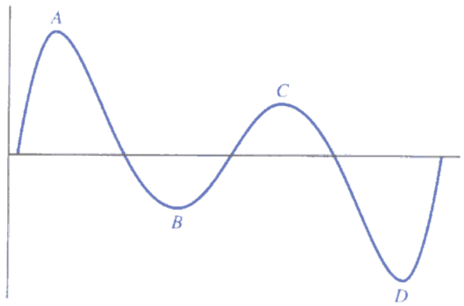
Introduction to Non-Linear Optimization

Concepts+

When we solve an NLP we sometimes get a “sub-optimal” solution. See the following figure from Winston and Albright, PMS, 3ed (figure 7.1):



A local optimum is better than all nearby points, but a global optimum is the best point in the entire feasible solution space. It is possible that we never find the global optimum.

Functions are defined as convex if its slope is always non-decreasing and concave is its slope is always non-increasing. (see figures from Winston)

Figure 7.2
A Convex Function with a Global Minimum

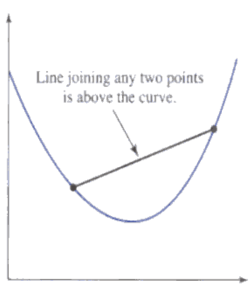


Figure 7.3
A Decreasing Convex Function

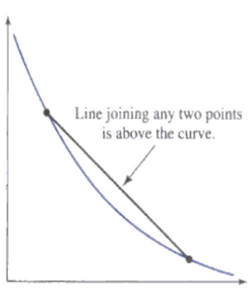


Figure 7.4
A Concave Function with a Global Maximum

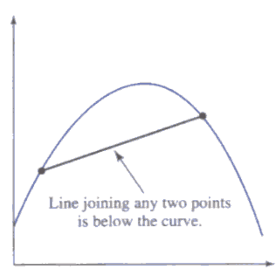
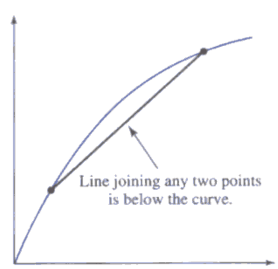


Figure 7.5
An Increasing Concave Function



Solvers always perform well if:

Maximization Problem:

- Objective function is concave
- Constraints are linear

Minimization Problem:

- Objective function is convex
- Constraints are linear

NLP Example: Pricing

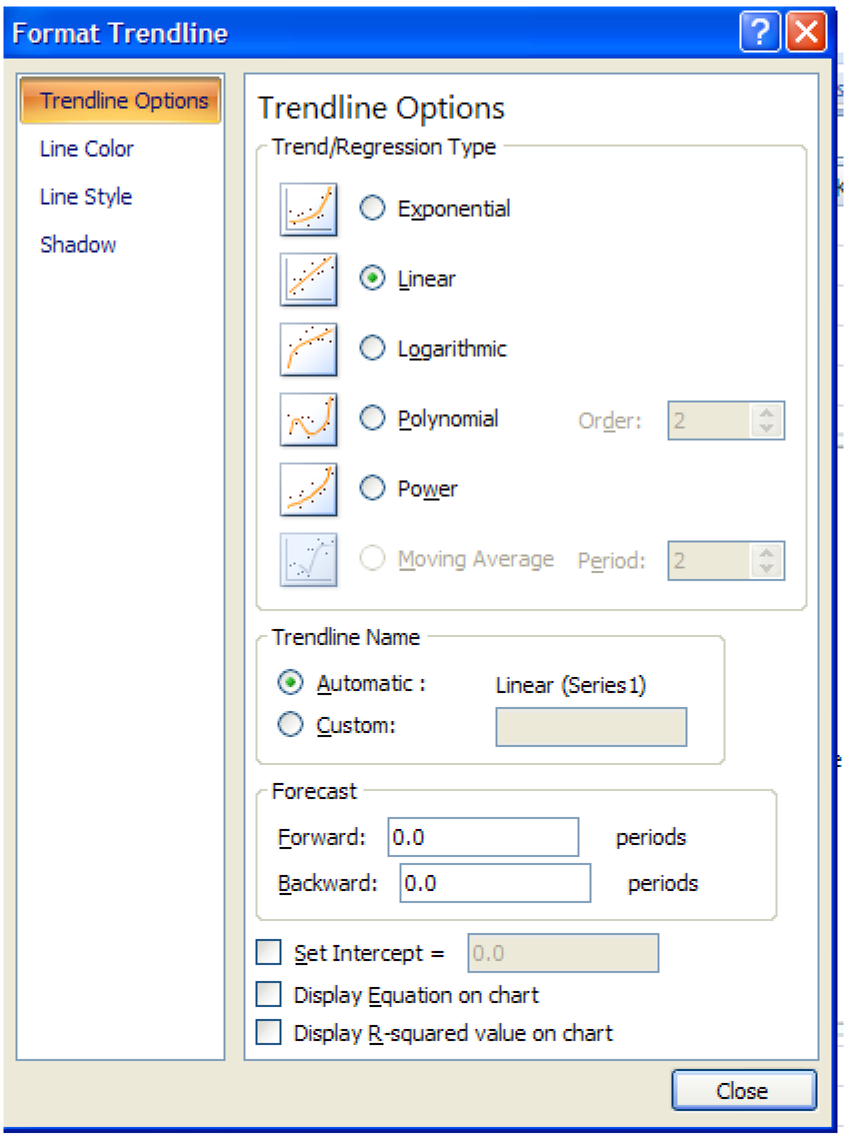
The Madison Company manufactures and retails a certain product. The company wants to determine the price that maximizes its profit from this product. The unit cost of producing and marketing the product is \$50. Madison will certainly charge at least \$50 for the product to ensure that it makes *some* profit. However, there is a very competitive market for this product, so that Madison's demand falls sharply when it increases its price. How should the company proceed?[4]

