Quantum States of Matter and Radiation

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Chapter 1

Postulates of QM

1.1 Postulate 1 - States of a quantum system

1. The possible states of a physical system correspond to normalized vectors (rays) of a Hilbert space $\mathcal{H}$.

2. A Hilbert space $\mathcal{H}$ is a complete vector space over $\mathbb{C}$ with a scalar product positive, linear, skew-symmetric.

3. The previous postulate implies the *superposition principle*.

   Superposition principle states that given two states, $|\psi_1\rangle$ and $|\psi_2\rangle$, the normalized state $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$ is also an admissible description of the quantum system under consideration.

4. A state $|\psi\rangle$ is an equivalence class of vectors of $\mathcal{H}$ where we neglect the overall phase: $|\psi\rangle \sim e^{i\alpha} |\psi\rangle$, $\alpha \in \mathbb{R}$.

   The overall phase α cannot be observed.

   A *relative phase* ϑ can instead be observed:

$$|\psi\rangle \;=\; |\psi_1\rangle + e^{i\vartheta} |\psi_2\rangle \tag{1.1}$$

1.2 Postulate 2 - Observables

An *observable* is a *property* of a physical system that in principle *can be measured.*

In QM observables are associated to self-adjoint operators A , for which:

$$A \;=\; A^{+} \tag{1.2}$$

which have *spectral decomposition*:

$$A = \sum_n a_n E_n \qquad a_n \in \mathbb{R} \tag{1.3}$$

where the a_n are the eigenvalues of the operator A and the operators E_n are called *orthogonal projector operators* and are defined by the properties:

$$E_n E_m = \delta_{nm} E_m \tag{1.4}$$

$$E_n{}^+ = E_n \tag{1.5}$$

1.3 Postulate 3 - Measurement

A *measurement* is a process in which information about the state is acquired by an observer.

In QM there are two ingredients for performing a measurement:

1. the state $|\psi\rangle$
2. the observables $A = \sum_n a_n E_n \qquad a_n \in \mathbb{R}$

The post measurement state is determined by combining (1) and (2); the outcomes of the measurement are related to the eigenvalues a_n of A given by (1.3).

The state $|\psi\rangle$ you measure gives the probability of getting a_n as result of the measurement procedure according to the *Born rule*:

$$Prob(a_n) = \langle\psi| E_n |\psi\rangle \tag{1.6}$$

$Prob(a_n)$ is the probability distribution of the possible outcomes.

After the measurement, the state will be:

$$\left|\psi^{PM}\right\rangle = \frac{E_n |\psi\rangle}{\sqrt{\langle\psi| E_n |\psi\rangle}} \tag{1.7}$$

A second measurement of A on the state $\left|\psi^{PM}\right\rangle$ gives a_n with probability 1. For example if I get a_1 in the 1^{st} measurement, I'll get a_1 again in the 2^{nd} measurement.

1.4 Postulate 4 - Dynamics

1. Dynamics describes how a quantum system evolves over time.
2. In QM the evolution of a closed system is described by a unitary operator: U s.t. $U^{+}U = \mathbb{I}$.
3. The Hamiltonian operator H generates the dynamics of a state through the *Schrodinger equation*:

$$\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = -iH\,|\psi(t)\rangle \tag{1.8}$$

$$|\psi(t)\rangle = U(t,0)\,|\psi(t=0)\rangle \qquad\qquad |\psi(t=0)\rangle \;\text{ initial state} \tag{1.9}$$

1.5 Postulate 5 - Composite systems

If the Hilbert space of a system A is $\mathcal{H}_A$ and the Hilbert space of a system B is $\mathcal{H}_B$ then the Hilbert space of the composite system AB is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Example:

If $\mathcal{H}_A$ and $\mathcal{H}_B$ are:

$$\mathcal{H}_A = \mathrm{span}(\{|i\rangle_A\}_{i=0}^{N_A-1}) \tag{1.10}$$

$$\mathcal{H}_B = \mathrm{span}(\{|j\rangle_B\}_{j=0}^{N_B-1}) \tag{1.11}$$

then:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathrm{span}(\;\underbrace{\{|i\rangle_A \otimes |j\rangle_B\}}_{N_A \cdot N_B}\;) \tag{1.12}$$

If O_A is an operator defined on $\mathcal{H}_A$. If we apply O_A to $\mathcal{H}$, it acts as follows:

$$(O_A \otimes \mathbb{I}_B)(|i\rangle_A \otimes |j\rangle_B) = O_A\,|i\rangle_A \otimes |j\rangle_B \tag{1.13}$$

1.6 Remarks

1. The evolution is determined by the Schrödinger equation (linear).
2. There is a mysterious dualism:
   - 2.1. Given an initial state $|\psi(t=0)\rangle$ the theory predicts deterministically the state at later times $t > 0$.
   - 2.2. Measurement is probabilistic and the the theory cannot predict deterministically what the outcome will be.

So, is the measurement process governed by a different type of evolution?

1.7 Example: qubit

A qubit is a 2-level system. The dimension of $\mathcal{H}$ is $dim(\mathcal{H}) = 2$:

Computational basis: $\{|0\rangle, |1\rangle\}$

States in $\mathcal{H}$: $|\psi\rangle = a|0\rangle + b|1\rangle$ s.t. $|a|^2 + |b|^2 = 1$

Observables:

- $Z = |0\rangle\langle 0| - |1\rangle\langle 1| \quad \Rightarrow \quad$ possible outcomes: $1, -1$
- $X = |0\rangle\langle 1| + |1\rangle\langle 0|$
- $Y = -iZX = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$
- $O = C_0\mathbb{I} + C_X X + C_Y Y + C_Z Z$ (general observables)

The states of the computational basis are represented in matrix form as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.14}$$

In the *computational basis* the previous operators have the following matricial representation:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.15}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.16}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.17}$$

$$O = \begin{pmatrix} C_0 + C_Z & C_X - iC_Y \\ C_X + iC_Y & C_0 - C_Z \end{pmatrix} \tag{1.18}$$

Let's measure Z on $|\psi\rangle = a|0\rangle + b|1\rangle$:

$$Z = \overbrace{|0\rangle\langle 0|}^{E_0} - \overbrace{|1\rangle\langle 1|}^{E_1} \tag{1.19}$$

$$Prob(1) = \langle\psi| E_0 |\psi\rangle = \langle\psi|0\rangle\langle 0|\psi\rangle = a^* a = |a|^2 \tag{1.20}$$

$$Prob(-1) = \langle\psi| E_1 |\psi\rangle = \langle\psi|1\rangle\langle 1|\psi\rangle = b^* b = |b|^2 \tag{1.21}$$

Let's measure X on $|\psi\rangle$.

In order to measure X we want to find a basis that diagonalizes X. The eigenstates and eigenvalues of X are:

Eigenstates	Eigenvalues
$\|+\rangle = \frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	+1
$\|-\rangle = \frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	-1

Table 1.1: Eigenstates and eigenvalues of X

X can be rewritten in the new diagonalized basis $\{|+\rangle, |-\rangle\}$:

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = \overbrace{|+\rangle\langle +|}^{E'_0} - \overbrace{|-\rangle\langle -|}^{E'_1} = E'_0 - E'_1 \tag{1.22}$$

We rewrite $|\psi\rangle$ in the new basis:

$$|\psi\rangle = a|0\rangle + b|1\rangle = a\frac{|+\rangle + |-\rangle}{\sqrt{2}} + b\frac{|+\rangle - |-\rangle}{\sqrt{2}} = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle \tag{1.23}$$

We measure the probability of getting the eigenvalue $+1$ and -1:

$$Prob(1) = \langle\psi| E'_0 |\psi\rangle = \langle\psi|+\rangle\langle +|\psi\rangle = \frac{1}{2}|a+b|^2 \tag{1.24}$$

$$Prob(-1) = \langle\psi| E'_1 |\psi\rangle = \langle\psi|-\rangle\langle -|\psi\rangle = \frac{1}{2}|a-b|^2 \tag{1.25}$$

So, the steps to measure a general operator O on a general state $|\psi\rangle$ are:

1. Diagonalize O
   1.1. find the eigenvectors and the eigenvalue of O
   1.2. define the *projectors* of O as the ket-bra product of the eigenvectors of O
2. express O and $|\psi\rangle$ in the basis of the eigenvectors of O
3. compute the probability of each eigenvalue of O evaluating the expectation value of the related projector on the state $|\psi\rangle$ (expressed in the basis of the eigenvectors of O) according to Born rule (1.6).

1.8 Bloch sphere

Let $|\psi\rangle = a|0\rangle + b|1\rangle$ s.t. $|a|^2 + |b|^2 = 1$.
We can represent this state $|\psi\rangle$ as a point in the Bloch sphere.
We define the parametrization:

$$\begin{cases} a = \cos\frac{\theta}{2} \\ b = e^{i\phi}\sin\frac{\theta}{2} \end{cases} \tag{1.26}$$

There is a correspondence between the states $|\psi\rangle = a|0\rangle + b|1\rangle$ and the points on the Bloch sphere: from a and b we compute the angles θ and ϕ, which we use to represent the point on the Bloch sphere using spherical coordinates.

$$\begin{cases} r_x = \sin\theta\cos\phi \\ r_y = \sin\theta\sin\phi \\ r_z = \cos\theta \end{cases} \tag{1.27}$$

For example, for the state $|\psi\rangle = |0\rangle$ we have $a = 1$, $b = 0$ thus $\theta = 0$, $\phi = 0$. This points sits on top of the z axis:

$$\begin{cases} r_x = 0 \\ r_y = 0 \\ r_z = 1 \end{cases}$$

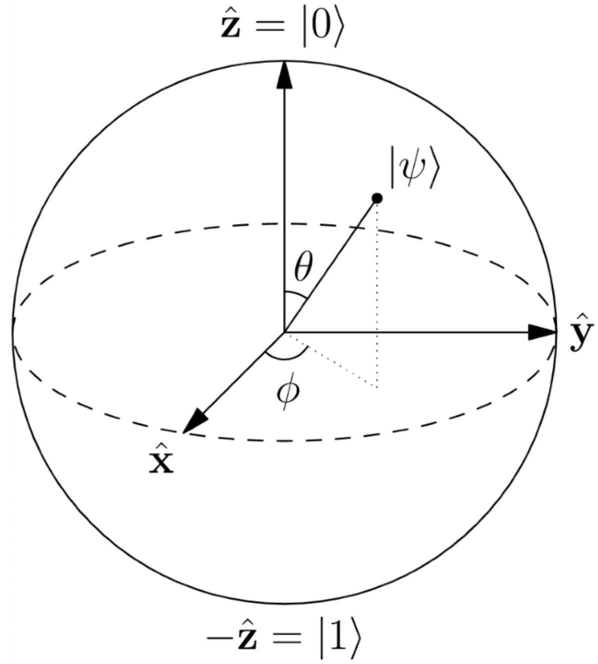


Figure 1.1: Bloch sphere.

1.8.1 General Hamiltonian for a qubit

The general Hamiltonian H for a qubit is a self-adjoint operator:

$$H = C_0\mathbb{I} + C_X X + C_Y Y + C_Z Z \tag{1.28}$$

The time evolution operator U is an unitary operator given by:

$$U = e^{-iHt} \tag{1.29}$$

Let's now substitute the general hamiltonian.

$$U_X = e^{-iC_X \frac{X}{2}} \tag{1.30}$$

$$U_Y = e^{-iC_Y \frac{Y}{2}} \tag{1.31}$$

$$U_Z = e^{-iC_Z \frac{Z}{2}} \tag{1.32}$$

The operators $U_{X,Y,Z}$ are rotations in the Bloch sphere around the respective axis of an angle $C_{X,Y,Z}$.

Recall that $X^2 = Y^2 = Z^2 = \mathbb{I}$. Let's compute the exponentials of the operators $U_{X,Y,Z}$:

$$U_X = e^{-iC_X \frac{X}{2}} = \cos\left(\frac{C_X}{2}\right)\mathbb{I} - i\sin\left(\frac{C_X}{2}\right)X \tag{1.33}$$

$$U_Y = e^{-iC_Y \frac{Y}{2}} = \cos\left(\frac{C_Y}{2}\right)\mathbb{I} - i\sin\left(\frac{C_Y}{2}\right)Y \tag{1.34}$$

$$U_Z = e^{-iC_Z \frac{Z}{2}} = \cos\left(\frac{C_Z}{2}\right)\mathbb{I} - i\sin\left(\frac{C_Z}{2}\right)Z \tag{1.35}$$

Chapter 2

Density matrices

2.1 Introduction to density matrices

How to describe the lack of information about a quantum system S? How can the 5 postulates deal with that lack of information?

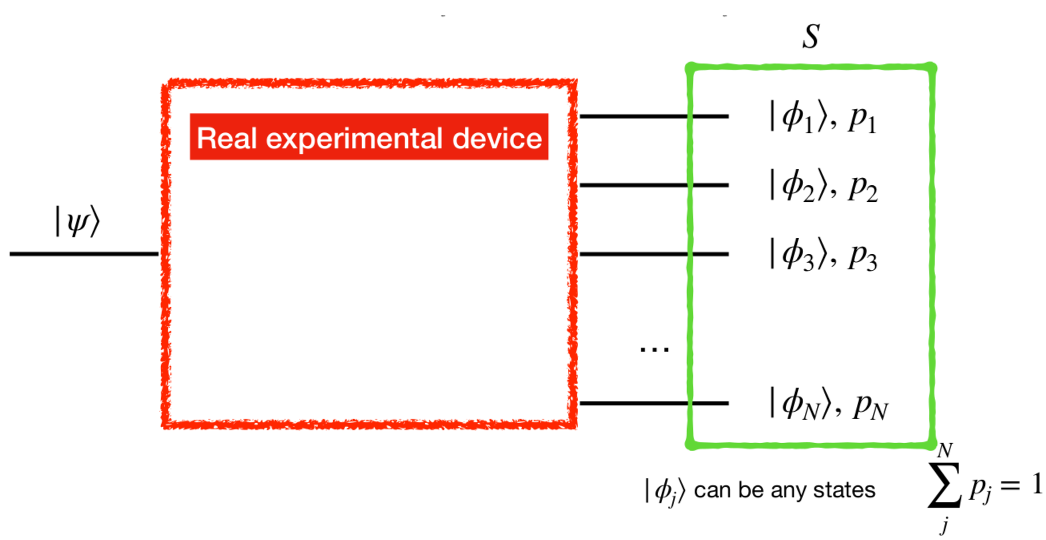


Figure 2.1: Real experimental device.

We consider a real experimental device that produces as output N quantum states $|\phi_j\rangle$, each one with a probability p_j.

We want to describe the state of the system S that the device produces in terms of quantum mechanics.

What is the probability that the final state of system S is the state $|\xi\rangle$?

Law of total probability: let $\{B_j\}_{j=1}^N$ be a collection of N events which partition the whole probability space[1], then:

$$\sum_{j=1}^{N} Prob(B_j) = 1 \tag{2.1}$$

Law of conditional probability: let A be an event, then

$$Prob(A) = \sum_{j=1}^{N} Prob(A \cap B_j) = \sum_{j=1}^{N} Prob(B_j) \cdot Prob(A|B_j) \tag{2.2}$$

We substitute A with the state $|\xi\rangle$ and B_j with the real device outputs $|\phi_j\rangle$. We rewrite eq. 2.2:

$$\begin{aligned}
Prob(|\xi\rangle) &= \sum_{j=1}^{N} Prob(|\xi\rangle \cap B_j) = \sum_{j=1}^{N} Prob(B_j) Prob(|\xi\rangle |B_j) \\
&= \sum_{j=1}^{N} p_j Prob(|\xi\rangle |B_j) = \sum_{j=1}^{N} p_j |\langle\xi|\phi_j\rangle|^2 \\
&= \sum_{j=1}^{N} p_j \langle\xi|\phi_j\rangle \langle\phi_j|\xi\rangle = \langle\xi| \sum_{j=1}^{N} p_j |\phi_j\rangle \langle\phi_j|\xi\rangle
\end{aligned} \tag{2.3}$$

where we used that conditional probability is given by the projection (which measures the superposition between states):

$$Prob(|\xi\rangle |B_j) = |\langle\xi|\phi_j\rangle|^2 \tag{2.4}$$

We define the mixed density matrix ρ_f to describe the final state as :

$$\rho_f = \sum_{j=1}^{N} p_j |\phi_j\rangle \langle\phi_j| \tag{2.5}$$

Thus, using ρ_f the probability $Prob(|\xi\rangle)$ eq. 2.3 can be written as:

$$Prob(|\xi\rangle) = \langle\xi| \rho_f |\xi\rangle \tag{2.6}$$

so ρ_f describes the ensemble of N states $\{|\phi_j\rangle, p_j\}_{j=1}^N$.

[1]the set $\{B_j\}_{j=1}^N$ is a partition of the whole probability space, i.e. the events B_j are mutually exclusive events and their union covers the whole probability space.

2.1.1 Alice and Bob problem

Let us consider the state $|\psi\rangle = a|0\rangle + b|1\rangle$ s.t. $|a|^2 + |b|^2 = 1$. We have 2 observers: Alice and Bob. They both know $|\psi\rangle$.

Alice measures $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$, then tells Bob that she measured Z without telling him the outcome. What are the post measurement (p.m.) states for Alice and Bob?

Alice will get $|0\rangle$ with probability $|a|^2$ and $|1\rangle$ with probability $|b|^2$. Since Bob doesn't know the outcome of Alice's measurement, for him the state is a probabilistic mixture of the two possible outcomes without quantum superposition i.e. the p.m. state:

$$\rho_{pm} = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| \tag{2.7}$$

e.g. for $a = b = \frac{1}{\sqrt{2}}$:

$$\rho_{pm} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \tag{2.8}$$

We now measure X on the state $|\psi\rangle$. Since $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$, we have:

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = |+\rangle\langle +| - |-\rangle\langle -| \tag{2.9}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle \tag{2.10}$$

The outcome of the measurement is $|+\rangle$ with $p = 1$.

We now measure X on ρ_{pm}, that we obtained by measuring Z. We need to change basis:

$$\rho_{pm} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \tag{2.11}$$

Using eq. 2.6, we get $|+\rangle$ with probability $\frac{1}{2}$ and $|-\rangle$ with probability $\frac{1}{2}$:

$$Prob(1) = \langle +|\rho_{pm}|+\rangle = \frac{1}{2} \tag{2.12}$$

$$Prob(-1) = \langle -|\rho_{pm}|-\rangle = \frac{1}{2} \tag{2.13}$$

There is a loss of information between the pure state $|\psi\rangle$ and the p.m. state ρ_{pm}.

