① Show that $F^n = \{(a_1, a_2, \dots a_n); a_i \in F\}$ is a vector space over F w.r.t addition and scalar multiplication defined component wise.

Soln.:

Given: $F^n = \{(a_1, a_2 \dots a_n), a_i \in F\}$

For all $u, v, w \in V$ and $\alpha, \beta \in F$

Let $u = a_1, a_2, \dots a_n$

$v = b_1, b_2, \dots b_n$

$w = c_1, c_2, \dots c_n$.

① $u+v = (a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n)$

$= (a_1+b_1, a_2+b_2, \dots a_n+b_n)$

$\in V$

Closure property is satisfied

② $u+v = (a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n)$

$= (a_1+b_1, a_2+b_2, \dots a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \dots b_n+a_n)$

$= (b_1, b_2, \dots b_n) + (a_1, a_2, \dots a_n)$

$= v+u$

Commutative property is satisfied.

(3) $u+(v+w) = (a_1, a_2, \ldots a_n) + ((b_1, b_2, \ldots b_n) + (c_1, c_2, \ldots c_n))$

$= (a_1, a_2, \ldots a_n) + (b_1+c_1, b_2+c_2 \ldots b_n+c_n)$

$= a_1+(b_1+c_1), a_2+(b_2+c_2), \ldots a_n+(b_n+c_n)$

$= (a_1+b_1)+c_1, (a_2+b_2)+c_2 \ldots (a_n+b_n)+c_n$

$= (u+v)+w$

Associative property is satisfied.

(4) There is an element $0 \in V$ and $u \in V$ such that

$u+0 = (a_1, a_2, \ldots a_n)+0$

$= a_1, a_2, \ldots a_n$

$= a$

$\therefore$ Identity property is satisfied

(5) There is an element $u \in V$ and $u \in V$ such that $u+(-u)$

$= (a_1, a_2, \ldots a_n) + (-a_1, -a_2, \ldots -a_n)$

$= 0$

$\therefore$ Inverse property is satisfied.

(6) $\alpha u = \alpha (a_1, a_2, \ldots a_n)$

$= \alpha a_1, \alpha a_2, \ldots \alpha a_n$

$\in V$

(7) $\alpha(u+v) = \alpha((a_1, a_2, \ldots a_n) + (b_1, b_2, \ldots b_n))$

$= \alpha(a_1+b_1, a_2+b_2, \ldots a_n+b_n)$

$= \alpha(a_1+b_1), \alpha(a_2+b_2), \ldots \alpha(a_n+b_n)$

$= \alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \ldots \alpha a_n + \alpha b_n$

$= \alpha(a_1, \alpha a_2, \ldots \alpha a_n) + (\alpha b_1, \alpha b_2, \ldots \alpha b_n)$

$= \alpha u + \alpha v$

$\therefore$ Property satisfied.

8) $(\alpha+\beta)(u) = (\alpha+\beta)(a_1, a_2)$

$= \alpha(a_1, a_2) + \beta(a_1, a_2)$

$= (a_1, 0) + \beta(a_1, a_2)$

$(\alpha+\beta)(u) \neq \alpha u + \beta u$

$V = \{(a_1, a_2);\ a_1, a_2 \in R\}$ is not a vector space over F.

4) Prove that the set of all ~~m~~ $m \times n$ matrices over F denoted by $M_{m\times n}(F)$ is a vector space over F w.r.t operations matrix addition and scalar multiplication of matrix.

Solution:

$M_{m\times n}(F)$

For all $u, v, w \in V$ and $\alpha, \beta \in F$

Let $u = [a_{ij}]$

$v = [b_{ij}]$

$w = [c_{ij}]$

1) $u+v \in V$

$= [a_{ij}] + [b_{ij}]$

$\in V$ $\therefore$ closure property

2) $u+v = v+u$

$= [a_{ij}] + [b_{ij}]$

$= [a_{ij}] + [b_{ij}] = [b_{ij}] + [a_{ij}]$

$= v+u$

$\therefore$ commutative property

3) $u+(v+w) = (u+v)+w$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= (u+v)+w$

$\therefore$ Associative property.

4) There is an element $0 \in V$ and $u \in V$ such that

$u+0 = u$

$= [a_{ij}] + 0$

$= [a_{ij}] = u$

$\therefore$ Identity property.

