Topology and Analysis Notes

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Contents

Part I

Topology

Administrative Matters

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1 Metric Spaces

Definition 1.1 (Metric Space). A metric on a set X is a function $d\colon X^2 \to \mathbb{R}_{\geq 0}$ such that

1. $d(x,y) = 0$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,z) \leq d(x,y) + d(y,z)$.

(X,d) is then called a metric space.

If condition 1 is omitted and replaced only with the condition that $d(x,x) = 0$, then d is called a semimetric (or quasimetric). If distances are allowed to be infinite, then d is called an extended metric.

If (X,d) is a semimetric space, then if we define a relation $\sim$ such that $x \sim y$ iff $d(x,y) = 0$, it is an equivalence relation. In this case, d "drops" to a metric on the equivalence classes $X/\sim$. To show this, we just need to prove that $\tilde{d}([x],[y]) = d(x,y)$ is a well-defined function, which utilizes the triangle inequality.

Example 1.1. If G is a connected weighted graph, then we can define a metric on G by defining the distance between two vertices to be the length of the minimal path connecting them.

If G is unconnected, by assigning the distance of infinity between points in separate components, we define a extended metric.

Definition 1.2 (Norm). If V is a vector space then $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ is a norm on V if

1. $\|v\| = 0$ iff $v = 0$
2. $\|\lambda v\| = |\lambda|\,\|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$

A norm on V clearly induces a metric on V by defining $d(v,w) = \|v - w\|$.

Example 1.2 (Euclidean metric). The real line $\mathbb{R}$ and the euclidean spaces $\mathbb{R}^n$ with the euclidean metric $d(x,y) = \|x - y\|_2$.

Example 1.3 (p-norms). The p-norms $\left\|(x_1,\ldots,x_n)\right\|_p = \left(\sum x_i^p\right)^{1/p}$ induce a metric on $\mathbb{R}^n$.

Definition 1.3 (Restricted metric). If (X,d) is a metric space and Y is a subset of X, then we can restrict d to Y and $d|_Y$ is a metric on Y.

It is common to consider the restriction of the Euclidean metric onto subsets of Euclidean space.

Example 1.4 (L^p norms). If $V = \mathscr{C}^0([0,1])$, then examples of norms on V include

1. $\|f\|_\infty = \sup |f(t)|$.
2. $\|f\|_1 = \int_0^1 |f(t)|\,\mathrm{d}t$
3. $\|f\|_2 = \left(\int_0^1 |f(t)|^2\,\mathrm{d}t\right)^{1/2}$
4. $\|f\|_p = \left(\int_0^1 |f(t)|^p\,\mathrm{d}t\right)^{1/p}$ for $1 \le p < \infty$.[1]

This generalizes to bounded closed regions of $\mathbb{R}^n$, where all continuous functions are integrable (due to compactness and the extreme value theorem). The proof that these are actually norms that satisfy the triangle inequality will come later (see the Minkowski and Hölder inequalities).

1.1 Topology of Metric Spaces

Categories

Categories contain objects and morphisms, that satisfy the following properties:

1. There is a morphism from any object A to itself, called the identity 1_A.
2. Morphisms can be composed.

Metric spaces can form a category under a few choices of collections of morphisms:

- Isometries: $i\colon (X,d_X) \to (Y,d_Y)$ such that $d_X(x,y) = d_Y(i(x),i(y))$ for all x,y.

  Surjective isometries form a subcategory of the above category, ,where every map is an isomorphism.

- Lipschitz maps: $f\colon (X,d_X) \to (Y,d_Y)$, where there exists a constant C (which may depend on f), such that $d_Y(f(x),f(y)) \le C d_X(x,y)$.

  We can also form the subcategory of bi-Lipschitz isomorphisms.

- Uniformly continuous maps: $f\colon (X,d_Y) \to (Y,d_Y)$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x,y) \le \delta$, then $d_Y(f(x),f(y)) < \varepsilon$.

- Continuous maps: $f\colon (X,d_Y) \to (Y,d_Y)$ such that for all $x \in X$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x,y) \le \delta$, then $d_Y(f(x),f(y)) < \varepsilon$.

These conditions get weaker as we go down, and continuity can be generalized to non-metric spaces which gives rise to general topology. The study of uniformly continuous maps is much less common, but gives rise to uniformities.

Definition 1.4 (Convergence). If $\{x_n\}$ is a sequence in a metric space (X,d), then $x_n \to x$ if $\forall \varepsilon > 0$, $\exists N$ such that $n > N$ implies that $d(x_n,x)$.

Definition 1.5 (Cauchy sequence). If $\{x_n\}$ is a sequence in a metric space (X,d), then $x_n \to x$ if $\forall \varepsilon > 0$, $\exists N$ such that $n,m > N$ implies that $d(x_n,x_m)$.

[1] The ∞-norm can be viewed as the limit of the p-norms as $p \to \infty$.

Definition 1.6 (Completeness). (X,d) is complete if every cauchy sequence converges to a point.

