UPC
Master in Computational Mechanics
Sem 1

Finite Elements

HOMEWORK # 1

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Problem 1a

Consider the following differential equation

$$-u'' = f \qquad \text{in} \qquad [0,1]$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$. The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \dots n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u_h.

2. Describe the linear system of equations to be solved.

3. Compute the FE approximation u_h for $n = 3$, $f = sinx$ and $\alpha = 3$. Compare it with the exact solution $u(x) = sinx + (3sin1)x$.

Solution

1. For this problem, we have that

$$A(u) := \frac{d^2u}{dx^2} + f = 0$$

$$B(u) := \begin{cases} u = 0 & \text{on} \quad x = 0 \\ u = \alpha & \text{on} \quad x = 1 \end{cases}$$

The unknown u is approximated as follows,

$$u(x) \simeq u_h(x) = \sum_{i=0}^{n} N_i(x) a_i$$

where $N_i(x)$ are the shape or interpolation functions and a_i, $i = 1, n$ are the unknown parameters. The application of the weight residual method choosing a finite set of weighted functions reads,

$$\int_\Omega W\ A(u)\, d\Omega + \int_\Gamma \bar{W}\ B(u)\, d\Gamma = 0 \Rightarrow \int_\Omega W_i(x)\ A(u_h)\ \ d\Omega + \int_\Gamma \bar{W}_i(x)\ B(u_h)\, d\Gamma = 0 \quad i = 1, 2, ..., n$$

In this particular problem we do not have any Neumann boundary Γ_q so in this case we will not have to calculate any integral over the boundary. This is because we are choosing our function u_h so that it satisfies the Dirichlet boundary conditions. Thus, integrating over our domain, it yields

$$\int_0^l W_i\ \left[\frac{d^2u_h}{dx^2} + f\right]\ dx = 0 \qquad (i = 1, 2, ..., n)$$

Applying now the integration by parts formula, we obtain

$$-\int_0^l \frac{dW_i}{dx}\frac{du_h}{dx}\ dx + \int_0^l W_i f\ dx + \left[W_i \frac{du_h}{dx}\right]_0^l = 0$$

Finally, using Galerkin's method $W_i(x) = N_i(x)$, and the definition of $u_h(x)$, we obtain the discretized weak form of the problem

$$-\int_0^l \frac{dN_i}{dx}\frac{dN_j}{dx} a_j\ dx + \int_0^l N_i\ f\ dx + \left[N_i \frac{dN_j}{dx} a_j\right]_0^l = 0$$

or

$$\int_0^l \frac{dN_i}{dx}\frac{dN_j}{dx}a_j \; dx = \int_0^l N_i \; f \; dx + \left[N_i \frac{dN_j}{dx} a_j\right]_0^l \tag{1}$$

2. The above equation can be rearranged into a linear matrix system of equations of the form $\mathbf{K}\;\mathbf{a} = \mathbf{f}$ where $\mathbf{K}$ is the *stiffness* matrix of the finite element, $\mathbf{a}$ is the nodal unknown vector and $\mathbf{f}$ is the *force* vector. The third term in equation (1) can be seen as the nodal "reaction" and could be computed a posteriori. We can now write,

$$K_{ij} = \int_0^l \frac{dN_i}{dx}\frac{dN_j}{dx} \; dx \quad \text{and} \quad f_i = \int_0^l N_i \; f \; dx + [N_i q]_l - [N_i q]_0 \tag{2}$$

where the notation $q = \frac{dN_j}{dx} a_j$ has been introduced for simplicity. In matrix notation,

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \quad \mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \tag{3}$$

being n the total number of nodes in the finite element mesh.

3. The equation above needs to be rewritten for each element of the discretization.

$$K_{ij}^{(e)} = \int_{l^{(e)}} \frac{dN_i^{(e)}}{dx}\frac{dN_j^{(e)}}{dx} \; dx \quad \text{and} \quad f_i^{(e)} = \int_{l^{(e)}} N_i^{(e)} f \; dx \tag{4}$$

If $n = 3$ then $h = 1/3$ which is the length of the element, and thus we have three elements and four nodes located at points $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$. The shape functions are locally defined as

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{l^{(e)}} \qquad \text{and} \qquad N_2^{(e)} = \frac{x - x_1^{(e)}}{l^{(e)}}$$

so that they take value 1 at the node at 0 otherwise. Moreover, it is easy to see that

$$dN_1^{(e)} = \frac{-1}{l^{(e)}} = \frac{-1}{h} = -3 \qquad \text{and} \qquad dN_2^{(e)} = \frac{1}{l^{(e)}} = \frac{1}{h} = 3$$

