Percolation in Complex Networks

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- No Hamiltonian
- No temperature
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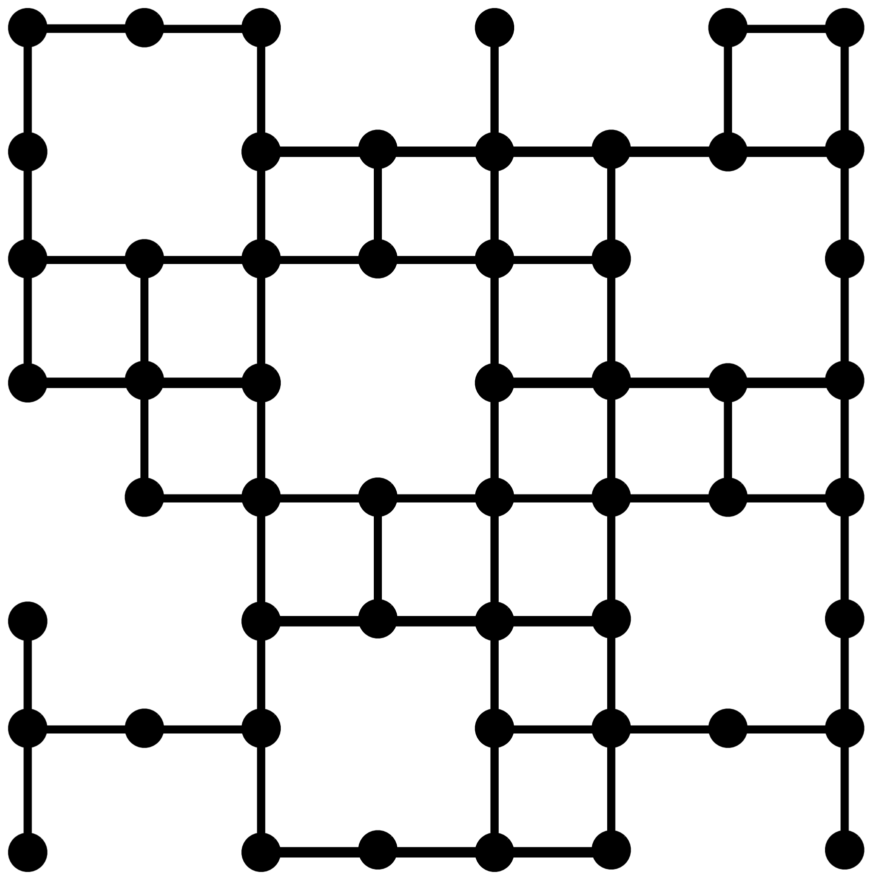

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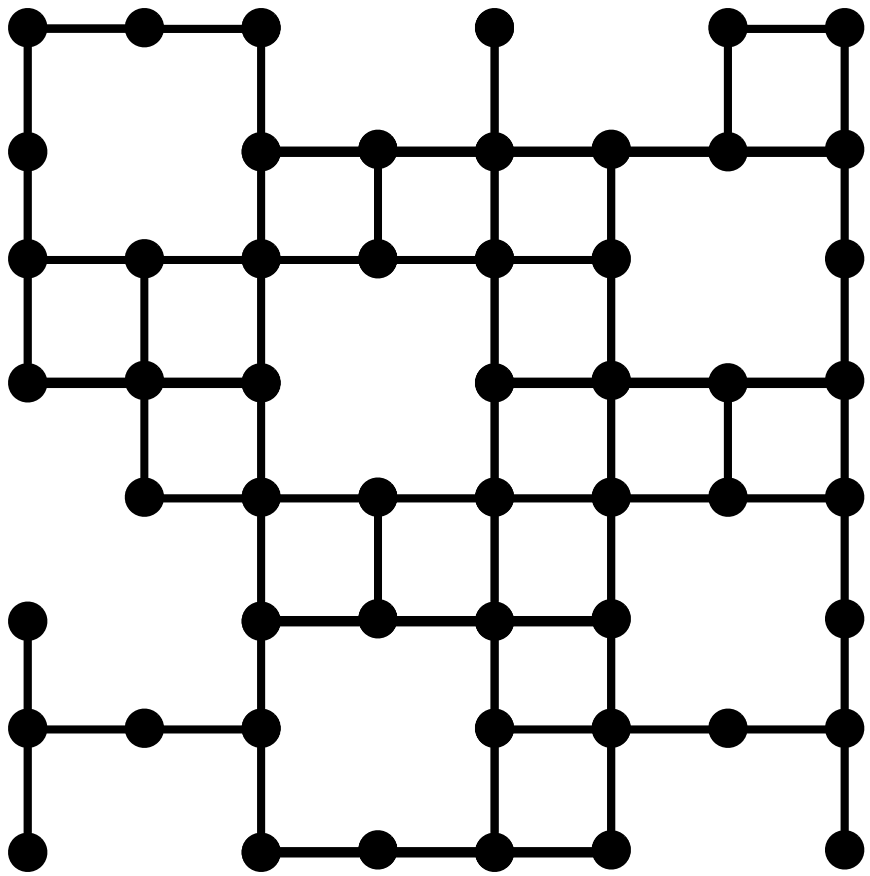

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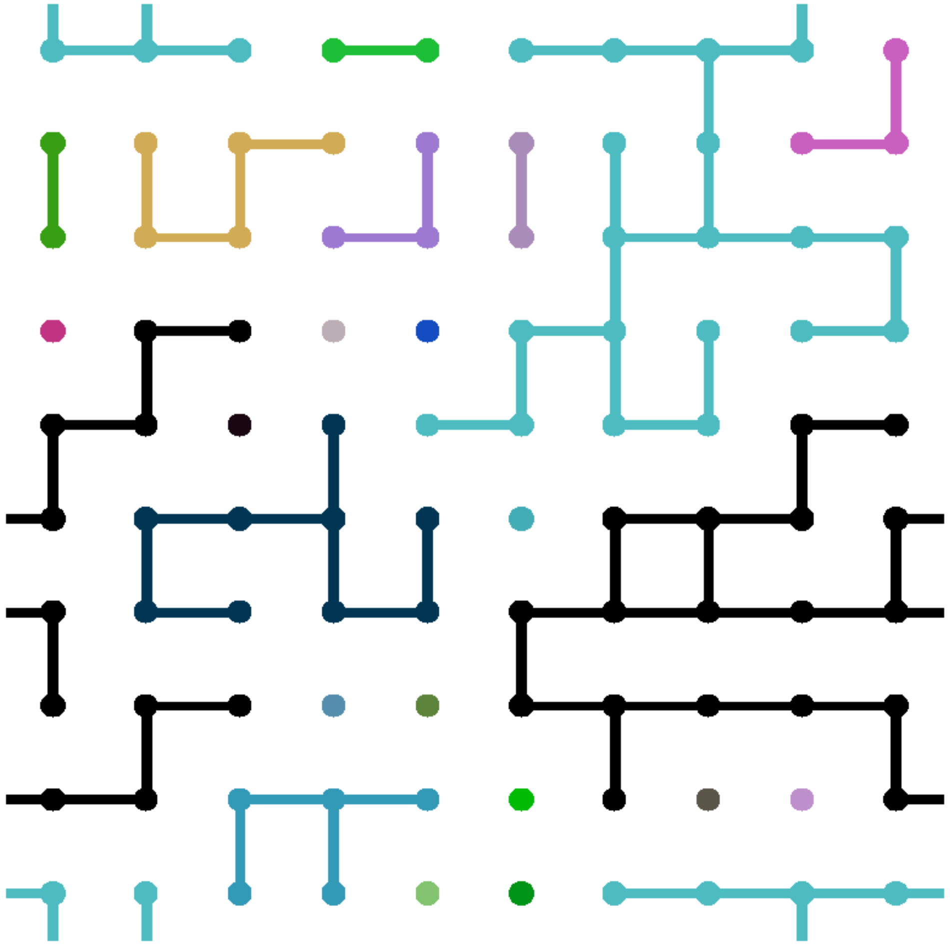
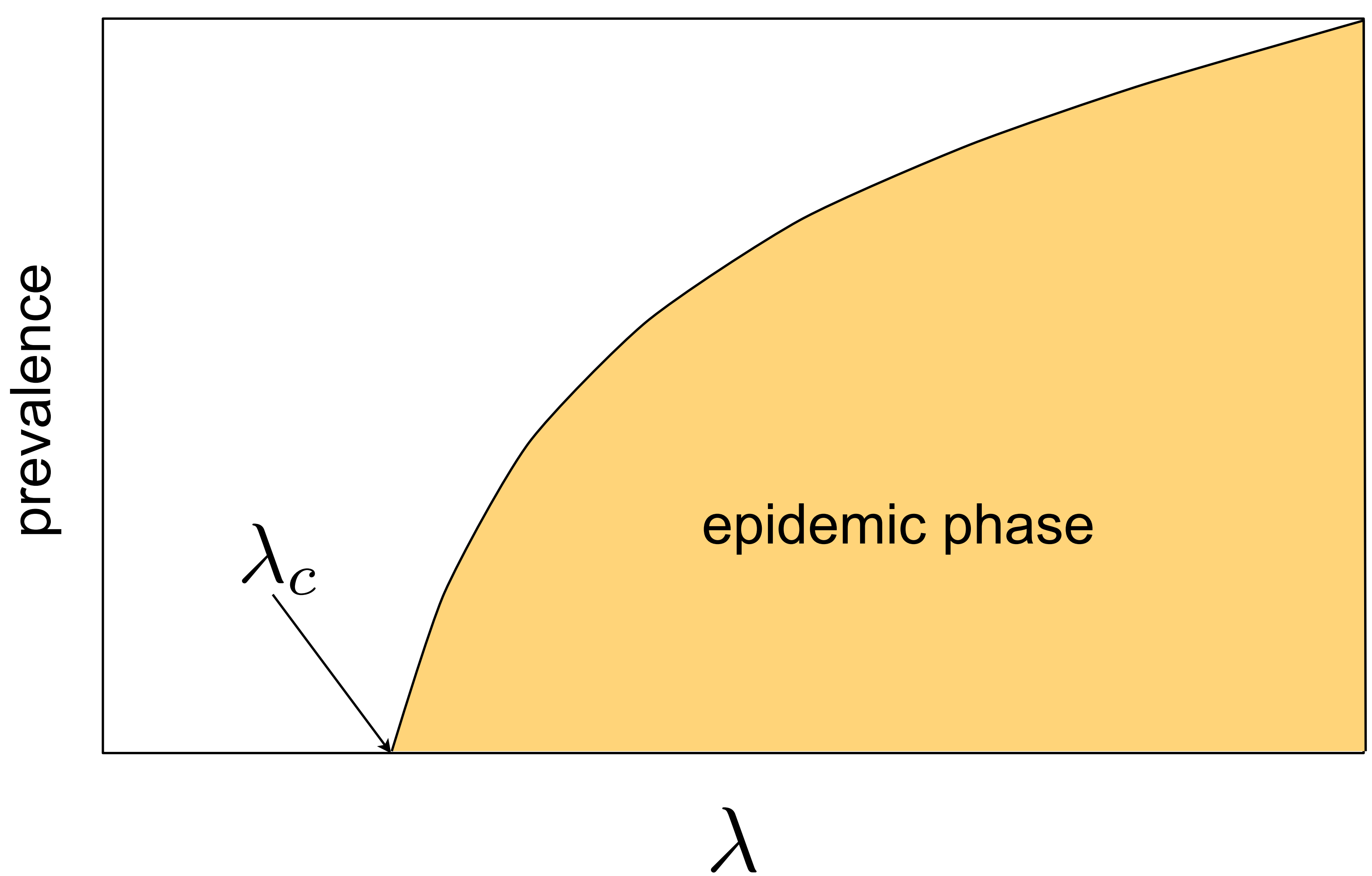


Bond percolation: each bond is kept with probability p

But there is a connection to epidemic models

$$S \xrightarrow{\lambda} I \xrightarrow{\delta} S \qquad\qquad S \xrightarrow{\lambda} I \xrightarrow{\delta} R$$



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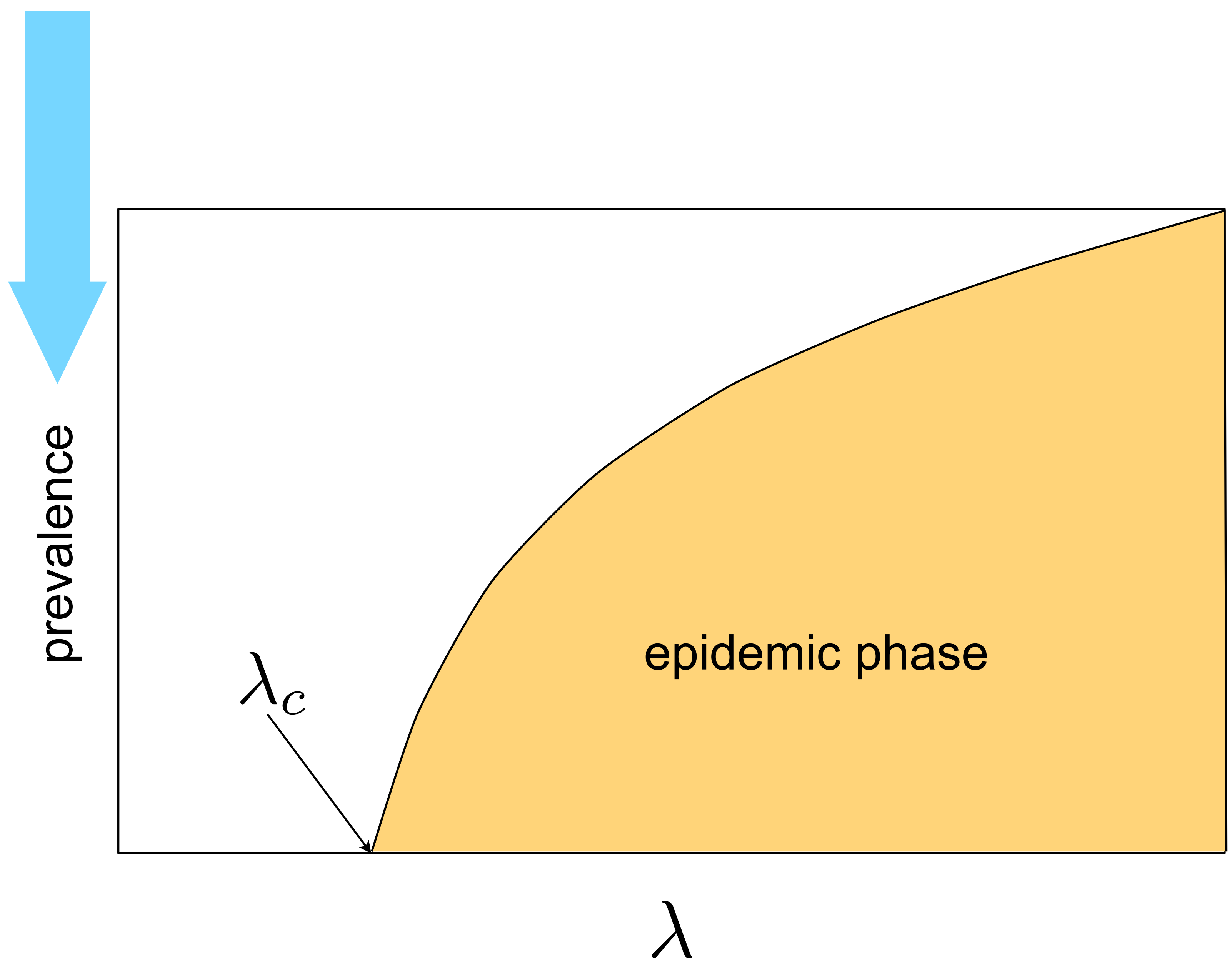

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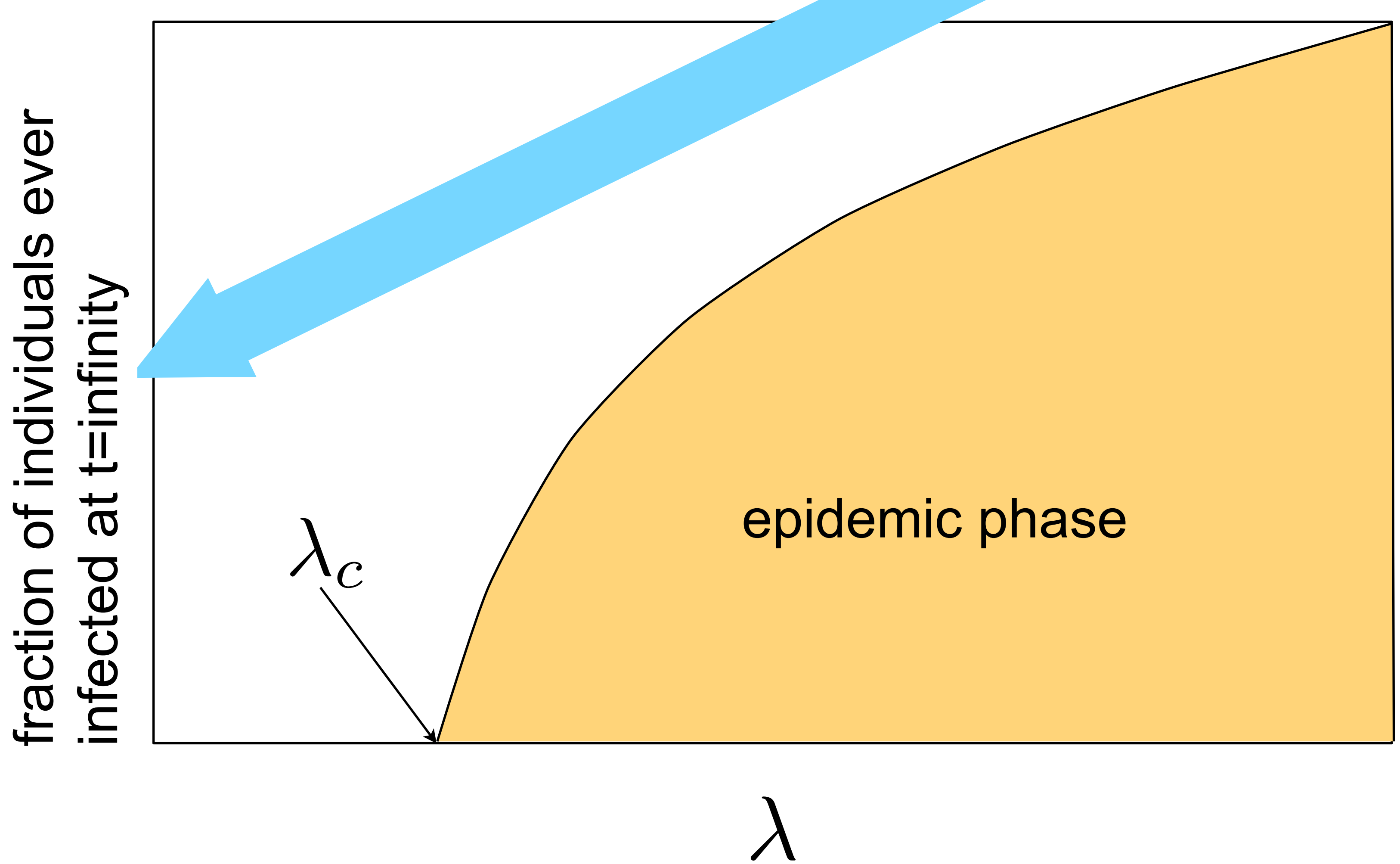

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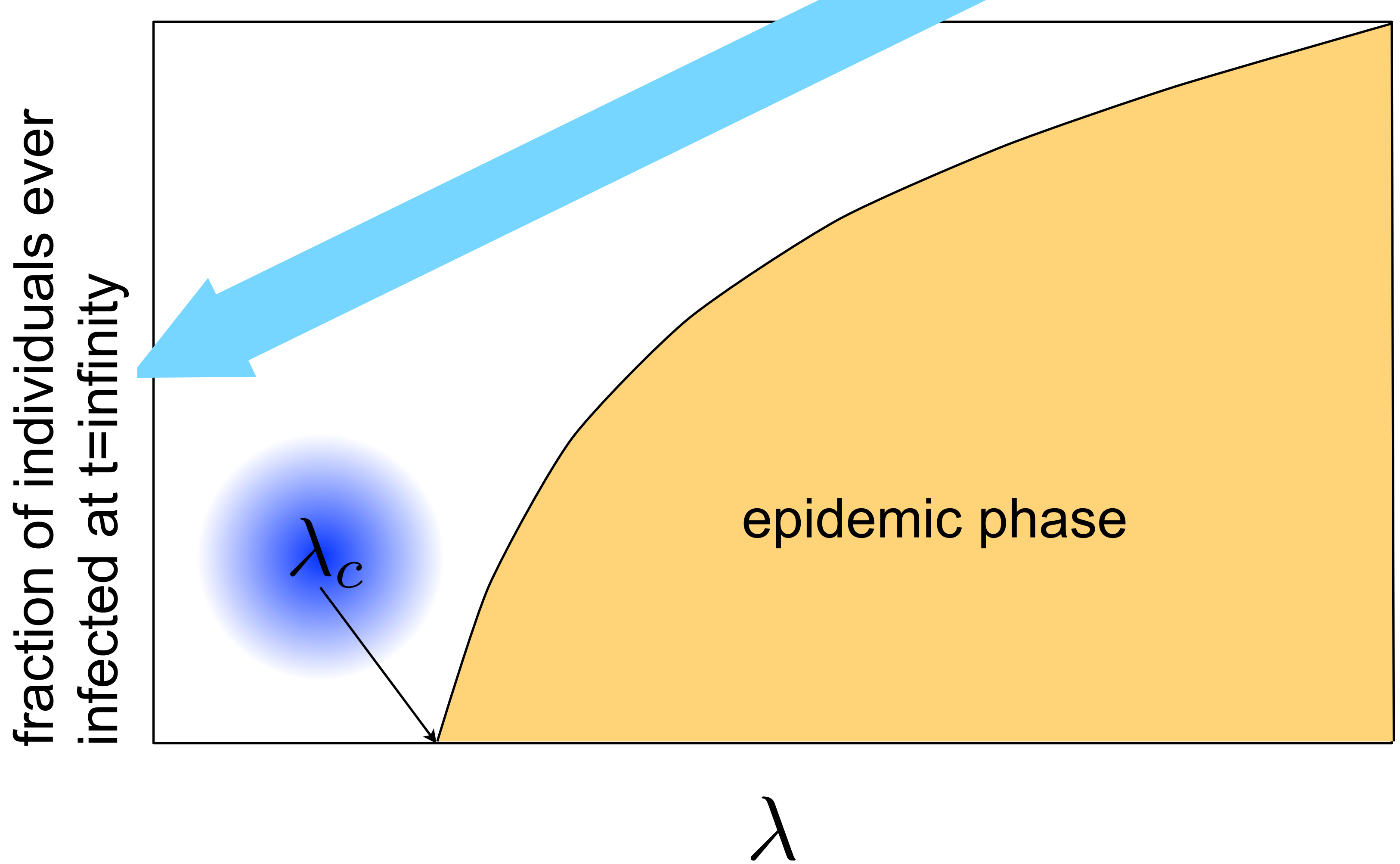

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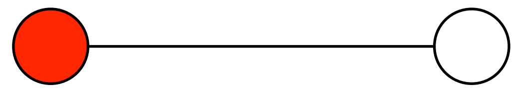
Beware!!!
The mapping to bond percolation may not be always exact

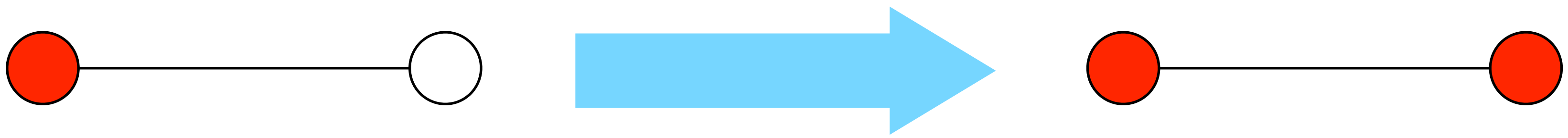
Mapping to bond percolation



M. E. J. Newman, Phys. Rev. E 66, 016128 (2002)

Mapping to bond percolation

probability of this event?