For any metric space that isn't complete, we want to form a "completion" of it.

Definition 1.7 (Density). If $S \subseteq X$, S is dense if for all x in X and all $\varepsilon < 0$ there exists $s \in S$ where $d(s,x) < \varepsilon$.

Definition 1.8 (Completion). A completion of a metric space (X,d) is a metric space $(\overline{X},\overline{d})$ with an inclusion isometry $i\colon X \to \overline{X}$ such that the image of i is dense in $\overline{X}$. In addition, $\overline{X}$ is a complete metric space.

One way to define a completion for any metric space is by defining a semimetric $\tilde{d}$ on the set of Cauchy sequences $\mathrm{cs}(X,d)$, where $\tilde{d}(\{x_n\},\{y_n\}) = \lim_{n\to\infty} d(x_n,y_n)$, then by dropping that to a metric on the equivalence classes induced by the semimetric.

We can just define $f\colon X \to \overline{X}$ as $f(x) = [\{x,x,x,\ldots\}]$. To show that $\overline{X}$ is complete, we consider Cauchy sequences of Cauchy sequences, approximate their elements by elements in X, and show that the sequence converges to the sequence generated by the approximation.

Example 1.5. If $f \in V = \mathscr{C}^0([0,1])$, the norm $\|f\|_\infty = \sup(|f(x)|)$ induces a complete metric space (due to the limit of uniformly continous functions converging to a continuous function).

However, in the L^1 norm $\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x$, V is not complete (you can construct a sequence of functions that converges to a step function).

Now to answer the question "What is the completion of V under the L^p norms", we must look to measure theory (2nd half of this course).

If (X,d_X) and (Y,d_Y) are metric spaces, if $f\colon X \to Y$, recall that if $x \in X$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x,y) \le \delta$, then $d_Y(f(x),f(y)) < \varepsilon$, then f is continuous.

Definition 1.9 (Open ball). The open ball of radius r, $B_r(x) = \{y \in X \mid d_X(x,y) < r\}$.

This allows us to refine our definition of continuity:

Definition 1.10 (Continuity). If $f\colon X \to Y$, then f is continuous if for all x, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Definition 1.11 (Open set). The set $A \subseteq X$ is open if for all $a \in A$, there exists an open ball around a that is contained within a.

Proposition 1.1. Open balls are open.

Proof. If $a \in B_r(x)$, then $B_{r-d_X(a,x)}(a)$ is contained within $B_r(x)$ due to the triangle inequality. □

Proposition 1.2. If f is continuous, then if U is open in Y such that $x \in X$, then there is an open ball around x that is contained within the preimage of U.

A direct corollary of this proposition is

Corollary 1.3. $f\colon X \to Y$ is continuous is equivalent to the following condition: If U is open in Y, then $f^{-1}(U)$ is open in X.

Now, we consider properties of the set $\mathscr{T}$ consisting of open sets in X:

Theorem 1.4.

1. $\varnothing, X \in \mathscr{T}$.
2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.
3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

Proof. Trivial. □

These properties give rise to the definition of a topology.

Definition 1.12 (Topology)**.** A topology on a set X is a collection $\mathscr{T} \subseteq 2^X$ such that

1. $\varnothing, X \in \mathscr{T}$.
2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.
3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

The sets in $\mathscr{T}$ are called open sets.

2 Topological Spaces

We recall the definition of a topology:

Definition 2.1 (Topology). A topology on a set X is a collection $\mathscr{T} \subseteq 2^X$ such that

1. $\varnothing, X \in \mathscr{T}$.
2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.
3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

The sets in $\mathscr{T}$ are called open sets.

We then extend the definition of continuity to general topologies.

Definition 2.2 (Continuity). If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces, then f is continuous if for all $U \in \mathscr{T}_Y$, $f^{-1}(U) \in \mathscr{T}_X$.

Topologies on a set X form a poset under the subset relation.

Definition 2.3 (Closed sets). The closed sets in $(X, \mathscr{T})$ are precisely the complement of open sets in X.

Proposition 2.1 (Properties of closed sets). 1. $\varnothing, X$ are closed.

2. Arbitrary intersections of closed sets are also closed.
3. Finite unions of closed sets are closed.

Definition 2.4 (Closure). The closure $\overline{A}$ of an arbitrary set A is the smallest closed set containing it.

Definition 2.5 (Density). Given $A \subseteq B$, we say A is dense in B if $B \subseteq \overline{A}$. Equivalently, $\overline{A} = \overline{B}$.

2.1 Bases and Subbases

Proposition 2.2. If X is a set, then the arbitrary intersection of topologies is a topology on X.

Proof. These details are routinely verified. □

Corollary 2.3. Given any collection $\mathscr{S}$ of subsets of X, there is a smallest topology containing $\mathscr{S}$.

