ECN
MSc Computational Mechanics
Sem 3

Model Reduction

Lab 5: Separated representation of vector field problems

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Contents

List of Figures

1 Introduction

Elasticity Problem

We start directly with the weak formulation of the plane-stress elasticity problem in a body Ω, which is given as follows:

$$\int_\Omega \varepsilon^{*t}\mathbf{D}\varepsilon d\Omega = \int_\Omega \mathbf{u}^{*t}\mathbf{b} d\Omega$$

where $\bullet^t$ stands for the transpose. The definitions for the terms in the weak form are given as follows:

$$\varepsilon = \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \qquad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix}b_x\\ b_y\end{bmatrix}$$

No surface tractions are considered; appropriate Dirichlet boundary conditions will be introduced later; $\mathbf{b}$ is the body force vector; E is the Young modulus; ν is the Poisson coefficient. Starting from the weak form , the left hand side in terms of primary variables u and v

$$\int_\Omega \varepsilon^{*t}\mathbf{D}\varepsilon d\Omega = \frac{E}{1-\nu^2}\int_\Omega \left[\varepsilon_x^*\varepsilon_x + \nu\varepsilon_x^*\varepsilon_y + \nu\varepsilon_y^*\varepsilon_y + \frac{1-\nu}{2}\gamma_{xy}^*\gamma_{xy}\right] d\Omega$$

$$= \frac{E}{1-\nu^2}\int_\Omega \Big[\frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} + \nu\frac{\partial u^*}{\partial x}\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\frac{\partial v^*}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial v^*}{\partial y}\cdots$$

$$\cdots\frac{1-\nu}{2}\frac{\partial u}{\partial y}\frac{\partial u^*}{\partial y} + \frac{1-\nu}{2}\frac{\partial v}{\partial x}\frac{\partial u^*}{\partial y} + \frac{1-\nu}{2}\frac{\partial u}{\partial y}\frac{\partial v^*}{\partial x} + \frac{1-\nu}{2}\frac{\partial v}{\partial y}\frac{\partial v^*}{\partial y}\Big] d\Omega \tag{1.1}$$

Finding the tensor structure

We want to find the corresponding matrix in order to fill the cell array `AA`. The u, u^*, v and v^* and the respective partial differentials with respect to x and y, are given as follows:

$$u = X_uY_u \qquad u^* = X_u^*Y_u + X_uY_u^*$$

$$v = X_vY_v \qquad v^* = X_v^*Y_v + X_vY_v^*$$

$$\frac{\partial u}{\partial x} = \frac{\partial X_u}{\partial x}Y_u \qquad \frac{\partial u}{\partial y} = X_u\frac{\partial Y_u}{\partial y}$$

$$\frac{\partial u^*}{\partial x} = \frac{\partial X_u^*}{\partial x}Y_u + \frac{\partial X_u}{\partial x}Y_u^* \qquad \frac{\partial u^*}{\partial y} = X_u^*\frac{\partial Y_u}{\partial y} + X_u\frac{\partial Y_u^*}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial X_v}{\partial x}Y_v \qquad \frac{\partial v}{\partial y} = X_v\frac{\partial Y_v}{\partial y}$$

$$\frac{\partial v^*}{\partial x} = \frac{\partial X_v^*}{\partial x}Y_v + \frac{\partial X_v}{\partial x}Y_v^* \qquad \frac{\partial v^*}{\partial y} = X_v^*\frac{\partial Y_v}{\partial y} + X_v\frac{\partial Y_v^*}{\partial y}$$

Now using the above definitions, we can get the terms for `AA` by solving for each term in equation 1.1 as follows. The example for first term is given in the problem statement:

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial u^*}{\partial x} d\Omega = \int_\Omega \frac{\partial X_u}{\partial x}Y_u\Big(\frac{\partial X_u^*}{\partial x}Y_u + \frac{\partial X_u}{\partial x}Y_u^*\Big) d\Omega$$

