\cos 4x + \cos 2x\right) + C$ (b) $\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x + C$ (c) $-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x + C$ (d) none of the above

SOLUTION

$$\int \sin 3x\cos x \, dx$$

Using the trigonometric identity

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Comparing $\sin A\cos B$ with $\sin 3x\cos x$
$A = 3x$, $B = x$, Thus

$$\sin 3x\cos x = \frac{1}{2}[\sin(3x+x) + \sin(3x-x)]$$
$$= \frac{1}{2}[\sin 4x + \sin 2x]$$

Therefore,

$$\int \sin 3x\cos x \, dx = \int \frac{1}{2}(\sin 4x + \sin 2x)dx$$
$$= \frac{1}{2}\int \sin 4x\, dx + \frac{1}{2}\int \sin 2x\, dx$$
$$= \frac{1}{2}\left(-\frac{\cos 4x}{4}\right) + \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$
$$= -\frac{1}{2}\left(\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x\right) + C$$

INTEGRAL OF EXPONENTIAL FORM

Given $\int a^{f(x)} dx$; we will obtain

$$\frac{a^{f(x)}}{f'(x)\ln a} + c$$ on integration

EXAMPLE 1: 2010/2011 EXAM QUESTION 69

69. What is the integral of 8^{3x} with respect to x. (a) $\frac{8^{3x}}{3\log_e 8} + C$ (b) $3\log_e 8\,(8^{3x}) + C$ (c) $\frac{8^{3x}}{8\log_e 3} + C$ (d) $8^{3x}\log_e 8 + C$

SOLUTION

$$\int 8^{3x} dx$$

let $u = 3x$, $\frac{du}{dx} = 3$

Note: $\ln x = \log_e x$

$$dx = \frac{du}{3}$$
$$\int 8^{3x} dx = \int 8^u \cdot \frac{du}{3}$$
$$= \frac{1}{3}\int 8^u du = \frac{8^u}{\ln u} + c$$

Recall that; $u = 3x$

$$\Rightarrow \int 8^{3x} dx = \frac{8^{3x}}{\log_e 3x}$$

EXAMPLE 2: 2014/2015 EXAM QUESTION 9

9. Evaluate $\int x5^x dx$ (a) $\frac{x5^x}{\ln 5} + c$ (b) $\frac{-5^x}{\ln 5} + c$ (c) $\frac{x5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2} + c$ (d) $\frac{25x5^x}{5} + c$ (e) $\frac{5^x}{(\ln 5)^2} + \frac{x5^x}{\ln 5} + c$

SOLUTION

$$\int x5^x dx$$

Let $u = 5^x$; Taking ln of both sides

$\ln u = \ln 5^x$; $\ln u = x\ln 5$

$x = \frac{\ln u}{\ln 5}$; $dx = \frac{du}{u\ln 5}$; substituting

$$\Rightarrow \int \frac{\ln u}{\ln 5}\cdot u \cdot \frac{du}{u\ln 5} = \frac{1}{(\ln 5)^2}\int \ln u\, du$$
$$= \frac{1}{(\ln 5)^2}(u\ln u - u) + c$$

Note: $\ln 5^x = x\ln 5$

Recall that $u = 5^x$

$$\Rightarrow \int x5^x dx = \frac{1}{(\ln 5)^2}[5^x\ln(5^x) - 5^x] + c$$
$$= \frac{x5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2} + c$$

$\int \ln x\, dx = x\ln x - x + c$

INTEGRALS OF STANDARD FORM

Evaluation of some integrals in MTH 101 require the knowledge of some standard integrals.

EXAMPLE 1: 2013/2014 EXAM QUESTION 60

60. Evaluate $\int \frac{dx}{\sqrt{49-x^2}}$ $A.\cos^{-1}\left(\frac{x}{7}\right) + c$ $B.\frac{x}{7} + c$ $C.\sin^{-1}\left(\frac{x}{7}\right) + c$ $D.\tan^{-1}\left(\frac{x}{7}\right) + c$ $E.\sec^{-1}\left(\frac{x}{7}\right) + c$

SOLUTION

$$\int \frac{dx}{\sqrt{49-x^2}} = \int \frac{dx}{\sqrt{7^2-x^2}}$$

Comparing with the standard integral form;

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c$$

$$\int \frac{dx}{\sqrt{49-x^2}} = \int \frac{dx}{\sqrt{7^2-x^2}} = \sin^{-1}(x/7) + c$$

EXAMPLE 2: 2012/2013 EXAM QUESTION 20

20. Evaluate $\int_0^2 \frac{dx}{\sqrt{x^2+2x+3}}$ A. $\sinh^{-1}\frac{3}{\sqrt{2}} - \sinh^{-1}\frac{3}{\sqrt{2}}$ B. $\sinh^{-1}(\frac{1}{\sqrt{2}})$ C. $\sinh^{-1}\frac{3}{\sqrt{2}}$ D. $\sinh^{-1}\frac{3}{\sqrt{2}} - \sinh^{-1}\frac{1}{\sqrt{2}}$

SOLUTION

$$\int_0^2 \frac{dx}{\sqrt{x^2+2x+3}}$$

using the standard integral form

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)+c$$

Let's make x^2+2x+3 in the form 'x^2-a^2'

making x^2-2x a perfect square

$$\Rightarrow x^2+2x+(1)^2$$

$$x^2+2x+1 = (x+1)^2$$

but $(x^2+2x+1)+2 = x^2+2x+3 = (x+1)^2+2$

Therefore;

$$\int_0^2 \frac{dx}{\sqrt{x^2+2x+3}} = \int_0^2 \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= \left[\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^2$$

$$= \sinh^{-1}\frac{3}{\sqrt{2}} - \sinh^{-1}\frac{1}{\sqrt{2}}$$

EXAMPLE 3: 2008/2009 EXAM QUESTION 9

9. Integrate $\frac{1}{(x^2-10x+18)}$ $w.r.t\ x$ (a) $\left(\frac{1}{2\sqrt{7}}\right)\tan\left(\frac{(x-5-\sqrt{7})}{(x-5+\sqrt{7})}\right)+C$ (b) $\left(\frac{1}{2\sqrt{7}}\right)\left(\frac{(x-5-\sqrt{7})}{(x-5-\sqrt{7})}\right)+C$ (c) $\left(\frac{1}{2\sqrt{7}}\right)\ln\left(\frac{(x-5-\sqrt{7})}{(x-5+\sqrt{7})}\right)+C$ (d) $2\sqrt{7}\ln\left(\frac{(x-5-\sqrt{7})}{(x-5+\sqrt{7})}\right)+C$

SOLUTION

$$\int \frac{1}{x^2-10x+18}\,dx$$

Comparing with the standard integral

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c$$

Now, let's make x^2-10x a perfect square

We have; $x^2-10x+25$

$$\Rightarrow \int \frac{1}{x^2-10x+18}\,dx = \int \frac{1}{(x^2-10x+25)-7}\,dx$$

$$= \int \frac{1}{(x-5)^2-(\sqrt{7})^2}\,dx =$$

$$= \frac{1}{2\sqrt{7}}\ln\left(\frac{x-5-\sqrt{7}}{x-5+\sqrt{7}}\right)$$

EXAMPLE 4: 22008/2009 EXAM QUESTION 23

23. Solve $\int \frac{ds}{\sqrt{4-s^2}}$ (a) $\cos^{-1}\left(\frac{x}{2}\right)+C$ (b) $\tan^{-1}\left(\frac{x}{2}\right)+C$ (c) $\sec^{-1}\left(\frac{x}{2}\right)+C$ (d) $\sin^{-1}\left(\frac{x}{2}\right)+C$

SOLUTION

$$\int \frac{ds}{\sqrt{4-s^2}}$$

using the standard integral form

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)+c$$

where $a=2$, $x=s$

$$\Rightarrow \int \frac{ds}{\sqrt{4-s^2}} = \int \frac{dx}{\sqrt{2^2-s^2}}$$

$$= \sin^{-1}(s/2)+c$$

EXAMPLE 5: 2010/2011 EXAM QUESTION 61

61. Evaluate $\int \frac{dx}{\sqrt{5-x^2}}$ (a) $\sin^{-1}\left(\frac{\sqrt{5}}{x}\right)+C$ (b) $\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$ (c) $\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$ (d) $\sec\left(\frac{x}{\sqrt{5}}\right)+C$

SOLUTION

$$\int \frac{dx}{\sqrt{5-x^2}}$$

Comparing with the standard integral;

$$\int \frac{dx}{\sqrt{a-x^2}} = \sin^{-1}(x/a)+c$$

$$a=\sqrt{5};\quad \int \frac{dx}{\sqrt{(\sqrt{5})^2-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$$

DEFINITE INTEGRALS

When $f(x)$ is bounded on an interval, say $[a,b]$, then the definite integral is given by;

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Again; $-\int_a^b f(x)\,dx = F(a)-F(b)$

EXAMPLE 1: 2012/2013 EXAM QUESTION 52

52. Evaluate $\int_{-1}^{5}(5x^2-4x+1)dx$ $A.\ 168$ $B.\ 104$ $C.\ 125$ $D.\ 150$

SOLUTION

$$\int_{-1}^{5}(5x^2-4x+1)dx$$

$$\left[\frac{5x^{2+1}}{3} - \frac{4x^{1+1}}{2} + \frac{1x^{0+1}}{1}\right]_{-1}^{5}$$

$$\left[\frac{5x^3}{3} - \frac{4x^2}{2} + \frac{x^1}{1}\right]_{-1}^{5}$$

$$\left[\frac{5}{3}x^3 - 2x^2 + x\right]_{-1}^{5}$$

Taking the limit, we substitute the upper limit first, then the lower limit.

upper limit, $x=5$

$$\left[\frac{5}{3}(5)^3 - 2(5)^2 + 5\right] = \frac{625}{3} - 50 + 5 = \frac{490}{3}$$

lower limit, $x=-1$

$$\left[\frac{5}{3}(-1)^3 - 2(-1)^2 + (-1)\right] = -\frac{5}{3} - 2 - 1 = \frac{-14}{3}$$

Upper limit result − lower limit result yields;

$$\frac{490}{3} - \left(-\frac{14}{3}\right) = \frac{490}{3} + \frac{14}{3} = \frac{504}{3} = 168$$

EXAMPLE 2: 2014/2015 EXAM QUESTION 13

13. Evaluate $\int_0^{\pi/4}\cos(-4x)\,dx$ (a) 3 (b) 4 (c) 0 (d) 1 (e) 2

SOLUTION

$$\int_0^{\pi}\cos(-4x)dx$$

let $u=-4x$; $\frac{du}{dx}=-4$; when $x=0, u=0$; $x=\pi ; u=-4\pi$

$dx=\frac{du}{-4}$; substituting

$$\int_0^{\pi}\cos(-4x)dx = -\frac{1}{4}\int_0^{-4\pi}\cos u\cdot du$$

$$= \left[-\frac{1}{4}\sin u\right]_0^{-4\pi}$$ Recall that $\pi=180$

$$= \left[-\frac{1}{4}\sin(-4\pi)\right]-\left[-\frac{1}{4}\sin(0)\right]$$

$$= \left[-\frac{1}{4}\sin(-720)\right]-\left[-\frac{1}{4}\sin 0\right]$$

$$= \left[-\frac{1}{4}(0)\right]-\left[-\frac{1}{4}(0)\right]$$

$$= -\frac{1}{4}(0)+\frac{1}{4}(0)$$

$$= 0+0=0$$

EXAMPLE 3: 2012/2013 EXAM QUESTION 15

15. Evaluate

$\int_0^{\pi}\frac{e^{3x}}{1+e^{3x}}dx$ A. $\frac{1+e^{3x}}{2}$ B. $\frac{1}{2}\ln\left(\frac{1+e^{3x}}{2}\right)$ C. $\ln\frac{1+e^{3x}}{2}$ D. $1+e^{3x}$

SOLUTION

$$\int_0^{\pi}\frac{e^{3x}}{1+e^{3x}}dx$$

let $u=1+e^{3x}$; $\frac{du}{dx}=3e^{3x}$

$dx=\frac{du}{3e^{3x}}$; substituting into the given integral

$$\int_0^{\pi}\frac{e^{3x}}{1+e^{3x}}dx = \int_0^{\pi}\frac{e^{3x}}{u}\cdot\frac{du}{3e^{3x}}$$

$$= \frac{1}{3}\int_0^{\pi}\frac{1}{u}du = \left[\frac{1}{3}\ln u\right]_0^{\pi}$$

$$= \left[\frac{1}{3}\ln(1+e^{3x})\right]_0^{\pi} = \frac{\ln(1+e^{3x})}{3}\Big|_0^{\pi}$$

EXAMPLE 4: 2013/2014 EXAM QUESTION 1

1. Find the value of $\int_0^{\pi}\frac{\cos^2\theta-1}{\sin^2\theta}d\theta$ A. π B. $\frac{\pi}{2}$ C. $-\frac{\pi}{2}$ D. 0 E. $-\pi$

SOLUTION

$$\int_0^{\pi}\frac{\cos^2\theta-1}{\sin^2\theta}d\theta$$

from trignometry

$\sin^2\theta+\cos^2\theta=1$

$\cos^2\theta-1=-\sin^2\theta$

$$\int_0^{\pi}\frac{\cos^2\theta-1}{\sin^2\theta}d\theta = \int -\frac{\sin^2\theta}{\sin^2\theta}d\theta$$

$$\int_0^{\pi}-1\,d\theta = [-\theta]_0^{\pi} = -\pi-0$$

$$= -\pi$$

EXAMPLE 5: 2013/2014 EXAM QUESTION 16

16. Evaluate $\int_{-1}^{0}x^2e^{x^3+1}dx$ A. $\frac{1}{3}$ B. $\frac{e}{3}$ C. $\frac{1}{3}(e-1)$ D. $\frac{1}{3}(1-e)$ E. $\frac{1}{e}$

SOLUTION

$$\int_{-1}^{0}x^2e^{x^3+1}dx$$

let $u=x^3+1$; $\frac{du}{dx}=3x^2$; $dx=\frac{du}{3x^2}$

Substituting into the given integral

$$\int_{-1}^{0}x^2e^{x^3+1}dx = \int_{-1}^{0}x^2e^{u}\cdot\frac{du}{3x^2}$$

$$= \int_{-1}^{0}\frac{e^u du}{3} = \frac{1}{3}\int_{-1}^{0}e^u du = \left[\frac{1}{3}e^u\right]_{-1}^{0}$$

$$= \left[\frac{1}{3}e^{x^3+1}\right]_{-1}^{0} = \left(\frac{1}{3}e^{0^3+1}\right)-\left(\frac{1}{3}e^{(-0^3+1)}\right) = \frac{1}{3}(e-1)$$

EXAMPLE 6: 2013/2014 EXAM QUESTION 44

44. Evaluate $\int_1^{e}\frac{5}{x}dx$ A. e B. 5 C. 0 D. 1 E. -1

SOLUTION

$$\int_1^{e}\frac{5}{x}dx = 5\int_1^{e}\frac{1}{x}dx$$

$$= [5\ln x]_1^{e} = [5\ln e]-[5\ln(1)]$$

$$= 5-0$$ $\ln e = 1$

$$= 5$$

EXAMPLE 7: 2013/2014 EXAM QUESTION 22

22. Evaluate $\int_1^{e}x\ln dx$ A. $\frac{1}{2}(e^2-1)$ B. $\frac{1}{2}(1+e)$ C. $\frac{1}{4}(e^2+1)$ D. $\frac{1}{8}(1-e^2)$ E. e^2-2

SOLUTION

$$\int_1^{e}x\ln x\,dx$$

using integration by parts

where $u=\ln x$, $dv=x$

$du=\frac{1}{x}$, $v=\frac{x^2}{2}$

$$\int_1^{e}x\ln x\,dx = \frac{x^2}{2}\ln x\Big|_1^{e}-\int_1^{e}\frac{x^2}{2}\cdot\frac{1}{x}$$

$$= \frac{x^2}{2}\ln x\Big|_1^{e}-\int_1^{e}\frac{x}{2}$$

$$= \frac{x^2}{2}\ln x\Big|_1^{e}-\frac{1}{2}\int_1^{e}x$$

$$= \frac{x^2}{2}\ln x\Big|_1^{e}-\frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^{e}$$

$$= \left[\frac{x^2}{2}\ln x-\frac{x^2}{4}\right]_1^{e}$$

$$= \left(\frac{e^2}{2}\ln e-\frac{e^2}{4}\right)-\left(\frac{1}{2}\ln 1-\frac{1}{4}\right) = \frac{1}{4}(e^2+1)$$

EXAMPLE 8: 2012/2013 EXAM QUESTION 30

30. Evaluate $\int_{-\pi/2}^{\pi/2}\cos x\,dx$ A. 0 B. 1 C. 2 D. 3

SOLUTION

$$\int_{-\pi/2}^{\pi/2}\cos x\,dx = [\sin x]_{-\pi/2}^{\pi/2}$$

π in radian $= 180°$

$\left[\sin\left(\frac{180}{2}\right)\right] - \left[\sin\left(-\frac{180}{2}\right)\right]$

$\left[\sin(90)\right] - \left[\sin(-90)\right]$

$1 - (-1) = 1+1 = 2$

EXAMPLE 9: 2012/2013 EXAM QUESTION 25

25. Evaluate $\int_0^{\frac{\pi}{4}} \sec^2\theta d\theta$ A. 1 B. 2 C. 3 D. 4

SOLUTION

$\int_0^{\pi/4} \sec^2\theta\, d\theta = \left[\tan\theta\right]_0^{\pi/4}$

$= \left[\tan\left(\frac{180}{4}\right)\right] - \left[\tan(0)\right]$

$= \tan(45) - \tan(0)$

$= 1 - 0 = 1$

EXAMPLE 10: 2009/2010 EXAM QUESTION 40

40. Evaluate $\int_0^4 \sqrt{y}.dy$ (a) $\frac{16}{3}$ (b) $\frac{3}{16}$ (c) 16 (d) 3

SOLUTION

$\int_0^4 \sqrt{y}\, dy$

$= \int_0^4 y^{1/2}\, dy$

$= \left[\frac{y^{3/2}}{3/2}\right]_0^4 = \left[\frac{2y^{3/2}}{3}\right]_0^4$

$= \left[\frac{2(4)^{3/2}}{3}\right] - \left[\frac{2(0)^{3/2}}{3}\right]$

$= \frac{2(8)}{3} = \frac{16}{3}$

EXAMPLE 11: 2009/2010 EXAM QUESTION 44

44. Solve the integra $\int_0^{\frac{\pi}{3}} \sin 3x dx$ (a) $-\frac{2}{3}$ (b) 3 (c) $\frac{2}{3}$ (d) 4

SOLUTION

$\int_0^{\pi/3} \sin 3x\, dx$

let $u = 3x$; $\frac{du}{dx} = 3$; $dx = \frac{du}{3x}$

$\Rightarrow \int_0^{\pi/3} \sin 3x\, dx = \int_0^{\pi/3} \sin u \cdot \frac{du}{3}$

$= \frac{1}{3}\int_0^{\pi/3} \sin u\, du$

$= \frac{1}{3}(-\cos u)\Big|_0^{\pi/3}$

$= -\frac{1}{3}\cos(3x)\Big|_0^{\pi/3}$

$= \left(-\frac{1}{3}\cos 3\pi/3\right) - \left(-\frac{1}{3}\cos 0\right)$

$= \left(-\frac{1}{3}\cos\pi\right) - \left(-\frac{1}{3}\cos 0\right)$

$= -\frac{1}{3}(-1) + \frac{1}{3}$

$= \frac{2}{3}$

EXAMPLE 12: 2009/2010 EXAM QUESTION 71

71. The value of the integral $\int_0^{\frac{\pi}{2}} x^2\cos x dx$ is (a) $\frac{\pi^2}{4} - 2$ (b) $\frac{\pi^2}{4} + 2$ (c) $\frac{\pi^2}{2} - 4$ (d) $\frac{\pi^2}{2} - 2$

SOLUTION

$\int_0^{\pi/2} x^2 \cos x\, dx$

Using integration by parts

$\int u dv = uv - \int v du$

Where $u = x^2$, $dv = \cos x$, $du = 2x$, $v = \sin x$

$\Rightarrow \int_0^{\pi/2} x^2 \cos x\, dx = x^2\sin x - 2\int x\sin x$

let $I = \int x\sin x\, dx$

Using integration by parts again

Where $u = x$, $dv = \sin x$, $du = 1$, $v = \int \sin x = -\cos x$

$\Rightarrow I = -x\cos x - \int -\cos x\,(1)$

$= -x\cos x + \int \cos x$

$= x\cos x + \sin x$

Therefore $\int_0^{\pi/2} x^2\cos x\, dx = x^2\sin x - 2I\Big|_0^{\pi/2}$

Where $I = \int x\sin x\, dx = -x\cos x + \sin x$

$\Rightarrow \int_0^{\pi/2} x^2\cos x\, dx = \left[x^2\sin x - 2(-x\cos x + \sin x)\right]_0^{\pi/2}$

taking the limits $= \left[x^2\sin x + 2x\cos x + 2\sin x\right]_0^{\pi/2}$

$\Rightarrow \left[\left(\frac{\pi}{2}\right)^2(1) + \pi(0) - 2(1)\right] - \left[(0)^2 + 2(0)(\cos 0) + 2(0)\right] = \frac{\pi^2}{4} - 2$