Proof. This topology is the intersection of all the topologies that contain $\mathscr{S}$. We say that $\mathscr{S}$ generates this topology. □

Definition 2.6 (Subbase). Let $\mathscr{T}$ be a topology on X, and let $\mathscr{S} \subseteq 2^X$, with $X = \bigcup \mathscr{S}$. We say that $\mathscr{S}$ is a subbase of $\mathscr{T}$ if $\mathscr{T}$ is the topology generated by $\mathscr{S}$.

Subbases can be useful for studying topologies in the same way that a basis is useful for studying vector spaces. For example, see Proposition 2.7.

We can explicitly characterize the topology generated by a set $\mathscr{S}$. We first state a lemma:

Lemma 2.4. If $\mathscr{I} \subseteq 2^X$ be closed under finite intersections and $\bigcup \mathscr{I} = X$, then the topology generated by $\mathscr{I}$ consists of the arbitrary unions of sets in $\mathscr{I}$ (with union over 0 sets being $\varnothing$).

Proof. The only part that is not immediately obvious is the closure under finite intersections. If $\mathscr{O}_1 \bigcup_A U_\alpha$ and $\mathscr{O}_2 = \bigcup_B V_\beta$, then

$$\mathscr{O}_1 \cap \mathscr{O}_2 = \bigcup_{A,B} U_\alpha \cap V_\beta.$$

□

Proposition 2.5. If $\mathscr{S} \subseteq 2^X$ whose union is X, then the topology $\mathscr{T}$ generated by $\mathscr{S}$ consist of the arbitrary unions of $\mathscr{I}^{\mathscr{S}}$ which are defined as the set of finite intersections of sets in $\mathscr{S}$.

Definition 2.7 (Base). A collection $\mathscr{B} \subseteq 2^X$ is called a base for a topology $\mathscr{T}$ on X if $\mathscr{T}$ consists of the arbitrary unions of elements in $\mathscr{B}$.

Example 2.1.

1. The collection of finite intersections $\mathscr{I}^{\mathscr{S}}$ of a subbase $\mathscr{S}$.
2. The set of open balls in a metric space.
3. The set of intervals of the form $(-\infty, a)$ and (b, ∞) in $\mathbb{R}$ under the standard topology.

Notice how the second and third examples are not closed under finite intersections.

Proposition 2.6. For any base $\mathscr{B}$, any finite intersection of sets in $\mathscr{B}$ must be a union of elements in $\mathscr{B}$.

Proposition 2.7. If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces, and let $f\colon X \to Y$. If $\mathscr{S}_Y$ is a subbase of $\mathscr{T}_Y$, then f is continuous iff for all $U \in \mathscr{S}_Y$, we have $f^{-1}(U)$ is open.

Proof. First show that the preimages of sets in $\mathscr{I}_{\mathscr{S}_Y}$ are open, and then show that the preimages of all sets in $\mathscr{T}_Y$ are open. □

2.2 New Topologies from Old

Definition 2.8 (Initial Topologies). If X is a set, $(Y_\alpha, \mathscr{T}_\alpha)$ are topological spaces, and for each α, $f_\alpha\colon X \to Y_\alpha$. Then, the initial topology is the smallest topology on X that makes all f_α continuous.

This initial topology is just the topology generated from the preimages of open sets in Y_α under f_α. This gives rise to a few important methods of constructing new topological spaces from old ones.

Definition 2.9 (Relative topology). If $X \subseteq Y$ and $(Y, \mathscr{T}_Y)$ is a topological space, and if j is the inclusion map from X to Y, then the initial topology generated by j consists of the intersection of open sets with X, which is also called the relative (or subspace) topology.

Definition 2.10 (Finite product topology). If $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ are topological spaces then if $X = X_1 \times X_2$. We have a couple of "natural" maps, $X \xrightarrow{p_1} X_1$, $X \xrightarrow{p_2} X_2$. We see that $p_1^{-1}(U) = U \times X_2$, and $p_2^{-1}(V) = X_1 \times V$, so it is easy to see that the product topology has a basis consisting of sets of the form $U \times V$, where U is open in X_1 and V is open in X_2.

This trivially generalizes to any **FINITE** number of topological spaces and their **FINITE** cartesian product. However, when we generalize to infinite products, the following case occurs:

Definition 2.11 (Product topology). Suppose we have topological spaces $(X_\alpha, \mathscr{T}_\alpha)$ over a (possibly infinite) indexing set A, where $X = \prod X_\alpha$. Then, p_α is the natural projection $X \to X_\alpha$, and $p_\alpha^{-1}(U)$ is $\prod U_\beta$, where $U_\beta = X_\beta$ when $\beta \neq \alpha$ and $U_\alpha = U$. These sets form a subbase of this topology. Now, since topologies only allow finite intersections, then the product topology is generated by products $\prod_\alpha U_\alpha$, where only a finite number of U_α are allowed to be not equal to X_α.

Example 2.2. If $X_i = \{0,1\}$ with the discrete topology, then $\prod_0^\infty X_i$ is compact.

