

**Math 202A**

# **TOPOLOGY AND ANALYSIS NOTES**

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# CONTENTS

I	TOPOLOGY	1
1	METRIC SPACES	3
1.1	Topology of Metric Spaces . . . . .	4
2	TOPOLOGICAL SPACES	7
2.1	Bases and Subbases . . . . .	7
2.2	New Topologies from Old . . . . .	8
3	FUNCTIONS INTO THE REALS	11
3.1	Urysohn's Lemma . . . . .	11
3.2	Tietze Extension Theorem . . . . .	12
3.3	Completion of Metric Spaces . . . . .	13
4	COMPACTNESS	14
4.1	Tychonoff's Theorem . . . . .	15
4.2	Totally Bounded Sets . . . . .	16
4.3	Locally Compact Spaces . . . . .	17
4.4	A Quick $C^*$ -algebra Interlude . . . . .	18
II	MEASURE AND INTEGRATION	20
5	BUILDING MEASURES	21
5.1	Families and Measures . . . . .	21
5.2	Prerings and Premeasures . . . . .	22
5.3	Caratheodory's Extension Theorem . . . . .	24
5.4	Properties of Extensions . . . . .	26

# PART I

## TOPOLOGY

# ADMINISTRATIVE MATTERS

Midterm date:

1. Wed Oct 19

Weekly psets due Friday midnight.

# 1 METRIC SPACES

**Definition 1.1** (Metric Space). A metric on a set  $X$  is a function  $d: X^2 \rightarrow \mathbb{R}_{\geq 0}$  such that

1.  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

$(X, d)$  is then called a metric space.

If condition 1 is omitted and replaced only with the condition that  $d(x, x) = 0$ , then  $d$  is called a semimetric (or quasimetric). If distances are allowed to be infinite, then  $d$  is called an extended metric.

If  $(X, d)$  is a semimetric space, then if we define a relation  $\sim$  such that  $x \sim y$  iff  $d(x, y) = 0$ , it is an equivalence relation. In this case,  $d$  “drops” to a metric on the equivalence classes  $X/\sim$ . To show this, we just need to prove that  $\tilde{d}([x], [y]) = d(x, y)$  is a well-defined function, which utilizes the triangle inequality.

**Example 1.1.** If  $G$  is a connected weighted graph, then we can define a metric on  $G$  by defining the distance between two vertices to be the length of the minimal path connecting them.

If  $G$  is unconnected, by assigning the distance of infinity between points in separate components, we define an extended metric.

**Definition 1.2** (Norm). If  $V$  is a vector space then  $\|\cdot\|: V \rightarrow \mathbb{R}_{\geq 0}$  is a norm on  $V$  if

1.  $\|v\| = 0$  iff  $v = 0$
2.  $\|\lambda v\| = |\lambda| \|v\|$
3.  $\|v + w\| \leq \|v\| + \|w\|$

A norm on  $V$  clearly induces a metric on  $V$  by defining  $d(v, w) = \|v - w\|$ .

**Example 1.2** (Euclidean metric). The real line  $\mathbb{R}$  and the euclidean spaces  $\mathbb{R}^n$  with the euclidean metric  $d(x, y) = \|x - y\|_2$ .

**Example 1.3** ( $p$ -norms). The  $p$ -norms  $\|(x_1, \dots, x_n)\|_p = (\sum x_i^p)^{1/p}$  induce a metric on  $\mathbb{R}^n$ .

**Definition 1.3** (Restricted metric). If  $(X, d)$  is a metric space and  $Y$  is a subset of  $X$ , then we can restrict  $d$  to  $Y$  and  $d|_Y$  is a metric on  $Y$ .

It is common to consider the restriction of the Euclidean metric onto subsets of Euclidean space.

**Example 1.4** ( $L^p$  norms). If  $V = C^0([0, 1])$ , then examples of norms on  $V$  include

1.  $\|f\|_\infty = \sup |f(t)|$ .
2.  $\|f\|_1 = \int_0^1 |f(t)| dt$
3.  $\|f\|_2 = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}$
4.  $\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}$  for  $1 \leq p < \infty$ .<sup>1</sup>

This generalizes to bounded closed regions of  $\mathbb{R}^n$ , where all continuous functions are integrable (due to compactness and the extreme value theorem). The proof that these are actually norms that satisfy the triangle inequality will come later (see the Minkowski and Hölder inequalities).

## 1.1 TOPOLOGY OF METRIC SPACES

### CATEGORIES

Categories contain objects and morphisms, that satisfy the following properties:

1. There is a morphism from any object  $A$  to itself, called the identity  $1_A$ .
2. Morphisms can be composed.

Metric spaces can form a category under a few choices of collections of morphisms:

- Isometries:  $i: (X, d_X) \rightarrow (Y, d_Y)$  such that  $d_X(x, y) = d_Y(i(x), i(y))$  for all  $x, y$ .  
Surjective isometries form a subcategory of the above category, where every map is an isomorphism.
- Lipschitz maps:  $f: (X, d_X) \rightarrow (Y, d_Y)$ , where there exists a constant  $C$  (which may depend on  $f$ ), such that  $d_Y(f(x), f(y)) \leq C d_X(x, y)$ .

We can also form the subcategory of bi-Lipschitz isomorphisms.

- Uniformly continuous maps:  $f: (X, d_X) \rightarrow (Y, d_Y)$  such that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that when  $d_X(x, y) \leq \delta$ , then  $d_Y(f(x), f(y)) < \varepsilon$ .
- Continuous maps:  $f: (X, d_X) \rightarrow (Y, d_Y)$  such that for all  $x \in X$ , for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that when  $d_X(x, y) \leq \delta$ , then  $d_Y(f(x), f(y)) < \varepsilon$ .

These conditions get weaker as we go down, and continuity can be generalized to non-metric spaces which gives rise to general topology. The study of uniformly continuous maps is much less common, but gives rise to uniformities.

**Definition 1.4** (Convergence). If  $\{x_n\}$  is a sequence in a metric space  $(X, d)$ , then  $x_n \rightarrow x$  if  $\forall \varepsilon > 0, \exists N$  such that  $n > N$  implies that  $d(x_n, x) < \varepsilon$ .

**Definition 1.5** (Cauchy sequence). If  $\{x_n\}$  is a sequence in a metric space  $(X, d)$ , then  $x_n \rightarrow x$  if  $\forall \varepsilon > 0, \exists N$  such that  $n, m > N$  implies that  $d(x_n, x_m) < \varepsilon$ .

---

<sup>1</sup>The  $\infty$ -norm can be viewed as the limit of the  $p$ -norms as  $p \rightarrow \infty$ .

**Definition 1.6** (Completeness).  $(X, d)$  is complete if every cauchy sequence converges to a point.

For any metric space that isn't complete, we want to form a "completion" of it.

**Definition 1.7** (Density). If  $S \subseteq X$ ,  $S$  is dense if for all  $x$  in  $X$  and all  $\varepsilon < 0$  there exists  $s \in S$  where  $d(s, x) < \varepsilon$ .

**Definition 1.8** (Completion). A completion of a metric space  $(X, d)$  is a metric space  $(\bar{X}, \bar{d})$  with an inclusion isometry  $i: X \rightarrow \bar{X}$  such that the image of  $i$  is dense in  $\bar{X}$ . In addition,  $\bar{X}$  is a complete metric space.

One way to define a completion for any metric space is by defining a semimetric  $\tilde{d}$  on the set of Cauchy sequences  $\text{cs}(X, d)$ , where  $\tilde{d}(\{x_n\}, \{y_n\}) = \lim_{n \rightarrow \infty} d(x_n, y_n)$ , then by dropping that to a metric on the equivalence classes induced by the semimetric.

We can just define  $f: X \rightarrow \bar{X}$  as  $f(x) = [\{x, x, x, \dots\}]$ . To show that  $\bar{X}$  is complete, we consider Cauchy sequences of Cauchy sequences, approximate their elements by elements in  $X$ , and show that the sequence converges to the sequence generated by the approximation.

**Example 1.5.** If  $f \in V = C^0([0, 1])$ , the norm  $\|f\|_\infty = \sup(|f(x)|)$  induces a complete metric space (due to the limit of uniformly continuous functions converging to a continuous function).

However, in the  $L^1$  norm  $\|f\|_1 = \int_0^1 |f(x)| dx$ ,  $V$  is not complete (you can construct a sequence of functions that converges to a step function).

Now to answer the question "What is the completion of  $V$  under the  $L^p$  norms", we must look to measure theory (2nd half of this course).

If  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces, if  $f: X \rightarrow Y$ , recall that if  $x \in X$ , for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that when  $d_X(x, y) \leq \delta$ , then  $d_Y(f(x), f(y)) < \varepsilon$ , then  $f$  is continuous.

**Definition 1.9** (Open ball). The open ball of radius  $r$ ,  $B_r(x) = \{y \in X \mid d_X(x, y) < r\}$ .

This allows us to refine our definition of continuity:

**Definition 1.10** (Continuity). If  $f: X \rightarrow Y$ , then  $f$  is continuous if for all  $x$ , for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ .

