Physics Notes

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November 7, 2022

Contents

Part I

Classical Mechanics

1 Newton's Formalism of Mechanics

Mechanics is the study of motion of objects. This started with the Greeks, but their ideas are antiquated and of little use. Newton and Galileo were the first to do mechanics as we know it today, and Newton's formulation is taught in introductory physics courses. However, two alternative formulations, Lagragian and Hamiltonian mechanics, are equivalent to Newtonian mechanics, and sometimes provides more elegant solutions to problems.

In this chapter, we will discuss only Newton's formalism of mechanics.

Classical mechanics is about studying the motion of macroscopic particles when $v \ll c$.

1.1 Space and Time

In this section, we follow the approach of Arnold to describe

Space can be described $\mathbb{A}^3$, affine 3-dimensional space. After picking a origin and orthonormal basis, we can induce a bijection $\mathbb{A}^3 \to \mathbb{R}^3$. We can carry over the standard inner product structure on $\mathbb{R}^3$ as well.

Time is can be described as $\mathbb{A}^1$, affine one-dimensional space. After picking an origin and a basis, similarly to space, we can induce a bijection $\mathbb{A}^1 \to \mathbb{R}^1$.

Now, a **reference frame** is just a choice of these origins and bases, and most of the time when we do physics, we either implicity or explicitly define a reference frame.

1.2 Newton's First Law

Newton's first law isn't so much a law as it is a definition.

2 Energy

2.1 Kinetic Energy and the Work-Energy Theorem

The **kinetic energy** T of a particle is given by the quadratic form $m\dot{r}^2/2$. The derivative of the kinetic energy is given by $\dot{T} = m\dot{r} \cdot \ddot{r} = F(r) \cdot \dot{r}$. Integrating this on both sides, it can be seen that the change of kinetic energy as a particle moves along a path $r(t)$ from time t_0 to t_f while subjected to a force $F(r)$ is equal to the **work** done on that particle,

$$\int_{r(t_0)}^{r(t_f)} F(r)\,\mathrm{d}r.$$

This equivalence is called the **work-energy theorem**.

2.2 Potential Energy

Suppose we have a system governed by a force field $F(r)$. We call F a **conservative force field** if F has a potential, that is a function $U(r)$ such that $F = -\nabla U$. Poincare's lemma tells us that this is equivalent to the condition that $\nabla \times F = 0$.

The function U is called **potential energy** of the system. It is easy to see that in one dimension, there is always a potential, namely $U(r) = -\int_c^r F(r)\,\mathrm{d}r$, where c is any constant. In higher dimensions, it is necessary to verify that the curl is zero.

For conservative fields it can be seen that the equations of motion can be rewritten as $\ddot{r} = -\nabla U(r)$.

Example 2.1 (Gravity near the surface of the earth). Gravity on a particle on mass m near the surface of the earth can be described as a force field $F(h) = -mg$, which results in a potential dependant on the height h of the particle $U(h) = mgh$.

Why is the negative sign there? If we consider the physical system governed only by gravity, $F(r) = -g$, we want a particle at a higher point in space to have more "potential," as it can accumulate more kinetic energy as it falls.

In fact, for any system governed by a conservative force field, it holds that $T + U$ remains constant. This is because

$$\frac{\mathrm{d}}{\mathrm{d}t} T + U = F(r) \cdot \dot{r} + -F(r) \cdot \dot{r} = 0$$

by application of the work-energy theorem and the chain rule. This result is called the **conservation of energy**.

3 Motion of a Single Particle

3.1 Phase Portraits

3.2 Projectiles

In the absence of forces other than gravity, the 2D equations of motion of a particle according to Newton's laws are $\ddot{x} = 0, \ddot{y} = -g$, where $g \approx 9.8\,\mathrm{N\,kg^{-1}} = 9.8\,\mathrm{m\,s^{-2}}$ is an approximation of the gravitational field near the surface of the Earth.

From this we can derive the equations $x(0) = x(0) + \dot{x}(0)t$, $y = y_0 + \dot{y}(0) - gt^2/2$. From there we can derive the range formula and other properties about this type of motion.

3.3 Resistive Forces

Resistanve isn't always simple to describe: The exact force caused by a resistive medium is often a complicated function of the velocity $f(v)$, but here we are only concerned with the lower-order terms.

3.3.1 Linear Mediums

Here, we consider physical systems where the resistive force $f(v) \approx -kv$. These

3.3.2 Quadratic Mediums

3.4 Charged Particle in a Magnetic Field

The force on charged particle in a magnetic field is described by the Lorentz law $F = qv \times B$.

3.5 Oscillations

4 Oscillation

4.1 Simple Harmonic Oscillation

Oscillations are omnipresent in physics. The equations of motion of a point connected to the end of a spring is determined by **Hooke's law**, which states that $F(x) = -kx$, where x is the displacement from the equilibrium position. Equivalently, the potential energy is given by $U(x) = kx^2/2$. The constant k is known as the spring constant.