) there is an element $u \in V$ and $-u \in V$ such that $u+(-u)=0$

$= [a_{ij}] + [-a_{ij}]$

$= 0$

$\therefore$ Inverse property.

6) $\alpha u \in F$

$\alpha [a_{ij}] \in F$

7) $\alpha(u+v) = \alpha u + \alpha v$

$= \alpha([a_{ij}] + [b_{ij}])$

$= \alpha [a_{ij}] + \alpha [b_{ij}]$

$= \alpha u + \alpha v$

8) $(\alpha+\beta) u = \alpha u + \beta u$

$= (\alpha+\beta) [a_{ij}]$

$= \alpha [a_{ij}] + \beta [a_{ij}]$

$= \alpha u + \beta u$

$\alpha(\beta u) = \alpha\beta(u)$

9) $\alpha(\beta u) = \alpha(\beta [a_{ij}])$

$= (\alpha\beta)[a_{ij}]$

$= \alpha\beta(u)$

10) $1 . u = u$

$1 . [a_{ij}] = [a_{ij}]$

$= u.$

$\therefore$ Hence $M_{m \times n}(F)$ is vector space over F wrt operations matrix addition and scalar multiplication of matrix.

State and prove ~~translation~~ cancellation law for addition.

STATEMENT:

In a vector space V over F is $u+w=v+w$ then

$u+v$

PROOF:

There is an element $0 \in V$ and $u \in V$

such that $0+u=u$ (Identity property)

there is an element $w \in V$ and $-w \in V$

such that $w+(-w)=0$

Given $u+w = v+w$

$0+u = u$ $\quad [0 = w+(-w)]$

$(w+(-w))+u = u$

Associative property

$(w+v)+(-w) = u$ $\quad [\because$ given $u+w = v+w]$

$(v+w)+(-w) = u$

$v+(w+(-w)) = u$

$\boxed{v=u}$

3) Let $S = \{0,1\}$ and $F(S,R)$ be the set of all functions from $S \to R$. If $f(t) = 2t+1$, $g(t) = 1+4t-2t^2$, $h(t) = 5^t+1$. Show that (i) $f=g$ ii) $f+g=h$

PROOF

Given $S = \{0,1\}$, $f(t) = 2t+1$, $g(t) = 1+4t-2t^2$ and $h(t) = 5^t+1$

For all $t \in S$

To prove : $f=g$

$f(t) = 2t+1$

$f(0) = 2\times0+1$
$= \boxed{1}$

$f(1) = 2\times1+1$
$= \boxed{3}$

$g(t) = 1+4t-2t^2$

$g(0) = 1+4\times0-2\times(0)^2$
$= \boxed{1}$

$g(1) = 1+4-2$
$= \boxed{3}$

$\boxed{f=g}$

To prove : $f+g=h$

$(f+g)(t) = f(t)+g(t)$
$= 2t+1+1+4t-2t^2$
$= 6t+2-2t^2$

$(f+g)(0) = \boxed{2}$

$(f+g)(1) = 6+2-2 = \boxed{6}$

$h(t) = 5^t+1$

$h(0) = 5^0+1 = 1+1$
$= \boxed{2}$

$(f+g)(1) = 5^1+1 = 5+1$
$= \boxed{6}$

$\therefore \boxed{f+g=h}$

Q) Using Generating function solve the recurrence relation $a_{n+2} - a_{n+1} - 6a_n = 0$ with $a_0 = 2$, $a_1 = 1$.

SOLN: $a_{n+2} - a_{n+1} - 6a_n = 0$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots$$

gn $a_{n+2} - a_{n+1} - 6a_n = 0$

Xply by x^n on both sides

$$a_{n+2} x^n - a_{n+1} x^n - 6 a_n x^n = 0$$

Taking $\sum_{n=0}^{\infty}$ on both sides.

$$\sum_{n=0}^{\infty} a_{n+2} x^n - \sum_{n=0}^{\infty} a_{n+1} x^n - 6 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$= \frac{1}{x^2} \sum_{n=0}^{\infty} a_{n+2} x^{n+2} - \frac{1}{x} \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 6 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$= \frac{1}{x^2}\left[G(x) - a_0 x^0 - a_1 x^1\right] - \frac{1}{x}\left[G(x) - a_0 x^0\right]$$