Table 7.1 Variables and Constraints for Madison's Pricing Model

Input variables	Unit cost, demand function (or points on demand function)
Decision variables (changing cells)	Unit price to charge
Objective (target cell)	Profit
Other output variables	Revenue, cost
Constraints	Unit price is greater than or equal to Unit cost

More specifically, if Madison charges P dollars per unit, then its profit is $(P - 50)D$, where D is the number of units demanded. The problem, however, is that D depends on P. As the price P increases, the demand D decreases. Therefore, the first step is to estimate how D varies with P—that is, we have to estimate the demand function. In fact, this is the first step in almost all pricing problems. We illustrate two possibilities: a *linear* demand function of the form $D = a - bP$, and a *constant elasticity* demand function of the form $D = aP^b$.

	A	B	C	D	E	F
1	**Madison pricing model - finding demand functions**					
2						
3	Two points on the demand curve (as estimated by Madison)					
4		Price	Demand			
5		$70	400			
6		$80	300			
7						
8						



	A	B	C	D	E	F
1	**Madison pricing model with linear demand function**					
2						
3	Unit cost	$50				
4						
5	Parameters of linear demand function (from first sheet)					
6		Intercept	Slope			
7						
8						
9	**Pricing model**					
10	Price					
11	Demand					
12	Profit					
13						
14	**Verification with a data table and corresponding chart**					
15	Price	Profit				
16						
17						
18						

	A	B	C	D	E	F	G
1	**Madison pricing model with constant elasticity demand function**						
2							
3	Unit cost	$50					
4							
5	Parameters of constant elasticity demand function (from first sheet)						
6		Constant	Elasticity				
7							
8							
9	**Pricing model**						
10	Price						
11	Demand						
12	Profit						
13							
14	**Verification with a data table and corresponding chart**						
15	Price	Profit					
16							
17							
18							
19							

NLP Example: Revenue Maximization

Coastal Telephone Company (Powell & Baker, p. 203-205 | worksheet 8.7)

Coastal Telephone Company (CTC) is a regional supplier of long-distance telephone services. CTC is trying to determine the optimal pricing structure for its daytime and evening long-distance calling rates. The daytime price applies from 8:00 a.m. to 6:00 p.m. and the evening price applies the rest of the time. With the help of a consultant, the company has estimated the average demand for phone lines (per minute) as follows:

Daytime Lines Demanded = 600 – 5,000 * Day Price + 300 * Evening Price

Evening Lines Demanded = 400 + 600 * Day Price – 2,500 * Evening Price

CTC wants to find prices that will maximize its revenue. These are the decision variables: DP (daytime price) and EP (evening price).