We can represent ρ_{pm} in matrix form and compare it to the density matrix of the state before the measurement $\rho = |\psi\rangle\langle\psi|$.

$$\rho_{pm} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.14}$$

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.15}$$

The elements in the antidiagonal are called *coherences*.

ρ_{pm} describes a state which can be with equal probability in state $|0\rangle$ or $|1\rangle$. It is a mixed state. In a *mixed state* there is no superposition of states: it is a classical state. The coherences are zero.

On the other hand, ρ describes a quantum superposition of the states $|0\rangle$ and $|1\rangle$ with equal probability amplitudes. It is a *pure state*. The coherences are not zero. Unlike the mixed state, this superposition can display quantum interference.

Recap:

1. A state $|\psi\rangle = a|0\rangle + b|1\rangle$ with quantum superposition is a pure state and is described by a pure density matrix (with coherences):

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \tag{2.16}$$

2. A post-measurement state ρ^{PM} is a mixed state and is described by a mixed density matrix (without coherences):

$$\rho^{PM} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{2.17}$$

Unlike a pure state, the mixed state ρ^{PM} cannot be written as ket-bra product of one state.

2.1.2 Trace of an operator

In the case of a matrix A the trace is the sum of diagonal elements: $tr(A) = \sum_i A_{ii}$.

Def 1 (Trace of operator) *In the case of an operator A the trace $tr(A)$ is defined as:*

$$tr(A) = \sum_n \langle n| A |n\rangle \tag{2.18}$$

where $\{|n\rangle\}$ is an orthonormal basis.

Properties of the trace

1. Linear
2. Cyclic
3. Invariant under unitary transformations
4. $tr(|\psi\rangle \langle\psi| A) = \langle\psi| A |\psi\rangle$

2.2 Density matrix

Def 2 (Density matrix) *Given a set of vectors $\{|\phi_j\rangle\}_{j=1}^N$ in a Hilbert space $\mathcal{H}$, the* ***density matrix*** *ρ is defined as:*

$$\rho = \sum_{j=1}^N \lambda_j |\phi_j\rangle \langle\phi_j| \tag{2.19}$$

with the properties:

1. *Self-adjoint:* $\rho = \rho^+$
2. *Positive semi-definite:* $\langle\xi| \rho |\xi\rangle \geq 0 \qquad \forall\ |\xi\rangle \in \mathcal{H}$
3. *Unitary trace:* $tr(\rho) = 1$

Remark:

Since ρ is self-adjoint, it can be diagonalized by choosing an orthonormal basis $\{|j\rangle\}$:

$$\rho = \sum_j p_j |j\rangle \langle j| \tag{2.20}$$

where p_j is the probability of the state $|j\rangle$.

From the unitarity of the trace we get that:

$$tr(\rho) = 1 \Leftrightarrow \quad \sum_j p_j = 1 \tag{2.21}$$

Def 3 (Pure density matrix) *A density matrix ρ is said* ***pure*** *if there exists a state $|\psi\rangle$ in $\mathcal{H}$ s.t. ρ can be written as the outer product:*

$$\rho = |\psi\rangle \langle\psi| \tag{2.22}$$

Def 4 (Mixed density matrix) *A density matrix ρ is said* ***mixed*** *if it's* ***not*** *pure.*

Remark:

This means that a mixed density matrix ρ can't be written as a ket-bra product of a *single* state $|\psi\rangle$: a mixed ρ is given by eq. 2.19 with $N \geq 2$.

Th 1 (Characterization of purity) *Given a density matrix ρ, the following properties are equivalent:*

1. *ρ is* ***pure***
2. *$\rho^2 = \rho$*
3. *$tr(\rho^2) = tr(\rho) = 1$*

Proof:

1. $\Leftrightarrow$ 2. :

Let $|\psi\rangle$ be a state in $\mathcal{H}$. Its density matrix ρ is:

$$\rho = |\psi\rangle \langle\psi| \tag{2.23}$$

which is pure. By computing the square of ρ we see that it is equal to ρ itself:

$$\rho^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = |\psi\rangle \langle\psi| = \rho \tag{2.24}$$

2. $\Leftrightarrow$ 3. :

$\rho^2 = \rho$, therefore:

$$tr(\rho^2) = tr(\rho) = 1 \tag{2.25}$$

3. $\Leftrightarrow$ 1. :

If a density matrix satisfies eq. 2.25 it is a pure density matrix and it describes a pure state.

Let us now consider the density matrix ρ defined as:

$$\rho = \sum_j \lambda_j |\phi_j\rangle \langle\phi_j| \tag{2.26}$$

We can diagonalize the operator by choosing an orthonormal basis $\{|j\rangle\}$:

$$\rho = \sum_j p_j |j\rangle \langle j| \tag{2.27}$$

The density matrix satisfies:

$$tr(\rho) = 1 \tag{2.28}$$

Therefore, knowing that ρ is positive semi-definite i.e. $p_j \geq 0$:

$$\sum_j p_j = 1 \quad \Rightarrow \quad 0 \leq p_j \leq 1 \tag{2.29}$$

$$tr(\rho^2) = \sum_j p_j^2 \leq \sum_j p_j = 1 \tag{2.30}$$

But since $tr(\rho^2) = tr(\rho)$ by hypothesis (3.), only one of the p_j is 1 and the others are 0. Choose $p_{j=1} = 1$, then:

$$\rho = \sum_j p_j |j\rangle \langle j| = p_1 |1\rangle \langle 1| = |1\rangle \langle 1| \tag{2.31}$$

So, ρ is pure. ■

2.2.1 Test of purity

We can distinguish a pure state and a mixed state by looking at the density matrix. Looking at eq. (2.30), we have that:

$$tr(\rho^2) \leq tr(\rho) = 1 \tag{2.32}$$

So:

1. if $tr(\rho^2) = tr(\rho) = 1$, ρ is **pure**
2. if $tr(\rho^2) < tr(\rho) = 1$, ρ is **mixed**

$tr(\rho^2)$ is called **purity**.

2.3 Reduced density matrix

Def 5 (Partial trace) *Consider an o.n. basis for the system B: $\{|j\rangle_B\}$. The **partial trace** of the state $|\psi\rangle_{AB}$ over B is defined as:*

$$\rho_A = tr_B(\ |\psi\rangle_{AB} \langle\psi|_{AB}\ ) = \sum_j \langle j|_B |\psi\rangle_{AB} \langle\psi|_{AB} |j\rangle_B \tag{2.33}$$

We have two quantum systems A and B in a state $|\psi\rangle_{AB}$. The two systems have interacted but only A is accessible.

Def 6 (Reduced density matrix) *The reduced density matrix ρ_A is defined as the partial trace of $\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$ over B.*

$$\rho_A = tr_B(\ |\psi\rangle_{AB} \langle\psi|_{AB} \ ) \tag{2.34}$$

The state of the system A is described by the reduced density matrix ρ_A.

The reduced density matrix has the same properties of the density matrix.

We want to describe a system where only one of its parts is accessible: a bipartite system AB where we only have access to A.

We consider a 2-qubit system:

$$\mathcal{H}_A = \text{span}(\{|0\rangle_A, |1\rangle_A\}) \tag{2.35}$$

$$\mathcal{H}_B = \text{span}(\{|0\rangle_B, |1\rangle_B\}) \tag{2.36}$$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \text{span}(\{|0\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B\}) \tag{2.37}$$

Let us consider the following state of system AB:

$$|\psi\rangle = a\,|0\rangle_A |0\rangle_B + b\,|1\rangle_A |1\rangle_B \tag{2.38}$$

We measure $Z_A = Z_A \otimes \mathbb{I}_B$:

$$Z_A |\psi\rangle = \begin{cases} |0\rangle_A |0\rangle_B & p = |a|^2 \\ |1\rangle_A |1\rangle_B & p = |b|^2 \end{cases} \tag{2.39}$$

Now we measure a general operator $M_A = M_A \otimes \mathbb{I}_B$:

$$\begin{aligned}
\langle\psi| M_A |\psi\rangle &= (\ a^* \langle 00| + b^* \langle 11| \ ) \ M_A \otimes \ \mathbb{I}_B \ (\ a\,|00\rangle + b\,|11\rangle \ ) \\
&= |a|^2 \langle 00| M_A \otimes \ \mathbb{I}_B |00\rangle + a^* b \langle 00| M_A \otimes \ \mathbb{I}_B |11\rangle + \\
&\quad + b^* a \langle 11| M_A \otimes \ \mathbb{I}_B |00\rangle + |b|^2 \langle 11| M_A \otimes \ \mathbb{I}_B |11\rangle \\
&= |a|^2 \langle 0|_A M_A |0\rangle_A + |b|^2 \langle 1|_A M_A |1\rangle_A \\
&= tr \left\{ (\ |a|^2 |0\rangle_A \langle 0|_A + |b|^2 |1\rangle_A \langle 1|_A \ ) M_A \right\} \\
&= tr(\rho_A M_A)
\end{aligned}$$

So:

$$\langle\psi| M_A |\psi\rangle = tr(\rho_A M_A) \tag{2.40}$$

ρ_A is the reduced density matrix for the system A, it is defined as:

$$\rho_A = |a|^2 \; |0\rangle_A \langle 0|_A + |b|^2 \; |1\rangle_A \langle 1|_A \tag{2.41}$$

It is the description of system AB when we do not have access to B.

In order to compute ρ_A from $|\psi\rangle_{AB}$ we have to choose a base for B and then take the partial trace with respect to B.

In this case a basis for B is:

$$\mathcal{H}_B = \{|0\rangle_B , |1\rangle_B\} \tag{2.42}$$

We take the partial trace over B:

$$tr_B(|\psi\rangle_{AB} \langle\psi|_{AB}) = \sum_{j=0,1} \langle j|_B |\psi\rangle_{AB} \langle\psi|_{AB} |j\rangle_B \tag{2.43}$$

In a bipartite system AB, the description of system A is given in terms of the reduced density matrix ρ_A by tracing out the states of system B.

2.4 Density matrix of a qubit

The Hilbert space for a qubit is $\mathcal{H}_{qubit} = \mathbb{C}^2$. A basis of $\mathcal{H}_{qubit}$ is given by the Pauli matrices.

The density matrix of the system $\mathcal{H}_{qubit}$ is:

$$\rho = \frac{1}{2}(\alpha_0\ \mathbb{I} + \alpha_x\sigma^x + \alpha_y\sigma^y + \alpha_z\sigma^z) \tag{2.44}$$

$$\rho = \frac{1}{2}\begin{pmatrix} \alpha_0 + \alpha_z & \alpha_x - i\alpha_y \\ \alpha_x + i\alpha_y & \alpha_0 - \alpha_z \end{pmatrix} \tag{2.45}$$

Since ρ is a density matrix we impose the trace to be 1 and the determinant to be non-negative (positive semi-definite operator).

$$tr(\rho) = 1 \ \Rightarrow \ \alpha_0 = 1 \tag{2.46}$$

$$det(\rho) \geq 0 \ \Rightarrow \ \frac{1}{4}(1 - |\vec{\alpha}|^2) \geq 0 \ \Rightarrow \ |\vec{\alpha}|^2 \leq 1 \tag{2.47}$$

In order to represent this state on the Bloch sphere, we rewrite ρ as:

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{\alpha} \cdot \vec{\sigma}) \tag{2.48}$$

$\vec{\alpha}$ is the point on the Bloch sphere associated to ρ.
$|\vec{\alpha}|^2$ is the squared distance from the center of the sphere.
$\vec{\sigma}$ is the vector constituted by the Pauli matrices: $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

$$\begin{cases} |\vec{\alpha}|^2 = 1 & \text{pure state} \rightarrow \text{lies on the surface of the sphere} \\ |\vec{\alpha}|^2 < 1 & \text{mixed state} \rightarrow \text{is inside the sphere} \end{cases} \tag{2.49}$$

For example, let's consider the state:

$$\rho = \frac{1}{2}\mathbb{I} = \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right| \tag{2.50}$$

This state lies in the center of the sphere, it is a maximally mixed state.

2.5 Singular value decomposition (SVD)

Given a rectangular matrix M of dimensions $N_A \cdot N_B$ ($n_{rows} \cdot n_{columns}$).
M can be decomposed into:

$$M = USV^+ \tag{2.51}$$

- U
  - U is a matrix of dimensions $N_A \cdot min(N_A, N_B)$.
  - The columns of U are the left singular vectors.
  - The columns of U are orthonormal, $U^+U = \mathbb{I}$.
  - If $N_A \leq N_B \rightarrow UU^+ = \mathbb{I}$, U is unitary.
- S
  - S is a square matrix of dimensions $min(N_A, N_B) \cdot min(N_A, N_B)$.
  - The diagonal non negative entries S_{aa} are the singular values of S.
- V^+
  - V^+ is a matrix of dimensions $min(N_A, N_B) \cdot N_B$.
  - The columns of V are right singular vectors.
  - The rows of V^+ are orthonormal, $V^+V = \mathbb{I}$.
  - If $N_A \geq N_B \rightarrow VV^+ = \mathbb{I}$, V is unitary.

2.5.1 How to compute SVD - matrix

$$M = USV^+ \tag{2.52}$$

$$\begin{aligned} M^+M &= (USV^+)^+(USV^+) \\ &= VS^+U^+USV^+ \\ &= VS^2V^+ \end{aligned} \tag{2.53}$$

We call $X = M^+M$. X is hermitian and positive semidefinite, so we can diagonalize it. Since $X = VS^2V^+$, V is the matrix with the eigenvectors of X as columns.

Let's solve for U:

$$\begin{aligned} M &= USV^+ \\ MV &= USV^+V \\ MV &= US \end{aligned} \tag{2.54}$$

S^{-1} may not be defined if there are zero singular values.

2.5.2 How to compute SVD - matrix elements

$$M = USV^+ \tag{2.55}$$

$$M_{ij} = \sum_a U_{ia} S_{aa} V^+_{aj}$$

$$= \sum_a U_{ia} S_{aa} V^*_{ja} \tag{2.56}$$

Let's apply SVD to a bipartite quantum system.

$$|\psi\rangle_{AB} = \sum_{ij} \psi_{ij} |i\rangle_A \otimes |j\rangle_B \tag{2.57}$$

$|i\rangle_A$ is an orthonormal basis of A .
$|j\rangle_B$ is an orthonormal basis of B .
ψ_{ij} is a rectangular matrix $N_A \cdot N_B$.

Let's apply SVD to ψ_{ij}:

$$\psi = USV^+$$

$$\psi_{ij} = \sum_{a=1}^{min(N_A,N_B)} U_{ia} S_{aa} V^*_{ja}$$

$$|\psi\rangle_{AB} = \sum_{ij} \sum_{a=1}^{min(N_A,N_B)} U_{ia} S_{aa} V^*_{ja} (|i\rangle_A \otimes |j\rangle_B)$$

$$|\psi\rangle_{AB} = \sum_{a=1}^{min(N_A,N_B)} S_{aa} \sum_i U_{ia} |i\rangle_A \otimes \sum_j V^*_{ja} |j\rangle_B$$

$$|\psi\rangle_{AB} = \sum_{a=1}^{min(N_A,N_B)} S_{aa} |a\rangle_A \otimes |a\rangle_B \tag{2.58}$$

where

$$|a\rangle_A = \sum_i U_{ia} |i\rangle_A \tag{2.59}$$

$$|a\rangle_B = \sum_j V^*_{ja} |j\rangle_B \tag{2.60}$$

We show that $\{|a\rangle_A\}$ is an orthonormal set of $\mathcal{H}_A$ (they may not form a basis). Since $U^+U = \mathbb{I}$:

$$\langle a|_A |a'\rangle_A = \sum_{ij} U^+_{ia} U_{ja'} \langle i|_A |j\rangle_A = \sum_{ij} U^+_{ia} U_{ja'} \delta_{ij}$$
$$= \sum_{i} U^+_{ia} U_{ia'} = \delta_{aa'} \tag{2.61}$$

The same argument works for $\{|a\rangle_B\}$.
To sum up:

$$|\psi\rangle_{AB} = \sum_{ij} \psi_{ij} |i\rangle_A \otimes |j\rangle_B$$
$$= \sum_{a=1}^{min(N_A,N_B)} S_{aa} |a\rangle_A \otimes |a\rangle_B \tag{2.62}$$

2.5.3 Schmidt decomposition

Consider a singular value decomposition:

$$|\psi\rangle_{AB} = \sum_{a=1}^{min(N_A,N_B)} S_{aa} |a\rangle_A \otimes |a\rangle_B \tag{2.63}$$

The **Schmidt decomposition** is:

$$|\psi\rangle_{AB} = \sum_{a=1}^{r} S_{aa} |a\rangle_A \otimes |a\rangle_B \tag{2.64}$$

The **Schmidt number** r is the number of non zero singular values.

Reduced density matrix for A:

$$\rho_A = tr_B(|\psi\rangle_{AB} \langle\psi|_{AB})$$
$$= \sum_{a=1}^{r} S^2_{aa} |a\rangle_A \langle a|_A \tag{2.65}$$

Reduced density matrix for B:

$$
\begin{aligned}
\rho_B &= tr_A(|\psi\rangle_{AB} \langle\psi|_{AB}) \\
&= \sum_{a=1}^{r} S_{aa}^2 \, |a\rangle_B \, \langle a|_B
\end{aligned}
\tag{2.66}
$$

The eigenvalues of ρ_A and ρ_B are S_{aa}^2 . ρ_A and ρ_B have the same eigenvalues but different eigenvectors $\{|a\rangle_A\}$ and $\{|a\rangle_B\}$.

Chapter 3

Entanglement

3.1 Entangled and separable states

Considering the Schmidt decomposition:

$$|\psi\rangle_{AB} = \sum_{a=1}^{r} S_{aa} |u_a\rangle_A \otimes |v_a\rangle_B \tag{3.1}$$

1. if $r = 1$ the state $|\psi\rangle_{AB}$ is ***separable***
2. if $r > 1$ the state $|\psi\rangle_{AB}$ is ***entangled***

This property depends on the state and on the bipartition.

