Topology and Analysis Notes

Bryan Li

Fall 2022

Contents

Administrative Matters

Midterm date options:

1. Fri Oct 14

2. Mon Oct 17

3. Wed Oct 19

Weekly psets due Friday midnight.

1 GENERAL TOPOLOGY

1.1 LECTURE 1: METRICS

Definition 1.1.1 (Metric Space). A metric on a set X is a function $d\colon X^2 \to \mathbb{R}_{\geq 0}$ such that

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$.

3. $d(x, z) \leq d(x, y) + d(y, z)$.

(X, d) is then called a metric space.

If condition 1 is omitted and replaced only with the condition that $d(x, x) = 0$, then d is called a semimetric (or quasimetric). If distances are allowed to be infinite, then d is called an extended metric.

If (X, d) is a semimetric space, then if we define a relation $\sim$ such that $x \sim y$ iff $d(x, y) = 0$, it is an equivalence relation. In this case, d "drops" to a metric on the equivalence classes $X/\sim$. To show this, we just need to prove that $\tilde{d}([x], [y]) = d(x, y)$ is a well-defined function, which utilizes the triangle inequality.

Example 1.1.1. If G is a connected weighted graph, then we can define a metric on G by defining the distance between two vertices to be the length of the minimal path connecting them.

If G is unconnected, by assigning the distance of infinity between points in separate components, we define a extended metric.

Definition 1.1.2 (Norm). If V is a vector space then $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ is a norm on V if

1. $\|v\| = 0$ iff $v = 0$

2. $\|\lambda v\| = |\lambda| \|v\|$

3. $\|v + w\| \leq \|v\| + \|w\|$

A norm on V clearly induces a metric on V by defining $d(v, w) = \|v - w\|$.

Example 1.1.2 (Euclidean metric). The real line $\mathbb{R}$ and the euclidean spaces $\mathbb{R}^n$ with the euclidean metric $d(x,y) = \left\| x - y \right\|_2$.

Example 1.1.3 (p-norms). The p-norms $\left\| (x_1, \ldots, x_n) \right\|_p = \left(\sum x_i^p \right)^{1/p}$ induce a metric on $\mathbb{R}^n$.

Definition 1.1.3 (Restricted metric). If (X, d) is a metric space and Y is a subset of X, then we can restrict d to Y and $d|_Y$ is a metric on Y.

It is common to consider the restriction of the Euclidean metric onto subsets of Euclidean space.

Example 1.1.4 (L^p norms). If $V = \mathscr{C}^0([0,1])$, then examples of norms on V include

1. $\left\| f \right\|_\infty = \sup \left| f(t) \right|$.

2. $\left\| f \right\|_1 = \int_0^1 \left| f(t) \right| \, \mathrm{d}t$

3. $\left\| f \right\|_2 = \left(\int_0^1 \left| f(t) \right|^2 \, \mathrm{d}t \right)^{1/2}$

4. $\left\| f \right\|_p = \left(\int_0^1 \left| f(t) \right|^p \, \mathrm{d}t \right)^{1/p}$ for $1 \le p < \infty$.[1]

This generalizes to bounded closed regions of $\mathbb{R}^n$, where all continuous functions are integrable (due to compactness and the extreme value theorem). The proof that these are actually norms that satisfy the triangle inequality will come later (see the Minkowski and Hölder inequalities).

1.2 Lectures 2, 3: Topology in Metric Spaces

Categories

Categories contain objects and morphisms, that satisfy the following properties:

1. There is a morphism from any object A to itself, called the identity 1_A.

2. Morphisms can be composed.

Metric spaces can form a category under a few choices of collections of morphisms:

- Isometries: $i \colon (X, d_X) \to (Y, d_Y)$ such that $d_X(x,y) = d_Y(i(x), i(y))$ for all x, y.

  Surjective isometries form a subcategory of the above category, ,where every map is an isomorphism.

---

[1] The ∞-norm can be viewed as the limit of the p-norms as $p \to \infty$.

- Lipschitz maps: $f\colon (X, d_X) \to (Y, d_Y)$, where there exists a constant C (which may depend on f), such that $d_Y(f(x), f(y)) \le C d_X(x, y)$.

  We can also form the subcategory of bi-Lipschitz isomorphisms.

- Uniformly continuous maps: $f\colon (X, d_Y) \to (Y, d_Y)$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x, y) \le \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

- Continuous maps: $f\colon (X, d_Y) \to (Y, d_Y)$ such that for all $x \in X$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x, y) \le \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

These conditions get weaker as we go down, and continuity can be generalized to non-metric spaces which gives rise to general topology. The study of uniformly continuous maps is much less common, but gives rise to uniformities.