The objective is to maximize revenue. Total revenue consists of a daytime component and an evening component. The daytime component per minute is DD * DP, where DD represents daytime demand. The evening component per minute is ED * EP, where ED represents evening demand. We observe there are 600 minutes in the daytime and 800 minutes in the evening, giving us an objective function:

Maximize Revenue = 600DD * DP + 800ED * EP

Subject to:

DD =   600 – 5,000*DP + 300*EP

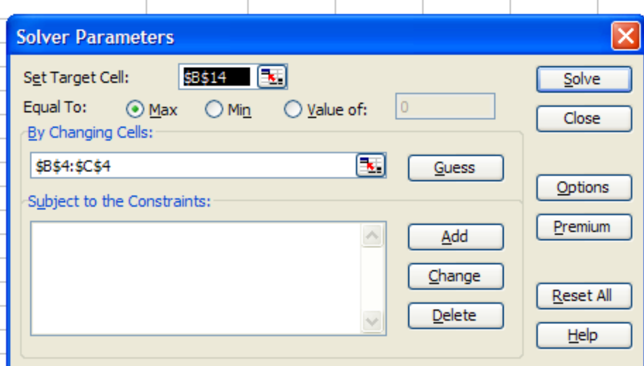
ED =   400 + 600*DP – 2,500*EP

Our objective function is nonlinear because it is a product of decision variables.

	A	B	C
1	**Coastal Telephone Company**		
2			
3	**Decisions**	Day Price	Eve Price
4		0.070	0.091
5			
6	**Demand**		
7	parameters	600	400
8		-5000	-2500
9		300	600
10	demand	275.303	213.582
11			
12	**Objective**		
13	revenue	11,633	16,410
14	total	28,044	

	A	B	C
1	**Coastal Telephone Company**		
2			
3	**Decisions**	Day Price	Eve Price
4		0.0704275367866058	0.0914696266678934
5			
6	**Demand**		
7	parameters	600	400
8		-5000	-2500
9		300	600
10	demand	=B7+B8*B4+B9*C4	=C7+C8*C4+C9*B4
11			
12	**Objective**		
13	revenue	=600*B4*B10	=840*C4*C10
14	total	=B13+C13	

	A	B	C	D	E	F	G
1	**Coastal Telephone Company**						
2							
3	**Decisions**	Day Price	Eve Price				
4		0.070	0.091				
5							
6	**Demand**						
7	parameters	600	400				
8		-5000	-2500				
9		300	600				
10	demand	275.303	213.582				
11							
12	**Objective**						
13	revenue	11,633	16,410				
14	total	28,044					



Additional NLP Application Examples

Cost Modeling Example (Powell & Baker, Exercise 8-1, p. 253)

General Widget has collected data on daily output and daily production cost. The company believes that daily output (DO) and daily production cost (PC) ought to be linearly related. Thus for some numbers a and b:

PC = a + b*DO

Study Data:

Day	Output	Production Cost
1	5,045	2,542
2	6,127	2,812
3	6,360	2,776
4	6,645	3,164
5	7,220	4,102
6	9,537	4,734
7	9,895	4,238
8	10,175	4,524
9	10,334	4,869
10	10,855	4,421

We might set up our spreadsheet like this:

	A	B	C	D	E	F	G
1	**Cost Modeling**						
2							
3	**Decisions**			a			Sum Sq Diff
4	**& Objective**			b			
5							
6	**Data**	**Day**	**Output**	**Cost**	Model	Difference	Sq. Difference
7		1	5,045	2,542			
8		2	6,127	2,812			
9		3	6,360	2,776			
10		4	6,645	3,164			
11		5	7,220	4,102			
12		6	9,537	4,734			
13		7	9,895	4,238			
14		8	10,175	4,524			
15		9	10,334	4,869			
16		10	10,855	4,421			