3.2 Quantum correlation

$$|\phi\rangle = |\uparrow\rangle_A \otimes |\uparrow\rangle_B \tag{3.2}$$

The state $|\phi\rangle$ is separable. Spin A and spin B have a classical correlation. If we measure S_{z_A}, z component of the spin, we get:

$$S_{z_A} |\phi\rangle = |\uparrow\rangle_A |\uparrow\rangle_B \tag{3.3}$$

Let us now consider the state $|\psi\rangle$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\ |\uparrow\rangle_A \otimes |\uparrow\rangle_B \ + \ |\downarrow\rangle_A \otimes |\downarrow\rangle_B \ ) \tag{3.4}$$

The state $|\psi\rangle$ is entangled. Spin A and spin B have a quantum correlation. If we measure S_{z_A}, we get:

$$S_{z_A} |\psi\rangle = \begin{cases} |\uparrow\rangle_A |\uparrow\rangle_B & p = \frac{1}{2} \\ |\downarrow\rangle_A |\downarrow\rangle_B & p = \frac{1}{2} \end{cases} \tag{3.5}$$

How can we prepare the states $|\phi\rangle$ and $|\psi\rangle$?

Let's start with $|\phi\rangle$:

$U_A |\alpha\rangle_A = |\uparrow\rangle_A$

$|\alpha\rangle_A$ is a random initial state

U_A rotates $|\alpha\rangle$ until spin is directed along +z

We do the same for B and we couple A and B :

$$A + B = |\uparrow\rangle_A \otimes |\uparrow\rangle_B \tag{3.6}$$

Let's now prepare $|\psi\rangle$.

We try rotating the state $|\alpha\rangle_A$ and $|\beta\rangle_B$ locally:

$$U_A |\alpha\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A + |\downarrow\rangle_A) \tag{3.7}$$

$$U_B |\beta\rangle_B = \frac{1}{\sqrt{2}}(|\uparrow\rangle_B + |\downarrow\rangle_B) \tag{3.8}$$

We now couple the states that we obtained:

$$\begin{aligned} |\psi'\rangle &= \frac{1}{\sqrt{2}}(\ |\uparrow\rangle_A + |\downarrow\rangle_A\ )\ \frac{1}{\sqrt{2}}(\ |\uparrow\rangle_B + |\downarrow\rangle_B\ ) \\ &= \frac{1}{2}(\ |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\ ) \end{aligned} \tag{3.9}$$

The state $|\psi'\rangle$ is different from the state $|\psi\rangle$ we are trying to prepare.

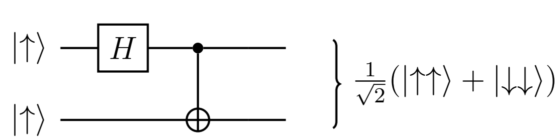
To prepare $|\phi\rangle = |\uparrow\uparrow\rangle$ we performed local rotations (indipendently on A and on B) and we prepared a separable state.

To prepare an entangled state we cannot do local rotations: we have to let the initial states interact with each other (exchange of information).

We want to find a non local unitary operator U such that

$$U |\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \tag{3.10}$$

U is $4 \cdot 4$ unitary matrix. For the calculation see eq. 7.25.



$$
\begin{aligned}
|\uparrow\uparrow\rangle \xrightarrow{H_A} (\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}) \otimes |\uparrow\rangle &= \frac{1}{\sqrt{2}} |\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\uparrow\rangle \\
&\xrightarrow{CNOT} \frac{1}{\sqrt{2}} |\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle
\end{aligned} \tag{3.11}
$$

$$
U = CNOT \cdot (H \otimes \mathbb{I}_2) \tag{3.12}
$$

3.3 Space of density matrices

Suppose we have two density matrices ρ_1, ρ_2 :

$$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2 \qquad 0 \leq \lambda \leq 1 \tag{3.13}$$

The segment $\rho(\lambda)$ that connects ρ_1, ρ_2 is a set of density matrices. Thus, the space of density matrices is a convex set.

There are infinite ways to represent a mixed density matrix:

$$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2 = \lambda'\rho_1' + (1-\lambda')\rho_2' \tag{3.14}$$

We now consider a pure density matrix $\rho = |\psi\rangle\langle\psi|$.

Th 2 *Given a pure density matrix $\rho = |\psi\rangle\langle\psi|$. It is not possible to find λ s.t.*

$$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2 \qquad 0 < \lambda < 1 \tag{3.15}$$

Proof:

Consider $|\psi'\rangle$ s.t $\langle\psi'|\psi\rangle = 0$.

$$0 = \langle\psi'|\psi\rangle\langle\psi|\psi'\rangle = \lambda\langle\psi'|\rho_1|\psi'\rangle + (1-\lambda)\langle\psi'|\rho_2|\psi'\rangle \tag{3.16}$$

$$\underbrace{\lambda}_{0<\lambda<1}\underbrace{\langle\psi'|\rho_1|\psi'\rangle}_{\geq 0} = \underbrace{(\lambda-1)}_{-1<\lambda<0}\underbrace{\langle\psi'|\rho_2|\psi'\rangle}_{\geq 0} \tag{3.17}$$

Since we require $0 < \lambda < 1$, the equation is satisfied only if:

$$\rho_1 = \rho_2 = |\psi\rangle\langle\psi| \qquad \blacksquare \tag{3.18}$$

There is only one way to prepare pure states: it is impossible to write a pure state as a convex sum of two other states. Pure states are at the boundary of the set of density matrices.

In the case of the Bloch sphere the points on the surface of the sphere represents pure states.

Chapter 4

Purification

4.1 Purification

Let ρ_A be a mixed state:

$$\rho_A = \sum_j p_j \left|\phi_j\right\rangle \left\langle\phi_j\right| \qquad \left|\phi_j\right\rangle \in \mathcal{H}_A \tag{4.1}$$

$\left|\phi_j\right\rangle$ are normalized states and not necessarily orthogonal.

Example

$$\rho_A = \frac{1}{3}\left|0\right\rangle\left\langle 0\right| + \frac{1}{3}\left|+\right\rangle\left\langle +\right| + \frac{1}{3}\left|-\right\rangle\left\langle -\right| \tag{4.2}$$

Def 7 (Purification) *A* ***purification*** *of a density operator ρ_A is a pure bipartite system $\left|\phi\right\rangle_{AB}$ on a reference system B and the original system A s.t. the reduced state on system A is equal to ρ_A:*

$$\rho_A = tr_B(\left|\phi\right\rangle_{AB}\left\langle\phi\right|_{AB}) \tag{4.3}$$

In fact, given a density operator ρ_A, there always exists an Hilbert space $\mathcal{H}_B$ — big enough to contain all the states $\left|\phi_j\right\rangle$ — and a pure state $\left|\phi_{AB}\right\rangle$ defined on $\mathcal{H}_A \otimes \mathcal{H}_B$ s.t.:

1. The reduced density matrix ρ_A is:

$$\rho_A = tr_B(\left|\phi\right\rangle_{AB}\left\langle\phi\right|_{AB}) = \sum_{j=1}^{N} p_j \left|\phi_j\right\rangle\left\langle\phi_j\right| \tag{4.4}$$

2. The purification of ρ_A is:

$$\left|\phi\right\rangle_{AB} = \sum_{j=1}^{N} \sqrt{p_j}\left|\phi_j\right\rangle_A \otimes \left|\alpha_j\right\rangle_B \tag{4.5}$$

where $\left\{\left|\alpha_j\right\rangle_B\right\}$ is an *orthonormal set* for the Hilbert space $\mathcal{H}_B$ of the system B.

Th 3 *The purification of ρ_A is a pure state $|\phi\rangle_{AB} = \sum_{j=1}^{N} \sqrt{p_j}\, |\phi_j\rangle_A \otimes |\alpha_j\rangle_B$ s.t. tracing out the component of the system B from $|\phi\rangle_{AB}$ one recovers ρ_A.*

Proof:

Let's choose for $\mathcal{H}_B$ an *orthonormal* basis $\{|\alpha_j\rangle_B\}$.

$$
\begin{aligned}
\rho_A &= tr_B(|\phi\rangle_{AB} \langle\phi|_{AB}) \\
&= tr_B(\sum_j \sqrt{p_j}\, |\phi_j\rangle_A \otimes |\alpha_j\rangle_B \sum_l \sqrt{p_l}\, \langle\phi_l|_A \otimes \langle\alpha_l|_B) \\
&= \sum_{j,l} \sqrt{p_j}\sqrt{p_l}\, |\phi_j\rangle_A \langle\phi_l|_A \; tr_B(|\alpha_j\rangle_B \langle\alpha_l|_B) \\
&= \sum_{j,l} \sqrt{p_j}\sqrt{p_l}\, |\phi_j\rangle_A \langle\phi_l|_A \; \delta_{jl} \\
&= \sum_j p_j\, |\phi_j\rangle_A \langle\phi_j|_A \qquad \blacksquare
\end{aligned}
\tag{4.6}
$$

Purification is equivalent to think that the density operator ρ_A arises from the entanglement between the system A and B, and lack of access to system B.

Purification is a way to prepare an ensemble.

The $\{|\alpha_j\rangle_B\}$ can be considered as coming from some orthogonal projectors $|\alpha_j\rangle \langle\alpha_j|$ on $\mathcal{H}_B$ that are used for measurement.

The post-measurement state if I measure $|\alpha_j\rangle_B \langle\alpha_j|_B$ will be:

$$
|\phi_j\rangle_A \otimes |\alpha_j\rangle_B \qquad \text{with probability } \sqrt{p_j}^{\,2} = p_j \tag{4.7}
$$

4.1.1 Realization of an ensemble through purification

To realize the ensemble $\{p_j, |\phi_j\rangle\}$ on A: we couple A with an auxiliary system B performing the purification on system A and we get $|\phi\rangle_{AB}$:

$$|\phi\rangle_{AB} = \sum_{j=1}^{N} \sqrt{p_j}\, |\phi_j\rangle_A \otimes |\alpha_j\rangle_B \tag{4.8}$$

where $\left\{|\alpha_j\rangle_B\right\}$ is an *orthonormal set* for the Hilbert space $\mathcal{H}_B$ of the reference system B.

We measure the projectors $|\alpha_j\rangle_B \langle\alpha_j|_B$ on B and get ρ_A as p.m. state (see Fig. 4.1).

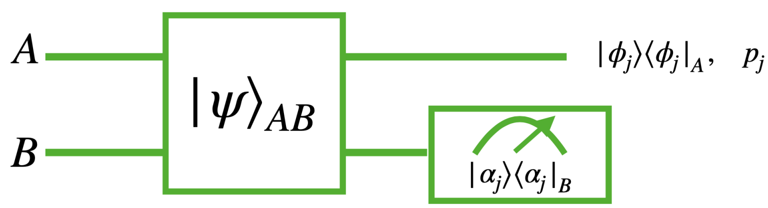


Figure 4.1: Realization of an ensemble through purification.

Any ensemble ρ_A can be written in infinite ways by using pure density matrices:

$$\begin{aligned} \rho_A &= \sum_{j=1}^{N_1} p_j\, |\phi_j\rangle \langle\phi_j| = \\ &= \sum_{\mu=1}^{N_2} q_\mu\, |\psi_\mu\rangle \langle\psi_\mu| \qquad N_1 \neq N_2 \end{aligned} \tag{4.9}$$

Let's consider an Hilbert space $\mathcal{H}_B$ and 2 orthogonal basis $\left\{|\alpha_j\rangle_B\right\}$, $\left\{|\beta_\mu\rangle_B\right\}$.
Consider the 2 purifications of ρ_A :

$$|\phi_1\rangle = \sum_{j=1}^{N_1} \sqrt{p_j}\, |\phi_j\rangle_A \otimes |\alpha_j\rangle_B \tag{4.10}$$

$$|\phi_2\rangle = \sum_{\mu=1}^{N_2} \sqrt{q_\mu}\, |\psi_\mu\rangle_A \otimes |\beta_\mu\rangle_B \tag{4.11}$$

How to relate these two purifications? We use the Schrodinger-HJW theorem.

4.2 Schrodinger-HJW theorem

Th 4 (Schrodinger-HJW theorem) *There is only one purification up to unitary transformation U defined only on the auxiliary space B.*

Proof:

Let's consider a density matrix ρ_A. ρ_A is self-adjoint, so it can be diagonalized.

$$\rho_A = \sum_k \lambda_k |k\rangle \langle k| \tag{4.12}$$

where $\{|k\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$. Purifications can be written as:

$$|\phi_1\rangle = \sum_k \sqrt{\lambda_k} |k\rangle_A \otimes \left|k^{(1)}\right\rangle_B \tag{4.13}$$

$$|\phi_2\rangle = \sum_k \sqrt{\lambda_k} |k\rangle_A \otimes \left|k^{(2)}\right\rangle_B \tag{4.14}$$

where $\left\{\left|k^{(1)}\right\rangle\right\}$ and $\left\{\left|k^{(2)}\right\rangle\right\}$ are orthonormal basis for $\mathcal{H}_B$. Since we begin with the same state ρ_A, if we trace out B we need to get ρ_A, therefore:

$$tr_B(|\phi_1\rangle \langle \phi_1|) = tr_B(|\phi_2\rangle \langle \phi_2|) \tag{4.15}$$

$$\begin{aligned} &tr_B(\sum_k \sqrt{\lambda_k} |k\rangle_A \otimes \left|k^{(1)}\right\rangle_B \sqrt{\lambda_k} \langle k|_A \otimes \left\langle k^{(1)}\right|_B) = \\ &tr_B(\sum_k \sqrt{\lambda_k} |k\rangle_A \otimes \left|k^{(2)}\right\rangle_B \sqrt{\lambda_k} \langle k|_A \otimes \left\langle k^{(2)}\right|_B) \end{aligned} \tag{4.16}$$

$$\sum_k \lambda_k^2 |k\rangle_A \langle k|_A \otimes tr_B(\left|k^{(1)}\right\rangle_B \left\langle k^{(1)}\right|_B) = \sum_k \lambda_k^2 |k\rangle_A \langle k|_A \otimes tr_B(\left|k^{(2)}\right\rangle_B \left\langle k^{(2)}\right|_B) \tag{4.17}$$

Since we require 4.15 and the trace is invariant up to unitary transformations we have:

$$\left|k^{(1)}\right\rangle_B = U \left|k^{(2)}\right\rangle_B \tag{4.18}$$

Chapter 5

Generalized measurement

5.1 Von Neumann measurment scheme for projective measurement

Quantum measurements are described by a collection $\{E_m\}$ of *measurement operators.*

Def 8 (Measurement operators) ***Measurement operators*** *are operators with the property of being:*

1. orthogonal projectors: $E_m\, E_n \;=\; \delta_{m,n} E_n \qquad E_n^+ \;=\; E_n$
2. positive semi-definite: $Prob(E_m) \;=\; \langle\psi| E_m |\psi\rangle \geq 0 \qquad \forall\; |\psi\rangle \in \mathcal{H}$
3. satisfy the *completeness relation*[1]: $\sum_m E_m^+ E_m = \mathbb{I} \quad \Leftrightarrow \quad \sum_m Prob(E_m) = 1$

This type of measurement is called *Projection-Valued Measure* (PVM).

According to the *Born rule* (1.7), after the measurement, the system will be in the state:

$$\left|\psi^{PM}\right\rangle \;=\; \frac{E_a |\psi\rangle}{\sqrt{\langle\psi| E_a |\psi\rangle}} \tag{5.1}$$

We need to measure an operator M of a system A. Let $\{|a\rangle\}$ be a basis for the Hilbert space $\mathcal{H}_A$ of the system A, formed by the *eigenvectors* $|a\rangle$ of the operator M. Then, the state $|\psi\rangle_A$ of the system A is described by:

$$|\psi\rangle_A = \sum_a c_a |a\rangle \tag{5.2}$$

[1]They also satisfy $\sum_m E_m = \mathbb{I}$

and the measurement operator M is:

$$M = \sum_a M_a |a\rangle \langle a| = \sum_a M_a E_a \tag{5.3}$$

To perform the measurement, we need to couple A to an auxiliary system B, that is the *meter* or *pointer* that will give the result of the measurement.

The coupling with B will establish entanglement between the states $|a\rangle$ of A and some distinguishable classical system that you can read on the pointer.

Let's model the pointer as a free particle of mass m, moving in 1 dimension and initially prepared in a wavepacket $|\psi(x)\rangle$.

We need to turn on the coupling between A and B. There will be an evolution ruled by the coupling and the pointer will move accordingly to the value of M we measure.

The total Hamiltonian is:

$$H = H_0 + \frac{p^2}{2m} + \lambda(t)\ M \otimes P \tag{5.4}$$

- H_0 is the unperturbed Hamiltonian of system A.
- $\frac{p^2}{2m}$ hamiltonian of the pointer (free particle of mass m).
- $M \otimes P$ is the coupling of the observable M with the momentum of the pointer P.
- $\lambda(t)$ is the coupling constant that will be turned on/off when we want to start/stop the measurement (measure/not measure).

We assume:

$$[M, H_0] = 0 \tag{5.5}$$

since we don't want M to change with the evolution H_0. Thus, the evolution will be given by:

$$H_{int} = \lambda(t) M \otimes P \tag{5.6}$$

The coupling $\lambda(t)$ is turned on/off in a very short time T, so we consider $\lambda(t)$ to be time independent.

The time evolution is given by:

$$\begin{aligned} U(T) &= e^{-i\lambda T M \otimes P} \\ &= \sum_a e^{-i\lambda T M_a P} |a\rangle \langle a| \end{aligned} \tag{5.7}$$

The initial state for the pointer is $|\psi(x)\rangle \equiv |x\rangle$.

Since the operator $P = -i\hbar\frac{\partial}{\partial x}$ is the *generator of translations*, the evolution of the pointer is:

$$e^{-i\lambda T M_a P} |\psi(x)\rangle = |\psi(x - \lambda T M_a)\rangle \tag{5.8}$$

We now consider the system $A + B$:

$$|\phi\rangle_{AB} = \sum_a c_a |a\rangle_A \otimes |\psi(x)\rangle_B \tag{5.9}$$

We apply the evolution operator:

$$\begin{aligned} U(T)|\phi\rangle_{AB} &= e^{-i\lambda T M \otimes P}(\sum_a c_a |a\rangle_A \otimes |\psi(x)\rangle_B) \\ &= \sum_a c_a e^{-i\lambda T M_a P} |a\rangle_A \otimes |\psi(x)\rangle_B \\ &= \sum_a c_a |a\rangle_A \otimes |\psi(x - \lambda T M_a)\rangle_B \end{aligned} \tag{5.10}$$

We obtain an *entangled* state between A and B.