Definition 1.2.1 (Convergence). If $\{x_n\}$ is a sequence in a metric space (X, d), then $x_n \to x$ if $\forall \varepsilon > 0,\ \exists N$ such that $n > N$ implies that $d(x_n, x)$.

Definition 1.2.2 (Cauchy sequence). If $\{x_n\}$ is a sequence in a metric space (X, d), then $x_n \to x$ if $\forall \varepsilon > 0,\ \exists N$ such that $n, m > N$ implies that $d(x_n, x_m)$.

Definition 1.2.3 (Completeness). (X, d) is complete if every cauchy sequence converges to a point.

For any metric space that isn't complete, we want to form a "completion" of it.

Definition 1.2.4 (Density). If $S \subseteq X$, S is dense if for all x in X and all $\varepsilon < 0$ there exists $s \in S$ where $d(s, x) < \varepsilon$.

Definition 1.2.5 (Completion). A completion of a metric space (X, d) is a metric space $(\overline{X}, \overline{d})$ with an inclusion isometry $i\colon X \to \overline{X}$ such that the image of i is dense in $\overline{X}$. In addition, $\overline{X}$ is a complete metric space.

One way to define a completion for any metric space is by defining a semimetric $\tilde{d}$ on the set of Cauchy sequences $\mathrm{cs}(X, d)$, where $\tilde{d}(\{x_n\}, \{y_n\}) = \lim_{n \to \infty} d(x_n, y_n)$, then by dropping that to a metric on the equivalence classes induced by the semimetric.

We can just define $f\colon X \to \overline{X}$ as $f(x) = [\{x, x, x, \dots\}]$. To show that $\overline{X}$ is complete, we consider Cauchy sequences of Cauchy sequences, approximate their elements by elements in X, and show that the sequence converges to the sequence generated by the approximation.

Example 1.2.1. If $f \in V = \mathscr{C}^0([0, 1])$, the norm $\|f\|_\infty = \sup(|f(x)|)$ induces a complete metric space (due to the limit of uniformly continous functions converging to a continuous function).

However, in the L^1 norm $\|f\|_1 = \int_0^1 |f(x)|\,\mathrm{d}x$, V is not complete (you can construct a sequence of functions that converges to a step function).

Now to answer the question "What is the completion of V under the L^p norms", we must look to measure theory (2nd half of this course).

If (X, d_X) and (Y, d_Y) are metric spaces, if $f\colon X \to Y$, recall that if $x \in X$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $d_X(x, y) \le \delta$, then $d_Y(f(x), f(y)) < \varepsilon$, then f is continuous.

Definition 1.2.6 (Open ball). The open ball of radius r, $B_r(x) = \{y \in X \mid d_X(x, y) < r\}$.

This allows us to refine our definition of continuity:

Definition 1.2.7 (Continuity). If $f\colon X \to Y$, then f is continuous if for all x, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Definition 1.2.8 (Open set). The set $A \subseteq X$ is open if for all $a \in A$, there exists an open ball around a that is contained within a.

Proposition 1.2.1. Open balls are open.

Proof. If $a \in B_r(x)$, then $B_{r-d_X(a,x)}(a)$ is contained within $B_r(x)$ due to the triangle inequality. $\square$

Proposition 1.2.2. If f is continuous, then if U is open in Y such that $x \in X$, then there is an open ball around x that is contained within the preimage of U.

A direct corollary of this proposition is

Corollary 1.2.3. $f\colon X \to Y$ is continuous is equivalent to the following condition: If U is open in Y, then $f^{-1}(U)$ is open in X.

Now, we consider properties of the set $\mathscr{T}$ consisting of open sets in X:

Theorem 1.2.4. 1. $\emptyset, X \in \mathscr{T}$.

2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.

3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

Proof. Trivial. $\square$

These properties give rise to the definition of a topology.

Definition 1.2.9 (Topology). A topology on a set X is a collection $\mathscr{T} \subseteq 2^X$ such that

1. $\emptyset, X \in \mathscr{T}$.

2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.

3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

The sets in $\mathscr{T}$ are called open sets.

1.3 GENERAL TOPOLOGY

We recall the definition of a topology:

Definition 1.3.1 (Topology)**.** A topology on a set X is a collection $\mathscr{T} \subseteq 2^X$ such that

1. $\emptyset, X \in \mathscr{T}$.