For this problem, the interpolating functions can be defined as,

recorte1.png

Figure 0.1: Global (left) and local (right) definition of the shape functions

$$N_0 = \begin{cases} N_1^{(1)} = \frac{1/3 - x}{1/3} = 1 - 3x \quad \text{on} \quad 0 \le x \le 1/3 \\ 0 \quad \text{on} \quad 1/3 < x \le 1 \end{cases}$$

$$N_1 = \begin{cases} N_2^{(1)} = \frac{x-0}{1/3} = 3x \quad \text{on} \quad 0 \le x \le 1/3 \\ N_1^{(2)} = \frac{2/3 - x}{1/3} = 2 - 3x \quad \text{on} \quad 1/3 < x \le 2/3 \\ 0 \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

$$N_2 = \begin{cases} 0 \quad \text{on} \quad 0 \le x \le 1/3 \\ N_2^{(2)} = \frac{x-1/3}{1/3} = 3x-1 \quad \text{on} \quad 1/3 < x \le 2/3 \\ N_1^{(3)} = \frac{1-x}{1/3} = 3-3x \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

$$N_3 = \begin{cases} 0 \quad \text{on} \quad 0 \le x \le 2/3 \\ N_2^{(3)} = \frac{x-2/3}{1/3} = 3x-2 \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

As we discussed in the class, the global stiffness matrix $\mathbf{K}$ and the global equivalent "nodal flux" vector $\mathbf{f}$ can be obtained assembling the contributions for each element. For our mesh of 3 2-noded elements, the resulting matrix expression is

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \cdot \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} - q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_3^{(1)} \\ f_2^{(3)} + q_l \end{Bmatrix} \tag{5}$$

But for this particular case, we already know the values $a_0 = u(0) = 0$ and $a_3 = u(1) = \alpha = 3$. Therefore, we can get rid of these terms in the matrix equation, and simply solve

$$\begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} \end{bmatrix} \cdot \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_3^{(1)} \end{Bmatrix} - \begin{Bmatrix} k_{12}^{(1)} \cdot a_0 \\ k_{12}^{(3)} \cdot a_3 \end{Bmatrix} \tag{6}$$

Let's compute now the components of the matrices in equation (6).

$$k_{22}^{(1)} + k_{11}^{(2)} = \int_0^{1/3} \frac{dN_2^{(1)}}{dx} \frac{dN_2^{(1)}}{dx} dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx} \frac{dN_1^{(2)}}{dx} dx = (-3)\cdot(-3)\cdot(1/3) + (-3)\cdot(-3)\cdot(1/3) = 6$$

$$k_{22}^{(2)} + k_{11}^{(3)} = 6$$

$$k_{12}^{(2)} = k_{21}^{(2)} = \int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx} \frac{dN_1^{(2)}}{dx} dx = (3) \cdot (-3) \cdot (1/3) = -3 = k_{ij}^{(e)}, \quad \forall i \neq j$$

$$k_{11}^{(3)} = \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx} \frac{dN_1^{(3)}}{dx} dx = 3 = k_{ij}^{(e)}, \quad \forall i = j$$

$$f_2^{(1)} + f_1^{(2)} = \int_0^{1/3} 3x \cdot \sin x \; dx + \int_{1/3}^{2/3} (2-3x) \cdot \sin x \; dx = 3\left[-x \cdot \cos x + \sin x\right]_0^{1/3}$$

$$+2 \int_{1/3}^{2/3} \sin x \; dx - 3 \int_{1/3}^{2/3} x \cdot \sin x \; dx = 3\left[-x \cdot \cos x + \sin x\right]_0^{1/3} - 2\left[\cos x\right]_{1/3}^{2/3} - 3\left[-x \cdot \cos x + \sin x\right]_{1/3}^{2/3} =$$

$$= 6 \cdot \sin(1/3) - 3 \cdot \sin(2/3)$$

$$f_2^{(2)} + f_1^{(3)} = \int_{1/3}^{2/3} (3x-1) \cdot \sin x \; dx + \int_{2/3}^{1} (3-3x) \cdot \sin x \; dx = 3 \int_{1/3}^{2/3} x \cdot \sin x \; dx - \int_{1/3}^{2/3} \sin x \; dx$$

$$+ 3 \int_{2/3}^{1} \sin x \; dx + 3 \int_{2/3}^{1} x \cdot \sin x \; dx = 3\left[-x \cdot \cos x + \sin x\right]_{1/3}^{2/3} + \left[\cos x\right]_{1/3}^{2/3}$$

$$-3\,[\cos x]_{2/3}^{1} + 3\,[-x\cdot\cos x + \sin x]_{2/3}^{1} = 6\cdot\sin(2/3) - 3\cdot\sin(1/3) - 3\cdot\sin(1)$$

