Quantum States of Matter and Radiation - Superfluids and Superconductors

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a.y. 2021-2022

Contents

Chapter 1

Superconductors

Def 1 (Superconductors) *Superconductors are materials for which there exist a critical temperature T_c, below which the resistivity drops to* 0 *(fig. 1.1).*

Superconductivity was first observed in 1911.

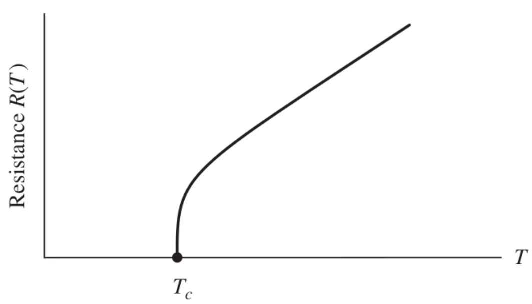


Figure 1.1: Superconductor, mercury, $T_c = 4.3K$.

1.1 Recap of electromagnetism

$\vec{H}$ is the external magnetic field, $\vec{B}$ is the magnetic induction field and $\vec{M}$ is the magnetization field. They are related as:

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \tag{1.1}$$

1.2 Meissner effect

Def 2 (Meissner effect) *Below the critical temperature T_c, superconductors expel completely the magnetic field from their inside (provided H not to strong).*

Superconductors can be thought as perfect diamagnet.

The external magnetic field is $\vec{H} \neq 0$ but the total magnetic induction field inside the superconductor is $\vec{B} = 0$, due to magnetization $\vec{M}$ produced by the screening currents. Superconductivity is destroyed for $H > H_c(T)$. The critical magnetic field $H_c(T)$ vanishes for $T \to T_c$: $H_c(T)$ is maximum for $T = 0$, $H_c(T) = 0$ for $T \geq T_c$ since there is no Meissner effect and screening currents (fig. 1.3).

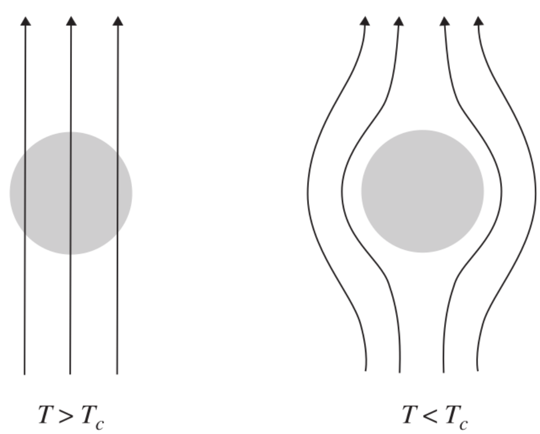


Figure 1.2: The Meissner effect. Magnetic field is expelled when the system is superconducting for $T \leq T_c$.

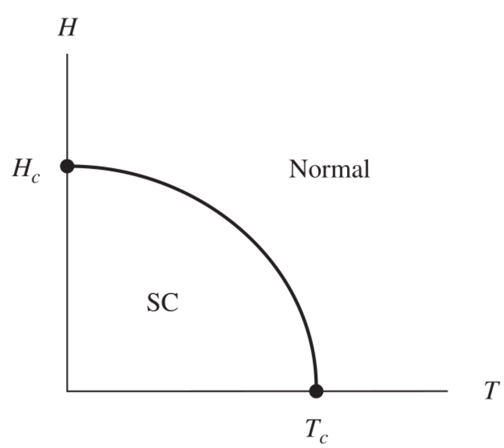


Figure 1.3: Phase diagram of a type-I superconductor. Temperature dependence of the critical field $H_c(T)$ for a superconductor like mercury in the $H - T$ plane.

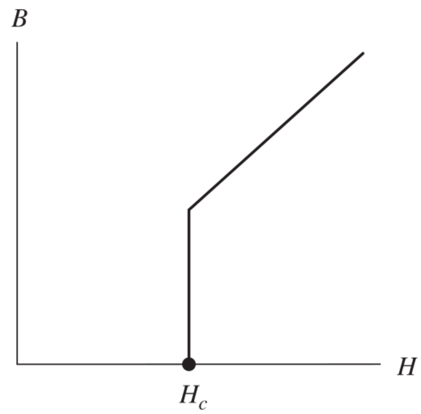


Figure 1.4: The magnetic induction field $\vec{B}$ in a superconductor like mercury.

Set a reference frame s.t. the origin $x = 0$ is on the superconductor surface and the x axis points towards the center of the superconductor. Thus, x is the distance from the surface of the superconductor. The penetration of the magnetic field is given by:

$$B(x) = B_0 \; e^{-\frac{x}{\lambda}} \tag{1.2}$$

where λ is the decay length. Experimentally λ is:

$$\lambda_{exp}^2 = \frac{\epsilon_0 m_e c^2}{e^2 n_s} \tag{1.3}$$

where n_s is the density of superconducting carriers.

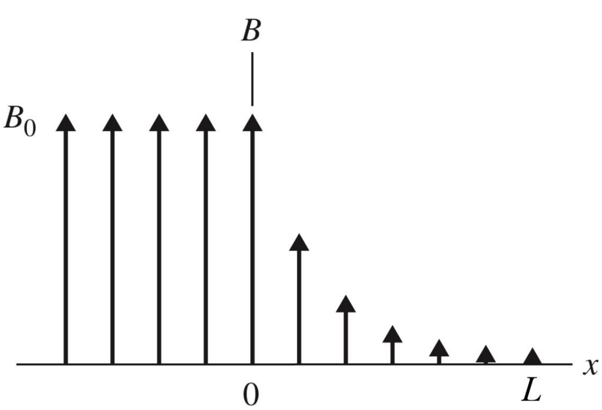


Figure 1.5: Penetration of a magnetic field into a superconductor. $x = 0$ is the metal surface. The decay length is λ.

1.3 Two fluid model for superconductors

We can model the superconductor as a fluid of charged particles. Below T_c the fluid separates in 2 components:

1. A normal component with particle density ρ_n
2. A superfluid component with particle density ρ_s

$$\rho = \rho_n + \rho_s \tag{1.4}$$

Clearly $\rho_s < \rho$.