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time this node takes to get infected

$$T = \int_0^\infty d\tau \phi_r(\tau) \int_0^\tau d\tau' \psi_i(\tau')$$

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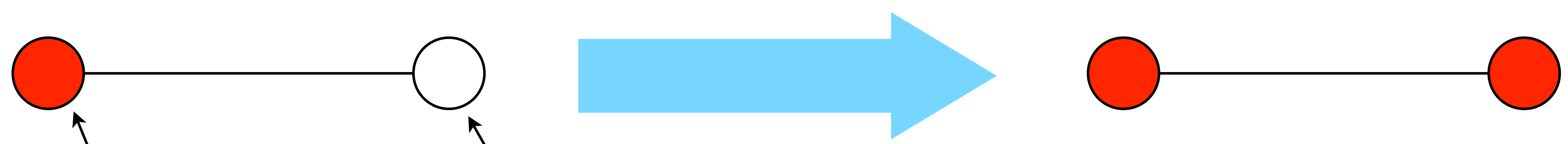

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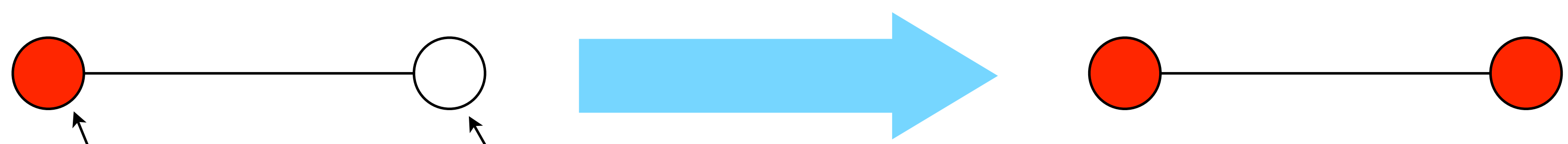

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T is mapped to bond percolation probability *p*

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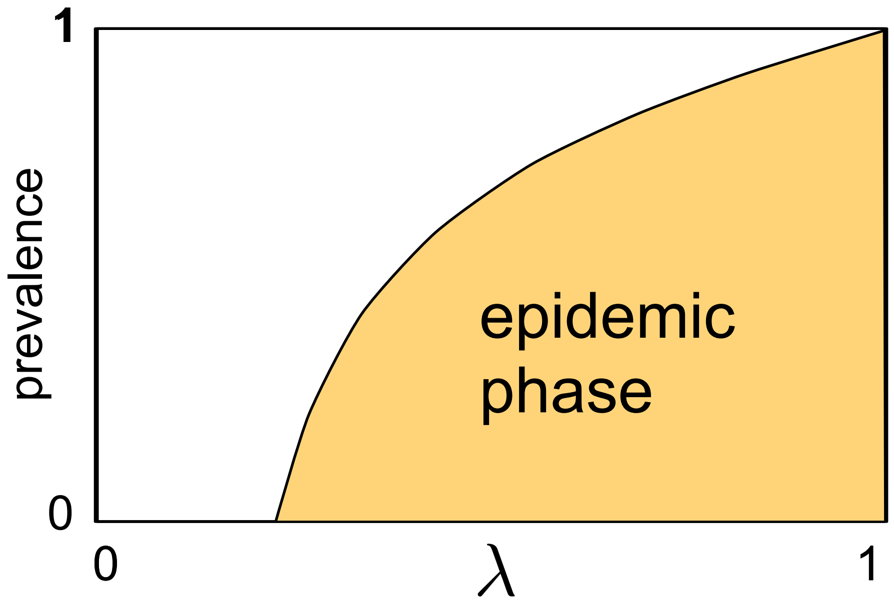
Classic results

epidemic threshold in maximally random graphs

(SIS) $$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$$

(SIR) $$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$



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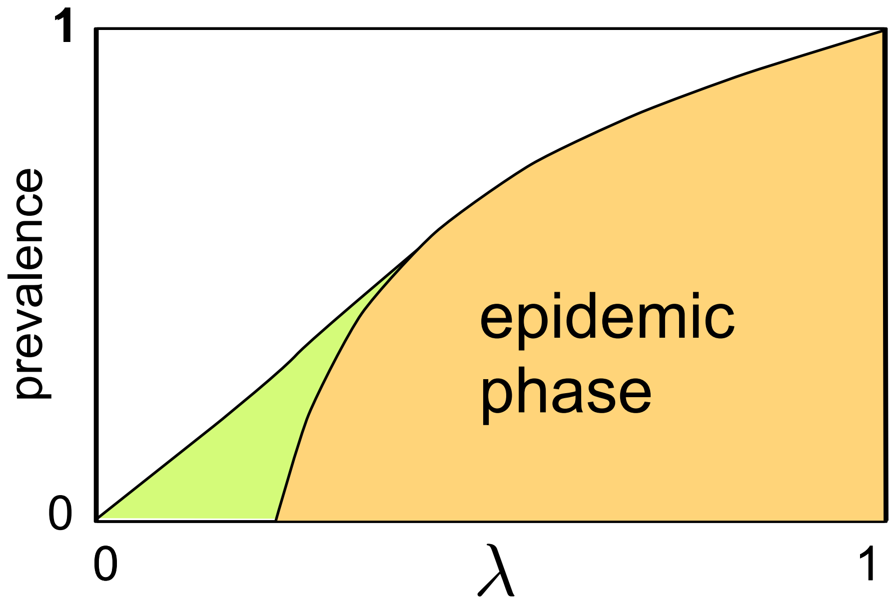

In Scale-free networks we have $P(k) \sim k^{-\gamma}$

$$\langle k^2 \rangle \rightarrow \infty \qquad \Longrightarrow \qquad \lambda_c \rightarrow 0$$

$$2 < \gamma \leq 3$$

Random SF networks can propagate any infective agent

R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. 86, 3200 (2001)

Outlook

- Basic results on percolation theory on lattices
  - Scaling theory
  - Relation between critical exponents
  - Site percolation on Cayley trees
  - Finite Size Scaling
- Bond percolation on networks
  - Undirected random graphs
  - Directed random graphs
  - Clustered graphs
  - Self-similar graphs

Basic bibliography

- Introduction To Percolation Theory, Dietrich Stauffer
- Networks: An introduction, M. E. J. Newman
- Evolution of Networks: From Biological Nets to the Internet and WWW, S. N. Dorogovtsev and J. F. F. Mendes
- Lectures on complex networks, S. N. Dorogovtsev

Basics on percolation theory

There is a critical occupation probability p_c separating two different topological phases

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At $p=p_c$ → Clusters of all sizes
Fractal structure

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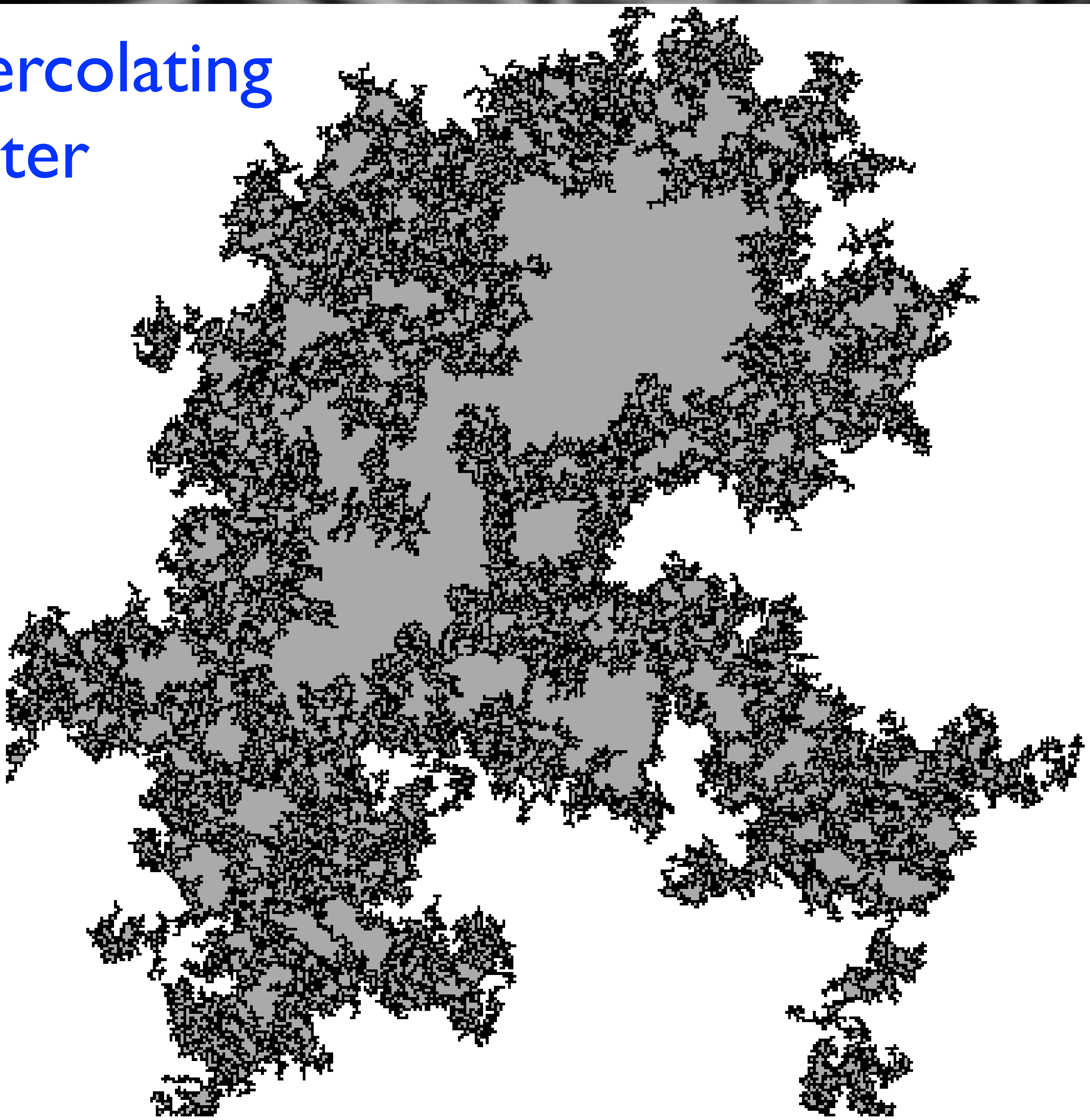
Critical percolating cluster



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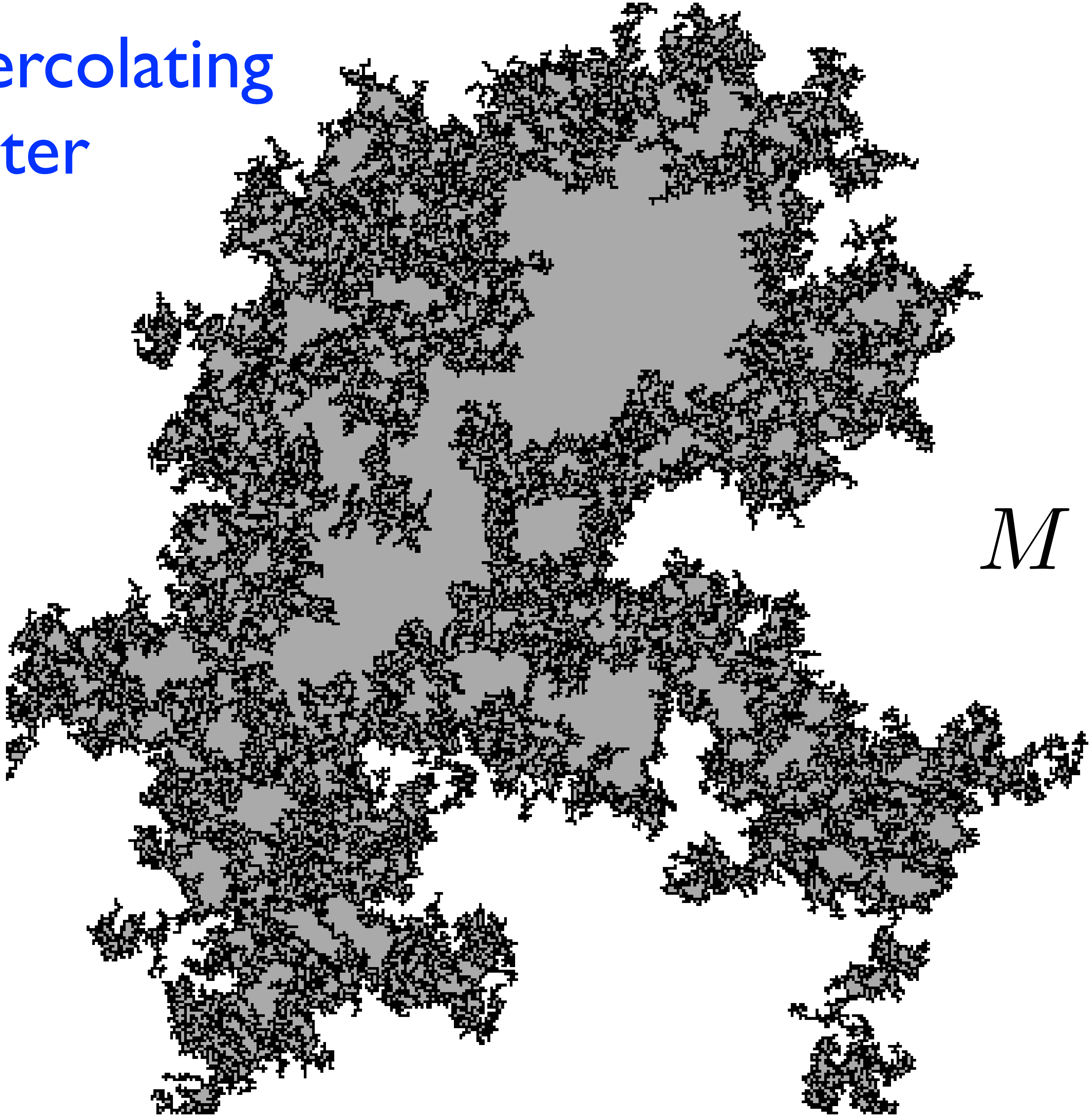

$$M \sim L^{d_f}$$

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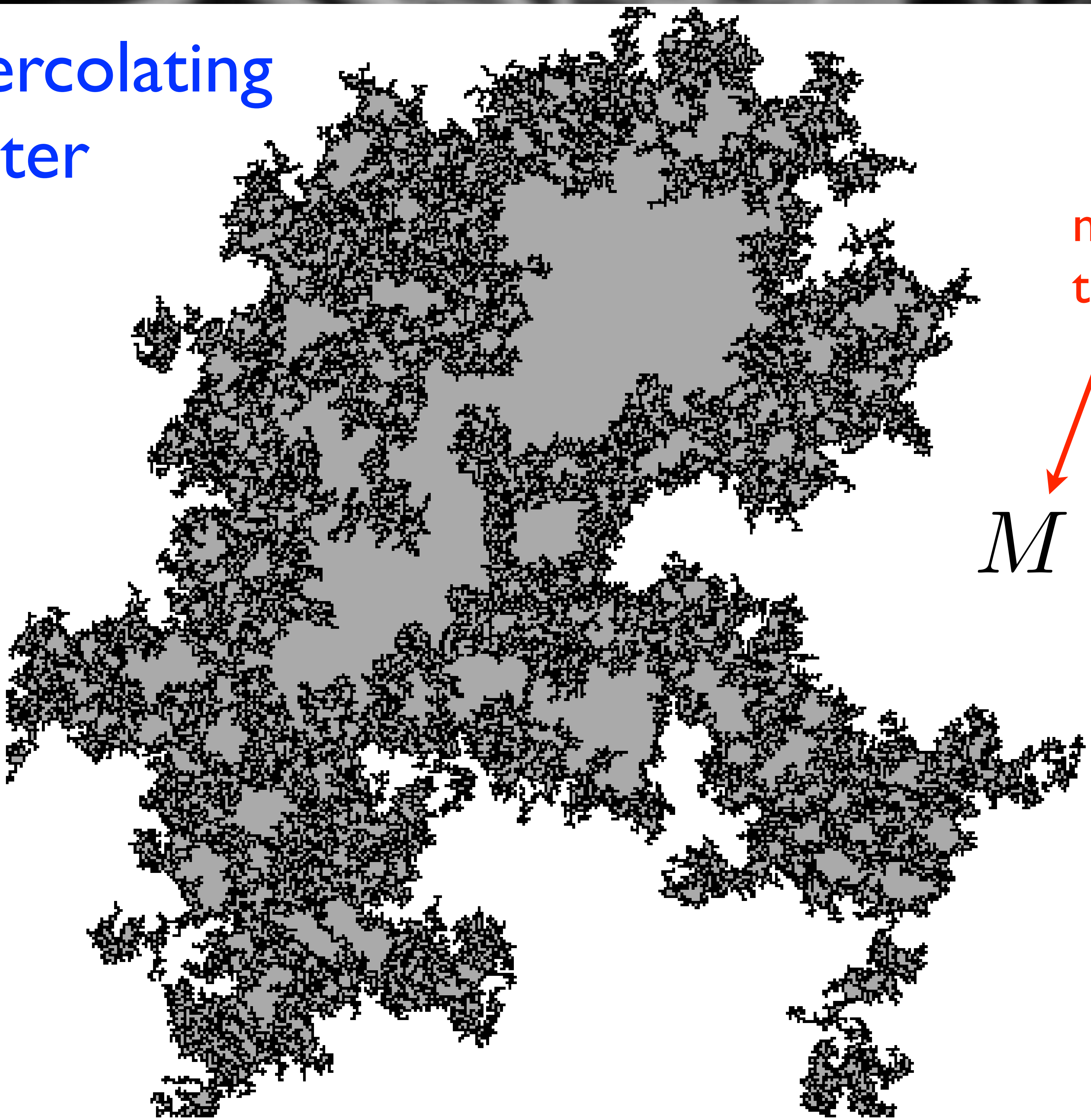

mass of the cluster

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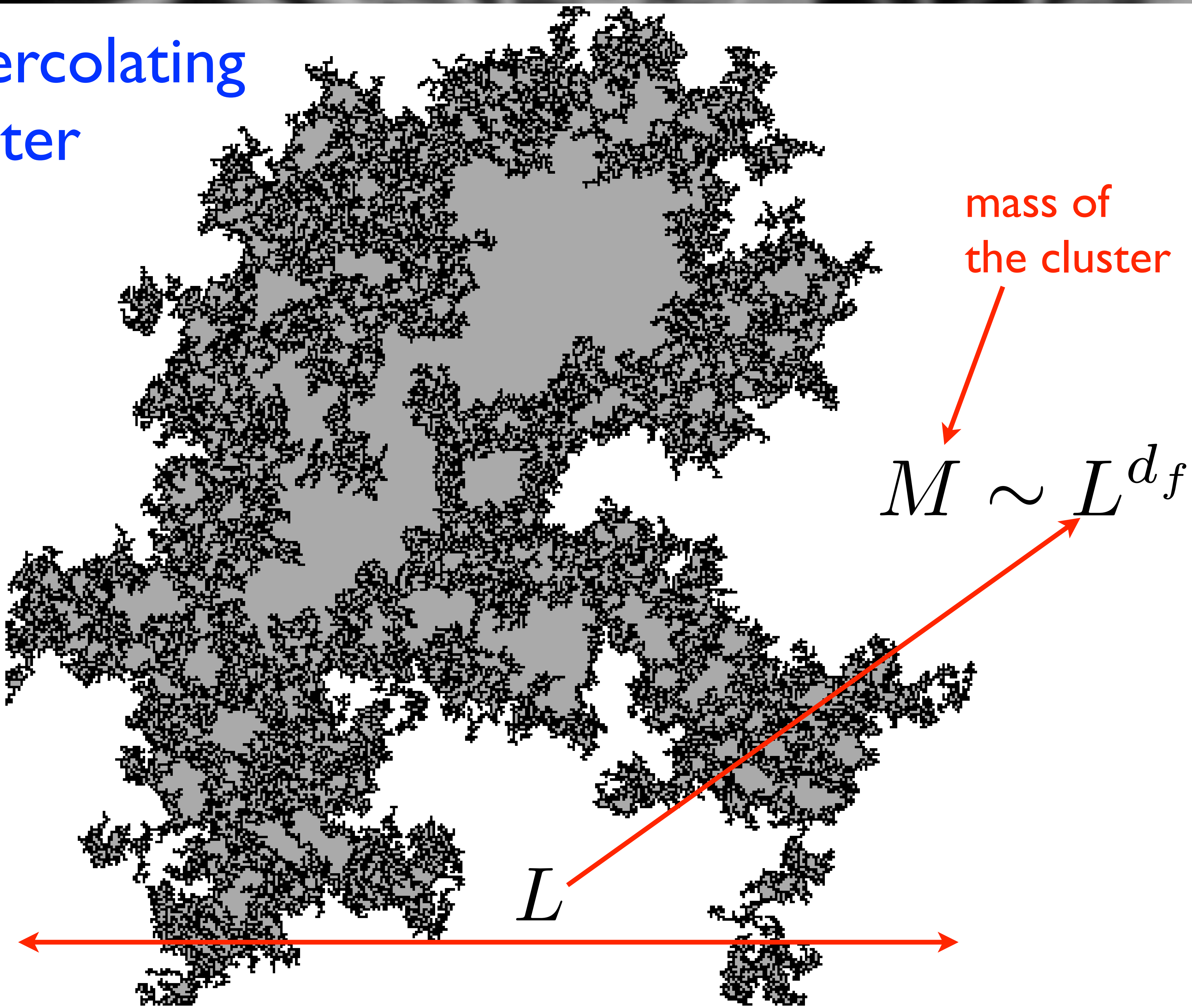

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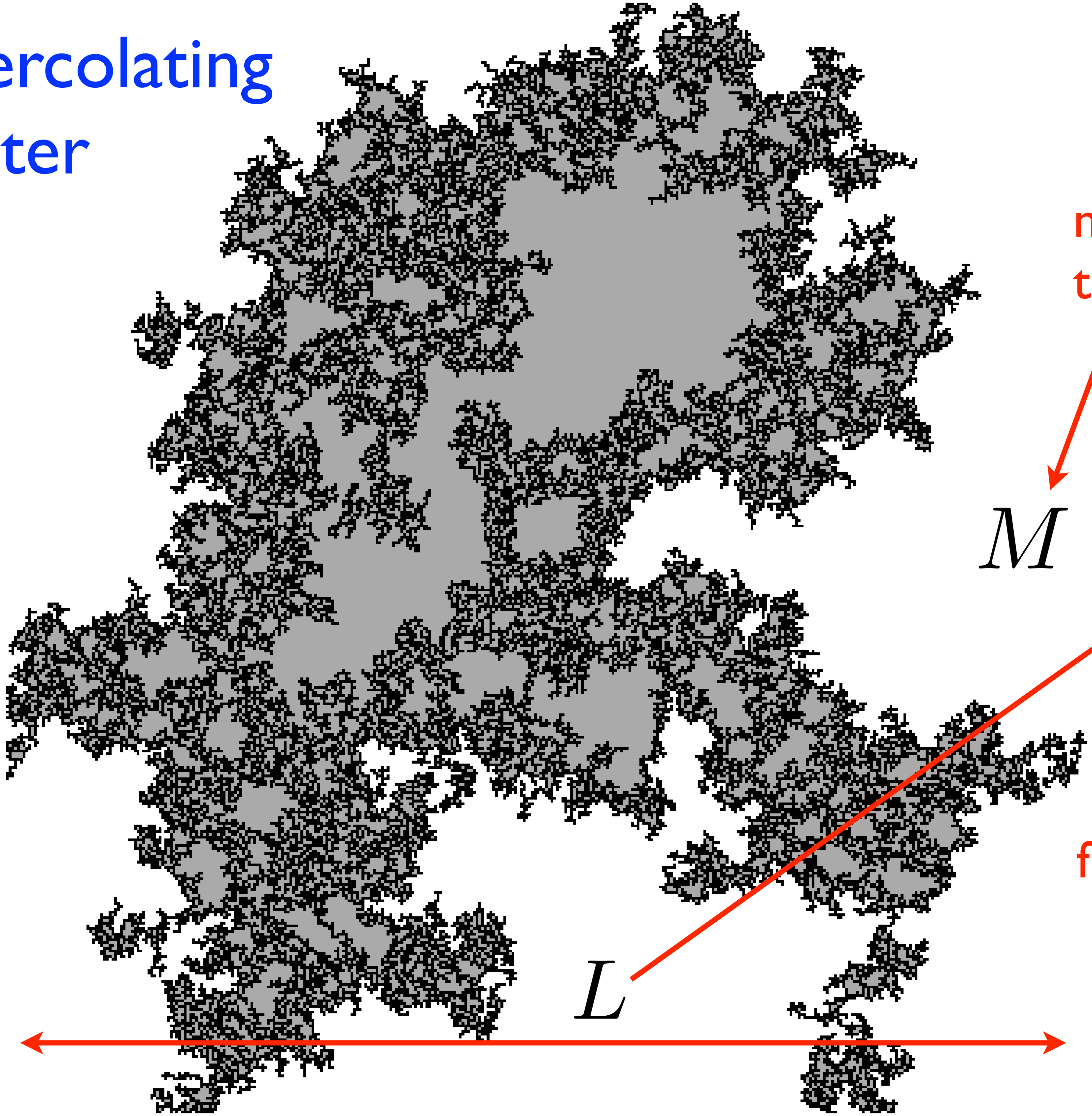