$\boxed{\text{Using I.C}}$ $- 6\, G(x) = 0.$

$$\frac{1}{x^2}\left[G(x) - 2 - x\right] - \frac{1}{x}\left[G(x) - 2\right] - 6\,G(x) = 0.$$

$$= G(x)\left[\frac{1}{x^2} - \frac{1}{x} - 6\right] - \frac{2}{x^2} - \frac{x}{x^2} + \frac{2}{x} = 0.$$

$$= G(x)\left[\frac{1 - x - 6x^2}{x^2}\right] = \frac{2}{x^2} + \frac{1}{x} - \frac{2}{x}.$$

$$= G(x)\left[\frac{1 - x - 6x^2}{x^2}\right] = \frac{2}{x^2} - \frac{1}{x}$$

$$= G(x)\left[\frac{1 - x - 6x^2}{\cancel{x^2}}\right] = \frac{2 - x}{\cancel{x^2}}$$

$$G(x) = \frac{2-x}{1-x-6x^2}$$

$-6x^2 - x + 1$

$$G(x) = \frac{2-x}{(1+2x)(1-3x)}$$

Using partial fraction,

$$G(x) = \frac{2-x}{(1+2x)(1-3x)} = \frac{A}{(1+2x)} + \frac{B}{(1-3x)}$$

$$= 2-x = A(1-3x) + B(1+2x)$$

Put $x = 1/3$

$$\frac{2 \times 3}{1 \times 3} - \frac{1}{3} = A\left(1 - \cancel{3}\frac{1}{\cancel{3}}\right) + B\left(\frac{1 \times 3}{1 \times 3} + 2\left(\frac{1}{3}\right)\right)$$

$$= \frac{6-1}{3} = B\left(\frac{3+2}{3}\right)$$

$$= \frac{5}{3} = B\left(\frac{5}{3}\right)$$

$$\boxed{B = 1}$$

Put $x = -1/2$

$$2-\left(\frac{-1}{2}\right) = A\left(1 - 3\left(\frac{-1}{2}\right)\right) + B\left(1 + \cancel{2}\left(\frac{-1}{\cancel{2}}\right)\right)$$

$$= \frac{2 \times 2}{1 \times 2} + \frac{1}{2} = A\left(\frac{1 \times 2}{1 \times 2} + \frac{3}{2}\right) + 0$$

$$= \frac{5}{2} = A\left(\frac{5}{2}\right)$$

$$\boxed{A = 1}$$

$$\therefore G(x) = \frac{1}{1+2x} + \frac{1}{1-3x} = (1+2x)^{-1} + (1-3x)^{-1}$$

$$G(x) = \frac{1}{2x}\cancel{(1+2x)^{-1}} + \cancel{(1-3x)^{-1}}$$

$$\frac{1}{\cancel{(2x+1)}} + \frac{1}{\cancel{-(3x-1)}}$$

$$= \cancel{(2x+1)^{-1}} - \cancel{(3x-1)^{-1}}$$

$$= (1+2x)^{-1} + (1-3x)^{-1}$$

$\sum_{n=0} a_n x^n = G(x) = 1 + (2x)^1 + (2x)^2 - (2x)^3 + \dots + (2x)^n$ $[(x+a)^{-1} + (x-a)^{-1}] =$

$\dots + (a a a a)$

$\dots + (1 + (3x) + (3x^2) + (3x)^3 + \dots + (3x)^n + \dots$

coeff of x^n

$\boxed{a_n = 2^n + 3^n}$

$\boxed{a_n = (-1)\, 2^n + 3^n}$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

x^3

$(1-x)^2 = 1 - 2x + 2x^2 - 4x^3 +$

$5x^2 -$

(2) $(1-x)^{-2} = 1 + 2x + 3x^2 +$

$4x^2 + \dots$

in Generating function

(10) Solve the recurrence relation $a_{n+1} - a_n = 3n^2 - n$ for $n \geq 0$ and $a_0 = 3$

Given :