All current is carried by ρ_s, the normal component will remain at rest. Only ρ_s is able to screen $\vec{H}$. By measuring λ we measure the fraction of electrons which contribute to the screening.

1.4 Flux quantization

Consider a ring shaped superconductor with a supercurrent flowing around the ring, i.e. along the dotted line in fig. 1.6 . S is the surface enclosed by the ring. C is a ring well inside the superconductor. Inside the superconductor $B = 0$.

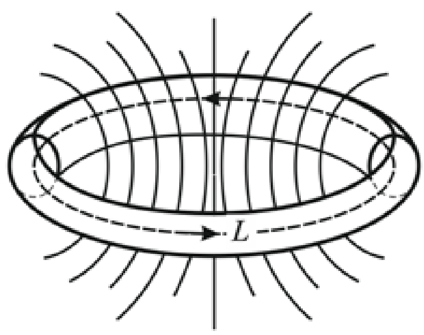


Figure 1.6: Ring shaped superconductor with a supercurrent flowing.

We observe experimentally (fig. 1.7) that the flux of the magnetic field on the surface S is quantized:

$$\Phi_S(\vec{B}) = \int_S \vec{B} \cdot d\vec{S} = n\phi_0 \qquad n \in \mathbb{Z} \tag{1.5}$$

The quantum of flux is:

$$\phi_0 = \frac{h}{2e} \tag{1.6}$$

The factor 2 suggests that we should consider pairs of electrons.

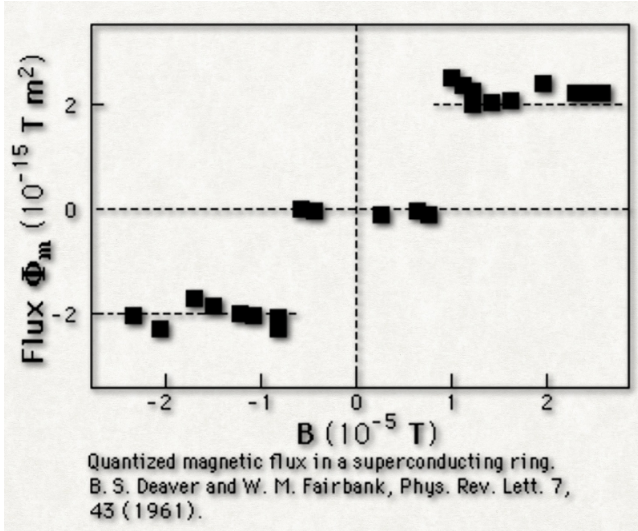


Figure 1.7: Quantized magnetic flux in a superconducting ring.

1.5 Types of superconductors

There are 2 types of superconductors, while there is only 1 type of superfluid.

Def 3 (Type 1) *A type-I superconductor has a single critical magnetic field in its phase diagram (fig. 1.8a).*

Def 4 (Type 2) *A type-II superconductor is a superconductor that exhibits an intermediate phase of mixed ordinary and superconducting properties at intermediate temperature and fields above the superconducting phases (fig. 1.8b).*

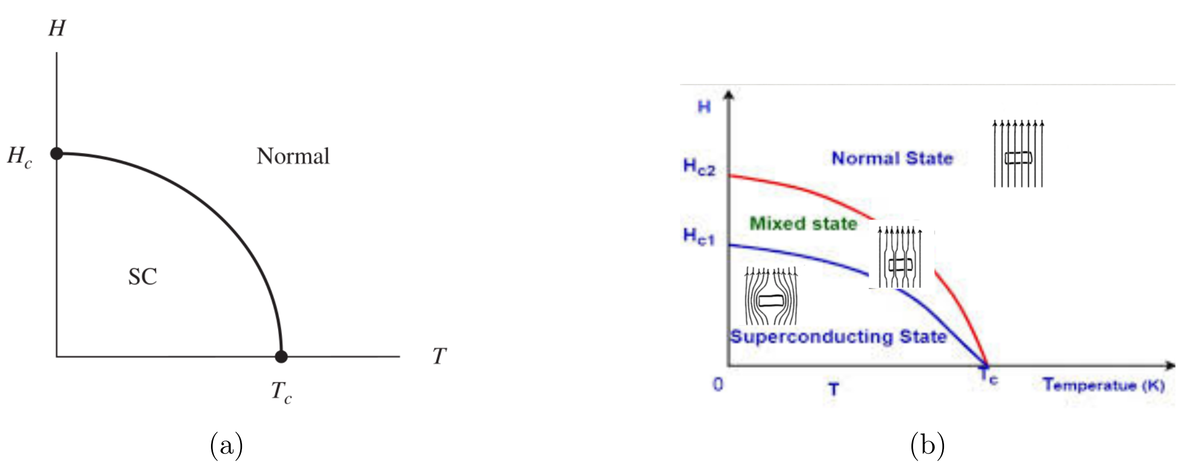


Figure 1.8: Phase diagram of superconductor of type-I (a) and of type-II (b).

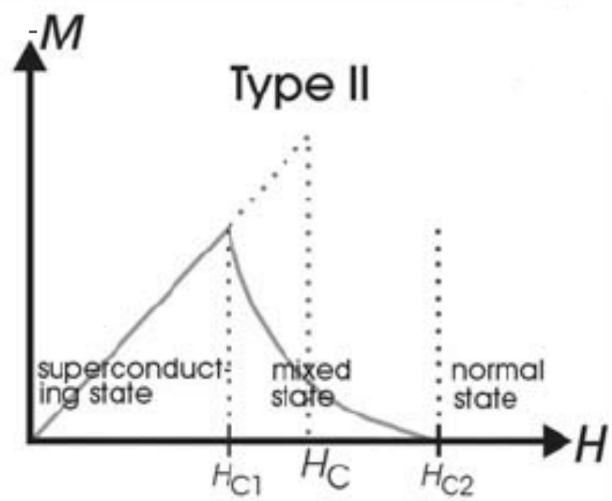


Figure 1.9: Magnetization field $-M$ in type-II superconductor.