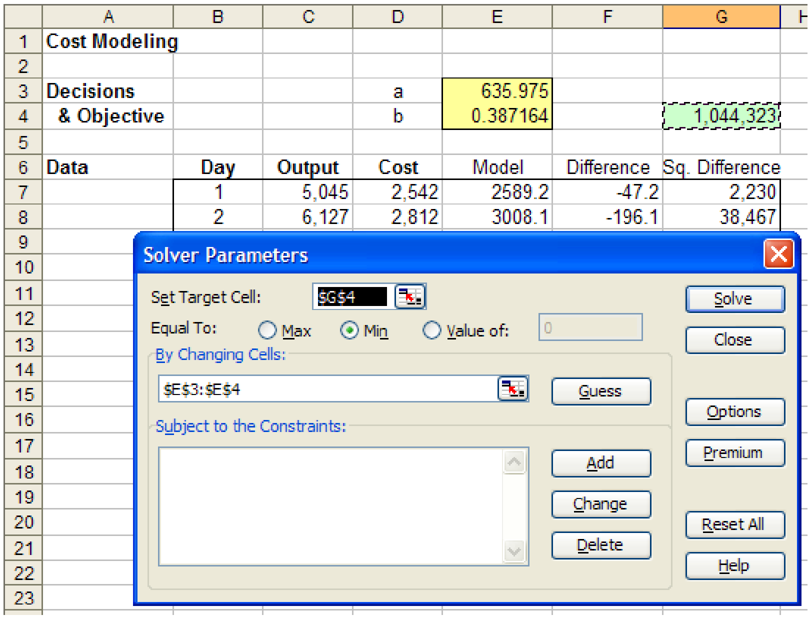
Our "model" cell is to predict the cost as a function of output using the relationships:

PC = a + b*DO. (=$E$3+$E$4*C7),

Difference is cost - model. (=D7-E7),

Sq. Difference is just the difference squared (=F7^2), and

Sum Sq Diff is the sum of Sq Difference column.



Options: Assume non-negative.

	A	B	C	D	E	F	G
1	**Cost Modeling**						
2							
3	**Decisions**			a	592.839		
4	**& Objective**			b	0.386355		2,429.21
5							
6	**Data**	**Day**	**Output**	**Cost**	Model	Difference	Abs. Diff.
7		1	5,045	2,542	2542.0	0.0	0
8		2	6,127	2,812	2960.0	-148.0	148
9		3	6,360	2,776	3050.1	-274.1	274
10		4	6,645	3,164	3160.2	3.8	4
11		5	7,220	4,102	3382.3	719.7	720
12		6	9,537	4,734	4277.5	456.5	456
13		7	9,895	4,238	4415.8	-177.8	178
14		8	10,175	4,524	4524.0	0.0	0
15		9	10,334	4,869	4585.4	283.6	284
16		10	10,855	4,421	4786.7	-365.7	366
17							

Abs Diff: =ABS(F7)

	A	B	C	D	E	F	G
1	**Cost Modeling**						
2							
3	**Decisions**			a	1.961		
4	**& Objective**			b	0.839206		2,429.9
5							
6	**Data**	**Day**	**Output**	**Cost**	Model	Difference	Abs. Diff.
7		1	5,045	2,542	2511.0	31.0	31
8		2	6,127	2,812	2955.7	-143.7	144
9		3	6,360	2,776	3049.7	-273.7	274
10		4	6,645	3,164	3164.0	0.0	0
11		5	7,220	4,102	3392.2	709.8	710
12		6	9,537	4,734	4284.7	449.3	449
13		7	9,895	4,238	4419.3	-181.3	181
14		8	10,175	4,524	4524.0	0.0	0
15		9	10,334	4,869	4583.3	285.7	286
16		10	10,855	4,421	4776.4	-355.4	355
17							
18		Best model is roughly Cost = 1.961*Output^(0.25)					
19			Parameters:		1.961		
20					0.839206		

Model: =$E$3*C7^$E$4

Portfolio Optimization Models

Weighted sum of random variables:

$$Expected\ return: R_p = \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_n x_n$$
$$Variance\ of\ R_p = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \ldots + \sigma_n^2 x_n^2 + \sum_{i \neq j} \rho_{ij} \sigma_i \sigma_j x_i x_j$$

Use sample mean, sample standard deviation and sample correlations instead of the population parameters.

Can use covariance instead of correlation, for example, if c_{ij} is the estimated covariance between stocks i and j, then

$$c_{ij} = r_{ij} s_i s_j$$

And the variance can alternately be calculated as:

$$Estimated\ Variance\ of\ R_p = \sum_{ij} c_{ij} x_i x_j$$

This formulation lends itself to calculating portfolio variance with Excel's matrix function.