**Definition 1.11** (Open set). The set  $A \subseteq X$  is open if for all  $a \in A$ , there exists an open ball around  $a$  that is contained within  $A$ .

**Proposition 1.1.** Open balls are open.

*Proof.* If  $a \in B_r(x)$ , then  $B_{r-d_X(a,x)}(a)$  is contained within  $B_r(x)$  due to the triangle inequality.  $\square$

**Proposition 1.2.** If  $f$  is continuous, then if  $U$  is open in  $Y$  such that  $x \in X$ , then there is an open ball around  $x$  that is contained within the preimage of  $U$ .

A direct corollary of this proposition is

**Corollary 1.3.**  $f: X \rightarrow Y$  is continuous is equivalent to the following condition: If  $U$  is open in  $Y$ , then  $f^{-1}(U)$  is open in  $X$ .

Now, we consider properties of the set  $\mathcal{T}$  consisting of open sets in  $X$ :

**Theorem 1.4.** 1.  $\emptyset, X \in \mathcal{T}$ .

2. Arbitrary unions of sets in  $\mathcal{T}$  are also in  $\mathcal{T}$ .

3. Finite intersections of sets in  $\mathcal{T}$  are in  $\mathcal{T}$ .

*Proof.* Trivial. □

These properties give rise to the definition of a topology.

**Definition 1.12** (Topology). A topology on a set  $X$  is a collection  $\mathcal{T} \subseteq 2^X$  such that

1.  $\emptyset, X \in \mathcal{T}$ .

2. Arbitrary unions of sets in  $\mathcal{T}$  are also in  $\mathcal{T}$ .

3. Finite intersections of sets in  $\mathcal{T}$  are in  $\mathcal{T}$ .

The sets in  $\mathcal{T}$  are called open sets.



# 2 TOPOLOGICAL SPACES

We recall the definition of a topology:

**Definition 2.1** (Topology). A topology on a set  $X$  is a collection  $\mathcal{T} \subseteq 2^X$  such that

1.  $\emptyset, X \in \mathcal{T}$ .
2. Arbitrary unions of sets in  $\mathcal{T}$  are also in  $\mathcal{T}$ .
3. Finite intersections of sets in  $\mathcal{T}$  are in  $\mathcal{T}$ .

The sets in  $\mathcal{T}$  are called open sets.

We then extend the definition of continuity to general topologies.

**Definition 2.2** (Continuity). If  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are topological spaces, then  $f$  is continuous if for all  $U \in \mathcal{T}_Y$ ,  $f^{-1}(U) \in \mathcal{T}_X$ .

Topologies on a set  $X$  form a poset under the subset relation.

**Definition 2.3** (Closed sets). The closed sets in  $(X, \mathcal{T})$  are precisely the complement of open sets in  $X$ .

**Proposition 2.1** (Properties of closed sets). 1.  $\emptyset, X$  are closed.

2. Arbitrary intersections of closed sets are also closed.
3. Finite unions of closed sets are closed.

**Definition 2.4** (Closure). The closure  $\bar{A}$  of an arbitrary set  $A$  is the smallest closed set containing it.

**Definition 2.5** (Density). Given  $A \subseteq B$ , we say  $A$  is dense in  $B$  if  $B \subseteq \bar{A}$ . Equivalently,  $\bar{A} = \bar{B}$ .

## 2.1 BASES AND SUBBASES

**Proposition 2.2.** If  $X$  is a set, then the arbitrary intersection of topologies is a topology on  $X$ .

*Proof.* These details are routinely verified. □

**Corollary 2.3.** Given any collection  $\mathcal{S}$  of subsets of  $X$ , there is a smallest topology containing  $\mathcal{S}$ .

*Proof.* This topology is the intersection of all the topologies that contain  $\mathcal{S}$ . We say that  $\mathcal{S}$  generates this topology. □

**Definition 2.6** (Subbase). Let  $\mathcal{T}$  be a topology on  $X$ , and let  $\mathcal{S} \subseteq 2^X$ , with  $X = \bigcup \mathcal{S}$ . We say that  $\mathcal{S}$  is a subbase of  $\mathcal{T}$  if  $\mathcal{T}$  is the topology generated by  $\mathcal{S}$ .

Subbases can be useful for studying topologies in the same way that a basis is useful for studying vector spaces. For example, see [Proposition 2.7](#).

We can explicitly characterize the topology generated by a set  $\mathcal{S}$ . We first state a lemma:

**Lemma 2.4.** If  $\mathcal{G} \subseteq 2^X$  be closed under finite intersections and  $\bigcup \mathcal{G} = X$ , then the topology generated by  $\mathcal{G}$  consists of the arbitrary unions of sets in  $\mathcal{G}$  (with union over 0 sets being  $\emptyset$ ).

*Proof.* The only part that is not immediately obvious is the closure under finite intersections. If  $\mathcal{O}_1 \cup_A U_\alpha$  and  $\mathcal{O}_2 = \bigcup_B V_\beta$ , then

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \bigcup_{A,B} U_\alpha \cap V_\beta.$$

□

**Proposition 2.5.** If  $\mathcal{S} \subseteq 2^X$  whose union is  $X$ , then the topology  $\mathcal{T}$  generated by  $\mathcal{S}$  consist of the arbitrary unions of  $\mathcal{G}^{\mathcal{S}}$  which are defined as the set of finite intersections of sets in  $\mathcal{S}$ .

**Definition 2.7** (Base). A collection  $\mathcal{B} \subseteq 2^X$  is called a base for a topology  $\mathcal{T}$  on  $X$  if  $\mathcal{T}$  consists of the arbitrary unions of elements in  $\mathcal{B}$ .

**Example 2.1.** 1. The collection of finite intersections  $\mathcal{G}^{\mathcal{S}}$  of a subbase  $\mathcal{S}$ .

2. The set of open balls in a metric space.

3. The set of intervals of the form  $(-\infty, a)$  and  $(b, \infty)$  in  $\mathbb{R}$  under the standard topology.

Notice how the second and third examples are not closed under finite intersections.

**Proposition 2.6.** For any base  $\mathcal{B}$ , any finite intersection of sets in  $\mathcal{B}$  must be a union of elements in  $\mathcal{B}$ .

**Proposition 2.7.** If  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are topological spaces, and let  $f: X \rightarrow Y$ . If  $\mathcal{S}_Y$  is a subbase of  $\mathcal{T}_Y$ , then  $f$  is continuous iff for all  $U \in \mathcal{S}_Y$ , we have  $f^{-1}(U)$  is open.

*Proof.* First show that the preimages of sets in  $\mathcal{G}_{\mathcal{S}_Y}$  are open, and then show that the preimages of all sets in  $\mathcal{T}_Y$  are open. □

## 2.2 NEW TOPOLOGIES FROM OLD

**Definition 2.8** (Initial Topologies). If  $X$  is a set,  $(Y_\alpha, \mathcal{T}_\alpha)$  are topological spaces, and for each  $\alpha$ ,  $f_\alpha: X \rightarrow Y_\alpha$ . Then, the initial topology is the smallest topology on  $X$  that makes all  $f_\alpha$  continuous.

This initial topology is just the topology generated from the preimages of open sets in  $Y_\alpha$  under  $f_\alpha$ . This gives rise to a few important methods of constructing new topological spaces from old ones.

**Definition 2.9** (Relative topology). If  $X \subseteq Y$  and  $(Y, \mathcal{T}_Y)$  is a topological space, and if  $j$  is the inclusion map from  $X$  to  $Y$ , then the initial topology generated by  $j$  consists of the intersection of open sets with  $X$ , which is also called the relative (or subspace) topology.

**Definition 2.10** (Finite product topology). If  $(X_1, \mathcal{T}_1)$  and  $(X_2, \mathcal{T}_2)$  are topological spaces then if  $X = X_1 \times X_2$ . We have a couple of “natural” maps,  $X \xrightarrow{p_1} X_1$ ,  $X \xrightarrow{p_2} X_2$ . We see that  $p_1^{-1}(U) = U \times X_2$ , and  $p_2^{-1}(V) = X_1 \times V$ , so it is easy to see that the product topology has a basis consisting of sets of the form  $U \times V$ , where  $U$  is open in  $X_1$  and  $V$  is open in  $X_2$ .

This trivially generalizes to any **FINITE** number of topological spaces and their **FINITE** cartesian product. However, when we generalize to infinite products, the following case occurs:

**Definition 2.11** (Product topology). Suppose we have topological spaces  $(X_\alpha, \mathcal{T}_\alpha)$  over a (possibly infinite) indexing set  $A$ , where  $X = \prod X_\alpha$ . Then,  $p_\alpha$  is the natural projection  $X \rightarrow X_\alpha$ , and  $p_\alpha^{-1}(U)$  is  $\prod U_\beta$ , where  $U_\beta = X_\beta$  when  $\beta \neq \alpha$  and  $U_\alpha = U$ . These sets form a subbase of this topology. Now, since topologies only allow finite intersections, then the product topology is generated by products  $\prod_\alpha U_\alpha$ , where only a finite number of  $U_\alpha$  are allowed to be not equal to  $X_\alpha$ .