Suppose that Y_u is known and that $\Omega = [0, L] \times [0, H]$ is a Cartesian domain. Then: the part of the equation is given as follows:

$$\int_\Omega \frac{\partial X_u^*}{\partial x} Y_u \frac{\partial X_u}{\partial x} Y_u d\Omega \equiv \int_0^L \frac{\partial X_u^*}{\partial x}\frac{\partial X_u}{\partial x} dx \int_0^H Y_u Y_u dy \equiv (\mathbf{X}_u^{*t}\mathbf{K}_x\mathbf{X}_u) \otimes (\mathbf{Y}_y^t\mathbf{M}_y\mathbf{Y}_u) \tag{1.2}$$

$$\int_\Omega \frac{\partial X_u}{\partial x} Y_u^* \frac{\partial X_u}{\partial x} Y_u d\Omega \equiv \int_0^L \frac{\partial X_u}{\partial x}\frac{\partial X_u}{\partial x} dx \int_0^H Y_u^* Y_u dy \equiv (\mathbf{X}_u^{t}\mathbf{K}_x\mathbf{X}_u) \otimes (\mathbf{Y}_y^{*t}\mathbf{M}_y\mathbf{Y}_u) \tag{1.3}$$

Observe that eq 1.2 and eq 1.3 are essentially equivalent, the only thing that changes is the position of the test function. Therefore, one can choose one or other indistinctly when the object is to find the matrices in order to fill `AA`. Up until this point we have done what we did in lab 4. If the problem was scalar valued, then we could say that $\texttt{AA}\{1,1\} = \mathbf{K}_x$ and $\texttt{AA}\{2,1\} = \mathbf{M}_y$. However, we are solving a vector field problem, and not a scalar field. We want to compute not only X_u and Y_u but also X_v and Y_u. Therefore, we need to recast eq 1.2 as follows:

$$(\mathbf{X}_u^{*t}\mathbf{K}_x\mathbf{X}_u) \otimes (\mathbf{Y}_y^t\mathbf{M}_y\mathbf{Y}_u) \equiv \left(\begin{bmatrix} X_u^{*t} & X_u^{*t} \end{bmatrix} \begin{bmatrix} K_x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_u \\ X_v \end{bmatrix} \right) \otimes \left(\begin{bmatrix} Y_u^t & Y_u^t \end{bmatrix} \begin{bmatrix} M_y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_u \\ Y_v \end{bmatrix} \right) \tag{1.4}$$

Of course we do not repeat the operation for 1.3 because it is completely useless. According to eq 1.4 the first column of the cell array `AA` is defined as follows:

$$\texttt{AA}\{1,1\} = \begin{bmatrix} K_x & 0 \\ 0 & 0 \end{bmatrix} \qquad \texttt{AA}\{2,1\} = \begin{bmatrix} M_y & 0 \\ 0 & 0 \end{bmatrix}$$

Here the K_x, M_x, K_y, M_y, C_{x1}, C_{x2}, C_{y1} and C_{y2}, are defined by the function provided `FEM_mat_1D(x,op1,op2)` and setting `op1, op2` values as 0 or 1 depending on presence or absence of the differentiation. For K matrix both the terms are differentiated , hence for K $op1 = op2 = 1$, for the mass matrix M both the terms are non-differentiated hence we set $op1 = op2 = 0$. Similarly, depending on the position of the differential, given as follows:

$$C_{x_i 1} = \int_{x_i} \frac{\partial N_i}{\partial x_i} N_j dx_i = \texttt{FEM\_mat\_1D}(x_i, 1, 0)$$

$$C_{x_i 2} = \int_{x_i} N_i \frac{\partial N_j}{\partial x_i} dx_i = \texttt{FEM\_mat\_1D}(x_i, 0, 1)$$