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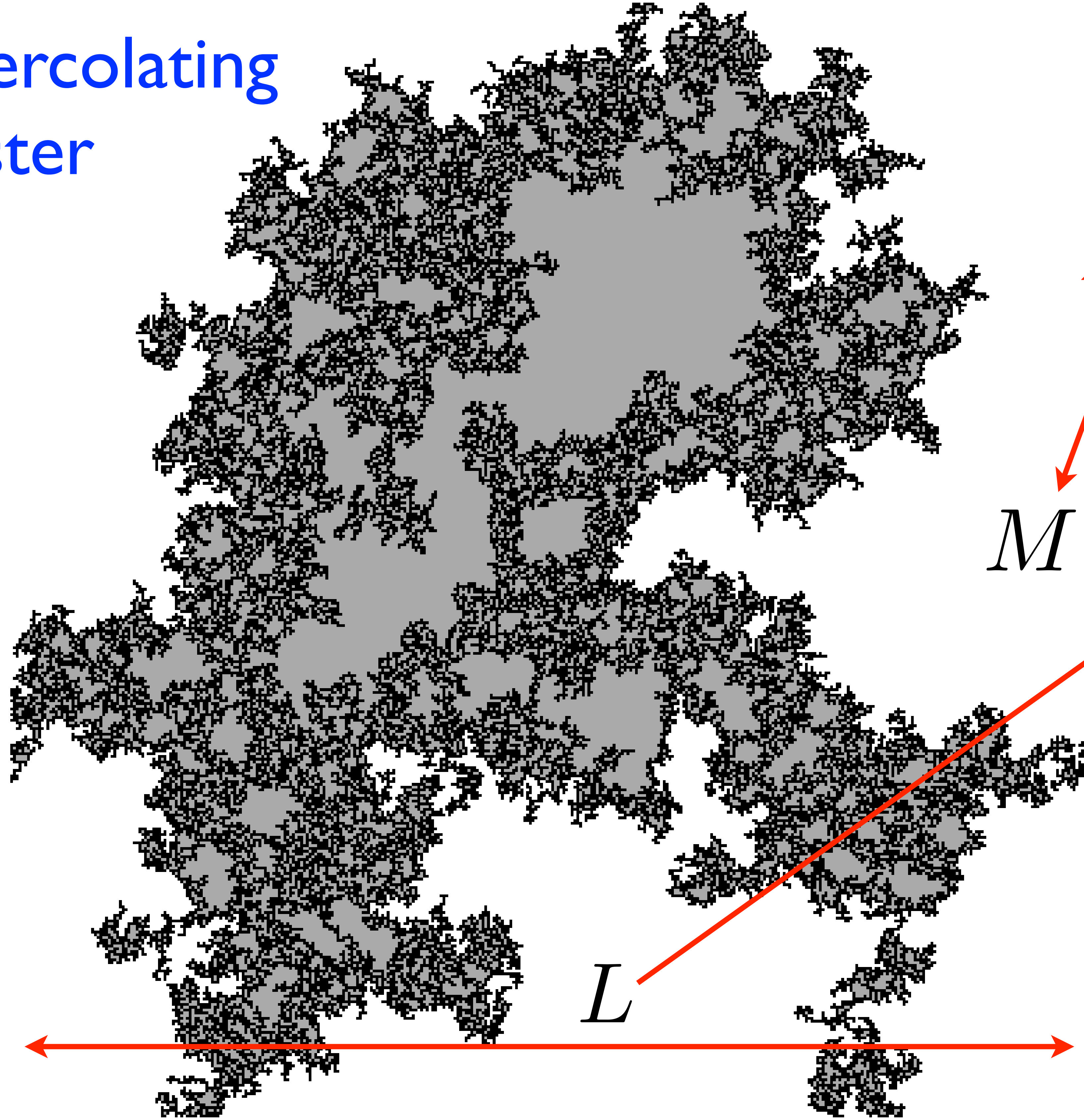

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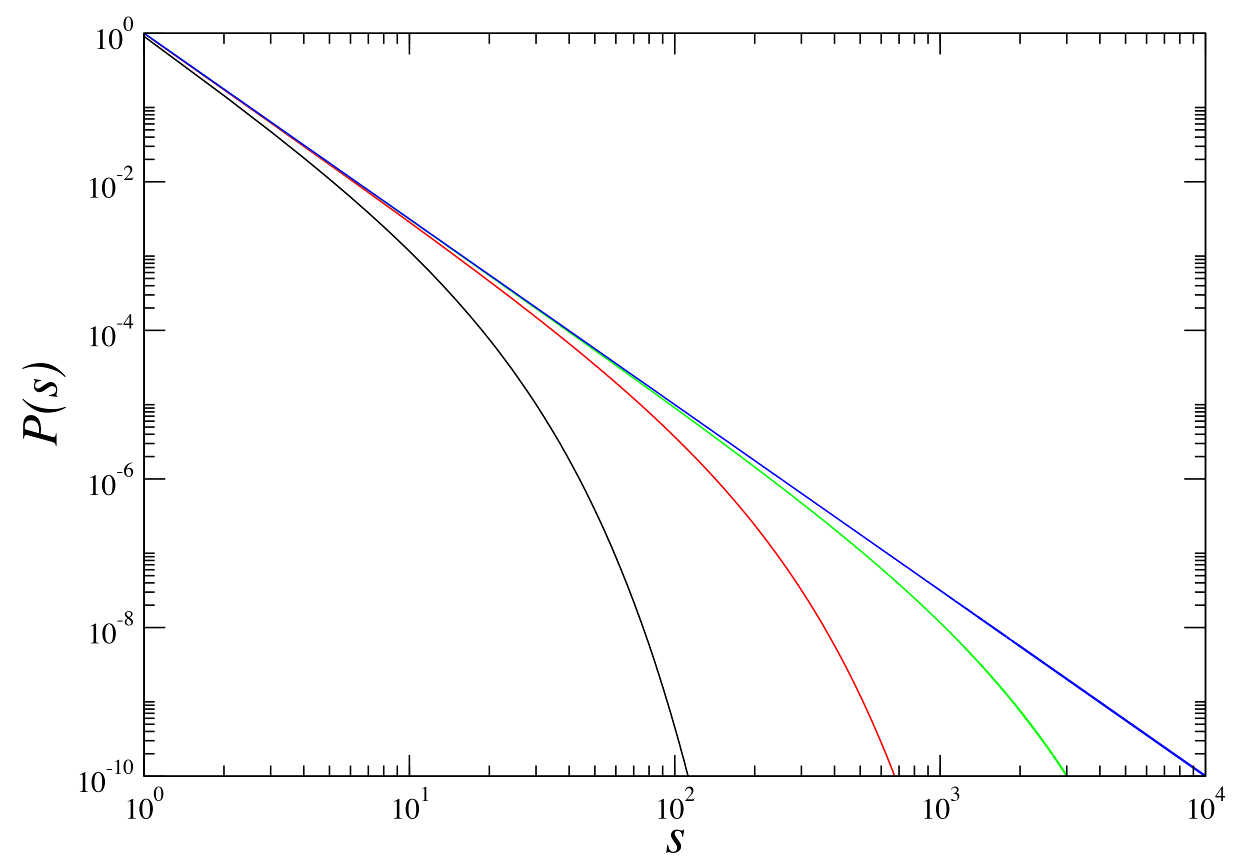

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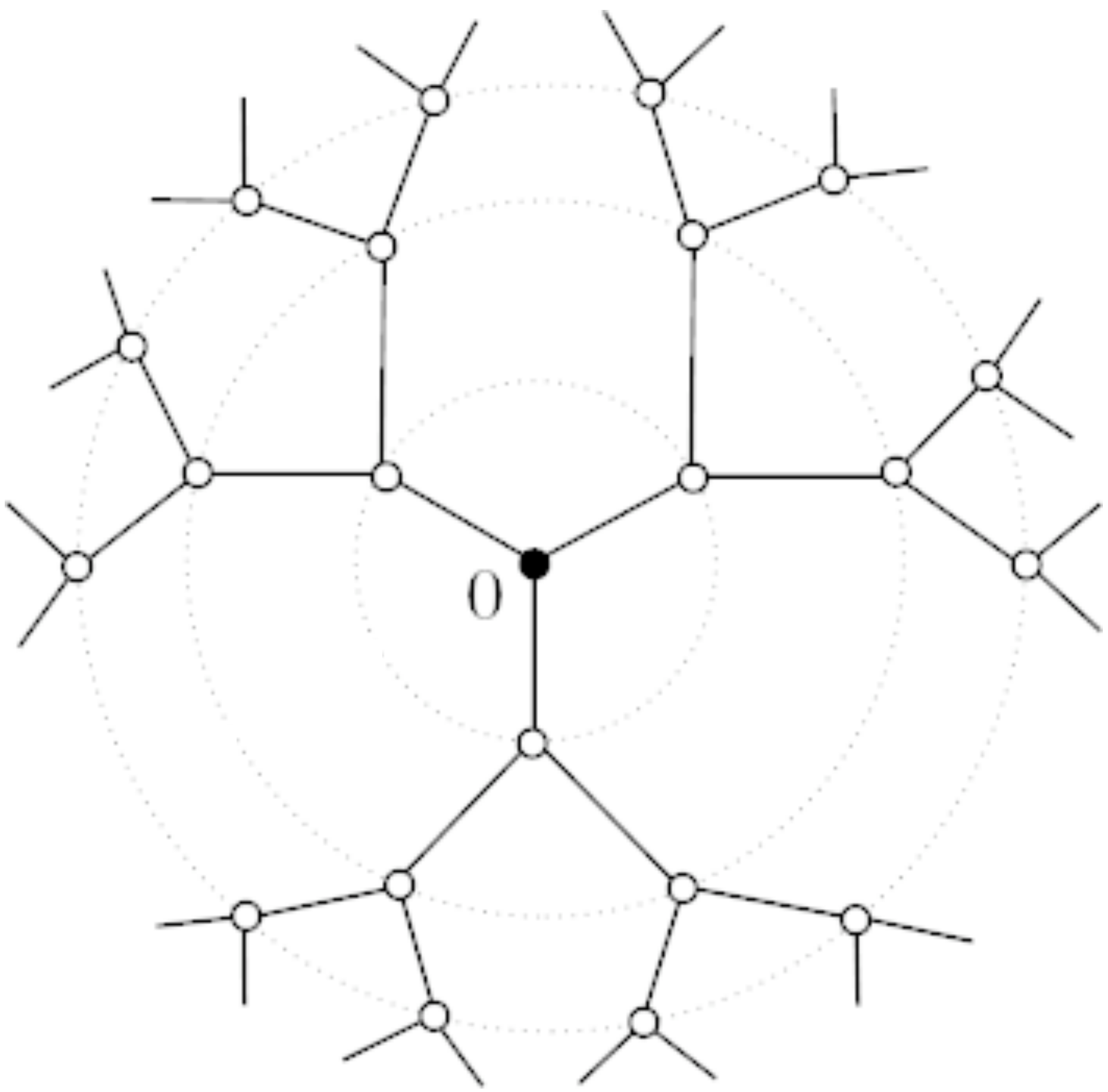
Site percolation on a Cayley tree (Bethe lattice)



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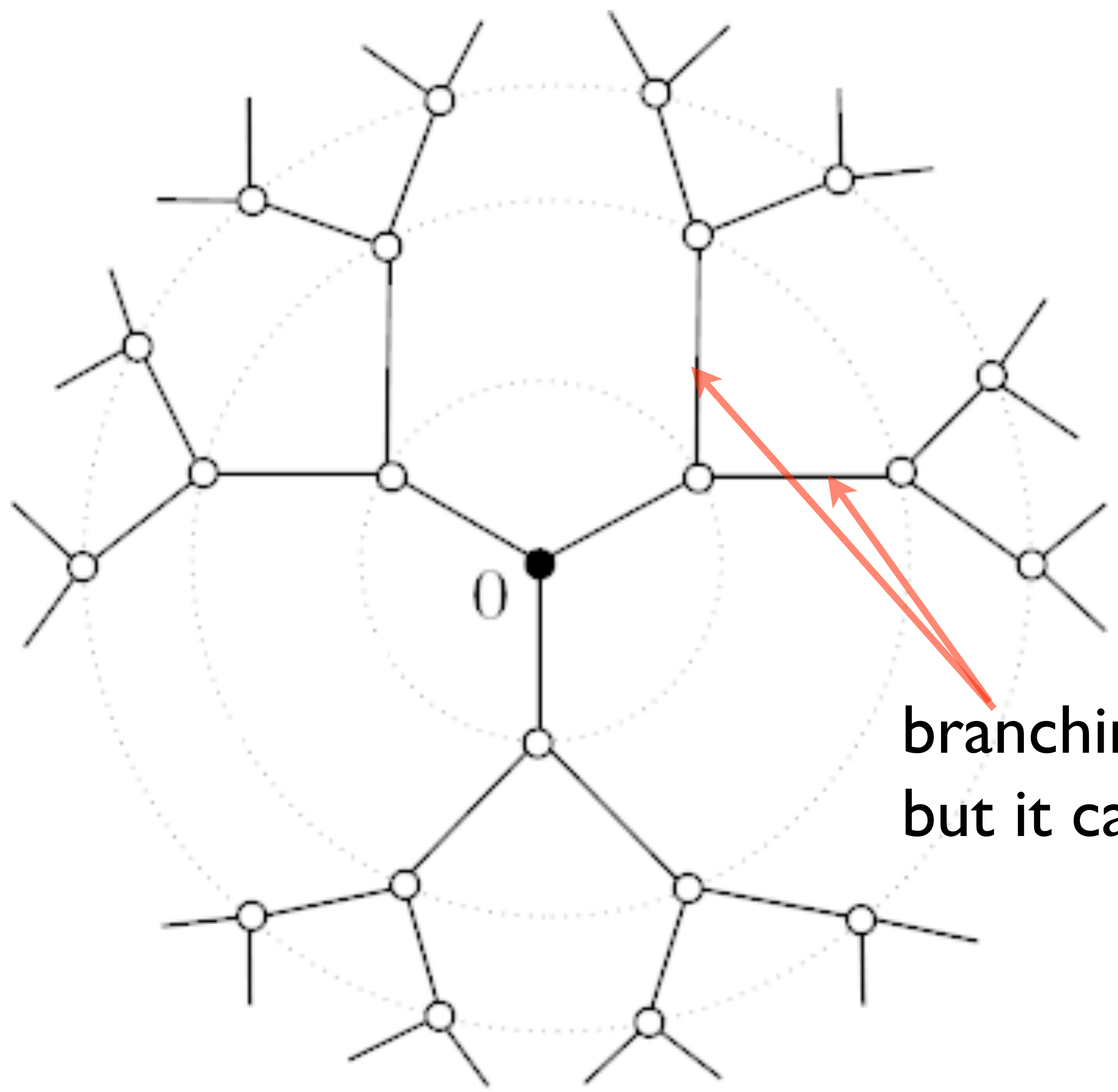
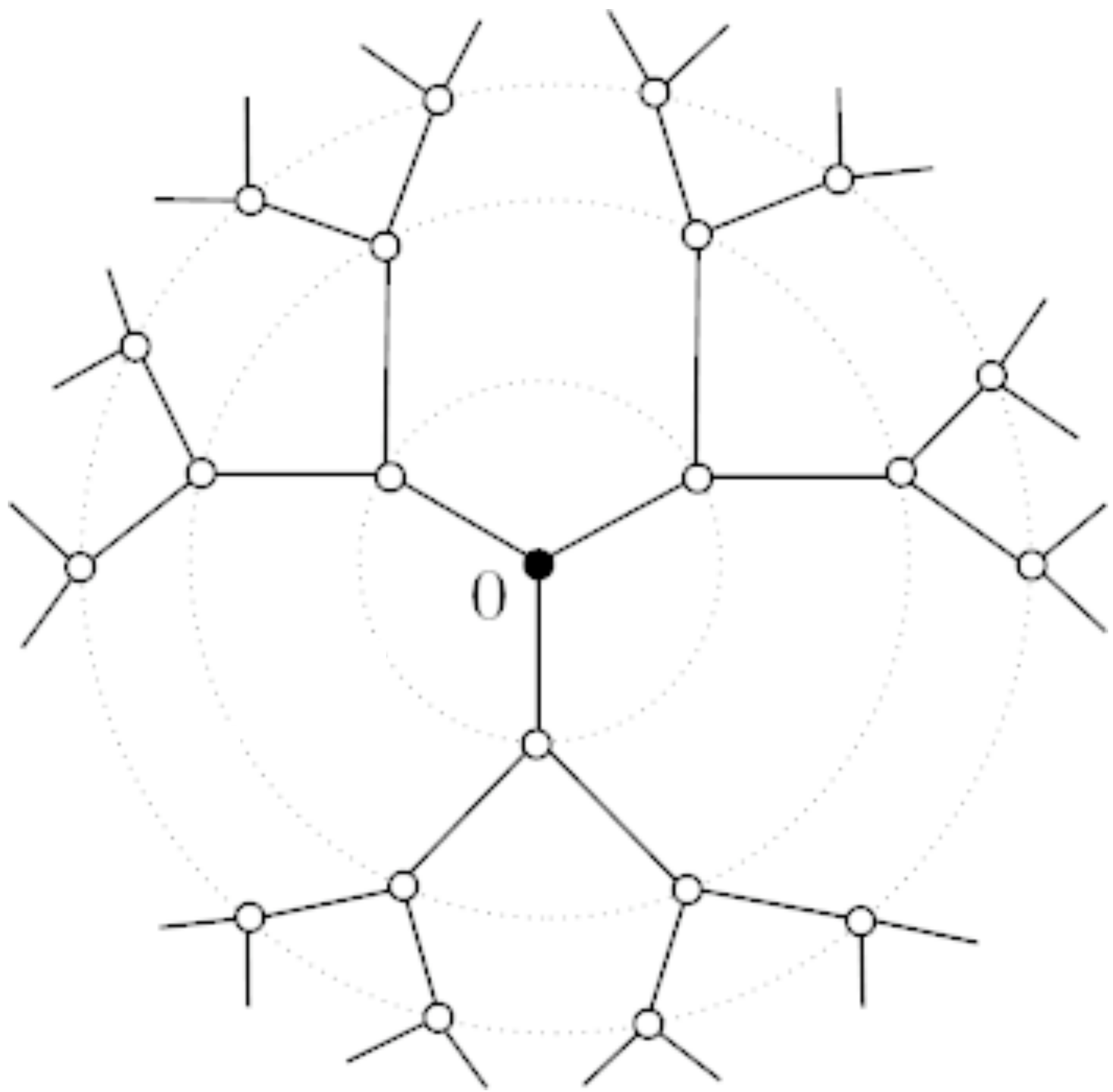


branching factor q=2
but it can be any integer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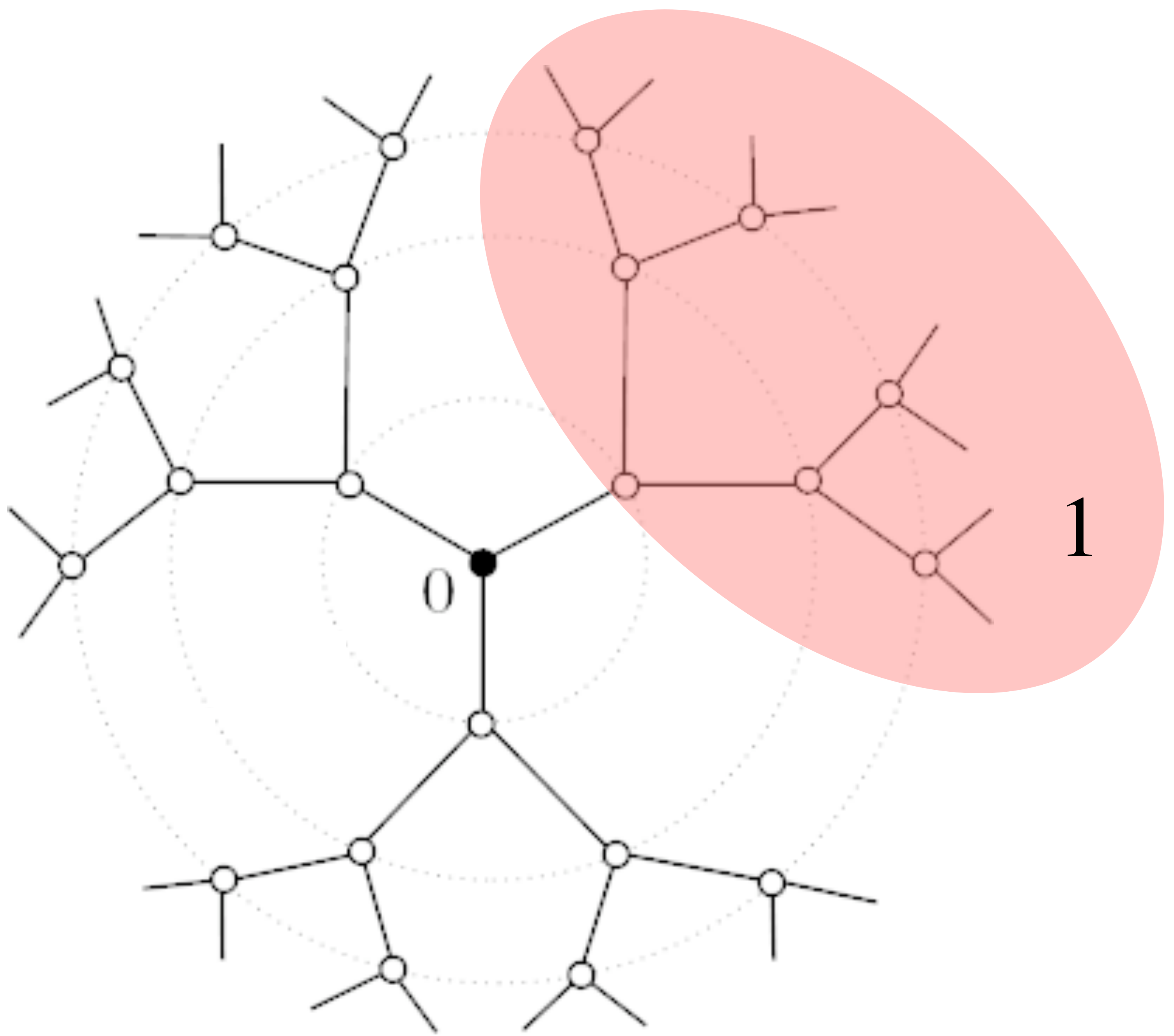
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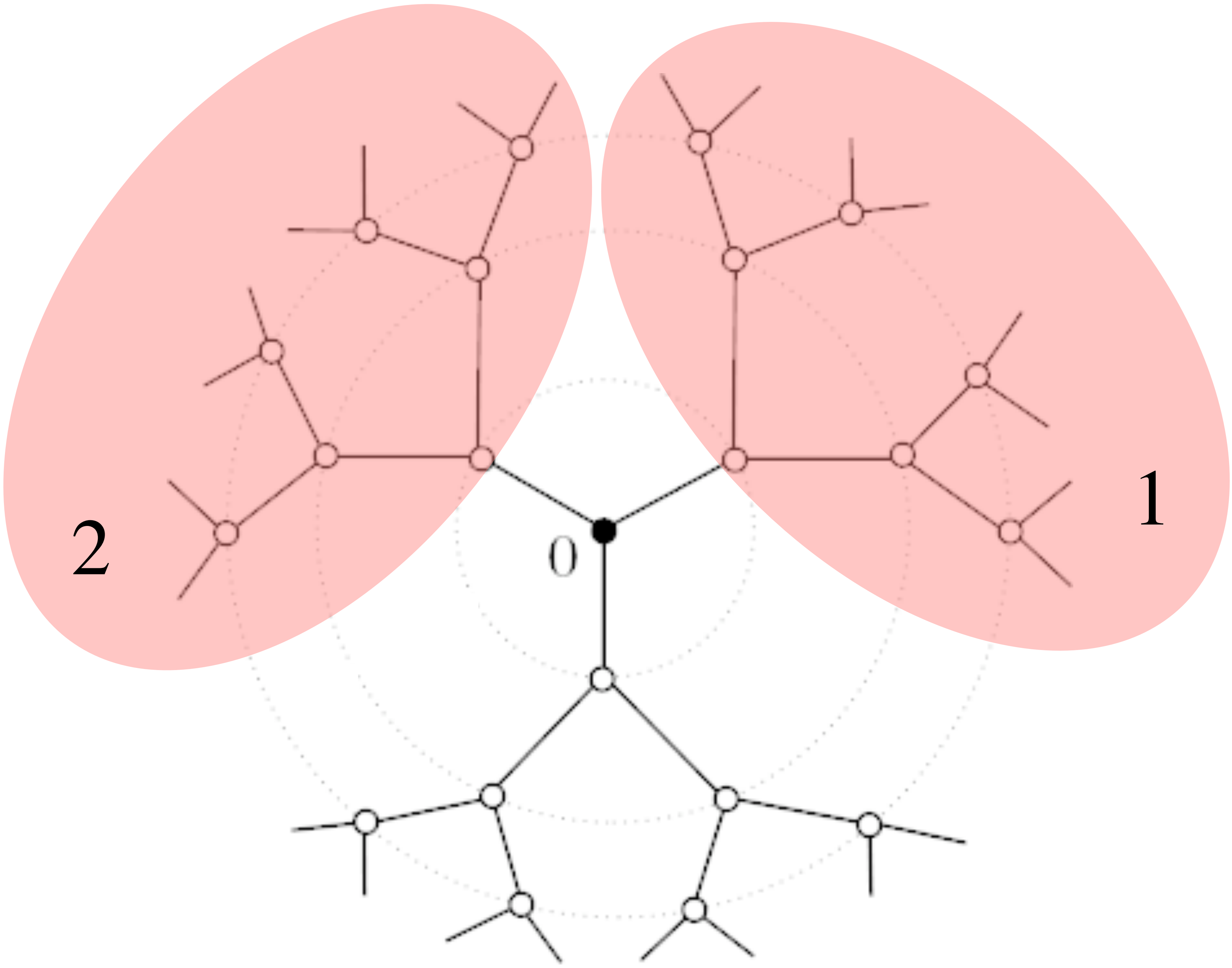
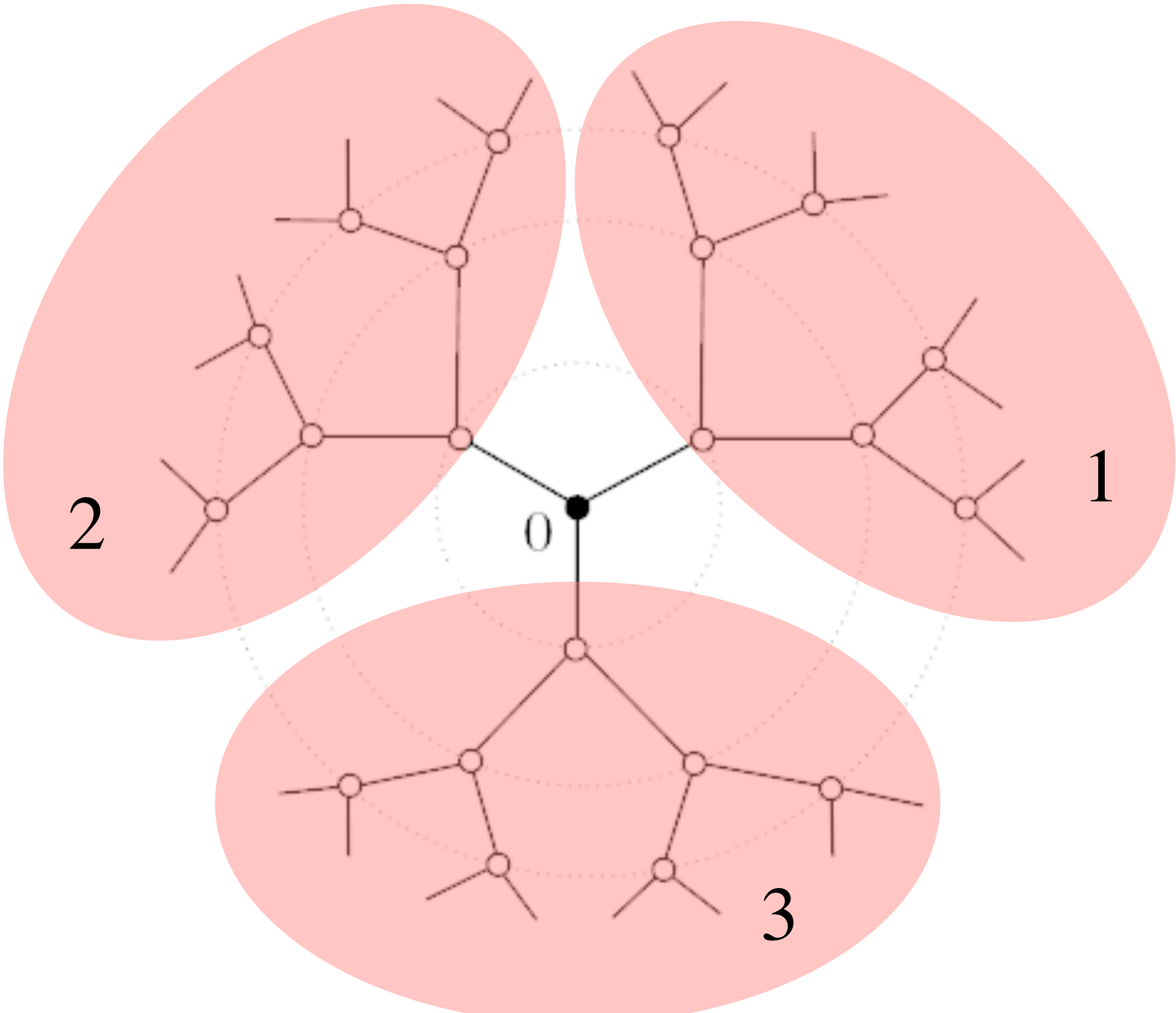


