LINEAR ALGEBRA

LECTURE 9: SINGULAR VALUE DECOMPOSITION

Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Advanced eigendecomposition
- Singular value decomposition

Singular Value Decomposition (SVD)

- Given a rectangular matrix $A \in \mathbb{R}^{m \times n}$,
  its singular value decomposition is written as
$$A = U\Sigma V^T$$

where

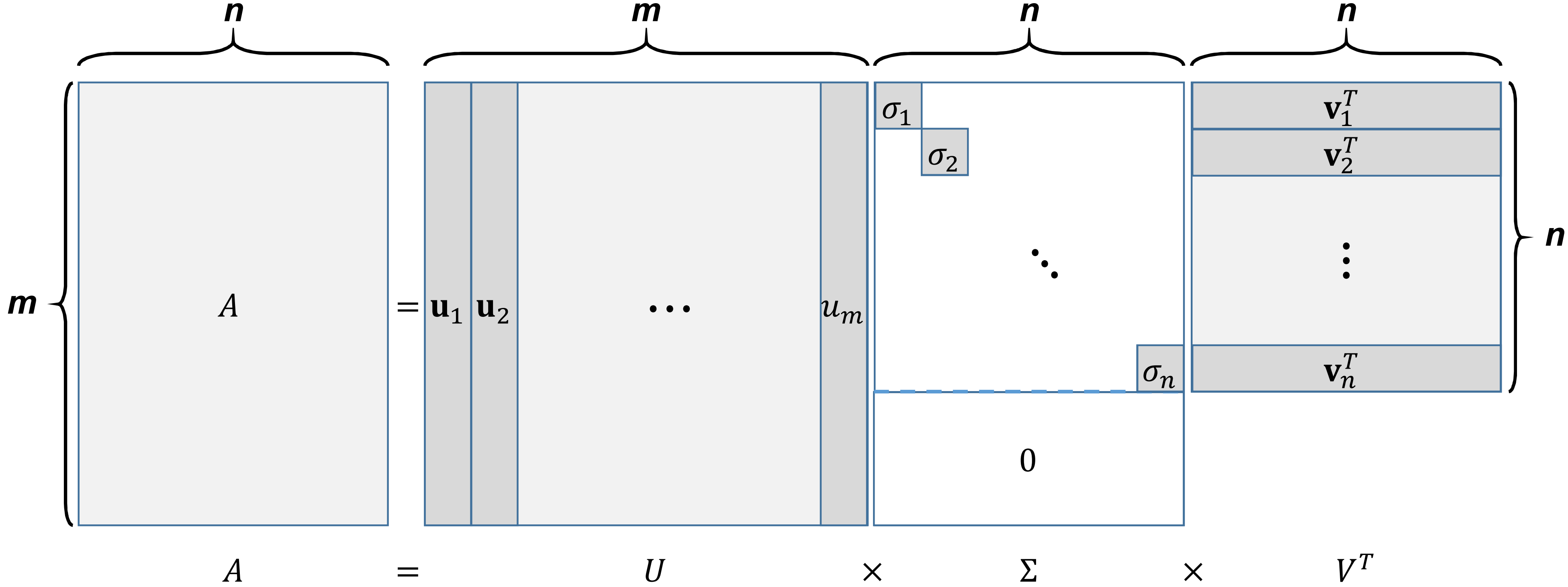
- $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$: matrices with orthonormal columns, providing an orthonormal basis of Col A and Row A, respectively

- $\Sigma \in \mathbb{R}^{m \times n}$: a diagonal matrix whose entries are in a decreasing order, i.e., $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$

