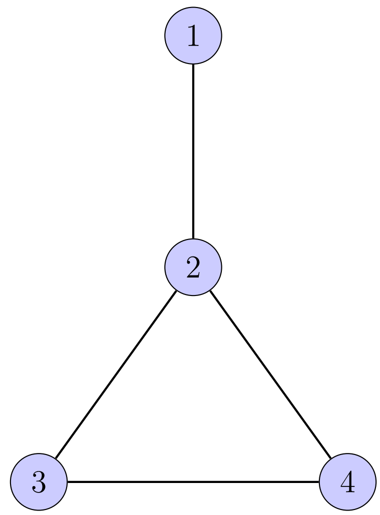
Undirected Graphs and Their Associated Matrices

In this lecture we'll look at a method to encode an undirected graph in a special matrix called an *adjacency matrix*.

Consider the following undirected graph, G,



Def: The adjacency matrix A of an undirected graph G, is a square $n \times n$ matrix where n is the number of vertices in the graph. The entries of A are given by

$$A_{ij} = \begin{cases} 1 & \text{if } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$

This says that the (i, j)-entry of the adjacency matrix is 1 is there is an edge connecting vertices i and j and 0 if there is not.

Example 1: For the graph G pictured above, the adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fact 1: The adjacency matrix A of an undirected graph is always symmetric.

Def: The **degree** of a vertex is the number of vertices that it is connected to (or alternatively, the number of edges that it is a part of). We denote the degree of vertex i by d_i.

Example 2: For the graph G pictured above the vertex degrees are

$$d_1 = 1 \qquad d_2 = 3 \qquad d_3 = 2 \qquad d_4 = 2$$

Fact 2: The degree of vertex i is equal to the sum of row i.

Def: A walk is a list of vertices that can be traversed along edges in the graph. Note that there is no restriction on the number of times a vertex is touched during a walk.

Example 3. Denote the i^{th} vertex by v_i. Then the walk the walk that goes from vertex 1 to vertex 2 to vertex 4 is denoted by $v_1 - v_2 - v_4$. The walk that goes from vertex 1 to vertex 2 to vertex 3 and then back to vertex 2 is denoted by $v_1 - v_2 - v_3 - v_2$.

Fact 3: The (i, j)-entry of the matrix power A^k tell you how many walks of length k exist in the graph that start at vertex i and end at vertex j.

Example 4: We'll look at powers of the adjacency matrix corresponding to the the graph G pictured above. For the first power of A we have

$$A^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

This is of course just the adjacency matrix of G itself. According to Fact 3, since $A_{12} = 1$ there is exactly one walk that starts at v_1 and ends at v_2. This is of course true because there is an edge between v_1 and v_2. Similarly, $A_{13} = 0$, indicating that there are no length-1 walks between v_1 and v_3.

Squaring the matrix A, we have

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

According to Fact 3, since $A_{14} = 1$ there is exactly one length-2 walk that starts at v_1 and ends at v_4. From the graph we see that this is the walk $v_1 - v_2 - v_4$. Notice that it is impossible to traverse a different sequences of vertices to get from v_1 to v_4. Similarly, since $A_{22} = 3$ there are exactly three length-2 walks that start at v_2 and end at v_2. These are $v_2 - v_1 - v_2$, $v_2 - v_3 - v_2$, and $v_2 - v_4 - v_2$.

Finally, cubing A we have

$$A^3 = \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{bmatrix}$$

Since $A_{42} = 4$ there are exactly four length-3 paths that start at v_4 and end at v_2. Looking at the graph we see that these walks are $v_4 - v_2 - v_1 - v_2$, $v_4 - v_2 - v_3 - v_2$, $v_4 - v_2 - v_4 - v_2$, and $v_4 - v_3 - v_4 - v_2$.

Def: A graph is called *connected* if there exists a walk from v_i to v_j for all pairs i and j.

Fact 4: A graph G is connected if there exists some integer $k > 0$ such that $B_k = I + A + A^2 + \cdots + A^k$ has all positive entries.

Essentially this works because the entries of the powers of A give the number of walks of length k that connect each of the vertices. If B_k has all positive entries, then there exists a walk of at most length k between any two vertices.