We use Excel MMULT and TRANSPOSE (see Web for file)

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Matrix multiplication in Excel**												
2													
3	**Typical multiplication of two matrices**							**Multiplication of a matrix and a column**					
4	Matrix 1	1	2	3				Column 1	2				
5		2	4	5					3				
6									4				
7	Matrix 2	1	2										
8		3	4					Matrix 1 times Column 1, with formula =MMULT(B4:D5,I4:I6)					
9		5	6					Select range with 2 rows, 1 column, enter formula, press Ctrl-Shift-Enter					
10													
11	Matrix 1 times Matrix 2, with formula =MMULT(B4:D5,B7:C9)												
12	Select range with 2 rows, 2 columns, enter formula, press Ctrl-Shift-Enter.												
13								**Multiplication of a row and a matrix**					
14								Row 1	4	5			
15													
16	**Multiplication of a quadratic form (row times matrix times column)**							Row 1 times Matrix 1, with formula =MMULT(I14:J14,B4:D5)					
17	Matrix 3	2	1	3				Select range with 1 row, 3 columns, enter formula, press Ctrl-Shift-Enter					
18		1	-1	0									
19		3	0	4									
20								**Multiplication of a row and a column**					
21	Transpose of Column 1 times Matrix 3 times Column 1							Row 2	1	6	3		
22	Formula is =MMULT(TRANSPOSE(I4:I6),MMULT(B17:D19,I4:I6))												
23	Select range with 1 row, 1 column, enter formula, press Ctrl-Shift-Enter							Row 2 times Column 1, with formula =MMULT(I22:K22,I4:I6)					
24								Select range with 1 row, 1 column, enter formula, press Ctrl-Shift-Enter					
25													
26	Notes on quadratic form example:												
27	Two MMULT's are required because MMULT works on only two ranges at a time.												
28	TRANSPOSE is needed to change a column into a row.												

(Source: Figure 7.33, Winston and Albright, PMS, 3ed.)

Portfolio Optimization at Perlman and Brothers (Example 7.9, Winston and Albright, PMS, 3ed.)

Perlman & Brothers, an investment company, intends to invest a given amount of money in three stocks. From past data, the means and standard deviations of annual returns have been estimated as shown in Table 7.7. The correlations among the annual returns on the stocks are listed in Table 7.8. The company wants to find a minimum-variance portfolio that yields an expected annual return of at least 0.12.

Table 7.7 Estimated Means and Standard Deviations of Stock Returns

Stock	Mean	Standard Deviation
1	0.14	0.20
2	0.11	0.15
3	0.10	0.08

Table 7.8 Estimated Correlations Among Stock Returns

Combination	Correlation
Stocks 1 and 2	0.6
Stocks 1 and 3	0.4
Stocks 2 and 3	0.7

	A	B	C	D	E	F	G
1	**Portfolio selection model**						
2							
3	**Stock input data**						
4		Stock 1	Stock 2	Stock 3			
5	Mean return	0.14	0.11	0.1			
6	StDev of return	0.2	0.15	0.08			
7							
8	Correlations	Stock 1	Stock 2	Stock 3			
9	Stock 1	1	0.6	0.4			
10	Stock 2	0.6	1	0.7			
11	Stock 3	0.4	0.7	1			
12							
13							
14	Covariances	Stock 1	Stock 2	Stock 3			
15	Stock 1						
16	Stock 2						
17	Stock 3						
18							
19							
20	**Investment decisions**						
21		Stock 1	Stock 2	Stock 3	Total		Required
22	Fractions to invest				0	=	1
23							
24	**Constraint on expected portfolio return**						
25		Actual		Required			
26			>=	0.12			
27							
28	**Portfolio variance**						
29	Portfolio stdev						

CovarMat	=StudentPortExmpl!$B$15:$D$17
ExpReturn	=StudentPortExmpl!$B$26
Invested	=StudentPortExmpl!$B$22:$D$22
LTable	=StudentPortExmpl!$B$4:$D$6
MeanReturns	=StudentPortExmpl!$B$5:$D$5
PortVar	=StudentPortExmpl!$B$28
ReqdReturn	=StudentPortExmpl!$D$26
TotInvested	=StudentPortExmpl!$E$22

Formulas to compute covariance matrix:

	A	B
13		
14	Covariances	Stock 1
15	Stock 1	=HLOOKUP($A15,LTable,3)*B9*HLOOKUP(B$14,LTable,3)
16	Stock 2	=HLOOKUP($A16,LTable,3)*B10*HLOOKUP(B$14,LTable,3)
17	Stock 3	=HLOOKUP($A17,LTable,3)*B11*HLOOKUP(B$14,LTable,3)
18		

	C
13	
14	Stock 2
15	=HLOOKUP($A15,LTable,3)*C9*HLOOKUP(C$14,LTable,3)
16	=HLOOKUP($A16,LTable,3)*C10*HLOOKUP(C$14,LTable,3)
17	=HLOOKUP($A17,LTable,3)*C11*HLOOKUP(C$14,LTable,3)

	D
13	
14	Stock 3
15	=HLOOKUP($A15,LTable,3)*D9*HLOOKUP(D$14,LTable,3)
16	=HLOOKUP($A16,LTable,3)*D10*HLOOKUP(D$14,LTable,3)
17	=HLOOKUP($A17,LTable,3)*D11*HLOOKUP(D$14,LTable,3)

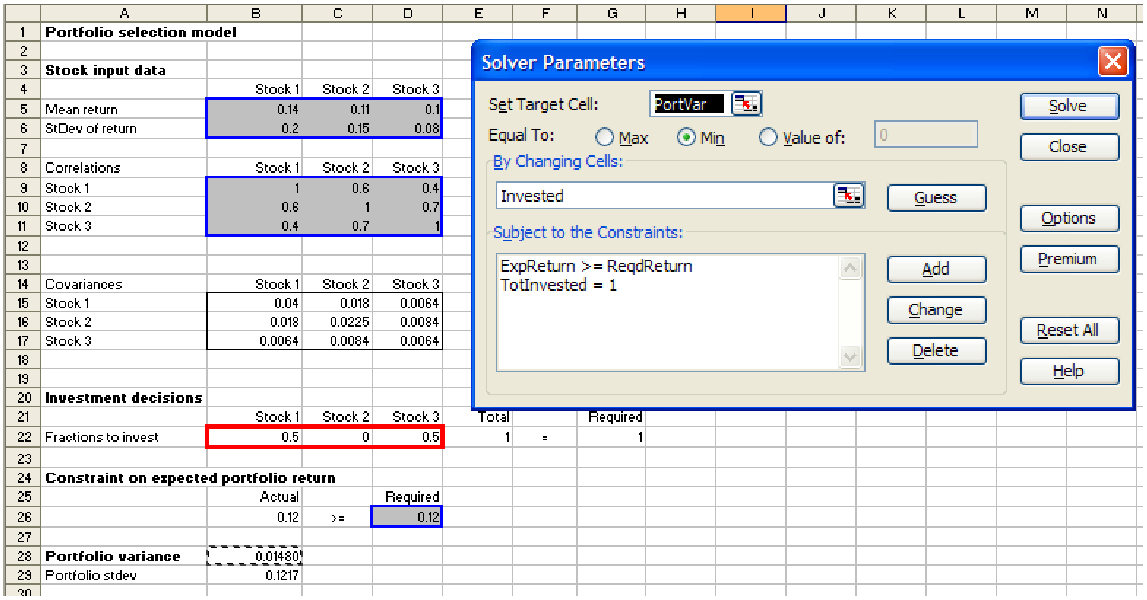
Expected Return and Variance:

PortVar {=SUMPRODUCT(MMULT(Invested,CovarMat),Invested)}

	A	B
22	Fractions to invest	0.500000064281812
23		
24	**Constraint on expected**	
25		Actual
26		=SUMPRODUCT(MeanReturns,Invested)
27		
28	**Portfolio variance**	=SUMPRODUCT(MMULT(Invested,CovarMat),Invested)
29	Portfolio stdev	=SQRT(PortVar)

Note the matrix formula in the variance calculation. You enter this through CTL-SHIFT-ENTER.