2
0
1



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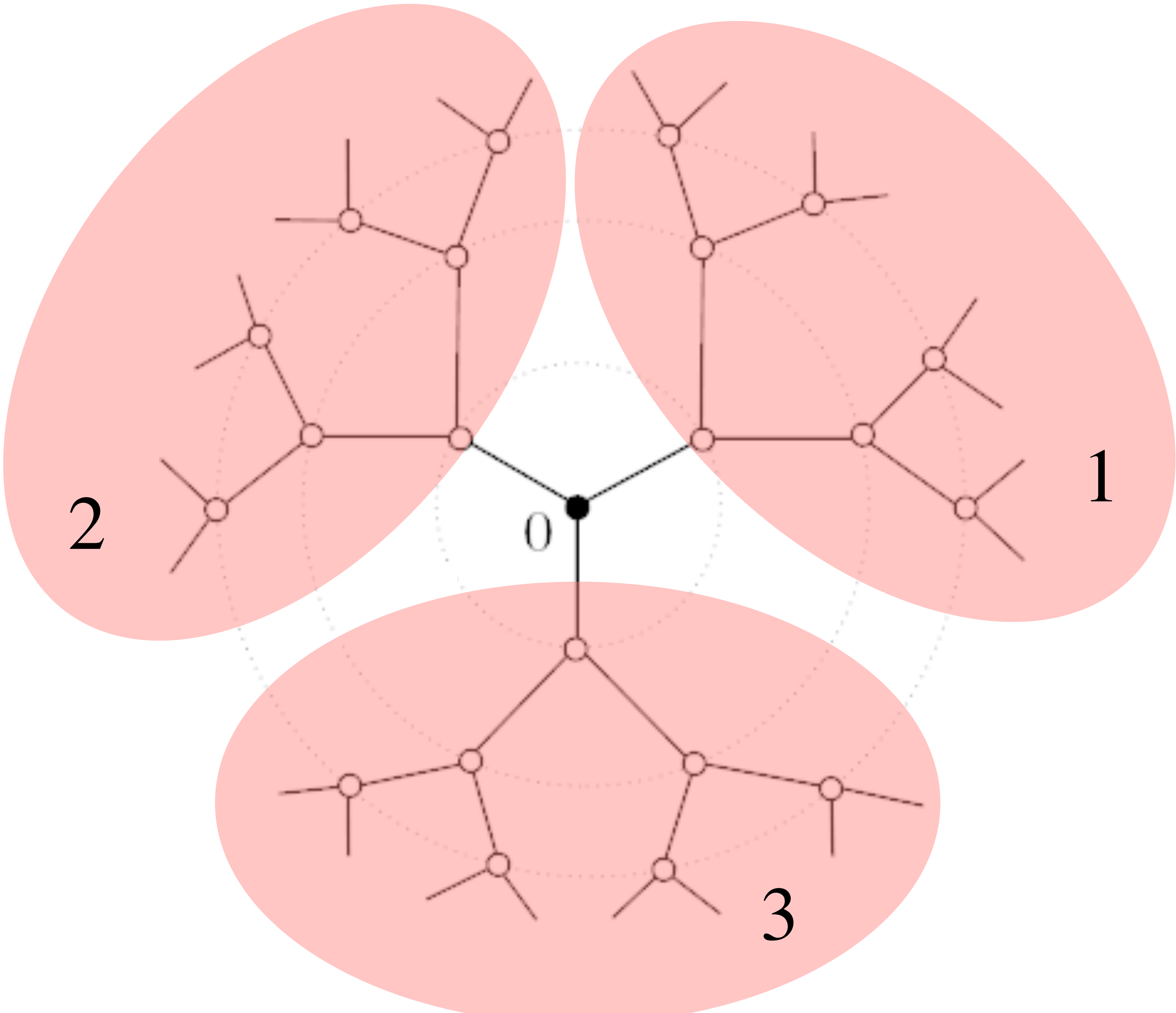
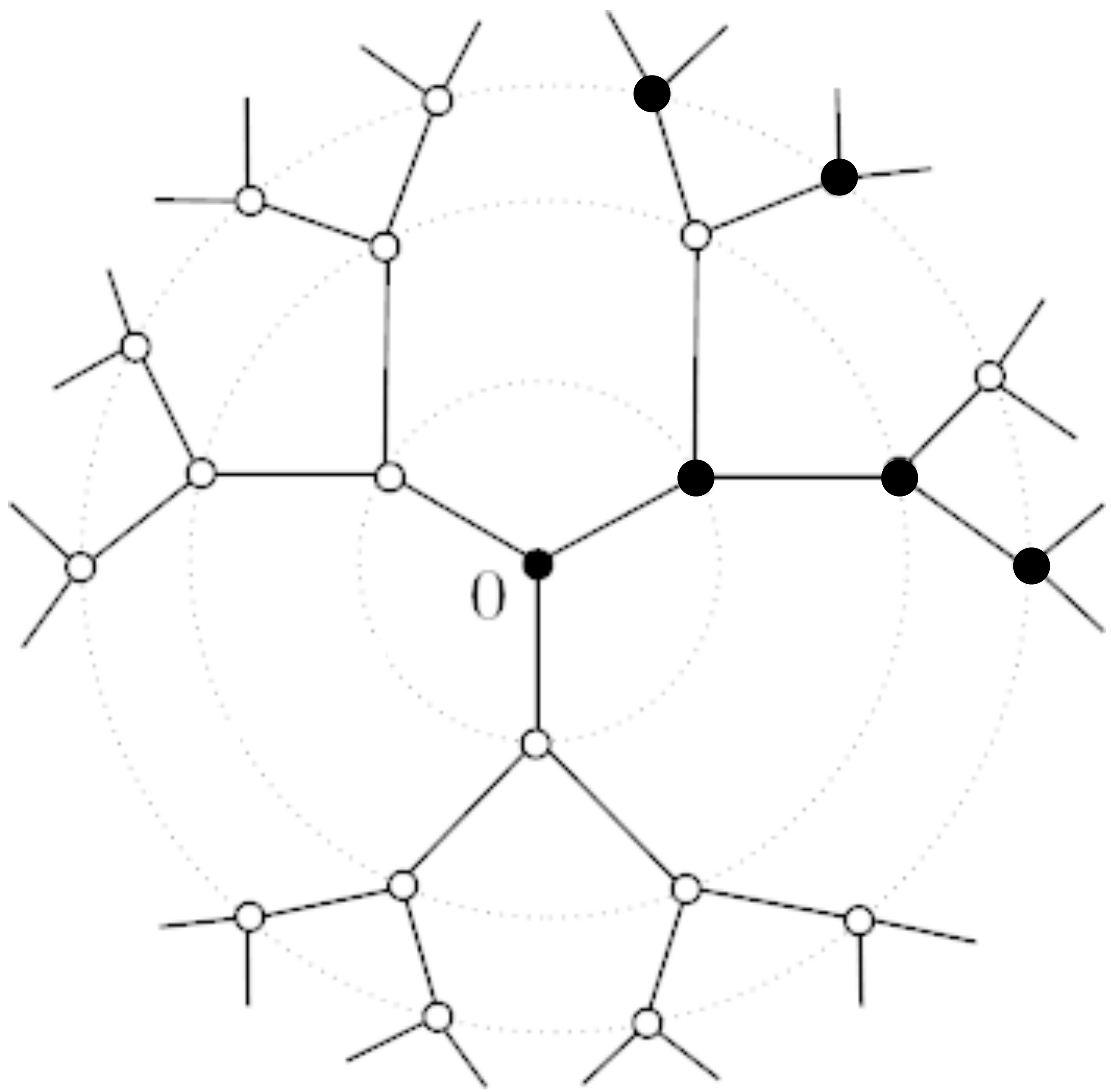


size of cluster 0=1+(size of branch 1)+(size of branch 2)+(size of branch 3)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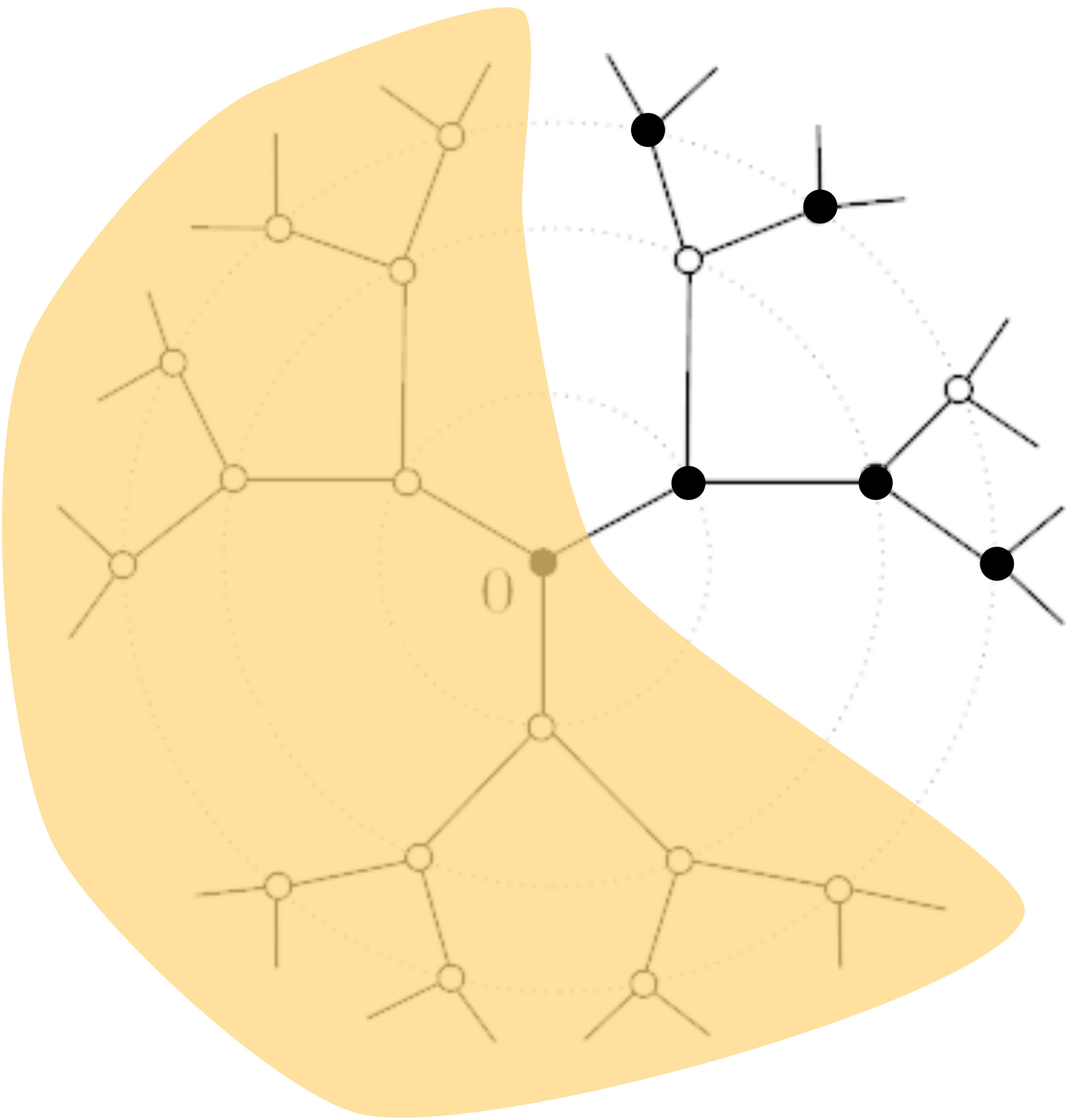
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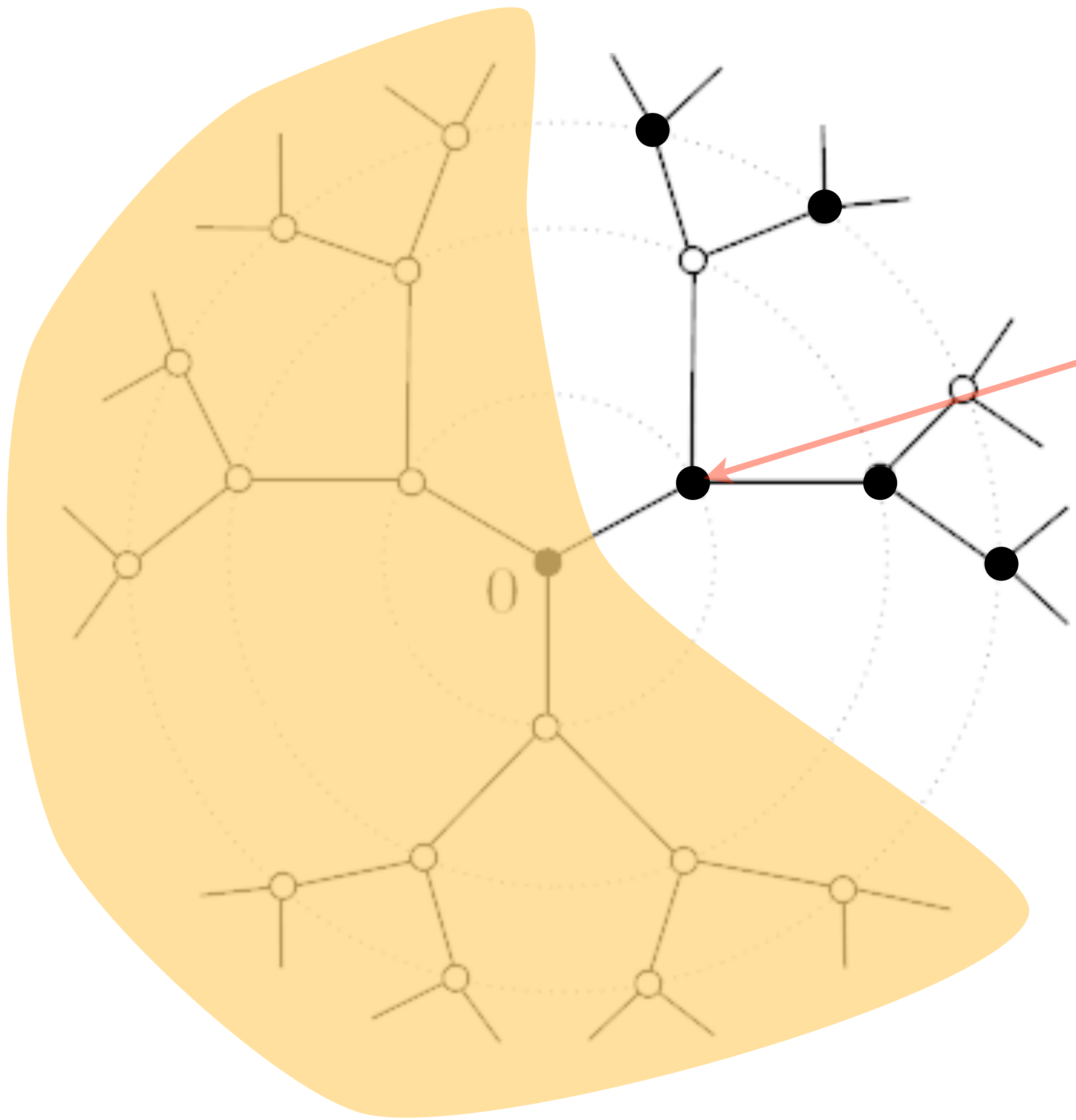
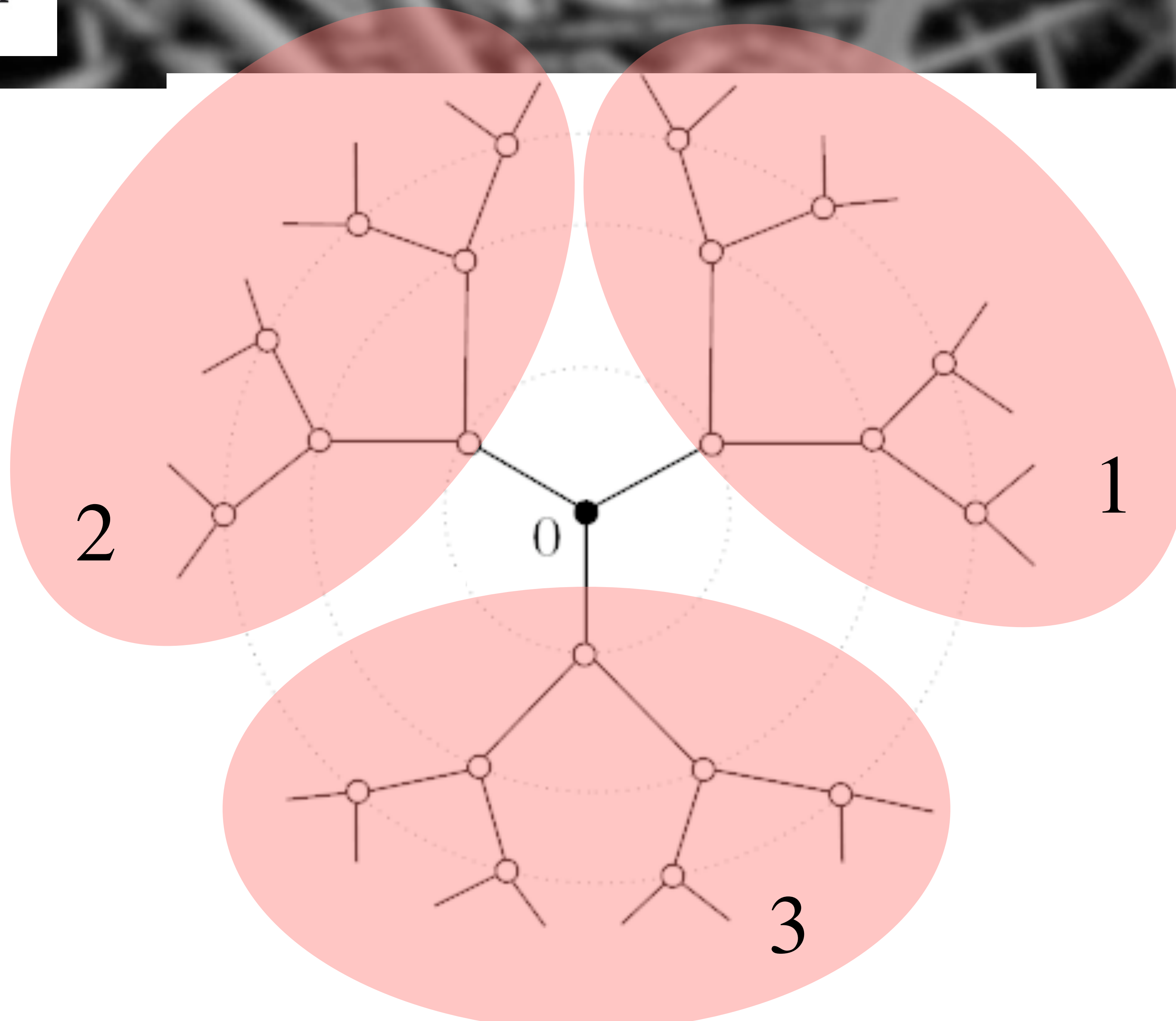
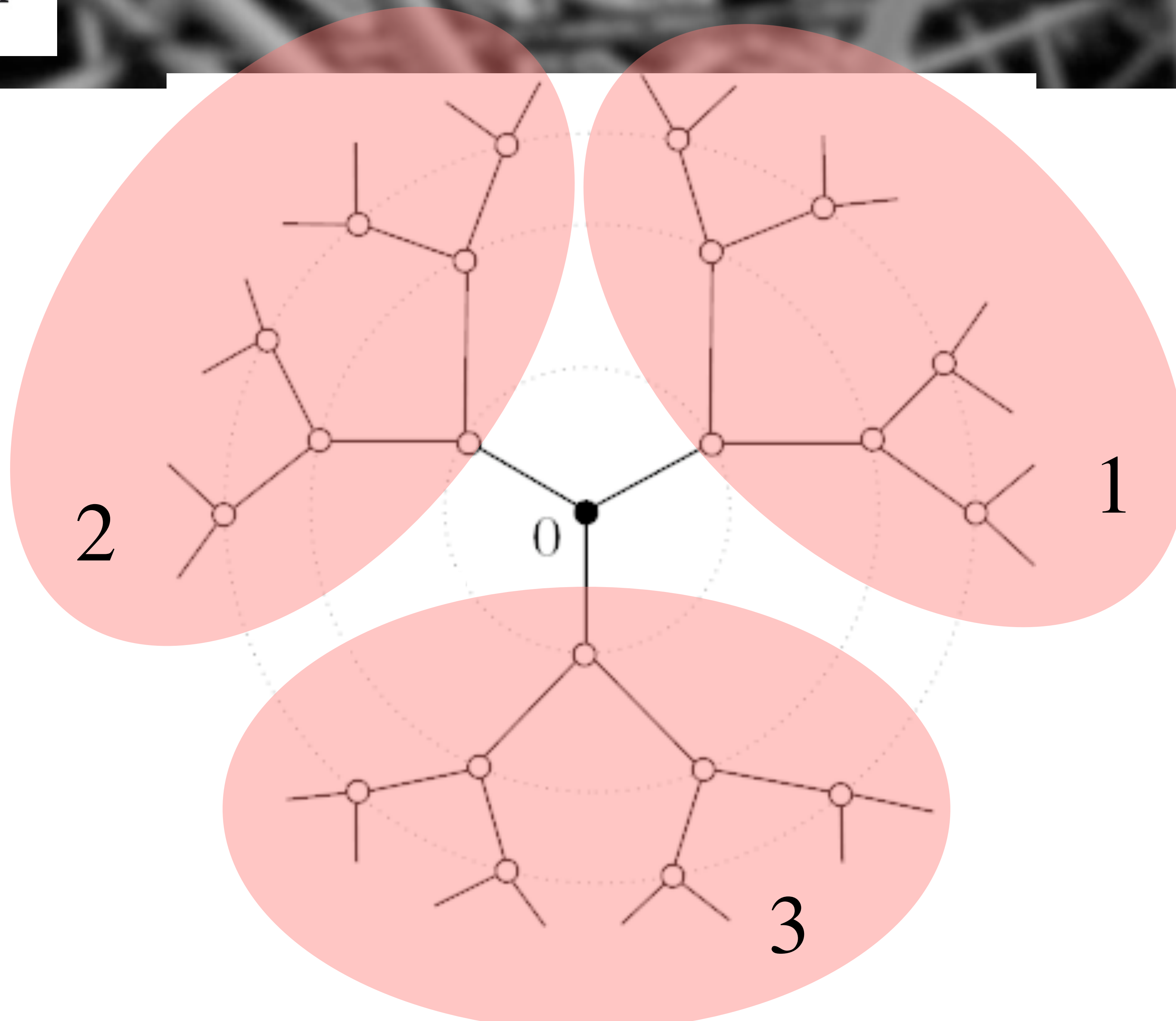
$G(s)$

probability that this branch has s reachable active nodes





$$P(s) = (1-p)\delta_{s,0} + p(1-\delta_{s,0}) \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \cdots \sum_{s_{q+1}=0}^{\infty} G(s_1)G(s_2)\cdots G(s_{q+1})$$