If we measure the position of the pointer in $x = \lambda T M_a$, then the system will be projected in $|a\rangle$ with probability $|c_a|^2$. We prepared the system in $|a\rangle$ by observing the pointer in $x = \lambda T M_a$.

We now introduce an Hilbert space $\mathcal{H}_B$ for the pointer:

$$\mathcal{H}_B = \text{span}(|a\rangle_n, n = 0, .., N-1) \tag{5.11}$$

of the N possible outcomes that describe the state of the particle when we measure M on the system A. We consider the spectral decomposition of operator M, being E_a the projectors:

$$M = \sum_a M_a E_a \tag{5.12}$$

$$U = \sum_{a,b=0}^{N-1} E_a \otimes |a + b\rangle_B \langle b|_B \tag{5.13}$$

U is unitary. U acts on the pure state $\rho_A = |\psi\rangle_A \langle\psi|_A$ as:

$$U |\psi\rangle_A \otimes |0\rangle_B = \sum_a E_a |\psi\rangle_A \otimes |a\rangle_B \tag{5.14}$$

If we measure the pointer we will get the outcome $|a\rangle_B$:

$$|a\rangle_B \ : \begin{cases} Prob(E_a) & = \langle\psi| E_a |\psi\rangle \\ |\psi^{PM}\rangle & = \frac{E_a|\psi\rangle}{\sqrt{\langle\psi|E_a|\psi\rangle}} \end{cases} \tag{5.15}$$

If we don't measure the pointer the system remains entangled.
If we forget the outcome of the measurement, we get a mixed state:

$$\rho_{mix} = \sum_a Prob(E_a) \frac{E_a |\psi\rangle \langle\psi| E_a^+}{\langle\psi| E_a |\psi\rangle} \tag{5.16}$$

$$= \sum_a E_a |\psi\rangle \langle\psi| E_a^+ \tag{5.17}$$

5.2 Kraus representation of a physical map

If the set of operators $\{E_a\}$ satisfies:

$$\sum_a E_a^+ E_a = \mathbb{I} \tag{5.18}$$

then the action of an operator U on a state $\rho = |\psi\rangle\langle\psi|$ can always be written as:

$$U(\rho) = \sum_a E_a \ \rho \ E_a^+ \tag{5.19}$$

$U(\rho)$ is the most general map that represents the evolution of a state $\rho = |\psi\rangle\langle\psi|$. The operators E_a are called Kraus operators.

5.2.1 Example of Kraus representation on a qubit

We want to measure the Z operator on a qubit (system A):

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{5.20}$$

We consider the auxiliary system B, which is a qubit. The number of outcomes of B is $N = 2$, so $a, b = 0, 1$.

$$\begin{aligned} U &= \sum_{a,b=0}^{N-1} E_a \otimes |a+b\rangle_B \langle b|_B \\ &= E_0 \otimes (\ |0\rangle\langle 0|) + |1\rangle\langle 1|\ ) + E_1 \otimes (\ |1\rangle\langle 0| + |1+1=0\rangle\langle 1|\ ) \\ &= E_0 \otimes \mathbb{I}_B + E_1 \otimes X_B \end{aligned} \tag{5.21}$$

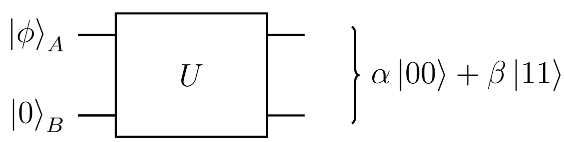
We performed a sum modulo $N = 2$, so $1 + 1 = 0$.

Let's measure the qubit (system A):

$$|\phi\rangle_A = \alpha\,|0\rangle + \beta\,|1\rangle \tag{5.22}$$

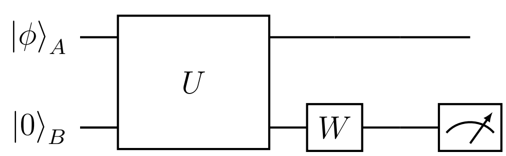
The initial state for system B is $|0\rangle$.

$$\begin{aligned}
U\,|\phi\rangle_A \otimes |0\rangle_B &= (E_0 \otimes \mathbb{I}_B + E_1 \otimes X_B)[(\alpha\,|0\rangle_A + \beta\,|1\rangle_A) \otimes |0\rangle_B] \\
&= (|0\rangle_A \langle 0|_A \otimes \mathbb{I}_B + |1\rangle_A \langle 1|_A \otimes X_B)[(\alpha\,|0\rangle_A + \beta\,|1\rangle_A) \otimes |0\rangle_B] \\
&= \alpha\,|0\rangle_A\,|0\rangle_B + \beta\,|1\rangle_A\,|1\rangle_B
\end{aligned} \tag{5.23}$$



Now we do a change of basis of system B and we measure in the new basis:

$$W: \quad |0\rangle\,,|1\rangle \longrightarrow |+\rangle\,,|-\rangle \tag{5.24}$$



The operator that changes the base is the Hadamard gate W:

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.25}$$

$$W_B(\alpha\,|0\rangle_A\,|0\rangle_B + \beta\,|1\rangle_A\,|1\rangle_B) = \frac{\alpha\,|0\rangle_A + \beta\,|1\rangle_A}{\sqrt{2}} \otimes |+\rangle_B + \frac{\alpha\,|0\rangle_A - \beta\,|1\rangle_A}{\sqrt{2}} \otimes |-\rangle_B \tag{5.26}$$

$$= \frac{|\phi\rangle_A}{\sqrt{2}}\,|+\rangle_B + \frac{Z_A\,|\phi\rangle_A}{\sqrt{2}}\,|-\rangle_B \tag{5.27}$$

The possible outcomes of measuring $|+\rangle\,,|-\rangle$ on the pointer are $|+\rangle\,,|-\rangle$. The post measurement state for system A is:

$$\begin{cases} \frac{\alpha|0\rangle+\beta|1\rangle}{\sqrt{2}} & prob = \frac{1}{2} \\ \frac{\alpha|0\rangle-\beta|1\rangle}{\sqrt{2}} & prob = \frac{1}{2} \end{cases} \tag{5.28}$$

Repeating the measurement does not give the same post-measurement state. This measurement scheme is not repeatable.

The action of U and W on $|\phi\rangle_A |0\rangle_B$ can be written as:

$$WU |\phi\rangle_A |0\rangle_B = M_0 |\phi\rangle_A |+\rangle_B + M_1 |\phi\rangle_A |-\rangle_B \tag{5.29}$$

The generalized measurement is:

$$WU |\phi\rangle = WU |\phi\rangle_A |0\rangle_B = \sum_a M_a |\phi\rangle_A |a\rangle_B \tag{5.30}$$

$$\sum_a M_a^+ M_a = \mathbb{I} \tag{5.31}$$

In this example $E_a = M_a^+ M_a$. The probability of measuring M_a is:

$$Prob(M_a) = |M_a |\phi\rangle|^2 = \langle\phi| M_a^+ M_a |\phi\rangle \tag{5.32}$$

Let's suppose we did the first measurement and got M_a, what is the probability that a second measurement will give M_b?

$$Prob(M_b|M_a) = \frac{Prob(M_b \cap M_a)}{Prob(M_a)} = \frac{|M_b M_a |\phi\rangle|^2}{|M_a |\phi\rangle|^2} \tag{5.33}$$

5.3 POVM - Positive operator valued measurement

Def 9 (POVM) *A POVM is a collection of operators $\{E_a\}$ s.t.:*

1. *E_a is self-adjoint* [2]*: $E_a^+ = E_a$*
2. *E_a is positive semi-definite: $\langle\phi| E_a |\phi\rangle \geq 0$*
3. *E_a satisfies a completeness relation*[3]*: $\sum_a E_a = \mathbb{I}$*

What is the post measurement state $|\phi^{PM}\rangle$ after a POVM?

Since $E_a = M_a^+ M_a$, the square root of E_a is $M_a = \sqrt{E_a}$ but it is not unique so we write it as:

$$\sqrt{E_a} = U_a M_a \tag{5.34}$$

The p.m. state is:

$$|\phi^{PM}\rangle = \frac{U_a M_a |\phi\rangle}{\sqrt{\langle\phi| M_a |\phi\rangle}} \tag{5.35}$$

After a POVM there is an ambiguity in the definition of the post measurement state.

5.3.1 POVM: distinguish quantum states

Let us consider two orthogonal states $\{|0\rangle, |1\rangle\}$. The POVM that distinguishes the two states is:

$$E_0 = |0\rangle\langle 0| \tag{5.36}$$

$$E_1 = |1\rangle\langle 1| \tag{5.37}$$

$$\text{initial state } |0\rangle \xrightarrow{measure} \begin{cases} E_0 & prob(E_0) = \langle 0| E_0 |0\rangle = 1 \\ E_1 & prob(E_1) = \langle 0| E_1 |0\rangle = 0 \end{cases} \tag{5.38}$$

$$\text{initial state } |1\rangle \xrightarrow{measure} \begin{cases} E_0 & prob(E_0) = \langle 1| E_0 |1\rangle = 0 \\ E_1 & prob(E_1) = \langle 1| E_1 |1\rangle = 1 \end{cases} \tag{5.39}$$

Since $\{|0\rangle, |1\rangle\}$ are orthogonal states they can be distinguished with certainty.

Let us now consider two non orthogonal states $|\psi_1\rangle, |\psi_2\rangle$.

[2] Unlike the PVM operators, POVM $\{E_a\}$ are not necessarily orthogonal or idempotent.
[3] POVM do not satisfy the PVM completeness relation $\sum_m E_m^+ E_m = \mathbb{I}$

Th 5 *It is possible to distinguish quantum states with certainty only if they are orthogonal.*

Proof:

We need a set of POVM s.t.

$$P_1 |\psi_2\rangle = 0 \tag{5.40}$$

$$P_2 |\psi_1\rangle = 0 \tag{5.41}$$

and because they are POVM, we have $P_1 + P_2 = \mathbb{I}$.

$$\langle\psi_1| P_1 + P_2 |\psi_2\rangle = \langle\psi_1| \mathbb{I} |\psi_2\rangle = \langle\psi_1|\psi_2\rangle \tag{5.42}$$

$$\langle\psi_1| P_1 + P_2 |\psi_2\rangle = \langle\psi_1| P_1 |\psi_2\rangle + \langle\psi_1| P_2 |\psi_2\rangle = 0 \tag{5.43}$$

$$\rightarrow \langle\psi_1|\psi_2\rangle = 0 \qquad \blacksquare \tag{5.44}$$

There is a POVM that allows to distinguish generic quantum states, although not with certainty. We want to distinguish $|\psi_1\rangle, |\psi_2\rangle$. We introduce $|\psi_1^\perp\rangle$ and $|\psi_2^\perp\rangle$ s.t. :

$$|\psi_1^\perp\rangle \perp |\psi_1\rangle$$

$$|\psi_2^\perp\rangle \perp |\psi_2\rangle$$

and we define the *POVM operators* E_1 , E_2 , $E_?$ as:

$$E_1 = a |\psi_2^\perp\rangle \langle\psi_2^\perp| \tag{5.45}$$

$$E_2 = a |\psi_1^\perp\rangle \langle\psi_1^\perp| \tag{5.46}$$

$$E_? = \mathbb{I} - E_1 - E_2 \tag{5.47}$$

We perform the measurement to distinguish $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$\text{initial state } |\psi_1\rangle \xrightarrow{measure} \begin{cases} E_1 & prob(E_1) = \langle\psi_1| E_1 |\psi_1\rangle \neq 0 \\ E_2 & prob(E_2) = \langle\psi_1| E_2 |\psi_1\rangle = 0 \\ E_? & prob(E_?) = \langle\psi_1| E_? |\psi_1\rangle = 1 - prob(E_1) \neq 0 \end{cases} \tag{5.48}$$

$$\text{initial state } |\psi_2\rangle \xrightarrow{measure} \begin{cases} E_1 & prob(E_1) = \langle\psi_2| E_1 |\psi_2\rangle = 0 \\ E_2 & prob(E_2) = \langle\psi_2| E_2 |\psi_2\rangle \neq 0 \\ E_? & prob(E_?) = \langle\psi_2| E_? |\psi_2\rangle = 1 - prob(E_2) \neq 0 \end{cases} \tag{5.49}$$

POVM is the best we can do to distinguish generic states.

Can we tune a to minimize the probability of $E_?$?
a depends on the way the initial states are chosen:

$$a = \frac{1}{1 + |\langle\psi_1|\psi_2\rangle|} \tag{5.50}$$

5.3.2 Example of POVM - Faulty measurement device

Consider a faulty measurement device that measures the states of a qubit. Sometimes the device does not work and gives uninformative measurements.

How can we describe it?

If it was ideal:

$$E_0 = |0\rangle\langle 0| \tag{5.51}$$

$$E_1 = |1\rangle\langle 1| \tag{5.52}$$

Since it is not ideal:

$$E_0 = (1-p)\,|0\rangle\langle 0| \tag{5.53}$$

$$E_1 = (1-p)\,|1\rangle\langle 1| \tag{5.54}$$

$$E_2 = p\ \mathbb{I} \tag{5.55}$$

where p is the probability of error.

The measurement apparatus works in the following way:

$$\text{state of the qubit } |0\rangle \longrightarrow \begin{cases} |0\rangle & prob = (1-p) \\ |1\rangle & prob = p \end{cases} \tag{5.56}$$

$$\text{state of the qubit } |1\rangle \longrightarrow \begin{cases} |1\rangle & prob = (1-p) \\ |0\rangle & prob = p \end{cases} \tag{5.57}$$

The operators E_a are not projectors, since $E_a \neq E_a^2$. The operators E_a are self-adjoint, positive semi-definite and satisfy a completeness relation so they are POVM.

We want to modify the projectors E_0 and E_1 of the ideal case to describe our system.

$$E_0' = (1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right| \tag{5.58}$$

$$E_1' = (1-p)\left|1\right\rangle\left\langle 1\right| + p\left|0\right\rangle\left\langle 0\right| \tag{5.59}$$

The new operators are not projectors anymore, they are POVM.

Chapter 6

Quantum channels

6.1 Quantum channel

Def 10 (Quantum channel) *A* ***quantum channel*** $\mathcal{E}$ *is a map from the space of density matrices to the space of density matrices:*

$$\rho \overset{\mathcal{E}}{\longmapsto} \mathcal{E}(\rho)$$

with the properties:

1. $\mathcal{E}$ *is* ***linear****:* $\mathcal{E}(a_1\rho_1 + a_2\rho_2) = a_1\mathcal{E}(\rho_1) + a_1\mathcal{E}(\rho_2)$
2. *preserves* ***hermiticity****:* $\rho^+ = \rho \ \Rightarrow \ \mathcal{E}(\rho^+) = \mathcal{E}(\rho)^+$
3. *preserves* ***trace****:* $tr(\rho) = 1 \ \Rightarrow \ tr\left(\mathcal{E}(\rho)\right) = 1$
4. *is* ***completely positive***[1]*:* $\rho \geq 0 \quad \Rightarrow \quad (\mathbb{I} \otimes \mathcal{E})(\rho) \geq 0$

Being a **quantum channel** $\mathcal{E}$ an operator acting on a space of operator ρ , $\mathcal{E}$ is called *super-operator*.
$\mathcal{E}$ as the whole $\sum_k E_k \rho E_k^+$ is *non invertible* in general.

E_k can be *invertible*.

The quantum channel $\mathcal{E}$ is *invertible* iff each $E_k = \alpha_k W$, being α_k a constant and W an *unitary matrix* that doesn't depend on k.

6.2 Model an open quantum system

Quantum channels describe the dynamics of an open quantum system by a density operator ρ_A.

How can we model an open quantum system?

[1]Complete positivity implies positivity: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$

Through a *bipartite system* AB. Tracing out (forgetting) the system B from system AB, we get a description of the evolution of A **alone**.

How U becomes when we forget about B?

Let the initial state be:

$$\rho_{AB} = \rho_A \otimes |e_0\rangle_B \langle e_0|_B \tag{6.1}$$

$|e_0\rangle_B \langle e_0|_B$ is a *pure quantum state* for B.

We apply the evolution operator U to system A + environment B according to Postulte (1.4):

$$\rho'_{AB} = U\rho_A \otimes |e_0\rangle_B \langle e_0|_B U^+ \tag{6.2}$$

Forget about the environment B, i.e. perform a *partial trace* of ρ'_{AB} over B:

$$\rho'_A = tr_B(\rho'_{AB}) \tag{6.3}$$

For $tr_B()$, we need to choose an *orthonormal* basis for B:

$$\{|e_k\rangle_B\}_{k=0}^{\infty} \tag{6.4}$$

$|e_0\rangle_B$ is part of the o.n. basis. Then:

$$\rho'_A = tr_B(\rho'_{AB}) = \sum_{k=0}^{\infty} \langle e_k|_B \rho'_{AB} |e_k\rangle_B = \sum_{k=0}^{\infty} \langle e_k|_B U\rho_A \otimes |e_0\rangle_B \langle e_0|_B U^+ |e_k\rangle_B \tag{6.5}$$

and we recognize:

$$E_k^+ = \langle e_0|_B U^+ |e_k\rangle_B \tag{6.6}$$

$|e_k\rangle_B$ is in $\mathcal{H}_B$ and U^+ is defined on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, so E_k^+ is an operator on $\mathcal{H}_A$.

In (6.5) we recognize E_k in $\langle e_k|_B U\rho_A |e_0\rangle_B$ since ρ_A doesn't act on the state of $\mathcal{H}_B$:

$$E_k = \langle e_k|_B U |e_0\rangle_B \qquad \text{defined on } \mathcal{H}_A \tag{6.7}$$

So the action of tracing out B from ρ'_{AB} allows us to write ρ'_A as:

$$\rho'_A = \sum_k E_k \; \rho_A \; E_k^+ \tag{6.8}$$

which is the general way of writing an *evolution operator* of the system A. The system B is hidden in E_k but E_k is an operator on $\mathcal{H}_A$.

The E_k are *Kraus operators.*

The operation which brings ρ_A to $\rho'_A = \sum_k E_k \; \rho_A \; E_k^+$ can be seen as a *map* from an operator space to an operator space:

$$\rho_A \; \stackrel{\mathcal{E}}{\longmapsto} \mathcal{E}(\rho_A) = \rho'_A = \sum_k E_k \; \rho_A \; E_k^+ \tag{6.9}$$

Remark:

The Kraus operators E_k are not necessarily orthogonal projectors.