Basic Form of SVD

- Given a matrix $A \in \mathbb{R}^{m \times n}$ where $m > n$, SVD gives

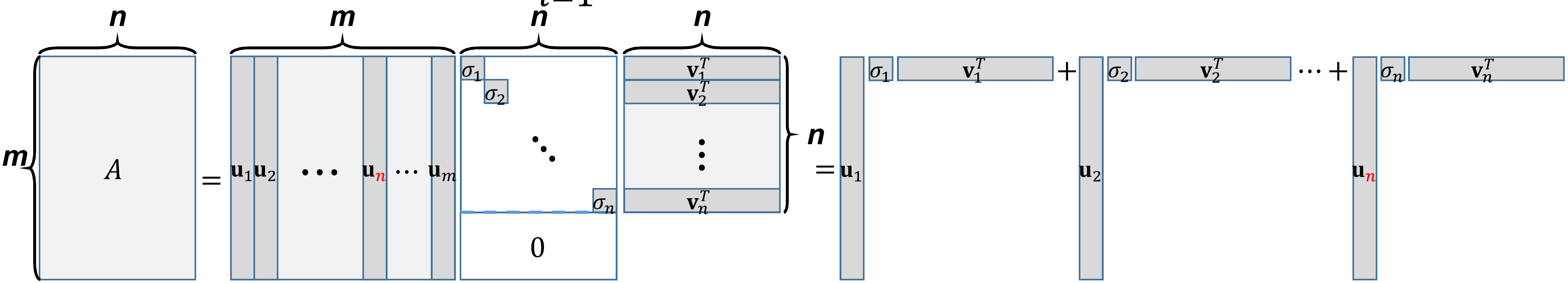
$$A = U \Sigma V^T$$



SVD as Sum of Outer Products

- A can also be represented as the sum of outer products

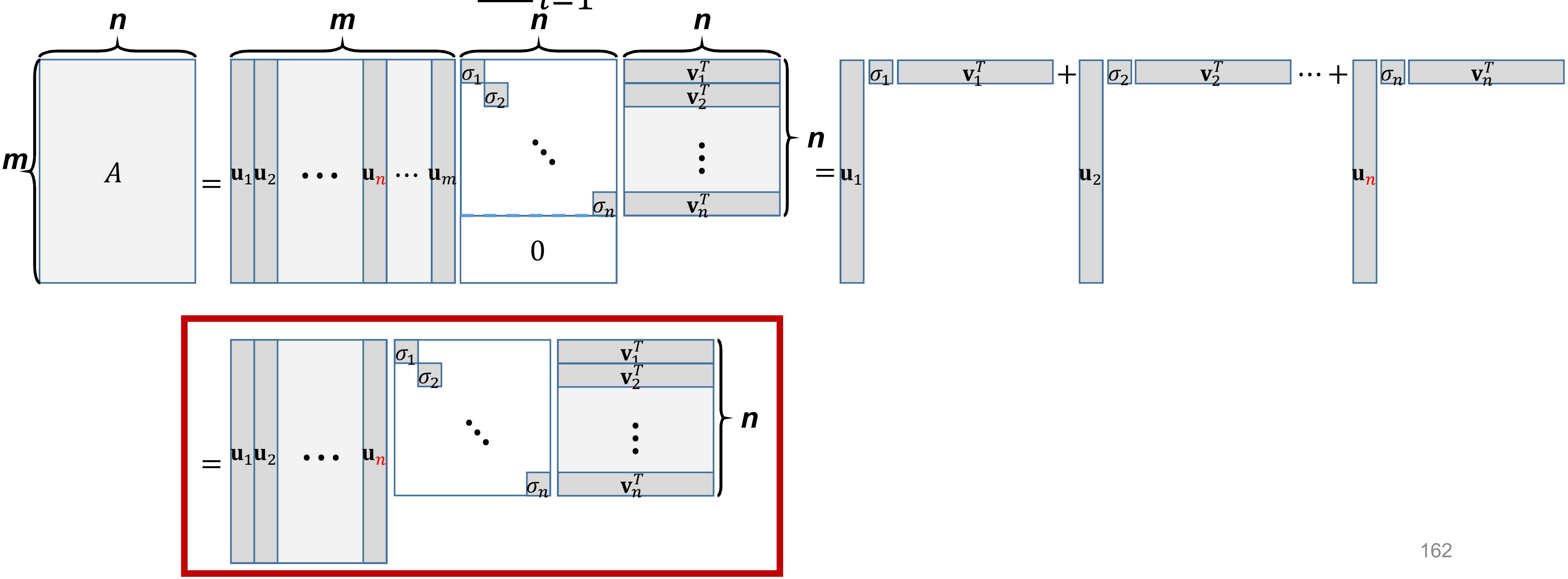
$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$



Reduced Form of SVD

- A can also be represented as the sum of outer products

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad \text{where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$$



The Singular Values of an *m* by *n* Matrix (1 of 3)

- **Theorem 9** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy $\lambda_1 \geq \cdots \geq \lambda_n$, and suppose *A* has *r* nonzero singular values. Then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for Col *A*, and rank $A = r$.