EXAMPLE 13: 2008/2009 EXAM QUESTION 10

10. Evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x\sin x dx$ (a) 5 (b) -5 (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

SOLUTION

$\int_0^{\frac{\pi}{2}} \cos^4 x\sin x\, dx$

let $u = \cos^5 x$; $\frac{du}{dx} = -5\cos^4 x\sin x$

$dx = \frac{du}{-5\cos^4 x\sin x}$

$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \cos^4 x\sin x \cdot \frac{du}{-5\cos^4 x\sin x}$

$= \int_0^{\frac{\pi}{2}} -\frac{du}{5} = -\frac{1}{5}\int_0^{\pi/2} du$

$= -\frac{1}{5}u\Big|_0^{\pi/2}$

$= -\frac{1}{5}\cos^5 x\Big|_0^{\pi/2}$

$= \left[-\frac{1}{5}\cos\frac{\pi}{2}\right] - \left[-\frac{1}{5}\cos 0\right]$

$= -\frac{1}{5}(-1) = \frac{1}{5}$

EXAMPLE 14: 2008/2009 EXAM QUESTION 25

25. Evaluate $\int_0^2 \frac{x\,dx}{1+x^2}$ (a) $2\ln 5$ (b) $-2\ln 5$ (c) $-\frac{1}{2}\ln 5$ (d) $\frac{1}{2}\ln 5$

SOLUTION

$$\int_0^2 \frac{x}{1+x^2}\,dx$$

let $u = 1+x^2$; $\frac{du}{dx} = 2x$; $dx = \frac{du}{2x}$

$$\Rightarrow \int_0^2 \frac{x}{1+x^2}dx = \int_0^2 \frac{x}{u}\cdot\frac{du}{2x}$$

$$= \frac{1}{2}\int_0^2 \frac{du}{u} = \left[\frac{1}{2}\ln(u)\right]_0^2$$

$$= \left[\frac{1}{2}\ln(1+x^2)\right]_0^2$$

$$= \left[\frac{1}{2}\ln(1+2^2)\right] - \left[\frac{1}{2}\ln(1+0^2)\right]$$

$$= \frac{1}{2}\ln 5 - 0$$

$$= \frac{1}{2}\ln 5$$

EXAMPLE 15: 2010/2011 EXAM QUESTION 45

45. Evaluate $\int_0^1 8x(x^2+1)^3\,dx$ (a) 12 (b) 15 (c) 5 (d) 16

SOLUTION

$$\int_0^1 8x(x^2+1)^3\,dx$$

let $u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

When $x = 1$; $u = 1^2+1 = 2$

$x = 0$; $u = 0^2+1 = 1$

$$\int_1^2 8x(x^2+1)^3dx = \int_1^2 8x\cdot u^3\cdot\frac{du}{2x}$$

$$= \int_1^2 4u^3du = u^4\Big|_1^2$$

$$= 2^4 - 1^4$$

$$= 16 - 1$$

$$= 15$$

APPLICATION OF INTEGRATION

There are some basic application of integration; In MTH 102, We shall learn the following;

1. AREA UNDER CURVE

The area bounded by a Curve $y = f(x)$ and the x-axis with co-ordinates $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b y\,dx$$

EXAMPLE 1: 2012/2013 EXAM QUESTION 18

18. Find the area enclosed between the curve $y = x^3$ and the straight line $y = x$ in $[0,1]$in $[0,1]\frac{1}{2}unit^2$ B. $-\frac{1}{4}unit^2$ C. $\frac{1}{4}unit^2$ D. $-\frac{1}{2}unit^2$

SOLUTION

$y = x$ and $y = x^3$

$\Rightarrow x^3 = x$

$x^3 - x = 0$

$$A = \int_0^1 (x^3-x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{1^4}{4} - \frac{1^2}{2}\right] - \left[\frac{0^4}{4} - \frac{0^2}{2}\right]$$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - 0$$

$$= -\tfrac{1}{4}\text{ unit}^2$$

EXAMPLE 2: 2013/2014 EXAM QUESTION 64

64. Calculate the area bounded by the curve $y=x^3$, *the x-axis and the lines* $x = -1$ *and* $x = 1$. A. 1 B. 2 C. 0 D. $\frac{1}{4}$ E. $\frac{1}{2}$

SOLUTION

$$\text{Area} = \int_{-1}^1 x^3\,dx$$

The integral of an odd function on the interval of a symmetrical portion is zero e.g. -4 to 4

$$= \left[\frac{x^4}{4}\right]_{-1}^1 = \left[\frac{(1)^4}{4}\right] - \left[\frac{(-1)^4}{4}\right]$$

$$= \frac{1}{4} - \frac{1}{4} = 0$$

EXAMPLE 3: 2012/2013 EXAM QUESTION 61

61. To find the area between the curve $y = 5x - 2x^2$ *and the line* $y = x$. *what will be the lower and* upper limits? A. 0,2 B. 2,0 C. 0, $\sqrt{2}$ D. $\sqrt{2}$,0

SOLUTION

$y = 5x - 2x^2$ and $y = x$

$\Rightarrow 5x - 2x^2 = x$

Arranging in the form $ax^2+bx+c=0$

$\Rightarrow 2x^2 - 5x + x = 0$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$2x = 0$ or $x - 2 = 0$

$x = \frac{0}{2}$ or $x = 2$

$x = 0$ or $x = 2$

Hence the lower limit is $x = 0$ and upper limit is $x = 2$

EXAMPLE 4: 2013/2014 EXAM QUESTION 52

52. Find the function f(x) whose derivative is sin x and whose graph passes through the point (0,2) A. 2+cos x B. 3 − cos x C. − cos x D. 3 + cos x E. 2 − cos x

SOLUTION

$y' = \sin x$; $y = \int \sin x\,dx = -\cos x + c$

The graph passes through (0,2)

$\Rightarrow$ When $x = 0$, $y = 2$, where $y = -\cos x + c$

$2 = -\cos 0 + c$; $c = 2 + \cos 0$; $c = 3$

Therefore; $y = -\cos x + 3$; $y = 3 - \cos x$

EXAMPLE 5: 2010/2011 EXAM QUESTION 46

46. Find the area of the region bounded by the lines $y = 2x$ and $x = 2$, and the x-axis (a) 4 (b) 2 (c) 10 (d) 12

SOLUTION

$$Area = \int_0^2 2x = x^2\Big|_0^2$$

$$= (2)^2 - (0)^2$$

$$= 4$$

EXAMPLE 6: 2012/2013 EXAM QUESTION 34

34. Find the area bounded by curves $y = 4 - x^2$ and $y = 2x + 1$ A. $10\frac{1}{3}$ sq units. B. $10\frac{2}{3}$ sq units. C. $20\frac{1}{3}$ sq units. D. $20\frac{2}{3}$ sq units.

SOLUTION

$y = 4 - x^2$; $y = 2x+1$

$\Rightarrow 4 - x^2 = 2x + 1$

Arranging in the form $ax^2 + bx + c = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

Therefore the boundary conditions is from -3 to 1

$$A = \int_{-3}^{1} (x^2 + 2x + 1)dx$$

$$= 10^2/_3 \text{ sq unit}$$

EXAMPLE 7: 2008/2009 EXAM QUESTION 27

27. Find the area of the finite region bounded by the line, $x + 1$, the x – axis and the ordinates at $x = 1$ and $= 3$. (a) 7.5 sq. units (b) 6 sq. units (c) 1.5 sq. units (d) 12 sq. units

SOLUTION

The area of the finite region bounded by the line $x+1$, the x-axis and the ordinates at abscissa $x=1$ and $x=3$ is given by;

$$A = \int_1^3 (x+1)dx = \left[\frac{x^2}{2} + x\right]_1^3$$

$$= \left(\frac{3^2}{2} + 3\right) - \left(\frac{1^2}{2} + 1\right)$$

$$= \left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} + 1\right)$$

$$= \frac{8}{2} + 2 = 6 \text{ sq units}$$

EXAMPLE 8: 2012/2013 EXAM QUESTION 55

55. Determine the area of the region enclosed by $y = e^x$ and $y = \sqrt{x}$ on the internal $[0,1]$ A. e B. e$-\frac{5}{3}$ C. $\frac{5}{3}$ D. $\frac{\sqrt{5}}{2}$

SOLUTION

$y = \sqrt{x}$ and $y = e^x$

$\Rightarrow \sqrt{x} = e^x$; $e^x - \sqrt{x} = 0$

$$A = \int_0^1 (e^x - \sqrt{x})dx$$

$$= \int_0^1 (e^x - x^{1/2})dx$$

$$= \left[e^x - \frac{x^{1/2+1}}{3/2}\right]_0^1$$

$$= \left[e^x - \frac{2x^{3/2}}{3}\right]_0^1$$

Taking the limit;

$$A = \left[e^1 - \frac{2(1)^{3/2}}{3}\right] - \left[e^0 - \frac{2(0)^{3/2}}{3}\right]$$

$$= (e - 2/3) - (1 - 0)$$

$$= e - 2/3 - 1$$

$$= e - 5/3$$

2. KINEMATICS

Given a function of acceleration with respect to time, $a(t)$, then

$V = \int a(t)\,dt$; and given a function of velocity with respect to time $V(t)$; Then

$S = \int V(t)\,dt$; Where S is distance

EXAMPLE 1: 2008/2009 EXAM QUESTION 24

24. The velocity of an automobile at time t (*in seconds*) given by $v(t) = 4t + 5$, $v(t)$ is measured in meters per second. Determine the distance function $s(t)$ assuming that the automobile is at rest when $t = 0$. (a) $4t^2 + 5x$ (b) $t^2 + 5x$ (c) $2t^2 + 5x$ (d) $2t^2 + 5x + 1$

SOLUTION

The velocity of an automobile at time t,

$V(t) = 4t + 5$; but $S(t) = \int V(t)\,dt$

$\Rightarrow S(t) = \int (4t+5)\,dt$

$= 2t^2 + 5t + c$

The automobile is at rest when $t = 0$; Therefore $S(t) = 0 + c$ where $S(t) = 0$; Hence $c = 0$

$\Rightarrow S(t) = 2t^2 + 5t$

EXAMPLE 2: 2011/2012 EXAM QUESTION 42

42. A particle moves long a line with velocity $V(t) = (t^2 - 5)m/s$. Find the distance travelled from time t =3s to time t= 5s (a) $\frac{23}{3}m$ (b) $\frac{40}{3}m$ (c) $383m$ (d) $\frac{68}{3}m$

SOLUTION

The distance travelled from time $t = 3s$ to time 5s is given by

$$S = \int_3^5 V(t)\,dt$$

N 64
he x –
B.2 C.0 D.
I 61
and
2
x B.

$$S = \int_3^5 (t^2 - 5)\,dt$$
$$= \frac{t^3}{3} - 5t \Big|_3^5$$
$$= \left(\frac{5^3}{3} - 5(5)\right) - \left(\frac{3^3}{3} - 5(3)\right)$$
$$= \left(\frac{125}{3} - 25\right) - (9 - 15)$$
$$= \frac{50}{3} - (-6)$$
$$= \frac{50}{3} + 6 = \frac{68}{3}\,m$$

EXAMPLE 3: 2010/2011 EXAM QUESTION 37

37. The acceleration of a particle at any time t seconds which moves along x axis is given by $a(t) = \frac{7}{(t+1)^8}$ calculate the velocity given that when $t = 0$, the velocity is 10m/s. (a) $11t^2 + C$ (b) $(t+1)^8 + C$ (c) $-(t+1)^{-7} + 11$ (d) $(t+1)^{-8} + 11$

SOLUTION

$$a(t) = \frac{7}{(t+1)^8}$$

$a(t) = \frac{dv(t)}{dt}$, thus $v = \int a\,dt$

$$v = \int \frac{7}{(t+1)^8}\,dt = 7\int (t+1)^{-8}\,dt$$
$$= \frac{7(t+1)^{-7}}{-7} + c$$
$$v = -(t+1)^{-7} + c = -\frac{1}{(t+1)^7} + c$$

Recall that at $t = 0$, $v = 10m/s$

$$\Rightarrow 10 = -\frac{1}{(0+1)^7} + c$$
$$c = 10 + 1;\ c = 11$$

Hence $v = -(t+1)^{-7} + 11$

3. MEAN VALUE OF A FUNCTION

Given a function $y = f(x)$, then the mean value 'M' is given by

$$M = \frac{1}{b-a}\int_a^b y\,dx = \frac{\text{Area}}{b-a}$$

EXAMPLE 1: 2015/2016 EXAM QUESTION 27

27. Find the mean value m of the function $f(x) = x^2$ between the limits $x = 2$ and $x = 4$ a. 28/3 b. 3/28 c. 64/3 d. 8/3 e. 3/64

SOLUTION

$f(x) = x^2$, $a = 2$, $b = 4$

Mean value of the function is given by

$$M = \frac{1}{b-a}\int_a^b y\,dx = \frac{\text{Area}}{b-a}$$

$$M = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_2^4$$

$$= \frac{1}{6}\left[4^3 - 2^3\right] = \frac{56}{6}$$
$$= \frac{28}{3}$$

4. AREA OF LINE ROTATED ABOUT X- AXIS

Given a line $y = f(x)$, the axis with co-ordinate $x = a$ and $x = b$, the area of the line rotated about x axis is given by;

$$\text{Area} = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

EXAMPLE 1: 2015/2016 EXAM QUESTION 28

28. Find the surface area of a cone formed about the x axis the line y= 2x between x=0 and x= h a. $2\sqrt{5\pi}h^2$ b. $25\sqrt{\pi}h^2$ c. $2\sqrt{5}\pi h^2$ d. $5\sqrt{2}\pi h^2$ e. $2h^2\sqrt{2\pi}$

SOLUTION

Area of a line rotated about x-axis is given by

$$A = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Where $y = 2x$; $\frac{dy}{dx} = 2$; $a = 0$; $b = h$

$$\Rightarrow A = \int_0^h \left(4\pi x \cdot \sqrt{1+4}\right)dx = \int_0^h 4\pi x\sqrt{5}\,dx$$
$$= \left[2x^2\pi\sqrt{5}\right]_0^h = (2h^2\pi\sqrt{5}) - 0 = 2\sqrt{5}\pi h^2$$

5. VOLUME OF REVOLUTION

When a line $y = f(x)$ is rotated about the x-axis with ordinates $x = a$ and $x = b$, the volume of a solid is generated.. The volume formed is given by;

$$V = \int_a^b \pi y^2\,dx$$

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

SCHOOL OF PHYSICAL SCIENCES DEPARTMENT OF MATHEMATICS

2017/2018 MTH 102 TEST TIME: 1 HR DATE: 02/08/2018 INSTRUCTION: ANSWER ALL QUESTIONS

1. Evaluate $\lim_{x \to 0} \frac{x}{3 - \sqrt{9+x}}$
2. Obtain the inverse $f^{-1}(x)$ of $f(x) = \frac{5x+3}{4x-7}$
3. Determine the domain and range of the function $g(x) = \frac{\log(2x-1)}{x+1}$
4. Find the value(s) of t if $f(x) = x^2 + 5x + 3$ and $f(t) = -3$
5. Find $y'(2,0)$ if $5x^2 - x^3\sin y + 5xy = 10$
6. Evaluate $\lim_{x \to \pi} \frac{\sin^2 x}{1 + \cos x}$
7. Find $(g \circ f)(x)$ if $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$
8. Evaluate $f'(x)$ if $f(x) = \sqrt{2x^2 + 3x}$. Hence, find $f'(1)$
9. Find the domain and range of $g(x) = (x^2 - 2x - 3)^{-1}$
10. For what value(s) of k is $f(x)$ continuous if $f(x) = \begin{cases} k, & x \le 1 \\ 2x + 4, & x > 1 \end{cases}$

2017/2018 TEST QUESTION AND SOLUTION

FUNCTIONS

2. Obtain the inverse $f^{-1}(x)$ of $f(x) = \frac{5x+3}{4x-7}$

SOLUTION

$F(x) = \frac{5x+3}{4x-7}$

Let $y = F(x)$

$\Rightarrow y = \frac{5x+3}{4x-7}$ Making x the Subject Formula

$y(4x-7) = 5x+3$

$4xy - 7y = 5x+3$

$4xy - 5x = 7y+3$

$x(4y-5) = 7y+3$

$x = \frac{7y+3}{4y-5}$

Replacing x with $F^{-1}(x)$ and y with x

$\Rightarrow F^{-1}(x) = \frac{7x+3}{4x-5}$

4. Find the value(s) of t if $f(x) = x^2 + 5x + 3$ and $f(t) = -3$

SOLUTION

$F(x) = x^2 + 5x + 3$

$\Rightarrow F(t) = t^2 + 5t + 3$

from the question, $F(t) = -3$

$\Rightarrow t^2 + 5t + 3 = -3$

$t^2 + 5t + 3 + 3 = 0$

$t^2 + 5t + 6 = 0$

factorising;

$(t+2)(t+3) = 0$

$t = -2$ or $t = -3$

7. Find $(gof)(x)$ if $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$

SOLUTION

$F(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$

$g \circ F = g[F(x)]$

$= \sqrt{[F(x)] + 1}$

$= \sqrt{(x^2-1)+1}$

$= \sqrt{x^2}$

$= x$

DOMAIN AND RANGE

3. Determine the domain and range of the function $g(x) = \frac{\log(2x-1)}{x+1}$

SOLUTION

$g(x) = \frac{\log(2x-1)}{x+1}$

g(x) will be defined if and only if; $2x-1 > 0$ and $x \neq -1$; solving we have;

$2x - 1 > 0$ and $x \neq -1$

$2x > 1$

$x > \frac{1}{2}$

Domain of $g(x) = (\frac{1}{2}, \infty)$

Range of $g(x)$ = All real numbers $= (-\infty, \infty)$ or $\mathbb{R}$

9. Find the domain and range of $g(x) = (x^2 - 2x - 3)^{-1}$

SOLUTION

$g(x) = (x^2 - 2x - 3)^{-1}$

$g(x) = \frac{1}{x^2 - 2x - 3} = \frac{1}{(x-3)(x+1)}$

g(x) is a Rational function and will be defined; if $x - 3 \neq 0$ and $x + 1 \neq 0$

$\Rightarrow x \neq 3$ and $x \neq -1$

Dom $g = \mathbb{R} - \{-1, 3\}$

LIMITS

1. Evaluate $\lim_{x\to 0} \frac{x}{3-\sqrt{9+x}}$

SOLUTION

$\lim_{x\to 0} \frac{x}{3-\sqrt{9+x}} = \frac{0}{3-\sqrt{9+0}} = \frac{0}{3-3} = \frac{0}{0}$

Applying L' Hopital's theorem;

$\lim_{x\to 0} \frac{x}{3-\sqrt{9+x}} = \lim_{x\to 0} \frac{1}{-\frac{1}{2\sqrt{9+x}}}$

$= \lim_{x\to 0} -2\sqrt{9+x} = -2\sqrt{9+0}$

$= -2(3) = -6$

6. Evaluate $\lim_{x\to \pi} \frac{\sin^2 x}{1+\cos x}$

SOLUTION

$\lim_{x\to \pi} \frac{\sin^2 x}{1+\cos x} = \frac{(\sin \pi)^2}{1+\cos(\pi)} = \frac{0^2}{1+(-1)} = \frac{0}{0}$

Applying L' Hopital's rule

$\lim_{x\to \pi} \frac{\sin^2 x}{1+\cos x} = \lim_{x\to \pi} \frac{2\sin x \cos x}{-\sin x}$

$= \lim_{x\to \pi} -2\cos x = -2\cos \pi$

$= -2(-1) = 2$

CONTINUITY

10. For what value(s) of k is $f(x)$ continuous if $f(x) = \begin{cases} k, & x \le 1 \\ 2x+4, & x > 1 \end{cases}$

SOLUTION

$F(x) = \begin{cases} k; & x \le 1 \\ 2x+4; & x > 1 \end{cases}$

$F(1) = k$

$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} k = k$

$\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} 2x+4 = 2(1)+4 = 6$

For the function to be continuous at $x = 1$

$\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} F(x)$

$\Rightarrow k = 6$

DERIVATIVE

5. Find $y'(2,0)$ if $5x^2 - x^3 \sin y + 5xy = 10$

SOLUTION

If $5x^2 - x^3 \sin y + 5xy = 10$

$y' = \frac{dy}{dx} = -\left(\frac{10x - 3x^2\sin y + 5y}{-x^3\cos y + 5x}\right)$

$= \frac{10x - 3x^2\sin y + 5y}{x^3\cos y - 5x}$

At $(2, 0)$; $x = 2$, $y = 0$

$y' = \frac{10(2) - 3(2)^2\sin(0) + 5(0)}{(2^3)\cos(0) - 5(2)}$

$= \frac{20 - 0 + 0}{8 - 10} = \frac{20}{-2} = -10$

The above short method of implicit differentiation will be discussed in the LFR class by UNLEASH