**Example 2.2.** If  $X_i = \{0, 1\}$  with the discrete topology, then  $\prod_0^\infty X_i$  is compact.

**Definition 2.12** (Weak topology). For a normed vector space  $(V, \|\cdot\|)$ . If  $V^*$  is the vector space of all continuous linear functionals. The initial topology generated by  $V^*$  is the weak topology on  $V$ .

**Example 2.3.** For example, if we have  $V = \mathcal{C}^0([0, 1])$ , with

$$\phi_g: V \rightarrow \mathbb{R}, f \mapsto \int_0^1 f(x)g(x) dx.$$

Then, the initial topology generated by  $\phi_g$  defines “a” weak topology on  $V$ .

As opposed to a initial topology, we can define final topologies:

**Definition 2.13** (Final topology). If  $(X_\alpha, \mathcal{T}_\alpha)$  are topological spaces and  $f_\alpha: X_\alpha \rightarrow Y$ , then the final topology  $\mathcal{T}$  is the strongest topology making all  $f_\alpha$  continuous.

To construct this topology, we can consider  $\{A \in 2^Y \mid f_\alpha^{-1}(A) \in \mathcal{T}_\alpha\}$ , which is a topology on  $Y$  for each  $\alpha$ . We can just take the intersection of these topologies to obtain the strongest topology that makes all the  $f_\alpha$  continuous.

**Example 2.4** (Quotient topology). Any function  $f: X \rightarrow Y$  induces an equivalence relation with the equivalence classes being the preimages  $f^{-1}(\{y\})$ . These equivalence classes partition  $X$ . It turns out any equivalence relation  $\sim$  induces equivalence classes that partition  $X$ , which are denoted  $X/\sim$ . If  $X$  has a topology  $\mathcal{T}_X$ , then the final topology from the natural mapping  $f: X \rightarrow X/\sim$  is called the quotient topology on  $X/\sim$ .

We can construct a bunch of new, interesting topologies using quotients. First we define what it means for topological spaces to be “the same.” I will also start dropping the topology and use individual letters  $X$  to denote the topological spaces  $(X, \mathcal{T}_X)$ , depending on context.

**Definition 2.14** (Homeomorphism). A function  $f: X \rightarrow Y$  is a homeomorphism if it is an invertible continuous mapping, whose inverse is also continuous. Two spaces where there exists a homeomorphism between them are called “homeomorphic.”

**Example 2.5** (Circle). If  $I = [0, 1]$  with the equivalence relation such that only  $0 \sim 1$ , then  $X/\sim$  is homeomorphic to the 1-sphere (circle),  $S^1$ . One homeomorphism from  $I \rightarrow S^1$  is the mapping  $t \mapsto e^{2\pi it}$ .

**Example 2.6** (Cut-off cylinder). If  $X = [0, 2] \times [0, 1]$  is endowed with the relation  $(0, r) = (2, r)$  then the quotient  $X/\sim$  is homeomorphic to a cylinder. If we instead identified  $(0, r) \sim (2, 1 - r)$  then  $X/\sim$  will be homeomorphic to a Möbius strip.

**Example 2.7** (Projective space). If  $X$  is the unit sphere in  $\mathbb{R}^3$ , then if we define the equivalence  $v \sim -v$ , then  $X/\sim$  is the real projective space  $\mathbb{RP}^2$ .

**Example 2.8.** It can be seen that homeomorphisms from an object from itself can form (quite complicated) groups. These groups are the automorphism groups  $\text{Aut}(X)$  of topological spaces. Recall that a group action on  $X$  is a homomorphism  $\alpha: G \rightarrow \text{Aut}(X)$ . Examples of group actions on spaces include  $\mathbb{Z}$  acting on  $\mathbb{R}$  by translation, or  $\mathbb{Z}_2$  acting on  $S^2$  by the antipodal map.

However, recall that group actions induce equivalence classes (orbits) on the topological space and thus induces a quotient topology  $A/\alpha$ . The quotient topology of the examples of the actions above give rise to  $S^1$  and  $\mathbb{RP}^2$ .

# 3 FUNCTIONS INTO THE REALS

The following are properties of topological spaces that can kind of categorize how “nice” they are.

**Definition 3.1** (Hausdorff ( $T_2$ )). A space  $X$  is Hausdorff if for all  $x \neq y$ , there exist disjoint open sets  $U, V$  such that  $x \in U$  and  $y \in V$ .

**Definition 3.2** (Normal ( $T_4$ )). A space  $X$  is Hausdorff if for all pairs of disjoint closed subsets  $C_1, C_2$ , there exist disjoint open sets  $O_1, O_2$  such that  $C_1 \subseteq O_1$  and  $C_2 \subseteq O_2$ .

## 3.1 URYSOHN’S LEMMA

It can easily be seen that all metric spaces are Hausdorff, but it is also the case that metric spaces are normal. How do we prove this? We first consider the following important theorem:

**Theorem 3.1** (Urysohn’s Lemma). If  $X$  is a normal space, then for any disjoint closed subsets  $C_0, C_1$  there exists a continuous function  $f: X \rightarrow [0, 1]$  where  $f(C_0) = 0$  and  $f(C_1) = 1$ .

The converse of [Urysohn’s Lemma](#) also holds, we simply take  $O_0 = f^{-1}([0, 1/3))$  and  $O_1 = f^{-1}((2/3, 1])$ . We prove that metric spaces satisfy this property:

**Proposition 3.2.** Metric spaces are normal.

**Definition 3.3** (Distance from a set). For a metric space  $(X, d)$ , we define  $d_A(x)$  or  $d(x, A)$  where  $A \subseteq X$ , as  $\inf_{a \in A} d(x, a)$ .

It is easily seen that this function is Lipschitz with constant  $\leq 1$ .

Now, we can proceed with a proof of [Proposition 3.2](#).

*Proof.* Let  $C_0, C_1$  be disjoint closed subsets in  $(X, d)$ , then define

$$f: X \rightarrow [0, 1], x \mapsto \frac{d(x, C_0)}{d(x, C_0) + d(x, C_1)}.$$

It is clear that  $f(C_0) = 0$  and  $f(C_1) = 1$ , and that  $f$  is continuous. □

Now, we continue with a proof of [Urysohn’s Lemma](#).

**Lemma 3.3.** If  $X$  is normal, then for any closed set  $C$  and open set  $O$  containing  $C$ , there is an open set  $U$  “between”  $C$  and  $O$ , that is,  $C \subseteq U \subseteq \bar{U} \subseteq O$ .

*Proof.* If  $D = X \setminus O$ , then  $D$  is closed and disjoint from  $C$  and then we can find disjoint open sets  $U, V$  such that  $U$  contains  $C$  and  $V$  contains  $D$ . In addition, the closure of  $C$  must be contained within  $X \setminus V$  which is contained by  $U$ .  $\square$

*Proof of Urysohn's Lemma.* Given  $X$  normal and disjoint closed sets  $C_0, C_1$ , set  $O_1 = C_1'$ , then apply the preceding lemma to get a set  $O_{1/2}$  "between"  $C_0$  and  $O_1$ . Applying the lemma again, we then get a two open sets  $O_{1/4}, O_{3/4}$  s.t.  $C_0 \subseteq O_{1/4} \subseteq \overline{O_{1/4}} \subseteq O_{1/2} \subseteq \overline{O_{1/2}} \subseteq O_{3/4} \subseteq \overline{O_{3/4}} \subseteq O_1$ . We can repeat this process to choose  $O_{1/8}, O_{3/8}, O_{5/8}, O_{7/8}$ , and then repeat the process over and over again.

Now, we are able to define  $O_d$ , for all  $d \in \Delta$  (rational numbers of the form  $k/2^n$  where  $k \leq 2^n$ ), such that if  $r < s$ ,  $\overline{O_r} \subseteq O_s$ . In addition, for all  $r$ ,  $C_0 \subseteq O_r$ . Define  $f$  on  $X$  by  $f(x) = 1$  if  $x \in C_1$ , otherwise, we set  $f(x) = \inf \{r \in \Delta : x \in O_r\}$ . We can see that  $f(C_0) = 0$  in this case, since  $r$  is dense in  $[0, 1]$ .

Now, to prove continuity, we choose a subbase of  $[0, 1]$  consisting of sets of the form  $(0, a]$  and  $(b, 1]$ . We then claim that

$$f^{-1}([0, a)) = \bigcup_{r < a} O_r, \text{ and } f^{-1}((b, 1]) = \bigcup_{s > b} (\overline{O_s})'$$

This can be seen to be true due to density of  $\Delta$ , as you can always find a dyadic rational between  $f(x)$  and either  $a$  or  $b$ .  $\square$

### 3.2 TIETZE EXTENSION THEOREM

If  $X$  is a set and  $(Y, d)$  is a metric space, then let  $B(X, Y)$  be the set of bounded functions from  $X$  to  $Y$ . Then, we can define a metric  $\tilde{d}$  on  $B(X, Y)$  as  $\tilde{d}(f, g) = \sup_x |f(x) - g(x)|$ .