2 Exercise 1 and Exercise 2

Now following the same procedure for all the terms in the eq 1.1, and acquiring the remaining terms of the `AA` matrix. The `AA` matrix will have 2 rows, meaning two dimensions for the x and y,and the equations to be solve are 8, hence it will have 8 columns. The repeated second term can be ignored, but just to give the complete integrals of the terms all the terms all included. In the eq 1.1, their are 8 terms, the first term is already solved in the problem statement, and shown above. The rest of the terms are given below as follows

$$Term2 = \nu \int \frac{\partial u^*}{\partial x}\frac{\partial v}{\partial y} d\Omega = \nu(\mathbf{X}_u^{*t}\mathbf{C}_x\mathbf{X}_v) \otimes (\mathbf{Y}_u^t\mathbf{C}_y\mathbf{Y}_v) + \nu(\mathbf{X}_u^t\mathbf{C}_x\mathbf{X}_v)(\mathbf{Y}_u^t\mathbf{C}_y\mathbf{Y}_v)$$

$$\texttt{AA}\{1,2\} = \begin{bmatrix} 0 & C_{x1} \\ 0 & 0 \end{bmatrix} \qquad \texttt{AA}\{2,2\} = \begin{bmatrix} 0 & C_{y2} \\ 0 & 0 \end{bmatrix}$$

$$Term3 = \nu \int \frac{\partial u}{\partial x}\frac{\partial v^*}{\partial y} d\Omega = \nu(\mathbf{X}_u^t\mathbf{C}_x\mathbf{X}_v^*) \otimes (\mathbf{Y}_u^t\mathbf{C}_y\mathbf{Y}_v) + \nu(\mathbf{X}_u^t\mathbf{C}_x\mathbf{X}_v)(\mathbf{Y}_u^t\mathbf{C}_y\mathbf{Y}_v)$$

$$\texttt{AA}\{1,3\} = \begin{bmatrix} 0 & 0 \\ C_{x2} & 0 \end{bmatrix} \qquad \texttt{AA}\{2,3\} = \begin{bmatrix} 0 & 0 \\ C_{y1} & 0 \end{bmatrix}$$

$$Term4 = \int \frac{\partial v}{\partial y}\frac{\partial v^*}{\partial y} d\Omega = (\mathbf{X}_v^t \mathbf{M}_x \mathbf{X}_v) \otimes (\mathbf{Y}_v^t \mathbf{K}_y \mathbf{Y}_v) + (\mathbf{X}_v^t \mathbf{M}_x \mathbf{X}_v)(\mathbf{Y}_v^t \mathbf{K}_x \mathbf{Y}_v)$$

$$\texttt{AA}\{1,4\} = \begin{bmatrix} 0 & 0 \\ 0 & M_x \end{bmatrix} \qquad \texttt{AA}\{2,4\} = \begin{bmatrix} 0 & 0 \\ 0 & K_y \end{bmatrix}$$

To simplify, let us assume $a = \dfrac{1-\nu}{2}$

$$Term5 = a\int \frac{\partial u}{\partial y}\frac{\partial u^*}{\partial y} d\Omega = a(\mathbf{X}_u^t \mathbf{M}_x \mathbf{X}_u) \otimes (\mathbf{Y}_u^t \mathbf{K}_y \mathbf{Y}_u) + a(\mathbf{X}_u^t \mathbf{M}_x \mathbf{X}_u)(\mathbf{Y}_u^t \mathbf{K}_y \mathbf{Y}_u^*)$$

$$\texttt{AA}\{1,5\} = \begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \qquad \texttt{AA}\{2,5\} = \begin{bmatrix} K_y & 0 \\ 0 & 0 \end{bmatrix}$$

$$Term6 = a\int \frac{\partial v}{\partial x}\frac{\partial u^*}{\partial y} d\Omega = a(\mathbf{X}_v^t \mathbf{C}_x \mathbf{X}_u) \otimes (\mathbf{Y}_v^t \mathbf{C}_x \mathbf{Y}_u) + a(\mathbf{X}_v^t \mathbf{C}_x \mathbf{X}_u)(\mathbf{Y}_v^t \mathbf{C}_y \mathbf{Y}_u)$$