8. Evaluate $f'(x)$ if $f(x) = \sqrt{2x^2 + 3x}$. Hence, find $f'(1)$

SOLUTION

$F(x) = \sqrt{2x^2+3x}$

let $y = F(x)$ and $u = 2x^2+3x$

$\Rightarrow y = \sqrt{u}$; $y = u^{1/2}$

$\frac{dy}{du} = \frac{1}{2}u^{-1/2}$

$\frac{du}{dx} = 4x + 3$

Applying chain rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow (\frac{1}{2}u^{-1/2})(4x+3)$

$= \frac{1}{2} \cdot \frac{1}{u^{1/2}} \cdot 4x+3$

putting the value of $u = 2x^2+3x$

$y' = \frac{4x+3}{2\sqrt{2x^2+3x}}$; $F'(x) = \frac{4x+3}{2\sqrt{2x^2+3x}}$

$F'(1) = \frac{4(1)+3}{2\sqrt{2(1)^2+3(1)}} = \frac{7}{2\sqrt{5}}$

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI

SCHOOL OF PHYSICAL SCIENCES DEPARTMENT OF MATHEMATICS

2017/2018 RAIN SEMESTER EXAMINATIONS

MTH 102: ELEMENTARY MATHEMATICS II DATE: 13/09/2018 TIME: 3 HOURS

INSTRUCTION: ANSWER ALL QUESTION, NOTA DENOTES "NONE OF THE ABOVE", DNE DENOTES "DOES NOT EXIST", AOTA DENOTES "ALL OF THE ABOVE"

1. Evaluate the limit $\lim_{x\to\infty}$ [illegible] (a) 0 (b) ∞ (c) 35 (d) $\frac{7}{5}$ (e) $\frac{3}{5}$.
2. Find the equation of the normal to the curve $y = x^3 - 4x^2 + 5x + 7$ at the point $(2, 1)$. (a) $y = x + 1$ (b) $y = 2x$ (c) $y = -x + 3$ (d) $y = x - 1$ (e) $y = 5x + 7$.
3. A particle moves along a straight line so that the distance travelled after time t secs is given by $S(t) = 3t^3 - 8t^2 + 4t - 5$. Find the acceleration of the body at time $t = 2$ secs. (a) $20ms^{-2}$ (b) $-8ms^{-2}$ (c) $-50ms^{-2}$ (d) $75ms^{-2}$ (e) $-28ms^{-2}$.
4. Given $f(x) = 5x - 2$ and $g(x) = \frac{x+2}{5}$, find respectively $(f \circ g)(x)$ and $(g \circ f)(x)$. (a) $-x, x$ (b) $x, -x$ (c) x, x (d) $x, 2x$ (e) NOTA
5. Obtain the minimum point of the function $y = x^3 - 6x^2 + 9x + 10$ (a) $(3, 10)$ (b) $(3, -17)$ (c) $(3, 5)$ (d) $(1, 2)$ (e) $(1, 14)$.
6. Find the vertical asymptote(s) of $f(x) = \frac{x^2+3x-28}{x^2-32}$. (a) -5 (b) 2 (c) -5, 5 (d) 0 (e) 3.
7. Obtain the domain of the function $f(x) = \frac{2x+7}{\sqrt{x^2-81}}$ (a) $(-\infty, -9) \cup (-9, \infty)$ (b) $(-\infty, -2)$ (c) $(-\infty, 9)$ (d) $(-\infty, -9) \cup (9, \infty)$ (e) $(2, \infty)$
8. Find $\lim_{x\to 0} \frac{\sin mx}{m \sin nx}$. (a) 0 (b) m (c) $\frac{1}{n}$ (d) $\frac{m}{n}$ (e) $\frac{1}{m}$
9. For what value of k is the function $f(x) = \begin{cases} x + 5, & x \le 1 \\ 5 + kx^2, & x > 1 \end{cases}$ continuous at $x = 1$. (a) 0 (b) -1 (c) 1 (d) -2 (e) 2.
10. Find the rate of change of the volume of a spherical balloon if the radius is increasing at the rate of $0.5cms^{-1}$ given that the radius is given by $r = 5cm$. (a) $4\pi cm^3s^{-1}$ (b) $50\pi cm^3s^{-1}$ (c) $90\pi cm^3s^{-1}$ (d) $25\pi cm^3s^{-1}$ (e) $100\pi cm^3s^{-1}$.
11. Evaluate $\int x \ln x\, dx$ (a) $\frac{x^2}{2}\ln x - \frac{x^2}{12} + c$ (b) $\frac{x^2}{2}\ln x - \frac{x^2}{16} + c$ (c) $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ (d) $x^2 \ln x + c$
12. Evaluate $\lim_{x\to\infty}(1 + \frac{1}{4x})^{2x}$ (a) e^2 (b) e (c) $e^{\frac{1}{2}}$ (d) $e^{\frac{1}{4}}$ (e) $e^{\frac{1}{8}}$.
13. Find $\frac{dy}{dx}$ if $y(t) = 1 + \sec 2t$ and $x(t) = \tan 2t$. (a) $\tan 2t$ (b) $\sec 2t$ (c) $\sin 2t$ (d) $-\tan 2t$ (e) $\csc 2t$
14. Find y' and y'' at $(1, 1)$ of $x^3y + xy^3 = 2$. (a) $-1, 0$ (b) $1, 1$ (c) $-1, -1$ (d) $1, 0$ (e) $-1, 1$
15. Evaluate $\lim_{x\to 1} x^{\frac{2}{x-1}}$ (a) $\frac{1}{2}$ (b) e (c) e^2 (d) e^{-2} (e) NOTA
16. Evaluate $\lim_{x\to 2} \frac{4-x^2}{3-\sqrt{x^2+5}}$ (a) 0 (b) 2 (c) 4 (d) 6 (e) 8
17. Determine the domain of $f(x) = \sqrt{\frac{x}{2-x}}$ (a) $\mathbb{R} \setminus \{2\}$ (b) $[0, 2)$ (c) $[0, \infty)$ (d) $(-\infty, 2)$ (e) NOTA
18. Find y' if $y = \sin(x + y)$ (a) $\frac{\cos(x+y)}{1-\cos(x+y)}$ (b) $\frac{-\cos(x+y)}{1+\cos(x+y)}$ (c) $\frac{\cos(x+y)}{1+\cos(x+y)}$ (d) DNE (e) NOTA
19. Given $y = \frac{x}{\sqrt{x-1}}$, find $y''(0)$. (a) 4 (b) -4 (c) 0 (d) -2 (e) DNE
20. Evaluate $\int_0^{5} |6 - 2x| dx$ (a) -8 (b) 8 (c) 10 (d) 12 (e) 0
21. Given $a \neq 0$. Determine $\int \frac{1}{a^2 - x^2} dx$ (a) $\frac{1}{2a}\ln|\frac{a+x}{a-x}| + c$ (b) $\frac{1}{a}\ln|\frac{a+x}{a-x}| + c$ (c) $-\frac{1}{a}\ln|\frac{a+x}{a-x}| + c$ (d) $\frac{2}{a}\ln|\frac{a+x}{a-x}| + c$ (e) NOTA
22. A body moves along a straight line according to the law $S(t) = \frac{t^3}{3}$. Determine the acceleration after 2 secs. (a) $2ms^{-2}$ (b) $3ms^{-2}$ (c) $4ms^{-2}$ (d) $5ms^{-2}$ (e) [illegible]
23. Determine the relative extreme of $g(x) = (x - 2)^{\frac{2}{3}}$ (a) -1 (b) 0 (c) 1 (d) 2 (e) [illegible]
24. Which of the following is true about $f(x) = \frac{|x|}{x}$. (a) $\lim_{x\to 0^+} f(x) = 0$ (b) [illegible] (c) $f(x)$ is discontinuous at $x = 0$ (d) $\lim_{x\to 0^-} f(x) = 1$ (e) NOTA
25. Evaluate $\int_0^{1} x^3(x^2 + 2)^2 dx$ (a) 3 (b) 6 (c) 9 (d) 18 (e) 27
26. Determine $\lim_{x\to 0} \frac{1-\cos x}{x}$ (a) Undefined (b) 1 (c) 0 (d) -1 (e) $\frac{1}{2}$
27. Find the area of the region bounded by $f(x) = |x - 2|$ and x-axis for $-1 \le x \le$ [illegible] (a) 9 (b) 4 (c) 6 (d) 8 (e) NOTA
28. Evaluate $\int_1^{e} \frac{\ln x}{x} dx$ (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{7}{2}$ (e) 2
29. Find $k \in [0, 3]$ that satisfies the Mean Value Theorem for $f(x) = x^3 + 3x^2 + 1$ (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{7}{2}$ (e) 3
30. All except one are odd functions (a) $3x^2 + 4\cos x$ (b) $\tan x$ (c) $\sin x$ (d) $x^3 + 8x$ (e) $|x| \sin x$
31. Obtain $\delta(\epsilon)$ for which $\lim_{x\to 1} 2x^2 + x - 4 = -1$ (a) $\frac{\epsilon}{3}$ (b) $\frac{\epsilon}{4}$ (c) $\frac{\epsilon}{5}$ (d) $\frac{\epsilon}{6}$ (e) [illegible]
32. Find the horizontal asymptotes of $f(x) = \frac{x^2+2x-3}{x^2-1}$ (a) ± 2 (b) -3 (c) 3 (d) 1 (e) [illegible]
33. Solve $\lim_{x\to 1} \frac{x^2-1}{x-1}$ (a) 3 (b) 2 (c) 1 (d) 0 (e) DNE
34. Evaluate $\int \frac{dx}{\sqrt{x^2-9}}$ (a) $\frac{1}{3}\cos^{-1}(\frac{x}{3}) + c$ (b) $\frac{1}{3}\sin^{-1}(\frac{x}{3}) + c$ (c) $\sinh^{-1}(\frac{x}{3}) + c$ (d) $\frac{1}{3}\cosh^{-1}(\frac{x}{3}) + c$ (e) $\cosh^{-1}(\frac{x}{3}) + c$
35. Find the range of $g(x) = (2x^2 - 3x)^{-\frac{1}{2}}$. (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{0\}$ (c) $(0, \infty)$ (d) $[0, \infty)$ (e) [illegible]
36. Find the domain of $f(x) = \frac{\sin(x^2-1)}{x^2+1}$. (a) $\mathbb{R} \setminus \{\pm 1\}$ (b) $(1, \infty)$ (c) $\mathbb{R}$ (d) $\mathbb{R} \setminus \{1\}$ (e) [illegible]
37. Evaluate $\lim_{x\to\infty}(1 + \frac{1}{2x})^x$ (a) 0 (b) $\frac{1}{2}$ (c) e (d) $\sqrt{e}$ (e) NOTA
38. Find the domain of $h(x) = (2x^2 - 3x)^{-\frac{1}{2}}$. (a) $\mathbb{R}$ (b) $(-\infty, 0) \cup (\frac{3}{2}, \infty)$ (c) [illegible] (d) $(0, \infty)$ (e) $(-\infty, 0]$
39. Evaluate $\int x^2 e^{x^3-2} dx$. (a) $\frac{1}{3}e^{x^3-2}$ (b) e^{x^3-2} (c) $3e^{x^3-2}$ (d) $3xe^{x^3-2}$ (e) NOTA
40. Which of these best describes the continuity of $f(x)$ in $[a, b]$? (a) $f(x)$ is continuous at every point of (a, b) (b) $\lim_{x\to a^+} f(x) = f(a)$ (c) $\lim_{x\to b^-} f(x) = f(b)$ (d) $f(x)$ exists (e) AOTA
41. Evaluate $\int_0^{2} \frac{1}{2}\sqrt{2}\,dx$ (a) $\sqrt{2}$ (b) 2 (c) $\sqrt{6}$ (d) $\sqrt{3}$ (e) $\frac{3}{2}$
42. The minimum point of $z = x^3 - 2x^2 + x + 4$ is? (a) $(1, 4)$ (b) $(\frac{1}{3}, \frac{112}{27})$ (c) [illegible] (e) $(3, 27)$
43. Evaluate $\lim_{x\to 0^+} x \cot x$ (a) 0 (b) 1 (c) ∞ (d) -1 (e) $\frac{1}{2}$
44. If $f(x) = k$, then $Dom(f)$ and $Rng(f)$ are (a) $\mathbb{R}, k$ (b) $\mathbb{R}, \{k\}$ (c) $k, \mathbb{R}$ (d) $\{k\}, \mathbb{R}$ (e) [illegible]
45. If $x^2 + y^2 - 2x - 6y + 10 = 0$, find $y'(2, 2)$ (a) 5 (b) -5 (c) 4 (d) 2 (e) -4
46. Evaluate $\int \sinh x \cosh x\, dx$ (a) $\frac{\sinh^2 x}{2} + c$ (b) $\frac{\cosh^2 x}{2} + c$ (c) $\tanh x + c$ (d) [illegible]
47. Evaluate $\lim_{x\to 0^+} \frac{\sin x}{x^2}$ (a) 0 (b) 1 (c) ∞ (d) -1 (e) 2
48. Which of these is false? (a) Not all relations are functions (b) Asymptote is the tangent to a curve at infinity (c) The range of every function is $\mathbb{R}$ (d) A bijective mapping [illegible] linear functions have inverse.
49. If $f(x) = \frac{\sqrt{x^2-9}}{x+5}$, then $f(x)$ is continuous at? (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{-5\}$ (c) $(-\infty, -5) \cup (-5, -3] \cup [3, \infty)$ (d) $(-5, -3] \cup [3, \infty)$ (e) $[-\infty, -3] \cup [3, \infty)$
50. If $f(x) = 2x$ and $g(x) = 4x^2 + 3$, then $g \circ f$ is? (a) $8x^2 + 6$ (b) $8x^2 + 6$ (c) [illegible] (d) $16x^2 + 3$ (e) $16x^2 + 6$
51. If $f(x) = \frac{x+2}{2x+1}$, find $Dom(f)$. (a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \{-\frac{1}{2}\}$ (c) $\mathbb{R} \setminus \{\frac{1}{2}\}$ (d) $\mathbb{R}_+$ (e) [illegible]
52. Evaluate $\lim_{x\to 0} \frac{\sinh x - x^2}{x}$ (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1 (e) -1
53. Evaluate $\int_2^{3} \frac{3x^2-2x+5}{(x-1)(x^2+5)} dx$ (a) $\ln\frac{28}{9}$ (b) $\frac{28}{9}$ (c) $\ln\frac{28}{3}$ (d) $\frac{28}{3}$ (e) NOTA
54. Determine the critical point(s) of the following function $f(x) = 12x^4 - 28x^3 + 24x^2 -$ [illegible] (a) $0, -1, 1$ (b) $0, 1, \frac{1}{2}$ (c) $0, -1, 2$ (d) $1, \frac{1}{2}, \frac{1}{4}$ (e) $0, 1, 2$
55. Evaluate $\lim_{x\to a} \frac{|x-a|}{x-a}$. (a) -1 (b) 1 (c) undefined (d) 0 (e) NOTA
56. Obtain $\frac{dy}{dx}$ if $y = e^{\sin x}$ (a) $e^{\cos x}$ (b) $\cos x e^{\sin x}$ (c) [illegible] (d) $\cos x e^{\cos x}$ (e) NOTA
57. $f^{-1}(x)$ of $f(x) = x^2 + 2$ is (a) $\sqrt{x-2}$ (b) $\sqrt{2-x}$ (c) $\pm\sqrt{x-2}$ (d) $\sqrt{x+2}$ (e) [illegible]
58. The total cost of manufacturing x units of an article is given by $y = \frac{1}{3}x^3 + \frac{8}{x}$. Find the number of units of the article for which the cost of manufacturing is least. (a) 1 (b) 2 (c) 3 (d) 4 (e) 5
59. Find $\frac{d^3y}{dx^3}$ if $y = \cos 3x$ (a) $-27\sin 3x$ (b) $27\sin 3x$ (c) $-27\cos 3x$ (d) $27\cos 3x$
60. Find $dom(f)$ if $f(x) = \ln(7x - x^2 - 10)$ (a) $\mathbb{R}$ (b) $[2, 5]$ (c) $(2, 5]$ (d) $[2, 5)$ (e) $(2, 5)$
61. Evaluate $\int_0^{\pi} \cos(-4x)dx$ (a) 0 (b) 1 (c) 2 (d) 3 (e) 4
62. If a function is continuous at some point, then it is differentiable at that point. (a) Contrary (b) True and False (c) False (d) True (e) Neither True nor False
63. If $f(x) = 3^{x-1}$, Find $f(x + 3) - f(x)$ in terms of $f(x)$. (a) $\frac{13}{2}f(x)$ (b) $\frac{13}{3}f(x)$ (c) $26f(x)$ (d) $13f(x)$ (e) $15f(x)$
64. Solve $\lim_{x\to 1} \frac{x^3-\sqrt{x}}{x-1}$ (a) 0.5 (b) 1 (c) 2.5 (d) 1.5 (e) 3.5
65. Find $\frac{dy}{dx}$ given that $y = \frac{2+3t}{1+t}$ and $x = \frac{2-3t}{1+t}$. (a) $\frac{1}{5}$ (b) $\frac{1}{1+t}$ (c) $-\frac{1}{5}$ (d) $(1 + t)^2$ (e) NOTA
66. Obtain the Maclaurin's series of $f(x) = e^{-x}$. (a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (c) $\sum_{n=0}^{\infty} -\frac{x^n}{n!}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^n}{n!}$ (e) $\sum_{n=0}^{\infty} \frac{x^{-n}}{n!}$
67. Solve $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 x} dx$. (a) $\ln 2$ (b) 2 (c) -2 (d) 0 (e) 1
68. Find $y'(2)$ if $y(x) = \frac{x^2+1}{x^2-1}$ (a) $-\frac{8}{9}$ (b) $\frac{8}{9}$ (c) $\frac{4}{9}$ (d) $-\frac{4}{9}$ (e) $\frac{2}{9}$
69. Pick the odd one (a) $f(x) = |x|$ is an even function (b) $g(x) = e^x$ is a positive function (c) At local minimum $f'(x)$ changes from sign from $-ve$ to $+ve$ [illegible] (d) Every linear function is injective (e) Every function is a relation
70. Solve $\lim_{x\to 0}(\csc x - \frac{1}{x})$. (a) 2 (b) undefined (c) 1 (d) -1 (e) NOTA

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2017/2018 EXAM QUESTION AND SOLUTION

FUNCTION

4. Given $f(x) = 5x - 2$ and $g(x) = \frac{x+2}{5}$, find respectively $(f\circ g)(x)$ and $(g\circ f)(x)$. (a) $-x, x$ (b) $x, -x$ (c) x, x (d) $x, 2x$ (e) NOTA

SOLUTION

$F(x) = 5x - 2$, $g(x) = \frac{x+2}{5}$

$(F\circ g)(x) = F[g(x)]$
$= 5[g(x)] - 2$
$= 5\left(\frac{x+2}{5}\right) - 2$
$= x + 2 - 2 = x$

$(g\circ F)(x) = g[F(x)]$
$= \frac{F(x)+2}{5}$

putting the $F(x) = 5x - 2$
$= \frac{(5x-2)+2}{5}$
$= \frac{5x}{5} = x$ (C)

$(F\circ g)(x)$ and $(g\circ F)(x)$ is x, x

10. If $f(x) = 2x$ and $g(x) = 4x^2 + 3$, then $g\circ f$ is? (a) $8x^2+6$ (b) $8x^2+6$ (c) $16x^4+3$ (d) $16x^2+3$ (e) $16x^3+6$

SOLUTION

$F(x) = 2x$ and $g(x) = 4x^2+3$

$g\circ F = g[F(x)]$
$= 4[F(x)]^2 + 3$ (D)
$= 4(2x)^2 + 3$
$= 4(4x^2) + 3$
$= 16x^2 + 3$

51. $f^{-1}(x)$ of $f(x) = x^2 + 2$ is (a) $\sqrt{x-2}$ (b) $\sqrt{2-x}$ (c) $\pm\sqrt{x-2}$ (d) $\sqrt{x+2}$ (e) DNE

SOLUTION

$F(x) = x^2 + 2$

let $y = F(x)$

$\Rightarrow y = x^2 + 2$

making x the Subject

$y - 2 = x^2$

$x = \pm\sqrt{y-2}$

Replacing x with $F^{-1}(x)$ and y with x; (C)

$\Rightarrow F^{-1}(x) = \pm\sqrt{x-2}$

63. If $f(x) = 3^{x-1}$. Find $f(x+3) - f(x)$ in terms of $f(x)$. (a) $\frac{15}{2}f(x)$ (b) $\frac{13}{2}f(x)$ (c) $26f(x)$ (d) $13f(x)$ (e) $15f(x)$

SOLUTION

$F(x) = 3^{x-1}$

$F(x+3) = 3^{(x+3)-1} = 3^{x+2}$
$= 3^{(x-1)+3}$
$= 3^{x-1}\cdot 3^3$

$F(x+3) - F(x) = 3^3(3^{x-1}) - 3^{x-1}$
$= 3^{x-1}(27-1)$
$= 26(3^{x-1})$ (C)
$= 26F(x)$

5. Pick the odd one (a) $f(x) = |x|$ is an even function (b) $g(x) = e^x$ is a positive function (c) At local minimum $f'(x)$ changes from sign from +ve to -ve as x increases ... is a relation.