**Proposition 3.4.** If  $Y$  is complete, then  $B(X, Y)$  is complete.

*Proof.* If  $f_n$  is a Cauchy sequence then  $f_n(x)$  is a Cauchy sequence for all  $x$ , and we can define  $f(x) = \lim f_n(x)$ .  $\square$

**Proposition 3.5.** If we assume that  $X$  is instead of a topological space, then the set of continuous bounded functions  $BC(X, Y)$  is complete if  $Y$  is complete.

The following is an important result using [Urysohn's Lemma](#).

**Theorem 3.6** (Tietze Extension Theorem). If  $X$  is a normal space and  $A$  is a closed subset of  $X$ , and if  $f: A \rightarrow \mathbb{R}$  is continuous, then there exists a continuous extension  $g: X \rightarrow \mathbb{R}$  such that  $g|_A = f$ . Furthermore, if  $f$  maps  $A$  to  $[a, b]$ , then  $g$  maps  $X$  to that same interval.

Note that this theorem uses special properties of  $\mathbb{R}$ . For example, there is no continuous extension of the identity function from the unit circle to itself to the closed unit ball. To prove this theorem, we first prove a lemma.

**Lemma 3.7.** If  $X$  is normal and  $A$  is a closed subset of  $X$ , if there is an  $h$  mapping  $A$  to  $[0, r]$ , then there is a  $g: X \rightarrow [0, r/3]$  such that  $0 \leq h - g|_A \leq 2r/3$ .

*Proof.* If  $B = \{x \in A \mid h(x) \leq r/3\}$ , and  $C = \{x \in A \mid h(x) \geq 2r/3\}$ . Then, we can find a function  $g: X \rightarrow [0, r/3]$  such that  $g|_B = 0$  and  $g|_C = r/3$  by [Urysohn's Lemma](#).  $\square$

*Proof of Tietze Extension Theorem.* We prove it for the special case  $[0, 1]$ , since all finite intervals are homeomorphic to the unit interval, this will hold for all finite intervals. To start, we can find a function  $g^1: [0, 1/3]$  such that  $0 \leq f - g|_A \leq 2/3$ .

We then apply the lemma repeatedly: We apply it to  $f - g|_A$  to get a function  $g^2: X \rightarrow [0, 1/3(2/3)]$  such that  $0 \leq f - g|_A - g|_A^2 \leq 4/9$ .

Repeat with  $f - g|_A - g|_A^2$  to get  $g^3$ , and in general we repeat this process with  $f - g|_A - \dots - g|_A^n: A \rightarrow [0, (\frac{2}{3})^n]$  to get a function  $g^{n+1}: [0, \frac{1}{3}(\frac{2}{3})^n]$  such that  $f - g|_A - \dots - g|_A^{n+1}: A \rightarrow [0, (2/3)^{n+1}]$ .

Now we have  $\|g^n\|_\infty \leq \frac{1}{3}(\frac{2}{3})^{n-1}$ , so  $\sum_{k=1}^n g^k$  is a Cauchy sequence and thus converges to  $g = \sum g^n$ , and we get that  $\|g\|_\infty \leq 1$  by summing up the geometric series. We see that  $g = f$  because

$$\|f - g|_A\| = \lim \left\| f - \sum_{k=1}^n g|_A^k \right\|.$$

$\square$

For infinite intervals, we use the homeomorphism  $h$  from  $\mathbb{R}$  to  $(0, 1)$ , and we set  $g = h \circ f: A \rightarrow (0, 1)$ . Now if we apply Tietze's extension theorem we get  $\tilde{g}: X \rightarrow [0, 1]$ , and let  $D = \tilde{g}^{-1}(0)$ , then  $D$  is disjoint from  $A$  and closed. We can take  $U = \tilde{g}^{-1}([0, 1/4)) \cap A'$ . By [Urysohn's Lemma](#) we can find a function that takes  $k: X \rightarrow [0, 1]$  where  $k(D) = 1/8$  and  $k(U') = 0$ , and we may add  $\tilde{g} + k$  to obtain a continuous function whose output is not 0. Repeat this process for the point 1 and we get a final function that maps into  $(0, 1)$ .

### 3.3 COMPLETION OF METRIC SPACES

Recall that if  $X$  is a topological space then  $BC(X, \mathbb{R})$  with the supremum norm is complete. This is an example of the following.

**Definition 3.4** (Banach space). A complete normed vector space is called a Banach space.

We can use this fact to define a completion of a metric space  $Y$ .

**Lemma 3.8.** If  $(Y, d_Y)$  and  $(M, d_M)$  are metric spaces and  $M$  is complete, then if  $f: Y \rightarrow M$  is an isometry, it can be seen that  $\overline{f(Y)}$  is a completion of  $M$ .

**Theorem 3.9.** All metric spaces have a completion.

*Proof.* Given  $(Y, d)$ , we let  $f_y(y') = d(y, y')$ . This is unbounded, but if we choose  $y_0 \in Y$  be any base point, we can define  $h_y = f_y - g$ , where  $g = f_{y_0}$ . By the triangle inequality,  $h_y$  is bounded.

We now show that  $y \mapsto h_y$  is an isometry, as  $(h_{y_1} - h_{y_2})(y) = (f_{y_1} - f_{y_2})(y)$  so the norm is at most  $d(y_1, y_2)$ . In addition, if we plug in  $y_1$ , we get  $R(f_{y_1} - f_{y_2})(y_1) = d(y_1, y_2)$ , so  $\|h_{y_1} - h_{y_2}\| = d(y_1, y_2)$ . According to the above lemma, we have a completion of  $Y$ .  $\square$

# 4 COMPACTNESS

**Definition 4.1** (Open Cover, Subcover). If  $X$  is a topological space, by an open cover we mean a collection  $\mathcal{C} = \{U_\alpha\}$  of open sets such that  $\bigcup \mathcal{C} = X$ .

By a subcover of  $\mathcal{C}$ , we mean a collection  $\mathcal{D} \subseteq \mathcal{C}$  where  $\bigcup \mathcal{D} = X$ .

**Definition 4.2** (Compactness).  $X$  is compact if and only if every open cover has a finite subcover. If  $A \subseteq X$ , then  $A$  is compact if  $A$  is compact in its relative topology.

**Proposition 4.1.**  $A$  is compact iff for any collection  $\mathcal{E}$  of open sets such that  $\bigcup \mathcal{E} \supseteq A$ , there is a finite subset  $\mathcal{S} \subseteq \mathcal{E}$  that also covers  $A$ , that is,  $\bigcup \mathcal{S} \supseteq A$ .

**Proposition 4.2.** Any closed subset of a compact space is compact.

*Proof.* If  $\mathcal{E}$  covers  $A \subseteq X$ , then  $\mathcal{E} \cup \{A'\}$  covers  $X$ , so we find a finite subcover and take out  $A'$ . □

Note that the converse is not true: Any subset of a space with the indiscrete topology is compact. However, when  $X$  is Hausdorff, then this holds:

**Proposition 4.3.** If  $X$  is Hausdorff, then if  $A \subseteq X$  is compact, then for all  $x \notin A$ , there exist disjoint open sets  $U$  and  $V$  such that  $A \subseteq U$  and  $x \in V$ .

*Proof.* If  $x$  is in the complement of  $A$ , for all  $y \in A$  we can find disjoint open sets  $U_y, V_y$  with  $x \in U_y$  and  $y \in V_y$ . Then  $\{V_y\}$  covers  $A$ , and we can find a finite subcover  $V_{y_k}, 1 \leq k \leq n$ . Then, we have  $\bigcap U_{y_k}$  is an open set that contains  $x$  and does not intersect  $\bigcup V_{y_k}$ , which is an open set containing  $A$ . □

We get two immediate corollaries:

**Corollary 4.4.** Any compact subset of a Hausdorff space is closed.

**Corollary 4.5.** Any compact Hausdorff space is regular ( $T_3$ ), i.e. for any closed set  $A \subset X$ , and any  $x \notin A$ , we have disjoint open sets  $U, V$  such that  $A \subseteq U$  and  $x \in V$ .

We can prove a slightly stronger statement:

**Corollary 4.6.** Any compact Hausdorff space is normal ( $T_4$ ).

*Proof.* If  $A$  and  $B$  are disjoint closed sets in a compact Hausdorff space  $X$ , then for any  $y \in B$  there are disjoint open sets  $U_y, V_y$  such that  $A \subseteq U_y$  and  $y \in V_y$ , and we can find a finite subcover  $V_{y_k}$  of the open cover  $\{V_y\}$  of  $B$ . Now, we let  $U = \bigcap U_{y_k}$  and  $V = \bigcup V_{y_k}$  and we have  $U$  disjoint from  $V$ , where  $B \subseteq V$  and  $A \subseteq U$ . □

Now, we examine maps on compact spaces.



**Proposition 4.7.** If  $X$  is compact and  $f: X \rightarrow Y$ , then  $f(X)$  is compact.

*Proof.* WLOG assume that  $Y = f(X)$ . If  $\mathcal{C}$  is an open cover of  $f(Y)$ , then if  $\mathcal{C}'$  is the set of preimages of sets in  $\mathcal{C}$ , then  $\mathcal{C}'$  is an open cover of  $X$ , which we can take a finite subcover  $\mathcal{S}' = \{f^{-1}(O_1), \dots, f^{-1}(O_n)\}$ , then we have that  $\mathcal{S} = \{O_1, \dots, O_n\}$  is a finite subcover of  $Y$ .  $\square$

**Corollary 4.8** (Extreme Value Theorem). If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous on an interval  $[a, b]$ , then  $f$  has a maximum on  $[a, b]$  and attains it on  $[a, b]$ .

