Topology and Analysis Notes

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Contents

Administrative Matters

Midterm date options:

1. Fri Oct 14
2. Mon Oct 17
3. Wed Oct 19

Weekly psets due Friday midnight.

I General Topology

1.1 Metric Spaces

Definition 1.1.1 (Metric Space)**.** A metric on a set X is a function $d\colon X^2 \to \mathbb{R}_{\geq 0}$ such that

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

(X, d) is then called a metric space.

If condition 1 is omitted and replaced only with the condition that $d(x, x) = 0$, then d is called a semimetric (or quasimetric). If distances are allowed to be infinite, then d is called an extended metric.

If (X, d) is a semimetric space, then if we define a relation $\sim$ such that $x \sim y$ iff $d(x, y) = 0$, it is an equivalence relation. In this case, d "drops" to a metric on the equivalence classes $X/\sim$. To show this, we just need to prove that $\tilde{d}([x], [y]) = d(x, y)$ is a well-defined function, which utilizes the triangle inequality.

Example 1.1.1. If G is a connected weighted graph, then we can define a metric on G by defining the distance between two vertices to be the length of the minimal path connecting them.

If G is unconnected, by assigning the distance of infinity between points in separate components, we define a extended metric.

Definition 1.1.2 (Norm)**.** If V is a vector space then $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ is a norm on V if

1. $\|v\| = 0$ iff $v = 0$
2. $\|\lambda v\| = |\lambda| \|v\|$
3. $\|v + w\| \leq \|v\| + \|w\|$

A norm on V clearly induces a metric on V by defining $d(v, w) = \|v - w\|$.

Example 1.1.2 (Euclidean metric). The real line $\mathbb{R}$ and the euclidean spaces $\mathbb{R}^n$ with the euclidean metric $d(x, y) = \|x - y\|_2$.

Example 1.1.3 (p-norms). The p-norms $\|(x_1, \dots, x_n)\|_p = \left(\sum x_i^p\right)^{1/p}$ induce a metric on $\mathbb{R}^n$.

Definition 1.1.3 (Restricted metric). If (X, d) is a metric space and Y is a subset of X, then we can restrict d to Y and $d|_Y$ is a metric on Y.

It is common to consider the restriction of the Euclidean metric onto subsets of Euclidean space.

Example 1.1.4 (L^p norms). If $V = \mathscr{C}^0([0, 1])$, then examples of norms on V include

1. $\|f\|_\infty = \sup |f(t)|$.
2. $\|f\|_1 = \int_0^1 |f(t)| \, \mathrm{d}t$
3. $\|f\|_2 = \left(\int_0^1 |f(t)|^2 \, \mathrm{d}t\right)^{1/2}$
4. $\|f\|_p = \left(\int_0^1 |f(t)|^p \, \mathrm{d}t\right)^{1/p}$ for $1 \leq p < \infty$.[1]

This generalizes to bounded closed regions of $\mathbb{R}^n$, where all continuous functions are integrable. The proof that these are actually norms that satisfy the triangle inequality will come later (see the Minkowski and Hölder inequalities).

[1] The ∞-norm can be viewed as the limit of the p-norms as $p \to \infty$.