SOLUTION

Option A: $F(x) = |x|$ is an even Function; TRUE

Option B: $g(x) = e^x$ is a positive function; TRUE
The range of e^x is $(0, \infty)$

Option C: At local minimum $F'(x)$ changes Sign from +ve to -ve as x increases; FALSE

Note: At local minimum $F'(x)$ changes sign from -ve to +ve as x increases While at local maximum $F'(x)$ changes sign from +ve to -ve

Option D: Every linear function is injective; TRUE

Note: Every linear function is mapped one-to-one. Injective means one-to-one

Option E: Every function is a relation; TRUE

Note: Every function is a relation, but not all relation are function.

Hence the odd option is (C)

DOMAIN AND RANGE

7. Obtain the domain of the function $f(x) = \frac{2x+7}{\sqrt{x^2-81}}$ (a) $(-\infty,-9)\cup(-9,\infty)$ (b) $(-\infty,-2...$ (c) $(-\infty,9)$ (d) $(-\infty,-9)\cup(9,\infty)$ (e) $(2,\infty)$

SOLUTION

$F(x) = \frac{2x+7}{\sqrt{x^2-81}}$

$F(x)$ will be defined, if $x^2 - 81 > 0$

$x^2 - 9^2 > 0$

$(x+9)(x-9) > 0$; $x = -9$ or $x = 9$

Truth table

	$x<-9$	$-9<x<9$	$x>9$
$(x+9)$	$-$	$+$	$+$
$(x-9)$	$-$	$-$	$+$
	$+$	$-$	$+$

The domain of $F(x)$ is given by the Solution set; Dom $F = (x<-9)\cup(x>9)$ which can be denoted on interval notation; Dom $F = (-\infty,-9)\cup(9,\infty)$ (D)

17. Determine the domain of $f(x) = \sqrt{\frac{x}{2-x}}$ (a) $\mathbb{R}\setminus\{2\}$ (b) $[0,2)$ (c) $[0,\infty)$ (d) $(-\infty,2)$ (e) NOTA

SOLUTION

$F(x) = \sqrt{\frac{x}{2-x}}$

The function will be defined, if

$\frac{x}{2-x} \geq 0$; $x(2-x) \geq 0$ and $x \neq 2$

Truth Table

	$x<0$	$0<x<2$	$x>2$
x	$-$	$+$	$+$
$2-x$	$+$	$+$	$-$
	$-$	$+$	$-$

The Domain is given by the Solution set $0 \leq x < 2$ $[0,2)$ (B)

35. Find the range of $g(x) = (2x^2 - 3x)^{-\frac{1}{2}}$ (a) $\mathbb{R}$ (b) $\mathbb{R}\setminus\{0\}$ (c) $(0,\infty)$ (d) $[0,\infty)$ (e) $(-\infty,0]$

SOLUTION

$g(x) = (2x^2 - 3x)^{-1/2}$

$g(x) = \frac{1}{\sqrt{(2x^2-3x)}}$

Range $g(x)$ is the possible outcome of $g(x)$, which must be positive. Since the numerator is a denominator can't output negative

Range $g(x) = (0, \infty)$

Note: Zero is not included.

36. Find the domain of $f(x) = \frac{\sin(x^2-1)}{x^2+1}$. (a) $\mathbb{R}\setminus\{\pm1\}$ (b) $(1,\infty)$ (c) $\mathbb{R}$ (d) $\mathbb{R}\setminus\{1\}$ (e)

SOLUTION

$$f(x) = \frac{\sin(x^2-1)}{x^2+1}$$

Domain of the numerator $[\sin f(x)] = \mathbb{R}$

Domain of the non-zero denominator $= \mathbb{R}$.

Computing the intersection;

$\text{Dom}_f = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$ (C)

38. Find the domain of $h(x) = (2x^2 - 3x)^{-\frac{1}{2}}$. (a) $\mathbb{R}$ (b) $(-\infty,0)\cup(\frac{3}{2},\infty)$ (c) $(-\infty,0)$ (d) $(0,\infty)$ (e) $(-\infty,0]$

SOLUTION

$$h(x) = (2x^2 - 3x)^{-1/2}$$

$$h(x) = \frac{1}{\sqrt{(2x^2-3x)}}$$

h(x) will be defined, if

$2x^2 - 3x > 0$

$x(2x-3) > 0$

Truth Table

	$x<0$	$0<x<3/2$	$x>3/2$
x	$-$	$+$	$+$
$2x-3$	$-$	$-$	$+$
	$+$	$-$	$+$

The Domain is given by the solution set; $\text{Dom}_f = x<0 \cup x>3/2$

using interval notation; (B)

$\text{Dom } f = (-\infty, 0)\cup(3/2, \infty)$

44. If $f(x) = k$, then $Dom(f)$ and $Rng(f)$ are (a) $\mathbb{R}, k$ (b) $\mathbb{R}, \{k\}$ (c) $k, \mathbb{R}$ (d) $\{k\}, \mathbb{R}$ (e) NOTA

SOLUTION

$F(x) = k$, Then $\text{Dom } F = \mathbb{R}$ (A)

$R_f = k$

Note: k is a polynomial function i.e kx^0 and Domain of polynomials is $\mathbb{R}$, while the range of a constant is the same constant.

48. Which of these is false? (a) Not all relations are functions (b) Asymptote is the tangent to a curve at infinity (c) The range of every function is $\mathbb{R}$ (d) A bijection a mapping. (e) All linear functions have inverse.

SOLUTION

The range of every function is not all real numbers. Hence it false. (C)

49. If $f(x) = \frac{\sqrt{x^2-9}}{x+5}$, then $f(x)$ is continuous at? (a) $\mathbb{R}$ (b) $\mathbb{R}\setminus\{-5\}$ (c) $(-\infty,-5)\cup(-5,-3]\cup[3,\infty)$ (d) $(-5,-3]\cup[3,\infty)$ (e) $(-\infty,-5)\cup[3,\infty)$

SOLUTION

$$F(x) = \frac{\sqrt{x^2-9}}{x+5}$$

The function will be continuous where it is defined.

F(x) is defined if $x+5 \neq 0$ and $x^2 - 9 \geq 0$

$\Rightarrow x^2 - 3^2 \geq 0$

$(x+3)(x-3) \geq 0$

Truth Table

	$x<-3$	$-3<x<3$	$x>3$
$x+3$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
	$+$	$-$	$+$

Domain of F(x) is given by the Solution Set: $x \leq -3 \cup x \geq 3$, where $x \neq -5$; Hence on interval notation,

$\text{Dom} F = (-\infty,-5)\cup(-5,-3]\cup[3,\infty)$

51. If $f(x) = \frac{x+3}{2x+1}$, find $Dom(f)$. (a) $\mathbb{R}$ (b) $\mathbb{R}\setminus\{-\frac{1}{2}\}$ (c) $\mathbb{R}\setminus\{\frac{1}{2}\}$ (d) $\mathbb{R}_+$ (e)

SOLUTION

$$F(x) = \frac{x+3}{2x+1}$$

F(x) is a rational function, Hence will be defined; if $2x+1 \neq 0$

$2x+1 \neq 0$

$2x \neq -1$

$x \neq -\frac{1}{2}$ (B)

$\text{Dom } F = \mathbb{R}\setminus\{-1/2\}$

60. Find $dom(f)$ if $f(x) = \ln(7x - x^2 - 10)$ (a) $\mathbb{R}$ (b) $[2,5]$ (c) $(2,5)$ (d) $[2,5)$ (e)

SOLUTION

$F(x) = \ln(7x - x^2 - 10)$

F(x) will be defined if $7x - x^2 - 10 > 0$

$\Rightarrow$ Factorising; $x^2 - 7x + 10 < 0$

$(x-5)(x-2) < 0$

Truth table

	$x<2$	$2<x<5$	$x>5$
$x-5$	$-$	$-$	$+$
$x-2$	$-$	$+$	$+$
	$+$	$-$	$+$

(C)

Domain is given by the solution set $2<x<5$, which is given on interval notation as $(2, 5)$

LIMITS

1. Evaluate the limit $\lim_{x\to\infty} \frac{35x^6+3x^2+8x}{45x^6-2x^2+3x}$ (a) 0 (b) ∞ (c) 35 (d) $\frac{7}{9}$ (e)

SOLUTION

$$\lim_{x\to\infty} \frac{35x^6+3x^2+8x}{45x^6-2x^2+3x}$$

Dividing all through by the highest power x^6

$$\lim_{x\to\infty} \frac{\frac{35x^6}{x^6}+\frac{3x^2}{x^6}+\frac{8x}{x^6}}{\frac{45x^6}{x^6}-\frac{2x^2}{x^6}+\frac{3x}{x^6}}$$

$$\lim_{x\to\infty} \frac{35+\frac{3}{x^4}+\frac{8}{x^5}}{45-\frac{2}{x^4}+\frac{3}{x^5}}$$

Taking the limit:

$$\Rightarrow \frac{35+\frac{3}{\infty^4}+\frac{8}{\infty^5}}{45-\frac{2}{\infty^4}+\frac{3}{\infty^5}}$$

$$= \frac{35+0+0}{45-0+0}$$

$$= \frac{35}{45} = \frac{7}{9} \quad (D)$$

$\frac{k}{\infty^n} = 0$ where k is any number

8. Find $\lim_{x\to0}\frac{\sin mx}{m\sin nx}$. (a) 0 (b) m (c) $\frac{1}{n}$ (d) $\frac{m}{n}$ (e) $\frac{1}{m}$

SOLUTION

$$\lim_{x\to0}\frac{\sin mx}{m\sin nx} = \frac{\sin m(0)}{m\sin n(0)} = \frac{\sin 0}{m\sin 0} = \frac{0}{0}$$

Applying L'Hopital's rule

$$\lim_{x\to0}\frac{\sin mx}{m\sin nx} = \lim_{x\to0}\frac{m\cos mx}{mn\cos nx}$$

Taking the limit again; (C)

$$\lim_{x\to0}\frac{m\cos mx}{mn\cos nx} = \frac{m\cos 0}{mn\cos 0} = \frac{m}{mn} = \frac{1}{n}$$

12. Evaluate $\lim_{x\to\infty}(1+\frac{1}{9x})^{2x}$ (a) e^2 (b) e (c) $e^{\frac{2}{9}}$ (d) $e^{\frac{1}{9}}$ (e) $e^{\frac{9}{2}}$

SOLUTION

$$\lim_{x\to\infty}\left(1+\frac{1}{9x}\right)^{2x}$$

Comparing with the Holger's theorem

$$\lim_{x\to\infty}\left(1+\frac{\boxed{A}}{\boxed{B}x}\right)$$

d) R_+ (e) $(-\infty,0]$

) [2, 5) (e) (2, 5)

$\frac{7}{9}$ (c) $\frac{2}{3}$.

... $\lim_{x\to\infty}(1+\frac{1}{9x})^{2x}$ (a) e^2 (b) e (c) $e^{\frac{1}{9}}$ (d) $e^{\frac{2}{9}}$ (e) $e^{\frac{1}{3}}$

SOLUTION

$\lim_{x\to\infty} (1+\frac{1}{9x})^{2x} = (1+\frac{1}{\infty})^{\infty} = 1^{\infty}$

let $y = (1+\frac{1}{9x})^{2x}$ ---- (*)

$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} (1+\frac{1}{9x})^{2x}$

Taking Natural logarithm of both sides of (*)

$\ln y = \ln(1+\frac{1}{9x})^{2x}$

$\Rightarrow \ln y = 2x \ln(1+\frac{1}{9x})$

now $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{9x})}{\frac{1}{2x}}$

$= \frac{\ln(1+\frac{1}{\infty})}{\frac{1}{\infty}} = \frac{\ln 1}{0} = \frac{0}{0}$

Applying L' Hopital's rule;

$\lim_{x\to\infty} \ln y = \frac{\frac{d}{dx}[\ln(1+\frac{1}{9x})]}{\frac{d}{dx}(\frac{1}{2x})}$

$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \left(\frac{-\frac{1}{9x^2}}{1+\frac{1}{9x}}\right) \Big/ \left(-\frac{1}{2x^2}\right)$

$= \lim_{x\to\infty} \left(\frac{-\frac{1}{9x^2}}{\frac{9x+1}{9x}}\right) \Big/ \left(\frac{-1}{2x^2}\right)$

$= \lim_{x\to\infty} \left(-\frac{1}{9x^2} \times \frac{9x}{9x+1}\right) \cdot \left(-\frac{2x^2}{1}\right)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2x}{9x+1} = \frac{\infty}{\infty}$

Applying L' Hopital's rule

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 2/9 = 2/9$

Taking exponential of the Curves

Hence; $\lim_{x\to\infty} y = \lim_{x\to\infty} (1+\frac{1}{9x})^{2x} = e^{2/9}$

15. Evaluate $\lim_{x\to1} x^{\frac{1}{x-1}}$ (a) $\frac{1}{e}$ (b) e (c) e^2 (d) e^{-2} (e) NOTA

SOLUTION

$\lim_{x\to1} x^{\frac{1}{x-1}} = 1^{\frac{1}{1-1}} = 1^{\infty}$ (Indeterminate form).

let $y = x^{\frac{1}{x-1}}$

Taking Natural logarithm of both side

$\ln y = \frac{1}{x-1}\ln x$

$\lim_{x\to1} \ln y = \lim_{x\to1} \frac{1}{x-1}\ln x$

$\lim_{x\to1} \ln y = \lim_{x\to1} \frac{\ln x}{x-1} = \frac{0}{0}$

Applying L' Hopital's rule

$\lim_{x\to1} \frac{\ln x}{x-1} = \lim_{x\to1} \frac{\frac{1}{x}}{1} = \lim_{x\to1} \frac{1}{x} = 1$

$\lim_{x\to1} \ln y = 1$

Taking exponent of both side

$\lim_{x\to1} e^{\ln y} = \lim_{x\to1} e^1$ (B)

$\lim_{x\to1} y = e$; $\lim_{x\to1} x^{\frac{1}{x-1}} = e$

16. Evaluate $\lim_{x\to2} \frac{4-x^2}{3-\sqrt{x^2+5}}$ (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

SOLUTION

$\lim_{x\to2} \frac{4-x^2}{3-\sqrt{x^2+5}} = \frac{4-(2)^2}{3-\sqrt{(2)^2+5}} = \frac{4-4}{3-3} = \frac{0}{0}$

Applying L' Hopital's rule

$\lim_{x\to2} \frac{4-x^2}{3-\sqrt{x^2+5}} = \lim_{x\to2} \frac{-2x}{\frac{-2x}{2\sqrt{x^2+5}}}$

$= \lim_{x\to2} \frac{2x}{1} \times \frac{2\sqrt{x^2+5}}{2x}$

$= \lim_{x\to2} 2\sqrt{x^2+5} = 2\sqrt{2^2+5}$

$= 2\sqrt{9}$

$= 2(3) = 6$ (D)

26. Determine $\lim_{x\to0} \frac{1-\cos x}{x}$ (a) Undefined (b) 1 (c) 0 (d) -1 (e) $\frac{1}{2}$

SOLUTION

$\lim_{x\to0} \frac{1-\cos x}{x} = \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$

Applying L' Hopital's theorem

$\boxed{\lim_{x\to0} \frac{1-\cos x}{x} = \lim_{x\to0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}}$

$\lim_{x\to0} \frac{1-\cos x}{x} = \lim_{x\to0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}$

$= \lim_{x\to0} \frac{\sin x}{1} = \frac{\sin 0}{1} = 0$ (C)

31. Obtain $\delta(\varepsilon)$ for which $\lim_{x\to1} 2x^2+x-4 = -1$ (a) $\frac{\varepsilon}{3}$ (b) $\frac{\varepsilon}{4}$ (c) $\frac{\varepsilon}{5}$ (d) $\frac{\varepsilon}{6}$ (e) $\frac{\varepsilon}{7}$

SOLUTION

$\lim_{x\to1} 2x^2+x-4 = -1$

using Epsilon - delta definition of limit.

$|f(x)-L| < \varepsilon$ whenever $0<|x-x_0|<\delta$

Where $f(x) = 2x^2+x-4$, $x_0=1$, $L=-1$

$\Rightarrow |(2x^2+x-4)-(-1)| < \varepsilon$ whenever $0<|x-1|<\delta$

$|2x^2+x-3| < \varepsilon$ " " "

Transforming in terms of δ

$|2(x-1)^2+4x-2+(x-1)+1-3| < \varepsilon$ " "

$|2(x-1)^2+4x+(x-1)-4| < \varepsilon$ " "

Transforming in term of δ again.

$|2(x-1)^2+4(x-1)+4+(x-1)-4| < \varepsilon$

$|2(x-1)^2+4(x-1)+(x-1)| < \varepsilon$

$|2\delta^2+4\delta+\delta| < \varepsilon$

but $\delta^2 < \delta$ (E)

$\Rightarrow |2\delta+4\delta+\delta| < \varepsilon$

$|7\delta| < \varepsilon$; $\delta \le \frac{\varepsilon}{7}$

33. Solve $\lim_{x\to1} \frac{x^3-1}{x-1}$ (a) 3 (b) 2 (c) 1 (d) 0 (e) DNE

SOLUTION

$\lim_{x\to1} \frac{x^3-1}{x-1} = \frac{1^3-1}{1-1} = \frac{0}{0}$ (Indeterminate form)

Applying L' Hopital's theorem

$\lim_{x\to1} \frac{x^3-1}{x-1} = \lim_{x\to1} \frac{\frac{d}{dx}(x^3-1)}{\frac{d}{dx}(x-1)}$ (A)

$= \lim_{x\to1} \frac{3x^2}{1} = 3(1)^2 = 3$

37. Evaluate $\lim_{x\to\infty}(1+\frac{1}{2x})^x$ (a) 0 (b) $\frac{1}{e}$ (c) e (d) $\sqrt{e}$ (e) NOTA

SOLUTION

$\lim_{x\to\infty} (1+\frac{1}{2x})^x$; A = 1, B = 2

Applying Holgee's theorem

$\lim_{x\to\infty} (1+\frac{A}{Bx})^x = e^{A/B}$

Note: This theorem is only for multiple choice exam

(D)

$\lim_{x\to\infty} (1+\frac{1}{2x})^x = e^{1/2} = \sqrt{e}$

43. Evaluate $\lim_{x\to 0^+} x\cot x$ (a) 0 (b) 1 (c) ∞ (d) -1 (e) 2

SOLUTION

$\lim_{x\to 0^+} x\cot x = \lim_{x\to 0^+} \frac{x}{\tan x}$

Note: $\cot x = \frac{1}{\tan x}$

$\lim_{x\to 0^+} \frac{x}{\tan x} = \frac{0}{0}$

Applying L' Hopital's rule

$= \lim_{x\to 0^+} \frac{x}{\tan x} = \lim_{x\to 0^+} \frac{1}{\sec^2 x} = \lim_{x\to 0^+} \cos^2 x$

$= \cos^2(0) = (1)^2 = 1$ (B)

Note: $\frac{1}{\sec^2 x} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x$

47. Evaluate $\lim_{x\to 0^+} \frac{\sin x}{x^2}$ (a) 0 (b) 1 (c) ∞ (d) -1 (e) 2

SOLUTION

$\lim_{x\to 0^+} \frac{\sin x}{x^2} = \frac{\sin 0}{0^2} = \frac{0}{0}$

Applying L' Hopital's (C)

$\lim_{x\to 0^+} \frac{\cos x}{2x} = \frac{\cos 0}{2(0)} = \frac{1}{0} = \infty$

52. Evaluate $\lim_{x\to 0} \frac{\cosh x - e^x}{x}$ (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1 (e) -1

SOLUTION

$\lim_{x\to 0} \frac{\cosh x - e^x}{x} = \frac{\cosh(0) - e^0}{0} = \frac{1-1}{0} = \frac{0}{0}$

Applying L' Hopital's rule

$\lim_{x\to 0} \frac{\cosh x - e^x}{1} = \frac{\cosh(0) - e^0}{1}$

$= \frac{1-1}{1} = \frac{0}{1} = 0$ (C)

55. Evaluate $\lim_{x\to a} \frac{|x-a|}{x-a}$. (a) -1 (b) 1 (c) undefined (d) 0 (e) NOTA

SOLUTION

$\lim_{x\to a} \frac{|x-a|}{x-a}$

Spliting the absolute value into piecewise definition; $|x-a| = \begin{cases} (x-a); & x \geq a \\ -(x-a); & x < a \end{cases}$

$\lim_{x\to a^-} -(x-a) = -2a$

$\lim_{x\to a^+} (x-a) = 0$

Since $\lim_{x\to a^-} F(x) \neq \lim_{x\to a^+} F(x)$ (C)

The limit as $x\to a$ of $F(x)$ is undefined and does not exist (DNE)

64. Solve $\lim_{x\to 1} \frac{x^2-\sqrt{x}}{x-1}$ (a) 0.5 (b) 1 (c) 2.5 (d) 1.5 (e) 3.5

SOLUTION

$\lim_{x\to 1} \frac{x^2-\sqrt{x}}{x-1} = \frac{1^2-\sqrt{1}}{1-1} = \frac{0}{0}$

Applying L' Hopital's rule;

$\lim_{x\to 1} \frac{x^2-\sqrt{x}}{x-1} = \lim_{x\to 1} \frac{2x - \frac{1}{2\sqrt{x}}}{1}$

$= \lim_{x\to 1} 2x - \frac{1}{2\sqrt{x}} = 2(1) - \frac{1}{2(1)}$

$= 2 - \frac{1}{2} = 3/2 = 1.5$ (D)

70. Solve $\lim_{x\to 0}(\csc x - \frac{1}{x})$. (a) -2 (b) undefined (c) 1 (d) -1 (e) NOTA

SOLUTION

$\lim_{x\to 0} (\operatorname{cosec} x - \frac{1}{x})$

$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$

$\Rightarrow \lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \frac{0-\sin 0}{(0)\sin 0} = \frac{0}{0}$

Applying L' Hopital's rule

$\Rightarrow \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x} = \frac{1-\cos 0}{0\cos 0 + \sin 0} = \frac{0}{0}$

Applying L' Hopital's rule again;

$\lim_{x\to 0} \frac{-(-\sin x)}{-x\sin x + \cos x + \cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x}$

$= \frac{\sin 0}{2\cos 0 - 0\sin 0} = \frac{0}{2(1)-0} = \frac{0}{2} = 0$ (E)

CONTINUITY

9. For what value of k is the function $f(x) = \begin{cases} x+5, & x \leq 1 \\ 5+kx^2, & x > 1 \end{cases}$ continuous at x = 1

(a) 0 (b) -1 (c) 1 (d) -2 (e) 2.