Example 5: For the adjacency matrix for the example graph, we have

$$B_1 = I + A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

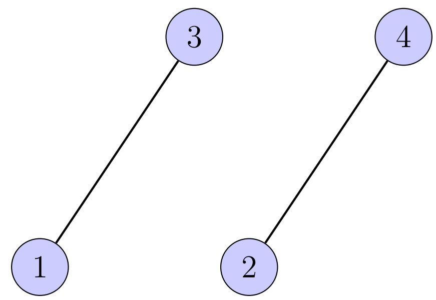
Since there are zero entries in B_1 it is not the case that you can get from every vertex to every other in walks of length 1 (for instance, it's impossible to get from v_1 to v_3 in one step).

We then have

$$B_2 = I + A + A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 2 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

Since B_2 has all positive entries we know that G is connected, and you can get from any vertex to any other in a walk of length 2 or less. This is clearly verified by looking at the graph.

Example 6: Consider the following disconnected graph



The associated adjacency matrix is $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Taking powers of A we have

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly this pattern repeats. Thus for any integer k the matrix B_k has the form

$$B_k = I + A + A^2 + \cdots + A^k = \begin{bmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{bmatrix}$$

Since B_k will not have all positive entries for any value of k the graph is not connected.

Example 7: Note that the analysis in Example 6 tells us that the associated graph has multiple disconnected components, but it doesn't really tell us where they are. It turns out that we can determine the disconnected components of a graph by examinging the eigenvectors of a new graph matrix: the so-called **Graph Laplacian Matrix**. First we need a diagonal matrix D called the degree matrix of an undirected graph. It has the following form

$$D_{ij} = \begin{cases} d_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Note that this is simply a diagonal matrix whose i^{th} main diagonal entry is the degree of vertex i. The degree matrix D together with the adjacency matrix A allows us to construct the graph Laplacian, which we call L. We have

$$L = D - A$$

For the graph in Example 1, we have

$$L = D - A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Notice that the $-1's$ on the off-diagonals of L correspond to entries in the adjacency matrix A (i.e. they tell us which vertices are connected) and the diagonal entries of L tell us the degree of each vertex. Notice in particular that each row of L sums to zero. This must be true since the -1's in row i correspond to the vertices that are connected to vertex i, and the diagonal element is the degree of vertex i (i.e. the number of nodes the vertex is connected to).

Since the rows of L sum to zero, it is very easy to find an eigenvector of L corresponding to a zero eigenvalue. Any nonzero constant vector (e.g. a vector of all 1's) will work. Define $\mathbb{1}$ to be the vector of all 1's, then

$$L\mathbb{1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So we have $\mathbf{v} = \mathbb{1}$ and $\lambda = 0$ is an eigenpair of L. For this particular graph, this is the only eigenvector associated with $\lambda = 0$. The other eigenvalues are $\lambda_2 = 1$, $\lambda_3 = 3$ and $\lambda_4 = 4$.

Example 8: Now consider the graph Laplacian of the graph in Example 6. We have

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Notice that, as expected, the constant vector $\mathbb{1}$ is in the nullspace of L, and therefore is an eigenvector of L with associated eigenvalue $\lambda = 0$. It turns out though, that this graph Laplacian has a second zero eigenvalue (or we could say, $\lambda = 0$ is an eigenvalue with multiplicity 2). If you compute the eigenvalues of L (by hand, or with Matlab), you find that the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$, and $\lambda_4 = 2$. We are interested in the eigenvectors of L associated with the zero eigenvalue. Solving for those eigenvectors, we have

$$L - 0I = L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding the two nullspace vectors (eigenvectors), we have

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

OK, now think of the entries of $\mathbf{v}_1$ and $\mathbf{v}_2$ as corresponding to the vertices on the graph. Notice that for $\mathbf{v}_1$, the nonzero entries are in the first and third positions. If you look at the graph in Example 6, you'll see that vertices v_1 and v_3 are connected to each other, but diconnected from vertices v_2 and v_4. Similarly, the nonzero entries in $\mathbf{v}_2$ correspond to vertices v_2 and v_4. Vertices v_2 and v_4 are connected to each other, but not connected to vertices v_1 and v_2. In other words, each eigenvector associated with $\lambda = 0$ corresponds to a particular disconnected component of the graph.

Furthermore, the graph corresponding to Examples 1 and 7 had only one eigenvector associated with $\lambda = 0$. This is because the graph from Examples 1 and 7 is connected.

Fact: The multiplicity of the $\lambda = 0$ eigenvalue of the graph Laplacian tells you the number of disconnected components in a graph. The nonzero entries in each eigenvector corresponding to $\lambda = 0$ tells you which vertices are in the corresponding component.