Definition 2.12 (Weak topology). For a normed vector space $(V, \|\cdot\|)$. If V^* is the vector space of all continuous linear functionals. The initial topology generated by V^* is the weak topology on V.

Example 2.3. For example, if we have $V = \mathscr{C}^0([0,1])$, with

$$\phi_g \colon V \to \mathbb{R}, f \mapsto \int_0^1 f(x) g(x) \, \mathrm{d}x .$$

Then, the initial topology generated by ϕ_g defines "a" weak topology on V.

As opposed to a initial topology, we can define final topologies:

Definition 2.13 (Final topology). If $(X_\alpha, \mathscr{T}_\alpha)$ are topological spaces and $f_\alpha \colon X_\alpha \to Y$, then the final topology $\mathscr{T}$ is the strongest topology making all f_α continuous.

To construct this topology, we can consider $\left\{A \in 2^Y \mid f_\alpha^{-1}(A) \in \mathscr{T}_\alpha\right\}$, which is a topology on Y for each α. We can just take the intersection of these topologies to obtain the strongest topology that makes all the f_α continuous.

Example 2.4 (Quotient topology). Any function $f \colon X \to Y$ induces an equivalence relation with the equivalence classes being the preimages $f^{-1}(\{y\})$. These equivalence classes partition X. It turns out any equivalence relation $\sim$ induces equivalence classes that partition X, which are denoted $X/\sim$. If X has a topology $\mathscr{T}_X$, then the final topology from the natural mapping $f \colon X \to X/\sim$ is called the quotient topology on $X/\sim$.

We can construct a bunch of new, interesting topologies using quotients. First we define what it means for topological spaces to be "the same." I will also start dropping the topology and use individual letters X to denote the topological spaces $(X, \mathscr{T}_X)$, depending on context.

Definition 2.14 (Homeomorphism). A function $f \colon X \to Y$ is a homeomorphism if it is an invertible continuous mapping, whose inverse is also continuous. Two spaces where there exists a homeomorphism between them are called "homeomorphic."

Example 2.5 (Circle). If $I = [0,1]$ with the equivalence relation such that only $0 \sim 1$, then $X/\sim$ is homeomorphic to the 1-sphere (circle), S^1. One homeomorphism from $I \to S^1$ is the mapping $t \mapsto e^{2\pi i t}$.

Example 2.6 (Cut-off cylinder). If $X = [0,2] \times [0,1]$ is endowed with the relation $(0,r) = (2,r)$ then the quotient $X/\sim$ is homeomorphic to a cylinder. If we instead identified $(0,r) \sim (2, 1-r)$ then $X/\sim$ will be homeomorphic to a Möbius strip.

Example 2.7 (Projective space). If X is the unit sphere in $\mathbb{R}^3$, then if we define the equivalence $v \sim -v$, then $X/\sim$ is the real projective space $\mathbb{R}\mathrm{P}^2$.

Example 2.8. It can be seen that homeomorphisms from an object from itself can form (quite complicated) groups. These groups are the automorphism groups $\mathrm{Aut}(X)$ of topological spaces. Recall that a group action on X is a homomorphism $\alpha\colon G \to \mathrm{Aut}(X)$. Examples of group actions on spaces include $\mathbb{Z}$ acting on $\mathbb{R}$ by translation, or $\mathbb{Z}_2$ acting on S^2 by the antipodal map.

However, recall that group actions induce equivalence classes (orbits) on the topological space and thus induces a quotient topology A/α. The quotient topology of the examples of the actions above give rise to S^1 and $\mathbb{R}\mathrm{P}^2$.

3 Functions into the Reals

The following are properties of topological spaces that can kind of categorize how "nice" they are.

Definition 3.1 (Hausdorff (T_2)). A space X is Hausdorff if for all $x \neq y$, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

Definition 3.2 (Normal (T_4)). A space X is Hausdorff if for all pairs of disjoint closed subsets C_1, C_2, there exist disjoint open sets O_1, O_1 such that $C_1 \subseteq O_1$ and $C_2 \subseteq C_2$.

3.1 Urysohn's Lemma

It can easily be seen that all metric spaces are Hausdorff, but it is also the case that metric spaces are normal. How do we prove this? We first consider the following important theorem:

Theorem 3.1 (Urysohn's Lemma). If X is a normal space, then for any disjoint closed subsets C_0, C_1 there exists a continuous function $f\colon X \to [0,1]$ where $f(C_0) = 0$ and $f(C_1) = 1$.

The converse of Urysohn's Lemma also holds, we simply take $O_0 = f^{-1}([0,1/3))$ and $O_1 = f^{-1}((2/3,1])$. We prove that metric spaces satisfy this property:

Proposition 3.2. Metric spaces are normal.

Definition 3.3 (Distance from a set). For a metric space (X,d), we define $d_A(x)$ or $d(x,A)$ where $A \subseteq X$, as $\inf_{a \in A} d(x,a)$.

It is easily seen that this function is Lipschitz with constant ≤ 1.