$$\texttt{AA}\{1,6\} = \begin{bmatrix} 0 & C_{x2} \\ 0 & 0 \end{bmatrix} \qquad \texttt{AA}\{2,6\} = \begin{bmatrix} 0 & C_{y1} \\ 0 & 0 \end{bmatrix}$$

$$Term7 = a\int \frac{\partial u}{\partial y}\frac{\partial v^*}{\partial x} d\Omega = a(\mathbf{X}_u^t \mathbf{C}_x \mathbf{X}_v^*) \otimes (\mathbf{Y}_u^t \mathbf{C}_y \mathbf{Y}_v) + a(\mathbf{X}_u^t \mathbf{C}_x \mathbf{X}_v)(\mathbf{Y}_u^t \mathbf{C}_y \mathbf{Y}_v^*)$$

$$\texttt{AA}\{1,7\} = \begin{bmatrix} 0 & 0 \\ C_{x1} & 0 \end{bmatrix} \qquad \texttt{AA}\{2,7\} = \begin{bmatrix} 0 & 0 \\ C_{y2} & 0 \end{bmatrix}$$

$$Term8 = a\int \frac{\partial v}{\partial x}\frac{\partial v^*}{\partial x} d\Omega = a(\mathbf{X}_v^t \mathbf{K}_x \mathbf{X}_v^*) \otimes (\mathbf{Y}_v^t \mathbf{M}_y \mathbf{Y}_v) + a(\mathbf{X}_v^t \mathbf{K}_x \mathbf{X}_v)(\mathbf{Y}_v^t \mathbf{M}_y \mathbf{Y}_v^*)$$

$$\texttt{AA}\{1,8\} = \begin{bmatrix} 0 & 0 \\ 0 & K_x \end{bmatrix} \qquad \texttt{AA}\{2,8\} = \begin{bmatrix} 0 & 0 \\ 0 & M_y \end{bmatrix}$$

3 Exercise 3

A unit square domain, $L = 1$ and $H = 1$, discretized with $N_x = 21$ and $N_y = 21$ nodes in horizontal and vertical directions respectively. The Young's Modulus $E = 10^4$ and Poisson's ration $\nu = 0.3$. Symmetry boundary conditions are imposed on both bottom and left edges i.e in the bottom edge the vertical component of the displacement is zero, while in the left edge horizontal component of the displacement is zero as shown in fig 1.

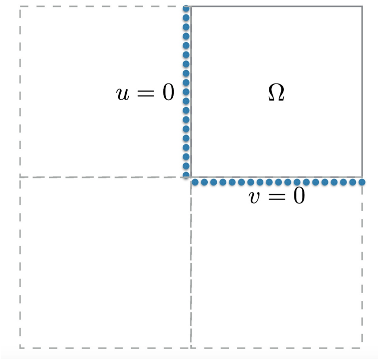

Figure 1: Domain with definition of the boundary conditions

3.1 Defining BB

The RHS of the weak form is defined so as to find the RHS matrix `BB`.

$$RHS = \int_\Omega \mathbf{u}^{*t}\mathbf{b} d\Omega = \int_\Omega \begin{bmatrix} u^* & v^* \end{bmatrix} \begin{bmatrix} b_x(X,Y) \\ b_y(X,Y) \end{bmatrix} d\Omega = \int_\Omega u^* b_x(X,Y) d\Omega + \int_\Omega v^* b_x(X,Y) d\Omega$$

We substitute the u^* and v^* as defined in earlier section, also segregating the body force $\mathbf{b}$ in of X and Y direction, and obtain the following:

$$RHS = \int_\Omega X_u^* Y_u b_x(X) b_x(Y) d\Omega + \int_\Omega X_u Y_u^* b_x(X) b_x(Y) d\Omega + \int_\Omega X_v^* Y_v b_y(X) b_y(Y) d\Omega + \int_\Omega X_v Y_v^* b_y(X) b_y(Y) d\Omega$$