Properties of a type-II superconductor

- A type-II superconductor has 2 critical values of magnetic field: H_{c_1}, H_{c_2}.
- Below H_{c_2} there is superconductivity (zero resistance).
- Only below H_{c_1} there is a complete Meissner effect. $B = 0$ in the bulk of the superconductor.

In the mixed state, the magnetic field penetrates across the superconductor in flux tubes. Flux tubes organize in a triangular lattice. Inside the flux tubes we observe normal behavior, outside superconductive behavior.

Around the core of each flux tube there is a current, similarly to a vortex in a fluid. Flux tubes are called vortices (fig. 1.10). Each flux tube carries a single quantum of flux ϕ_0.

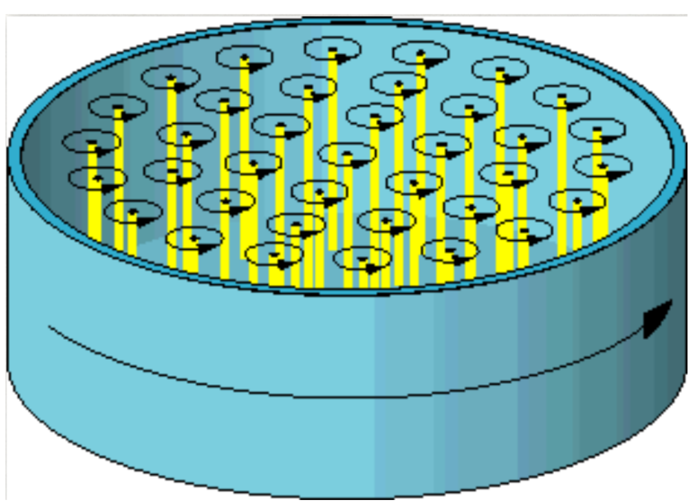


Figure 1.10: Vortices.

1.6 Specific heat of superconductors

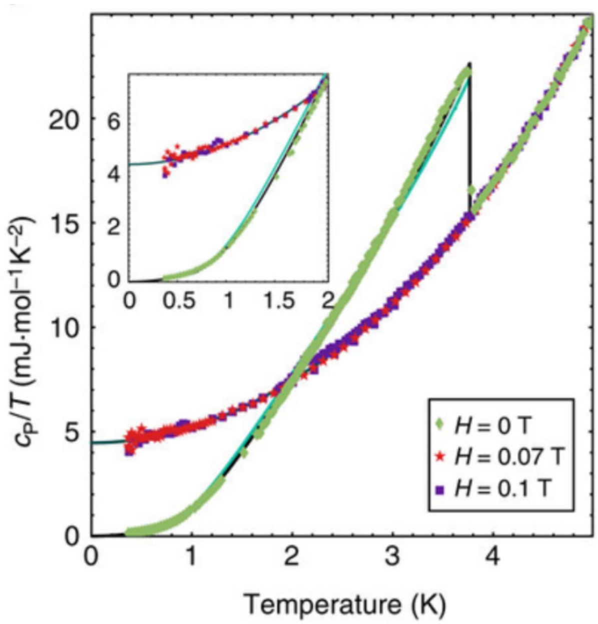


Figure 1.11: Specific heat c_P of a superconductor as a function of temperature for 3 different values of external magnetic field H.

The specific heat c_P of a superconductor material as a function of temperature is shown in fig. 1.11 for 3 different values of external magnetic field H . For $H = 0$ the material behaves as a superconductor for $0 < T \leq T_c$; at $T = T_c$ specific heat c_P exhibits a discontinuity (green line). For $H > H_c$, independently from the value of H , the material exhibits a continuously increasing behaviour (blue and red lines).

For $T \geq T_c$, independently from the value of H , the material has no more superconductive behaviour and the 3 lines coincide.

When should we expect an exponential behavior of the specifc heat c_p ?

$$c_P \sim e^{-\frac{\Delta}{k_B T}} \tag{1.7}$$

$$< E > \sim \frac{\sum_n E_n e^{-\frac{E_n}{k_B T}}}{Z} \tag{1.8}$$

If $\Delta := E_1 - E_0$ is the gap between *ground state* and 1^{st} *excited state*, at low temperature T :

$$< E > \sim E_0 + E_1 e^{-\frac{\Delta}{k_B T}} \tag{1.9}$$

The finite gap Δ causes the *exponential behavior* of the specifc heat c_P .

1.7 Macroscopic wavefunction

We can explain many properties of superconductors and superfluids in terms of macroscopic occupation of the same single particle wavefunction $\psi_0(\vec{x})$.

For a single particle whose state is described by the wavefunction $\psi_0(\vec{x})$, the probability of finding the particle in a small volume of space $d\vec{x}$ around $\vec{x}$ is:

$$p(\vec{x}) = |\psi_0(\vec{x})|^2 \tag{1.10}$$

and it is normalized:

$$\int d^3x \; |\psi_0(\vec{x})|^2 = 1 \tag{1.11}$$

We assume that a macroscopic number of particles are described each by the same wavefunction $\psi_0(\vec{x})$.

So in the case of N_0 particles we define a total wavefunction $\psi(\vec{x})$ as:

$$\psi(\vec{x}) = \sqrt{N_0} \; \psi_0(\vec{x}) \tag{1.12}$$

in such a way that the normalization of ψ gives the total number of particles N_0:

$$\int d^3x \; |\psi(\vec{x})|^2 = N_0 \tag{1.13}$$

The quantity $n_0(\vec{x})$ defined as:

$$n_0(\vec{x}) \coloneqq |\psi(\vec{x})|^2 = N_0 \; |\psi_0(\vec{x})|^2 \tag{1.14}$$

represents the density of particles in a small volume $d^3\vec{x}$ around $\vec{x}$. $n_0(\vec{x})$ is a density of particles in the sense that is a number of particles per unit volume, but has also a QM-probabilistic interpretation by this meaning that n_0 is the QM-statistical average number of particles we measure in a small volume $d^3\vec{x}$ around $\vec{x}$.