Let $dim(\mathcal{H}_A) = N_A$, $dim(\mathcal{H}_B) = N_B$. The Kraus operators E_k are in number N_B nut they act (i.e. are defined) on $\mathcal{H}_A$ which is a space of dimension N_A . In general $N_A \neq N_B$ so the Kraus operators are not necessarily linearly independent.

Th 6 $\rho'_A = \sum_k E_k \; \rho_A \; E_k^+$ *preserves the trace.*

Proof:

We have that:

$$tr_A(\rho_A) \;=\; 1 \tag{6.10}$$

$$tr_{AB}(\rho_A \otimes |e_0\rangle_B \langle e_0|_B) \;=\; 1 \tag{6.11}$$

Since the trace is inviariant under unitary transformations we also have:

$$tr_{AB}(U \rho_A \otimes |e_0\rangle_B \langle e_0|_B U^+) \;=\; 1 \tag{6.12}$$

We want to prove that:

$$tr_A(\rho'_A) \;=\; 1 \tag{6.13}$$

We write ρ'_A in Kraus representation:

$$\begin{aligned} tr_A(\rho'_A) &= tr_A(\sum_k E_k\ \rho_A\ E_k^+) = \sum_k tr_A(E_k\ \rho_A\ E_k^+) = \sum_k tr_A(E_k^+\ E_k\ \rho_A) \\ &= tr_A(\sum_k\ E_k^+\ E_k\ \rho_A) = tr_A(\mathbb{I}\ \rho_A) \\ &= 1 \quad \text{holds for } \forall \rho_A \end{aligned} \tag{6.14}$$

We used the *linearity* and *cyclicity* of the trace, and the *completeness relation* of Kraus operators E_k:

$$\sum_k E_k\ \rho_A\ E_k^+\ =\ \mathbb{I} \qquad \blacksquare \tag{6.15}$$

So, we have the *quantum channel* $\mathcal{E}$:

$$\mathcal{E}(\rho_A) = \rho'_A = \sum_k E_k\ \rho_A\ E_k^+ \tag{6.16}$$

which, as all quantum channels, is *trace-preserving*:

$$tr(\mathcal{E}(\rho_A)) = 1 \tag{6.17}$$

Quantum channels describe the dynamics of quantum systems, isolated or open, without considering the properties of the environment.

1. We entangle the system A with environment B via evolution operator U.
2. Because of the entanglement some information leak from A to B.
3. When we trace out B, we neglect the information about B. This information cannot be recovered.
4. The loss of information of system A that leaked into B which was neglected (trace out B) is called *decoherence*.

Def 11 (Decoherence) ***Decoherence*** *is the loss of coherence. A system is said to be coherent as long as there exists a definite phase relation between different states.*

$\mathcal{E}$ can start from a *pure state* $|\psi\rangle_A \langle\psi|_A$ and make it into a *mixed state*, due to a loss of coherence:

$$\mathcal{E}(|\psi\rangle_A \langle\psi|_A) = \sum_k E_k\ \ |\psi\rangle_A \langle\psi|_A\ \ E_k^+ \tag{6.18}$$

Decoherence arises because by entangling A and B via U we had some **information leaking** from $A \longrightarrow B$ which cannot be recovered since we forgot about B by tracing it out.

6.2.1 Example: 1-qubit system

Consider a 1-qubit system as system B and choose as o.n. basis $\{|e_k\rangle_B\}_k$ the usual *computational basis*:

$$\mathcal{H}_B \;=\; span\left\{|0\rangle_B\,, |1\rangle_B\right\} \tag{6.19}$$

System A is described by a density matrix ρ_A.
We entangle the system A with environment B via evolution operator U.
$|0\rangle_B$ is the initial state for system B.

$$\rho'_{AB} = U\rho_A \otimes |0\rangle_B \langle 0|_B U^+ \tag{6.20}$$

Then we trace out system B.

$$\rho'_A = \sum_{k=0}^{1} \langle e_k|_B U\rho_A \otimes |0\rangle_B \langle 0|_B U^+ |e_k\rangle_B \tag{6.21}$$

U was defined in eq. 5.13. $b = 0$ because we chose $|0\rangle_B$ as the initial state for system B.

$$\begin{aligned} U &= \sum_{a,b=0}^{N-1} E_a \otimes |a+b\rangle_B \langle b|_B \\ &= E_0 \otimes |0\rangle_B \langle 0|_B + E_1 \otimes |1\rangle_B \langle 0|_B \\ &= |0\rangle_A \langle 0|_A \otimes |0\rangle_B \langle 0|_B + |1\rangle_A \langle 1|_A \otimes |1\rangle_B \langle 0|_B \\ &= |0\rangle_A \langle 0|_A \otimes \mathbb{I}_B + |1\rangle_A \langle 1|_A \otimes X_B \end{aligned} \tag{6.22}$$

Now we compute the Kraus operators:

$$E_k = \langle e_k|_B U |e_0\rangle_B \qquad \text{defined on } \mathcal{H}_A \tag{6.23}$$

$$\begin{aligned} E_0 &= \langle 0|_B U |0\rangle_B \\ &= \langle 0|_B \left(|0\rangle_A \langle 0|_A \otimes \mathbb{I}_B + |1\rangle_A \langle 1|_A \otimes X_B\right) |0\rangle_B \\ &= |0\rangle_A \langle 0|_A \end{aligned} \tag{6.24}$$

$$\begin{aligned}E_1 &= \langle 1|_B \, U \, |0\rangle_B \\ &= \langle 1|_B \left(|0\rangle_A \langle 0|_A \otimes \mathbb{I}_B + |1\rangle_A \langle 1|_A \otimes X_B \right) |0\rangle_B \\ &= |1\rangle_A \langle 1|_A\end{aligned} \tag{6.25}$$

Now we can write the quantum channel $\mathcal{E}$ in Kraus representation:

$$\mathcal{E}(\rho_A) = E_0 \rho_A E_0^{+} + E_1 \rho_A E_1^{+} \tag{6.26}$$

If we choose a pure state ρ_A:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{6.27}$$

$$\rho_A = |\psi\rangle \langle\psi| = |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \alpha^*\beta |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1| \tag{6.28}$$

Then:

$$\begin{aligned}\mathcal{E}(\rho_A) &= |0\rangle_A \langle 0|_A \, \rho_A \, |0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A \, \rho_A \, |1\rangle_A \langle 1|_A \\ &= |\alpha|^2 |0\rangle_A \langle 0|_A + |\beta|^2 |1\rangle_A \langle 1|_A\end{aligned} \tag{6.29}$$

Decoherence takes a *quantum pure state* $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ into a *classical mixed state* $\rho = |\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|$.

6.3 Postulates of QM

6.3.1 Postulate 1 - States of a quantum system

There is a Hilbert space associated to any isolated physical system. The system is completely described by its density operator, which is a positive self-adjoint operator ρ with trace one.

6.3.2 Postulate 2 - Observables

An observable is a property of a physical system that in principle can be measured. In QM observable are associated to self-adjoint operators.

6.3.3 Postulate 3 - Measurements

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are self-adjoint and positive semi-definite operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment.

If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by:

$$Prob(m) = tr(M_m^+ M_m \rho) \tag{6.30}$$

and the state of the system after the measurement is:

$$\rho^{PM} = \frac{M_m \rho M_m^+}{tr(M_m^+ M_m \rho)} \tag{6.31}$$

The measurement operators satisfy the completeness equation:

$$\sum_m M_m^+ M_m = \mathbb{I} \tag{6.32}$$

Note: Projector-valued measures are a particular case of the above.

6.3.4 Postulate 4 - Dynamics

The evolution of a quantum system is described by a quantum channel $\mathcal{E}$: the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by an operator sum representation:

$$\rho' = \mathcal{E}(\rho) = \sum_n \; E_n \; \rho E_n^+ \tag{6.33}$$

where the operators E_n (called operation elements) satisfy

$$\sum_n E_n^+ E_n = \mathbb{I} \tag{6.34}$$

6.3.5 Postulate 5 - Composite systems

If the Hilbert space of a system A is $\mathcal{H}_A$ and the Hilbert space of a system B is $\mathcal{H}_B$ then the Hilbert space of the composite system AB is:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{6.35}$$

and the state of the system will be:

$$\rho = \rho_A \otimes \rho_B \tag{6.36}$$

6.4 Qubit channels

6.4.1 Bit-flip channel

The bit-flip flips a qubit:

$$\begin{aligned} |0\rangle &\longrightarrow |1\rangle \\ |1\rangle &\longrightarrow |0\rangle \end{aligned}$$

We couple the system A with environment B.

With probability p, the outcome is:

$$\begin{aligned} |0\rangle_A |0\rangle_B &\longrightarrow |1\rangle_A |1\rangle_B \\ |1\rangle_A |0\rangle_B &\longrightarrow |0\rangle_A |1\rangle_B \end{aligned}$$

With probability $1 - p$ the outcome is:

$$\begin{aligned} |0\rangle_A |0\rangle_B &\longrightarrow |0\rangle_A |0\rangle_B \\ |1\rangle_A |0\rangle_B &\longrightarrow |1\rangle_A |0\rangle_B \end{aligned}$$

So, the description of the evolution U is:

$$|\phi\rangle_A \otimes |0\rangle_B \xrightarrow{U} \sqrt{1-p}\,|\phi\rangle_A \otimes |0\rangle_B + \sqrt{p}X_A\,|\phi\rangle_A \otimes |1\rangle_B \tag{6.37}$$

$$U = \sqrt{1-p}\;\mathbb{I}_{AB} + \sqrt{p}\;X_{AB} \tag{6.38}$$

where $\mathbb{I}_{AB} = \mathbb{I}_A \otimes \mathbb{I}_B$ and $X_{AB} = X_A \otimes X_B$.
We compute the Kraus operators:

$$E_0 = \langle 0|_B\,U\,|0\rangle_B = \sqrt{1-p}\;\mathbb{I}_A \tag{6.39}$$

$$E_1 = \langle 1|_B\,U\,|0\rangle_B = \sqrt{p}\;X_A \tag{6.40}$$

Then we compute the channel:

$$\begin{aligned}\mathcal{E}(\rho_A) &= E_0\rho_A E_0^{+} + E_1\rho_A E_1^{+} \\ &= (1-p)\rho_A + pX_A\rho_A X_A \end{aligned}\tag{6.41}$$

Let's see how the channel behaves. We choose the pure state $\rho_A = |0\rangle_A\langle 0|_A$:

$$\begin{aligned}\mathcal{E}(\rho_A) &= (1-p)\,|0\rangle_A\langle 0|_A + pX_A\,|0\rangle_A\langle 0|_A\,X_A \\ &= (1-p)\,|0\rangle_A\langle 0|_A + p\,|1\rangle_A\langle 1|_A \end{aligned}\tag{6.42}$$

Now we choose the pure state $\rho_A = |+\rangle_A\langle +|_A$ [2]:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{6.43}$$

$$\begin{aligned}\mathcal{E}(\rho_A) &= (1-p)\,|+\rangle_A\langle +|_A + pX_A\,|+\rangle_A\langle +|_A\,X_A \\ &= (1-p)\,|+\rangle_A\langle +|_A + p\,|+\rangle_A\langle +|_A \\ &= |+\rangle_A\langle +|_A \end{aligned}\tag{6.44}$$

$\rho_A = |+\rangle_A\langle +|_A$ is invariant under the bit-flip channel.

[2] $|+\rangle$ is an eigenstate of X.

6.4.2 Phase flip channel

We do the same for the phase flip operator Z:

$$\mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z \tag{6.45}$$

6.4.3 Phase flip + bit-flip channel

We do the same for the phase flip + bit-flip operator $Y = iXZ$:

$$\mathcal{E}(\rho) = (1-p)\rho + pY\rho Y \tag{6.46}$$

6.4.4 Depolarising channel

The depolarising channel leaves intact the qubit with probability $1-p$, it applies X with probability $p/3$, Y with probability $p/3$, Z with probability $p/3$:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) \tag{6.47}$$

The Kraus operators:

$$E_0 = \sqrt{1-p}\ \mathbb{I} \tag{6.48}$$

$$E_1 = \sqrt{\frac{p}{3}}X \tag{6.49}$$

$$E_2 = \sqrt{\frac{p}{3}}Y \tag{6.50}$$

$$E_3 = \sqrt{\frac{p}{3}}Z \tag{6.51}$$

Let's see the action of $\mathcal{E}(\rho)$ on the Bloch sphere:

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{n}\cdot\vec{\sigma}) \tag{6.52}$$

where $\vec{n} = (n_x, n_y, n_z)$, $\vec{\sigma} = (X, Y, Z)$, $|\vec{n}| \leq 1$.

$$\mathcal{E}(\rho) = \mathcal{E}\left(\frac{1}{2}(\mathbb{I} + \vec{n}\cdot\vec{\sigma})\right) = \frac{1}{2}\mathcal{E}(\mathbb{I}) + \frac{1}{2}\vec{n}\cdot\mathcal{E}(\vec{\sigma}) \tag{6.53}$$

where

$$\mathcal{E}(\vec{\sigma}) = (\mathcal{E}(X), \mathcal{E}(Y), \mathcal{E}(Z)) \tag{6.54}$$

Let's compute.

$$\begin{aligned}
\mathcal{E}(\mathbb{I}) &= (1-p)\mathbb{I} + \frac{p}{3}(X\mathbb{I}X + Y\mathbb{I}Y + Z\mathbb{I}Z) \\
&= (1-p)\mathbb{I} + \frac{p}{3}(3\mathbb{I}) \\
&= \mathbb{I}
\end{aligned} \tag{6.55}$$

$$\begin{aligned}
\mathcal{E}(X) &= (1-p)X + \frac{p}{3}(XXX + YXY + ZXZ) \\
&= (1-p)X - \frac{p}{3}X \\
&= \left(1 - \frac{4}{3}p\right) X
\end{aligned} \tag{6.56}$$

where we used that X, Y, Z anticommute. Similarly:

$$\mathcal{E}(Y) = \left(1 - \frac{4}{3}p\right) Y \tag{6.57}$$

$$\mathcal{E}(Z) = \left(1 - \frac{4}{3}p\right) Z \tag{6.58}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\mathcal{E}(\mathbb{I}) + \frac{1}{2}\vec{n} \cdot \mathcal{E}(\vec{\sigma}) = \frac{1}{2}\mathbb{I} + \frac{1}{2}\left(1 - \frac{4}{3}p\right) \vec{n} \cdot \vec{\sigma} \tag{6.59}$$

The block sphere shrinks.

For $p = \frac{3}{4}$, $\mathcal{E}(\rho) = \frac{\mathbb{I}}{2}$. This is a maximally mixed state, it is at the center of the Bloch sphere.

Equivalent definition

We have that for every density matrix ρ:

$$\frac{\mathbb{I}}{2} = \frac{\mathbb{I}}{2}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right) \tag{6.60}$$

The quantum channel 6.59 can be written as:

$$\mathcal{E}(\rho_A) = (1-q)\rho_A + q\,\frac{\mathbb{I}}{2} \tag{6.61}$$

where $q = \frac{4}{3}p$ and $0 \leq p \leq \frac{3}{4}$.

Leaves the qubit state untouched with probability $1 - q$.

Replace the qubit state with the maximally mixed state $\frac{\mathbb{I}}{2}$ with probability q.

6.4.5 Phase damping channel

The phase damping channel models decoherence.

Decoherence is the loss of relative phases among the quantum states that form superpositions.

Example

Faulty apparatus takes as input a pure state $|\psi\rangle$, applies a random rotation $R_z(\theta)$ (can be due to a non precise magnetic field).

θ follows a gaussian probability distribution (variance $= 2\lambda^2$, mean $= 0$):

$$p(\theta) = \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} \tag{6.62}$$

The input state of the machine is the pure state ρ:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \tag{6.63}$$

$$\rho = |\psi\rangle\,\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \tag{6.64}$$

The output state of the machine is a continuous ensemble:

$$\begin{aligned}
\rho' &= \int d\theta\ p(\theta)\ \ |\phi(\theta)\rangle\,\langle\phi(\theta)| \\
&= \int d\theta\ \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} R_z(\theta)\,|\psi\rangle\,\langle\psi|\,R_z^{+}(\theta) \\
&= \int d\theta\ \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} R_z(\theta)\rho R_z^{+}(\theta)
\end{aligned} \tag{6.65}$$

being:

$$R_z(\theta) = e^{-i\frac{\theta}{2}Z} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \tag{6.66}$$

so:

$$R_z(\theta)\rho R_z^{+}(\theta) = \begin{pmatrix} |a|^2 & ab^*\ e^{-i\theta} \\ a^*b\ e^{i\theta} & |b|^2 \end{pmatrix} \tag{6.67}$$

then ρ' becomes:

$$\rho' = \begin{pmatrix} |a|^2 & ab^* \int d\theta \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} e^{-i\theta} \\ a^*b \int d\theta \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} e^{i\theta} & |b|^2 \end{pmatrix}$$
$$= \begin{pmatrix} |a|^2 & ab^* \, e^{-\lambda^2} \\ a^*b \, e^{-\lambda^2} & |b|^2 \end{pmatrix} \tag{6.68}$$

Since:

$$\int d\theta \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} e^{-i\theta} = e^{-\lambda^2}$$
$$\int d\theta \frac{1}{\sqrt{4\pi}\lambda} e^{-\frac{\theta^2}{4\lambda^2}} e^{+i\theta} = e^{-\lambda^2} \tag{6.69}$$

where we have used:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c} \tag{6.70}$$

The off diagonal terms decay exponentially as the variance $2\lambda^2$ increases.
For $\lambda \gg 1$, we see that:

$$\rho' \approx \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{6.71}$$

i.e. ρ' loses coherence.

Phase damping from scattering

States of the atom:

$$\{|0\rangle_A, |1\rangle_A\} \tag{6.72}$$

States of the photon:

$$\{|0\rangle_B, |1\rangle_B, |2\rangle_B\} \tag{6.73}$$

The photon scatters to different final states depending on the internal states of the atom.

With probability $1-p$ nothing happens.