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Characteristic funtions

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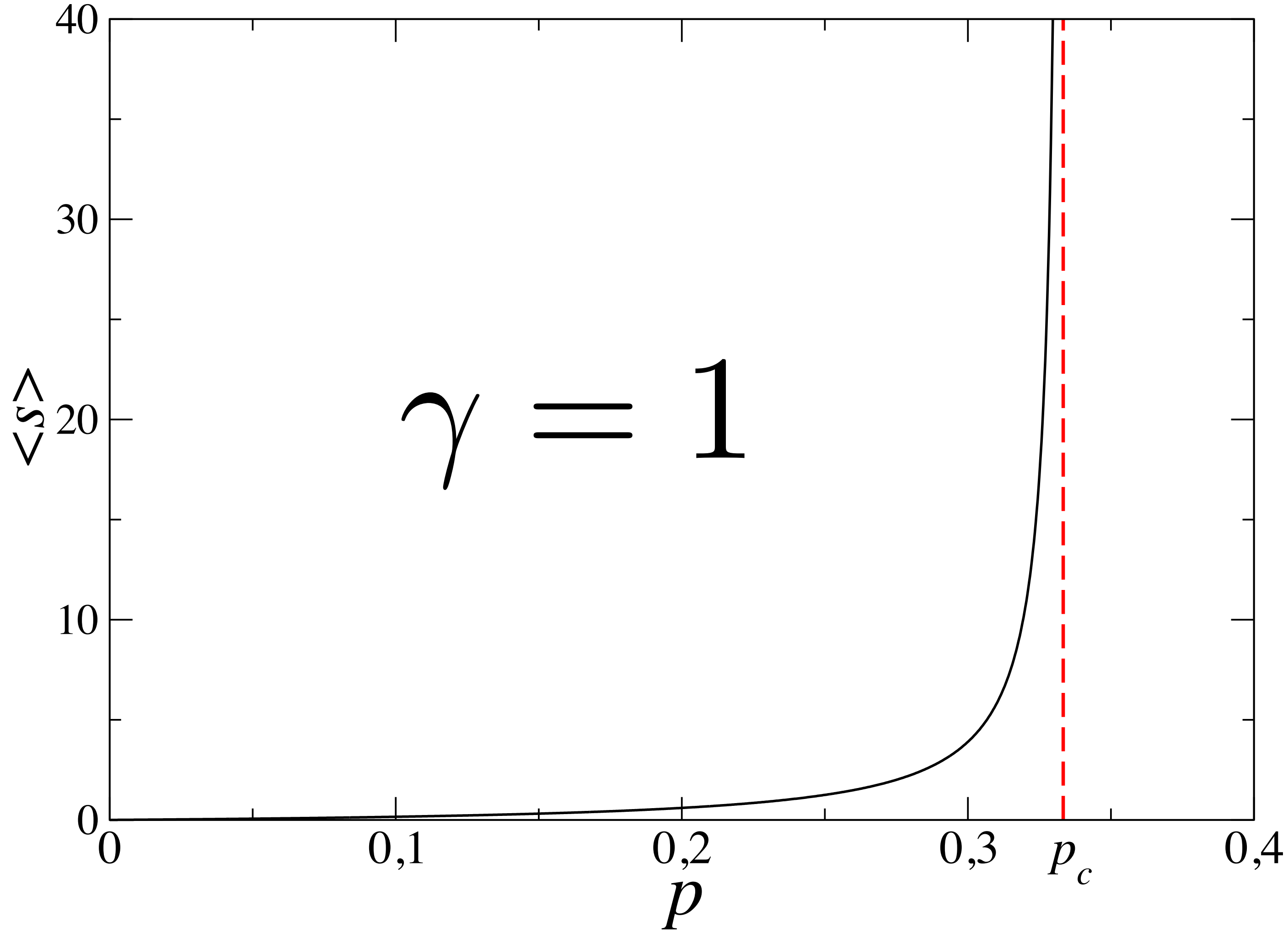
also, from the first eq. $\left.\frac{d\hat{G}(z)}{dz}\right|_{z=1} = p + pq\left.\frac{d\hat{G}(z)}{dz}\right|_{z=1}$

Finally

$$\langle s \rangle = \frac{p(1+p)}{1-pq} \sim (p_c - p)^{-1} \text{ with } p_c = q^{-1}$$



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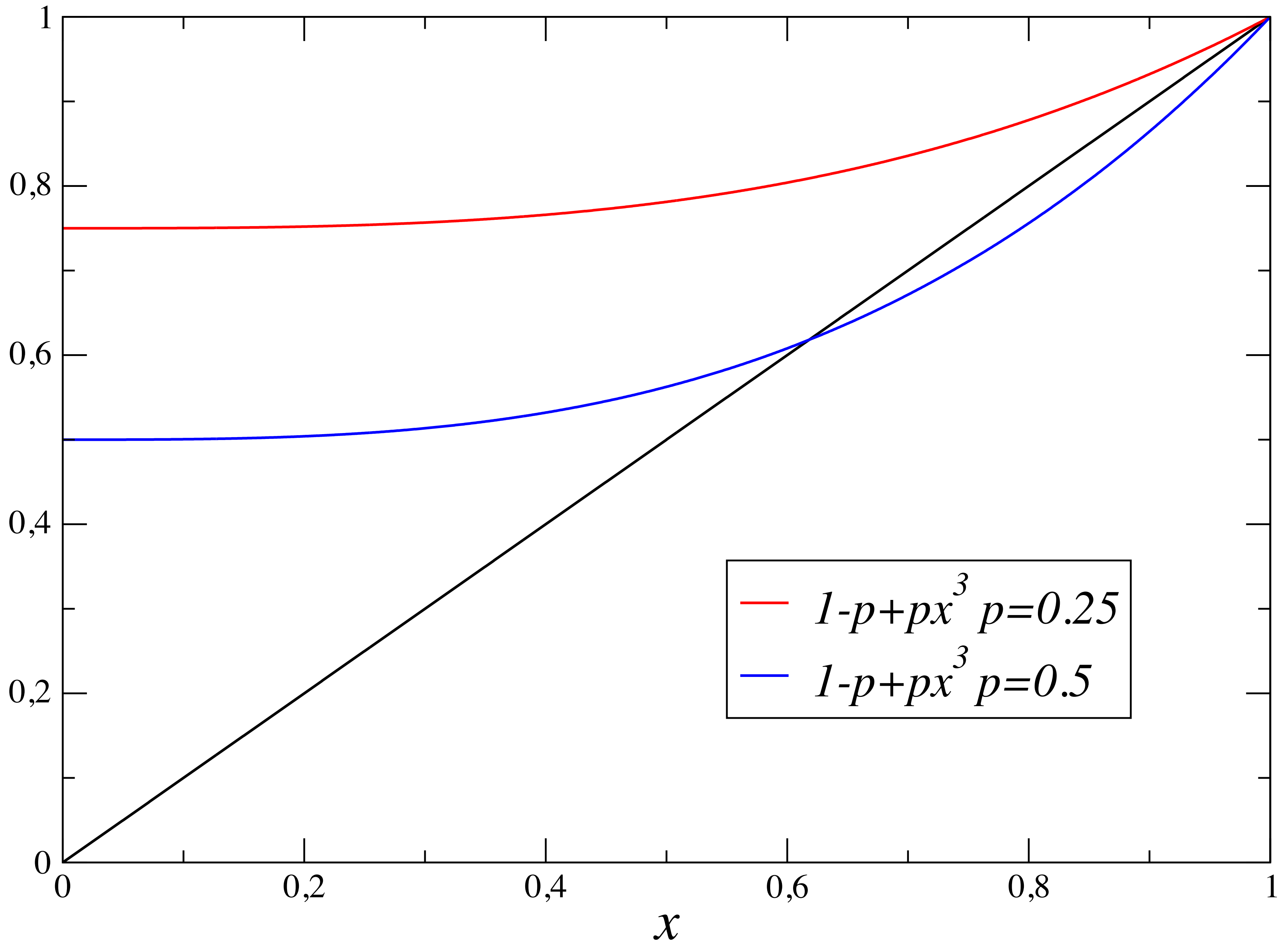

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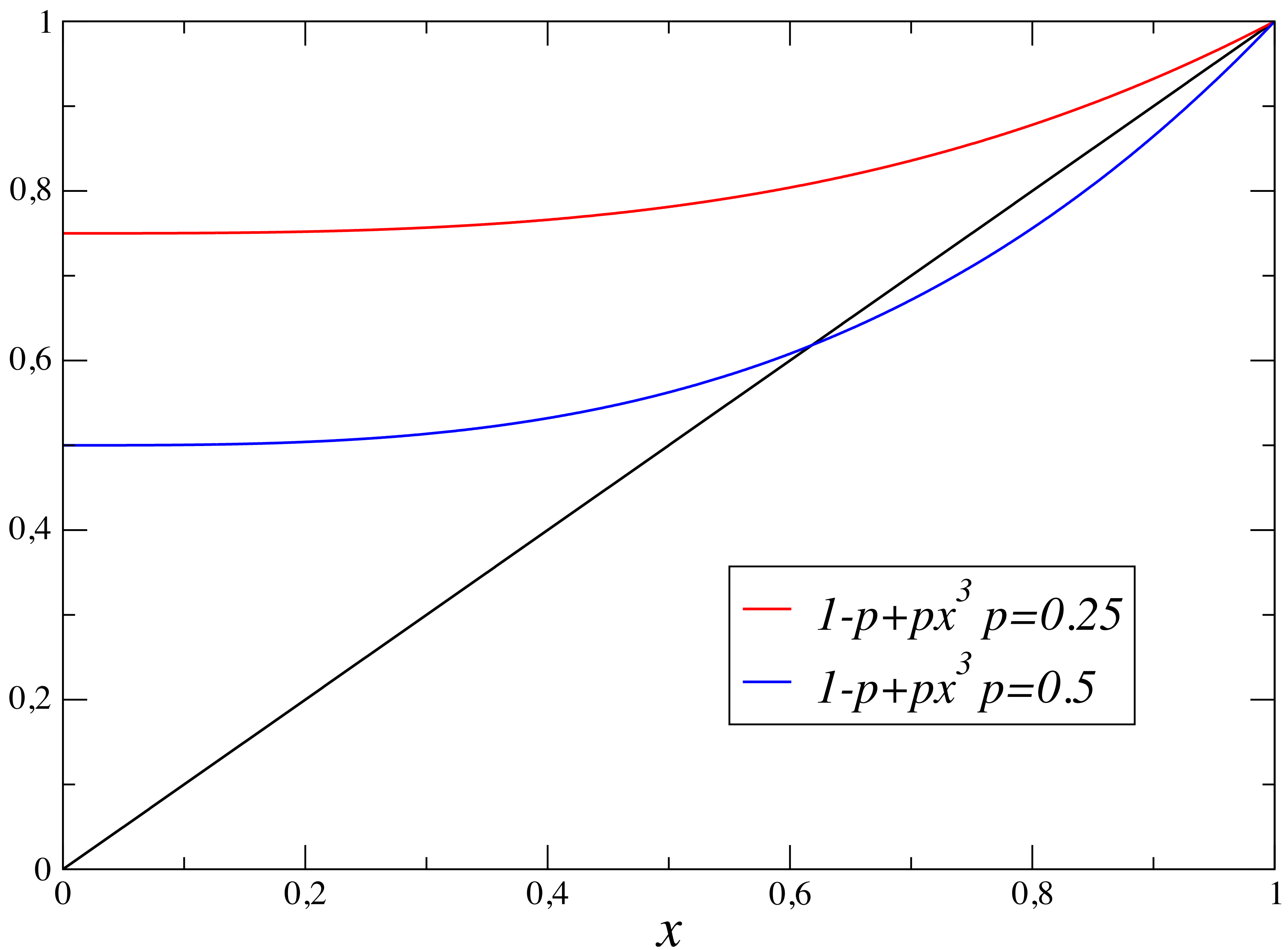

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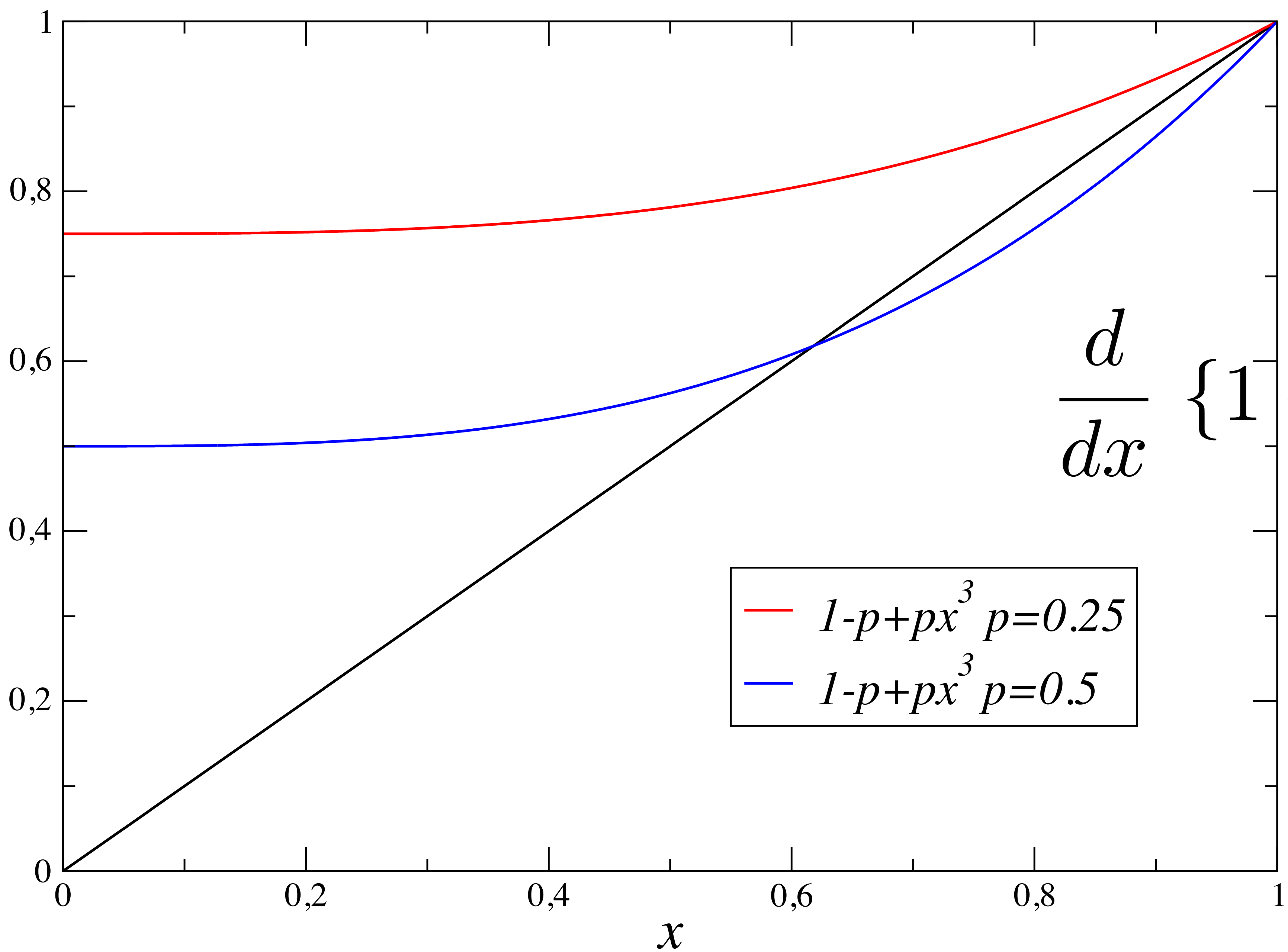

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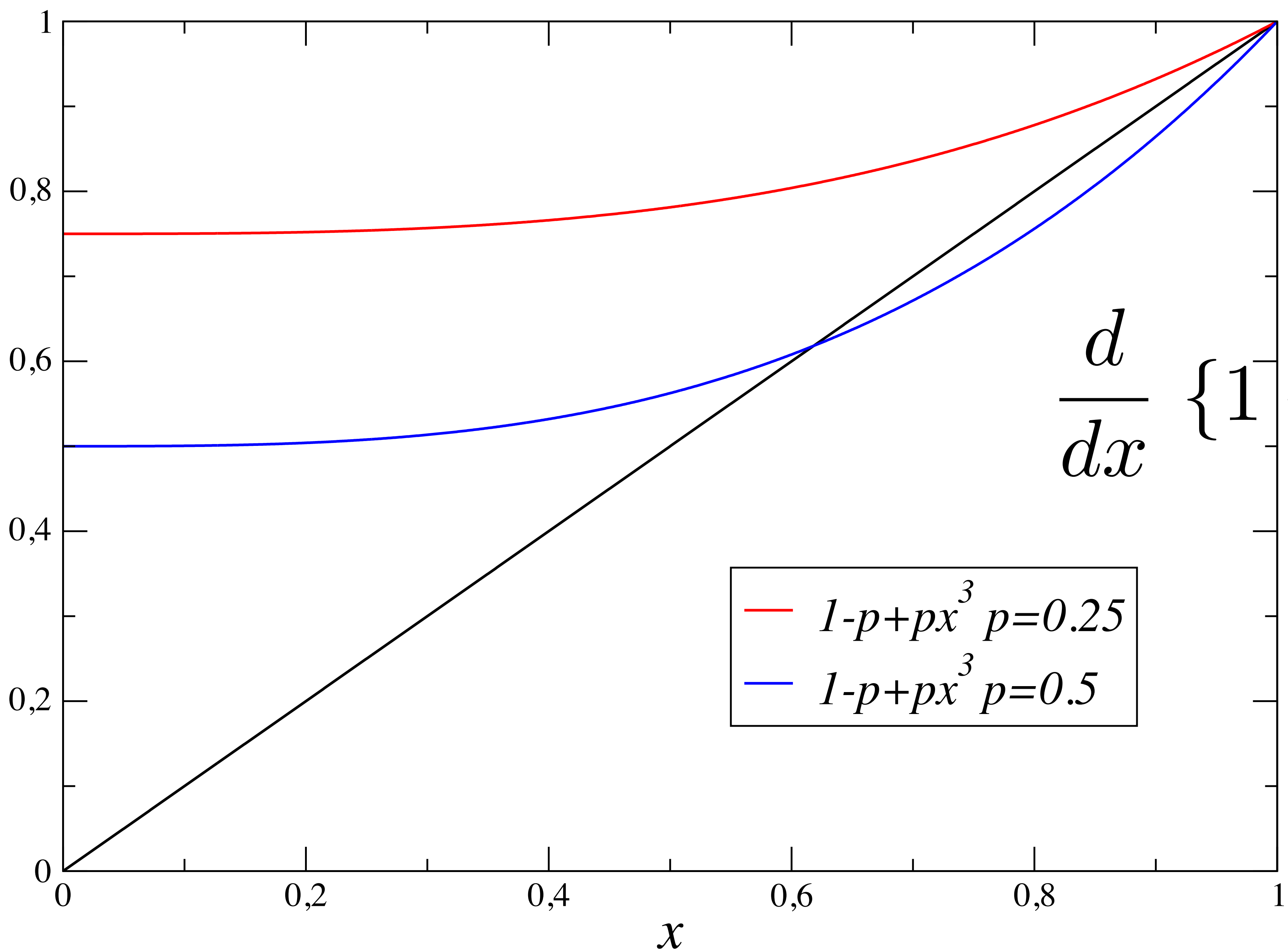

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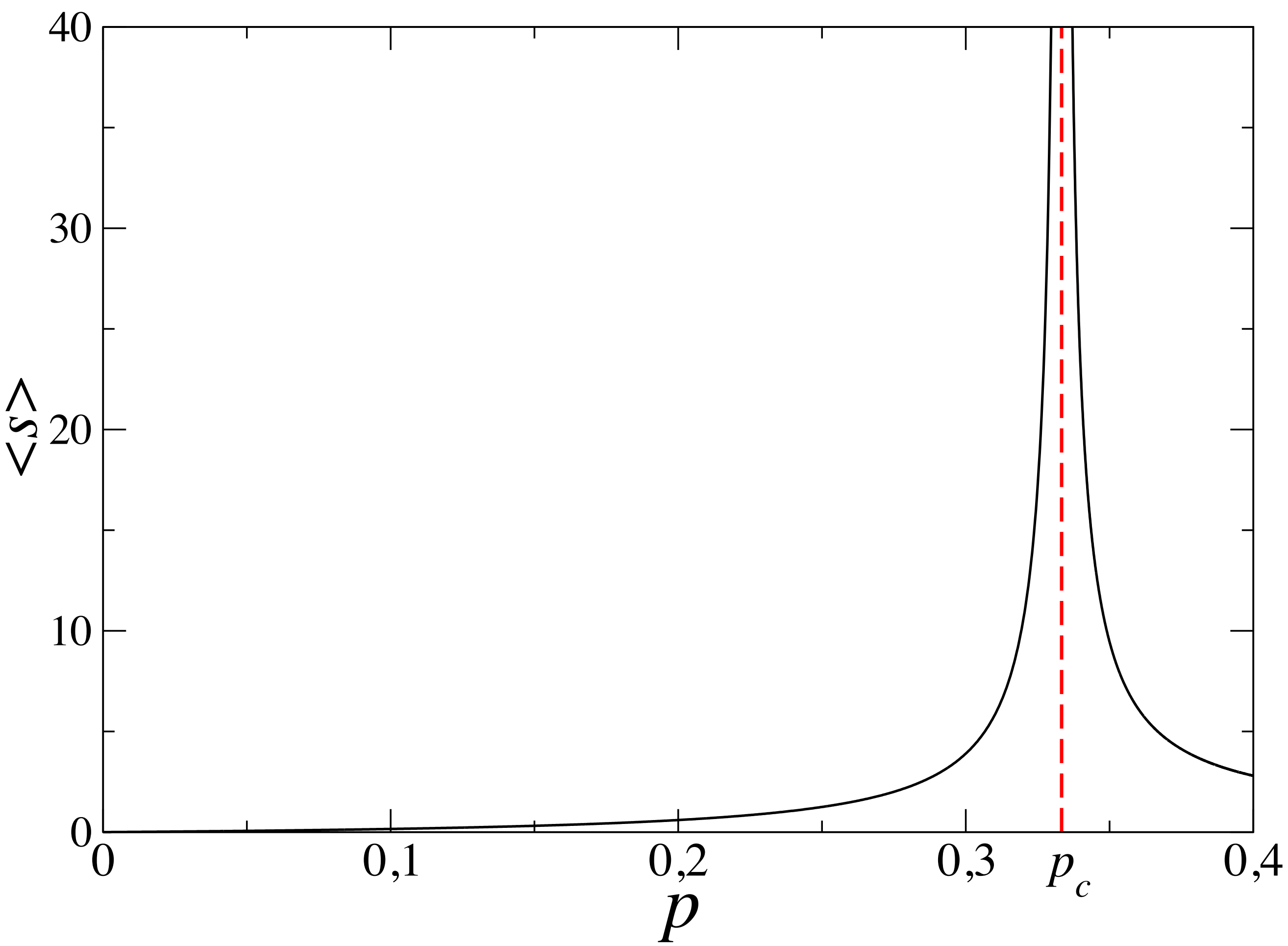
$$\boxed{\tau = \frac{5}{2} \qquad \sigma = \frac{1}{2}}$$