Now, we can proceed with a proof of Proposition 3.2.

Proof. Let C_0, C_1 be disjoint closed subsets in (X,d), then define

$$f\colon X \to [0,1], x \mapsto \frac{d(x,C_0)}{d(x,C_0) + d(x,C_1)}.$$

It is clear that $f(C_0) = 0$ and $f(C_1) = 1$, and that f is continuous. □

Now, we continue with a proof of Urysohn's Lemma.

Lemma 3.3. If X is normal, then for any closed set C and open set O containing C, there is an open set U "between" C and O, that is, $C \subseteq U \subseteq \overline{U} \subseteq O$.

Proof. If $D = X \setminus O$, then D is closed and disjoint from C and then we can find disjoint open sets U, V such that U contains C and V contains D. In addition, the closure of C must be contained within $X \setminus V$ which is contained by U. □

Proof of Urysohn's Lemma. Given X normal and disjoint closed sets C_0, C_1, set $O_1 = C_1'$, then apply the preceding lemma to get a set $O_{1/2}$ "between" C_0 and O_1. Applying the lemma again, we then get a two open sets $O_{1/4}, O_{3/4}$ s.t. $C_0 \subseteq O_{1/4} \subseteq \overline{O_{1/4}} \subseteq O_{1/2} \subseteq \overline{O_{1/2}} \subseteq O_{3/4} \subseteq \overline{O_{3/4}} \subseteq O_1$. We can repeat this process to choose $O_{1/8}, O_{3/8}, O_{5/8}, O_{7/8}$, and then repeat the process over and over again.

Now, we are able to define O_d, for all $d \in \Delta$ (rational numbers of the form $k/2^n$ where $k \leq 2^n$), such that if $r < s$, $\overline{O_r} \subseteq O_s$. In addition, for all r, $C_0 \subseteq O_r$. Define f on X by $f(x) = 1$ if $x \in C_1$, otherwise, we set $f(x) = \inf\{r \in \Delta \colon x \in O_r\}$. We can see that $f(C_0) = 0$ in this case, since r is dense in $[0, 1]$.

Now, to prove continuity, we choose a subbase of $[0, 1]$ consisting of sets of the form $(0, a]$ and $(b, 1]$. We then claim that

$$f^{-1}([0, a)) = \bigcup_{r<a} O_r, \text{ and } f^{-1}((b, 1]) = \bigcup_{s>b} (\overline{O_s})'$$

This can be seen to be true due to density of Δ, as you can always find a dyadic rational between $f(x)$ and either a or b. □

3.2 Tietze Extension Theorem

If X is a set and (Y, d) is a metric space, then let $B(X, Y)$ be the set of bounded functions from X to Y. Then, we can define a metric $\tilde{d}$ on $B(X, Y)$ as $\tilde{d}(f, g) = \sup_x |f(x) - g(x)|$.

Proposition 3.4. If Y is complete, then $B(X, Y)$ is complete.

Proof. If f_n is a Cauchy sequence then $f_n(x)$ is a Cauchy sequence for all x, and we can define $f(x) = \lim f_n(x)$. □

Proposition 3.5. If we assume that X is instead of a topological space, then the set of continuous bounded functions $BC(X, Y)$ is complete if Y is complete.

The following is an important result using Urysohn's Lemma.

Theorem 3.6 (Tietze Extension Theorem)**.** If X is a normal space and A is a closed subset of X, and if $f \colon A \to \mathbb{R}$ is continuous, then there exists a continuous extension $g \colon X \to \mathbb{R}$ such that $g|_A = f$. Furthermore, if f maps A to $[a, b]$, then g maps X to that same interval.

Note that this theorem uses special properties of $\mathbb{R}$. For example, there is no continuous extension of the identity function from the unit circle to itself to the closed unit ball. To prove this theorem, we first prove a lemma.

Lemma 3.7. If X is normal and A is a closed subset of X, if there is an h mapping A to $[0, r]$, then there is a $g \colon X \to [0, r/3]$ such that $0 \leq h - g|_A \leq 2r/3$.

Proof. If $B = \{x \in A \mid h(x) \le r/3\}$, and $C = \{x \in A \mid h(x) \ge 2r/3\}$. Then, we can find a function $g\colon X \to [0, r/3]$ such that $g|_B = 0$ and $g|_C = r/3$ by Urysohn's Lemma. □

Proof of Tietze Extension Theorem. We prove it for the special case $[0,1]$, since all finite intervals are homeomorphic to the unit interval, this will hold for all finite initervals. To start, we can find a function $g^1\colon [0, 1/3]$ such that $0 \le f - g|_A^1 \le 2/3$.

We then apply the lemma repeatedly: We apply it to $f - g|_A^1$ to get a function $g^2\colon X \to [0, 1/3(2/3)]$ such that $0 \le f - g|_A^1 - g|_A^2 \le 4/9$.