**Proposition 4.9.** If  $f: X \rightarrow Y$  for  $X$  compact and  $Y$  Hausdorff, then if  $f$  is bijective, then it is a homeomorphism.

*Proof.* Images of closed sets are closed due to [Proposition 4.7](#) and [Corollary 4.4](#).  $\square$

We can give an alternative characterization of compactness using closed sets.

**Proposition 4.10.**  $X$  is compact iff for any collection of closed sets has the finite intersection property, the intersection is nonempty.

## 4.1 TYCHONOFF'S THEOREM

We first state Tychonoff's theorem:

**Theorem 4.11** (Tychonoff's Theorem). If  $\{X_\alpha\}$  are a collection of compact spaces, then the space  $\prod_\alpha X_\alpha$  is compact.

It is clear that this requires the Axiom of Choice, as  $\prod_\alpha X_\alpha$  may be empty without it. It is also interesting that Tychonoff's Theorem implies AC, although Tychonoff's Theorem for Hausdorff compact spaces does not imply AC.

The proof of this theorem also requires AC, in the form of

**Theorem 4.12** (Zorn's Lemma). If  $P$  is a poset, then if any chain  $C$  (totally ordered subset) in  $P$  has an upper bound in  $P$ , then  $P$  has a maximal element. (We call  $P$  inductively ordered).

[Zorn's Lemma](#) has wide applications across many fields, such as in the proof that every vector space has a basis, or that every proper ideal is contained in a maximal ideal.

**Definition 4.3.** For any set  $X$ , a filter on  $X$  is a collection  $\mathcal{F}$  of nonempty subsets that is closed under intersections, and where all supersets of a set in the filter is also in the filter.

An ultrafilter is a maximal filter.

*Proof of [Tychonoff's Theorem](#).* If  $\alpha \in \mathcal{A}$  indexes a family of compact spaces  $X_\alpha$ , then if  $X = \prod X_\alpha$ , then let  $\mathcal{C}$  be a collection of closed subsets of  $X$  with the finite intersection property. If  $W_{\mathcal{C}}$  is the collection of families  $\mathcal{D}$  of subsets of  $X$  such that  $\mathcal{C} \subseteq \mathcal{D}$  and such that  $\mathcal{D}$  has FIP.

Now, we show that  $W_{\mathcal{C}}$  is inductively ordered: If  $\Phi$  is a chain in  $W_{\mathcal{C}}$ , then if  $\mathcal{D}_\Phi = \bigcup \Phi$ , then let  $A_1, \dots, A_n \in \mathcal{D}_\Phi$ , then  $A_i \in \mathcal{D}_{\alpha_i}$ , which we can just find the maximal index out of the  $\alpha_i$ , which we call  $\alpha_0$ . Then,  $A_i \in \mathcal{D}_{\alpha_0}$  for all  $i$ , which proves that  $\bigcap A_i$  has nonempty intersection.

We can then use Zorn's Lemma to obtain a maximal element  $\mathcal{D}^*$  in  $W_{\mathcal{C}}$ . Now, we can see that  $\mathcal{D}^*$  is closed under finite intersections because it is maximal. In addition, we also have that if

$B \cap A \neq \emptyset$  for all  $A \in \mathcal{D}^*$ , then  $B \in \mathcal{D}^*$ , again due to maximality (we can use the previous property to prove this statement).

Now, for any  $\alpha \in \mathcal{A}$ , we have the continuous projection  $\pi_\alpha : X \rightarrow X_\alpha$ , now we claim that for any  $\mathcal{D} \in W_c$ , the set of  $\{\pi_\alpha(A) \mid A \in \mathcal{D}\}$  has FIP since if  $x \in \bigcap A_j$ , then  $\pi_\alpha(x) \in \bigcap \pi_\alpha(A_j)$ . We also trivially have that the closures  $\overline{\pi_\alpha(A)}$  has the FIP. Now, applying to  $\mathcal{D}^*$ , we have that  $\bigcap_{A \in \mathcal{D}^*} \overline{\pi_\alpha(A)}$  is nonempty since  $X_\alpha$  is compact.

Now, applying AC, we can pick  $x \in X$  where  $\pi_\alpha(x) \in \bigcap_{A \in \mathcal{D}^*} \overline{\pi_\alpha(A)}$ . Now, we prove that  $x \in \overline{A}$  for all  $A \in \mathcal{D}^*$ , using the subbase of the product topology. If we let  $U = \pi_\alpha^{-1}(U_\alpha)$ , then  $x \in U$  implies that  $U \cap A \neq \emptyset$  for all  $A \in \mathcal{D}^*$ , since we have  $x_\alpha \in \overline{\pi_\alpha(A)}$  and  $U_\alpha$ , then  $\pi_\alpha(A) \cap U_\alpha$  must be nonempty otherwise  $\overline{\pi_\alpha(A)}$  would not contain  $x_\alpha$ . This implies that  $\pi^{-1}(U_\alpha)$  has nonempty intersection with any set in  $\mathcal{D}^*$ , and thus belongs to  $\mathcal{D}^*$ .

Now, by the first property of  $\mathcal{D}^*$ , then we have that finite intersections  $\bigcap \pi_\alpha^{-1}(U_{\alpha_i})$  are in  $\mathcal{D}^*$ , thus  $\mathcal{D}^*$  contains a neighborhood base of  $\mathcal{D}^*$ . Now, if  $x \in U$  open in  $X$ , then there exists an open set  $V$  in  $\mathcal{D}^*$ . Thus, for all  $A \in \mathcal{D}^*$ , we have that  $x \in \overline{A}$ . Finally, this results in the fact that  $x$  is contained in  $\bigcap \mathcal{C}$ .  $\square$

This theorem has applications in 202B, such as the Aoglu theorem (unit ball in weak-\* topology is compact), and when considering locally compact spaces and  $\beta$ -compactifications.

Now, we consider the following interesting result

**Theorem 4.13** (Kelley). Tychonoff's Theorem is equivalent to AC.

*Proof.* The reverse implication was proven above, and now if we assume Tychonoff's theorem, we need to show that  $\prod X_\alpha$  is nonempty. We can take  $\omega = \bigcup X_\alpha$  which is not in  $\bigcup X_\alpha$  due to regularity. Now, if  $Y_\alpha = X_\alpha \cup \{\omega\}$ , we can take  $\mathcal{T}_\alpha = \{Y_\alpha, \emptyset, T_\alpha, \{\omega\}\}$ , which is compact since there are finitely many open sets. The product  $Y = \prod Y_\alpha$  is compact due to Tychonoff's theorem. If  $F_\alpha = \pi_\alpha^{-1}(X_\alpha)$ , then  $F_\alpha$  is closed, and has the FIP, since for finite intersections  $\bigcap F_{\alpha_i}$ , we can consider  $x_{\alpha_i} = F_{\alpha_i}$ , and  $x_\alpha = \omega$  otherwise. Thus due to compactness we have  $\bigcap F_\alpha = \prod X_\alpha \neq \emptyset$ , which proves the equivalence.  $\square$

## 4.2 TOTALLY BOUNDED SETS

**Definition 4.4.** We say a metric space  $(X, d)$  is **totally bounded** if for all  $\varepsilon > 0$ , there exists a finite set  $\{x_i\}$  such that

$$X = \bigcup_{i=1}^n B_\varepsilon(x_i).$$

It is trivial that compactness implies totally bounded. We now prove a partial converse:

**Theorem 4.14.** Complete and totally bounded metric spaces are compact.

*Proof.* If  $X$  is totally bounded, then if  $\{x_n\}$  is a sequence in  $X$ , we have for any  $\varepsilon > 0$  that  $X = \bigcup B_\varepsilon(x_i)$ . Due to finiteness, for at least one  $x_i$ , we have infinitely many points in the sequence in  $B_\varepsilon(x_i)$ . Taking  $\varepsilon = 1$ , we have  $A_1 = B_1(x_1)$  which has infinitely many points of  $x_n$  inside, which means we can choose  $A_{1/2} = B_{1/2}(x_2) \cap A_1$  with infinitely many points of  $x_n$  inside, and we

can repeat this process to obtain a sequence  $A_1 \supseteq A_{1/2} \supseteq A_{1/4} \supseteq \dots$  with successively decreasing diameters. By completeness, there exists an (unique) point in the intersection, and thus we have found an accumulation point.  $\square$

**Corollary 4.15** (Heine-Borel Theorem). A subset of Euclidean space is compact iff it is closed and bounded.

We can now study compact sets in  $BC(X, M)$  with the  $d_\infty$  metric, where  $X$  is a topological space and  $M$  is a complete metric space. These correspond precisely to the totally bounded sets.