So finally, the system we need to solve is

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\cdot\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 6\cdot\sin(1/3) - 3\cdot\sin(2/3) \\ 6\cdot\sin(2/3) - 3\cdot\sin(1/3) - 3\cdot\sin(1) \end{Bmatrix} - \begin{Bmatrix} 0 \\ -9 \end{Bmatrix} \Rightarrow \begin{cases} a_1 = 1.0467 \\ a_2 = 2.0574 \end{cases} \tag{7}$$

Our FEM solution at the nodes is therefore

$u_h(x=0) = a_0 = 0$; $u_h(x=1/3) = a_1 = 1.0467$; $u_h(x=2/3) = a_2 = 2.0574$; $u_h(x=1) = a_3 = 3$

For the analytical solution, we obtain the values

$$u(x=0) = \sin(0) + (3-\sin(1))\cdot(0) = 0$$

$$u(x=1/3) = \sin(1/3) + (3-\sin(1))\cdot(1/3) = 1.0467$$

$$u(x=2/3) = \sin(2/3) + (3-\sin(1))\cdot(2/3) = 2.0574$$

$$u(x=1) = \sin(1) + (3-\sin(1))\cdot(1) = 3$$

As we can see, our FEM solution obtains the same values at the nodes. At any other point, the value is not going to be equal since we are building our approximation with linear functions. In order to compute the FE approximation $u_h(x)$, we know that

$$u_h(x) = \underbrace{N_0(x)a_0}_{=0} + N_1(x)a_1 + N_2(x)a_2 + N_3(x)a_3$$

and we also need to remember the definition of $N_1(x)$, $N_2(x)$ and $N_3(x)$

$$N_1 = \begin{cases} N_2^{(1)} = 3x \quad \text{on} \quad 0 \le x \le 1/3 \\ N_1^{(2)} = 2-3x \quad \text{on} \quad 1/3 < x \le 2/3 \\ 0 \quad \text{on} \quad 2/3 < x \le 1 \end{cases} \quad ; \quad N_2 = \begin{cases} 0 \quad \text{on} \quad 0 \le x \le 1/3 \\ N_2^{(2)} = 3x-1 \quad \text{on} \quad 1/3 < x \le 2/3 \\ N_1^{(3)} = 3-3x \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

$$N_3 = \begin{cases} 0 \quad \text{on} \quad 0 \le x \le 2/3 \\ N_2^{(3)} = 3x-2 \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

Therefore,

$$u_h(x) = \begin{cases} 3x\cdot a_1 \quad \text{on} \quad 0 \le x \le 1/3 \\ (2-3x)\cdot a_1 + (3x-1)\cdot a_2 \quad \text{on} \quad 1/3 < x \le 2/3 \\ (3-3x)\cdot a_2 + (3x-2)\cdot a_3 \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

and finally, introducing the values of a_1, a_2 and a_3 and rearranging terms, it yields

$$u_h(x) = \begin{cases} 3.1401x \quad \text{on} \quad 0 \le x \le 1/3 \\ 0.036 + 3.0321x \quad \text{on} \quad 1/3 < x \le 2/3 \\ 0.1722 + 2.8278x \quad \text{on} \quad 2/3 < x \le 1 \end{cases}$$

Problem 1b

Consider the 1-D heat conduction problem defined in the figure governed by the following equation

$$2\phi'' + Q = 0$$

where ϕ is the temperature and Q is the heat source. Consider the following boundary conditions

$$\phi = 0°C \quad \text{on} \quad x = 0$$

$$q = 0Nm/s \quad \text{on} \quad x = 8$$

problem1b.png

1. Calculate the temperature distribution if the domain is discretized with 2 2-noded elements.

2. Calculate the temperature distribution if the domain is discretized with 4 2-noded elements.

3. Plot the previous results in comparison with the analytical solution.

Solution

For this 1-D heat conduction problem, we have that

$$A(\phi) := 2\frac{d^2\phi}{dx^2} + Q = 0 \quad ; \quad Q = \begin{cases} 10 & \text{on} \quad 0 \le x \le 4 \\ 0 & \text{on} \quad 4 < x \le 8 \end{cases} \quad ; \quad k = 2\frac{W}{m°C}$$

$$B(\phi) := \begin{cases} \phi = \bar{\phi} = 0 & \text{on} \quad x = 0 \\ \bar{q} = 0 & \text{on} \quad x = 1 \end{cases}$$