SOLUTION

$F(x) = \begin{cases} x+5 & ; x \leq 1 \\ 5+kx^2 & ; x > 1 \end{cases}$

$F(1) = 1+5 = 6$

$\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} 5+kx^2 = 5+k(1)^2$

$\lim_{x\to 1^+} F(x) = 5+k$

$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} x+5 = 1+5 = 6$

Since $F(x)$ is continous at $x=1$

$\Rightarrow \lim_{x\to 1} F(x)$ exists, hence $5+k = 6$

Solving $5+k=6$; $k = 6-5$, $k = 1$ (C)

34. Which of the following is true about $f(x) = \frac{|x|}{x}$. (a) $\lim_{x\to 0^+} f(x) = 0$ (b) $f(x)$ is cont... (c) $f(x)$ is discontinuous at $x = 0$ (d) $\lim_{x\to 0^-} f(x) = 1$ (e) NOTA

SOLUTION

$F(x)$ is discontinous at $x=0$ because the limit does not exist.

i.e $\lim_{x\to 0^-} F(x) = -1$ and $\lim_{x\to 0^+} F(x) = 1$

Since $\lim_{x\to 0^-} F(x) \neq \lim_{x\to 0^+} F(x)$

The limit does not exist.

40. Which of these best describes the continuity of $f(x)$ in $[a,b]$? (a) $f(x)$ is continuous at each point of (a,b) (b) $\lim_{x\to a} f(x) = f(a)$ (c) $\lim_{x\to b} f(x) = f(b)$ (d) $f(x)$ exists (e) NOTA

SOLUTION

$F(x)$ is continous at each point point (a,b) best describe the continuity of $F(x)$ (A)

ASYMPTOTES

6. Find the vertical asymptote(s) of $f(x) = \frac{x^2+2x-35}{x^2-25}$. (a) -5 (b) 2 (c) -5,5 (d) 0 (e) 3

SOLUTION

$F(x) = \frac{x^2+2x-35}{x^2-25}$

Vertical Asymptotes is given by the value of x for which the denominator of a rational function is equal to zero

$x^2 - 25 = 0$

$x^2 - 5^2 = 0$

$(x+5)(x-5) = 0$ (C)

$x = -5$ or $x = 5$

The vertical Asymptotes is at $-5, 5$

32. Find the horizontal asymptotes of $f(x) = \frac{x^2+2x-3}{x^2-4}$ (a)±2 (b) −3 (c) 2 (d) 1 (e) 3

SOLUTION

Horizontal Asymptote is given by $\lim_{x\to\infty} F(x) = H\cdot A$

$\Rightarrow \lim_{x\to\infty} \frac{x^2+2x-3}{x^2-4} = \frac{\infty^2+2(\infty)-3}{\infty^2-4} = \frac{\infty}{\infty}$

Dividing through by x^2

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} = \lim_{x\to\infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{4}{x^2}}$$

Taking the limit

$$\Rightarrow \lim_{x\to\infty} \frac{x^2+3x-2}{x^2-4} = \frac{1 + \frac{3}{\infty} - \frac{2}{\infty^2}}{1 - \frac{4}{\infty^2}}$$

$$= \frac{1+0-0}{1-0} = \frac{1}{1} = 1 \quad \text{(C)}$$

Hence the Horizontal Asymptote is $x = 1$

DIFFERENTIATION

Find y' if $y = \sin(x+y)$ (a) $\frac{\cos(x+y)}{1-\cos(x+y)}$ (b) $\frac{-\cos(x+y)}{1+\cos(x+y)}$ (c) $\frac{\sin(x+y)}{1-\cos(x+y)}$ (d) DNE (e) NOTA

SOLUTION

$y = \sin(x+y)$

$y' = (1+y')\cos(x+y)$

$y' = \cos(x+y) + y'\cos(x+y)$

$y' - y'\cos(x+y) = \cos(x+y)$

$y'[1-\cos(x+y)] = \cos(x+y)$

$y' = \frac{\cos(x+y)}{1-\cos(x+y)}$ (A)

Obtain $\frac{dy}{dx}$ if $y = e^{\sin x}$ (a) $e^{\cos x}$ (b) $\cos x e^{\sin x}$ (c) $\frac{\cos x}{e^{\sin x}}$ (d) $\cos x e^{\cos x}$ (e) NOTA

SOLUTION

$y = e^{\sin x}$

Let $u = \sin x$; $\frac{du}{dx} = \cos x$

$\Rightarrow y = e^u$; $\frac{dy}{du} = e^u$

using Chain rule;

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = e^u \cos x$

putting the value of u

$\Rightarrow \frac{dy}{dx} = e^{\sin x}\cos x$ (B)

$= \cos x\, e^{\sin x}$

59. Find $\frac{d^3y}{dx^3}$ if $y = \cos 3x$ (a) $-27\sin 3x$ (b) $27\sin 3x$ (c) $-27\cos 3x$ (d) $27\cos 3x$

SOLUTION

$y = \cos 3x$

$\frac{dy}{dx} = 3[-\sin(3x)] = -3\sin 3x$

$\frac{d^2y}{dx^2} = (-3)(3)\cos(3x) = -9\cos 3x$

(B)

$\frac{d^3y}{dx^3} = (-9)(3)(-\sin 3x) = 27\sin 3x$

62. If a function is continuous at some point, then it is differentiable at that point. (a) Contrary (b) True and False (c) False (d) True (e) Neither True nor False

SOLUTION

Continuity is not sufficient condition for a function to be differentiable. But if a function is differentiable at a point, then it is continous at that point. Hence it is False (C)

68. Find $y'(2)$ if $y(x) = \frac{x^2+1}{x^2-1}$ (a) $-\frac{2}{3}$ (b) $\frac{4}{9}$ (c) $\frac{1}{3}$ (d) $-\frac{8}{9}$ (e) $\frac{2}{3}$

SOLUTION

$y(x) = \frac{x^2+1}{x^2-1}$ ---- u / ---- v ; By Quotient rule

$y'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$

$y'(2) = \frac{(2^2-1)(4) - (2^2+1)(4)}{(2^2-1)^2}$

$= \frac{(3)(4) - 5(4)}{3^2}$

$= \frac{12-20}{9} = -\frac{8}{9}$

PARAMETRIC EQUATION

53. Find $\frac{dy}{dx}$ if $y(t) = 1 + \sec 2t$ and $x(t) = \tan 2t$ (a) $\tan 2t$ (b) $\sec 2t$ (c) $\sin 2t$ (d) $-\tan 2t$ (e) $\cos 2t$

SOLUTION

$y(t) = 1 + \sec 2t$ and $x(t) = \tan 2t$

$\frac{dy}{dt} = 2\sec 2t \tan 2t$

$\frac{dx}{dt} = 2\sec^2 2t$

$\frac{dt}{dx} = \frac{1}{2\sec^2 2t}$

using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2\sec 2t \tan 2t}{2\sec^2 2t}$

$= \frac{\tan 2t}{\sec 2t} = \frac{\frac{\sin 2t}{\cos 2t}}{\frac{1}{\cos 2t}}$

$= \frac{\sin 2t}{\cos 2t} \times \frac{\cos 2t}{1} = \sin 2t$

55. Find $\frac{dy}{dx}$ given that $y = \frac{3+2t}{1+t}$ and $x = \frac{2-3t}{1+t}$. (a) $\frac{1}{5}$ (b) $\frac{1}{1+t}$ (c) $-\frac{1}{5}$ (d) $(1+t)^2$ (e) NOTA

SOLUTION

$y = \frac{3+2t}{1+t}$

$x = \frac{2-3t}{1+t}$

$\frac{dy}{dt} = \frac{(1+t)(2) - (3+2t)(1)}{(1+t)^2}$

$= \frac{2+2t-3-2t}{(1+t)^2} = -\frac{1}{(1+t)^2}$

$\frac{dx}{dt} = \frac{(1+t)(-3) - (2-3t)(1)}{(1+t)^2}$

$= \frac{-3-3t-2+3t}{(1+t)^2} = \frac{-5}{(1+t)^2}$

But $\frac{dt}{dx} = -\frac{(1+t)^2}{5}$

using Chain rule; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{(1+t)^2} \times \frac{(1+t)^2}{-5} = \frac{1}{5}$ (A)

IMPLICIT DIFFERENTIATION

14. Find y' and y'' at $(1,1)$ of $x^3y + xy^3 = 2$. (a) $-1,0$ (b) $1,1$ (c) $-1,-1$ (d) $1,0$ (e) $-1,1$

SOLUTION

$x^3y + xy^3 = 2$ at $(1,1)$

$\frac{dy}{dx} = -\left(\frac{3x^2y + y^3}{x^3 + 3xy^2}\right)$

at $(1,1)$; $x=1$, $y=1$

$\left.\frac{dy}{dx}\right|_{(1,1)} = -\left[\frac{3(1)^2(1) + (1)^3}{(1)^3 + 3(1)(1)^2}\right]$

$= -\left(\frac{3+1}{1+3}\right) = -1$

SOLUTION QUESTION 14 CONTD

$\frac{dy}{dx} = \frac{3x^2y + y^3}{-(x^3 + 3xy^2)} \cdots u \cdots v$

Applying Quotient rule

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{v\,du - u\,dv}{v^2}$

$= \frac{\left[-(x^3+3xy^2)(3y^2\frac{dy}{dx} + 3x^2\frac{dy}{dx} + 6xy)\right] - \left[-(y^3+3x^2y)(3x^2+6xy\frac{dy}{dx}+3y^2)\right]}{(x^3+3xy^2)^2}$

$\frac{d^2y}{dx^2}$ at (1,1) where $\frac{dy}{dx} = -1$, we have

$\frac{d^2y}{dx^2} = \frac{-(1+3)(-3-3+6) - (-1+3)(3-6+3)}{(1+3)^2}$

$= \frac{0-0}{4^2} = \frac{0}{16} = 0$

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(1,1) = -1, 0$ (A)

45. If $x^2 + y^2 - 2x - 6y + 10 = 0$, find $y'(3,2)$ (a) 5 (b) -5 (c) 4 (d) 2 (e) -4

SOLUTION

$x^2 + y^2 - 2x - 6y + 10 = 0$

Differentiating implicitly

$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} - 6\frac{dy}{dx} = 2 - 2x$

$\frac{dy}{dx}(2y - 6) = 2 - 2x$

$\frac{dy}{dx} = \frac{2-2x}{2y-6}$

at (3, 2); $x = 3$, $y = 2$

$\Rightarrow \frac{dy}{dx} = \frac{2-2(3)}{2(2)-6}$ (D)

$= \frac{2-6}{4-6} = \frac{-4}{-2} = 2$

APPLICATION OF DIFFERENTIATION (AOD)

ON TANGENT AND NORMAL

2. Find the equation of the normal to the curve $y = x^3 - 4x^2 + 5x + 7$ at the point (2, 1). (a) $y = x + 1$ (b) $y = 2x$ (c) $y = -x + 3$ (d) $y = x - 1$ (e) $y = 5x + 7$.

SOLUTION

$y = x^3 - 4x^2 + 5x + 7$

$\frac{dy}{dx} = 3x^2 - 8x + 5$

at (2,1); $x = 2$, $y = 1$

$M = \frac{dy}{dx} = 3(2)^2 - 8(2) + 5$

$= 3(4) - 16 + 5$

$= 1$

using equation of Normal given by

$\frac{y - y_0}{x - x_0} = -\frac{1}{m}$; where $y_0 = 1$, $x_0 = 2$, $M = 1$

$\Rightarrow \frac{y-1}{x-2} = -\frac{1}{1}$ (cross multiplying)

$y - 1 = -(x-2)$

$y - 1 = -x + 2$

$y = -x + 3$ (C)

AOD ON KINEMATICS

3. A particle moves along a straight line so that the distance travelled after time t secs is given by $S(t) = 3t^3 - 8t^2 + 4t - 5$. Find the acceleration of the body at time $t = 2$ secs. (a) $20ms^{-2}$ (b) $-8ms^{-2}$ (c) $-50ms^{-2}$ (d) $75ms^{-2}$ (e) $-28ms^{-2}$.

SOLUTION

$S(t) = 3t^3 - 8t^2 + 4t - 5$

$V = S'(t) = 9t^2 - 16t + 4$

$a = S''(t) = 18t - 16$

at $t = 2$secs (A)

$a = 18(2) - 16$

$= 36 - 16 = 20m/s^2$

22. A body moves along a straight line according to the law $S(t) = \frac{t^3 - 4t}{2}$. Determine its acceleration after 2 secs. (a) $2ms^{-2}$ (b) $3ms^{-2}$ (c) $4ms^{-2}$ (d) $5ms^{-2}$ (e) $6ms^{-2}$

SOLUTION

$S(t) = \frac{t^3 - 4t}{2}$

$V = S'(t) = \frac{3t^2}{2} - \frac{4}{2}$

$a = S''(t) = 3t$ (E)

at $t = 2$secs

$a = 3(2) = 6ms^{-2}$

AOD ON EXTREMEM

5. Obtain the minimum point of the function $y = x^3 - 6x^2 + 9x + 10$ (a) (3, 10) (b) (3, -17) (c) (3, 5) (d) (1, 3) (e) (1, 14).

SOLUTION

$y = x^3 - 6x^2 + 9x + 10$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

at turning point; $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

Dividing through by 3

$\Rightarrow x^2 - 4x + 3 = 0$; Factorising

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

using Second derivative test

$\frac{d^2y}{dx^2} = 6x - 12$

at $x = 1$; $\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0$

Hence the point $x = 1$ is minimum

when $x = 1$; $y = 1^3 - 6(1)^2 + 9(1) + 10$

$= 1 - 6 + 9 + 10$

$= 14$

The minimum point of the function is at (1, 14) (E)

42. The minimum point of $z = x^3 - 2x^2 + x + 4$ is? (a)(1,4) (b)$(\frac{1}{3}, \frac{112}{27})$ (c)(1,8) (d)(0,2) (e)(3,27)

SOLUTION

$z = x^3 - 2x^2 + x + 4$

$\frac{dz}{dx} = 3x^2 - 4x + 1$

at turning points $\frac{dz}{dx} = 0$

$\Rightarrow 3x^2 - 4x + 1 = 0$

factorising;

$(3x-1)(x-1) = 0$

$3x - 1 = 0$ or $x - 1 = 0$

$x = \frac{1}{3}$ or $x = 1$

Using Second derivative test

$\frac{d^2z}{dx^2} = 6x - 4$

at $x = \frac{1}{3}$

$\frac{d^2z}{dx^2} = 6(\frac{1}{3}) - 4 = -2 < 0$ (negative)

Hence, The function is minimum at $x = \frac{1}{3}$

when $x = \frac{1}{3}$; $z = (\frac{1}{3})^3 - 2(\frac{1}{3})^2 + (\frac{1}{3}) + 4$

$= \frac{112}{27}$

The minimum point of the function is $(\frac{1}{3}, \frac{112}{27})$ (B)

$\frac{t^2-11}{3}$. Determine its
(d) $5ms^{-2}$ (e) $6ms^{-2}$

EM

10 (a) (3, 10) (b) (3, −17)

:0

$\frac{112}{27}$) (c) (1,8) (d)(2,12

(a)

5

14

Determine the critical point(s) of the following function $f(x) = 12x^4 - 28x^3 + 21x^2 - 6x$
(a) $0, -1, 1$ (b) $0, 1, \frac{1}{2}$ (c) $0, -1, 2$ (d) $1, \frac{1}{2}, \frac{1}{4}$ (e) $0, 1, 2$

SOLUTION

$f(x) = 12x^4 - 28x^3 + 21x^2 - 6x$

$f'(x) = 48x^3 - 84x^2 + 42x - 6$

at Critical point; $F'(x) = 0$

$48x^3 - 84x^2 + 42x - 6 = 0$

Solving the Cubic polynomial (D)

$x = 1$ or $x = \frac{1}{4}$ or $x = \frac{1}{2}$

The total cost of manufacturing x units of an article is given by $y = \frac{5}{4}x^2 + \frac{20}{x}$. Find the number of units of the article for which the cost of manufacturing is least.
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

SOLUTION

$y = \frac{5}{4}x^2 + \frac{20}{x}$

$y = \frac{5}{4}x^2 + 20x^{-1}$

$\frac{dy}{dx} = 2(\frac{5}{4}x) - 1(20x^{-2})$

$\frac{dy}{dx} = \frac{5}{2}x - \frac{20}{x^2}$

at minimum point; the manufacturing cost will be least.

when; $\frac{dy}{dx} = 0$

$\Rightarrow \frac{5x}{2} - \frac{20}{x^2} = 0$

multiplying both sides by $2x^2$

$\Rightarrow 5x^3 - 40 = 0$; $x^3 - 8 = 0$ (B)

$x^3 = 8$; Hence $x = \sqrt[3]{8}$; $x = 2$

AOD ON RELATED RATES/ RATE OF CHANGE

Find the rate of change of the volume of a spherical balloon if the radius is increasing at the rate of $0.5cms^{-1}$ given that the radius is given by $r = 5$cm. (a) $4\pi cm^3s^{-1}$ (b) $50\pi cm^3s^{-1}$ (c) $90\pi cm^3s^{-1}$ (d) $25\pi cm^3s^{-1}$ (e) $100\pi cm^3s^{-1}$.