Repeat with $f - g|_A^1 - g|_A^2$ to get g^3, and in general we repeat this process with $f - g|_A^1 - \cdots - g|_A^n\colon A \to [0, \left(\frac{2}{3}\right)^n]$ to get a function $g^{n+1}\colon [0, \frac{1}{3}\left(\frac{2}{3}\right)^n]$ such that $f - g|_A^1 - \cdots - g|_A^{n+1}\colon A \to [0, (2/3)^{n+1}]$.

Now we have $\|g^n\|_\infty \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, so $\sum_{k=1}^n g^k$ is a Cauchy sequence and thus converges to $g = \sum g^n$, and we get that $\|g\|_\infty \le 1$ by summing up the geometric series. We see that $g = f$ because

$$\left\| f - g|_A \right\| = \lim \left\| f - \sum_{k=1}^{n} g|_A^k \right\|.$$

□

For infinite intervals, we use the homeomorphism h from $\mathbb{R}$ to $(0,1)$, and we set $g = h \circ f\colon A \to (0,1)$. Now if we apply Tietze's extension theorem we get $\tilde{g}\colon X \to [0,1]$, and let $D = \tilde{g}^{-1}(0)$, then D is disjoint from A and closed. We can take $U = \tilde{g}^{-1}([0, 1/4)) \cap A'$. By Urysohn's Lemma we can find a function that takes $k\colon X \to [0,1]$ where $k(D) = 1/8$ and $k(U') = 0$, and we may add $\tilde{g} + k$ to obtain a continuous function whose output is not 0. Repeat this process for the point 1 and we get a final function that maps into $(0,1)$.

3.3 Completion of Metric Spaces

Recall that if X is a topological space then $\mathrm{BC}(X, \mathbb{R})$ with the supremum norm is complete. This is an example of the following.

Definition 3.4 (Banach space). A complete normed vector space is called a Banach space.

We can use this fact to define a completion of a metric space Y.

Lemma 3.8. If (Y, d_Y) and (M, d_M) are metric spaces and M is complete, then if $f\colon Y \to M$ is an isometry, it can be seen that $\overline{f(Y)}$ is a completion of M.

Theorem 3.9. All metric spaces have a completion.

Proof. Given (Y, d), we let $f_y(y') = d(y, y')$. This is unbounded, but if we choose $y_0 \in Y$ be any base point, we can define $h_y = f_y - g$, where $g = f_{y_0}$. By the triangle inequality, h_y is bounded.

We now show that $y \mapsto h_y$ is an isometry, as $(h_{y_1} - h_{y_2})(y) = (f_{y_1} - f_{y_2})(y)$ so the norm is at most $d(y_1, y_2)$. In addition, if we plug in y_1, we get $R(f_{y_1} - f_{y_2})(y_1) = d(y_1, y_2)$, so $\left\| h_{y_1} - h_{y_2} \right\| = d(y_1, y_2)$. According to the above lemma, we have a completion of Y. □

4 Compactness

Definition 4.1 (Open Cover, Subcover). If X is a topological space, by an open cover we mean a collection $\mathscr{C} = \{U_\alpha\}$ of open sets such that $\bigcup \mathscr{C} = X$.

By a subcover of $\mathscr{C}$, we mean a collection $\mathscr{D} \subseteq \mathscr{C}$ where $\bigcup \mathscr{D} = X$.

Definition 4.2 (Compactness). X is compact if and only if every open cover has a finite subcover. If $A \subseteq X$, then A is compact if A is compact in its relative topology.

Proposition 4.1. *A is compact iff for any collection $\mathscr{E}$ of open sets such that $\bigcup \mathscr{E} \supseteq A$, there is a finite subset $\mathscr{S} \subseteq \mathscr{E}$ that also covers A, that is, $\bigcup \mathscr{S} \supseteq A$.*

Proposition 4.2. *Any closed subset of a compact space is compact.*

Proof. If $\mathscr{E}$ is covers $A \subseteq X$, then $\mathscr{E} \bigcup \{A'\}$ covers X, so we find a finite subcover and take out A'. □

Note that the converse is not true: Any subset of a space with the indiscrete topology is compact. However, when X is Hausdorff, then this holds:

Proposition 4.3. *If X is Hausdorff, then if $A \subseteq X$ is compact, then for all $x \notin A$, then there exist disjoint open sets U and V such that $A \in U$ and $x \in V$.*

Proof. If x is in the complement of A, for all $y \in A$ we can find disjoint open sets U_y, V_y with $x \in U_y$ and $y \in V_y$. Then V_y covers A, and we can find a finite subcover V_{y_k}, $1 \le k \le n$. Then, we have $\bigcap U_{y_k}$ is an open set that contains X and does not intersect $\bigcup V_{y_k}$, which is an open set containing x. □

We get two immediate corollaries:

Corollary 4.4. *Any compact subset of a Hausdorff space is closed.*

Corollary 4.5. *Any compact Hausdorff space is regular (T_3), i.e. for any closed set $A \subset X$, and any $x \notin A$, we have disjoint open sets U, V such that $A \subset U$ and $x \in V$.*

We can prove a slightly stronger statement:

Corollary 4.6. *Any compact Hausdorff space is normal (T_4).*

Proof. If A and B are disjoint closed sets in a compact Hausdorff space X, then for any $y \in B$ there are disjoint open sets U_y, V_y such that $A \subseteq V_y$ and $y \in U_y$, and we can find a finite subcover U_{y_k} of the open cover U_y of B, Now, we let $U = \bigcup U_{y_k}$ and $V = \bigcap V_{y_k}$ and we have U disjoint from V, where $B \subseteq U$ and $A \subseteq V$. □

Now, we examine maps on compact spaces.