With probability p:

$$\begin{cases} |0\rangle_B \to |1\rangle_B & \text{if atom is in } |0\rangle_A \\ |0\rangle_B \to |2\rangle_B & \text{if atom is in } |1\rangle_A \end{cases} \tag{6.74}$$

The operator U acts as:

$$|0\rangle_A |0\rangle_B \xrightarrow{U} \sqrt{p}\, |0\rangle_A |1\rangle_B + \sqrt{1-p}\, |0\rangle_A |0\rangle_B \tag{6.75}$$

$$|1\rangle_A |0\rangle_B \xrightarrow{U} \sqrt{p}\, |1\rangle_A |2\rangle_B + \sqrt{1-p}\, |1\rangle_A |0\rangle_B \tag{6.76}$$

We compute the Kraus operators:

$$E_0 = \langle 0|_B \, U \, |0\rangle_B = \sqrt{1-p}\; \mathbb{I}_A \tag{6.77}$$

$$E_1 = \langle 1|_B \, U \, |0\rangle_B = \sqrt{p}\, |0\rangle_A \langle 0|_A \tag{6.78}$$

$$E_2 = \langle 2|_B \, U \, |0\rangle_B = \sqrt{p}\, |1\rangle_A \langle 1|_A \tag{6.79}$$

E_0, E_1, E_2 are not independent in fact $E_0 \propto E_1 + E_2$.

The Kraus operators satisfy the completeness relation 6.34, so we can compute the quantum channel:

$$\mathcal{E}(\rho) = E_0 \rho E_0^+ + E_1 \rho E_1^+ + E_2 \rho E_2^+ \tag{6.80}$$

$$\mathcal{E}(\rho) = \mathcal{E}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix} \tag{6.81}$$

For $p = 1$:

$$\mathcal{E}(\rho) = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix} \tag{6.82}$$

$\mathcal{E}(\rho)$ has lost coherence.

If the state of the atom is $|\psi\rangle = a\, |0\rangle_A + b\, \langle 1|_A$:

$$\rho'_A = \mathcal{E}(\rho_{\mathcal{A}}) = |a|^2 \, |0\rangle_A \langle 0|_A + |b|^2 \, |1\rangle_A \langle 1|_A \tag{6.83}$$

6.4.6 Amplitude damping channel

The amplitude damping channel models the spontaneous decay of an atom from an excited state to the ground state of a 2-level system with the emission of a photon.

States of the atom:

$$\{|0\rangle_A, |1\rangle_A\} \tag{6.84}$$

States of the photon:

$$\{|0\rangle_B, |1\rangle_B\} \tag{6.85}$$

The atom decays from $|1\rangle_A$ to $|0\rangle_A$ and a photon is emitted i.e that state of the photon goes from $|0\rangle_B$ to $|1\rangle_B$ with probability p.

The process we want to model has 2 parts:

1. If the atom is in state $|0\rangle_A$ nothing happens.
2. If the atom is in state $|1\rangle_A$:

$$\begin{cases} |1\rangle_A \xrightarrow{\text{atom decays}} |0\rangle_A \text{ and } |0\rangle_B \xrightarrow{\text{photon is emitted}} |1\rangle_B & \text{with prob } p \\ |1\rangle_A \xrightarrow{\text{atom does not decay}} |1\rangle_A \text{ and } |0\rangle_B \xrightarrow{\text{photon is not emitted}} |0\rangle_B & \text{with prob } 1-p \end{cases}$$

The 2 evolution operators are:

1. $|00\rangle \xrightarrow{U_1} |00\rangle \implies U_1 = |00\rangle\langle 00|$
2. $|10\rangle \xrightarrow{U_2} \sqrt{p}\,|01\rangle + \sqrt{1-p}\,|10\rangle \implies U_2 = \sqrt{p}\,|01\rangle\langle 10| + \sqrt{1-p}\,|10\rangle\langle 10|$

The evolution operator of the whole process U is:

$$U = |00\rangle\langle 00| + \sqrt{1-p}\,|10\rangle\langle 10| + \sqrt{p}\,|01\rangle\langle 10| \tag{6.86}$$

We want to compute the quantum channel by tracing out the system B of the photon.

Let's compute the quantum channel in the Kraus representation. We start by computing the Kraus operators.

$$E_0 = \langle 0|_B U |0\rangle_B = |0\rangle_A \langle 0|_A + \sqrt{1-p}\,|1\rangle_A \langle 1|_A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \tag{6.87}$$

$$E_1 = \langle 1|_B U |0\rangle_B = \sqrt{p}\,|0\rangle_A \langle 1|_A = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{6.88}$$

The Kraus operators satisfy the completeness relation:

$$E_0^{+} E_0 + E_1^{+} E_1 = \mathbb{I} \tag{6.89}$$

By using the Kraus representation we compute the quantum channel $\mathcal{E}(\rho_{\mathcal{A}})$:

$$\mathcal{E}(\rho_{\mathcal{A}}) = \mathcal{E}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} + p\ \rho_{11} & \sqrt{1-p}\ \rho_{01} \\ \sqrt{1-p}\ \rho_{10} & (1-p)\ \rho_{11} \end{pmatrix} \tag{6.90}$$

We take the trace of the quantum channel:

$$tr(\mathcal{E}(\rho_{\mathcal{A}})) = \rho_{00} + p\rho_{11} + (1-p)\rho_{11} = \rho_{00} + \rho_{11} \tag{6.91}$$

So:

$$\rho_{00} + \rho_{11} = 1 \tag{6.92}$$

For $p = 1$, coherences go to 0:

$$\mathcal{E}(\rho_{\mathcal{A}}) = \begin{pmatrix} \rho_{00} + \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.93}$$

$$\mathcal{E}(\rho_{\mathcal{A}}) = |0\rangle \langle 0| \tag{6.94}$$

If the state of the atom is $\rho_A = |1\rangle_A \langle 1|_A$:

$$\rho_A' = \mathcal{E}(\rho_{\mathcal{A}}) = |0\rangle_A \langle 0|_A \tag{6.95}$$

Chapter 7

Quantum circuits

7.1 Introduction

State for 1 qubit:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{7.1}$$

State for N qubits:

$$|\psi\rangle = \sum_{x=0}^{2^N-1} c_x |x\rangle \tag{7.2}$$

where $|x\rangle$ is the binary (e.g. $|x\rangle = |0101110\rangle$).

Advantages of quantum computation:

1. Superposition
2. Entanglement
3. Interference: coherence between the states that form the superposition.

$$|\psi\rangle = c_{00} |00\rangle + e^{i\phi_{01}} c_{01} |01\rangle + e^{i\phi_{10}} c_{10} |10\rangle + e^{i\phi_{11}} c_{11} |11\rangle \tag{7.3}$$

where the relative phases $\phi_{01}, \phi_{10}, \phi_{11}$ are measurable.

Exercise

Think about an experimental protocol that is able to guess δ (with error).

$$|\psi\rangle = \frac{|0\rangle + e^{i\delta} |1\rangle}{\sqrt{2}} \tag{7.4}$$

Solution

If I measure in $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ basis:

$$\begin{cases} Prob(|0\rangle) & = |\langle 0|\psi\rangle|^2 = \frac{1}{2} \\ Prob(|1\rangle) & = |\langle 1|\psi\rangle|^2 = \frac{1}{2} \end{cases} \tag{7.5}$$

If I measure in $X = |+\rangle\langle +| - |-\rangle\langle -|$ basis:

$$|\psi\rangle = e^{i\frac{\delta}{2}} \cos\frac{\delta}{2} |+\rangle - e^{i\frac{\delta}{2}} \sin\frac{\delta}{2} |-\rangle \tag{7.6}$$

$$\begin{cases} Prob(|+\rangle) & = |\langle +|\psi\rangle|^2 = \cos^2\left(\frac{\delta}{2}\right) \\ Prob(|-\rangle) & = |\langle -|\psi\rangle|^2 = \sin^2\left(\frac{\delta}{2}\right) \end{cases} \tag{7.7}$$

There is a POVM which measures the relative phase δ: δ is an observable.

7.2 Examples of classical gates

NOT gate

in	out
0	1
1	0

XOR gate

The last column (out*) was added to make the gate reversible. We choose out* equal to the first input.

in	in	out	out*
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

AND gate

in	in	out	out*
a	b	ab	a
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

We added the last column out* to make the gate reversible, but it is still not reversible.

We define the Toffoli gate:

in	in	in	out	out	out
a	b	c	a	b	c*
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

where c* is $c^* = c \oplus a \cdot b$.

In classical computation we spend energy to erase information. For example if we give two bits $1, 0$ as input of an AND gate we get the bit 0, so the 1 bit is erased.

Landauer's principle states that any logically irreversible manipulation of information, such as the erasure of a bit must be accompanied by a corresponding entropy increase.

Example of Landauer principle:

We have a gas of 1 particle in a box divided in two parts.

When the particle is in the first half of the box it represents the bit state 1, when it is in the other half it represents the bit state 0. In order to erase the bit 1 we compress the gas towards the half of the box of the 0 bit at T constant. Now the box is not divided in two parts anymore so we have erased bit 1, the only possible state is 0.

The entropy increase associated to this irreversible transformation is:

$$\Delta S = k_B \log(2) \tag{7.8}$$

and the energy spent in the transformation is:

$$W = \Delta S\ T = k_B T \log(2) \tag{7.9}$$

which is the minimum energy for erasing classical information.

7.3 Quantum gates

Quantum gates are unitary operators U:

$$|\psi\rangle \longrightarrow U|\psi\rangle \tag{7.10}$$

Quantum gates have to be unitary because:

1. Preserve probability
2. $U(t) = e^{-iHt/\hbar}$, it's always easier to implement Hamiltonians. You need to evolve H for a time t.

We want to have unitary operators that are able to generate a generic unitary operator.

We need a basis set of unitary operators that act on a few qubits, but that can generate every unitary operator acting on N qubits (the equivalent of NAND in classical computation).

7.3.1 Examples of 1-qubit gates

- NOT gate: X
- phase-bit-flip gate: Y
- phase-flip gate: Z
- Hadamard gate (change of basis $|0\rangle, |1\rangle \longrightarrow |+\rangle, |-\rangle$) :

$$H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7.11}$$

- Rotations: $R_{\hat{\alpha}} = e^{-i\frac{\theta}{2}\hat{\alpha}}$

Th 7 *Every unitary operator U*

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.12}$$

can be written as:

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \tag{7.13}$$

where $\alpha, \beta, \gamma, \delta$ depend on a, b, c, d.

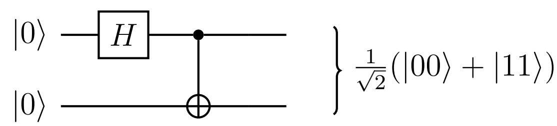
7.3.2 Example of 2-qubits gate

The CNOT gate flips the target qubit if the control qubit is in $|1\rangle$:

in	in	out	out
control	target	control	target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

$$CNOT = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7.14}$$

With the CNOT gate we can build entangled states.



$$|00\rangle \xrightarrow{H_1} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle \tag{7.15}$$

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \tag{7.16}$$

The final state is an entangled state: CNOT is an *entangling gate*.

In matrix form:

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{7.17}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1\begin{pmatrix}0\\1\end{pmatrix} \\ 0\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{7.18}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{7.19}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.20}$$

The first step of the circuit is to apply an H gate to the first qubit.

$$H_A = H \otimes \mathbb{I} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & -\mathbb{I}_2 \end{pmatrix} \tag{7.21}$$

The second step is to apply the CNOT gate: the first qubit is the control and the second is the target.

$$CNOT = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & X \end{pmatrix} \tag{7.22}$$

The whole circuit is represented by the unitary operator U:

$$U = CNOT \cdot H_A = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & X \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & -\mathbb{I}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & \mathbb{I}_2 \\ X & -X \end{pmatrix} \tag{7.23}$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \tag{7.24}$$

$$\begin{aligned} U|00\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \end{aligned} \tag{7.25}$$

7.3.3 Exercise: prepare the Bell states

By using $X, Y, Z, H, CNOT$, create a circuit to realize the four Bell states:

$$|\phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|{+}{+}\rangle + |{-}{-}\rangle}{\sqrt{2}} \tag{7.26}$$

$$|\phi_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{|{+}{+}\rangle - |{-}{-}\rangle}{\sqrt{2}} \tag{7.27}$$

$$|\phi_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{|{+}{-}\rangle + |{-}{+}\rangle}{\sqrt{2}} \tag{7.28}$$

$$|\phi_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|{-}{+}\rangle - |{+}{-}\rangle}{\sqrt{2}} \tag{7.29}$$

Solution

The Bell states $|\phi_{00}\rangle$, $|\phi_{01}\rangle$, $|\phi_{10}\rangle$, $|\phi_{11}\rangle$ are created by the same circuit 7.15 applied respectively to the inputs $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

7.3.4 Problem

Consider Alice, Bob, Charlie.
C prepares a state among the Bell states $|\phi_{ij}\rangle$.
C gives one of the qubits of $|\phi_{ij}\rangle$ to A and the other to B.
A and B are far apart.
How can A and B guess which one of the four states are given?

If A measures Z_A on $|\phi_{00}\rangle$ the p.m. state will be:

$$Z_A\,|\phi_{00}\rangle \ : \ \begin{cases} |00\rangle & p = \frac{1}{2} \\ |11\rangle & p = \frac{1}{2} \end{cases} \tag{7.30}$$

Now B will measure $|00\rangle$ with $p = 1$ or $|11\rangle$ with $p = 1$, because the 2 qubits of the Bell states are entagled.

There is a non-local operation that allows them to distinguish the states.
They can measure non-local operators $Z_A Z_B$, $X_A X_B$.

The two operators commute, so measuring $Z_A Z_B$ does not disturb $X_A X_B$.

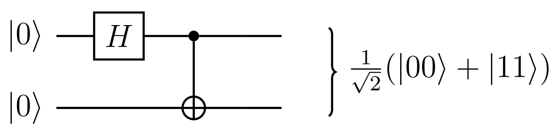
$$[Z_A Z_B, X_A X_B] = 0 \tag{7.31}$$

The outcomes of the measurements of the operators $Z_A Z_B$, $X_A X_B$ on the Bell states are:

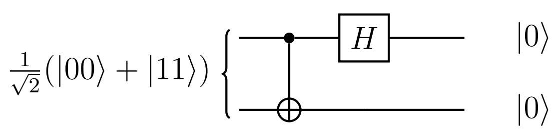
	$\|\phi_{00}\rangle$	$\|\phi_{01}\rangle$	$\|\phi_{10}\rangle$	$\|\phi_{11}\rangle$
$Z_A Z_B$	+1	-1	+1	-1
$X_A X_B$	+1	+1	-1	-1

This measurement must be done simultaneously so A and B cannot be far apart.
In order to perform the measurement at different times we need to disentagle the Bell pair and then measure $Z_A Z_B$.

The following circuit prepares the Bell state $|\phi_{00}\rangle$:



In order to disentagle the Bell state we need to invert the circuit which prepares it:

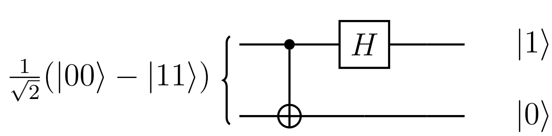


$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = |+\rangle \otimes |0\rangle$$

$$\xrightarrow{H_A} |0\rangle \otimes |0\rangle \tag{7.32}$$

This circuit transforms the entangled Bell state $|\phi_{00}\rangle$ into a separable state that can be measured to get information on the orginal Bell pair.
On this circuit I can measure $Z_A Z_B$ not simultaneously.

The Bell states $|\phi_{00}\rangle$, $|\phi_{01}\rangle$, $|\phi_{10}\rangle$, $|\phi_{11}\rangle$ are disentangled by the same circuit 7.32 giving as outputs respectively $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. E.g. for the Bell state $|\phi_{10}\rangle$:



$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle = |-\rangle \otimes |0\rangle \tag{7.33}$$

$$\xrightarrow{H_A} |1\rangle \otimes |0\rangle \tag{7.34}$$

7.3.5 EPR paradox

Consider an entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{7.35}$$

This system has **total spin** $J = 0$ and decays in 2 particles that fly very far away from each other:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{7.36}$$

$|\psi\rangle_{AB}$ is the Bell state $|\phi_{11}\rangle$.

Suppose Alice measures her system in Z basis; she will find:

$$\begin{aligned} &|0\rangle_A \qquad \text{with } p = \frac{1}{2} \\ &|1\rangle_A \qquad \text{with } p = \frac{1}{2} \end{aligned}$$

Since in Q.M. the total spin J is conserved, the 2 particles have opposite spins. Therefore, the p.m. states of the particles can only be:

$$\begin{aligned} \left|\psi_1^{PM}\right\rangle = |0\rangle_A |1\rangle_B \\ \left|\psi_2^{PM}\right\rangle = |1\rangle_A |0\rangle_B \end{aligned}$$

So the state received by Bob is known with $p = 1$ after the measurement of Alice.

What if Alice measures the system in X basis?

$$|\phi_{11}\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|{-}{+}\rangle - |{+}{-}\rangle) \tag{7.37}$$

$$\begin{aligned} &|-\rangle_A \qquad \text{with } p = \frac{1}{2} \\ &|+\rangle_A \qquad \text{with } p = \frac{1}{2} \end{aligned}$$

Since in Q.M. the total spin J is conserved, the 2 particles have opposite spins. Therefore, the p.m. states of the particles can only be:

$$\left|\phi_1^{PM}\right\rangle = \left|-\right\rangle_A \left|+\right\rangle_B$$
$$\left|\phi_2^{PM}\right\rangle = \left|+\right\rangle_A \left|-\right\rangle_B$$

Again, the state of Bob is decided by Alice which is far away from Bob.

Alice measures Z_A and she knows the eigenvalues of Z_B. Suppose that the p.m. state is $\left|\psi_1^{PM}\right\rangle$:

$$\left|\psi_1^{PM}\right\rangle = \left|0\right\rangle_A \left|1\right\rangle_B = \left|0\right\rangle_A \otimes \frac{\left|+\right\rangle_B - \left|-\right\rangle_B}{\sqrt{2}}$$

Bob will measure X_B and will find:

$$\begin{array}{lll} \left|+\right\rangle & \text{with} & p = \frac{1}{2} \\ \left|-\right\rangle & \text{with} & p = \frac{1}{2} \end{array}$$

We are able to know the eigenvalues of X and Z but in Q.M. they don't commute:

$$[X_B, Z_B] \neq 0 \tag{7.38}$$

In conclusion the Bell state $\left|\phi_{11}\right\rangle_{AB}$ can't be a *complete* description of reality.

The **hidden variable theory** was proposed to solve this problem.

7.3.6 Hidden variable theory

To remove the uncertainty coming from the measurement process in such a way that the measurement becomes deterministic.

In theory measurements are deterministic but they appear probabilistic because we don't have access to the full description of the state. There is a *d.o.f.* that is **hidden** to us, called ***hidden variable***.