Systems described by equations of these form are said to be **simple harmonic oscillation**. Many systems are closely approximable by these equations, such as the pendulum.

4.2 Damped Oscillation

When we add a linear resistive force we can still solve the differential equation using techniques from linear algebra. We rewrite the equation $m\ddot{x} + b\dot{x} + kx = 0$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2\beta & \omega_0^2 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

where we use $2\beta = b/m$, and $\omega_0 = \sqrt{k/m}$. and solve using the matrix exponential. In particular, we have the solutions $x = e^{rt}$ where r are the solutions of the quadratic $r^2 + 2\beta r + \omega_0^2$, which gives us $-\beta \pm \sqrt{\beta^2 - \omega_0^2}$.

4.3 Driven Damped Oscillation: Sinusoidal case

Any damped oscillator will eventually lose its energy, and this can be prevented by driving it with a periodic signal. We call this driving force $F(t)$. Now, we consider the equation $m\ddot{x} + b\dot{x} + kx = F(t)$.

We can simplify this equation by dividing by m: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$. It can be seen that $D := \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right) : x \mapsto \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$ is a linear operator. Now, the first isomorphism theorem shows us that if we have one *particular solution* x_p such that $Dx_p = f$ then any solution of the system $Dx = f$ is equal to $x_p + x_h$, where x_p is the particular solution and x_h is a solution to $Dx_0 = 0$.

If we consider the case where $f(t) = f_0 \cos(\omega t)$, then for any solution to $Dx = f_0 \cos(\omega t)$, there is a solution to $Dy = f_0 \sin(\omega t)$ where $y(t) = x(t + \pi/\omega)$. Then, we define the complex function

$z = x + iy$, which satisfies $Dz = f_0 e^{i\omega t}$. We guess a solution of the form $z(t) = Ce^{i\omega t}$, and indeed, substituting this in gives us $\ddot{z} + 2\beta\dot{z} + \omega_0^2 = (-\omega^2 + 2i\beta\omega + \omega_0^2)Ce^{i\omega t}$. Thus, for

$$C = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2},$$

$z = Ce^{i\omega t}$ is a solution to the system $Dz = f_0 e^{i\omega t}$, and thus $x = \Re(z)$ is a solution to the system $Dx = f_0 \cos(\omega t)$.

In order to make taking the real part easier, we attempt to write C in polar coordinates as $C = Ae^{-i\delta}$ where $A \in \mathbb{R}$. This is clearly given by

$$A = \sqrt{|C|^2} = \sqrt{\frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.$$

Now, to find the phase angle δ, we see that

$$Ae^{-i\delta} = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

which can be rearranged into $A(-\omega^2 + 2i\beta\omega + \omega_0^2) = e^{i\delta} f_0$. Using the inverse tangent on the imaginary and real parts to find the phase, we get

$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right).$$

Now, it is easy to express $x(t) = \mathrm{Re}(z) = A\cos(\omega t - \delta)$ in terms of the constants we found above. This is a particular solution to the system $Dx_p = f_0 \cos(\omega t)$. Now, we just add a homogenous solution x_h to x_p, which we found in the previous section. We now have the general form of a solution $x = A\cos(\omega t - \delta) + Be^{-\beta t}\cos(\omega_1 t - \delta_u)$. The second term in the solution is called the *transient*, and fades very quickly relative to the first term when $\beta \gg 0$.

5 Calculus of Variations

5.1 Basic Examples

Geodesic

In Euclidean space, the length of a curve $x(t), y(t)$ is defined as

$$\int_0^{t_f} \sqrt{x'(t) + y'(t)}\, \mathrm{d}t.$$

To find the geodesic between two points, i.e. the shortest curve between two points P and Q, we consider the constraints $(x(0), y(0)) = p$ and $(x(t_f), y(t_f)) = q$.

In general, on a Riemannian manifold (M, g), you can find the geodesic connecting two points p and q by minimizing $\int_0^{t_f} \|\gamma'(t)\|_g^2\, \mathrm{d}t$ subject the constraints $\gamma(0) = p$, $\gamma(1) = q$.

Catenary

Consider a bridge with a cable hanging from two posts. We want to find the shape of the cable, which is equivalent to minimizing the potential energy of the cable.

If the height of cable at position x is denoted by $h(x)$ and the cable's linear density is denoted by λ, then this is equivalent to minimizing

$$\int_0^{\ell} \lambda g h(x) \sqrt{1 + h'(x)}\, \mathrm{d}x$$

with the constraints $h(0) = h(\ell) = h_0$.

One can see that both examples involve constrained minimation of a *functional*.

5.2 Basic Techniques

In both cases, we have a functional $\Phi\colon F \to \mathbb{R}$, of the form $\int L(q, \dot{q}, t)$.

Index