Using the generalization of the solution for a mesh of 2-noded elements, the components of the matrices become

$$K_{ij}^{(e)} = \int_{l^{(e)}} \frac{dN_i^{(e)}}{dx} k \frac{dN_j^{(e)}}{dx} dx = (-1)^{i+j} \left(\frac{k}{l}\right)^{(e)} \quad ; \quad f_i^{(e)} = \int_{l^{(e)}} N_i^{(e)} Q dx = \frac{(Ql)^{(e)}}{2}$$

1. For a domain discretized with 2 2-noded elements the system of equations that we need to solve takes the form,

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \cdot \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} - \bar{q} \end{Bmatrix} \tag{8}$$

But since we already know that $\bar{q} = 0$ and $\phi_1 = \bar{\phi} = 0$, we just need to solve for ϕ_2 and ϕ_3.

$$\begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \cdot \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix}$$

The values of the components of the matrices are,

$$k_{ii}^{(e)} = \frac{1}{2} \quad ; \quad k_{12}^{(2)} = k_{21}^{(2)} = -\frac{1}{2} \quad ; \quad f_2^{(1)} = \frac{10 \cdot 4}{2} = 20$$

and

$$f_1^{(2)} = f_2^{(2)} = 0$$

since the heat source is zero within this element. Thus,

$$\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \end{Bmatrix} \Rightarrow \{\phi_2 = \phi_3 = 40$$

The end flux q_0, which is considered as a nodal "reaction", can be obtained by substituting ϕ_2 into the first equation giving

$$q_0 = k_{21}^{(1)} \phi_2 - f_1^{(1)} = -\frac{1}{2}(40) - 20 = -40$$

and the negative sign means that the flux is going out the domain in order to satisfy the balance equation. Analyzing the results, we see that the temperature remains constant within the second element ($4 < x \leq 8$) which makes sense due to the conditions of the problem. The approximate function $\hat{\phi}(x)$ for this 2 2-noded elements reads,

$$\hat{\phi}^{(1)}(x) = \underbrace{N_1^{(1)} \phi_1^{(1)}}_{=0} + N_2^{(1)} \phi_2^{(1)} = N_2^{(1)} \phi_2^{(1)} = \frac{x - x_1^{(1)}}{l^{(1)}} \phi_2^{(1)} = \frac{x}{4} \cdot 40 = 10x$$

$$\hat{\phi}^{(2)}(x) = N_1^{(2)} \phi_1^{(2)} + N_2^{(2)} \phi_2^{(2)} = \frac{x_2^{(2)} - x}{l^{(2)}} \phi_1^{(2)} + \frac{x - x_1^{(2)}}{l^{(2)}} \phi_2^{(2)} = \frac{8 - x}{4} \cdot 40 + \frac{x - 4}{4} \cdot 40 = 40$$

Therefore, the FEM solution is

$$\hat{\phi}(x) = \begin{cases} 10x & \text{on} \quad 0 \leq x \leq 4 \\ 40 & \text{on} \quad x > 4 \end{cases}$$

2. If the domain is now discretized with 4 2-noded elements, the system of equations that we need to solve takes the form,

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} & 0 \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} + k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \cdot \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_3^{(1)} \\ f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} - \bar{q} \end{Bmatrix} \tag{9}$$

Again, the value at the first node is known , $\phi_1 = 0$, and also $\bar{q} = 0$ so the system reduces to

$$\begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 & 0 \\ k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & 0 & 0 \\ 0 & k_{21}^{(3)} & k_{22}^{(3)} + k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \cdot \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_3^{(1)} \\ f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{Bmatrix}$$

The length of each finite element is now $l^{(e)} = 2$, and using

$$k_{ij}^{(e)} = (-1)^{i+j} \left(\frac{k}{l}\right)^{(e)} \quad ; \quad f_i^{(e)} = \frac{(Ql)^{(e)}}{2}$$

the values are now,

$$k_{22}^{(1)} = k_{11}^{(2)} = k_{22}^{(2)} = k_{11}^{(2)} = k_{22}^{(3)} = k_{11}^{(4)} = k_{22}^{(4)} = 1 \quad ; \quad f_2^{(1)} = f_1^{(2)} = f_2^{(2)} = \frac{10 \cdot 2}{2} = 10$$

and $f_1^{(3)} = f_2^{(3)} = f_1^{(4)} = 0$ since $Q = 0$ within the 3rd and 4th elements. Therefore, the system of equations becomes