Example

When we prepare the state $|0\rangle$, we are actually preparing $|0, \lambda\rangle$ where λ is the *hidden variable*.
We rotate $|0, \lambda\rangle$ and get:

$$|\psi, \lambda\rangle = \cos\frac{\theta}{2}\,|0, \lambda\rangle + \sin\frac{\theta}{2}\,|1, \lambda\rangle \tag{7.39}$$

and we measure the Z operator. We'll find:

$$\begin{array}{lcl} |0, \lambda\rangle & \text{if} & 0 \leq \lambda \leq \cos^2\frac{\theta}{2} \\ |1, \lambda\rangle & \text{if} & \cos^2\frac{\theta}{2} \leq \lambda \leq 1 \end{array}$$

The randomness of the measurement is in our ignorance about the *hidden variable* λ .

7.3.7 Bell theorem

Th 8 (Bell theorem) *A theory based on local hidden variables cannot reproduce the results of quantum mechanics.*

Proof:

Consider the Bell state $|\phi_{11}\rangle$ that we call $|\psi\rangle$:

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{7.40}$$

If Alice measures Z and gets $|1\rangle$, then Bob will get $|0\rangle$ and viceversa.
The outcomes of Alice and Bob are anticorrelated.

Alice now measures the spin along the axis $\hat{a}$: ($\hat{a}$ is a unit vector)
Alice measures $\sigma_A \cdot \hat{a}$.
Bob measures $\sigma_B \cdot \hat{b}$, where $\hat{a} \neq \hat{b}$.
The outcomes seem uncorrelated.

We define $R(\hat{a}) = \pm 1$ and $R'(\hat{b}) = \pm 1$ as the eigenvalues that Alice and Bob measure in their lab.

$$F^{QM}(\hat{a}, \hat{b}) = R(\hat{a})R'(\hat{b}) = \langle\psi|\, \sigma_A \cdot \hat{a}\ \sigma_B \cdot \hat{b}\, |\psi\rangle = -\hat{a} \cdot \hat{b} \tag{7.41}$$

In the case $\hat{a} = \hat{b} = \hat{z}$, we get $F = -1$. We recover the previous case where the outcomes are anticorrelated.

We can develop a hidden variable theory with λ: $R(\hat{a}, \lambda)$, $R'(\hat{b}, \lambda)$.
λ decides if the eigenvalues R and R' are ± 1.

$$F^{HV} = R(\hat{a}, \lambda)R'(\hat{b}, \lambda) = \int d\lambda P(\lambda) R R' \tag{7.42}$$

where $P(\lambda)$ is the probability distribution of λ.

Since the spins are antialigned, if we choose $\hat{a} = \hat{b}$ the outcomes will be anticorrelated.
This is a constraint for R, R':

$$R'(\hat{b}, \lambda) = -R(\hat{b}, \lambda) \tag{7.43}$$

Therefore:

$$F^{HV}(\hat{a}, \hat{b}) = -\int d\lambda P(\lambda) R(\hat{a}, \lambda) R(\hat{b}, \lambda) \tag{7.44}$$

Let's consider a vector $\hat{c}$.

$$
\begin{aligned}
F^{HV}(\hat{a},\hat{b}) - F^{HV}(\hat{a},\hat{c}) &= -\int d\lambda P(\lambda)[R(\hat{a},\lambda)R(\hat{b},\lambda) - R(\hat{a},\lambda)R(\hat{c},\lambda)] \\
&= -\int d\lambda P(\lambda)[R(\hat{a},\lambda)R(\hat{b},\lambda) - R(\hat{a},\lambda)R(\hat{b},\lambda)R(\hat{b},\lambda)R(\hat{c},\lambda)] \\
&= -\int d\lambda P(\lambda)R(\hat{a},\lambda)R(\hat{b},\lambda)[1 - R(\hat{b},\lambda)R(\hat{c},\lambda)] \qquad (7.45)
\end{aligned}
$$

We take the absolute value:

$$
\left|F^{HV}(\hat{a},\hat{b}) - F^{HV}(\hat{a},\hat{c})\right| = \left|\int d\lambda P(\lambda)R(\hat{a},\lambda)R(\hat{b},\lambda)[1 - R(\hat{b},\lambda)R(\hat{c},\lambda)]\right| \qquad (7.46)
$$

$$
\begin{aligned}
\left|F^{HV}(\hat{a},\hat{b}) - F^{HV}(\hat{a},\hat{c})\right| &\leq \int d\lambda P(\lambda)\cdot|R(\hat{a},\lambda)|\cdot\left|R(\hat{b},\lambda)\right|\cdot\left|1 - R(\hat{b},\lambda)R(\hat{c},\lambda)\right| \\
&= \int d\lambda P(\lambda)(1 - R(\hat{b},\lambda)R(\hat{c},\lambda)) \\
&= 1 - \int d\lambda P(\lambda)R(\hat{b},\lambda)R(\hat{c},\lambda)) \\
&= 1 + F^{HV}(\hat{b},\hat{c}) \qquad (7.47)
\end{aligned}
$$

We found the Bell inequality:

$$
\left|F^{HV}(\hat{a},\hat{b}) - F^{HV}(\hat{a},\hat{c})\right| \leq 1 + F^{HV}(\hat{b},\hat{c}) \qquad (7.48)
$$

The correlators $F^{QM}(\hat{a},\hat{b})$ should satisfy the Bell inequality.

$$
F^{QM}(\hat{a},\hat{b}) = -\hat{a}\cdot\hat{b} = -\cos\theta_{\hat{a},\hat{b}} \qquad (7.49)
$$

Let's consider the following example:

$$
\begin{aligned}
F(\hat{a},\hat{b}) &= -\cos 60^\circ = -\frac{1}{2} \\
F(\hat{b},\hat{c}) &= -\cos 60^\circ = -\frac{1}{2} \\
F(\hat{a},\hat{c}) &= -\cos 120^\circ = \frac{1}{2} \qquad (7.50)
\end{aligned}
$$

Let's substitute in the Bell inequality:

$$\left|-\frac{1}{2}-\frac{1}{2}\right| \leq 1-\frac{1}{2}$$
$$1 \leq \frac{1}{2} \tag{7.51}$$

The Bell inequality is violated, therefore QM cannot be described by a local hidden theory.

Nature is fundamentally non local.

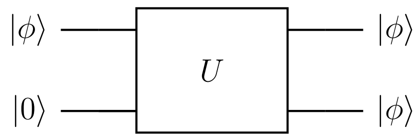
7.3.8 No cloning theorem

Th 9 (No cloning theorem) *It is impossible (there is no unitary operator U) to clone an arbitrary quantum state.*

This statement means that we cannot find a unique unitary operator U s.t. :

$$U \ |\phi\rangle \otimes |0\rangle = |\phi\rangle \otimes |\phi\rangle \tag{7.52}$$

being $|\phi\rangle$ the state we want to copy and $|0\rangle$ the slot where we want to put it.



Unique U means that U should be the same for every state $|\phi\rangle$.

Proof:

We consider a state $|\psi\rangle$ s.t. $|\psi\rangle \neq |\phi\rangle$:

$$U \ |\phi\rangle \otimes |0\rangle = |\phi\rangle \otimes |\phi\rangle \quad \text{copy } \ |\phi\rangle \tag{7.53}$$

$$U \ |\psi\rangle \otimes |0\rangle = |\psi\rangle \otimes |\psi\rangle \quad \text{copy } \ |\psi\rangle \tag{7.54}$$

Let's sum the 2 relations:

$$U\left(|\phi\rangle \otimes |0\rangle + |\psi\rangle \otimes |0\rangle\right) = U\left(\ (|\phi\rangle + |\psi\rangle) \otimes |0\rangle \ \right) = \underbrace{|\phi\rangle \otimes |\phi\rangle + |\phi\rangle \otimes |\psi\rangle}_{\text{2 states}} \tag{7.55}$$

Let's take a state $|\chi\rangle = |\phi\rangle + |\psi\rangle$:

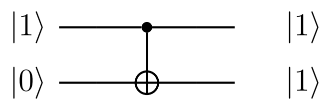
$$U \ \left(|\chi\rangle \otimes |0\rangle\right) = |\chi\rangle \otimes |\chi\rangle \quad \text{copy } \ |\chi\rangle \tag{7.56}$$

$$\begin{aligned} U\left(\ |\chi\rangle \otimes |0\rangle \ \right) &= |\chi\rangle \otimes |\chi\rangle = \\ &= \left(\ |\phi\rangle + |\psi\rangle \ \right) \otimes \left(\ |\phi\rangle + |\psi\rangle \ \right) = \\ &= \underbrace{|\phi\rangle \otimes |\phi\rangle + |\phi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes |\phi\rangle + |\psi\rangle \otimes |\psi\rangle}_{\text{4 states}} \end{aligned} \tag{7.57}$$

U cannot copy because it cannot satisfy (7.55) and (7.57).

Example of no-cloning

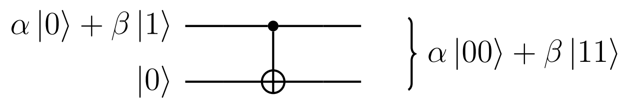
CNOT:



$$CNOT\ \ (|1\rangle \otimes |0\rangle) = |1\rangle \otimes |1\rangle \tag{7.58}$$

$|1\rangle$ has been copied.

CNOT will not work in general:



$$\alpha\,|00\rangle + \beta\,|11\rangle \neq (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\alpha\,|0\rangle + \beta\,|1\rangle) \tag{7.59}$$

7.3.9 Entanglement as a resource

Protocol for sharing randomness

Alice has created a random string by 0, 1:

$$01001101\ldots \tag{7.60}$$

and she wants to send it to Bob.

Initially Alice has a state $|0\rangle_A \otimes |0\rangle_A$, then she prepares a Bell pair $|\phi_{00}\rangle$ and gives the second qubit to Bob. Now Bob has one of the 2 qubits that form the Bell pair.

$$|\phi_{00}\rangle = \frac{|0\rangle_A\,|0\rangle_B + |1\rangle_A\,|1\rangle_B}{\sqrt{2}} \tag{7.61}$$

Alice measures Z:

$$\begin{array}{lll} |0\rangle_A\,|0\rangle_B & \quad\text{with} & \quad p = \dfrac{1}{2} \\ |1\rangle_A\,|1\rangle_B & \quad\text{with} & \quad p = \dfrac{1}{2} \end{array}$$

They can share a random bit if Alice measures in the Z basis. The *Bell pair* can be considered as a bit of entanglement or *ebit*.

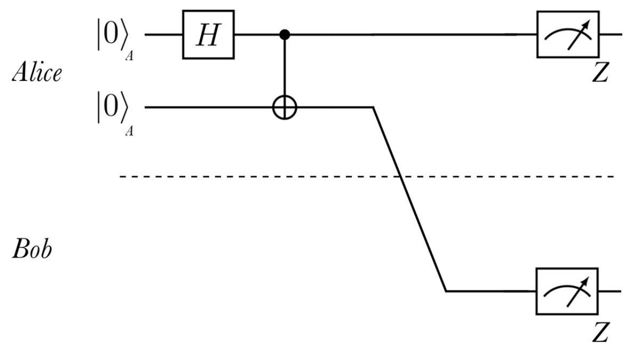


Figure 7.1: After Alice measures her qubit in the Z basis, Alice and Bob share a random bit.

By sharing an *ebit* and by doing only local measurements:

$$\text{Quantum resources } \geq \text{Classical resources}$$

Resources inequality:

$$[qq] \geq [cc]$$

$$\begin{array}{rcl} [qq] & = & \text{noiseless quantum resource } (ebit) \text{ shared by 2 parties} \\ [cc] & = & \text{noiseless classical resource (2 bits) shared by 2 parties} \end{array}$$

The inequality means that there exists a protocol that generates the resources on the right $[cc]$ by consuming the resources on the left $[qq]$.

The opposite is <u>not</u> true:

$$[cc] \geq [qq] \quad \text{WRONG} \quad \Longleftrightarrow \quad \text{Bell inequality}$$

7.3.10 Superdense coding

We can transmit 2 bit of information by sharing an *ebit*.

Alice prepares the Bell state $|\phi_{00}\rangle$. Then, she sends the second qubit of the pair to Bob. Now Alice has access to 1 qubit of a Bell pair and Bob has access to the other qubit of the same Bell pair.

Alice would like to transmit 2 classical bits b_0, b_1.

She applies to her qubit one among the operators $\mathbb{I}_A$, X_A , Z_A , $X_A Z_A$ depending on b_0 and b_1 according to the following rule and sends her qubit to Bob via quantum channel:

b_1	b_0	Op
0	0	$\mathbb{I}_A$
0	1	X_A
1	0	Z_A
1	1	$X_A Z_A$

Table 7.1: Correspondence table of the operators applied by Alice according to the 2 bits she wants to transmit.

The action of the four operators on $|\phi_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is:

$$\mathbb{I}_A |\phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\phi_{00}\rangle \tag{7.62}$$

$$X_A |\phi_{00}\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} = |\phi_{01}\rangle \tag{7.63}$$

$$Z_A |\phi_{00}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\phi_{10}\rangle \tag{7.64}$$

$$X_A \cdot Z_A |\phi_{00}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\phi_{11}\rangle \tag{7.65}$$

So Bob has one among the $|\phi_{ij}\rangle$. We note that the indeces of the Bell pair $|\phi_{ij}\rangle$ created by Alice coincide exactly with the 2 bits b_0 , b_1 she want to transmit s.t. one could write $|\phi_{b_1 b_0}\rangle$. He has to apply the inverse of the circuit for creating the *Bell pair* and measure in the Z basis to distinguish the 4 *Bell pairs*. This process is called *Bell measurement*.

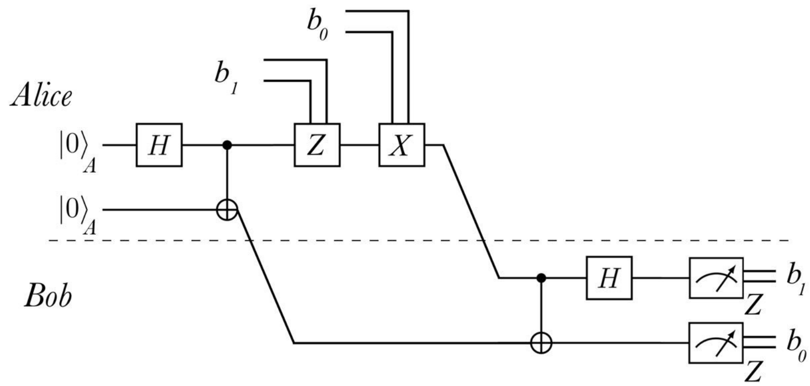


Figure 7.2: (1) Alice prepares the Bell state $|\phi_{00}\rangle$, (2) Alice sends 1 qubit to Bob, (3) Alice encodes the classical bits b_0, b_1 by acting on her qubit with the X, Z gates, (4) Alice sends her qubit to Bob and now he has an entire Bell state $|\phi_{ij}\rangle$, (5) Bob performs a Bell measurement on $|\phi_{ij}\rangle$ to get b_0 and b_1.

Example

If Alice sends to Bob the state $|\phi_{01}\rangle = \frac{|10\rangle+|01\rangle}{\sqrt{2}}$. Bob performs the Bell measurement:

$$|\phi_{01}\rangle \xrightarrow{CNOT} \frac{|11\rangle + |01\rangle}{\sqrt{2}} = \frac{|1\rangle + |0\rangle}{\sqrt{2}} \otimes |1\rangle = |+\rangle \otimes |1\rangle \xrightarrow{H_A} |0\rangle \otimes |1\rangle \tag{7.66}$$

Then Bob measures in the Z basis and gets the eigenvalues 1 and -1, so the 2 qubits are $|0\rangle$ and $|1\rangle$. Bob has recovered the value of the classical bits $(b_1, b_0) = (0, 1)$.

Resource inequality

$$[qq] + [q \to q] \geq 2[c \to c] \tag{7.67}$$

where:

- $[qq]$ quantum resource
- $[q \to q]$ channel that gives to Bob the entire Bell pair
- $[c \to c]$ classical channel

Trivial inequality:

$$2[qq] \geq 2[c \to c] \tag{7.68}$$

7.4 Deutsch algorithm

Toy algorithm (superposition, interference)

The algorithm helps us understand if a coin has 2 heads or 2 tails or if it is a fair coin.

Classically we need to do 2 observations. D.A. does the same with 1 observation.

Define a classical binary function f. It can be one of the 4 functions f_0, f_1, f_2, f_3 :

input	f_0	f_1	f_2	f_3
0	0	1	1	0
1	0	1	0	1

The functions f_0, f_1 represent a non fair coin (constant f).

The functions f_2, f_3 represent a fair coin (balanced f).

The D.A. algorithm distinguishes between constant f and balanced f by measuring only one time.

This is also true for n bits. In a classical setting we would need

$$N_{mesurements} = \frac{2^n}{2} + 1 \tag{7.69}$$

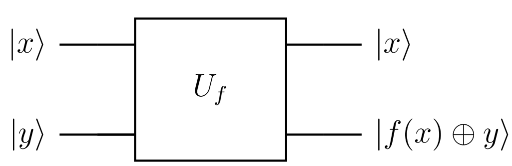
With D.A. algorithm we only need one measurement independently from the number n of qubits.

How do we measure the classical function with a quantum circuit ?

We need an oracle (black box) i.e. an operator U_f (depends on f) that acts on the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$$U_f |x\rangle |y\rangle = |x\rangle |f(x) \oplus y\rangle \tag{7.70}$$

where $x, y \in \{0, 1\}$. $|y\rangle$ is added to make the operator U_f invertible and unitary.



This is the quantum way to interrogate a classical binary function f. Every time we apply U_f, we measure f.

Exercise

Write the 4 oracles that implement the 4 binary functions f_0, f_1, f_2, f_3.