As done for the LHS, the above integrals can be split into independent domain integrals of x and y as follows:

$$RHS = \int_0^L X_u^* b_x(X) dx \int_0^H Y_u b_x(Y) dy + \int_0^L X_u b_x(X) dx \int_0^H Y_u^* b_x(Y) dy + \cdots$$

$$\cdots \int_0^L X_v^* b_y(X) dx \int_0^H Y_v b_y(Y) dy + \int_0^L X_v b_y(X) dx \int_0^H Y_v^* b_y(Y) dy$$

Knowing Y_u and Y_v, we have the updated RHS as:

$$RHS = \int_0^L X_u^* b_x(X) dx \int_0^H Y_u b_x(Y) dy + \int_0^L X_v^* b_y(X) dx \int_0^H Y_v b_y(Y) dy$$

Applying FE discretization to the above equation, we obtain

$$RHS = (\mathbf{X}_u^* \mathbf{M}_x \mathbf{b}_x(X)) \otimes (\mathbf{Y}_u^* \mathbf{M}_y \mathbf{b}_x(Y)) + (\mathbf{X}_v^* \mathbf{M}_x \mathbf{b}_y(X)) \otimes (\mathbf{Y}_v^* \mathbf{M}_y \mathbf{b}_y(Y))$$

$$RHS = \begin{bmatrix} \mathbf{X}_u^* & \mathbf{X}_v^* \end{bmatrix} \begin{bmatrix} \mathbf{M}_x \mathbf{b}_x(X) & 0 \\ 0 & \mathbf{M}_x \mathbf{b}_y(X) \end{bmatrix} \otimes \begin{bmatrix} \mathbf{Y}_u^* & \mathbf{Y}_v^* \end{bmatrix} \begin{bmatrix} \mathbf{M}_y \mathbf{b}_x(Y) & 0 \\ 0 & \mathbf{M}_y \mathbf{b}_y(Y) \end{bmatrix} \tag{3.1}$$

In this problem,for the case where $b_x = 1$ $b_y = 1$, it was implemented in `script` as

- `BB{1,1} = [Mx*ones(`N_x`,1) sparse(`N_x`,1);sparse(`N_x`,1) Mx*ones(`N_x`,1)];`
- `BB{2,1} = [My*ones(`N_y`,1) sparse(`N_y`,1);sparse(`N_y`,1) My*ones(`N_y`,1)];`

3.2 Defining Bords

Here there is symmetry boundary conditions, that is; $u(x=0,y) = 0$ and $v(x, y=0) = 0$. In the separated representation this is interpreted as $X_u = Y_v = 0$ and was implemented in `script` as `Bords{1,1} = 1; Bords{2,1} = N+1;`.

4 Exercise 4

With the parameters , boundary conditions and body forces defined in Exercise 3, the `easy_PGD` is executed through the script `Main_Lab5_ADHAV_AGARAWAL.m`, with `Nmode`= 20 and `Niter`= 10 and the following graphs were obtained for the constant body force terms.

Based on the structure of `AA` assembly, and solution vector `u(x,y)`, the displacement x-component defined as $u = X_u Y_u$ was computed as `Ux = FF{1,1}(1:N,:)*FF{2,1}(1:N,:)';` and the y-component defined as $v = X_v Y_v$ was computed as `Uy = FF{1,1}((N+1):end,:) *FF{2,1}((N+1):end,:)';`