SOLUTION

Volume of a Sphere is given by $V = \frac{4}{3}\pi r^3$ ----(*)

$\frac{dr}{dt} = 0.5$ where $r = 5$cm

using chain,

$\frac{dv}{dt} = \frac{dr}{dt} \times \frac{dv}{dr}$

From equation (*) $\frac{dv}{dr} = 4\pi r^2$

$\frac{dv}{dt} = 4\pi(5)^2 \times 0.5$

$= 50\pi cm^3 s^{-1}$

AOD ON MEAN VALUE THEOREM

29. Find $k \in [0, 3]$ that satisfies the Mean Value Theorem for $f(x) = x^3 + 2x^2 + 1$.
(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{5}{3}$ (d) $\frac{7}{3}$ (e) 3

SOLUTION

$F(x) = x^3 + 2x^2 + 1$ at $[0, 3]$

$a = 0$, $b = 3$

$F(a) = F(0) = 0^3 + 2(0)^2 + 1 = 1$

$F(b) = F(3) = 3^3 + 2(3)^2 + 1 = 46$

$F'(x) = 3x^2 + 4x$

$F'(k) = 3k^2 + 4k$

using mean value theorem

$F'(k) = \frac{F(b) - F(a)}{b - a}$

$\Rightarrow 3k^2 + 4k = \frac{46 - 1}{3 - 0}$

$3k^2 + 4k = 15$

$3k^2 + 4k - 15 = 0$

$(3k - 5)(k + 3) = 0$

$3k - 5 = 0$ or $k + 3 = 0$

$k = 5/3$ or $k = -3$ (C)

$k \in [0, 3]$ is $5/3$

26. Obtain the Maclaurin's series of $f(x) = e^{-x}$. (a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ (c) $\sum_{n=0}^{\infty} -\frac{x^n}{n!}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n!}$ (e) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

SOLUTION

$F(x) = e^{-x}$

maclaurin Series of e^x is given by;

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(-1 \cdot x)^n}{n!}$

$= \sum_{n=0}^{\infty} (-1)^n x^n$ (B)

INTEGRATION

11. Evaluate $\int x \ln x\, dx$ (a) $\frac{x^2}{2}\ln x - \frac{x^2}{2} + c$ (b) $\frac{x^2}{3}\ln x - \frac{x^2}{12} + c$ (c) $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ (d) $x^2 \ln x + c$

SOLUTION

$\int x \ln x\, dx$

using integration by parts

let $u = \ln x$, $dv = x$, $du = \frac{1}{x}$, $v = \frac{x^2}{2}$

where; $\int u\, dv = uv - \int v\, du$

$\Rightarrow \int x \ln x\, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x$ (C)

$= \frac{x^2}{2}\ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

20. Evaluate $\int_2^6 |6 - 2x|\, dx$ (a) -8 (b) 8 (c) 10 (d) 12 (e) 0

SOLUTION

$\int_2^6 |6 - 2x|\, dx$

$= \int_2^3 (6 - 2x)\, dx + \int_3^6 -(6 - 2x)\, dx$

$= 9 + 1$

$= 10$ (C)

21. Given $a \neq 0$, Determine $\int \frac{dx}{x^2 - a^2} dx$ (a) $\frac{1}{2a}\ln|\frac{x-a}{x+a}| + c$ (b) $\frac{1}{a}\ln|\frac{x-a}{x+a}| + c$ (c) $-\frac{1}{a}\ln|\frac{x-a}{x+a}| + c$ (d) $\frac{2}{a}\ln|\frac{x-a}{x+a}| + c$ (e) NOTA

SOLUTION

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$ (A)

It's a Standard integral

28. Evaluate $\int_e^{e^2} \frac{\ln x}{x} dx$ (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{7}{2}$ (e) 2

SOLUTION

$\int_e^{e^2} \frac{\ln x}{x}\, dx = \int_e^{e^2} \frac{1}{x} \cdot \ln x\, dx$

let $u = \ln x$; $\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

when $x = e$; $u = \ln e = 1$

$x = e^2$; $u = \ln e^2 = 2$

$$\int_e^{e^2} \frac{\ln x}{x}\,dx = \int_1^2 \frac{1}{x} \cdot u \cdot x\,du$$

$$= \int_1^2 u\,du$$

$$= \frac{u^2}{2}\Big|_1^2$$

$$= \frac{2^2}{2} - \frac{1^2}{2} = 2 - \tfrac{1}{2}$$

$$= \tfrac{3}{2} \quad (B)$$

25. Evaluate $\int_0^1 x^2(x^3+2)^2dx$ (a) 3 (b) 6 (c) 9 (d) 18 (e) 27

SOLUTION

$$\int_0^1 x^2(x^3+2)^2\,dx$$

let $u = x^3+2$; $\frac{du}{dx} = 3x^2$; $dx = \frac{du}{3x^2}$

when $x=0$; $u = 0^3+2 = 2$

$x = 1$; $u = 1^3+2 = 3$

Substituting into the given integral

$$\int_0^1 x^2(x^3+2)^2\,dx = \int_2^3 x^2 \cdot u^2 \cdot \frac{du}{3x^2}$$

$$= \frac{1}{3}\int_2^3 u^2\,du = \left[\frac{1}{3}\cdot\frac{u^3}{3}\right]_2^3$$

$$= \left(\frac{1}{3}\cdot\frac{3^3}{3}\right) - \left(\frac{1}{3}\cdot\frac{2^3}{3}\right)$$

$$= 3 - \tfrac{8}{9}$$

$$= \frac{19}{9}$$

34. Evaluate $\int \frac{dx}{\sqrt{x^2-9}}$ (a) $\frac{1}{3}\cos^{-1}(\frac{x}{3})+c$ (b) $\frac{1}{3}\sin^{-1}(\frac{x}{3})+c$ (c) $\sinh^{-1}(\frac{x}{3})+c$ (d) $\frac{1}{3}\cosh^{-1}(\frac{x}{3})+c$ (e) $\cosh^{-1}(\frac{x}{3})+c$

SOLUTION

$$\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{dx}{\sqrt{x^2-3^2}}$$

Comparing with the standard integral

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + c \quad \text{where } x > a$$

$$\Rightarrow \int \frac{dx}{\sqrt{x^2-3^2}} = \cosh^{-1}(x/3) + c \quad (E)$$

39. Evaluate $\int x^2e^{x^3-3}dx$. (a) $\frac{1}{3}e^{x^3-3}$ (b) e^{x^3-3} (c) $3e^{x^3-3}$ (d) $3xe^{x^3-3}$ (e) NOTA

SOLUTION

$$\int x^2 e^{x^3-3}\,dx$$

let $u = x^3-3$; $\frac{du}{dx} = 3x^2$, $dx = \frac{du}{3x^2}$

$$\Rightarrow \int x^2 \cdot e^u \cdot \frac{du}{3x^2} = \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + c$$

Putting $u = x^3-3$, we have

$$\int x^2 e^{x^3-3}\,dx = \frac{1}{3}e^{x^3-3} + c \quad (A)$$

41. Evaluate $\int_0^2 \frac{3}{2}\sqrt{x}\,dx$ (a) $\sqrt{2}$ (b) 2 (c) $\sqrt{8}$ (d) $\sqrt{3}$ (e) $\frac{3}{2}$

SOLUTION

$$\int_0^2 \frac{3}{2}\sqrt{x}\,dx = \frac{3}{2}\int_0^2 x^{1/2}\,dx$$

$$= \left[\frac{3}{2}\left(\frac{x^{3/2}}{3/2}\right)\right]_0^2 = \left[\frac{3}{2}\left(\frac{2x^{3/2}}{3}\right)\right]_0^2$$

$$= \left[x^{3/2}\right]_0^2 = 2^{3/2} - 0^{3/2}$$

$$= (\sqrt{2})^3$$

$$= (\sqrt{2})^2 \cdot \sqrt{2}$$

$$= 2\sqrt{2}$$

46. Evaluate $\int \sinh x \cosh x\,dx$ (a) $\frac{\sinh^2 x}{2}+c$ (b) $\frac{\cosh^2 x}{2}+c$ (c) $\tanh x+c$ (d) $\sinh x \tanh x+c$

SOLUTION

$$\int \sinh x \cosh x\,dx$$

Let $u = \sinh x$; $\frac{du}{dx} = \cosh x$

$dx = \frac{du}{\cosh x}$; Substituting:

$$= \int u \cdot \cosh x \cdot \frac{du}{\cosh x}$$

$$= \int u\,du$$

$$= \frac{u^2}{2} + c \quad (A)$$

putting the value of u;

$$\int \sinh x \cdot \cosh x\,dx = \frac{\sinh^2 x}{2} + c$$

53. Evaluate $\int_2^3 \frac{3x^2-2x+5}{(x-1)(x^2+5)}dx$ (a) $\ln\frac{28}{9}$ (b) $\frac{28}{9}$ (c) $\ln\frac{20}{9}$ (d) $\frac{20}{9}$ (e) NOTA

SOLUTION

$$\int_2^3 \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx$$

Expanding $(x-1)(x^2+5) = x^3+5x-x^2-5$

$= x^3-x^2+5x-5$

$$\int_2^3 \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx = \int_2^3 \frac{3x^2-2x+5}{x^3-x^2+5x-5}\,dx$$

let $u = x^3-x^2+5x-5$

$\frac{du}{dx} = 3x^2-2x+5$; $dx = \frac{du}{3x^2-2x+5}$

when $x = 2$; $u = 2^3-2^2+5(2)-5 = 9$

when $x = 3$; $u = 3^3-3^2+5(3)-5 = 28$

$$\int_2^3 \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx = \int_9^{28} \frac{3x^2-2x+5}{u}\cdot\frac{du}{3x^2-2x+5}$$

$$= \int_9^{28} \frac{1}{u}\,du = \left[\ln u\right]_9^{28}$$

$$= \ln 28 - \ln 9$$

$$= \ln\frac{28}{9} \quad (A)$$

61. Evaluate $\int_0^{\pi/2} \cos(-4x)dx$ (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

SOLUTION

$$\int_0^{\pi/2} \cos(-4x)\,dx$$

Let $u = -4x$, $\frac{du}{dx} = -4$

$dx = \frac{du}{-4}$

When $x = 0$; $u = -4(0) = 0$

When $x = \pi/2$, $u = -4(\pi/2) = -2\pi$

$$\int_0^{\pi/2} \cos(-4x)\,dx = \int_0^{-2\pi} \cos u \cdot \frac{du}{-4}$$

$= -\frac{1}{4}\int_0^{-2\pi} \cos u\,du$

$= \left[-\frac{1}{4}\sin u\right]_0^{-2\pi}$

$= \left[-\frac{1}{4}\sin(-360)\right] - \left[-\frac{1}{4}\sin(0)\right]$

$= -\frac{1}{4}(0) - \left[-\frac{1}{4}(0)\right]$ (A)

$= 0 - 0 = 0$

: (d) sinh x tanh x + c

31. Solve $\int_0^{\pi/2} \frac{\sin 2x}{1+\cos^2 x} dx$. (a) $\ln 2$ (b) 2 (c) -2 (d) 0 (e) 1

SOLUTION

$\int_0^{\pi/2} \frac{\sin 2x}{(1+\cos^2 x)} dx$

From Trignametry

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = \cos^2 x - (1-\cos^2 x)$

$\cos 2x = 2\cos^2 x - 1$

making $\cos^2 x$ the Subject

$\cos^2 x = \frac{\cos 2x + 1}{2}$

$\int_0^{\pi/2} \frac{\sin 2x}{(1+\cos^2 x)} dx = \int_0^{\pi/2} \frac{\sin 2x}{1+\frac{\cos 2x+1}{2}} dx$

$\int_0^{\pi/2} \left(\frac{\sin 2x}{1} \times \frac{2}{2+\cos 2x + 1}\right) dx$

$\int_0^{\pi/2} \frac{2\sin 2x}{2+\cos 2x + 1} dx$

$\int_0^{\pi/2} \frac{2\sin 2x}{3+\cos 2x} dx = -\ln(3+\cos 2x)\Big|_0^{\pi/2}$

$= -\ln[3+\cos\pi] - [-\ln(3+\cos 0)]$ (A)

$= -\ln 2 + \ln 4;\ \ln 4 - \ln 2;\ \ln\left(\frac{4}{2}\right) = \ln 2$

(c) NOTA

APPLICATION OF INTEGRATION

33. Find the area of the region bounded by $f(x) = |x-2|$ and x-axis for $-1 \le x \le 5$
(a) 9 (b) 4 (c) 6 (d) 8 (e) NOTA

SOLUTION

$\text{Area} = \int_{-1}^{5} f(x)\,dx$

$= \int_{-1}^{5} |x-2|\,dx$

The Transition point is at $x=2$

$\int |x-2|\,dx = \int_{-1}^{2} -(x-2)\,dx + \int_2^5 (x-2)$

$= \frac{9}{2} + \frac{9}{2} = \frac{18}{2}$

$= 9$ (A)

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI.

SCHOOL OF PHYSICAL SCIENCES DEPARTMENT OF MATHEEMNT

2016/2017 MTH 102 TEST TIME: 1HR INSTRUCTION ANSWER ALL QUESTION

1. Given $f(x) = -x^2 + 6x - 11$, find $f(t-3)$
2. Find the domain of function, $|x-6| - 3$
3. Find the range of function, $|x-6| - 3$
4. Find the domain of function $g(t)=100$
5. Find the range of function, $p(t)=12$
6. If $\lim f(x) = \lim f(x)$, then the normal limit exists. TRUE OR FALSE?
7. Given the functions f(x) and g(x), suppose we have $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) : L$, for some real numbers, c and L, then $\lim_{x\to c}\left(\frac{g(x)}{f(x)}\right) = \infty$, TRUE OR FALSE?
8. Using the two functions in question 7, if $L > 0$, then $\lim_{x\to c}[f(x)g(x)] = \infty$, TRUE OR FALSE?
9. If r is a positive rational number and c is any real number, find $\lim_{x\to\infty}\left(\frac{c}{x^r}\right)$
10. Evaluate the following limit. $\lim_{x\to -\infty}(e^{2-4x-8x^2})$

2016/2017 TEST QUESTION AND SOLUTION

FUNCTIONS

1. Given $f(x) = -x^2 + 6x - 11$, find $f(t-3)$

SOLUTION

$F(x) = -x^2 + 6x - 11$

$F(t-3) = -(t-3)^2 + 6(t-3) - 11$

$= -(t^2 - 6t + 9) + 6t - 18 - 11$

$= -t^2 + 6t - 9 + 6t - 18 - 11$

$= -t^2 + 12t - 38$

DOMAIN AND RANGE

2. Find the domain of function, $|x-6| - 3$

SOLUTION

Let $F(x) = |x-6| - 3$

Domain of absolute value function is All real numbers.

Hence Dom F $= (-\infty, \infty)$ or $\mathbb{R}$

2016/2017 Test Q3

3. Find the range of function, $|x-6| - 3$

SOLUTION

The Range of absolute value function is All positive real numbers including zero. But F(x) has an extension "-3" it will now begin from "-3"

$R_F = [-3, \infty)$ or $\{x : x \geq -3, x \in \mathbb{R}\}$

4. Find the domain of function $g(t)=100$

SOLUTION

$P(t) = 12$ is a polynomial function

Hence Dom $P = (-\infty, \infty)$ or $\mathbb{R}$

5. Find the range of function, $p(t)=12$

SOLUTION

$P(t) = 12$; The range of a Constant is the Constant. Hence $R_P = 12$

LIMITS

6. If $\lim f(x) = \lim f(x)$, then the normal limit exists.

TRUE OR FALSE?

SOLUTION

If the right hand limit; $\lim_{x\to a^+} F(x)$ is equal to the left hand limit; $\lim_{x\to a^-} F(x)$ Then the normal limit exist.

TRUE

7. Given the functions f(x) and g(x), suppose we have $\lim_{x\to c} f(x)=\infty$ and $\lim_{x\to c} g(x): L$, for some real numbers, c and L, then $\lim_{x\to c}\left(\frac{g(x)}{f(x)}\right)=\infty$, *TRUE OR FALSE?*

SOLUTION

Given f(x) and g(x), Suppose

$\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$

FALSE

Then $\lim_{x\to c} \frac{g(x)}{f(x)} = \frac{L}{\infty} = 0 \neq \infty$

L stands for a real number

Recall that a number divided by ∞ is "0".

8. Using the two functions in question 7, if L> 0, then $\lim_{x\to c}[f(x)g(x)]=\infty$, *TRUE OR FALSE?*

SOLUTION

If, $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$

Then $\lim_{x\to c} [f(x)\, g(x)] = \infty \cdot L = \infty$ TRUE

Note: ∞ multiplied by any number is ∞.

9. If r is a positive rational number and c is any real number, find $\lim_{x\to\infty}\left(\frac{c}{x^r}\right)$..

SOLUTION

$$\lim_{x\to\infty} \frac{c}{x^r} = \frac{c}{\infty^r} = \frac{c}{\infty}$$

A number divided by ∞ is "0"

$$\Rightarrow \lim_{x\to\infty} \frac{c}{x^r} = 0$$

10. Evaluate the following limit,

$\lim_{x\to\infty}(e^{2-4x-8x^2})$..

SOLUTION

$$\lim_{x\to\infty} \left(e^{2-4x-8x^2}\right) = e^{2-4(\infty)-8(\infty)^2}$$

$$= e^{-\infty-\infty} = e^{-\infty}$$

$$= \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI
SCHOOL OF PHYSICAL SCIENCES DEPARTMENT OF MATHEMATICS
2016/2017 RAIN SEMESTER EXAMINATION
MTH 102: ELEMENTRY MATHEMATICS II TIME 3 HOURS
INSTRUCTION : ANSWER ALL QUESTIONS