- **Proof** Because $\mathbf{v}_i$ and $\lambda_j \mathbf{v}_j$ are orthogonal for $i \neq j$,

$$\left(A\mathbf{v}_i\right)^T \left(A\mathbf{v}_j\right) = v_i^T A^T A\mathbf{v}_j = v_i^T \left(\lambda_j \mathbf{v}_j\right) = 0$$

- Thus $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is an orthogonal set.

The Singular Values of an *m* by *n* Matrix (2 of 3)

- Since the lengths of the vectors $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ are the singular values of *A*, and since there are *r* nonzero singular values, $A\mathbf{v}_i \neq \mathbf{0}$ if and only if $1 \leq i \leq r$.
- So $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ are linearly independent vectors, and they are in Col *A*.
- Finally, for any **y** in Col *A*—say, $\mathbf{y} = A\mathbf{x}$—we can write $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, and

$$\mathbf{y} = A\mathbf{x} = c_1A\mathbf{v}_1 + \cdots + c_rA\mathbf{v}_r + c_{r+1}A\mathbf{v}_{r+1} + \cdots + c_nA\mathbf{v}_n$$

The Singular Values of an *m* by *n* Matrix (3 of 3)

$$= c_1 A\mathbf{v}_1 + \cdots + c_r A\mathbf{v}_r + 0 + \cdots + 0$$

- Thus **y** is in Span $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$, which shows that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an (orthogonal) basis for Col *A*.
- Hence rank *A* = dim Col *A* = *r*.

The Singular Value Decomposition (1 of 11)

- **Theorem 10: The Singular Value Decomposition**
  Let *A* be an $m \times n$ matrix with rank *r*. Then there exists an $m \times n$ matrix Σ as in (3) for which the diagonal entries in *D* are the first *r* singular values of *A*, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ orthogonal matrix *U* and an $n \times n$ orthogonal matrix *V* such that

$$A = U \Sigma V^T$$

The Singular Value Decomposition (2 of 11)

- Any factorization $A = U\Sigma V^T$, with *U* and *V* orthogonal, Σ as in (3), and positive diagonal entries in *D*, is called a **singular value decomposition** (or **SVD**) of *A*.

- The columns of *U* in such a decomposition are called **left singular vectors** of *A*, and the columns of *V* are called **right singular vectors** of *A*.

- **Proof** Let λ_i and $\mathbf{v}_\mathrm{i}$ be as in Theorem 9, so that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_\mathrm{r}\}$ is an orthogonal basis for Col *A*.

The Singular Value Decomposition (3 of 11)

- Normalize each $A\mathbf{v}_{\mathrm{i}}$ to obtain an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{\mathrm{r}}\}$, where

$$\mathbf{u}_i = \frac{1}{\|A\mathbf{v}_i\|} A\mathbf{v}_i = \frac{1}{\sigma_1} A\mathbf{v}_i$$

- And

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \qquad (1 \le i \le r) \qquad (4)$$

- Now extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_{\mathrm{r}}\}$ to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of $\mathbb{R}^m$, and let

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_m \end{bmatrix} \text{ and } V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

- By construction, *U* and *V* are orthogonal matrices.

The Singular Value Decomposition (4 of 11)

- Also, from (4),

$$AV = \begin{bmatrix} A\mathbf{v}_1 & \dots & A\mathbf{v}_r & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \dots & \sigma_r\mathbf{u}_r & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

- Let D be the diagonal matrix with diagonal entries $\sigma_1, \dots, \sigma_r$, and let Σ be as in (3) above. Then

$$U\Sigma = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{bmatrix} \left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= \begin{bmatrix} \sigma_1\mathbf{u}_1 & \cdots & \sigma_r\mathbf{u}_r & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= AV$$

- Since V is an orthogonal matrix, $U\Sigma V^T = AVV^T = A$.