**Definition 4.5.** If  $X$  is a topological space and  $(M, d)$  is a metric space, then a family of functions  $\mathcal{F} \subseteq C(X, M)$  is **equicontinuous at a point**  $x$  if for any  $\varepsilon > 0$ , there exists an open neighborhood  $\mathcal{O}$  such that  $d(f(y), f(x)) < \varepsilon$  whenever  $y \in \mathcal{O}$ .

We say  $\mathcal{F}$  is **equicontinuous** if it is equicontinuous at all points in  $X$ .

**Theorem 4.16** (Arzela-Ascoli). Let  $X$  and  $M$  be as above, and let  $X$  be compact. If  $\mathcal{F} \subseteq BC(X, M)$  with  $\mathcal{F}$  pointwise totally bounded, and equicontinuous, then  $\mathcal{F}$  is totally bounded.

*Proof.* If  $\varepsilon > 0$  is given, for each  $x$  there is an open set  $O_x$  such that  $y \in O_x$ , then  $d(f(y), f(x)) < \varepsilon/4$ . By compactness, there are  $x_1, \dots, x_n$  such that

$$X = \bigcup_{i=1}^n O_{x_i}.$$

Now, for each  $j$ , the set  $I_j = \{f(x_j) \mid f \in \mathcal{F}\}$  is totally bounded, so if we choose  $S_j$  to be a finite subset of  $I_j$  that is  $\varepsilon/4$ -dense in  $I_j$ . We can then take  $S = \bigcup S_j$ . Now, if we let  $\Psi = \{\psi: \{1, \dots, n\} \rightarrow S\}$ , which is finite, then we can define  $\mathcal{F}_\psi = \{f \in \mathcal{F} \mid \forall j. d(f(x_j), \psi(j)) < \varepsilon/4\}$ . We thus have  $\mathcal{F} = \bigcup \mathcal{F}_\psi$ .

Now, we prove that  $\text{diam}(\mathcal{F}_\psi) < \varepsilon$ . For all  $y$ , we have that  $y \in O_{x_i}$  for some  $i$ . Thus, if  $f, g \in \mathcal{F}_\psi$ , then

$$d(f(y), g(y)) \leq d(f(y), f(x_i)) + d(g(y), g(x_i)) + d(f(x_i), \psi(x_i)) + d(g(x_i), \psi(x_i)) < 4(\varepsilon/4) = \varepsilon.$$

Then,  $\mathcal{F}_\psi$  is contained within  $B_\varepsilon(f_\psi)$  for any  $f_\psi \in \mathcal{F}_\psi$ , and so  $X = \bigcup B_\varepsilon(f_\psi)$ .  $\square$

### 4.3 LOCALLY COMPACT SPACES

**Definition 4.6.** A topological space  $X$  is **locally compact** if for all  $x \in X$ , there exists a open set  $x \in O$  such that  $\overline{O}$  is compact.

For now, we assume that  $X$  is Hausdorff for simplicity.

**Proposition 4.17.** If  $X$  is locally compact, then for any point and open set  $x \in O$ , there exists an open set  $x \in U \subseteq \overline{U} \subseteq O$ .

*Proof.* Let  $O' = O \cap O_1$ , where  $\overline{O_1}$  is compact. Now, due to regularity of  $\overline{O'}$  which is also compact, and disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $\partial O \subseteq V$ . Then  $\overline{U}$  does not contain any points in  $\partial O$  and thus  $\overline{U} \subseteq O' \subseteq O$ . Furthermore,  $\overline{U}$  is compact due to being a closed subset of  $\overline{O'}$ .  $\square$

A stronger proposition is the following:

**Proposition 4.18.** If  $X$  is locally compact, then for any compact subset of an open set  $C \subseteq O$ , there exists an open set  $C \subseteq U \subseteq \overline{U} \subseteq O$ .

*Proof.* This just follows by taking points in  $C$  and applying the previous proposition, then using compactness.  $\square$

**Definition 4.7.** For  $f \in C(X, V)$  where  $V$  is a vector space and  $X$  is locally compact, the **support** of  $f$ ,  $\text{supp}(f)$  is defined as the closure of the set  $\{x: f(x) \neq 0\}$ .

**Definition 4.8.** We say that  $f$  has **compact support** if  $\text{supp}(f)$  is compact. We denote the set of continuous compactly-supported functions by  $C_c(X, V)$ .

**Proposition 4.19.** For  $X$  locally compact and  $C$  compact,  $O$  open, then there is a function  $f \in C_c(X, \mathbb{R})$  where  $f(C) = 1$  and  $f(O') = 0$ .

*Proof.* Find  $U$  such that  $C \subseteq U \subseteq \overline{U} \subseteq O$  and apply Urysohn's Lemma.  $\square$

**Definition 4.9.**  $C_b(X, \mathbb{R})$  denotes the set of all bounded functions  $X \rightarrow \mathbb{R}$ . Using the  $L^\infty$  norm, we have that the closure of  $C_c(X, \mathbb{R})$  is  $C_\infty(X, \mathbb{R})$ , which are the set of functions that vanish at infinity.

## 4.4 A QUICK $C^*$ -ALGEBRA INTERLUDE

We start with the following proposition:

**Proposition 4.20.** If  $X$  is a compact Hausdorff space, then the only unital homomorphisms between  $C(X)$  and  $\mathbb{K}$  are the evaluation homomorphisms  $\varphi_x(f) = f(x)$ .

If  $A$  is any  $\mathbb{K}$ -algebra, then by  $\hat{A}$  we mean the set of unital homomorphisms  $A \rightarrow \mathbb{K}$ . If  $a \in A$ , we define  $\hat{a}$  to be the function  $\hat{a}: \hat{A} \rightarrow \mathbb{K}$  to be  $\hat{a}(\varphi) = \varphi(a)$ . We give  $\hat{A}$  the weakest topology that makes these functions continuous, that is, the initial topology generated by the  $\hat{a}$ .

**Proposition 4.21.** For  $A = C(X)$ , we have that  $x \mapsto \varphi_x$  is a homeomorphism from  $X$  to  $\hat{A}$ .

If  $K = \mathbb{C}$ , then  $A = C(X)$  has an involution, where  $f^*(x) = \overline{f(x)}$ . It is obvious that  $\|f^*f\|_\infty = \|f\|_\infty^2$ . This seemingly innocent property actually characterizes  $C(X)$  with the  $\infty$ -norm:

**Theorem 4.22** (Little Gelfand-Naimark theorem, 1943). If  $A$  is a commutative Banach algebra with involution. If  $\|a^*a\| = \|a\|^2$  for all  $a \in A$ , then  $A$  is isomorphic (isometric  $*$ -isomorphism) to  $C(\hat{A})$ .

Now if we consider  $X$  locally compact, then we let  $A = C_\infty(X)$ . Similarly to the cases above,  $C_\infty(X) \cong C_\infty(\hat{A})$  and  $X \cong \hat{A}$ .

**Definition 4.10.** A **Hilbert space** is a inner product space that is complete under its induced norm.

We write  $B(\mathcal{H})$  to be the set of all continuous operators on  $\mathcal{H}$ .

$B(\mathcal{H})$  has an involution, the adjoint, and  $\|T^*T\| = \|T\|^2$ . However, it isn't commutative, but this motivates the general form of the Gelfand-Naimark theorem, where any space satisfying this property is isomorphic in a sense to  $B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .

Algebras that satisfy this property are called  $C^*$ -algebras.

## PART II

## MEASURE AND INTEGRATION

# 5 BUILDING MEASURES

Previously, we've considered the  $L^p$  norms on  $C([0, 1])$  and shown that it is not complete under these norms. In order to complete  $C([0, 1])$ , we need to define the integral on characteristic functions  $\chi_E$  for many sets  $E \subseteq [0, 1]$ , as they can be approximated by trapezoids.

To do this, we want to be able measure the sizes of sets “consistently.” By “consistent”, we mean that it satisfies a few properties that are intuitively true. For example, if  $\mu$  is a function that “measures” the sizes of sets, then if  $E$  and  $F$  are two disjoint sets, then intuitively,  $\mu(E \sqcup F)$  should be  $\mu(E) + \mu(F)$ .

## 5.1 FAMILIES AND MEASURES

**Definition 5.1.** If  $X$  is a set, a **ring** of subsets is a collection  $R \subseteq 2^X$  such that

1. If  $E, F \in R$ , then  $E \cup F \in R$ .
2. If  $E, F \in R$ , then  $E \setminus F \in R$ .

This additionally implies that  $\emptyset$  and  $E \cap F = E \cup F \setminus (E \Delta F)$  is in  $R$ .

A **finitely additive measure** on  $R$  is a function  $\mu: X \rightarrow [0, \infty]$  where  $\mu(E \sqcup F) = \mu(E) + \mu(F)$ .

It is obvious that for a finitely-additive measure we have that  $\mu(\emptyset) = 0$ .

However, rings and finitely-additive measures aren't good enough, because of the key word *finite*. If  $E_n$  is a sequence of disjoint intervals  $[a, b)$ , we define  $\mu(E_n) = [a, b)$ . We consider that

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty$$

and define

$$F_n = \bigsqcup_{k=1}^n E_k,$$

we see that  $\chi_{F_n}$  is a Cauchy sequence, which “wants to” converge to  $\chi_E$ , where  $E = \bigcup E_k$ . Thus it makes sense to define

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n),$$

and by doing so, we introduce countable additivity.