Let's calculate the current of probability $\vec{j}(\vec{x})$:

$$\vec{j}(\vec{x}) = \frac{1}{2}[\psi^*(\vec{x})\frac{\hat{\vec{p}}}{m}\psi(\vec{x}) + \psi(\vec{x})\frac{\hat{\vec{p}}}{m}\psi^*(\vec{x})] \tag{1.15}$$

and remembering that $\hat{\vec{p}} = -i\hbar\vec{\nabla}$:

$$\vec{j}(\vec{x}) = \frac{\hbar}{m} \; \mathrm{Im} \; [\psi^*(\vec{x})\vec{\nabla}\psi(\vec{x})] \tag{1.16}$$

From eq. 1.14, I can rewrite $\psi(\vec{x})$ as:

$$\psi(\vec{x}) = \sqrt{n_0(\vec{x})}\ e^{i\theta(\vec{x})} \tag{1.17}$$

Then, the current becomes:

$$\begin{aligned}\vec{j}(\vec{x}) &= \frac{\hbar}{m}\ \mathrm{Im}\ [\sqrt{n_0(\vec{x})}\ e^{-i\theta(\vec{x})}\vec{\nabla}(\sqrt{n_0(\vec{x})}\ e^{i\theta(\vec{x})})\ ] \\ &= \frac{\hbar}{m}\ \mathrm{Im}\ [\sqrt{n_0(\vec{x})}\ \vec{\nabla}\sqrt{n_0(\vec{x})} + i\ n_0(\vec{x})\vec{\nabla}\theta\ ]\end{aligned} \tag{1.18}$$

Since $\sqrt{n_0(\vec{x})}\ \vec{\nabla}\sqrt{n_0(\vec{x})}$ is real:

$$\vec{j}(\vec{x}) = n_0(\vec{x})\ \frac{\hbar}{m}\vec{\nabla}\theta(\vec{x}) \tag{1.19}$$

We define the superfluid velocity:

$$\vec{v}_s(\vec{x}) = \frac{\hbar}{m}\vec{\nabla}\theta(\vec{x}) \tag{1.20}$$

Thus, the final expression for the probability current is:

$$\vec{j}(\vec{x}) = n_0(\vec{x})\ \vec{v}_s(\vec{x}) \tag{1.21}$$

1.8 Meissner effect - explanation

Basic assumption: charge carriers of charge q and mass m^* are described by the same single particle wavefunction, this implies that the carriers of charge must be *bosons*.

Charge density:

$$\rho_s(\vec{x}) = q|\psi(\vec{x})|^2 = q\ n_0(\vec{x}) \tag{1.22}$$

Charge current for $\vec{A} = 0$:

$$\vec{j}(\vec{x}) = \frac{q}{2}[\psi^*(\vec{x})\frac{\hat{\vec{p}}}{m^*}\psi(\vec{x}) + \psi(\vec{x})\frac{\hat{\vec{p}}}{m^*}\psi^*(\vec{x})] \tag{1.23}$$

Charge current for $\vec{A} \neq 0$:

$$\vec{j}(\vec{x}) = \frac{q}{2}[\psi^*(\vec{x})\frac{(\hat{\vec{p}} - q\vec{A})}{m^*}\psi(\vec{x}) + \psi(\vec{x})\frac{(\hat{\vec{p}} - q\vec{A})}{m^*}\psi^*(\vec{x})] \tag{1.24}$$

We have that:

$$\vec{j}(\vec{x}) = \frac{q\hbar}{m^*}\ \text{Im}\ (\psi^*\vec{\nabla}\psi) - \frac{q^2}{m^*}|\psi|^2\vec{A} \tag{1.25}$$

using the expression eq. 1.17 for ψ and identifying $n_s^* \equiv n_0 = |\psi|^2$[1] (the first part of eq. 1.25 comes from eq. 1.19), the current becomes:

$$\vec{j}(\vec{x}) = q\ n_s^*(\vec{x})\ \left(\frac{\hbar}{m^*}\vec{\nabla}\theta - \frac{q}{m^*}\vec{A}\right) = q\ n_s^*\ \vec{v}_s(\vec{x}) \tag{1.26}$$

where we defined the superfluid[2] velocity $\vec{v}_s$ as:

$$\vec{v}_s(\vec{x}) = \frac{\hbar}{m^*}\vec{\nabla}\theta - \frac{q}{m^*}\vec{A} \tag{1.27}$$

Inside the superconductor at equilibrium we have $\frac{\partial\vec{E}}{\partial t} = 0$ so:

$$\vec{\nabla} \times \vec{B} = \mu_0\vec{j} \tag{1.28}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0\vec{\nabla} \times \vec{j} \tag{1.29}$$

Recall:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2\vec{B} = -\nabla^2\vec{B} \tag{1.30}$$

and using that $\vec{\nabla} \cdot \vec{B} = 0$, we get:

$$\begin{aligned}
\nabla^2\vec{B} &= -\mu_0\vec{\nabla} \times \vec{j} \\
&= -\mu_0\vec{\nabla} \times q\ n_s^*(\vec{x})\ \left(\frac{\hbar}{m^*}\vec{\nabla}\theta - \frac{q}{m^*}\vec{A}\right) \\
&= \mu_0\frac{q^2}{m^*}n_s^*\ \vec{B}
\end{aligned} \tag{1.31}$$

where we used the fact that $\vec{\nabla} \times \vec{\nabla}\theta = 0$ and $\vec{B} = \vec{\nabla} \times \vec{A}$.

[1] the substitution of n_0 with n_s^* is made for generality. We will see that n_s^* will be the density of electron *pairs*.

[2] *superfluid* in the context of the *superfluid model*.

Set a reference frame s.t. the origin $x = 0$ is on the superconductor surface and the x axis points towards the center of the superconductor. Thus, x is the distance from the surface of the superconductor.

$$\frac{\mathrm{d}^2 B(x)}{\mathrm{d}x^2} = \mu_0 \frac{q^2}{m^*} n_s^* \, B(x) = \frac{B(x)}{\lambda^2} \tag{1.32}$$

$$B(x) = B_{ext} \; e^{-\frac{x}{\lambda}} \tag{1.33}$$

The only acceptable solution is $B(x)$ decays exponentially with length λ, at distance x from the surface of the superconductor. λ is called *decay lenght* and is given by:

$$\lambda^{-2} = \frac{\mu_0 q^2 n_s^*}{m^*} \tag{1.34}$$

The model predicts that B decreases exponentially going from the surface of the superconductor to the center which agrees with the fact that experimentally $\vec{B} = 0$ in the bulk of the superconductor.