Distribution of finite clusters above the critical point

$$P_f(s) = \frac{P(s)}{1 - P_\infty}$$

close to the critical point (from above)

$$\langle s \rangle_f \approx (p - p_c)^{-1}$$



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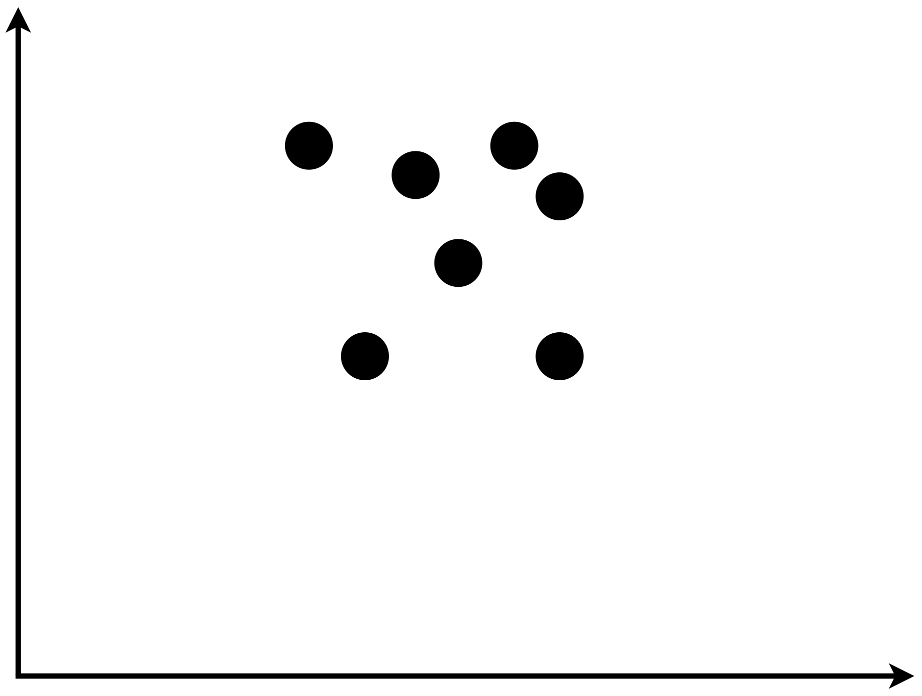

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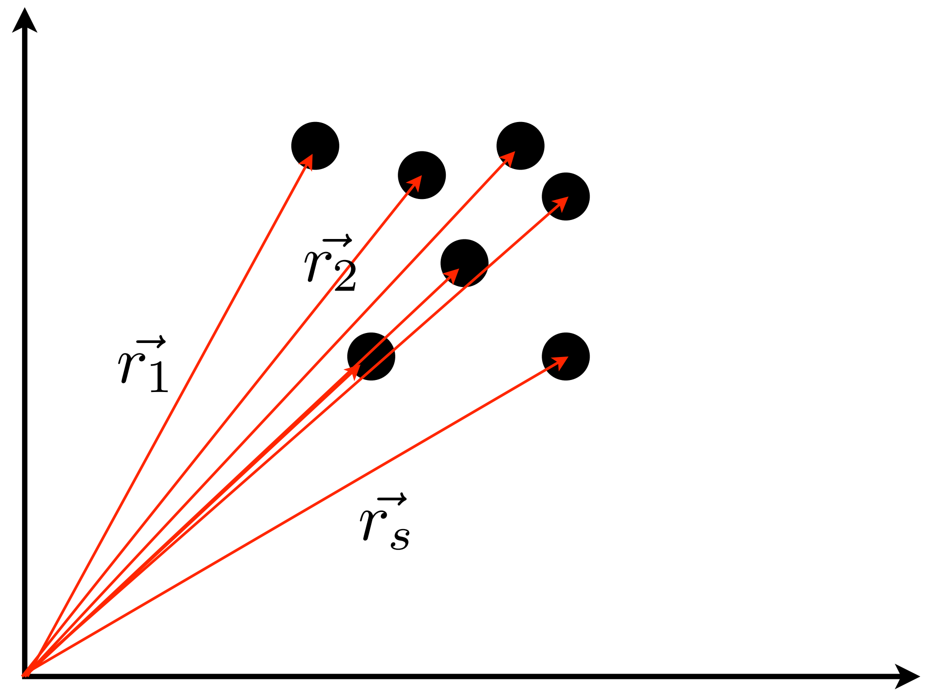

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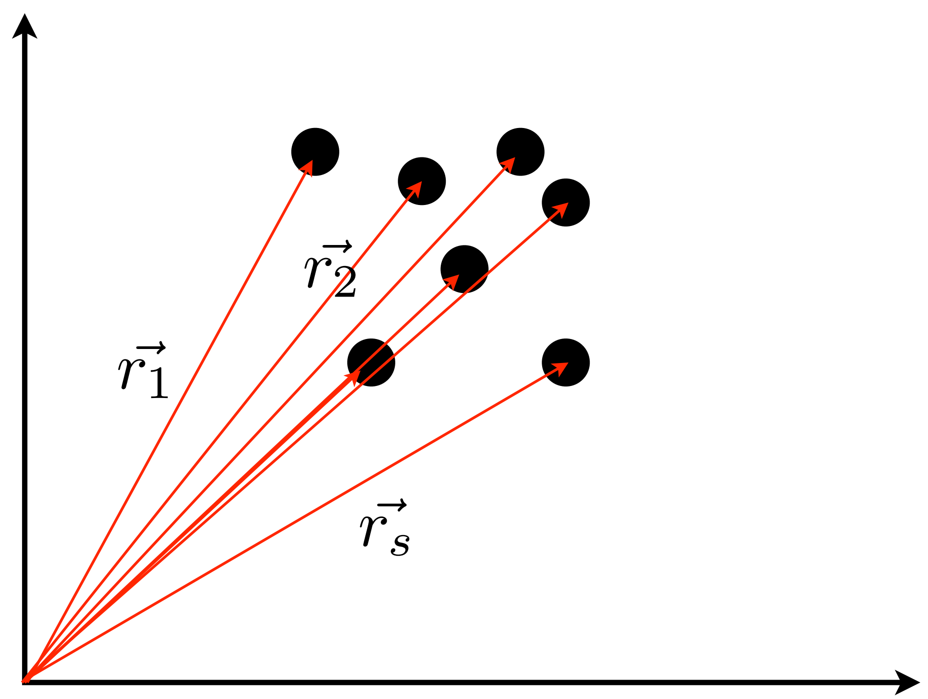
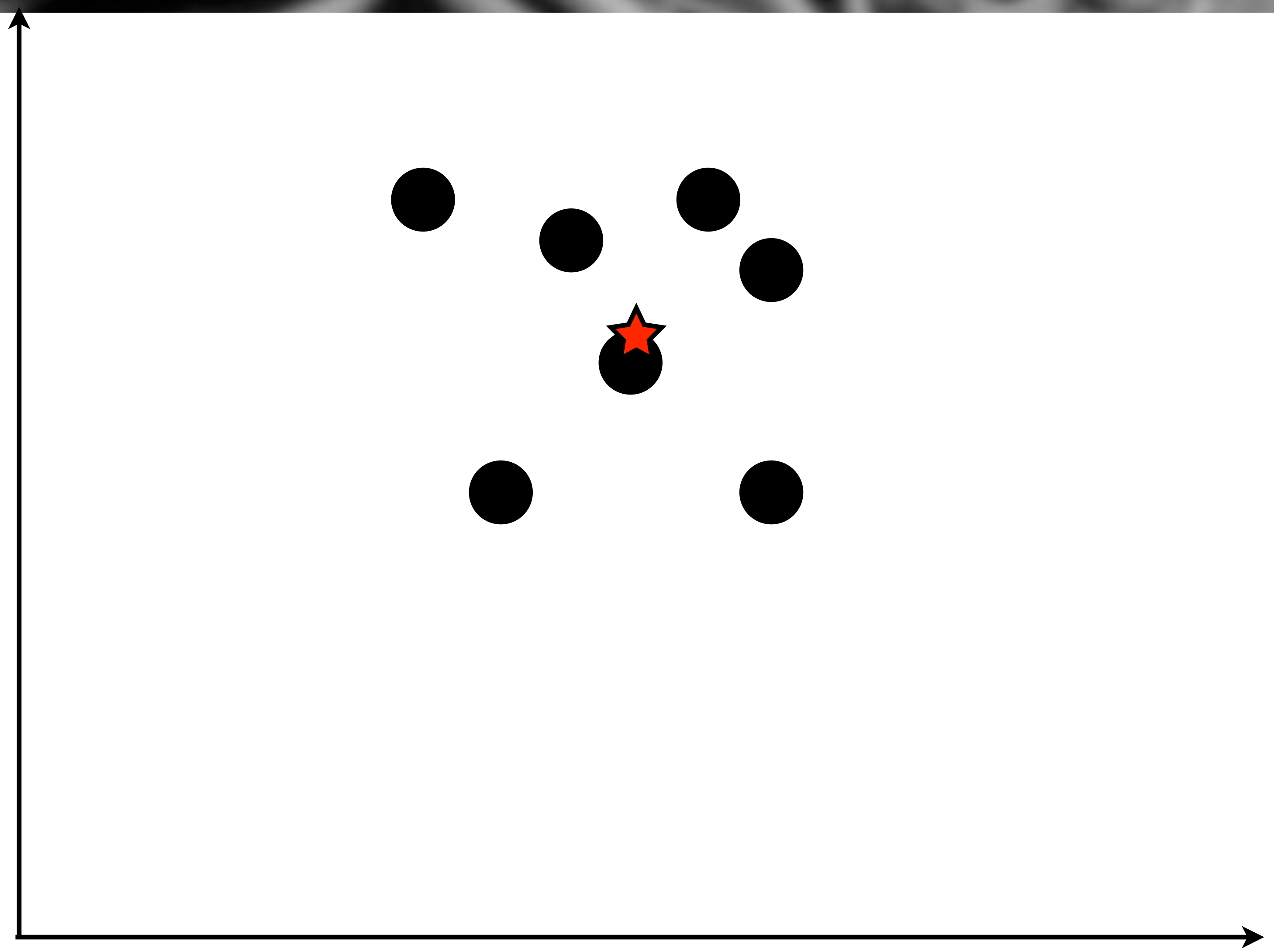
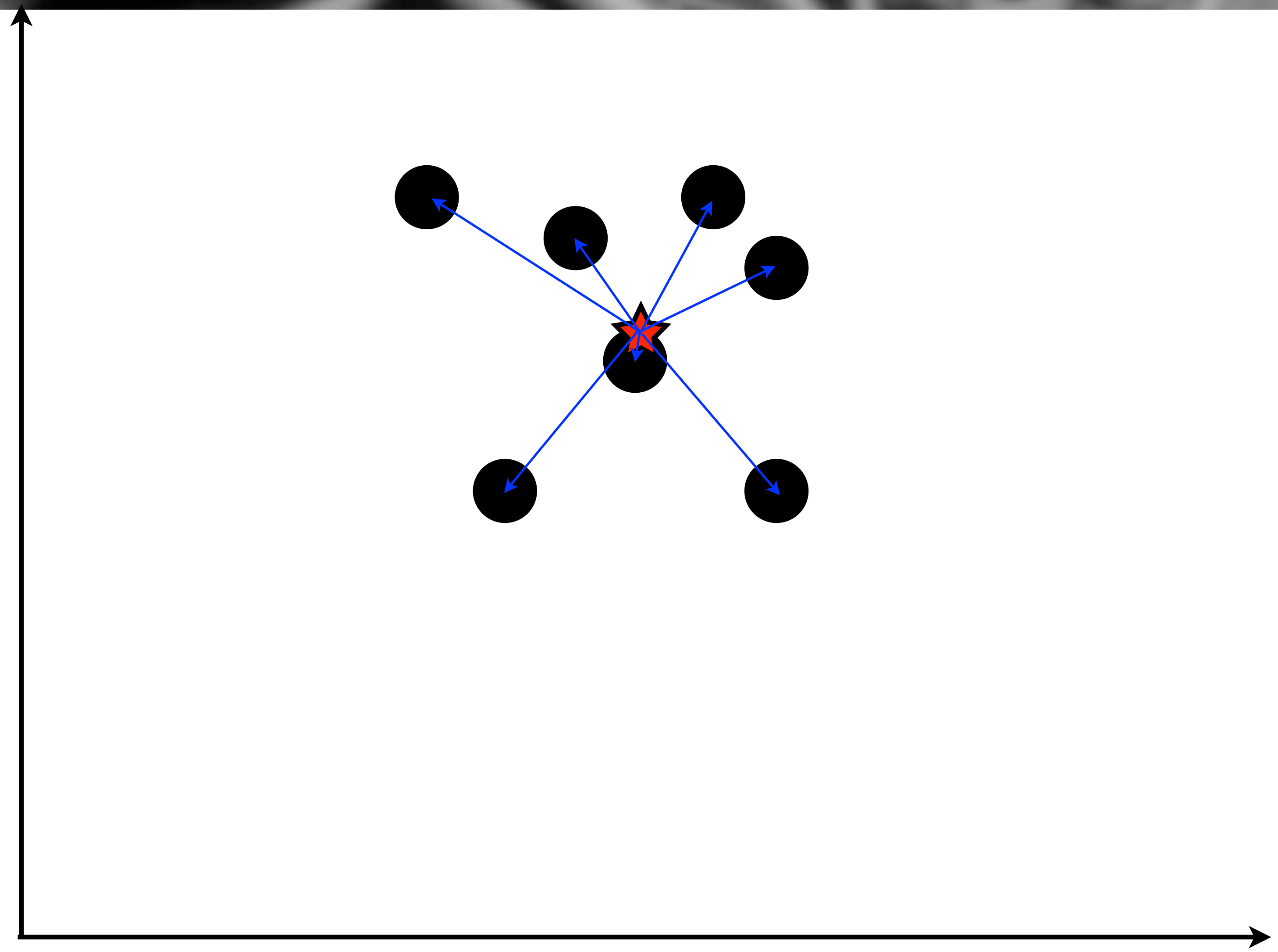
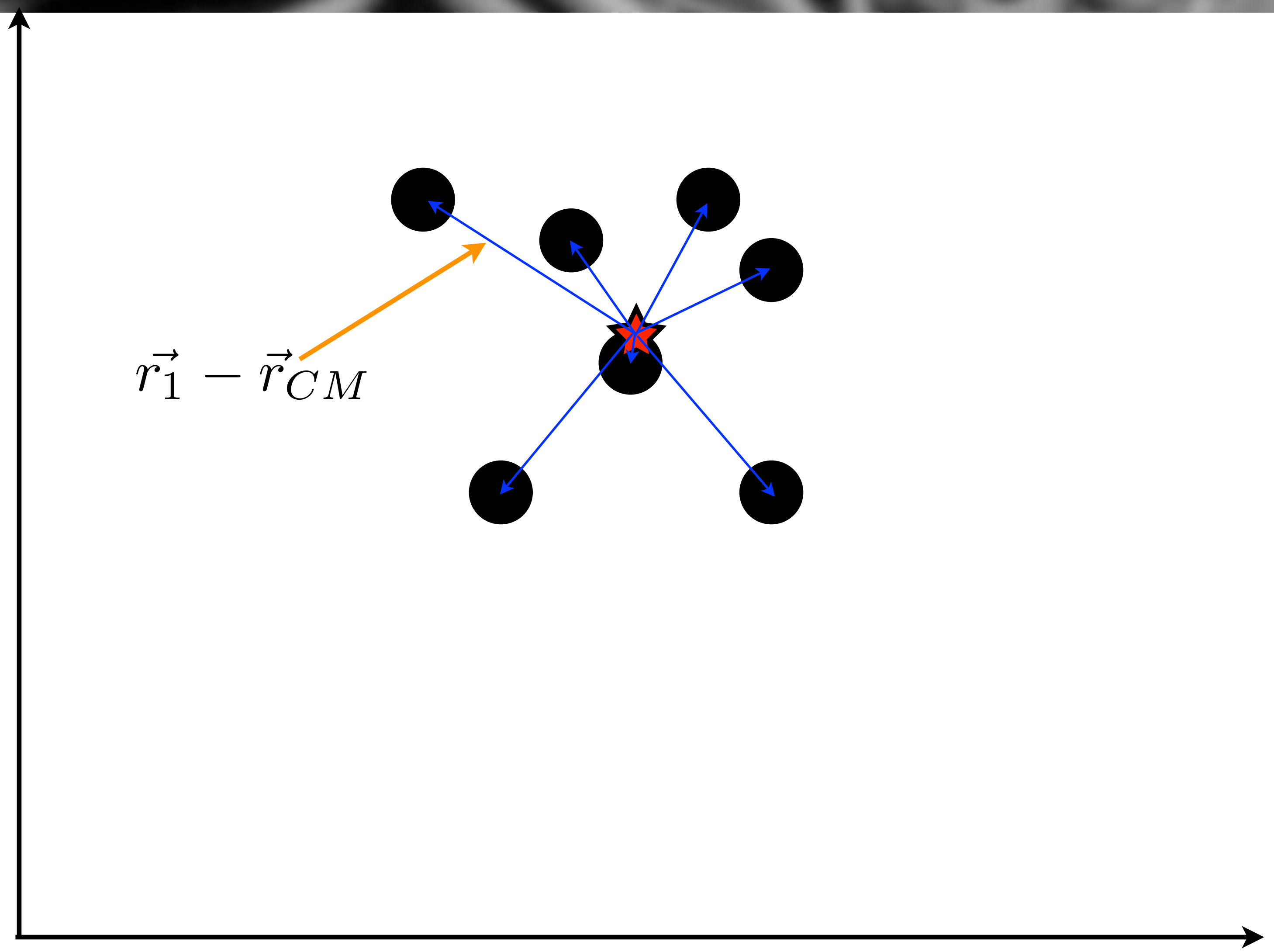


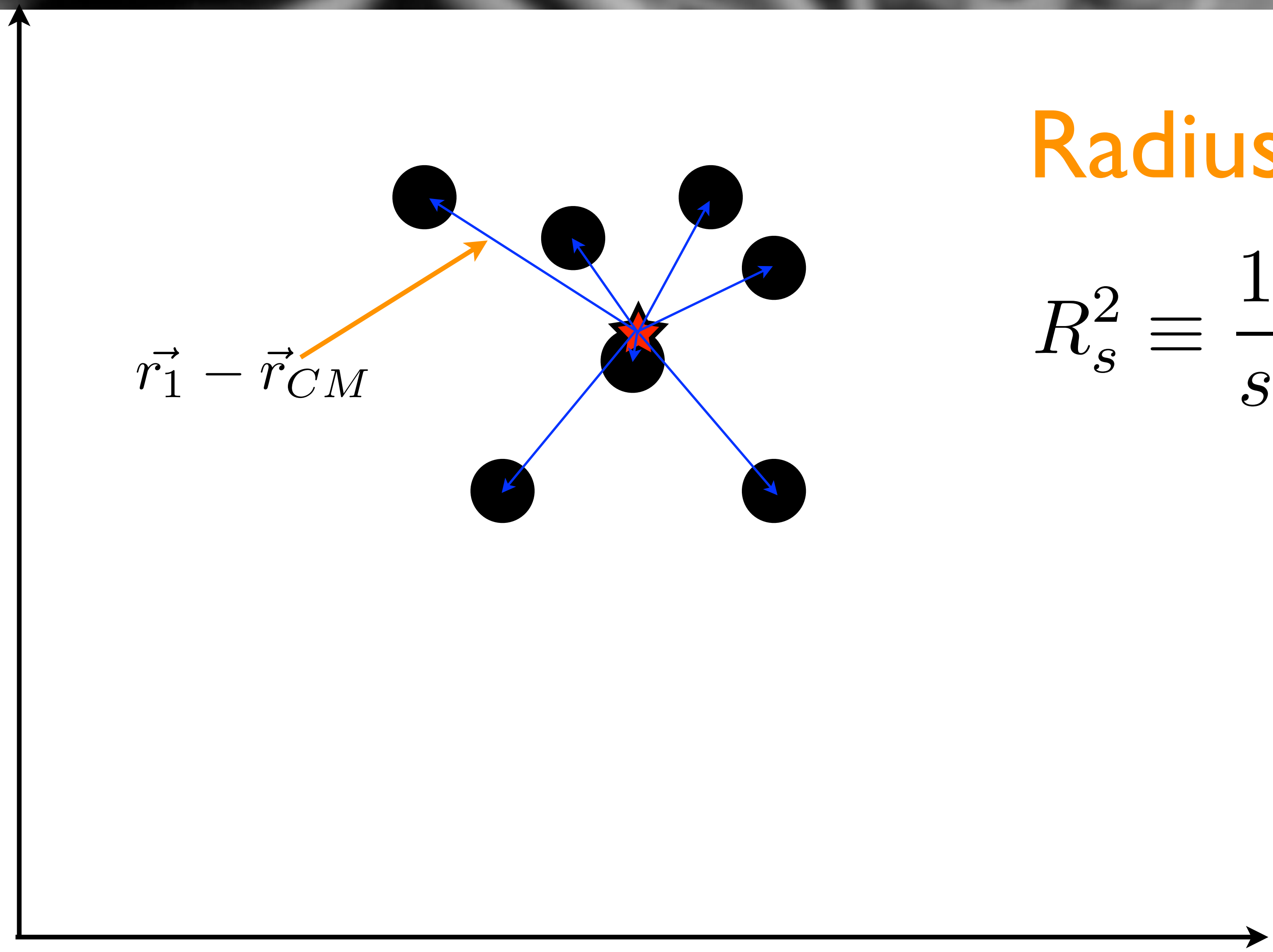
$$\vec{r}_{CM} \equiv \frac{1}{s}\sum_{i=1}^{s} \vec{r}_i$$



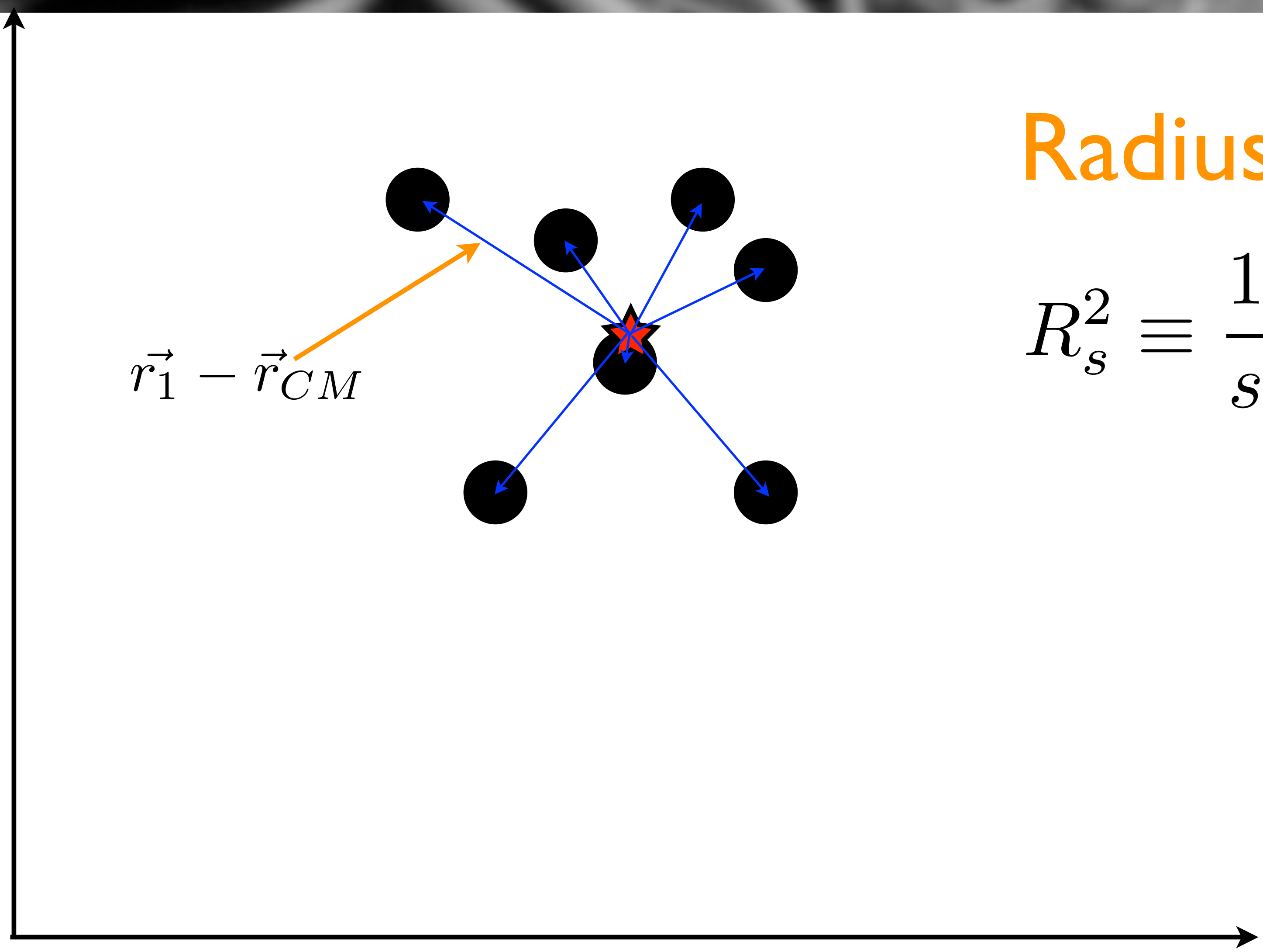



$\vec{r_1} - \vec{r}_{CM}$

Radius of gyration



$$R_s^2 \equiv \frac{1}{s} \sum_{i=1}^{s} |\vec{r_i} - \vec{r}_{CM}|^2$$



Radius of gyration

$$R_s^2 \equiv \frac{1}{s}\sum_{i=1}^{s} |\vec{r_i} - \vec{r}_{CM}|^2$$

You can also see that

$$2R_s^2 = \frac{1}{s^2}\sum_{i,j=1}^{s} |\vec{r_i} - \vec{r_j}|^2 \longrightarrow$$