Proposition 4.7. If X is compact and $f: X \to Y$, then $f(X)$ is compact.

Proof. WLOG assume that $Y = f(X)$. If $\mathscr{C}$ is an open cover of $f(Y)$, then if $\mathscr{C}'$ is the set of preimages of sets in $\mathscr{C}$, then $\mathscr{C}'$ is an open cover of X, which we can take a finite subcover $\mathscr{S}' = \{f^{-1}(O_1), \dots, f^{-1}(O_n)\}$, then we have that $\mathscr{S} = \{O_1, \dots, O_n\}$ is a finite subcover of Y. □

Corollary 4.8 (Extreme Value Theorem)**.** If $f: \mathbb{R} \to \mathbb{R}$ is continuous on an interval $[a, b]$, then f has a maximum on $[a, b]$ and attains it on $[a, b]$.

Proposition 4.9. If $f: X \to Y$ for X compact and Y Hausdorff, then if f is bijective, then it is a homeomorphism.

Proof. Images of closed sets are closed due to Proposition 4.7 and Corollary 4.4. □

We can give an alternative characterization of compactness using closed sets.

Proposition 4.10. X is compact iff for any collection of closed sets has the finite intersection property, the intersection is nonempty.

4.1 Tychonoff's Theorem

We first state Tychonoff's theorem:

Theorem 4.11 (Tychonoff's Theorem)**.** If $\{X_\alpha\}$ are a collection of compact spaces, then the space $\prod_\alpha X_\alpha$ is compact.

It is clear that this requires the Axiom of Choice, as $\prod_\alpha X_\alpha$ may be empty without it. It is also interesting that Tychonoff's Theorem implies AC, although Tychonoff's Theorem for Hausdorff compact spaces does not imply AC.

The proof of this theorem also requires AC, in the form of

Theorem 4.12 (Zorn's Lemma)**.** If P is a poset, then if any chain C (totally ordered subset) in P has an upper bound in P, then P has a maximal element. (We call P inductively ordered).

Zorn's Lemma has wide applications across many fields, such as in the proof that every vector space has a basis, or that every proper ideal is contained in a maximal ideal.

Definition 4.3. For any set X, a filter on X is a collection $\mathscr{F}$ of nonempty subsets that is closed under intersections, and where all supersets of a set in the filter is also in the filter.

An ultrafilter is a maximal filter.

Proof of Tychonoff's Theorem. If $\alpha \in \mathscr{A}$ indexes a family of compact spaces X_α, then if $X = \prod X_\alpha$, then let $\mathscr{C}$ be a collection of closed subsets of X with the finite intersection property. If $W_{\mathscr{C}}$ is the collection of families $\mathscr{D}$ of subsets of X such that $\mathscr{C} \subseteq \mathscr{D}$ and such that $\mathscr{D}$ has FIP.

Now, we show that $W_{\mathscr{C}}$ is inductively ordered: If Φ is a chain in W_C, then if $\mathscr{D}_\Phi = \bigcup \Phi$, then let $A_1, \dots, A_n \in \mathscr{D}_\phi$, then $A_i \in \mathscr{D}_{\alpha_i}$, which we can just find the maximal index out of the α_i, which we call α_0. Then, $A_i \in \alpha_0$ for all i, which proves that $\bigcap A_i$ has nonempty intersection.

We can then use Zorn's Lemma to obtain a maximal element $\mathscr{D}^*$ in $W_{\mathscr{C}}$. Now, we can see that $\mathscr{D}^*$ is closed under finite intersections because it is maximal. In addition, we also have that if

$B \cap A \neq \varnothing$ for all $A \in \mathscr{D}^*$, then $B \in \mathscr{D}^*$, again due to maximality (we can use the previous property to prove this statement).

Now, for any $\alpha \in \mathscr{A}$, we have the continuous projection $\pi_\alpha : X \to X_\alpha$, now we claim that for any $\mathscr{D} \in W_{\mathscr{C}}$, the set of $\{\pi_\alpha(A) \mid A \in \mathscr{D}\}$ has FIP since if $x \in \bigcap A_j$, then $\pi_\alpha(x) \in \bigcap \pi_\alpha(A_j)$. We also trivially have that the closures $\overline{\pi_\alpha(A)}$ has the FIP. Now, applying to $\mathscr{D}^*$, we have that $\bigcap_{A \in \mathscr{D}^*} \overline{\pi_\alpha(A)}$ is nonempty since X_α is compact.