$$a_{n+1} - a_n = 3n^2 - n \longrightarrow (1)$$

put $a_n = r^n$

$$r^{n+1} - r^n = 0$$

$$r^n \cdot r - r^n = 0$$

$$r^n (r-1) = 0$$

$$\boxed{r = 1}$$

$$\therefore CF = c_1 m_1^n$$

$$= c_1 (1)^n$$

$$\boxed{CF = c_1}$$

To find P.S $\Rightarrow$ particular solution

$$(3n^2 - n)(1)^n$$

$r = 1$ is a simple root

Hence the suitable trial solution for RHS is

$$a_n = a^n \, n [An^2 + Bn + C]$$

$$a_n = 1^n \, n [An^2 + Bn + C]$$

$$a_n = n [An^2 + Bn + C] \longrightarrow (2)$$

$$a_{n+1} = n+1 [A(n+1)^2 + B(n+1) + C]$$

$$= n+1 [A(n^2+2n+1) + Bn + B + C]$$

$$= (n+1)[An^2 + 2An + A + Bn + B + C]$$

sub a_n & a_{n+1} in ①

$$(n+1)[An^2 + 2An + A + Bn + B + C] - n[An^2 + Bn + C] - 3n^2 - n$$

$$[\cancel{An^3} + 2An^2 + An + \cancel{Bn^2} + Bn + \cancel{Cn} + An^2 + 2An + A + Bn + B + C]$$

$$- \cancel{An^3} - \cancel{Bn^2} - \cancel{Cn} = 3n^2 - n$$

$$= 3An^2 + 3An + 2Bn + A + B + C = 3n^2 - n$$

$$A[3n^2 + 3n + 1] + B[2n+1] + C = 3n^2 - n$$

$$3An^2 + n(3A + 2B) + (A + B + C) = 3n^2 - n$$

Equating the coeff of n^2, n and const

$3A = 3$ | $3A + 2B = -1$ | $A + B + C = 0$

$\boxed{A = 1}$ | $3 + 2B = -1$ | $1 - 2 + C = 0$

$2B = -4$ | $-1 + C = 0$

$\boxed{B = -2}$ | $\boxed{C = 1}$

sub A, B, C in equ ②

$$a_n = n[n^2 - 2n + 1] \longrightarrow ③$$

$$a_n = CF + P.S$$

$$\boxed{a_n = c_1 + n[n^2 - 2n + 1]}$$

using initial condition $a_0 = 3$ $n = 0$

$$a_0 = c_1 + 0[0]$$

$$\boxed{3 = c_1}$$

$$\therefore a_n = 3 + n[n^2 - 2n + 1]$$

③ Using Generating function, solve $y_{n+2} - 5y_{n+1} + 6y_n = 0$, $n \geq 2$ with $y_0 = 1$, $y_1 = 1$

SOLN:

Given:

$y_{n+2} - 5y_{n+1} + 6y_n = 0$ can be written as

$G(x) = \sum_{n=0}^{\infty} a_n x^n$ $\quad a_{n+2} - 5a_{n+1} + 6a_n = 0.$

$a_{n+2} x^n - 5a_{n+1} x^n + 6a_n x^n = 0$

Taking $\sum_{n=0}^{\infty}$ on both sides

$\Rightarrow \sum_{n=0}^{\infty} a_{n+2} x^n - 5\sum_{n=0}^{\infty} a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \frac{1}{x^2} \sum_{n=0}^{\infty} a_{n+2} x^{n+2} - \frac{1}{x} \sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0.$

$\Rightarrow \frac{1}{x^2} [G(x) - a_0 x^0 - a_1 x^1] - \frac{1}{x} [G(x) - a_0 x^0] + 6 [G(x)] = 0.$

Using IC —

$\frac{1}{x^2} [G(x) - 1 - x] - \frac{5}{x} [G(x) - 1] + 6 [G(x)] = 0$

$\frac{1}{x^2} G(x) - \frac{1}{x^2} - \frac{x}{x^2} - \frac{5}{x} G(x) + \frac{5}{x} + 6 G(x))$

$G(x) \left[\frac{1}{x^2} - \frac{5}{x} + 6 \right] - \frac{1}{x^2} + \frac{5}{x} - \frac{x}{x^2} = 0$

$= G(x) \left[\frac{1}{x^2} - \frac{5}{x} + 6 \right] - \left(\frac{1}{x^2} - \frac{5}{x} + \frac{1}{x} \right) = 0.$

$= G(x) \left[\frac{1 - 5x + 6x^2}{x^2} \right] = \frac{1}{x^2} - \frac{5}{x} + \frac{1}{x}$

$$G(x)\left[\frac{1-5x+6x^2}{x^2}\right] = \frac{1}{x^2} - \frac{4}{x}$$

$$G(x)\left[\frac{1-5x+6x^2}{x^2}\right] = \frac{1-4x}{x^2}$$

$$\therefore G(x) = \frac{1-4x}{1-5x+6x^2} = \frac{1-4x}{(1-2x)(1-3x)}$$

Using partial function $=$ $G(x) = \dfrac{1-4x}{(1-2x)(1-3x)} = \dfrac{A}{1-2x} + \dfrac{B}{1-3x}$

