Linear Systems Notes

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Part I

Linear Systems

1 STATE-SPACE SYSTEMS

1.1 BASIC DEFINITIONS

We first start with the definition of a state-space system:

Definition 1.1. A **continuous state-space linear system** is a system by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

x is called the **state**, y is called the **output**, and u is called the **input**.

If u is one-dimensional, then we call the system **single input (SI)**, and if y is one-dimensional, we call the system **single output (SO)**. If their dimensions are not one, then we call the system **multiple input (MI)** and/or **multiple output (MO)** respectively.

If A, B, and C are all zero, then we call the system **memoryless**. If A, B, C, D are constant, then we call this a **linear time invariant (LTI)**, we call the general case **linear time varying (LTV)**.

We will sometimes drop the time argument on the state, output, and input just to make the equations more concise, and if the system is LTI, then we will drop the argument for the matrices as well.

Definition 1.2. A **discrete state-space linear system** is a system by

$$x(t+1) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

which we shorten to

$$x^+ = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

and in the LTI case we drop the time argument on the matrices.

1.1.1 LINEARIZATION

Most systems in nature are nonlinear, but nonlinear systems are hard to work with. Thus, since nonlinear systems look linear locally, we can linearize them around a point.

More precisely, if $x^{\text{eq}}, u^{\text{eq}}$ is an equilibrium solution with output y^{eq}, then for input $u = u^{\text{eq}} + \delta u$ and initial condition $x(0) = x^{\text{eq}} + \delta x(0)$, then we attempt to approximate formulas for δx and δy. If

$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$

then $\dot{\delta x} = f(x^{\text{eq}} + \delta x(t), u^{\text{eq}} + \delta u(t))$ is approximated to the first order by its (total) derivative $\mathrm{d}f(\delta x(t), \delta u(t))$ and $\delta y \approx \mathrm{d}g(\delta x(t), \delta u(t))$, so in our linearized approximation, we have

$$A = \frac{\partial f}{\partial x} \qquad\qquad B = \frac{\partial f}{\partial u}$$
$$C = \frac{\partial g}{\partial x} \qquad\qquad D = \frac{\partial g}{\partial u}$$

A similar result holds for the discrete case.

Instead of linearizing around an equilibrium, we can instead linearize along a known trajectory. If $x^{\text{sol}}, u^{\text{sol}}$ are solutions to a nonlinear system $\dot{x} = f(x, u), y = g(x, u)$ then we consider solutions that satisfy coniditions $u = u^{\text{sol}} + \delta u$ and $x(0) = x^{\text{sol}}(0) + \delta x(0)$. Assuming that δx and δu are small, we can reasonably approximate the system as

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{(x^{\text{sol}}(t), u^{\text{sol}}(t))} \qquad\qquad B(t) = \left.\frac{\partial f}{\partial u}\right|_{(x^{\text{sol}}(t), u^{\text{sol}}(t))}$$
$$C(t) = \left.\frac{\partial g}{\partial x}\right|_{(x^{\text{sol}}(t), u^{\text{sol}}(t))} \qquad\qquad D(t) = \left.\frac{\partial g}{\partial u}\right|_{(x^{\text{sol}}(t), u^{\text{sol}}(t))}$$

where $\dot{\delta x} \approx A(t)(\delta x) + B(t)(\delta u)$ and $\delta y \approx C(t)(\delta x) + D(t)(\delta u)$.