Let us compare λ with λ_{exp} :

$$\lambda_{exp}^{-2} = \frac{\mu_0 e^2 n_s^e}{m_e} \tag{1.35}$$

We get $q = e$, $m^* = m_e$ and $n_s^* = n_s^e$, being n_s^e the density of superconducting electrons.

This is in contrast with the fact that carriers of charge are pairs.

An alternative solution is obtained by assuming that charge carriers are pairs:

$$q = 2e \qquad m^* = 2m_e \qquad n_s^* = \frac{n_s^e}{2} \tag{1.36}$$

being $n_s^* = \frac{n_s^e}{2}$ the density of superconducting electron *pairs*.

1.9 Flux quantization - explanation

Consider again a ring shaped superconductor with a supercurrent flowing (fig. 1.6). S is the surface enclosed by the ring. C is a ring well inside the superconductor. Inside the superconductor $B = 0$ due to Meissner effect.

The flux $\Phi_S(\vec{B})$ of the magnetic field on the surface S is:

$$\Phi_S(\vec{B}) = \int_S \vec{B} \cdot d\vec{S} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l} \tag{1.37}$$

From the stationary Ampere-Maxwell eq. 1.28:

$$\vec{B} = 0 \text{ along } C \quad \Rightarrow \quad \vec{j} = 0 \tag{1.38}$$

Note that currents in the superconductor are limited to the region of width λ where $\vec{B} \neq 0$.

If $\vec{j} = 0$:

$$\vec{v}_s(\vec{x}) = \frac{\hbar}{m^*}\vec{\nabla}\theta - \frac{q}{m^*}\vec{A} = 0 \quad \Rightarrow \quad \hbar\vec{\nabla}\theta = q\vec{A} \tag{1.39}$$

so (why is the phase of ψ the angle of the ring?):

$$\oint_C \vec{A} \cdot d\vec{l} = \frac{\hbar}{q} \oint_C \vec{\nabla}\theta(\vec{x}) \cdot d\vec{l} = \frac{h}{q} n \tag{1.40}$$

so the flux of the magnetic field is:

$$\Phi_S(\vec{B}) = \int_S \vec{B} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l} = \frac{h}{q} n \tag{1.41}$$

Experiments tell us:

$$\Phi_S(\vec{B}) = \frac{h}{2e} n \quad \Rightarrow \quad q = 2e \tag{1.42}$$

i.e. we have *pairs* of electrons. So the wavefunction $\psi(\vec{x})$ in a superconductor describes the single particle wavefunction of a *pair* of electrons, which realises a *boson*.

Below T_c , pairs of electrons called *Cooper pairs* undergo *Bose-Einstein condensation* (BEC).

Let Δ be the *binding energy* of each pair. Then, for low temperature T:

$$c_V \sim e^{-\frac{\Delta}{k_B T}} \tag{1.43}$$

because there is an *energy gap* between the *ground state* and 1^{st} *excited state*. This is consistent with experiments.

How to have attraction between electrons (of a Cooper pair)?

Coulomb repulsion is screened in a metal by redistribution of charge around a single negative charge:

?on the average, depletion of the negative charge of the single electron.?

Retarded attraction between electrons:

1. First, electron produces a cloud of positive charge

2. then, the electron moves away and leaves a cloud of positive charge

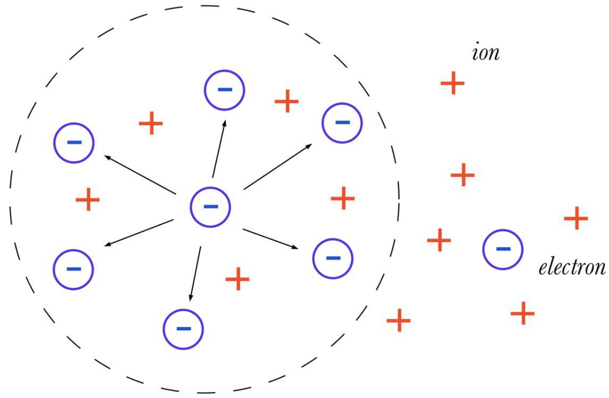


Figure 1.12: Positive ions screen the Coulomb repulsion between electrons.

3. another electron is attracted by the positive charge produced by the 1^{st} electron

so globally the first electron attracts another electron.

Balance between screened Coulomb interaction and retarded attraction produces a *small effective attraction* between electrons: so they form *Cooper pairs*.
The binding energy Δ is small.

If $k_B T > \Delta$, pairs will be broken by the *thermal energy*. This is why in standard superconductors T_c is very small.

It's possible to prove (see eq. 2.25) that for a type-II superconductor in an external magnetic field $\vec{H}$ s.t. $H_{c_1} < H < H_{c_2}$:

$$n_v = \frac{2e}{h} \frac{H}{\mu_0} = \frac{2e}{h} B_{ext} \tag{1.44}$$

being n_v the superficial density of vortices, i.e. the number of vortices per unit area.

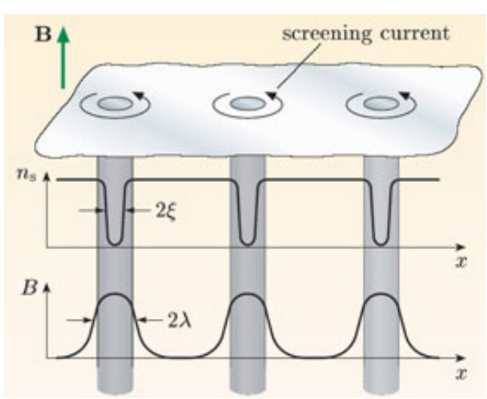


Figure 1.13: Density n_s of superconductor carriers and magnetic field B inside the superconductor.