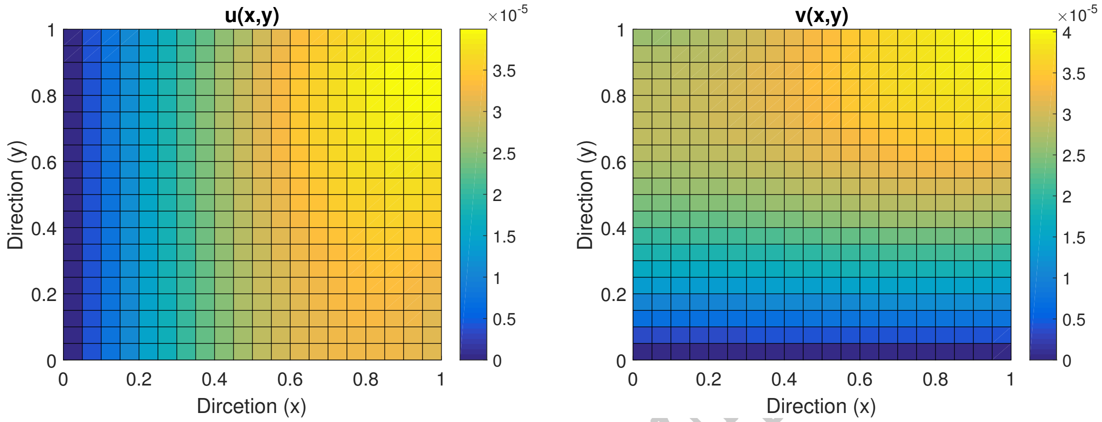

Figure 2: The horizontal component $u(x, y)$ and the vertical component $v(x, y)$ of the displacement

It can be observed from figure 2 and figure ?? that the displacement field verifies the symmetry boundary condition prescribed, as horizontal component has zero value at the left end and vertical component has zero at the bottom. The magnitude of the displacement computed as $\bar{u} = \sqrt{u^2 + v^2}$ and implemented in the matlab code as `U = sqrt(Ux.*Ux + Uy.*Uy);` is shown in figure 3. The deformed shape is scaled by factor $5.e3$ for visualization purposes and this scaling was achieved by adding the magnified x,y displacements on to the grid nodal values. This was implemented as follows in Matlab: `surf(X+5.e3*UU',Y+5.e3*VV',U')`. This is represented in figure 4.

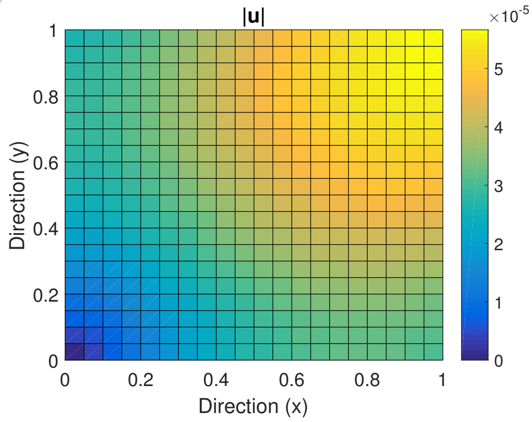

Figure 3: The magnitude of the displacement field

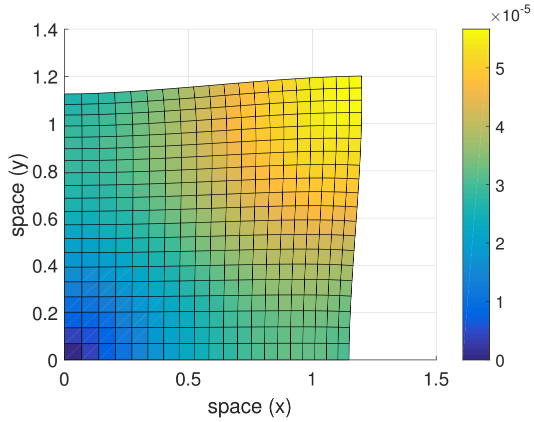

Figure 4: The deformed shape applying a scale factor of 5.10^3.