$= 4x$

$$G(x) = \frac{2}{1-2x} - \frac{1}{1-3x}$$

$= 1-4x = A(1-3x) + B(1-2x)$

$= x = +1/2 \quad A = 2$

$x = 1/3 \quad B = -1$

$$= 2(1-2x)^{-1} - (1-3x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2\left[1 + (2x) + (2x)^2 + \dots + (2x)^n + \dots\right] - \left[1 + (3x) + (3x)^2 + \dots + (3x)^n + \dots\right]$$

coeff of x^n

$$\boxed{a^n = 2(2^n) - 3^n}$$

9) Using generating function solve

$a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$, $a_0 = 2$, $a_1 = 8$.

Solution:

Given: $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$

$= a_n - 4a_{n-1} + 4a_{n-2} - 4^n = 0$

$G(x) = \sum_{n=0}^{\infty} a_n x^n$

$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$

Multiply x^n on both sides.

$a_n x^n - 4a_{n-1} x^n + 4a_{n-2} x^n - 4^n x^n = 0$.

Taking $\sum_{n=2}^{\infty}$.

$\sum_{2}^{\infty} a_n x^n - 4 \sum_{2}^{\infty} a_{n-1} x^n + 4 \sum_{2}^{\infty} a_{n-2} x^n - \sum_{2}^{\infty} 4^n x^n = 0$.

$\sum_{2}^{\infty} a_n x^n - 4x \sum_{2}^{\infty} a_{n-1} x^{n-1} + 4x^2 \sum_{2}^{\infty} a_{n-2} x^{n-2} - \sum_{2}^{\infty} \underbrace{4^n x^n}_{(4x)^n} = 0$.

$[G(x) - a_0 x^0 - a_1 x^1] - 4x[G(x) - a_0 x^0] + 4x^2 [G(x)] - [(4x)^2 + (4x)^3 + (4x)^4 + \ldots] = 0$.

Using initial condition,

$[G(x) - 2 - 8x] - 4x[G(x) - 2] + 4x^2 [G(x)] - (4x)^2 [1 + 4x + 4x^2 + \ldots] = 0$.

$1 + x + x^2 + x^3 + \ldots + x^n = (1-x)^{-1}$ Binomial form.

$\Rightarrow G(x)[1 - 4x + 4x^2] - 2 - \cancel{8x} + \cancel{8x} - (4x)^2 [(1-4x)^{-1}] = 0$

$\Rightarrow G(x)[1 - 4x + 4x^2] = 2 + \dfrac{(4x)^2}{(1-4x)}$

$\Rightarrow G(x)[1 - 4x + 4x^2] = \dfrac{2(1-4x) + 16x^2}{(1-4x)}$

$\Rightarrow G(x)[1 - 4x + 2x^2] = \dfrac{2 - 8x + 16x^2}{(1-4x)(1-4x+4x^2)} = \dfrac{16x^2 - 8x + 2}{(1-4x)(4x^2 - 4x + 1)}$

$$\Rightarrow \frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} \qquad \left[\frac{Ax+B}{x^n} = \frac{A}{x} + \frac{B}{x^m} + \frac{C}{x^n} \right]$$

$16x^2 - 8x + 2$
$= 8x^2 - 4x + 1$

$$= \frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} = \frac{A}{(1-4x)} + \frac{B}{(1-2x)} + \frac{C}{(1-2x)^2}$$

$$16x^2 - 8x + 2 \Rightarrow A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)$$

$1-4x$
$1=4x$
$x = \frac{1}{4}$

~~Put A $x = 1/4$~~ $\boxed{\text{Put } x = 1/4}$

$$16\left(\tfrac{1}{4}\right)^2 - 8\left(\tfrac{1}{4}\right) + 2 = A\left(1 - 2\left(\tfrac{1}{4}\right)\right)^2 + B\left(1 - 4\left(\tfrac{1}{4}\right)\right)\left(1 - 2\left(\tfrac{1}{4}\right)\right) + C\left(1 - 4\left(\tfrac{1}{4}\right)\right)$$