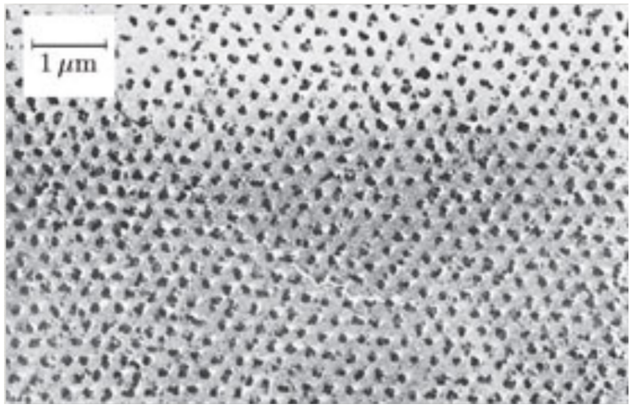


Figure 1.14: Vortices in a superconductor.

1.10 Feynman argument

How to explain zero resistance and zero friction with the single particle wavefunction assumption?

Superfluid flow is seen as a condensate of bosons with velocity $\vec{v}_s$. There is macroscopic occupation of the momentum state:

$$\vec{p_0} = m\vec{v}_s \tag{1.45}$$

All bosons have the same momentum.

Dissipation by friction requires some bosons to be scattered in a different momentum state.

$$W(\vec{p}_0 \to \vec{p}') \propto \left| \langle N_{p_0} - 1, N_{p'} + 1 | \, \hat{T} \, | N_{p_0}, N_{p'} \rangle \right|^2 \propto N_{p_0}(N_{p'} + 1) \; T_{p_0,p} \tag{1.46}$$

For $\vec{p}' = \vec{p}_0$:

$$W(\vec{p}_0 \to \vec{p}_0) \propto N_{p_0}^2 T \tag{1.47}$$

We have a macroscopic number of bosons in the state $\vec{p}_0$ ($\approx 10^{23}$) while a number of order 1 in a state $\vec{p}' \neq \vec{p}_0$.

$$\frac{W(\vec{p}_0 \to \vec{p}_0)}{W(\vec{p}_0 \to \vec{p}')} \propto N_{p_0} \approx 10^{23} \tag{1.48}$$

Zero probability of scattering to $\vec{p}' \implies$ no dissipation.

Chapter 2

Superfluids

Def 5 (Superfluids) *Superfluids are materials for which there exist a critical temperature T_c, below which the viscosity drops to* 0.

In a normal fluid, we consider the flow of a fluid in a narrow pipe. There is friction with the walls of the pipe. We observe a drop of the pressure along the flow of the fluid.

$$\Delta P = P_2 - P_1 \propto \frac{Lv}{R^2} \tag{2.1}$$

$$\Delta P = \eta \frac{Lv}{R^2} \tag{2.2}$$

Where L is the lenght of the tube, R is the radius, v is the velocity and η is the viscosity.

In a superfluid flowing in a narrow capillary, we observe $\Delta P = 0$ below T_c, so $\eta = 0$.

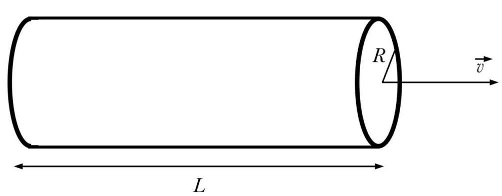


Figure 2.1: Fluid in a pipe.

2.1 Andronikashvili experiment

Consider a torsion pendulum formed by two closely spaced disks on top of each other and immersed in a superfluid. The period of oscillation T is:

$$T = \frac{1}{2\pi}\sqrt{\frac{I_{tot}}{k}} \tag{2.3}$$

where I_{tot} is the total momentum of inertia of the system of the two disks and of the fluid between them:

$$I_{tot} = I_{disks} + I_{fluid} \tag{2.4}$$

I_{fluid} is the momentum of inertia of the fluid which is dragged by the friction with the disks. I_0 is the maximum momentum of inertia of the fluid which corresponds to having no superfluid component. Thus, $I_{fluid} \xrightarrow{T \to T_c} I_0$.

For T approaching 0, the superfluid component grows so the momentum of inertia of the dragged fluid I_{fluid} decreases.

Analougosly with the superconductor, we define a density of superfluid particles ρ_s i.e. the number of superfluid particles per unit volume. ρ is the density of fluid particles i.e. the total number of particles per unit volume (total meaning that we consider both the normal and superfluid components).

$$\frac{\rho_s}{\rho} = 1 - \frac{I_{fluid}}{I_0} \tag{2.5}$$

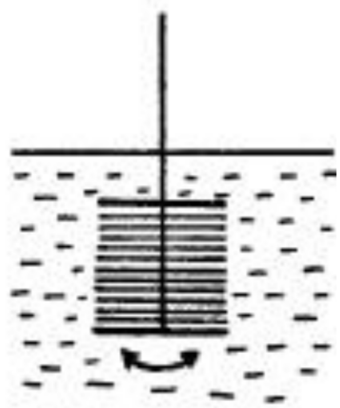


Figure 2.2: Closely spaced disks immersed in a fluid with superfluid behavior.

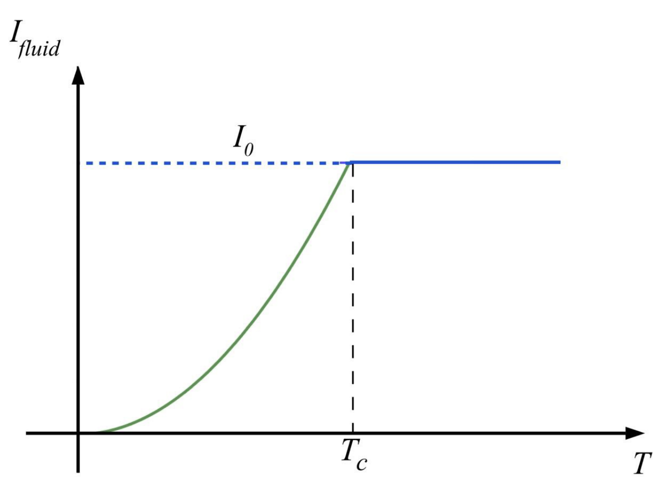


Figure 2.3: Momentum of inertia of the dragged fluid I_{fluid} as a function of T.