5 Exercise 5

The Exercise 4 is repeated but with the implementation of the body force defined as follows:

$$b_x(x,y) = x^2 y; \quad b_x(x) = x^2, b_y(y) = y \tag{5.1}$$

$$b_y(x,y) = (y-1)^2; \quad b_y(x) = 1, b_y(y) = (y-1)^2$$

The above values can be implemented in equation 3.1 to obtain the **BB** cell array.

The previous values of **AA** cell array and the above **BB** cell array was used to compute the $u(x,y)$ and $v(x,y)$ shown in the below figures.

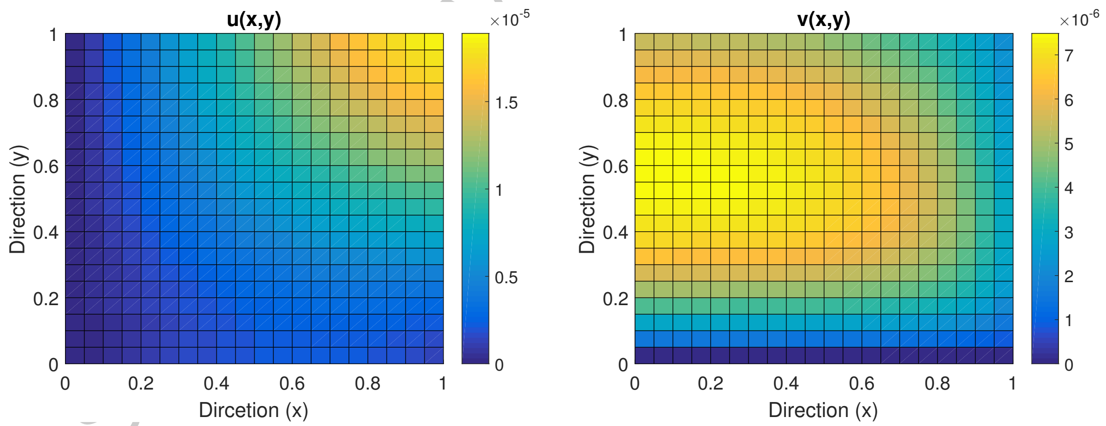

Figure 5: The horizontal component of the displacement $u(x,y)$ and the vertical component $v(x,y)$ of the displacement

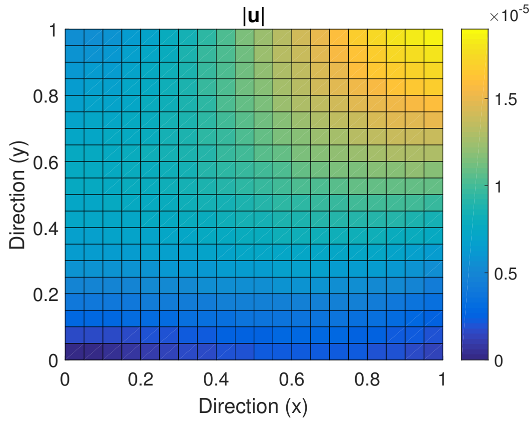

Figure 6: The magnitude of the displacement field

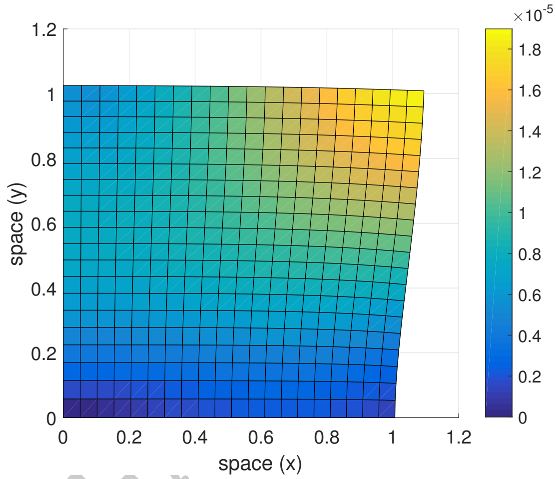

Figure 7: The deformed shape applying a scale factor of $5\dot{1}0^3$.