1. Find respectively the domain and range of the function $f(x)=\sqrt{9-x^2}$ A. (-3,3); (0,3) B. [-3,3]; (0,3) C. [-3,3]; [0,3] D. (0,3); (-3,3) E. None of the above.
2. Determine all the vertical asymptotes of the graph $f(x)=\frac{x^2+x-6}{x^2-9}$ A. -3 B. 2 C. -3,3 D. 0 E.3
3. Determine all the horizontal asymptotes of the function $f(x)=\frac{5x^2+7x-1}{x^2-25}$ A. 2 B. -2 C. -5,5 D. 25 E. 5
4. Evaluate $\int x\ln x\,dx$ A. $\frac{x^2}{2}\ln x-\frac{x^4}{12}+c$ B. $\frac{x^2}{2}\ln x-\frac{x^4}{15}+c$ C. $\frac{x^2}{2}\ln x+c$ D. $\ln x+c$ E. None of the above
5. Evaluate $\int\cos^2 x\,dx$ A. $\frac{x}{2}-\frac{\sin 2x}{4}+c$ B. $\frac{x}{2}+\frac{\sin 2x}{2}+c$ C. $\frac{x}{2}+\frac{\sin 2x}{4}+c$ D. $\frac{x}{2}-\frac{\cos 2x}{4}+c$ E. None of the above
6. Find the slope of the tangent line to the circle $x^2+y^2=25$ *at the point* (3,4) A. 3/4 *B.* $-3/4$ *C.* 1/4 *D.* $-5/4$ *E.* 1/2
7. Let f be defined by x^3+2x^2+1 Using the mean value theorem find any real number c in the interval [0,3] A. -3 B. 5/3 C. 2 D. 4/3 E. None of the above.
8. If $x^2-2y^2=4$ *Find* $\frac{dy}{dx}$ *at the point* (1,3) A. 3 B. 0 C. -3 D. $-1/3$ E. None of the above.
9. Evaluate $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ A. e^3 B. e C. 1 D. e^2 E. None of the above
10. Obtain the value of δ *if* $\lim_{x\to 2}(3x+7)=1$ A. $\frac{\varepsilon}{3}$ *B.* ε *C.* $\frac{\varepsilon}{2}$ *D.* $\frac{\varepsilon}{6}$ *E. None of the above*
11. A particle moves along a line so that at time t secs its position is $s(t)=t^3-6t^2+7t-2$ *metres. find the time* t_0 *at which acceleration equals zero. A.* 4secs B. 3secs C. 2secs D. 12secs E. 0.5secs
12. Obtain the maximum point of the function y= $x(x-1)^2$ A. (1,0) B. (1/3, 4/27) C. (1, 1/3) *D.* (1/3, 0) E. None of the above.
13. A function is given by $f(t)=\sin\pi t$. Find the instantaneous rate of change at $t=\frac{1}{4}$ *A.* $\frac{\pi}{2}$ *B.* π *C.* $\frac{\pi\sqrt{2}}{2}$ D. $\frac{\pi\sqrt{3}}{2}$ E. None of the above.
14. Find all the vertical asymptotes for the graph of $f(x)=\frac{x^2}{x-2}$ A. 0,2 B. ... 2,2 D. 0,-2 E. 0,-4
15. All these except one can be expressed $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. A. The derivative of f(x) at x_0 B. The average rate of change of $f(x)$ at x_0 C. The slope of the ... of $f(x)$ at x_0 D. The differential coefficient of f(x). E None of the above
16. A Giraffe 6ft tall walks at the rate of 4ft/sec. At what rate is the length of the shadow increasing? A. $\frac{64}{25}ft/sec$ B. $\frac{32}{7}ft/sec$ C. $\frac{32}{5}ft/sec$ D. ... $\frac{3}{32}ft/sec$
17. Find f^{-1} *if* $f(x)=4x+3$. A. $\frac{x-3}{4}$ B. $\frac{1}{4x+3}$ C. $\frac{3}{4x}$ D. $\frac{4}{4x+3}$ E. $\frac{4x+3}{4}$
18. Find the horizontal asymptotes of the graph $f(x)=\frac{(3x-7)(3x+7)}{(2x+3)(x+4)}$ A. 3/2 B. ... 0 D. 9/2 E. 2/3
19. Obtain the integral $\int x^3(x^2+7)^4\,dx$ A. $\frac{(x^2+7)^5}{10}+K$ B. $\frac{(x^2+7x)^4}{8}+K$ C. $\frac{x^4(x^2+7)^5}{12}+K$ D. $\frac{x^4(x^2+7)^5}{8}+K$ E. $\frac{(x^4+7x^3)^4}{4}+K$
20. Two functions f(x) and g(x) are defined on the set of real numbers by f(x)=2x+3 and g(x)=x^2+2x-7 *Find* $f(g(2))$. A.2 B. 3 C. 5 D. 6 E. ...
21. Determine respectively the domain and range of the function, f(x)=$\frac{1}{\sqrt{9-x^2}}$ *A.* $(-3,3); R^+$ *B.* $[-3,3]; R^+$ C. (-3,3);R D. [-3,3];R E. R;R
22. Find the inverse of the function f(x)= $\frac{1}{2}x-3$. A. 2(x+1) B. 2(x+2) C. ... D. 2(x+6) E. 2x+3
23. Find $\delta>0$ *such that* $|f(x)-L|<0.01$ *whenever* $0<|x-2|<\delta$ *for a limit* $\lim_{x\to 2}(3x+2)=8$ A. 0.033 B. 0.33 C. 0.00033 D. 0.0033 E. 0.000033
24. Evaluate the limit, $\lim_{x\to\infty}\frac{2x^2-3x+1}{3x^2+5x+1}$ A. 3/2 B. 2/3 C. 1/2 D. 2/5 E. 5/2
25. Determine the slope of the graph of $3(x^2+y^2)^2=100xy$ at the point (3,1) A. 9/13 B. -9/13 C. -13/9 D. 13/9 E. 2/13
26. Determine all vertical asymptotes of the graph of $f(x)=\frac{x^2+2x}{x^2-4}$ A. 2,-2 B. ... C. -2 D. 0 E. 2,2
27. Find the tangent line to the graph given by $x^2(x^2+y^2)=y^2$ at the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ A. y=3x-$\sqrt{2}$ B. y= 3x+$\sqrt{2}$ C. $\sqrt{2}x-3$ D. $\sqrt{2}x+3$ E. -3x-$\sqrt{2}$
28. Find $\frac{dy}{dx}$ *if* $y=\cos 2t, x=\sin t$ *A.* $4\sin t$ *B.* $-4\sin t$ *C.* $4\cos t$ *D.* $-4\cos t$ *E.* $4\sin 2t$
29. Find the greatest product of two numbers whose sum is 12. A. 36 B. 64 C. ... 6 E. 24
30. Determine the minimum points of the function $f(x)=x^4-6x^2+8x+...$ ... 0 B. 1,1,-2 C. 2 D. 1, -2 E. -2
31. Evaluate $\int\frac{7+x-3x^2}{2-x}dx$ A. $x^2+3x-\ln(2-x)+C$ B. $x^2+2x-\ln(2+x)+C$ C. $\frac{x^2}{2}+x-\ln(1+x)+C$ D. $\ln(1+x)+C$ E. $\ln(1-x)+C$
32. A body is projected vertically upward and the height h metres after time t secs is given by A = 196t-4.9t² Find the time taken to reach the greatest height A. 2 B. 12 C. 24 D. 20 E. 36
33. Given f(z) = $-z^2+6z-11$, *find* $f(4z-1)$. A. $16z^2+32z-18$ B. $16z^2-32z-18$ C. $-16z^2+32z-18$ D. $-16z^2+32z+18$ E. None of the above.
34. Find the domain of the function $y(z)=\sqrt{4-7z}$ *A.* $(-\infty,\infty)$ *B.* $(0,\infty)$ *C.* $\left(-\infty,\frac{4}{7}\right]$ *D.* $[0,\infty)$ E. None of the above
35. Find the range of the function $y(z)=\sqrt{4-7z}$ *A.* $(-\infty,\infty)$ *B.* $(0,\infty)$ *C.* $\left(-\infty,\frac{4}{7}\right]$ *D.* $[0,\infty)$ E. None of the above
36. Find the domain of the function $g(t)=|t-3|$-10 A.(-10,∞) *B.* $(-10,\infty]$ *C.* $(-\infty,\infty)$ *D.* $[-10,\infty)$. E. None of the above
37. Find the range of the function $g(t)=|t-3|$-10 A.(-10,∞) *B.* $(-10,\infty]$ *C.* $(-\infty,\infty)$ *D.* $[-10,\infty)$. E. None of the above
38. Given the functions f(x) and g(x), suppose we have $\lim_{x\to c} f(x)=\infty$ *and* $\lim_{x\to c} g(x)=L$, for some real numbers, c and L, find $\lim_{x\to c}[f(x)g(x)]$ A. 0 B. ∞ *C.* $-\infty$ *D.* $(-\infty,\infty)$ E. None of the above
39. Using the two functions in question 38, if L < 0. Find $\lim_{x\to c}[f(x)g(x)]$ A. 0 B. ∞ *C.* $-\infty$ *D.* $(-\infty,\infty)$ E. None of the above
40. Evaluate $\lim_{x\to\infty}(2x^4-x^2-8x)$. A. 0 B. ∞ *C.* $-\infty$ *D.* $(-\infty,\infty)$ E. ...
41. Evaluate the following limit $\lim_{t\to-\infty}(e^{t^4-5t^2+1})$ A. 0 B. ∞ *C.* $-\infty$ *D.* $(-\infty,\infty)$ E. undefined
42. Evaluate $\lim_{t\to-\infty}(\frac{1}{3}t^3+2t^3-t^2+8)$. A. 0 B. ∞ *C.* $-\infty$ *D.* $(-\infty,\infty)$ E. Undefined
43. Which of the following is not a transcendental function. A. $\cos 2x$ *B.* e^{2x} C. $\ln x$ *D.* $\ln(x^2+2)$ E. None of the above
44. Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ A. e B. 0 C. 1 D. e^x *E. None of the above*
45. Evaluate $\lim_{x\to 0}\frac{(2-2\cos x)}{4}$ A. -1 B. 0 C. 1 D. 2 E. Undefined
46. Given that y= $\frac{1}{\sqrt{(x-1)^3}}$ *find* $\frac{dy}{dx}$ *A.* $\frac{1}{2}(x-1)^{\frac{3}{2}}$ B. $-\frac{1}{2}(x-1)^{\frac{3}{2}}$ C. $\frac{1}{2}(x-1)^{...}$ D. $-\frac{1}{2}(x-1)^{-\frac{5}{2}}$ E. $\frac{1}{2}(x-1)^{\frac{5}{2}}$
47. Evaluate $\int k\,dx$ where k is a constant A. $0+C$ B. $K+C$ C. $Kx+C$ D. ... E. $x+C$
48. Evaluate $\int\frac{2x+3}{x^2+3x+6}dx$ A. $\ln(2x+3)+C$ B. $-\ln(2x+3)+C$ C. $\ln(x^2+3x+6)+C$ D. $-\ln(x^2+3x+6)+C$ E. $-\ln C(x^2+3x+6)$
49. Evaluate $\int xe^{x^2+3}dx$ *A.*
50. Given f(x)=$\log_e x$. *Evaluate* $\int f(x)dx$ *A.* $\frac{1}{x}+C$ *B.* e *C.* $\ln x-x+C$ D. $x\ln x-x+C$ E. $\ln x+x+C$

...ive) A. The derivative of ... C. The slope of the graph ... E. None of the above ... rate is the length of its ... ft/sec D. 26.4 ft/sec E.

...A. 3/2 B. 3 C.

... + K C.

...al numbers by ... 3 C. 5 D. 6 E. 25

...3],R E. R,R

... B. 2(x+2) C. 2(x+3)

...33 D. 0.0033 E.

...5 E. 5/2

...y at the point (3,1)

... A. 2,-2 B. 2

... y^2 at the point ... 3 E. $-3x\sqrt{2}$

... A. 36 B. 64 C. 2 D.

... $-6x^2+8x+10$ A.

... E. $\ln(1-x)+C$

...s after a time t ... he greatest height

... $z-18$ B. - ... E. None of the

...the above.

...the above.

...above.

...above.

... $[f(x) \pm g(x)]$

... $(x)]$ A. 0 B.

... $(-\infty,\infty)$ E.

...)E.

... $-1)^{-\frac{2}{3}}$ D.

... D. $1+C$

... $(x^2+$

... D.

2016/2017 EXAM QUESTION AND SOLUTION

FUNCTION

...of $f(x) = 4x+3$. A. $\frac{x-3}{4}$ B. $\frac{1}{4x+3}$ C. $\frac{1}{4x}$ D. $\frac{4}{4x+3}$ E. $\frac{x+3}{4}$

SOLUTION

$f(x) = 4x+3$
let $y = f(x)$
$\Rightarrow y = 4x+3$
making x the subject;
$y-3 = 4x$; $x = \frac{y-3}{4}$
interchange y with x and x with $f^{-1}(x)$
$\Rightarrow f^{-1}(x) = \frac{x-3}{4}$ (A)

Two functions f(x) and g(x) are defined on the set of real numbers by f(x)=2x+3 and g(x)=x^2+2x-7 *Find* $f(g(2))$. A.2 B. 3 C. 5 D. 6 E. 25

SOLUTION

$f(x) = 2x+3$, $g(x) = x^2+2x-7$
$g(2) = 2^2+2(2)-7 = 1$ (C)
$f[g(2)] = f(1) = 2(1)+3 = 5$

22. Find the inverse of the function f(x)= $\frac{1}{2}x-3$. A. 2(x+1) B. 2(x+2) C. 2(x+3) D. 2(x+6) E. 2x+3

SOLUTION

$f(x) = \frac{1}{2}x-3$; let $y = f(x)$
$\Rightarrow y = \frac{1}{2}x-3$; making x the subject
$y+3 = \frac{x}{2}$; $x = 2(y+3)$
interchanging: $f^{-1}(x) = 2(x+3)$ (C)

Given $f(z) = -z^2+6z-11$. *find* $f(4z-1)$. A. $16z^2+32z-18$ B. $-16z^2-32z-18$ C. $-16z^2+32z-18$ D. $-16z^2+32z+18$ E. None of the above.

SOLUTION

$f(z) = -z^2+6z-11$
$f(4z-1) = -(4z-1)^2+6(4z-1)-11$
$= -(4z-1)(4z-1)+6(4z-1)-11$
$= -[16z^2-4z-4z+1]+24z-6-11$
$= -16z^2+8z-1+24z-6-11$
$= -16z^2+32z-18$ (C)

DOMAIN AND RANGE

Find respectively the domain and range of the function $f(x) = \sqrt{9-x^2}$ A. (-3,3); (0,3) B. [-3,3]; (0,3) C. [-3,3]; [0,3] D. (0,3); (-3,3) E. None of the above.

SOLUTION

$f(x) = \sqrt{9-x^2}$
f(x) is defined if $9-x^2 \geq 0$
$\Rightarrow 3^2-x^2 \geq 0$
$(3+x)(3-x) \geq 0$; Turning points; $-3, 3$

Truth table

	$x<-3$	$-3<x<3$	$x>3$
$3-x$	+	+	−
$3+x$	−	+	+
Product	−	+	−

The Domain is the solution set
Dom f = $\{x: -3 \leq x \leq 3, x \in \mathbb{R}\}$
or
$= [-3, 3]$ (C)
The Range is $[0, 3]$

Determine respectively the domain and range of the function, f(x)=$\frac{1}{\sqrt{9-x^2}}$ A. (−3,3); R^+ B. [−3,3]; R^+ C. (-3,3);R D. [-3,3];R E. R;R

SOLUTION

$f(x) = \frac{1}{\sqrt{9-x^2}}$
f(x) is defined if $9-x^2 > 0$
$3^2-x^2 > 0$
$(3+x)(3-x) > 0$
The turning points are -3, 3

Truth Table

	$x<-3$	$-3<x<3$	$x>3$
$3-x$	+	+	−
$3+x$	−	+	+
Product	−	+	−

The domain is given by the solution set.
Dom f = $\{x: -3 < x < 3, x \in \mathbb{R}\}$
$= (-3, 3)$
Range $= \mathbb{R}^+$ (A)
$= (0, \infty)$

34. Find the domain of the function $y(z) = \sqrt{4-7z}$ A. $(-\infty,\infty)$ B. $(0,\infty)$ C. $\left(-\infty,\frac{4}{7}\right]$ D. $[0,\infty)$ E. None of the above.

SOLUTION

$y(z) = \sqrt{4-7z}$
$y(z)$ is defined if $4-7z \geq 0$
$4 \geq 7z$; $7z \leq 4$
Dividing both sides by 7
$z \leq 4/7$; which is given in interval form as
$(-\infty, 4/7]$ (C)

35. Find the range of the function $y(z) = \sqrt{4-7z}$ A. $(-\infty,\infty)$ B. $(0,\infty)$ C. $\left(-\infty,\frac{4}{7}\right]$ D. $[0,\infty)$ E. None of the above.

SOLUTION

$y(z) = \sqrt{4-7z}$
The range of every square root function begins with zero.
The maximum output of $y(z)$ is as large as possible.
Range $= [0, \infty)$

36. Find the domain of the function $g(t) = |t-3|-10$ A.$[-10,\infty)$ B. $(-10,\infty]$ C. $(-\infty,\infty)$ D. $[-10,\infty)$. E. None of the above.

SOLUTION

$g(t) = |t-3|-10$
Domain of every Absolute value function contain a polynomial is $\mathbb{R} = (-\infty, \infty)$

37. Find the range of the function $g(t) = |t-3|-10$ A.$[-10,\infty)$ B. $(-10,\infty]$ C. $(-\infty,\infty)$ D. $[-10,\infty)$. E. None of the above.

SOLUTION

$g(t) = |t-3|-10$
The range of absolute value function, say $f(x) = |x|$; is $[0, \infty)$
but $g(t) = |t-3|-10$ has an extension; hence it will begin from the extension -10
Range $= [-10, \infty)$

LIMITS

9. Evaluate $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ A. e^3 B. e C. 1 D. e^2 E. None of the above

SOLUTION

$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$
Applying Hodges theorem
$\lim_{x\to\infty}\left(1+\frac{A}{Bx}\right)^x = e^{A/B}$
where $A = 3$, $B = 1$
$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = e^{3/1} = e^3$ (A)

10. Obtain the value of δ *if* $\lim_{x\to -2}(3x+7) = 1$ A. $\frac{\varepsilon}{2}$ B. e C. $\frac{\varepsilon}{3}$ D. $\frac{\varepsilon}{6}$ E. None of the above

SOLUTION

$\lim_{x\to -2}(3x+7) = 1$
By Epsilon-delta definition of limit
$f(x) = 3x+7$, $x_0 = -2$, $L = 1$
$\Rightarrow |(3x+7)-1| < \varepsilon$ whenever $0 < |x+2| < \delta$
Transforming in terms of δ
$|3(x+2)-6+7-1| < \varepsilon$
$|3\delta| < \varepsilon$; Choose $\delta \leq \frac{\varepsilon}{3}$ (C)

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23. Find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - 2| < \delta$ for ϵ limit $\lim_{x\to 2}(3x + 2) = 8$ A. 0.011 B. 0.11 C. 0.00033 D. 0.0033 E. 0.000033

SOLUTION

$$\lim_{x\to 2}(3x+2) = 8$$

By epsilon - delta definition;

$F(x) = 3x+2,\ x_0 = 2,\ L = 8$

using $|F(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$

$\Rightarrow |(3x+2) - 8| < \epsilon$ whenever $0 < |x-2| < \delta$

Transforming in terms of δ, we have

$|3(x-2) + 6 + 2 - 8| < \epsilon$

$|3(x-2)| < \epsilon$

$|3\delta| < \epsilon$; Choose $\delta \leq \frac{\epsilon}{3}$

where $\epsilon = 0.01$; $\delta \leq 0.0033$ D

24. Evaluate the limit, $\lim_{x\to\infty}\frac{2x^2-3x+1}{3x^2+5x+1}$ A. 3/2 B. 2/3 C. 1/2 D. 2/5 E. 5/2

SOLUTION

$$\lim_{x\to\infty}\frac{2x^2-3x+1}{3x^2+5x+1}$$

Dividing through by the highest power

$$\lim_{x\to\infty}\frac{\frac{2x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}}{\frac{3x^2}{x^2} + \frac{5x}{x^2} + \frac{1}{x^2}}$$

$$\lim_{x\to\infty}\frac{2 - \frac{3}{x} - \frac{1}{x^2}}{3 + \frac{5}{x} + \frac{1}{x^2}}$$

Taking the limit B

$$= \frac{2 - \frac{3}{\infty} - \frac{1}{\infty^2}}{3 + \frac{5}{\infty} - \frac{1}{\infty^2}} = \frac{2-0-0}{3+0-0} = \frac{2}{3}$$

38. Given the functions f(x) and g(x), suppose we have $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$, for some real numbers, c and L, find $\lim_{x\to c}[f(x) \pm g(x)]$ A. 0 B. ∞ C. $-\infty$ D. $(-\infty,\infty)$ E. None of the above

SOLUTION

$\lim_{x\to c} F(x) = \infty$; $\lim_{x\to c} g(x) = L$

$\lim_{x\to c} F(x) \pm g(x) = \infty \pm L$

$= \infty$ B

Note: Infinity $\pm$ any real number will give infinity. ∞ is a very large number

39. Using the two functions in question 38, if $L < 0$. Find $\lim_{x\to c}[f(x)g(x)]$ A. 0 B. ∞ C. $-\infty$ D. $(-\infty,\infty)$ E. None of the above

SOLUTION

$\lim_{x\to c} F(x) = \infty$; $\lim_{x\to c} g(x) = L$

If $L < 0$, Then

$\lim_{x\to c} F(x)g(x) = \infty \cdot -L$

$= -\infty$

Note: If $L < 0$, Then L is a negative real number, also note that "∞" multiplied by a negative number is negative "∞"

40. Evaluate $\lim_{x\to\infty}(2x^4 - x^2 - 8x)$. A. 0 B. ∞ C. $-\infty$ D. $(-\infty,\infty)$ E. 2

SOLUTION

$$\lim_{x\to\infty}(2x^4 - x^2 - 8x)$$

$= 2(\infty)^4 - (\infty)^2 - 8(\infty)$

$= \infty^4 - \infty^2 - \infty$

$= \infty$ B

41. Evaluate the following limit $\lim_{x\to-\infty}(e^{x^4-5x^2+1})$ A. 0 B. ∞ C. $-\infty$ D. $(-\infty,\infty)$ E. undefined

SOLUTION

$$\lim_{x\to-\infty} e^{x^4} - 5x^2 + 1$$

$= e^{(-\infty)^4} - 5(-\infty)^2 + 1$

$= e^{\{\infty^4 - 5(\infty^2) + 1\}}$

$= \infty$ $= e^{\infty}$

42. Evaluate $\lim_{x\to-\infty}(\frac{1}{3}x^5 + 2x^3 - x^2 + 8)$. A. 0 B. ∞ C. $-\infty$ D. $(-\infty,\infty)$ E. Undefined

SOLUTION

$$\lim_{x\to-\infty}(\tfrac{1}{3}x^5 + 2x^3 - x^2 + 8)$$

$= \frac{1}{3}(-\infty)^5 + 2(-\infty)^3 - (-\infty)^2 + 8$

$= -\infty^5 - \infty^3 - \infty^2 + 8$

$= -\infty$ C

43. Which of the following is not a transcendental function. A. $\cos 2x$ B. e^{2x} C. $\ln x$ D. $\ln(x^2 + 2)$ E. None of the above

SOLUTION

Non - Transcendental Function are algebraic Function, Hence None of the option. E

44. Evaluate $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$ A. e B. 0 C. 1 D. e^x E. None of the above

SOLUTION

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$$

STANDARD LIMIT

45. Evaluate $\lim_{x\to 0}\frac{(2-2\cos x)}{x}$ A. -1 B. 0 C. 1 D. 2 E. Undefined

SOLUTION

$$\lim_{x\to 0}\frac{2-2\cos x}{x} = \frac{2-2\cos 0}{0} = \frac{2-2}{0} = \frac{0}{0}$$

Applying L'Hopital's rule B

$$\lim_{x\to 0}\frac{2\sin x}{1} = \frac{2(0)}{1} = \frac{0}{1} = 0$$

ASYMPTOTES

2. Determine all the vertical asymptotes of the graph $f(x) = \frac{x^2-x-6}{x^2-9}$ A. -3 B. 3 C. -3,3 D. 0 E. 3

SOLUTION

$$F(x) = \frac{x^2-x-6}{x^2-9} = \frac{(x-3)(x+2)}{(x-3)(x+3)}$$

$$= \frac{x+2}{x+3}$$

For vertical asymptote; we equate denominator to zero.

$\Rightarrow x+3 = 0$ A

$x = -3$

3. Determine all the horizontal asymptotes of the function $f(x) = \frac{5x^2+7x-1}{x^2-25}$ A. 1 B. -2 C. -5,5 D. 25 E. 5

SOLUTION

$$F(x) = \frac{5x^2+7x-1}{x^2-25}$$

To determine horizontal asymptote we take limit of the function as $x \to \infty$

$$\Rightarrow \lim_{x\to\infty}\frac{5x^2+7x-1}{x^2-25}$$

Dividing through by the highest power x^2

$$= \lim_{x\to\infty}\frac{\frac{5x^2}{x^2} + \frac{7x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{25}{x^2}} = \lim_{x\to\infty}\frac{5 + \frac{7}{x} - \frac{1}{x^2}}{1 - \frac{25}{x^2}}$$

$$= \frac{5 + \frac{7}{\infty} - \frac{1}{\infty^2}}{1 - \frac{25}{\infty}} = \frac{5+0-0}{1-0} = 5$$ E

C. $-\infty$ D. $(-\infty,\infty)$ E.

)E.