Solver:



<u>How do we get the "statistics" from raw data?</u>

	A	B	C	D	E	F
1		Portfolio Model	AT&T	GM	USS	Year
2			30.0%	22.5%	14.9%	1
3			10.3%	29.0%	26.0%	2
4			21.6%	21.6%	41.9%	3
5			-4.6%	-27.2%	-7.8%	4
6			-7.1%	14.4%	16.9%	5
7			5.6%	10.7%	-3.5%	6
8			3.8%	32.1%	13.3%	7
9			8.9%	30.5%	73.2%	8
10			9.0%	19.5%	2.1%	9
11			8.3%	39.0%	13.1%	10
12			3.5%	-7.2%	0.6%	11
13			17.6%	71.5%	90.8%	12
14		Average Return	8.91%	21.37%	23.46%	
15		Covariance Matrix	AT&T	GM	USS	
16		AT&T	0.0099	0.0114	0.0120	
17		GM	0.0114	0.0535	0.0508	
18		USS	0.0120	0.0508	0.0864	

	B	C	D	E
13		0.176	0.715	0.908
14	Average Return	=AVERAGE(C2:C13)	=AVERAGE(D2:D13)	=AVERAGE(E2:E13)
15	Covariance Matrix	AT&T	GM	USS
16	AT&T	=COVAR(C2:C13,$C$2:$C$13)	=COVAR(D2:D13,$C$2:$C$13)	=COVAR(E2:E13,$C$2:$C$13)
17	GM	=COVAR(C2:C13,$D$2:$D$13)	=COVAR(D2:D13,$D$2:$D$13)	=COVAR(E2:E13,$D$2:$D$13)
18	USS	=COVAR(C2:C13,$E$2:$E$13)	=COVAR(D2:D13,$E$2:$E$13)	=COVAR(E2:E13,$E$2:$E$13)

36. Add a new stock, stock 4, to the model in Example 7.9. Assume that the estimated mean and standard deviation of return for stock 4 are 0.125 and 0.175, respectively. Also, assume the correlations between stock 4 and the original three stocks are 0.3, 0.5, and 0.8. Run Solver on the modified model, where the required expected portfolio return is again 0.12. Is stock in the optimal portfolio? Then run SolverTable as in the example. Is stock 4 in any of the optimal portfolios on the efficient frontier?

38. The stocks in Example 7.9 are all *positively* correlated. What happens when they are *negatively* correlated? Answer for each of the following scenarios. In each case, two of the three correlations are the negatives of their original values. Discuss the differences between the optimal portfolios in these three scenarios.

a. Change the signs of the correlations between stocks 1 and 2 and between stocks 1 and 3. (Here, stock 1 tends to go in a different direction from stocks 2 and 3.)

b. Change the signs of the correlations between stocks 1 and 2 and between stocks 2 and 3. (Here, stock 2 tends to go in a different direction from stocks 1 and 3.)

c. Change the signs of the correlations between stocks 1 and 3 and between stocks 2 and 3. (Here, stock 3 tends to go in a different direction from stocks 1 and 2.)

39. The file **P07_39.xlsx** contains historical monthly returns for 28 companies. For each company, calculate the estimated mean return and the estimated variance of return. Then calculate the estimated correlations between the companies' returns. Note that "return" here means *monthly* return. (*Hint:* Make life easy for yourself by using StatTools' Summary Statistics capabilities.)

63. Consider three investments. You are given the following means, standard deviations, and correlations for the annual return on these three investments. The means are 0.12, 0.15, and 0.20. The standard deviations are 0.20, 0.30, and 0.40. The correlation between stocks 1 and 2 is 0.65, between stocks 1 and 3 is 0.75, and between stocks 2 and 3 is 0.41. You have $10,000 to invest and can invest no more than half of your money in any single stock. Determine the minimum-variance portfolio that yields an expected annual return of at least 0.14.

64. I have \$1000 to invest in three stocks. Let R_1 be the random variable representing the annual return on \$1 invested in stock i. For example, if $R_i = 0.12$, then \$1 invested in stock i at the beginning of a year is worth \$1.12 at the end of the year. The means are $E(R_1) = 0.14$, $E(R_2) = 0.11$, and $E(R_3) = 0.10$. The variances are $Var\ R_1 = 0.20$, $Var\ R_2 = 0.08$, and $Var\ R_3 = 0.18$. The correlations are $r_{12} = 0.8$, $r_{13} = 0.7$, and $r_{23} = 0.9$. Determine the minimum-variance portfolio that attains an expected annual return of at least 0.12.