2. Arbitrary unions of sets in $\mathscr{T}$ are also in $\mathscr{T}$.

3. Finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$.

The sets in $\mathscr{T}$ are called open sets.

We then extend the definition of continuity to general topologies.

Definition 1.3.2 (Continuity)**.** If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces, then f is continuous if for all $U \in \mathscr{T}_Y, f^{-1}(U) \in \mathscr{T}_X$.

Topologies on a set X form a poset under the subset relation.

Definition 1.3.3 (Closed sets)**.** The closed sets in $(X, \mathscr{T})$ are precisely the complement of open sets in X.

Proposition 1.3.1 (Properties of closed sets)**.**   1. $\emptyset, X$ are closed.

2. Arbitrary intersections of closed sets are also closed.

3. Finite unions of closed sets are closed.

Definition 1.3.4 (Closure)**.** The closure $\overline{A}$ of an arbitrary set A is the smallest closed set containing it.

Definition 1.3.5 (Density)**.** Given $A \subseteq B$, we say A is dense in B if $B \subseteq \overline{A}$. Equivalently, $\overline{A} = \overline{B}$.

1.4 Bases and Subbases

Proposition 1.4.1. If X is a set, then the arbitrary intersection of topologies is a topology on X.

Proof. These details are routinely verified. ☐

Corollary 1.4.2. Given any collection $\mathcal{S}$ of subsets of X, there is a smallest topology containing $\mathcal{S}$.

Proof. This topology is the intersection of all the topologies that contain $\mathcal{S}$. We say that $\mathcal{S}$ generates this topology. ☐

Definition 1.4.1 (Subbase). Let $\mathcal{T}$ be a topology on X, and let $\mathcal{S} \subseteq 2^X$, with $X = \bigcup \mathcal{S}$. We say that $\mathcal{S}$ is a subbase of $\mathcal{T}$ if $\mathcal{T}$ is the topology generated by $\mathcal{S}$.

Subbases can be useful for studying topologies in the same way that a basis is useful for studying vector spaces. For example, see [Proposition 1.4.6](#).

We can explicitly characterize the topology generated by a set $\mathcal{S}$. We first state a lemma:

Lemma 1.4.3. If $\mathcal{I} \subseteq 2^X$ be closed under finite intersections and $\bigcup \mathcal{I} = X$, then the topology generated by $\mathcal{I}$ consists of the arbitrary unions of sets in $\mathcal{I}$ (with union over 0 sets being $\emptyset$).

Proof. The only part that is not immediately obvious is the closure under finite intersections. If $\mathcal{O}_1 \bigcup_A U_\alpha$ and $\mathcal{O}_2 = \bigcup_B V_\beta$, then

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \bigcup_{A,B} U_\alpha \cap V_\beta.$$

☐

Proposition 1.4.4. If $\mathcal{S} \subseteq 2^X$ whose union is X, then the topology $\mathcal{T}$ generated by $\mathcal{S}$ consist of the arbitrary unions of $\mathcal{I}^{\mathcal{S}}$ which are defined as the set of finite intersections of sets in $\mathcal{S}$.

Definition 1.4.2 (Base). A collection $\mathcal{B} \subseteq 2^X$ is called a base for a topology $\mathcal{T}$ on X if $\mathcal{T}$ consists of the arbitrary unions of elements in $\mathcal{B}$.

Example 1.4.1.   1. The collection of finite intersections $\mathcal{I}^{\mathcal{S}}$ of a subbase $\mathcal{S}$.

2. The set of open balls in a metric space.

3. The set of intervals of the form $(-\infty, a)$ and (b, ∞) in $\mathbb{R}$ under the standard topology.

Notice how the second and third examples are not closed under finite intersections.

Proposition 1.4.5. For any base $\mathcal{B}$, any finite intersection of sets in $\mathcal{B}$ must be a union of elements in $\mathcal{B}$.

Proposition 1.4.6. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, and let $f \colon X \to Y$. If $\mathcal{S}_Y$ is a subbase of $\mathcal{T}_Y$, then f is continuous iff for all $U \in \mathcal{S}_Y$, we have $f^{-1}(U)$ is open.

Proof. First show that the preimages of sets in $\mathcal{T}_{\mathcal{S}_Y}$ are open, and then show that the preimages of all sets in $\mathcal{T}_Y$ are open. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.5 New Topologies from Old

Definition 1.5.1 (Initial Topologies)**.** If X is a set, $(Y_\alpha, \mathcal{T}_\alpha)$ are topological spaces, and for each α, $f_\alpha \colon X \to Y_\alpha$. Then, the initial topology is the smallest topology on X that makes all f_α continuous.