E

bove

B. 2 C.

A. 2

Find all the vertical asymptotes for the graph of $f(x) = \frac{x^2}{4-x^2}$ A. 0,2 B. 0,4 C. 2,-2 D. 0,-2 E. 0,-4

SOLUTION

$F(x) = \frac{x^2}{4-x^2}$

For vertical Asymptotes

$4 - x^2 = 0$

$2^2 - x^2 = 0$

$(2+x)(2-x) = 0$ (C)

$x = -2,\ x = 2$

Find the horizontal asymptotes of the graph $f(x) = \frac{(3x-2)(2x+3)}{(2x+1)(x+2)}$ A. 3/2 B. 3 C. 6 D. 9/2 E. 2/3

SOLUTION

$f(x) = \frac{6x^2+5x-6}{2x^2+5x+2}$ (B)

Horizontal Asymptote is given by

$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{6x^2+5x-6}{2x^2+5x+2} = \frac{6}{2} = 3$

Determine all vertical asymptotes of the graph of $f(x) = \frac{x^2+2x-8}{x^2-4}$ A. 2,-2 B. 2 C. -2 D. 0 E. 2,2

SOLUTION

$f(x) = \frac{x^2+2x-8}{x^2-4} = \frac{(x+4)(x-2)}{(x+2)(x-2)} = \frac{x+4}{x+2}$

For vertical Asymptote; we equate denominator to zero (C)

$\Rightarrow x+2 = 0;\ x = -2$

DIFFERENTIATION

1. A function is given by $f(t) = \sin \pi t$. Find the instantaneous rate of change at $t = \frac{1}{4}$ A. $\frac{\pi}{2}$ B. π C. $\frac{\pi\sqrt{3}}{2}$ D. $\frac{\pi\sqrt{2}}{2}$ E. None of the above.

SOLUTION

$F(t) = \sin \pi t$

$\frac{dF}{dt} = \pi \cos \pi t$; at $t = 1/4$

$\left.\frac{dF}{dt}\right|_{t=1/4} = \pi \cos\left(\frac{\pi}{4}\right)$

$= \pi\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{2}$ (D)

3. All these except one can be expressed $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, A. The derivative of f(x) at x_0 B. The average rate of change of $f(x)$ at x_0 C. The slope of the graph of $f(x)$ at x_0 D. The differential coefficient of f(x). E. None of the above.

SOLUTION

Differential Coefficient, Derivative

Slope, gradient, instantaneous rate of change at x_0 is given by

$\lim_{x\to h} \frac{F(x_0+h) - F(x_0)}{h}$

While Average rate of change is given by at x_0 is given by

$\frac{F(x_0+h) - F(x_0)}{h}$ (B)

Observe that, we don't take $\lim$ as $h \to 0$ in Average rate of change

8. Given that $y = \frac{1}{\sqrt{(x-1)^2}}$ find $\frac{dy}{dx}$ A. $\frac{1}{2}(x-1)^{\frac{3}{2}}$ B. $-\frac{1}{2}(x-1)^{\frac{3}{2}}$ C. $\frac{1}{2}(x-1)^{-\frac{3}{2}}$ D. $-\frac{1}{2}(x-1)^{-\frac{3}{2}}$ E. $\frac{1}{2}(x-1)^{\frac{2}{3}}$

SOLUTION

$y = \frac{1}{\sqrt{x-1}}$

$y = (x-1)^{-1/2}$

$\frac{dy}{dx} = -\frac{1}{2}(x-1)^{-1/2-1}$

$= -\frac{1}{2}(x-1)^{-3/2}$ (D)

IMPLICIT DIFFERENCIATION

6. Find the slope of the tangent line to the circle $x^2+y^2 = 25$ at the point (3,4) A. 3/4 B. −3/4 C. 1/4 D. −1/4 E. 1/2

SOLUTION

$x^2 + y^2 = 25$

Differentiating implicitly

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$; $\frac{dy}{dx} = \frac{-2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$; at (3,4); $x=3, y=4$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ (B)

8. If $x^2 - 2y^2 = 4$ Find $\frac{dy}{dx}$ at the point (1,3) A. 3 B. 0 C. -3 D. −1/3 E. None of the above.

SOLUTION

$x^2 - 2y^2 = 4$

Differentiating implicitly

$2x - 4y\frac{dy}{dx} = 0$

$-4y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{-4y} = \frac{x}{2y}$

at (1,3); $x = 1, y = 3$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2(3)} = \frac{1}{6}$ (E)

25. Determine the slope of the graph of $3(x^2+y^2)^2 = 100xy$ at the point (3,1). A. 9/13 B. -9/13 C. -13/9 D. 13/9 E. 2/13

SOLUTION

$3(x^2+y^2)^2 = 100xy$

$3(x^4 + 2x^2y^2 + y^4) = 100xy$

$3x^4 + 6x^2y^2 + 3y^4 - 100xy = 0$

$\frac{dy}{dx} = -\left(\frac{12x^3 + 12xy^2 - 100y}{12x^2y + 12y^3 - 100x}\right)$

$\frac{dy}{dx} = \frac{100y - 12xy^2 - 12x^3}{12x^2y + 12y^3 - 100x}$

at (3,1); $x = 3, y = 1$

$\frac{dy}{dx} = \frac{100(1) - 12(3)(1)^2 - 12(3)^3}{12(3)^2 + 12(1)^3 - 100(3)}$

$= \frac{100 - 36 - 324}{108 + 12 - 300} = \frac{-260}{-180}$

$= \frac{13}{9}$ (D)

PARAMETRIC EQUATIONS

28. Find $\frac{dy}{dx}$ if $y = \cos 2t, x = \sin t$ A. $4\sin t$ B. $-4\sin t$ C. $4\cos t$ D. $-4\cos t$ E. $4\sin 2t$

SOLUTION

$y = \cos 2t$; $x = \sin t$

$\frac{dy}{dt} = -2\sin 2t$; $\frac{dx}{dt} = \cos t$

Using Chain rule

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

but $\frac{dt}{dx} = \frac{1}{\cos t}$

$\Rightarrow \frac{dy}{dx} = \frac{-2\sin 2t}{\cos t}$

From Trigonometry: $\sin 2t = 2\sin t\cos t$

$\Rightarrow \frac{dy}{dx} = \frac{-2(2\sin t\cos t)}{\cos t} = \frac{-4\sin t\cos t}{\cos t}$

$= -4\sin t$ (B)

APPLICATION OF DIFFERENCIATION

5b. The radius of a sphere is increasing at the rate of 0.25cm/sec when r=3cm, find the rate at which the volume is increasing (a) 44cm³/sec (b) 45cm³/sec (c) 160cm³/sec (d) 156cm³/sec (e) 154cm³/sec

SOLUTION

The Volume of a Sphere is given by

$V = \frac{4}{3}\pi r^3$; $\frac{dV}{dr} = 4\pi r^2$

rate of change of radius is given by

$\frac{dr}{dt} = 0.25$

$\frac{dV}{dt} = 4\pi r^2 \times 0.25$

$= \pi r^2$; at $r = 7cm$

$\frac{dV}{dt} = \frac{22}{7} \times 7^2 = 154$ (E)

7. Let f be defined by $x^3 + 2x^2 + 1$ Using the mean value theorem find any real number c in the interval [0,1] A. -3 B. 5/3 C. 2 D. 4/3 E. None of the above.

SOLUTION

$f(x) = x^3 + 2x^2 + 1$

$f(0) = 0^3 + 2(0)^2 + 1 = 1$

$f(3) = 3^3 + 2(3) + 1 = 46$

$f'(x) = 3x^2 + 4x$

$f'(c) = 3c^2 + 4c$

By mean value theorem;

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 + 4c = \frac{f(3) - f(0)}{3 - 0}$

$3c^2 + 4c = \frac{46 - 1}{3}$

$3c^2 + 4c = 15$; $3c^2 + 4c - 15 = 0$

Solving the quadratic equation

$C = 5/3$ or $C = -3$

But $C = 5/3 \in [0, 3]$ (B)

16. A Giraffe 6ft tall walks at the rate of 4ft/sec. At what rate is the length of its shadow increasing? A. $\frac{60}{35} ft/sec$ B. $\frac{35}{7} ft/sec$ C. $\frac{22}{5} ft/sec$ D. $26.4 ft/sec$ E. $\frac{3}{22} ft/sec$

SOLUTION

If 4.4 ft tall : 1 ft/sec

Then, 6 ft tall : x

Using cross multiplication

$4.4x = 6$

$x = \frac{6}{4.4}$ ft/sec

$= \frac{3}{2.2}$ ft/sec (E)

27. Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ A. $y=3x-\sqrt{2}$ B. $y= 3x+\sqrt{2}$ C. $\sqrt{2}x - 3$ D. $\sqrt{2}x + 3$ E. $-3x-\sqrt{2}$

SOLUTION

$x^2(x^2 + y^2) = y^2$

$x^4 + x^2y^2 = y^2$

$x^4 + x^2y^2 - y^2 = 0$

Differentiating implicitly

$\frac{dy}{dx} = -\left(\frac{4x^3 + 2xy^2}{2x^2y - 2y}\right)$

at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\frac{dy}{dx} = -\left[\frac{4\left(\frac{\sqrt{2}}{2}\right)^3 + 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^2}{2\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)}\right]$

$= \frac{3\sqrt{2}}{4} \times \frac{4}{\sqrt{2}} = 3$

Equation of tangent is given by:

$\frac{y - y_0}{x - x_0} = m$

$\Rightarrow y - y_0 = m(x - x_0)$

Where $x_0 = \frac{\sqrt{2}}{2}$, $y_0 = \frac{\sqrt{2}}{2}$, $m = 3$

$\Rightarrow y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right)$

$y - \frac{\sqrt{2}}{2} = 3x - \frac{3\sqrt{2}}{2}$

multiply through by 2

$\Rightarrow 2y - \sqrt{2} = 6x - 3\sqrt{2}$

$2y = 6x - 3\sqrt{2} + \sqrt{2}$

$2y = 6x - 2\sqrt{2}$

$2y = 2(3x - \sqrt{2})$

Dividing both sides by 2

$y = 3x - \sqrt{2}$ (A)

29. Find the greatest product of two numbers whose sum is 12. A. 36 B. 6 C. 12 D. 6 E. 24

SOLUTION

Let the two numbers be x and y and "p" be the greatest product

$\Rightarrow x + y = 12$ --- (i)

$P = xy$ --- (ii)

From equation (i) $y = 12 - x$

$\Rightarrow P = x(12 - x)$

$P = 12x - x^2$

$\frac{dP}{dx} = 12 - 2x$

At maximum point (Greatest Product)

$\frac{dP}{dx} = 0$

$\Rightarrow 12 - 2x = 0$

$12 = 2x$; $x = 12/2$

$x = 6$

Hence $y = 12 - 6$

$= 6$

The two numbers are 6, 6 and the greatest product is $6 \times 6 = 36$ (A)

30. Determine the minimum points of the function $f(x) = x^4 - 6x^2 + 8x + 10$ A. 0 B. 1,1,-2 C. 2 D. 1, -2 E. -2

SOLUTION

$F(x) = x^4 - 6x^2 + 8x + 10$

$F'(x) = 4x^3 - 12x + 8$

at turning point $F'(x) = 0$

$\Rightarrow 4x^3 - 12x + 8 = 0$

$x = -2, 1, 1$

using Second derivative test.

$F''(x) = 12x^2 - 12$

at $x = -2$

$F''(x) = 12(-2)^2 - 12$

$= 12(4) - 12$

$= 48 - 12 = 36 > 0$ (positive) (E)

Hence, at $x = -2$, the function is minimum.

11. A particle moves along a line so that at time t secs its position is $s(t) = t^3 - 6t^2 + 7t - 2$ metres. find the time t_0 at which acceleration equals zero. A. 4secs B. 3secs C. 2secs D. 12secs E. 0.5secs

SOLUTION

$S(t) = t^3 - 6t^2 + 7t - 2$

$V = S'(t) = 3t^2 - 12t + 7$

$a = S''(t) = 6t - 12$

at $a = 0$

$\Rightarrow 6t_0 - 12 = 0$

$6t_0 = 12$; $t_0 = \frac{12}{6}$

$t_0 = 2secs$ (C)

12. Obtain the maximum point of the function $y = x(x-1)^2$ A. (1,0) B. (1/3, 4/27) C. (1, 1/3) D. (1/3, 0) E. None of the above.

SOLUTION

$y = x(x-1)^2$; $y = x(x^2 - 2x + 1)$

$y = x^3 - 2x^2 + x$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

at turning point $\frac{dy}{dx} = 0$

$3x^2 - 4x + 1 = 0$

Factorising: $(x-1)(3x-1) = 0$

$x = 1$ or $x = 1/3$

using second derivative test (B)

$\frac{d^2y}{dx^2} = 6x - 4$; at $x = 1/3$; we have

$6(1/3) - 4 = -2 < 0$ minimum; $(x, y) = (1/3, \frac{4}{27})$

32. A body is projected vertically upward and the height h metres after a time t secs is given by $h = 196t - 4.9t^2$ Find the time taken to reach the greatest height A. 2 B. 12 C. 24 D. 20 E. 36

SOLUTION

$h = 196t - 4.9t^2$

$v = \frac{dh}{dt} = 196 - 9.8t$

at the greatest $v = 0$

$\Rightarrow 196 - 9.8t = 0$; $196 = 9.8t$

$t = \frac{196}{9.8} = 20$ sec (D)

INTEGRATION

4. Evaluate $\int x\ln x\, dx$ A. $\frac{x^2}{2}\ln x - \frac{x^2}{12} + c$ B. $\frac{x^2}{3}\ln x - \frac{x^3}{15} + c$ C. $\frac{x^2}{2}\ln x + c$ D. $\ln x + c$ E. None of the above

SOLUTION

$\int x\ln x\, dx$

using integration by parts

let $u = \ln x$, $dv = x$ such that

$du = \frac{1}{x}$ and $v = \frac{x^2}{2}$

By parts: $\int u\,dv = uv - \int v\,du$

$\Rightarrow \int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

$= \frac{x^2}{2}(\ln x - 1/2) + c$ (E)

5. Evaluate $\int \cos^2 x\, dx$ A. $\frac{x}{2} - \frac{\sin 2x}{4} + c$ B. $\frac{x}{2} + \frac{\sin 2x}{2} + c$ C. $\frac{x}{2} + \frac{\sin 2x}{4} + c$ D. $\frac{x}{2} - \frac{\cos 2x}{4} + c$ E. None of the above

SOLUTION

$\int \cos^2 x\, dx$

using Trignometric identity

$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{1}{2}(\cos 2x + 1)$

$\Rightarrow \int \cos^2 x\, dx = \int \frac{1}{2}(\cos 2x + 1)dx$

$= \frac{1}{2}\int(\cos 2x + 1)dx$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + c$

$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$ (C)

$= \frac{x}{2} + \frac{\sin 2x}{4} + c$

19. Obtain the integral $\int x^2(x^3+7)^4\, dx$ A. $\frac{(x^3+7)^5}{15} + K$ B. $\frac{(x^3+7x^2)^5}{5} + K$ C. $\frac{x^2(x^3+7)^5}{15} + K$ D. $\frac{x^2(x^3+7)^5}{8} + K$ E. $\frac{(x^3+7x^2)^5}{4} + K$

SOLUTION

$\int x^2(x^3+7)^4 dx$

let $u = x^3 + 7$; $\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$; substituting

$\int x^2(x^3+7)^4 dx = \int x^2 \cdot u^4 \cdot \frac{du}{3x^2}$

$= \frac{1}{3}\int u^4 du = \frac{1}{3}\left(\frac{u^5}{5}\right) + K$

$= \frac{u^5}{15} + K$ (A)

Putting $u = x^3 + 7$

$\Rightarrow \int x^2(x^3+7)^4 dx = \frac{(x^3+7)^5}{15} + K$

31. Evaluate $\int \frac{7+x-2x^2}{2-x}dx$ A. $x^2 + 3x - \ln(2-x) + C$ B. $x^2 + 2x - \ln(2+x) + C$ C. $\frac{x^2}{2} + x - \ln(1+x) + C$ D. $\ln(1+x) + C$ E. $\ln(1-x) + C$

SOLUTION

$\int \frac{7+x-2x^2}{2-x}dx = \int \frac{-2x^2+x+7}{-x+2}dx$

By polynomial long division;

$-x+2 \overline{) -2x^2+x+7}$ quotient $2x+3$

$-2x^2 + 4x$

$-3x + 7$

$-3x + 6$

1

$\int \frac{7+x-2x^2}{2-x}dx = \int\left(2x+3 + \frac{1}{2-x}\right)dx$

$= \frac{2x^2}{2} + 3x - \ln(2-x) + c$

$= x^2 + 3x - \ln(2-x) + c$ (A)

47. Evaluate $\int k dx$ where k is a constant A. $0 + C$ B. $K + C$ C. $Kx + C$ D. $1 + C$ E. $x + C$

SOLUTION

$\int k\,dx = k\int 1\,dx$

$= k(x) + c$

$= kx + c$ (C)

48. Evaluate $\int \frac{2x+3}{x^2+3x+6}dx$ A. $\ln(2x+3) + C$ B. $-\ln(2x+3) + C$ C. $\ln(x^2 + 3x + 6) + C$ D. $-\ln(x^2 + 3x + 6) + C$ E. $-\ln(x^2 + 3x + 6)$

SOLUTION

$\int \frac{2x+3}{x^2+3x+6}dx$

let $u = x^2 + 3x + 6$; $\frac{du}{dx} = 2x + 3$

$dx = \frac{du}{2x+3}$

$\int \frac{2x+3}{x^2+3x+6}dx = \int \frac{2x+3}{u}\cdot\frac{du}{2x+3}$

$= \int \frac{1}{u}du = \ln u + c$ (C)

$= \ln(x^2 + 3x + 6) + c$.

49. Evaluate $\int xe^{x^2+3}dx$ A.

SOLUTION

$\int xe^{x^2+3}dx$

let $u = x^2 + 3$; $\frac{du}{dx} = 2x$; $dx = \frac{du}{2x}$

$\Rightarrow \int xe^{x^2+3}dx = \int xe^u \cdot \frac{du}{2x}$

$= \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c$ (C)

$= \frac{1}{2}e^{x^2+3} + c$

50. Given $f(x) = \log_e x$. Evaluate $\int f(x)dx$ A. $\frac{1}{x} + C$ B. e C. $\ln x - x + C$ D. $x\ln x - x + C$ E. $\ln x + x + C$

SOLUTION

$F(x) = \log_e x = \ln x$

$\int 1 \cdot \ln x\, dx$

let $u = \ln x$, $dv = 1$; $du = \frac{1}{x}$, $v = x$

Applying integration by parts

$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}$

$= x\ln x - \int 1$ (D)

$= x\ln x - x + c$

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STANDARD DERIVATIVES

Derivative	Integral (Antiderivative)
$\frac{d}{dx}n = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x\,dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \tan x \sec x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \cot x \csc x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\operatorname{arc\,cot} x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2}\,dx = \operatorname{arc\,cot} x + C$
$\frac{d}{dx}\operatorname{arc\,sec} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arc\,sec} x + C$
$\frac{d}{dx}\operatorname{arc\,csc} x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arc\,csc} x + C$

OBSERVATION: THE INTEGRAL OF A DERIVATIVE OF AFUNCTION IS THE FUNCTION.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} \quad (n \neq -1)$

$\int \frac{dx}{x} = \log x$

$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)$

$\int e^{mx}\,dx = \frac{e^{mx}}{m}$

$\int \sin mx\,dx = -\frac{\cos mx}{m}$

$\int \cos mx\,dx = \frac{\sin mx}{m}$

$\int \sec^2 mx\,dx = \frac{\tan mx}{m}$

$\int \operatorname{cosec}^2 mx\,dx = -\frac{\cot mx}{m}$

$\int \tan x\,dx = \log \sec x$
OR $-\log \cos x$

$\int \cot x\,dx = \log \sin x$
OR $-\log \operatorname{cosec} x$

$\int \sec x\,dx = \log(\operatorname{cosec} x + \tan x)$
OR $\log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$
OR $\log \tan \frac{x}{2}$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$
OR $\log(x + \sqrt{a^2+x^2})$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$
OR $\log(x + \sqrt{x^2-a^2})$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$
OR $\frac{1}{2a}\log\frac{a+x}{a-x} \quad (x^2 < a^2)$

$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\coth^{-1}\frac{x}{a}$
OR $\frac{1}{2a}\log\frac{x-a}{x+a} \quad (x^2 > a^2)$

$\int \sinh x\,dx = \cosh x$

$\int \cosh x\,dx = \sinh x$

$\int \operatorname{sech}^2 x\,dx = \tanh x$

$\int \operatorname{cosech}^2 x\,dx = -\coth x$

$\int \tanh x\,dx = \log \cosh x$
OR $-\log \operatorname{sech} x$

$\int \coth x\,dx = \log \sinh x$
OR $-\log \operatorname{cosech} x$

$\int \operatorname{sech} x\,dx = 2\tan^{-1}\left(\tanh\frac{x}{2}\right)$
OR $\tan^{-1}(\sinh x)$

$\int \operatorname{cosech} x\,dx = -\log(\coth x + \operatorname{cosech} x)$
OR $\log(\coth x - \operatorname{cosech} x)$
OR $\log\left(\tanh\frac{x}{2}\right)$

(NOTE : the arbitrary constant of integration has been omitted throughout)

THIS BOOK WAS CAREFULLY WRITTEN, COMPILED AND ARRANGED BY
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