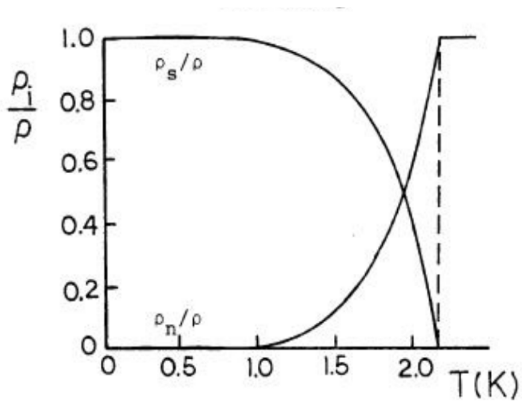


Figure 2.4: The decreasing curve is the ratio between the superfluid particle density and total particle density $\frac{\rho_s(T)}{\rho}$ and the increasing curve is the ratio between the normal particle density and total particle density $\frac{\rho_n(T)}{\rho}$.

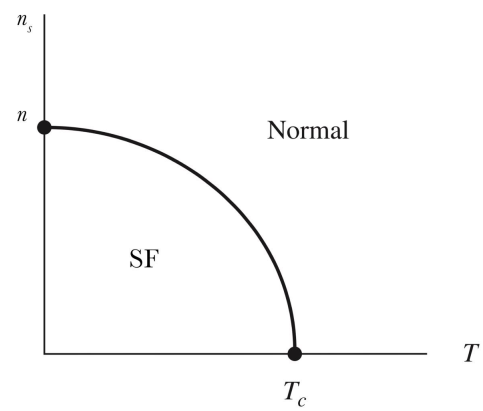


Figure 2.5: Density of superfluid n_s as a function of T. n_s is a continuous function.

2.2 Two fluid model

Below T_c the fluid separates in 2 components:

1. A normal component with particle density ρ_n
2. A superfluid component with particle density ρ_s

$$\rho = \rho_n + \rho_s \tag{2.6}$$

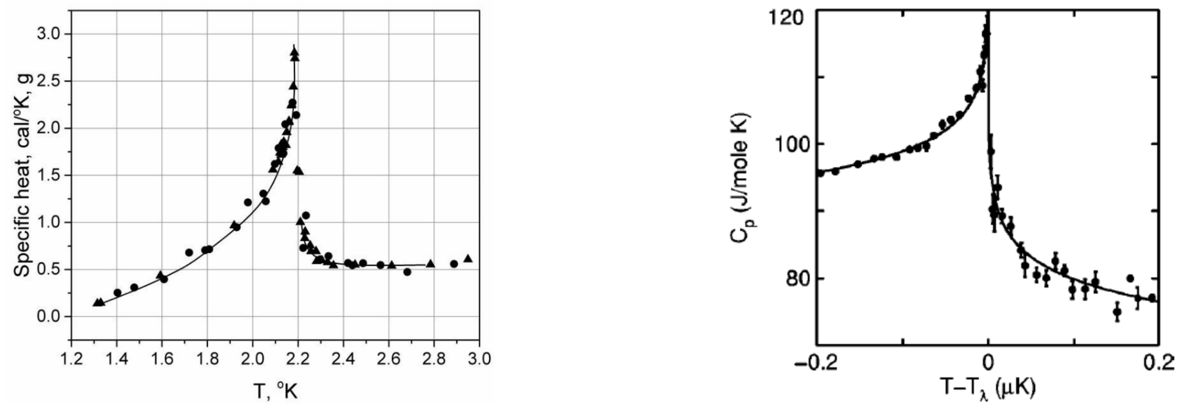
Clearly $\rho_s < \rho$.

In a flow through a capillary, all flow is carried by the superfluid component with particle density ρ_s, the normal component will remain at rest. In a drag experiment we measure only the normal component which is dragged.

$\rho_s = n_s$

2.3 Specific heat of a superfluid

Specific heat c_P of a superfluid as a function of temperature T exibits a *lambda transition* in corrispondence of T_λ, i.e. $c_P(T)$ has a cuspid for $T = T_\lambda$.



(a) (b)

Figure 2.6: Specific heat c_P of the superfluid ^4He as a function of temperature T .

$$c_p \propto |T - T_\lambda|^{0,0127} \tag{2.7}$$

2.4 Quantization of circulation

Consider a ring shaped pipe:

$$\oint_C \vec{v}_s \cdot d\vec{l} = n\frac{h}{m} \tag{2.8}$$

being $\vec{v}_s$ the local[1] velocity of the fluid and m the mass of the particles of the fluid.

Proof:

As in the case of superconductors, the velocity $\vec{v}_s$ is given by eq. (1.20):

$$\vec{v}_s(\vec{x}) = \frac{\hbar}{m}\vec{\nabla}\theta(\vec{x})$$

so, the circulation is:

$$\oint_C \vec{v}_s \cdot d\vec{l} = \frac{\hbar}{m}\oint_C \vec{\nabla}\theta(\vec{x}) \cdot d\vec{l} \tag{2.9}$$

[1] $\vec{v}_s$ is local in the sense that in a fixed (stationary) reference frame, the velocity $\vec{v}_s$ depends on the geometric point $\vec{x}$ of the reference frame and the time t: $\vec{v}_s = \vec{v}_s(\vec{x}, t)$.

but:

$$\oint_A^B \vec{\nabla}\theta(\vec{x}) \cdot d\vec{l} = \theta(\vec{x}_B) - \theta(\vec{x}_A) = \Delta\theta_{AB} \tag{2.10}$$

and for a closed loop $\vec{x}_B = \vec{x}_A$ and since $\theta(\vec{x})$ appears in the phase of the wavefunction $\psi(\vec{x})$ 1.17, which must be a *single-valued function*, the values $\theta(\vec{x}_A)$ and $\theta(\vec{x}_B)$ can differ only by an integer multiple of 2π:

$$\theta(\vec{x}_B) - \theta(\vec{x}_A) = 2\pi n \tag{2.11}$$

In the end:

$$\oint_C \vec{v}_s \cdot d\vec{l} = \frac{\hbar}{m} 2\pi n = n\frac{h}{m} \tag{2.12}$$