$$= 16\left(\frac{1}{16}\right) = A\left(\frac{2-1}{2}\right)^2 + B0$$

$$1 = A\left(\frac{1}{2}\right)^2$$

$$1 = A\left(\frac{1}{4}\right)$$

$$\boxed{A = 4}$$

Put $\boxed{x = 1/2}$

~~$2 = A(1)^2 + B + C$~~

~~$\Rightarrow 16(1/2)^2 - 8(1/2) + 2 = A(1 - 2(1/2))^2 + B(1 - 4(1/2))(1 - 2(1/2)) + C(1 - 4(1/2))$~~

$$\Rightarrow 4 - 4 + 2 = 0 + 0 - C$$

$$2 = -C$$

$$-C = 2$$

$$\boxed{C = -16} \quad -2$$

$x = 1/2$

$16(1/2)^2 - 8(1/2) + 2 =$
$A(1 - 2(1/2))^2 + B(1 - 4(1/2))(1 - 2(1/2)) + C(1 - 4(1/2))$

$= 16(1/4) - 8(1/2) + 2 =$
$A(0) + B(-1)(0) + C(-1)$

$= 4 - 4 + 2 = -C$

$= \boxed{C = -2}$

Put $\boxed{x = 0}$

$$\Rightarrow 2 = A(1)^2 + B + C$$

$$\Rightarrow 2 = 4 + B - 2$$

$$2 = B + 2$$

$$\boxed{B = 0}$$

$A = 4, \quad B = 0, \quad C = -2$ $\therefore$

$$G(x) = \frac{4}{1-4x} - \frac{2}{(1-2x)^2} \quad \left[(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \right]$$

$$= 4(1-4x)^{-1} - 2(1-2x)^{-2}$$

$$= 4\left[1 + (4x) + (4x)^2 + \ldots + 4x^n\right] - 2\left[1 + 2(2x)^1 + 3(2x)^2 + 4(2x)^3 \ldots + (n+1)(2x)^n\right]$$

$$\therefore a_n = 4^1 [4^n] - 2\,[(n+1)\, 2^n]$$

$$a_n = 4^{n+1} - 2^{n+1}(n+1)$$

12) Find the solution of recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, where $a_0 = 2$, $a_1 = 5$, $a_3 = 15$

SOLUTION :

Given :

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

put $a_n = r^n$

$$r^n - 6r^{n-1} + 11r^{n-2} - 6r^{n-3} = 0$$

$$r^n - \frac{6r^n}{r} + \frac{11r^n}{r^2} - \frac{6r^n}{r^3} = 0$$

$$r^n\left(1 - \frac{6}{r} + \frac{11}{r^2} - \frac{6}{r^3}\right) = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

$\cancel{r^3 +}$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$r^2 - 5r + 6 = 0$

$r = 1, 2, 3$

$CF = c_1 m_1^n + c_2 m_2^n + c_3 (m_3)^n$

$= c_1 (1)^n + c_2 (2)^n + c_3 (3)^n$

$a_n = c_1 1^n + c_2 2^n + c_3 3^n$

$n=0$

$a_0 = c_1 (1)^0 + c_2 (2)^0 + c_3 (3)^0$

$2 = c_1 + c_2 + c_3 \longrightarrow$ 1

$n=1$

$a_1 = c_1 (1)^1 + c_2 (2)^1 + c_3 (3)^1$

$5 = c_1 + 2c_2 + 3c_3 \longrightarrow$ 2

$n=2$

$a_2 = c_1 (1)^2 + c_2 (2)^2 + c_3 (3)^2$

$15 = c_1 + 4c_2 + 9c_3 \longrightarrow$ 3

Solve 1 & 2

$\cancel{c_1} + c_2 + c_3 = 2$

$\cancel{c_1} + 2c_2 + 3c_3 = 5$

$-c_2 - 2c_3 = -3$

$c_2 + 2c_3 = 3 \longrightarrow$ 4

solve 2 & 3

$\cancel{c_1} + 2c_2 + 3c_3 = 5$

$\cancel{c_1} + 4c_2 + 9c_3 = 15$

(-)

$-2c_2 - 6c_3 = -10$

$2c_2 + 6c_3 = 10 \longrightarrow$ 5

Solve 4 and 5

$2c_2 + 4c_3 = 6$

$2c_2 + 6c_3 = 10$

$-2c_3 = -4$

$\boxed{c_3 = 2}$

sub $c_3 = 2$ in equ 5

$2c_2 + 6(2) = 10$

$2c_2 = 10 - 12$

$c_2 = -2/2$

$\boxed{c_2 = -1}$

Sub c_2 and c_3 in eqn ①

$c_1 - 1 + 2 = 2$

$c_1 + 1 = 2$

$c_1 = 2 - 1$

$\boxed{c_1 = 1}$

$\boxed{\therefore a_n = 1^n - 2^n + 2(3)^n}$

(11) Solve $a_n - 4a_{n-1} + 4a_{n-2} = 3n + 2^n \longrightarrow (1)$ $a_0 = 1, a_1 = 1$