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 10 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} \phi_2 = 30 \\ \phi_3 = \phi_4 = \phi_5 = 40 \end{cases}$$

and the nodal reaction q_0 is again

$$q_0 = k_{12}^{(1)} \phi_2 - f_1^{(1)} = -40$$

Our function to approximate the solution can be computed knowing that,

$$\hat{\phi}^{(1)}(x) = \underbrace{N_1^{(1)} \phi_1^{(1)}}_{=0} + N_2^{(1)} \phi_2^{(1)} = \frac{x - x_1^{(1)}}{l^{(1)}} \phi_2^{(1)} = \frac{x}{2} \cdot 30 = 15x$$

$$\hat{\phi}^{(2)}(x) = N_1^{(2)} \phi_1^{(2)} + N_2^{(2)} \phi_2^{(2)} = \frac{x_2^{(2)} - x}{l^{(2)}} \phi_1^{(2)} + \frac{x - x_1^{(2)}}{l^{(2)}} \phi_2^{(2)} = \left(2 - \frac{x}{2}\right) \cdot (30) + \left(\frac{x}{2} - 1\right) \cdot (40) = 20 + 5x$$

$$\hat{\phi}^{(3)}(x) = \hat{\phi}^{(4)}(x) = 40$$

Thus,

$$\hat{\phi}(x) = \begin{cases} 15x & \text{on} \quad 0 \leq x \leq 2 \\ 20 + 5x & \text{on} \quad 2 < x \leq 4 \\ 40 & \text{on} \quad x > 4 \end{cases}$$

3. Let's find now the analytical solution for this problem. For $0 \leq x \leq 4$, we have that

$$\frac{d^2\phi}{dx^2} = \frac{-Q}{2} \quad \Rightarrow \quad \frac{d\phi}{dx} = \frac{-Q}{2} x + C_1 \quad \Rightarrow \quad \phi(x) = \frac{-Qx^2}{4} + C_1 x + C_2$$

Applying the boundary conditions,

$$\phi(x = 0) = 0 \quad \Rightarrow \quad C_2 = 0 \quad \Rightarrow \quad \phi(x) = \frac{-Qx^2}{4} + C_1 x$$

$$q(x = 4) = 0 \quad \Rightarrow \quad \left.\frac{d\phi}{dx}\right|_{x=4} = 0 \quad \Rightarrow \quad 0 = \frac{-10 \cdot 4}{2} + C_1 \quad \Rightarrow \quad C_1 = 20$$

Therefore,

$$\phi(x) = \frac{-5x^2}{2} + 20x \quad 0 \leq x \leq 4$$

For the second half of the bar, we have that

$$\frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \phi(x) = C_3 x + C_4$$

$$q(x = 8) = 0 \quad \Rightarrow \quad \left.\frac{d\phi}{dx}\right|_{x=8} = 0 \quad \Rightarrow \quad C_3 = 0 \quad \Rightarrow \quad \phi(x) = C_4$$

And since the temperature has to be the same at the midpoint of the bar, we can write

$$\phi = \frac{-5 \cdot 4^2}{2} + 20 \cdot 4 = 40 \quad \Rightarrow \quad \phi(x) = 40 \quad \forall x > 4$$

Finally, let's plot the three functions

$$\underbrace{\phi(x) = \begin{cases} \frac{-5x^2}{2} + 20x & 0 \leq x \leq 4 \\ 40 & x > 4 \end{cases}}_{Analytical\ solution} \quad \underbrace{\hat{\phi}_1(x) = \begin{cases} 10x & 0 \leq x \leq 4 \\ 40 & x > 4 \end{cases}}_{2\ 2-noded\ elements} \quad \underbrace{\hat{\phi}_2(x) = \begin{cases} 15x & 0 \leq x \leq 2 \\ 20 + 5x & 2 < x \leq 4 \\ 40 & x > 4 \end{cases}}_{4\ 2-noded\ elements}$$

plot_hw1-eps-converted-to.pdf

As we can see, the solution at the nodes for both analytical and FEM approximations coincide and it is easy to see how the approximated solution increases its accuracy as we introduce a finer discretization. Something remarkable in this particular problem is that the solution for the second half of the bar coincides in all cases. This is a consequence of the linear approximation chosen that coincides with the exact one.