1. Classical measurement - wrong way

We apply U_f to the states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

This is equivalent of implementing a classical measurement of f since we are using orthonormal states. Since this is the equivalent of a classical measurement we need to apply U_f twice.

$$U_f |00\rangle = |0\rangle |0 \oplus f(0)\rangle = |0\rangle |f(0)\rangle \tag{7.71}$$

$$U_f |01\rangle = |0\rangle |1 \oplus f(0)\rangle = |0\rangle \left|\bar{f}(0)\right\rangle \tag{7.72}$$

$$U_f |10\rangle = |1\rangle |0 \oplus f(1)\rangle = |1\rangle |f(1)\rangle \tag{7.73}$$

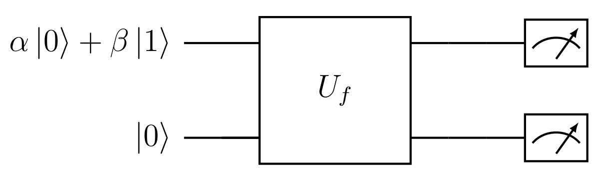
$$U_f |11\rangle = |1\rangle |1 \oplus f(1)\rangle = |1\rangle \left|\bar{f}(1)\right\rangle \tag{7.74}$$

where $\bar{f}(1) \equiv NOT f(1)$. The last step of the previous 4 eqs. is just an application of the XOR table.

We need to measure twice U_f : if we choose $|y\rangle = |0\rangle$ we need to perform the measurements $U_f |00\rangle$ and $U_f |10\rangle$; if we choose $|y\rangle = |1\rangle$ we need to perform the measurements $U_f |01\rangle$ and $U_f |11\rangle$.

2. Superposition - wrong way

Let's start with a superposition $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$.



We measure U_f on $|\psi\rangle$:

$$U_f(\ \alpha |0\rangle + \beta |1\rangle \otimes |0\rangle\ ) = U_f(\alpha |00\rangle) + U_f(\beta |10\rangle) = \alpha |0, f(0)\rangle + \beta |1, f(1)\rangle \tag{7.75}$$

$$\begin{cases} |0, f(0)\rangle & Prob = |\alpha|^2 \\ |1, f(1)\rangle & Prob = |\beta|^2 \end{cases} \tag{7.76}$$

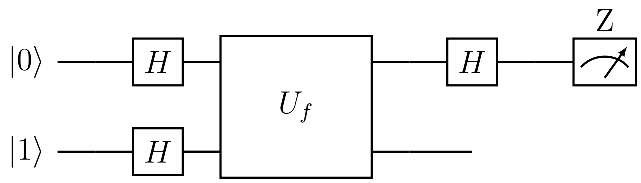
This is also ineffective.

3. Separate measurements - right way

The right way is to measure $f(0)$ and $f(1)$ separately.

$$f(0) \oplus f(1) = \begin{cases} 1 & \text{balanced} \\ 0 & \text{constant} \end{cases} \tag{7.77}$$

Consider the circuit:



$$\begin{aligned}
|01\rangle &\xrightarrow{H} |+-\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} \\
&\xrightarrow{U_f} \frac{|0, f(0)\rangle - \left|0, \bar{f}(0)\right\rangle + |1, f(1)\rangle - \left|1, \bar{f}(1)\right\rangle}{2} \\
&= \frac{|0\rangle}{2}[|f(0)\rangle - \left|\bar{f}(0)\right\rangle] + \frac{|1\rangle}{2}[|f(1)\rangle - \left|\bar{f}(1)\right\rangle]
\end{aligned} \tag{7.78}$$

If f is constant:

$$f(0) = f(1) \tag{7.79}$$

$$\bar{f}(0) = \bar{f}(1) \tag{7.80}$$

$$\begin{aligned}
|01\rangle &\xrightarrow{H,U_f} \frac{|0\rangle}{2} \left[|f(0)\rangle - \left|\bar{f}(0)\right\rangle\right] + \frac{|1\rangle}{2}[|f(1)\rangle - \left|\bar{f}(1)\right\rangle] \\
&= \frac{|0\rangle}{2}[|f(0)\rangle - \left|\bar{f}(0)\right\rangle] + \frac{|1\rangle}{2}[|f(0)\rangle - \left|\bar{f}(0)\right\rangle] \\
&= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|f(0)\rangle - \left|\bar{f}(0)\right\rangle}{\sqrt{2}} \\
&= |+\rangle \otimes \frac{|f(0)\rangle - \left|\bar{f}(0)\right\rangle}{\sqrt{2}} \\
&\xrightarrow{H_A} |0\rangle \otimes \frac{|f(0)\rangle - \left|\bar{f}(0)\right\rangle}{\sqrt{2}}
\end{aligned} \tag{7.81}$$

We measure the first qubit, if we get $|0\rangle$ f is constant.

If f is balanced:

$$f(0) = \bar{f}(1) \tag{7.82}$$

$$\bar{f}(0) = f(1) \tag{7.83}$$

$$\begin{aligned}
|01\rangle &\xrightarrow{H,U_f} \frac{|0\rangle}{2}[|f(0)\rangle - |\bar{f}(0)\rangle] + \frac{|1\rangle}{2}[|f(1)\rangle - |\bar{f}(1)\rangle] \\
&= \frac{|0\rangle}{2}[|f(0)\rangle - |f(1)\rangle] + \frac{|1\rangle}{2}[|f(1)\rangle - |f(0)\rangle] \\
&= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|f(0)\rangle - |f(1)\rangle}{\sqrt{2}} \qquad (7.84) \\
&= |-\rangle \otimes \frac{|f(0)\rangle - |f(1)\rangle}{\sqrt{2}} \\
&\xrightarrow{H_A} |1\rangle \otimes \frac{|f(0)\rangle - |f(1)\rangle}{\sqrt{2}} \qquad (7.85)
\end{aligned}$$

We measure the first qubit, if we get $|1\rangle$ f is balanced.

In this procedure we never measure the second qubit.
This procedure works because we never access directly the values $f(0)$, $f(1)$.

This procedure also works for a function f of n bits.

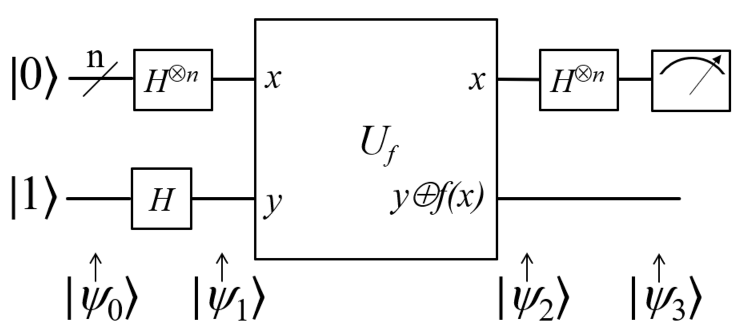


Figure 7.3: Quantum circuit of Deutsch-Jozsa algorithm.

We only need one measurement:
if we get $|0\rangle$ f is constant,
if we get $|1\rangle$ f is balanced.

7.5 Teleportation

We want to transmit an arbitrary quantum state $|\phi\rangle$ between two parties. We need an *e-bit*, i.e. *shared Bell pair*, and a classical communication channel.

There are 2 parties A, B that share the *Bell pair* $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

Alice has the state $|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ that wants to transmit to Bob.
Alice has 2 qubits:

- one is for the state $|\phi\rangle$
- one is for the qubit belonging to the Bell pair

Bob has the other qubit of the Bell pair; on this qubit he will get the state $|\phi\rangle$.

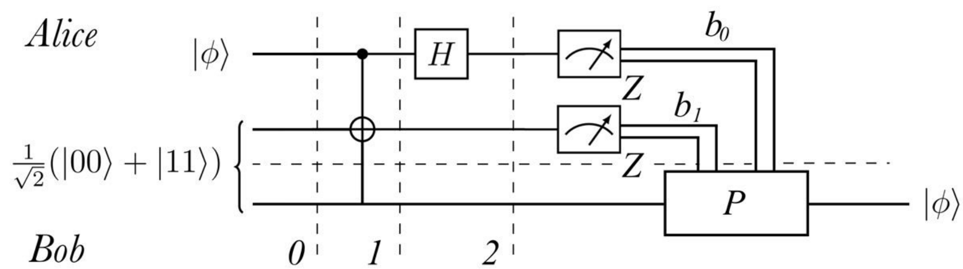


Figure 7.4: Teleportation circuit.

$$
\begin{aligned}
|\phi\rangle\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) \quad &= \quad (\alpha\,|0\rangle+\beta\,|1\rangle)\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = \\
\text{Step 0:} \quad &= \quad \frac{\alpha\,|0\rangle}{\sqrt{2}}(|00\rangle+|11\rangle)+\frac{\beta\,|1\rangle}{\sqrt{2}}(|00\rangle+|11\rangle) \\
\text{Step 1:} \quad &\xrightarrow{CNOT} \quad \frac{\alpha}{\sqrt{2}}|0\rangle(|00\rangle+|11\rangle)+\frac{\beta}{\sqrt{2}}|1\rangle(|10\rangle+|01\rangle) \\
\text{Step 2:} \quad &\xrightarrow{H} \quad \frac{\alpha}{\sqrt{2}}|+\rangle(|00\rangle+|11\rangle)+\frac{\beta}{\sqrt{2}}|-\rangle(|10\rangle+|01\rangle) \\
&= \quad \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}}\frac{|00\rangle+|11\rangle}{\sqrt{2}}+\beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}\frac{|10\rangle+|01\rangle}{\sqrt{2}} \\
&= \quad \frac{\alpha}{2}(|000\rangle+|011\rangle+|100\rangle+|111\rangle) \\
&+ \quad \frac{\beta}{2}(|010\rangle+|001\rangle-|110\rangle-|101\rangle)
\end{aligned}
\tag{7.86}
$$

After the H gate (step 2[1]), the final state $|\psi\rangle$ is:

$$|\psi\rangle = \frac{1}{2}\big(\;|00\rangle_A\,[\,\alpha\,|0\rangle + \beta\,|1\rangle\,]_B \; + |01\rangle_A\,[\,\alpha\,|1\rangle + \beta\,|0\rangle\,]_B \; +$$

$$|10\rangle_A\,[\,\alpha\,|0\rangle - \beta\,|1\rangle\,]_B \; + |11\rangle_A\,[\,\alpha\,|1\rangle - \beta\,|0\rangle\,]_B\big)$$

Alice measures the first 2 qubits and gets the outcome (with prob. $\frac{1}{4}$):

Alice		**Bob**
00	$\longrightarrow$	$[\,\alpha\,|0\rangle + \beta\,|1\rangle\,]_B$
01	$\longrightarrow$	$[\,\alpha\,|1\rangle + \beta\,|0\rangle\,]_B$
10	$\longrightarrow$	$[\,\alpha\,|0\rangle - \beta\,|1\rangle\,]_B$
11	$\longrightarrow$	$[\,\alpha\,|1\rangle - \beta\,|0\rangle\,]_B$

Alice's outcome is the state that Bob will get. Alice needs to tell Bob which one of the possible 4 outcomes she got. Depending on Alice's outcome, Bob has to apply an operator to recover the state $|\phi\rangle$. The operator which Bob has to apply depends on Alice's outcome according to the following rule:

Alice's outcome		**Bob's operator**
00	$\Rightarrow$	$\mathbb{I}$
01	$\Rightarrow$	X
10	$\Rightarrow$	Z
11	$\Rightarrow$	$Z \cdot X$

In the end, Bob will get $|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. The state $|\phi\rangle$ is transferred and not copied from Alice to Bob.

Two questions arise:

1. is *special relativity* violated? No.
2. is *no-cloning theorem* violated? No.

1. Special relativity

Special relativity is not violated since, even though the transfer of the state happens instantaneously, Bob doesn't have the state that Alice wanted to transmit in 3 cases out of 4. The final transfer happens only after Alice's communication with a classical channel and this cannot happen instantaneously, so special relativity is not violated.

[1]Remembering that: $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

2. No-cloning theorem

No-cloning theorem is not violated since Alice needs to measure her 2 qubits and any quantum state will be destroyed by this measurement.

Teleportation is used for transferring quantum states within quantum processes. It is a common process.

7.6 Entanglement measures for pure state

Consider the Schmidt decomposition of a bipartite state.

$$|\psi_{AB}\rangle = \sum_{a=1}^{r} S_{aa} |u_a\rangle_A \otimes |v_a\rangle_B \tag{7.87}$$

A bipartite state is separable if its Schmidt decomposition contains only one state. Therefore if $r = 1$, $|\psi\rangle_{AB}$ is separable, otherwise it's entangled.
For $r = 1$:

$$|\psi_{AB}\rangle = |\alpha\rangle_A \otimes |\beta\rangle_B \tag{7.88}$$

We want to quantify the degree of entanglement between A and B.
Recall that given a state ρ:

- if $tr(\rho^2) = 1 \implies \rho$ is a pure state
- if $tr(\rho^2) < 1 \implies \rho$ is a mixed state.

Suppose that $|\psi_{AB}\rangle$ is separable then:

$$\rho_A = tr_B(|\psi_{AB}\rangle \langle\psi_{AB}|) = |\alpha\rangle_A \langle\alpha|_A \tag{7.89}$$

$$\rho_B = tr_A(|\psi_{AB}\rangle \langle\psi_{AB}|) = |\beta\rangle_B \langle\beta|_B \tag{7.90}$$

If the purity of the parties is 1 (i.e. the traces of the squared reduced density matrices of system A and B are both 1) then the state $|\psi\rangle_{AB}$ is separable:

$$tr(\rho_A^2) = 1 \quad , \quad tr(\rho_B^2) = 1 \implies |\psi\rangle_{AB} \text{ is separable} \tag{7.91}$$

The purity is an indicator (proxy) of entanglement.
We want to quantify the entanglement constant on an arbitrary state $|\psi\rangle_{AB}$, we use Von Neumann entropy.

7.6.1 Von Neumann entropy

Def 12 (Von Neumann entropy) *Given the reduced density matrix ρ_A :*

$$\rho_A = tr_B(\ |\psi_{AB}\rangle \langle\psi_{AB}|\ ) \tag{7.92}$$

*the **Von Neumann entropy** $S(\rho_A)$ is defined as:*

$$S(\rho_A) = -tr_A(\ \rho_A \log \rho_A\ ) \tag{7.93}$$

How do we compute $\log(\rho_A)$?

Consider a function $f(A)$ of an hermitian operator A. We can do the spectral decomposition:

$$A = \sum_n a_n |a_n\rangle \langle a_n| \tag{7.94}$$

$$f(A) = \sum_n f(a_n) |a_n\rangle \langle a_n| \tag{7.95}$$

So for $f(A) = \log(A)$ and $A = \rho_A$ we have:

$$\log(\rho_A) = \sum_n \log(\lambda_n) |\lambda_n\rangle \langle\lambda_n| \tag{7.96}$$

where λ_n are the eigenvalues of ρ_A and $|\lambda_n\rangle$ are the eigenstates of ρ_A.

$$\begin{aligned} S(\rho_A) &= -tr_A(\rho_A \log(\rho_A)) \\ &= -tr_A(\rho_A \sum_n \log(\lambda_n) |\lambda_n\rangle \langle\lambda_n|) \\ &= -tr_A \left(\sum_n \lambda_n \log(\lambda_n) |\lambda_n\rangle \langle\lambda_n| \right) \\ &= -\sum_n \lambda_n \log(\lambda_n) \end{aligned} \tag{7.97}$$

We use the convention:

$$\lambda_n \log(\lambda_n) \xrightarrow{\lambda_n \to 0} 0 \implies 0 \log(0) = 0 \tag{7.98}$$

If we have the Schmidt decomposition:

$$S(\rho_A) = -\sum_{a=1}^{r} S_{aa}^2 \log\left(S_{aa}^2\right) \tag{7.99}$$

Properties

1.

$S(\rho_A) = 0$ if:

$$\begin{aligned} \rho_A \text{ is } \textit{pure} \ &\Leftrightarrow |\psi\rangle_{AB} \ \text{ is } \textit{separable} \\ &\Leftrightarrow r = 1 \text{ there is only 1 Schmidt coefficient} \end{aligned} \tag{7.100}$$

2.

There are some bounds on $S(\rho_A)$:

$$0 \leq S(\rho_A) \leq S_{max} \tag{7.101}$$

Let's consider a Hilbert space of 2 qubits.
The maximally mixed state is

$$|\psi\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{7.102}$$

$$\rho_A = \frac{1}{2} \cdot \mathbb{I}_A \tag{7.103}$$

The entropy of the maximally mixed state is the upper bound.

$$\begin{aligned} S_{max} = S(\rho_A) \ &= \ -tr(\ \rho_A \log \rho_A\ ) = \\ &= \ -tr(\ \frac{\mathbb{I}}{2} \log \frac{\mathbb{I}}{2}\ ) = \\ &= \ -\frac{1}{2} tr(\ \log \frac{\mathbb{I}}{2}\ ) = -\frac{2}{2} \log \frac{1}{2} = \log 2 \end{aligned}$$

In the general case when $\mathcal{H}_A$ has D qubits the maximally mixed state is:

$$\rho_A = \frac{\mathbb{I}}{D} \qquad tr(\rho_A) = 1 \tag{7.104}$$

thus:

$$S_{max} = S(\rho_A) = \log D \tag{7.105}$$

So, for S there are the bounds:

$$0 \leq S(\rho_A) \leq \log D \tag{7.106}$$

the minimum 0 is reached when $|\psi\rangle_{AB}$ is *separable* and the maximum $\log D$ when $|\psi\rangle_{AB}$ is *maximally entangled.*

3.

We consider system B:

$$\rho_B = tr_A(\ |\psi\rangle_{AB} \langle\psi|_{AB} \ ) \tag{7.107}$$

Since the Schimdt coefficients S_{aa} are the same for A and B, we have:

$$S(\rho_A) = S(\rho_B) \tag{7.108}$$

4. Concavity

For a mixture of ρ_j with weights α_j ($\sum_j \alpha_j = 1$), we have:

$$S\left(\sum_j \alpha_j \rho_j\right) \geq \sum_j \alpha_j \ S(\rho_j) \tag{7.109}$$

7.7 Separable mixed states

Def 13 (Separable mixed states)

$$\rho_{AB} = \sum_{ijkl} \rho_{ijkl} \left|a_i\right\rangle_A \left\langle a_j\right|_A \otimes \left|b_k\right\rangle \left\langle b_l\right| \tag{7.110}$$

ρ_{AB} is separable if :

$$\rho_{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B \tag{7.111}$$

ρ_j^A and ρ_j^B are, in general, not pure states.

Consider the purity of the state ρ_{AB}:

$$\rho_{AB} = \rho_A \otimes \left|\beta\right\rangle_B \left\langle\beta\right|_B \tag{7.112}$$

Let's compute the purity of the reduced density matrix ρ_A:

$$p = tr_A(\rho_A^2) < 1 \tag{7.113}$$

According to our definition, ρ_{AB} is separable, but ρ_A is not pure.