Now, applying AC, we can pick $x \in X$ where $\pi_\alpha(x) \in \bigcap_{A \in \mathscr{D}^*} \overline{\pi_\alpha(A)}$. Now, we prove that $x \in \overline{A}$ for all $A \in \mathscr{D}^*$, using the subbase of the product topology. If we let $U = \pi_\alpha^{-1}(U_\alpha)$, then $x \in U$ implies that $U \cap A \neq 0$ for all $A \in \mathscr{D}^*$, since we have $x_\alpha \in \overline{\pi_\alpha(A)}$ and U_α, then $\pi_\alpha(A) \cap U_\alpha$ must be nonempty otherwise $\overline{\pi_\alpha(A)}$ would not contain x_α. This implies that $\pi^{-1}(U_\alpha)$ has nonempty intersection with any set in $\mathscr{D}^*$, and thus belongs to $\mathscr{D}^*$.

Now, by the first property of $\mathscr{D}^*$, then we have that finite intersections $\bigcap \pi_\alpha^{-1}(U_{\alpha_i})$ are in $\mathscr{D}^*$, thus $\mathscr{D}^*$ contains a neighborhood base of $\mathscr{D}^*$. Now, if $x \in U$ open in X, then there exists an open set V in $\mathscr{D}^*$. Thus, for all $A \in \mathscr{D}^*$, we have that $x \in \overline{A}$. Finally, this results in the fact that x is contained in $\bigcap \mathscr{C}$. □

This theorem has applications in 202B, such as the Aloglu theorem (unit ball in weak-* topology is compact), and when considering locally compact spaces and β-compactifications.

Now, we consider the following interesting result

Theorem 4.13 (Kelley). Tychonoff's Theorem is equivalent to AC.

Proof. The reverse implication was proven above, and now if we assume Tychonoff's theorem, we need to show that $\prod X_\alpha$ is nonempty. We can take $\omega = \bigcup X_\alpha$ which is not in $\bigcup X_\alpha$ due to regularity. Now, if $Y_\alpha = X_\alpha \cup \{\omega\}$, we can take $\mathscr{T}_\alpha = \{Y_\alpha, \varnothing, T_\alpha, \{\omega\}\}$, which is compact since there are finitely many open sets. The product $Y = \prod Y_\alpha$ is compact due to Tychonoff's theorem. If $F_\alpha = \pi_\alpha^{-1}(X_\alpha)$, then F_α is closed, and has the FIP, since for finite intersections $\bigcap F_{\alpha_i}$, we can consider $x_{\alpha_i} = F_{\alpha_i}$, and $x_\alpha = \omega$ otherwise. Thus due to compactness we have $\bigcap F_\alpha = \prod X_\alpha \neq \varnothing$, which proves the equivalence. □

4.2 Totally Bounded Sets

Definition 4.4. We say a metric space (X,d) is **totally bounded** if for all $\varepsilon > 0$, there exists a finite set $\{x_i\}$ such that

$$X = \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

It is trivial that compactness implies totally bounded. We now prove a partial converse:

Theorem 4.14. Complete and totally bounded metric spaces are compact.

Proof. □

We can now study compact sets in $\mathrm{BC}(X,M)$ with the d_∞ metric, where X is a topological space and M is a complete metric space. These correspond precisely to the totally bounded sets.

Definition 4.5. If X is a topological space and (M, d) is a metric space, then a family of functions $\mathscr{F} \subseteq \mathrm{C}(X, M)$ is **equicontinuous at a point** x if for any $\varepsilon > 0$, there exists an open neighborhood $\mathscr{O}$ such that $d(f(y), f(x)) < \varepsilon$ whenever $y \in \mathscr{O}$.

We say $\mathscr{F}$ is **equicontinuous** if it is equicontinuous at all points in X.

Theorem 4.15 (Arzela-Ascoli)**.** Let X and M be as above, and let X be compact. If $\mathscr{F} \subseteq \mathrm{BC}(X, M)$ with $\mathscr{F}$ pointwise totally bounded, and equicontinuous, then $\mathscr{F}$ is totally bounded.

Proof. If $\varepsilon > 0$ is given, for each x there is an open set O_x such that $y \in O_x$, then $d(f(y), f(x)) < \varepsilon/4$. By compactness, there are $x_1, \ldots, x_n$ such that

$$X = \bigcup_{i=1}^{n} O_{x_i}.$$

Now, for each j, the set $I_j = \left\{ f(x_j) \mid f \in \mathscr{F} \right\}$ is totally bounded, so if we choose S_j to be a finite subset of I_j that is $\varepsilon/4$-dense in I_j. We can then take $S = \bigcup S_j$. Now, if we let $\Psi = \left\{ \psi \colon \{1, \ldots, n\} \to S \right\}$, which is finite, then $\square$