**Definition 5.2.** A  **$\sigma$ -ring** is a ring that is closed under countable unions.

A  **$\sigma$ -algebra** (or  $\sigma$ -field) is a  $\sigma$ -ring with the whole  $X$  in it.

Similarly to topologies, we can order  $\sigma$ -rings (or  $\sigma$ -algebras) by inclusion, and we have the following proposition:

**Proposition 5.1.** If  $\{\mathcal{S}_\alpha\}$  is any (arbitrarily large) collection of  $\sigma$ -rings ( $\sigma$ -algebras) on  $X$ , then the intersection  $\bigcap \mathcal{S}_\alpha$  is a  $\sigma$ -ring (resp.  $\sigma$ -algebra) on  $X$ .

As a corollary we have that

**Corollary 5.2.** Any collection  $\mathcal{C} \in 2^X$  has a smallest  $\sigma$ -ring (or  $\sigma$ -algebra) containing it.

We call this  $\sigma$ -ring  $\mathcal{S}(\mathcal{C})$ , and we say that it is generated by  $\mathcal{C}$ .

**Example 5.1.** If  $X$  is any set, the set of finite and countable subsets of  $X$  form a  $\sigma$ -ring on  $X$ . This is also the  $\sigma$ -ring generated by singletons.

**Definition 5.3.** A (countably additive) **measure** on a  $\sigma$ -ring  $\mathcal{S}$  is a function  $\mu: \mathcal{S} \rightarrow [0, \infty]$  such that for all disjoint families of sets  $E_1, E_2, \dots \in \mathcal{S}$ , we have

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

One may ask why we only consider countable additivity and not uncountable additivity. Intuitively, on the real line, the length (size) of any singleton  $\{x\}$  should be 0, as it is smaller than any interval of positive length. However, the interval  $[0, 1]$  is an uncountable disjoint union of these singletons, so uncountable additivity would imply that the length of  $[0, 1]$  would be 0.

Some examples of measures are below:

**Example 5.2 (Counting Measure).** For a space  $X$ , we can let  $\mathcal{S}$  be the set of all countable subsets of  $X$ , and we can define a measure on  $\mathcal{S}$  by  $\mu(E) = \#(E)$ , which is  $\infty$  if  $E$  has infinite cardinality.

**Example 5.3.** The counting measure is a special case of the measure defined as

$$\mu_f(E) = \sum_{x \in E} f(x)$$

where  $f: X \rightarrow [0, \infty]$ .

The reason why this is countably additive is due to the ability to switch the order of summation when it is absolutely convergent.

**Example 5.4.** Another special case is when  $X$  itself is countable and

$$\sum_{x \in X} f(x) = 1.$$

This gives rise to discrete probability measures and distributions, and  $f$  is in this case known as the probability mass function.

As an example, taking  $X = \{0, 1, \dots\}$  and  $f(n) = p(1-p)^{n-1}$  gives us the geometric distribution with parameter  $p$ .

## 5.2 PRERINGS AND PREMEASURES

However, directly defining measures for more complicated  $\sigma$ -rings is very difficult. For instance, it would be very difficult to explicitly define a measure on the  $\sigma$ -ring (which happens to be a  $\sigma$ -



algebra) generated by the intervals  $[a, b)$  (we will see why we choose one end to be open later) in  $\mathbb{R}$ .

Thus, we try to define measures on smaller objects and then extend them to the  $\sigma$ -rings generated by those objects, analogously to how we define a function on a basis of a vector space and extend it to the entire vector space. These smaller objects are called prerings.

**Definition 5.4.** A **prering** (or semiring) on a set  $X$  is a nonempty collection of subsets  $\mathcal{P} \subseteq 2^X$  satisfying the following properties:

1. If  $E, F \in \mathcal{S}$ , then  $E \cap F \in \mathcal{S}$ .
2. If  $E, F \in \mathcal{S}$ , then there exists disjoint sets  $E_1, \dots, E_n \in \mathcal{S}$  such that

$$E \setminus F = \bigsqcup_{i=1}^n E_i.$$

**Definition 5.5.** A countably additive set function  $\mu: \mathcal{P} \rightarrow [0, \infty]$  is called a **premeasure**.

**Example 5.5.** As above, the sets  $[a, b)$  form a prering on the reals. The proof of this is simple but tedious.

The function  $\mu([a, b)) = b - a$  is a premeasure. The proof of this is less simple and even more tedious.

**Example 5.6.** If  $\alpha: \mathbb{R} \rightarrow \mathbb{R}$  is a nondecreasing function, that is left-continuous, then  $\mu_\alpha([a, b)) = \alpha(b) - \alpha(a)$  is a premeasure. We see that the preceding example is a special case of this.

Another special case is when  $\alpha(x) \rightarrow 0$  as  $x \rightarrow -\infty$ , and  $\alpha(x) \rightarrow 1$  as  $x \rightarrow \infty$ . This gives rise to continuous probability distributions on  $\mathbb{R}$ .

**Proposition 5.3.**  $\mu_\alpha$  is an example of a premeasure.

*Proof.* To show that this is indeed a premeasure, we consider  $[a_0, b_0) = \bigsqcup [a'_n, b'_n)$  and for all  $\varepsilon$  we set  $\varepsilon_j = 2^{-n-1}\varepsilon$ . Then, we pick  $b' < b$  such that  $\alpha(b') + \varepsilon/2 > \alpha(b)$ . Now, for each  $j$ , choose  $a'_j < a_j$  such that  $\alpha(a'_j) + \varepsilon_j > \alpha(a_j)$ . Then,  $(a'_j, b'_j)$  form an open covering of  $[a_0, b'_0]$ . We can then find a finite subcover  $\mathcal{C}$ . We want to reorder and renumber inductively the sets in this interval, to make the sets disjoint members of the prering. As a first step, we choose  $S = (a'_1, b_1) \in \mathcal{C}$  such that  $a_0 \in S$ , and we call it  $S_1$ . Now, there's an interval containing  $b_1$ , and we do a similar process until we run out of sets. Now,

$$\sum_{j=1}^n \alpha(b_j) - \alpha(a'_j) = -\alpha(a_1) + \sum_{j=1}^{n-1} \alpha(b_j) - \alpha(a'_{j+1}) + \alpha(b_n) \geq \alpha(b_n) - \alpha(a'_1) > \alpha(b_0) - \alpha(a_0) - \varepsilon/2.$$

We also have that

$$\sum_{j=1}^n \alpha(b_j) - \alpha(a'_j) < \sum_{j=1}^n \alpha(b_j) - \alpha(a_j) + \varepsilon_j \leq \varepsilon/2 + \sum_{j=1}^{\infty} \alpha(b_j) - \alpha(a_j).$$

Combining these we get that

$$\mu_\alpha([a_0, b_0)) - \varepsilon/2 < \alpha(b_0) - \alpha(a_0) - \varepsilon/2 < \sum_{j=1}^n \alpha(b_j) - \alpha(a'_j) < \varepsilon/2 + \sum_{j=1}^{\infty} = \varepsilon/2 + \sum_{j=1}^n \mu_\alpha([a_j, b_j))$$

The fact that the sum of the measures is greater than or equal to the measure of the union follows from taking  $\varepsilon \rightarrow 0$ . The fact that the sum of the measures is at most the measure of the union follows from considering the partial sums.  $\square$

### 5.3 CARATHEODORY'S EXTENSION THEOREM

The main reason why we were interested in prerings and premeasures is because we can extend premeasures on prerings to measures on  $\sigma$ -rings.

The main star of the show is

**Theorem 5.4** (Caratheodory's Extension Theorem). If  $\mathcal{P}$  is a prering and  $\mu$  is a premeasure, then it extends to a measure  $\mu$  on  $\mathcal{S} = \mathcal{S}(\mathcal{P})$ .

A few lemmas first, before we can build up to the theorem:

**Lemma 5.5.** If  $E, F_1, \dots, F_n$  are all in  $\mathcal{P}$ , then  $E \setminus \bigcup F_k$  is the disjoint union of sets in  $\mathcal{P}$ .

*Proof.* Slightly messy, but pretty straightforward.  $\square$

This leads to a few important corollaries:

**Corollary 5.6.** If  $E \subseteq F$ , then  $\mu(E) \leq \mu(F)$ , i.e.  $\mu$  is monotone.

**Corollary 5.7.** If  $E = \bigcup F_k$ , then  $\mu(E) \leq \sum \mu(F_k)$ , i.e.  $\mu$  is (countably) subadditive.

*Proof.* Given a sequence  $F_k$  then we can “disjointize” it by defining

$$G_k = F_k \setminus \bigcup_{j=1}^{k-1} F_j.$$

Now we apply the preceding results.  $\square$

**Definition 5.6.** Let  $\mathcal{F}$  be any family of subsets of  $X$ , then  $\mathcal{H}(\mathcal{F})$  is the **hereditary**<sup>1</sup>  $\sigma$ -ring of sets that are countably covered by  $\mathcal{F}$ .