This initial topology is just the topology generated from the preimages of open sets in Y_α under f_α. This gives rise to a few important methods of constructing new topological spaces from old ones.

Definition 1.5.2 (Relative topology)**.** If $X \subseteq Y$ and $(Y, \mathcal{T}_Y)$ is a topological space, and if j is the inclusion map from X to Y, then the initial topology generated by j consists of the intersection of open sets with X, which is also called the relative (or subspace) topology.

Definition 1.5.3 (Finite product topology)**.** If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces then if $X = X_1 \times X_2$. We have a couple of "natural" maps, $X \xrightarrow{p_1} X_1, X \xrightarrow{p_2} X_2$. We see that $p_1^{-1}(U) = U \times X_2$, and $p_2^{-1}(V) = X_1 \times V$, so it is easy to see that the product topology has a basis consisting of sets of the form $U \times V$, where U is open in X_1 and V is open in X_2.

This trivially generalizes to any **FINITE** number of topological spaces and their **FINITE** cartesian product. However, when we generalize to infinite products, the following case occurs:

Definition 1.5.4 (Product topology)**.** Suppose we have topological spaces $(X_\alpha, \mathcal{T}_\alpha)$ over a (possibly infinite) indexing set A, where $X = \prod X_\alpha$. Then, p_α is the natural projection $X \to X_\alpha$, and $p_\alpha^{-1}(U)$ is $\prod U_\beta$, where $U_\beta = X_\beta$ when $\beta \neq \alpha$ and $U_\alpha = U$. These sets form a subbase of this topology. Now, since topologies only allow finite intersections, then the product topology is generated by products $\prod_\alpha U_\alpha$, where only a finite number of U_α are allowed to be not equal to X_α.

Example 1.5.1. If $X_i = \{0, 1\}$ with the discrete topology, then $\prod_0^\infty X_i$ is compact.

Definition 1.5.5 (Weak topology)**.** For a normed vector space $(V, \|\cdot\|)$. If V^* is the vector space of all continuous linear functionals. The initial topology generated by V^* is the weak topology on V.

Example 1.5.2. For example, if we have $V = \mathscr{C}^0([0, 1])$, with

$$\phi_g \colon V \to \mathbb{R}, f \mapsto \int_0^1 f(x)g(x)\, \mathrm{d}x\,.$$

Then, the initial topology generated by ϕ_g defines "a" weak topology on V.

As opposed to a initial topology, we can define final topologies:

Definition 1.5.6 (Final topology)**.** If $(X_\alpha, \mathscr{T}_\alpha)$ are topological spaces and $f_\alpha \colon X_\alpha \to Y$, then the final topology $\mathscr{T}$ is the strongest topology making all f_α continuous.

To construct this topology, we can consider $\{A \in 2^Y \mid f_\alpha^{-1}(A) \in \mathscr{T}_\alpha\}$, which is a topology on Y for each α. We can just take the intersection of these topologies to obtain the strongest topology that makes all the f_α continuous.

Example 1.5.3 (Quotient topology)**.** Any function $f \colon X \to Y$ induces an equivalence relation with the equivalence classes being the preimages $f^{-1}(\{y\})$. These equivalence classes partition X. It turns out any equivalence relation $\sim$ induces equivalence classes that partition X, which are denoted $X/\sim$. If X has a topology $\mathscr{T}_X$, then the final topology from the natural mapping $f \colon X \to X/\sim$ is called the quotient topology on $X/\sim$.

We can construct a bunch of new, interesting topologies using quotients. First we define what it means for topological spaces to be "the same." I will also start dropping the topology and use individual letters X to denote the topological spaces $(X, \mathscr{T}_X)$, depending on context.

Definition 1.5.7 (Homeomorphism)**.** A function $f \colon X \to Y$ is a homeomorphism if it is an invertible continuous mapping, whose inverse is also continuous. Two spaces where there exists a homeomorphism between them are called "homeomorphic."

Example 1.5.4 (Circle)**.** If $I = [0, 1]$ with the equivalence relation such that only $0 \sim 1$, then $X/\sim$ is homeomorphic to the 1-sphere (circle), S^1. One homeomorphism from $I \to S^1$ is the mapping $t \mapsto e^{2\pi i t}$.