Average quadratic distance between two cluster sites

Let's define now the following correlation function

$$g(r) \longrightarrow$$ Probability that a site at a distance r from an occupied site of a finite cluster is occupied and belongs to the same cluster

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$$s \sim R_s^{d_f} \longleftrightarrow R_s \sim s^{1/d_f}$$

$$\xi^2 = \frac{2\sum_s s^{2/d_f+1} P(s)}{\sum_s s P(s)}$$

$$\xi^2 = \frac{2\sum_s s^{2/d_f+1}P(s)}{\sum_s sP(s)} \sim \frac{(p-p_c)^{-\frac{1}{\sigma}(\frac{2}{d_f}+3-\tau)}}{(p-p_c)^{-\frac{1}{\sigma}(3-\tau)}}$$

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$$\xi \sim (p-p_c)^{-1/d_f\sigma}$$

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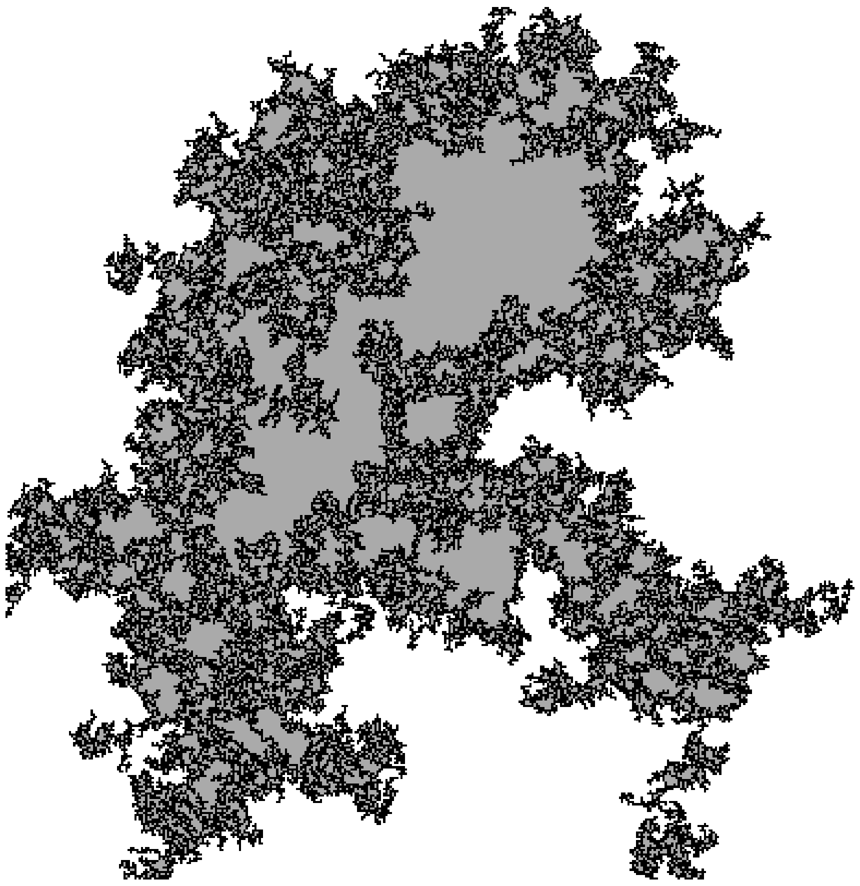

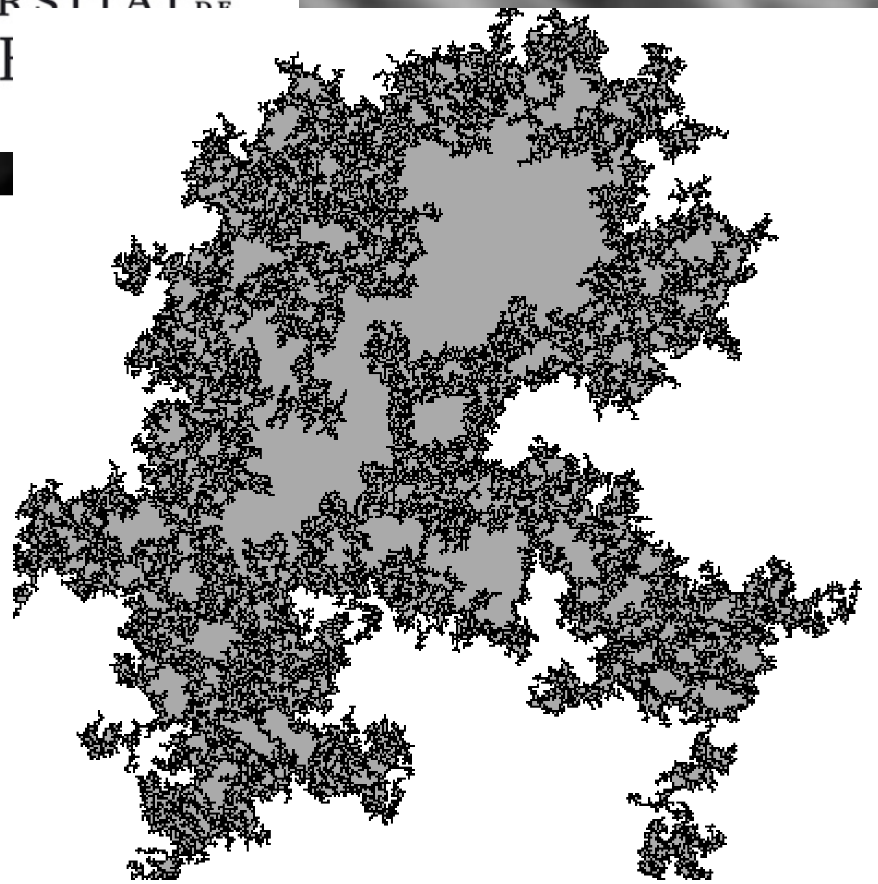

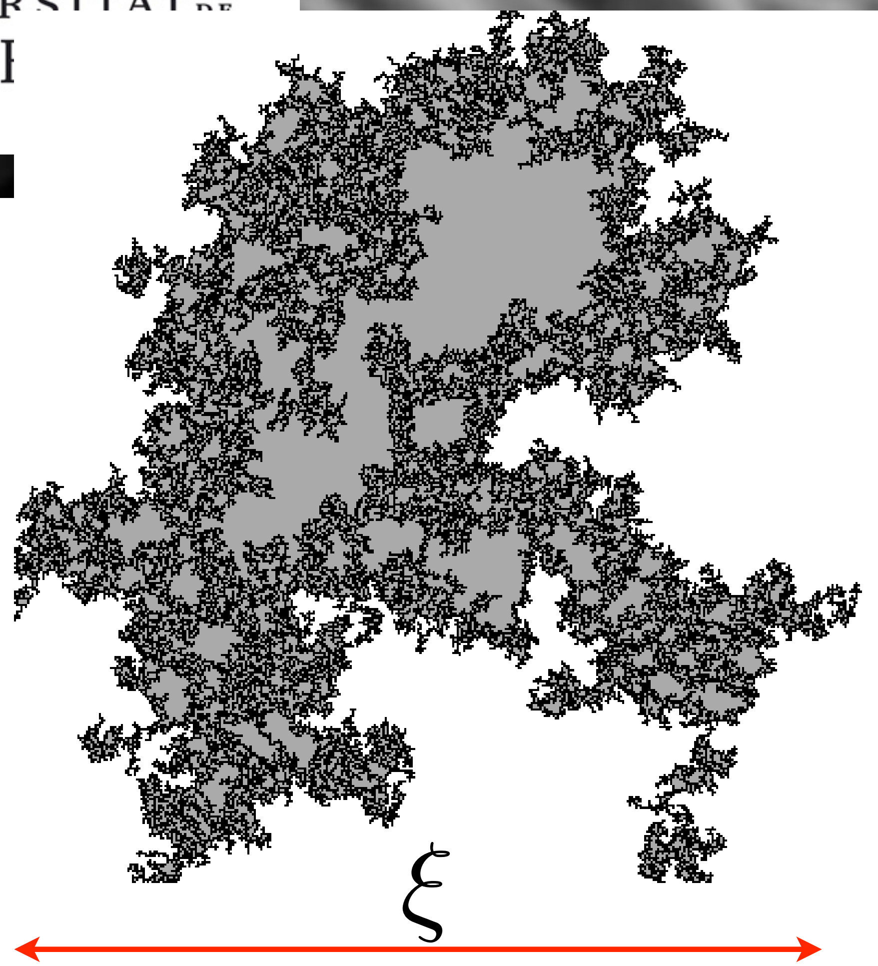
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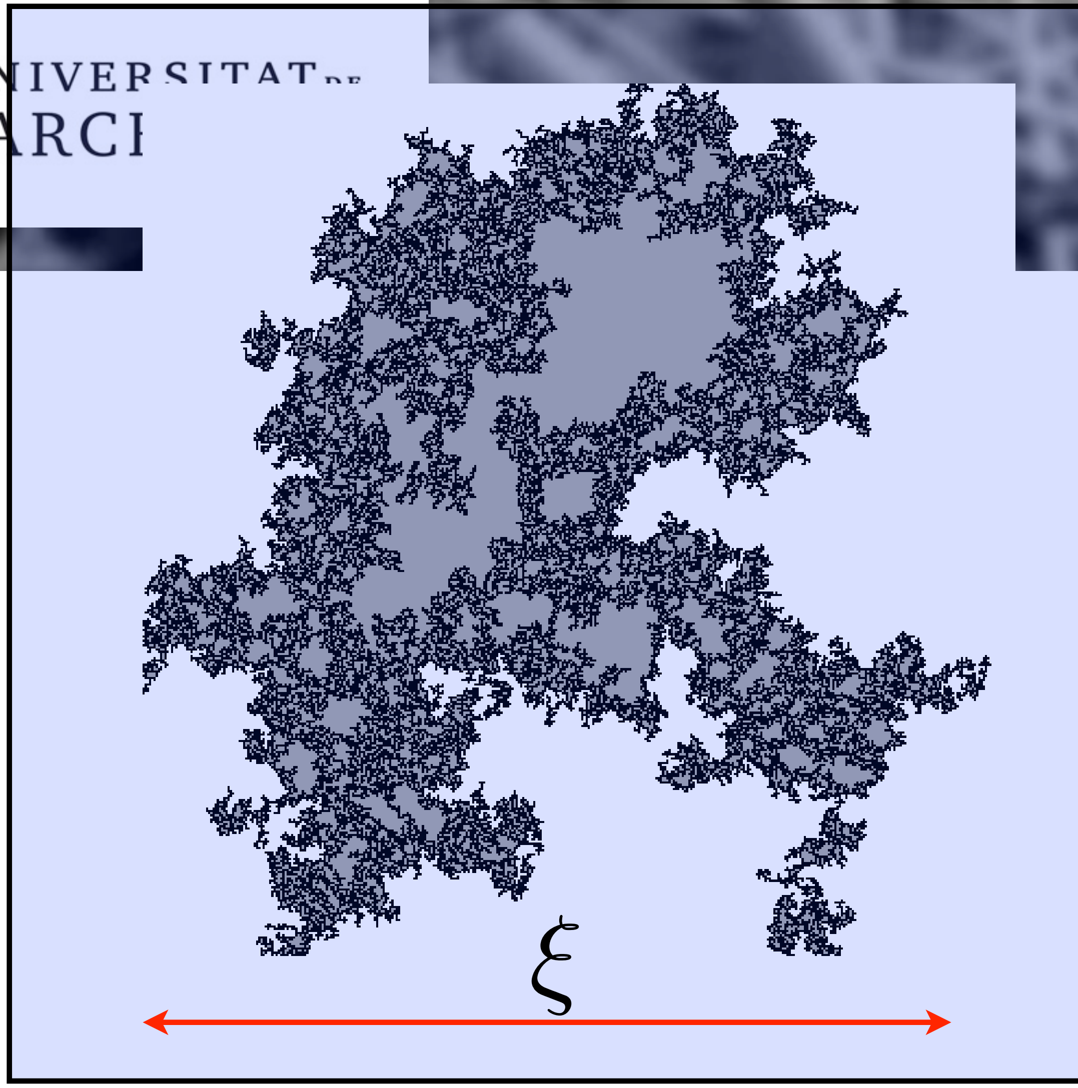
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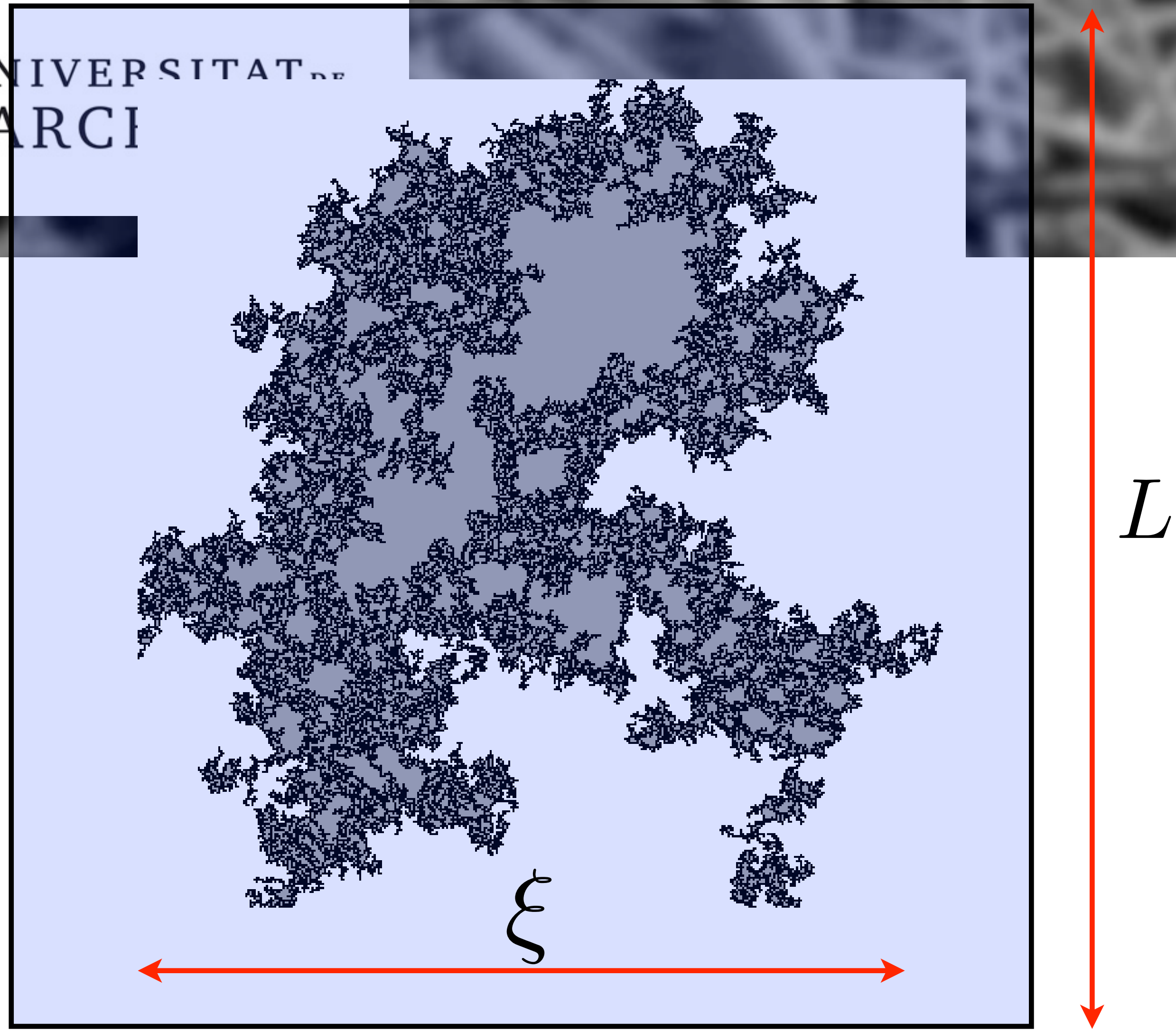
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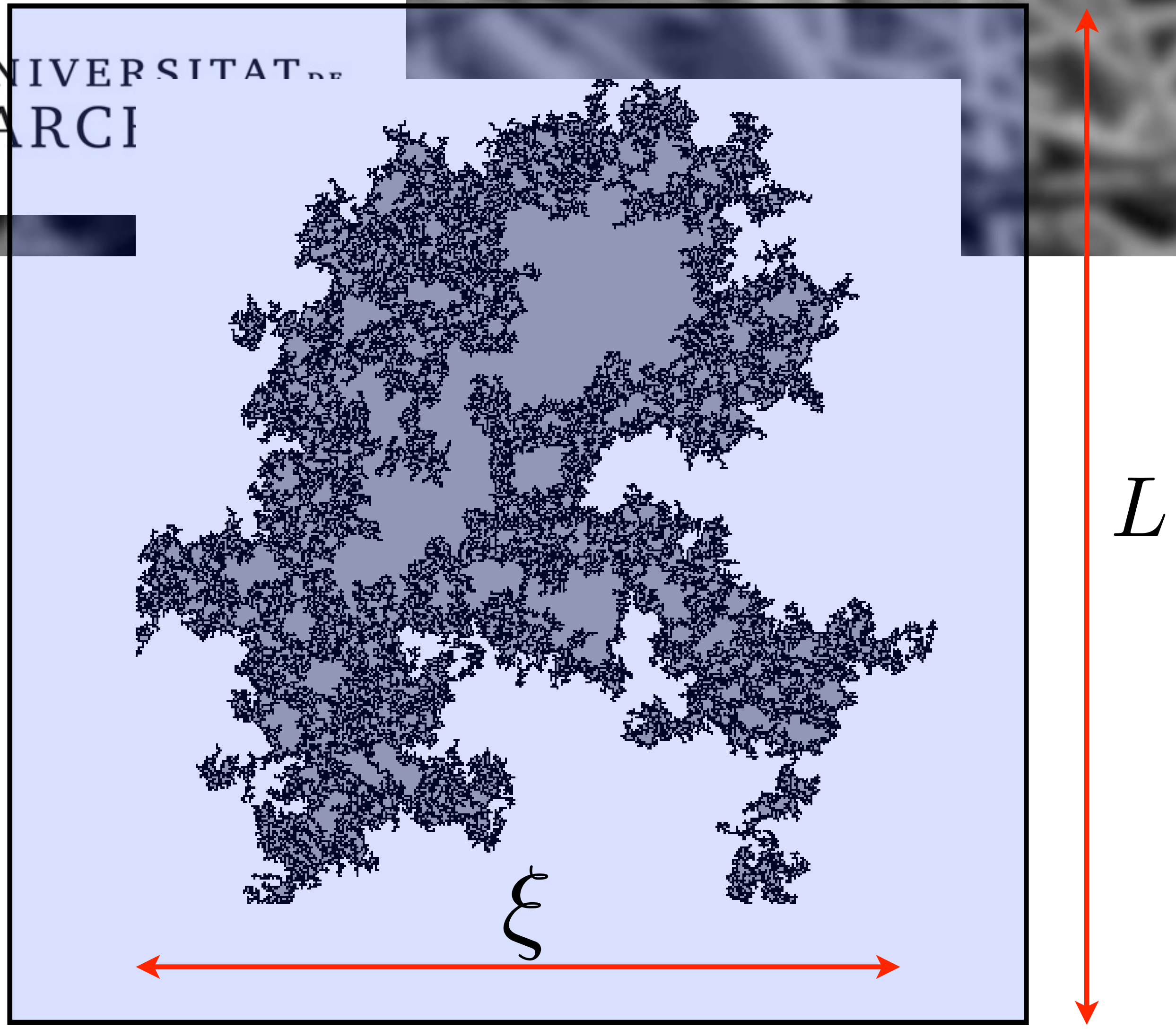
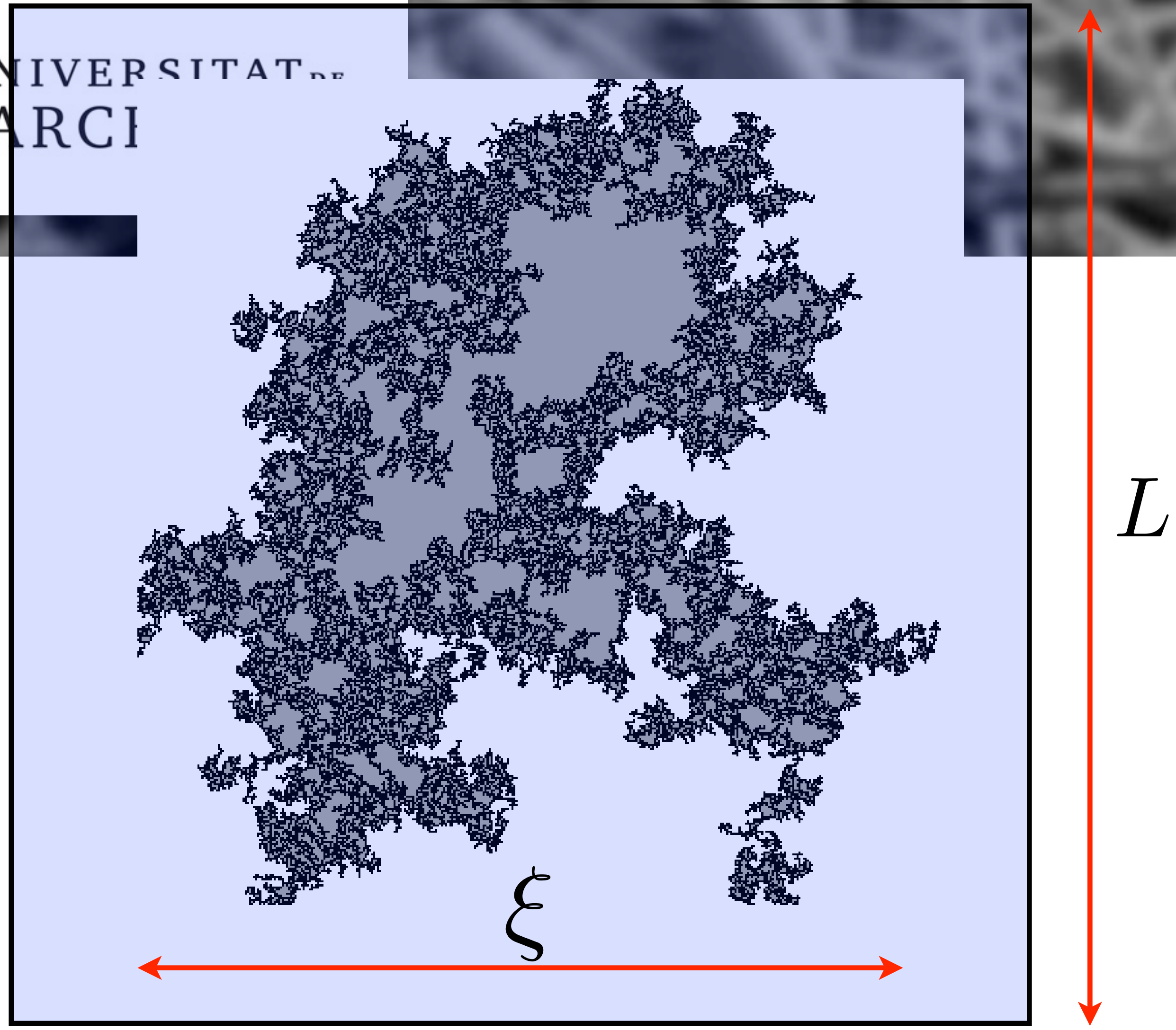


$M \sim P_\infty L^d \qquad \text{if} \qquad L > \xi$



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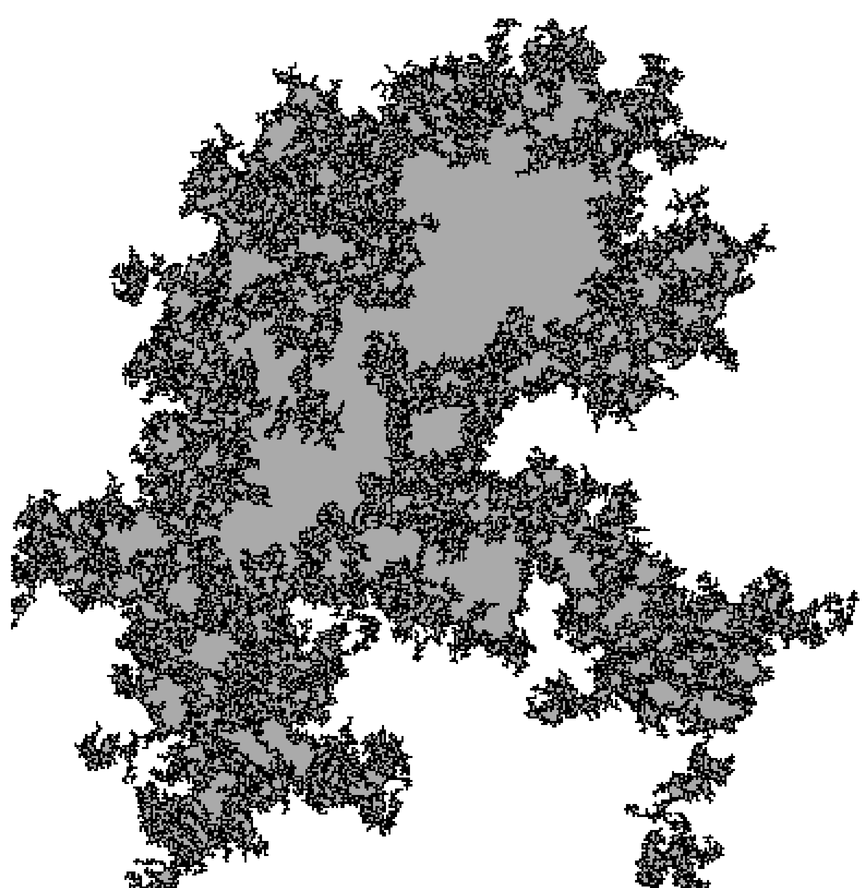