**Definition 5.7.** If  $\mathcal{F}$  is a family and  $\mu: \mathcal{F} \rightarrow [0, \infty]$ , then define  $\mu^*: \mathcal{H}(\mathcal{F}) \rightarrow [0, \infty]$  by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(E_j) : E_j \in \mathcal{F}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

For  $\mathcal{F}$  a prering and  $\mu$  a premeasure, then  $\mu^*$  is called the **outer measure** induced by  $\mu$ .

**Proposition 5.8.**  $\mu^*$  is countably subadditive and monotone, i.e.,  $\mu^*$  is an **outer measure**.

<sup>1</sup>Hereditary just means if  $E$  is in the family, then all subsets of  $E$  are also in the family.

*Proof.* Monotonicity is trivial, but for subadditivity, suppose  $B_1, \dots, B_k$  are in  $\mathcal{H}(\mathcal{F})$ , then for any  $\varepsilon$  we let  $\varepsilon_j = 2^{-j-1}\varepsilon$ . Now, for each  $B_j$  we have sets  $E_k^j$  such that  $B_j \subseteq \bigcup_k E_k^j$ , and  $\mu^*(B_j) + \varepsilon_j \geq \sum_k \mu(B_k^j)$ . Now, we have

$$\mu^*\left(\bigcup_j B_j\right) \leq \sum_j \sum_k \mu(E_k^j) \leq \sum_j \mu^*(B_j) + \varepsilon_j \leq \varepsilon + \sum_j \mu^*(B_j).$$

Since  $\varepsilon$  can be made arbitrarily small, we prove the desired inequality.  $\square$

However, while  $\mu^*$  is subadditive, it turns out that it isn't necessarily countably additive. If we want to make it countably additive, we have to restrict the domain, which leads to the following clever definition:

**Definition 5.8.** If  $\mathcal{H}$  is a hereditary  $\sigma$ -ring on  $X$  and  $\nu$  is an outer measure on  $\mathcal{H}$ , then we say a set  $E \in \mathcal{H}$  is  $\nu$ -**measurable** iff it “splits” every  $A \in \mathcal{H}$ , that is,  $\nu(A) = \nu(A \setminus E) + \nu(A \cap E)$ .

We denote the  $\nu$ -measurable subsets by  $M(\nu)$ .

**Proposition 5.9** (Caratheodory).  $M(\nu)$  is a  $\sigma$ -ring and  $\nu|_{M(\nu)}$  is a measure.

We first prove a lemma:

**Lemma 5.10.**  $M(\nu)$  is a ring and for any  $A \in \mathcal{H}$ ,  $E, F \in \mathcal{H}$  disjoint, we have  $\nu(A \cap (E \cup F)) = \nu(A \cap E) + \nu(A \cap F)$ .

*Proof.* We first prove that  $M(\nu)$  is closed under finite unions. If  $E, F$  are measurable, from subadditivity we have  $\nu(A \cap (E \cup F)) + \nu(A \setminus (E \cup F)) \geq \nu(A)$ . However, we also have that

$$\begin{aligned} \nu(A \cap (E \cup F)) + \nu(A \setminus (E \cup F)) &= \nu((A \cap E) \cup (A \setminus E) \cap F) + \nu((A \setminus E) \setminus F) \\ &\leq \nu(A \cap E) + \nu((A \setminus E) \cap F) + \nu((A \setminus E) \setminus F) \\ &= \nu(A \cap E) + \nu(A \setminus E) \quad (\text{because } F \text{ splits } A \setminus E) \\ &= \nu(A) \end{aligned}$$

thus  $\nu(A \cap (E \cup F)) + \nu(A \setminus (E \cup F)) = \nu(A)$ .

Now, we prove that  $M(\nu)$  is closed under set difference. If  $E, F$  are measurable, one direction of the inequality follows from subadditivity. Now, we have that

$$\begin{aligned} \nu(A \cap (E \setminus F)) + \nu(A \setminus (E \setminus F)) &= \nu((A \cap E) \setminus F) + \nu((A \setminus E) \cup (A \cap (E \cap F))) \\ &\leq \nu((A \cap E) \setminus F) + \nu(A \setminus E) + \nu((A \cap E) \cap F) \\ &= \nu(A \cap E) + \nu(A \setminus E) \\ &= \nu(A). \end{aligned}$$

Now, we prove the last property. Since  $E$  splits  $A \cap (E \cup F)$ , then we have  $\nu(A \cap (E \cup F) \cap E) + \nu((A \cap (E \cup F)) \setminus E) = \nu(A \cap E) + \nu(A \cap F) = \nu(A \cap (E \cup F))$ .  $\square$

We have a couple of corollaries:

**Corollary 5.11.**  $\nu|_{M(\nu)}$  is finitely additive.

**Corollary 5.12.** If  $E_k$  are a disjoint finite sequence of measurable sets, then  $\nu 1.(A \cap \bigsqcup E_k) = \sum \nu(\cap E_k)$

Now we prove that  $M(\nu)$  is a  $\sigma$ -ring and that  $\nu$  is countably additive on  $M(\nu)$ .

*Proof.* If  $E_j$  is a sequence of measurable sets, then we can disjointize the  $E_j$  by setting

$$F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j.$$

Now, each  $F_k$  is measurable due to  $M(\nu)$  being a ring, and now we prove that  $G = \bigsqcup F_j$  is measurable. Now, for all  $n$ ,

$$\begin{aligned} \nu(A) &= \nu\left(A \cap \bigsqcup_{j=1}^n F_k\right) + \nu\left(A \setminus \bigsqcup_{j=1}^n F_k\right) \\ &\geq \nu(A \setminus G) + \sum_{j=1}^n \nu(A \cap E_j). \end{aligned}$$

and thus as a corollary,

$$\nu(A) \geq \nu(A \setminus G) + \sum_{j=1}^{\infty} \nu(A \cap E_j) \geq \nu(A \setminus G) + \nu(A \cap G).$$

Now, to prove countable additivity, just take  $A = G$  in the previous proof.  $\square$

We have now proved Caratheodory's Extension Theorem in full, as if  $\nu$  is the outer measure induced by the premeasure  $\mu$ , it is easy to see that  $\nu$  is an extension of  $\mu$  to a  $\sigma$ -ring containing  $\mathcal{S}(\mathcal{P})$ .

## 5.4 PROPERTIES OF EXTENSIONS

Now, we prove a few properties of this constructed measure.

**Definition 5.9.** If  $\mathcal{F}$  is a hereditary family of subsets of  $X$ , and  $\nu: \mathcal{F} \rightarrow [0, \infty]$ , we say  $\nu$  is **complete** if when  $\nu(E) = 0$ , for all  $F \subseteq E$ , we have  $F \in \mathcal{F}$ , and  $\nu(F) = 0$ .

**Proposition 5.13.**  $\nu$  is complete.

*Proof.* If  $\nu(E) = 0$  and  $B \subseteq E$ , then for any set  $A$ ,  $\nu(A \cap B) = 0$ , and  $\nu(A \setminus B)$  must be equal to  $\nu(A)$  due to monotonicity and subadditivity.  $\square$

Note that a similar proof also implies that any set with outer measure 0 is measurable with measure 0.

We may assume extensions are unique, but this does not hold:

**Proposition 5.14.** Extensions are not necessarily unique.

*Proof.* Consider  $\mathbb{R}$  with the pre-ring  $\mathcal{P}$  consisting of all countable sets, and define  $\mu(P) = 0$  for all  $P \in \mathcal{P}$ . Then when extending this to  $S(\mathcal{P})$ , we can define the measure to be the zero measure or the measure that attains infinity on uncountable sets.  $\square$

Another example is given by:

**Example 5.7.** Consider  $\mathcal{P}$  to be the pre-ring of half open intervals  $[a, b)$  and define  $\mu$  to be infinite on all of these.

We notice that these all happen due to the extensions attaining infinite values, which motivates the following definition:

**Definition 5.10.** If  $(X, \mathcal{P}, \mu)$  is a premeasure space, then we say it is  **$\sigma$ -finite** if for all  $E \in \mathcal{P}$  there exists a sequence of  $E_j$  in  $\mathcal{P}$  such that  $E = \bigcup E_j$  and  $\mu(E_j) < \infty$ .

**Theorem 5.15.** If  $(X, \mathcal{P}, \mu)$  is  $\sigma$ -finite then  $\mu^*|_{\mathcal{S}}$  is the unique extension of  $\mu$  to  $\mathcal{S}$ .

*Proof.* If  $\nu$  is any extension of  $\mu$  to  $\mathcal{S}$ , if  $G \in \mathcal{S}$  and  $G \subseteq E \in \mathcal{P}$   $\square$

However,

**Proposition 5.16.** If  $(X, \mathcal{P}, \mu)$  is a premeasure, then for all  $\sigma$ -rings  $\mathcal{S} \subseteq M(\mu^*)$  containing  $\mathcal{P}$ , then  $\mu^*|_{\mathcal{S}}$  is the largest measure on  $\mathcal{S}$  extending  $\mu$ .