Example 1.5.5 (Cut-off cylinder)**.** If $X = [0, 2] \times [0, 1]$ is endowed with the relation $(0, r) = (2, r)$ then the quotient $X/\sim$ is homeomorphic to a cylinder. If we instead identified $(0, r) \sim (2, 1 - r)$ then $X/\sim$ will be homeomorphic to a Möbius strip.

Example 1.5.6 (Projective space)**.** If X is the unit sphere in $\mathbb{R}^3$, then if we define the equivalence $v \sim -v$, then $X/\sim$ is the real projective space $\mathbb{R}\mathrm{P}^2$.

Example 1.5.7. It can be seen that homeomorphisms from an object from itself can form (quite complicated) groups. These groups are the automorphism groups $\text{Aut}(X)$ of topological spaces. Recall that a group action on X is a homomorphism $\alpha\colon G \to \text{Aut}(X)$. Examples of group actions on spaces include $\mathbb{Z}$ acting on $\mathbb{R}$ by translation, or $\mathbb{Z}_2$ acting on S^2 by the antipodal map.

However, recall that group actions induce equivalence classes (orbits) on the topological space and thus induces a quotient topology A/α. The quotient topology of the examples of the actions above give rise to S^1 and $\mathbb{R}\text{P}^2$.

1.6 Properties of Topological Spaces

There are certain properties of topological spaces we can find, which are useful for distinguishing topological spaces from each other.

Definition 1.6.1 (Hausdorff (T_2)). A space X is Hausdorff if for all $x \neq y$, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$.

Definition 1.6.2 (Normal (T_4)). A space X is Hausdorff if for all pairs of disjoint closed subsets C_1, C_2, there exist disjoint open sets O_1, O_1 such that $C_1 \subseteq O_1$ and $C_2 \subseteq C_2$.

It can easily be seen that all metric spaces are Hausdorff, but it is also the case that metric spaces are normal. How do we prove this? We first consider the following important theorem:

Theorem 1.6.1 (Urysohn's Lemma). If X is a normal space, then for any disjoint closed subsets C_0, C_1 there exists a continuous function $f\colon X \to [0, 1]$ where $f(C_0) = 0$ and $f(C_1) = 1$.

The converse of Urysohn's Lemma also holds, we simply take $O_0 = f^{-1}([0, 1/3))$ and $O_1 = f^{-1}((2/3, 1])$. We prove that metric spaces satisfy this property:

Proposition 1.6.2. Metric spaces are normal.

Definition 1.6.3 (Distance from a set). For a metric space (X, d), we define $d_A(x)$ or $d(x, A)$ where $A \subseteq X$, as $\inf_{a \in A} d(x, a)$.

It is easily seen that this function is Lipschitz with constant ≤ 1.

Now, we can proceed with a proof of Proposition 1.6.2.

Proof. Let C_0, C_1 be disjoint closed subsets in (X, d), then define

$$f\colon X \to [0, 1], x \mapsto \frac{d(x, C_0)}{d(x, C_0) + d(x, C_1)}.$$

It is clear that $f(C_0) = 0$ and $f(C_1) = 1$, and that f is continuous. □

   Now, we continue with a proof of Urysohn's Lemma.

Lemma 1.6.3.   If X is normal, then for any closed set C and open set O containing C, there is an open set U "between" C and O, that is, $C \subseteq U \subseteq \overline{U} \subseteq O$.

Proof. If $D = X \setminus O$, then D is closed and disjoint from C and then we can find disjoint open sets U, V such that U contains C and V contains D. In addition, the closure of C must be contained within $X \setminus V$ which is contained by U. □

Proof of Urysohn's Lemma. Given X normal and disjoint closed sets C_0, C_1, set $O_1 = C_1'$, then apply the preceding lemma to get a set $O_{1/2}$ "between" C_0 and O_1. Applying the lemma again, we then get a two open sets $O_{1/4}, O_{3/4}$ s.t. $C_0 \subseteq O_{1/4} \subseteq \overline{O_{1/4}} \subseteq O_{1/2} \subseteq \overline{O_{1/2}} \subseteq O_{3/4} \subseteq \overline{O_{3/4}} \subseteq O_1$. We can repeat this process to choose $O_{1/8}, O_{3/8}, O_{5/8}, O_{7/8}$, and then repeat the process over and over again.

   Now, we are able to define O_d, for all $d \in \Delta$ (rational numbers of the form $k/2^n$ where $k \leq 2^n$). □