For vortices:

$$\vec{v}_s(\vec{x}) = \frac{\hbar}{m}\vec{\nabla}\theta(\vec{x}) \quad \Rightarrow \quad \vec{\nabla} \times \vec{v}_s(\vec{x}) = \frac{\hbar}{m}\vec{\nabla} \times \vec{\nabla}\theta(\vec{x}) = 0 \tag{2.13}$$

since the curl of a gradient is 0. So the velocity field $\vec{v}_s(\vec{x})$ is *irrotational.*

The fact that $\vec{v}_s(\vec{x})$ is irrotational implies that, for paths which enclose simply connected regions of space where the system is superfluid, the integral $\oint_C \vec{v}_s \cdot d\vec{l} = 0$ using Stokes theorem.

figura

In the core of the vortex the density of superfluid $n_s = 0$. If we compute the circulation around a vortex in the region outside of it, we will not get necessarily zero since the region of superfluid enclosed by the path is not simply connected.

For the same reasons as the ring:

$$\oint_C \vec{v}_s \cdot d\vec{l} = n\frac{h}{m} \tag{2.14}$$

We observe that $n = \pm 1$ because vortices with $|n| > 1$ cost too much energy. It's convenient to split them.

figura

What is the angular momentum associated to a single vortex?

$$\vec{\omega} = \omega\hat{z} \qquad \vec{L} = L_z\hat{z} \tag{2.15}$$

$$L_z = \int n_s(r) m r v_s(r) \; dV \tag{2.16}$$

from eq. 2.14, the velocity $v_s(r)$ as function of the radius r :

$$v_s(r) = \frac{\hbar}{mr} \tag{2.17}$$

$$L_z = \hbar \int n_s(r) \; dV = \hbar N_s \tag{2.18}$$

L_z is a small momentum.

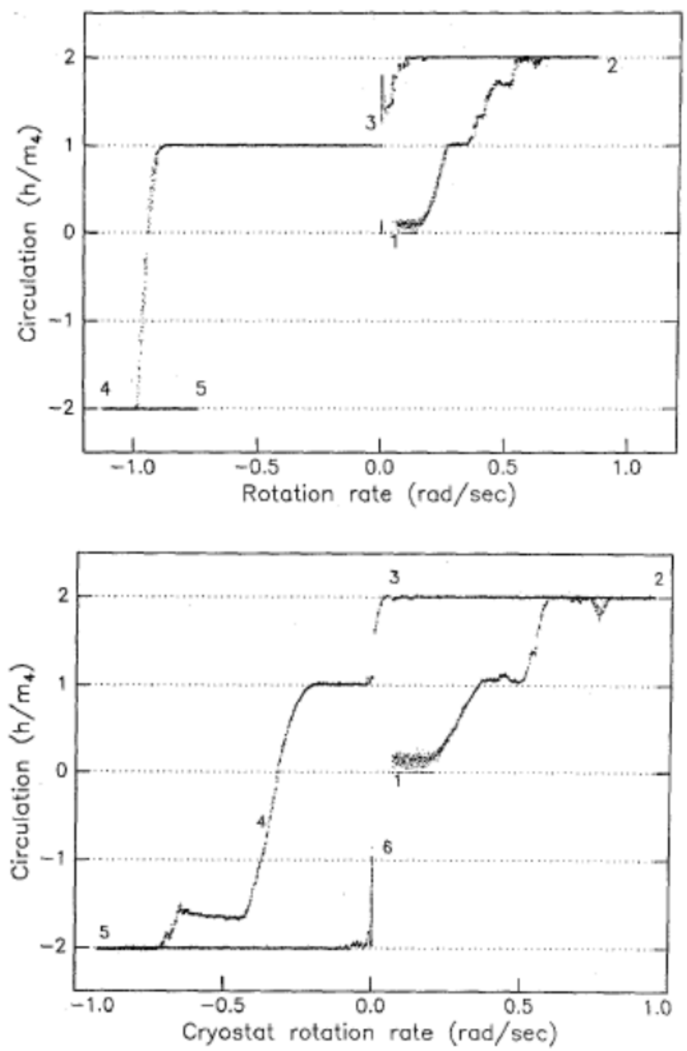


Fig. 7. Circulation versus rotation rate. The numbers in each plot show the time sequence in which the data was taken. Such hysteresis is usually observed.

Figure 2.7: Experimental observation of quantization of circulation.

2.5 Experimental observation of a superfluid in a rotating bucket

We consider a bucket filled with ^4He superfluid. The bucket rotates at angular speed ω. For ω not too small the superfluid apparently behaves as a normal fluid with the typical paraboloid shape of the surface.

Experiments suggest that the velocity $v(r)$ at distance r from the rotation axis should be given by the same expression as in classical physics:

$$v(r) = \omega\ r \tag{2.19}$$

Let us integrate along a circle around the axis circulation:

$$I(r) = \oint \vec{v} \cdot \vec{dl} = \int_0^{2\pi} v(r)\ r\ d\phi = 2\pi\omega r^2 \tag{2.20}$$

We should obtain the same result by summing the circulation of all vortices contained within the circle:

$$C(r) = N_v(r)\ \frac{h}{m} \tag{2.21}$$

being $N_v(r)$ the number of vortices inside a circular closed path of radius r. Imposing $I(r) = C(r)$ we get:

$$\frac{N_v}{\pi r^2} = \frac{2m\omega}{h} \tag{2.22}$$

We define the number of vortices per unit area $n_v(r)$ as:

$$n_v(r) = \frac{N_v(r)}{\pi r^2} \tag{2.23}$$

and we find that $n_v(r)$ is independent from r:

$$n_v = \frac{2m\omega}{h} \tag{2.24}$$

There is an analogous formula for a type-II superconductor in an external magnetic field H s.t. $H_{c_1} < H < H_{c_2}$.

$$n_v = \frac{2e}{h}\ \frac{H}{\mu_0} = \frac{2e}{h} B_{ext} \tag{2.25}$$