Given :

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

put $a_n = r^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$r^n - 4r^n \cdot r^{-1} + 4r^n \cdot r^{-2} = 0$$

$$r^n \left(1 - \frac{4}{r} + \frac{4}{r^2}\right) = 0$$

$$r^n \left(\frac{r^2 - 4r + 4}{r^2}\right) = 0$$

$$r^2 - 4r + 4 = 0$$

$a = 1$ $\quad b = -4$ $\quad c = 4$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 16}}{2}$$

$$= 2, 2$$

$$CF = (c_1 + c_2 n)\, m^n$$

$$\boxed{CF = (c_1 + c_2 n)\, 2^n}$$

To find PS

$$PS_1 = a_n = An + B$$

$a_{n-1} = A(n-1)\cancel{B}+B \Rightarrow An-A+B$

$a_{n-2} = A(n-2)+B \Rightarrow An-2n+B$

Sub a_n, a_{n-1}, a_{n-2} in eqn (1)

$(An+B)-4(An-A+B)+4(An-2A+B) = 3n$

$An+B-4\cancel{An}+4A-\cancel{4B}+4\cancel{An}-8A+\cancel{4B} = 3n$

$An-4A+B = 3n$

$An-4(A)+(B) = 3n$

Equating the coefficient of n and const

$\boxed{A=3}$

$-4(A)+B=0$

$-4(3)+B=0$

$\boxed{B=12}$

P.S1 $= \boxed{a_n = 3n+12} \longrightarrow (2)$

P.S2 $\Rightarrow a_n = An^2 a^n \Rightarrow An^2(2)^n$

$a_{n-1} = A(n-1)^2 a^{n-1}$

$= A(n^2-2n+1)\,(2)^{n-1}$

$= (An^2-2An+A)\,2^{n-1}$

$a_{n-2} = A(n-2)^2\,2^{n-2}$

$= A(n^2-4An+4)\cdot 2^{n-2}$

$a_{n-2} = (An^2-4An+4A)\cdot 2^{n-2}$

Sub a_n, a_{n-1}, a_{n-2} in equn (1)

$$An^2 2^n - 4\left[(An^2-2An+A)2^{n-1}\right]+4\left[(An^2-4An+4A\right]2^{n-2} = 2^n$$

$$An^2 2^n - \frac{\cancel{4}^{2}(An^2 2^n - 2An2^n + A2^n)}{\cancel{2}} + \frac{\cancel{4}(An^2 2^n - 4An2^n + 4A2^n)}{\cancel{4}} = 2^n$$

$$= \cancel{An^2 2^n} - 2\cancel{An^2 2^n} + \cancel{4An2^n} - 2A2^n + \cancel{An^2 2^n} - \cancel{4An2^n} + 4A2^n = 2^n$$

$$= 2A2^n = 2^n$$

equating the coeff

$$2A = 1$$

$$\boxed{A = \frac{1}{2}}$$

sub A in a_n

$$\boxed{P.S_2 = \frac{1}{2}n^2 2^n}$$

$$a_n = C.F + P.S_1 + P.S_2$$

$$a_n = (C_1 + C_2 n)2^n + (3n+12) + \frac{1}{2}n^2 2^n$$

$n = 0$

$$a_0 = C_1 + C_2(0)2^0 + 3(0)+12 + \frac{1}{2}(0)^2 2^0$$

$$1 = C_1 + 12 \Rightarrow \boxed{C_1 = -11}$$

$n=1$

$$a_1 = (C_1 + C_2(1))2^1 + 3(1)+12 + \frac{1}{2}1^2 \times 2^1$$

$$1 = (-11 + C_2)\,2 + 15$$

$$1 = -22 + 2C_2 + 15$$

$$1 = -7 + 2C_2$$

$$\frac{8}{2} = C_2$$

$$\boxed{C_2 = 4}$$

$$a_n = (-11 + 4n)\,2^n + 3n + 12 + \frac{1}{2} n^2 2^n$$