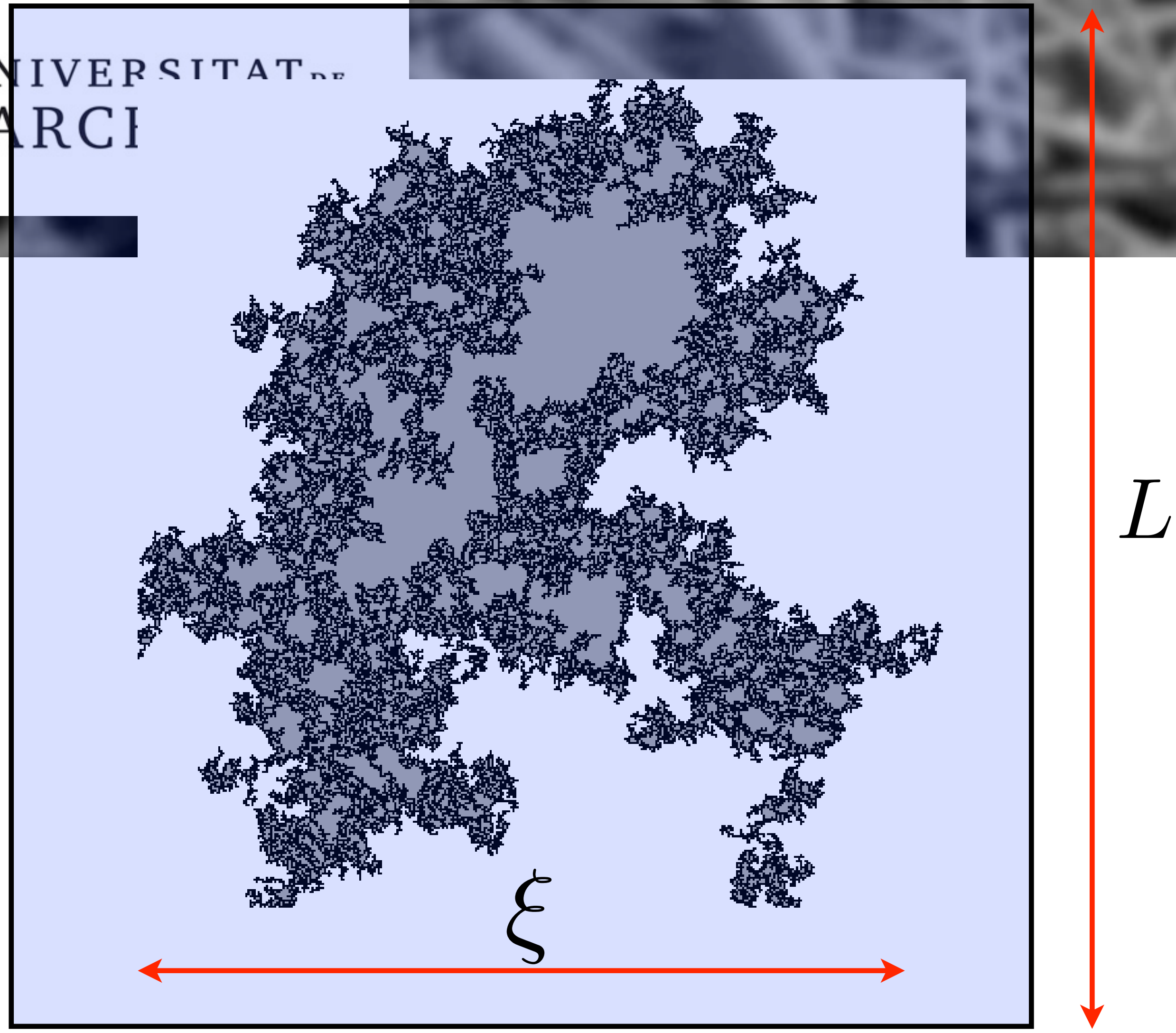

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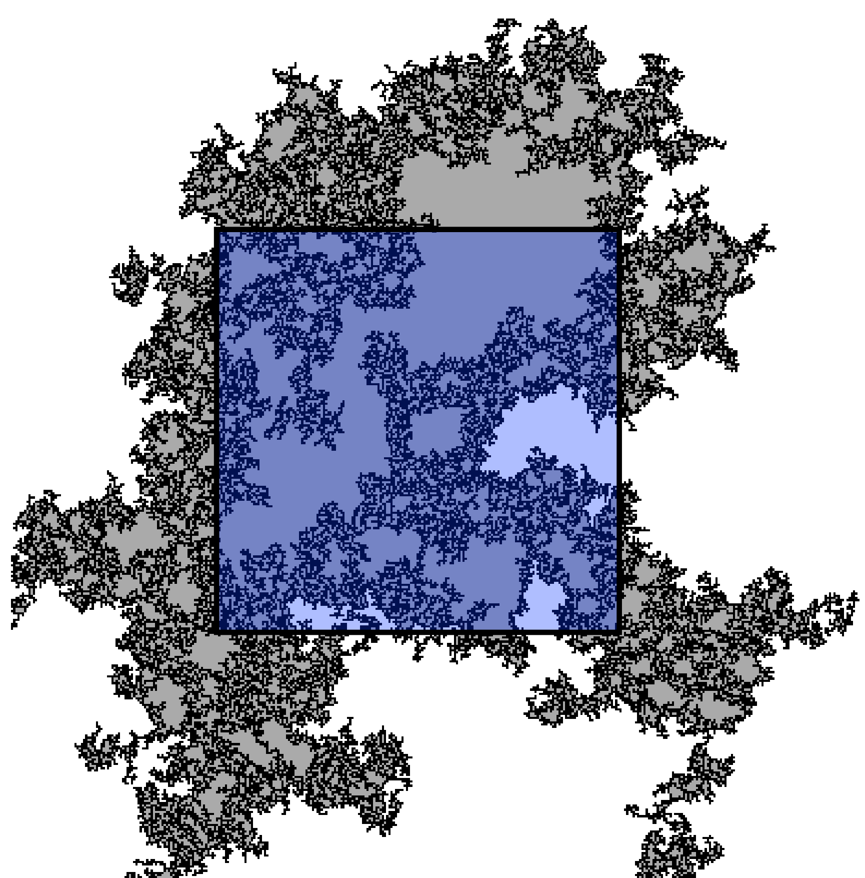

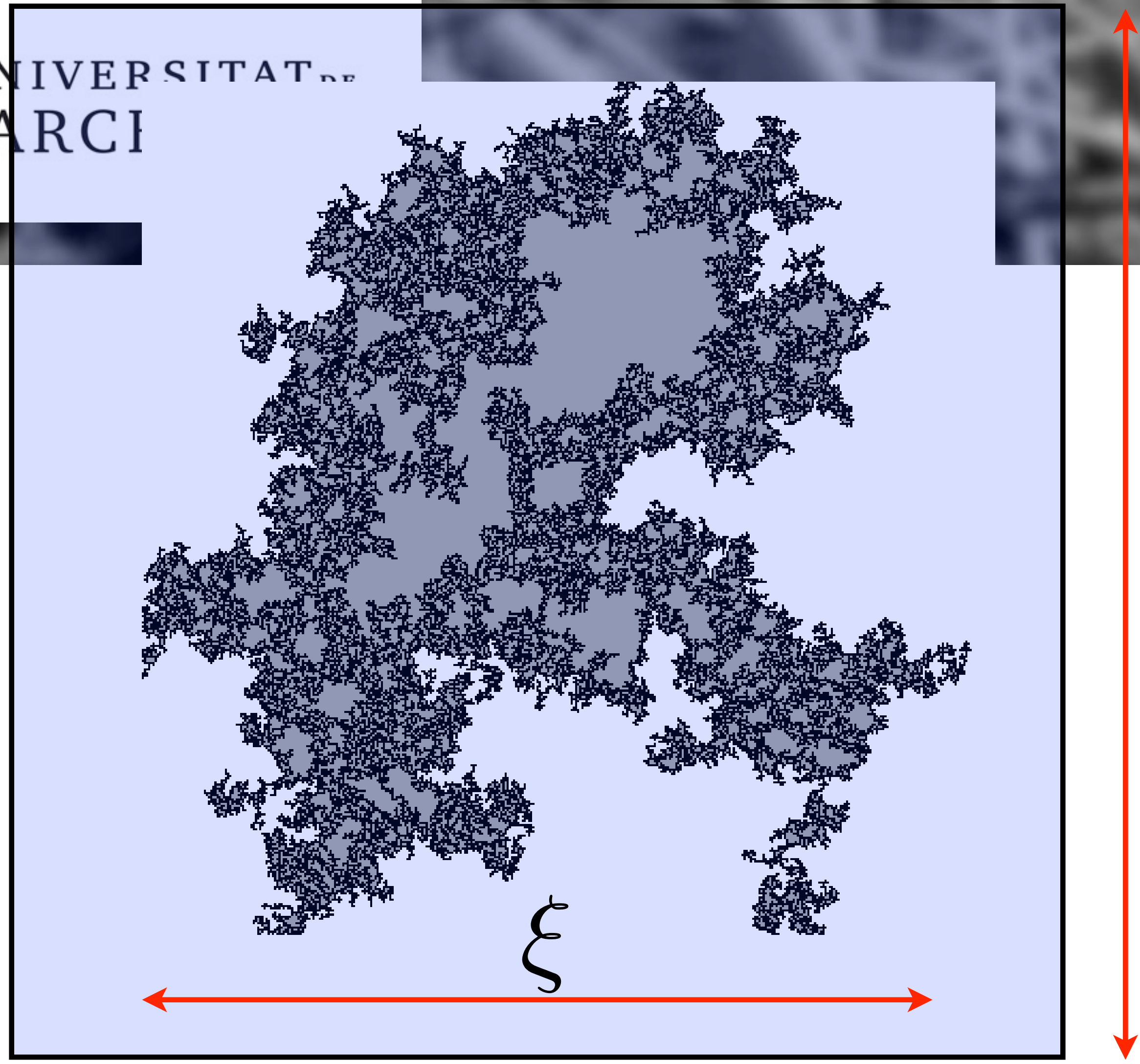

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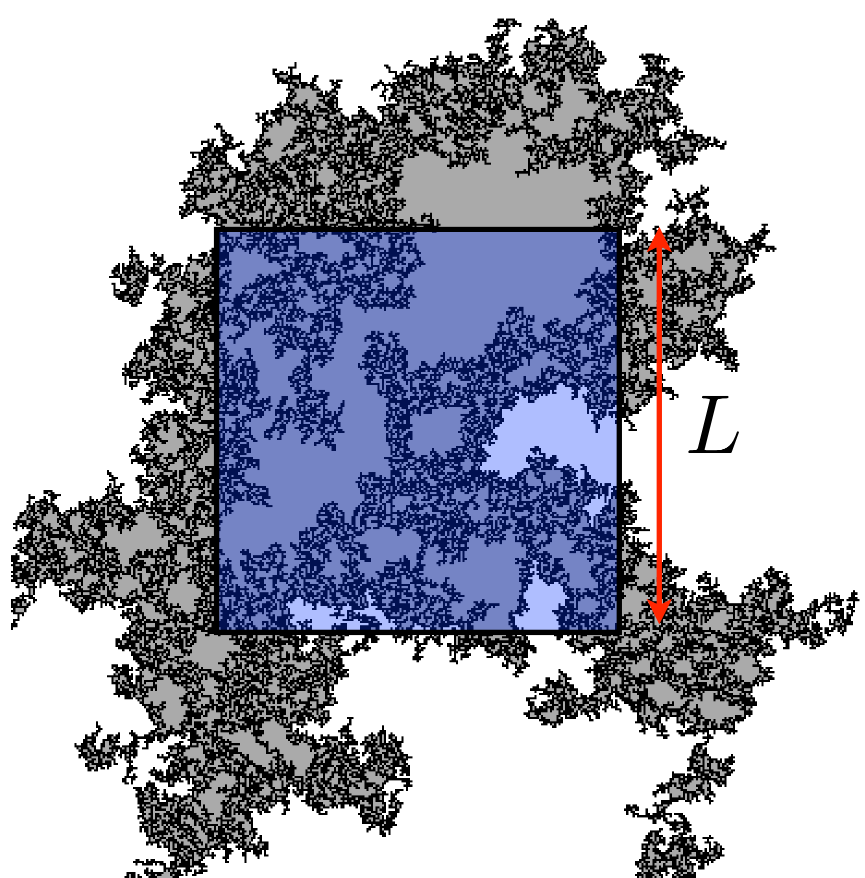

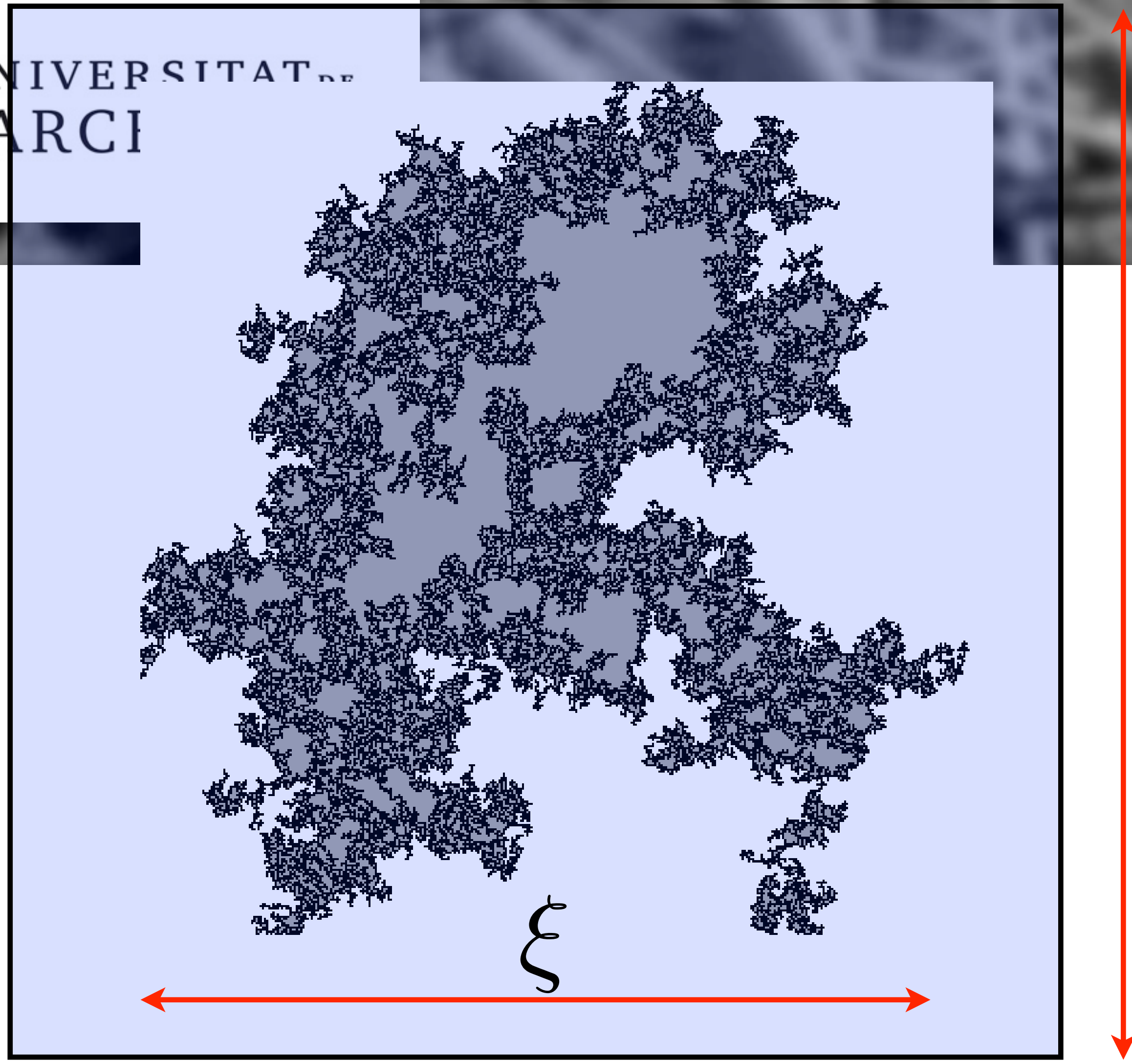

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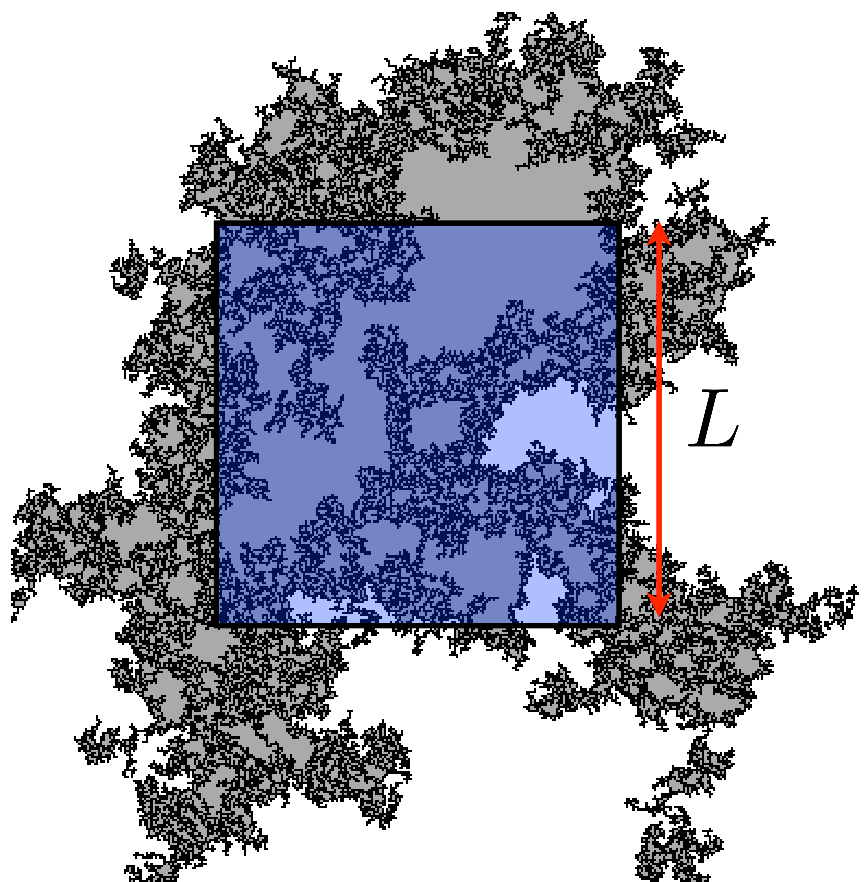

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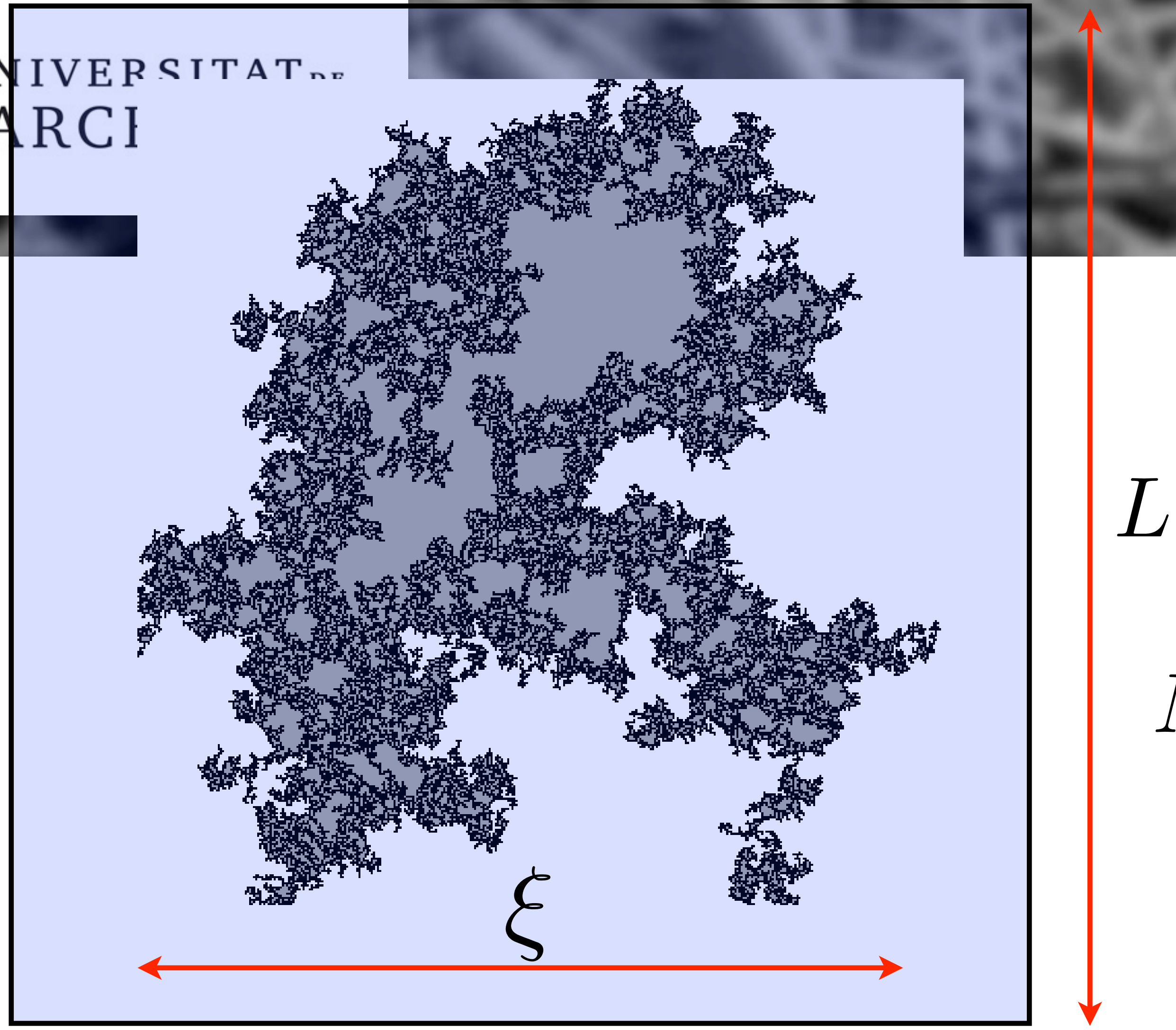

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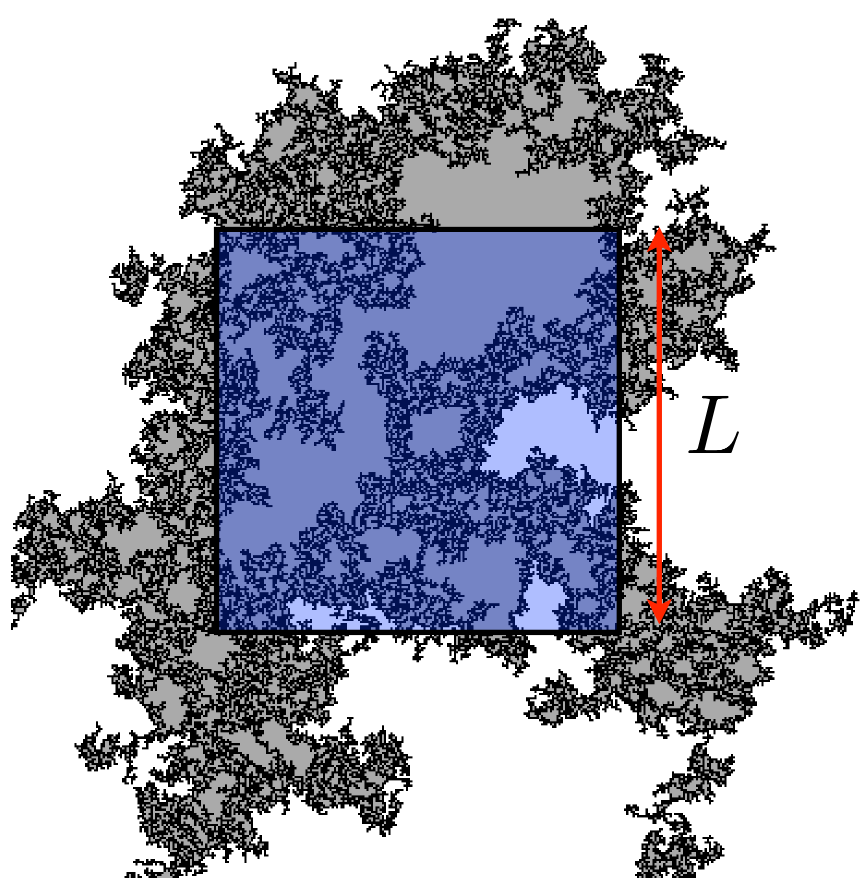

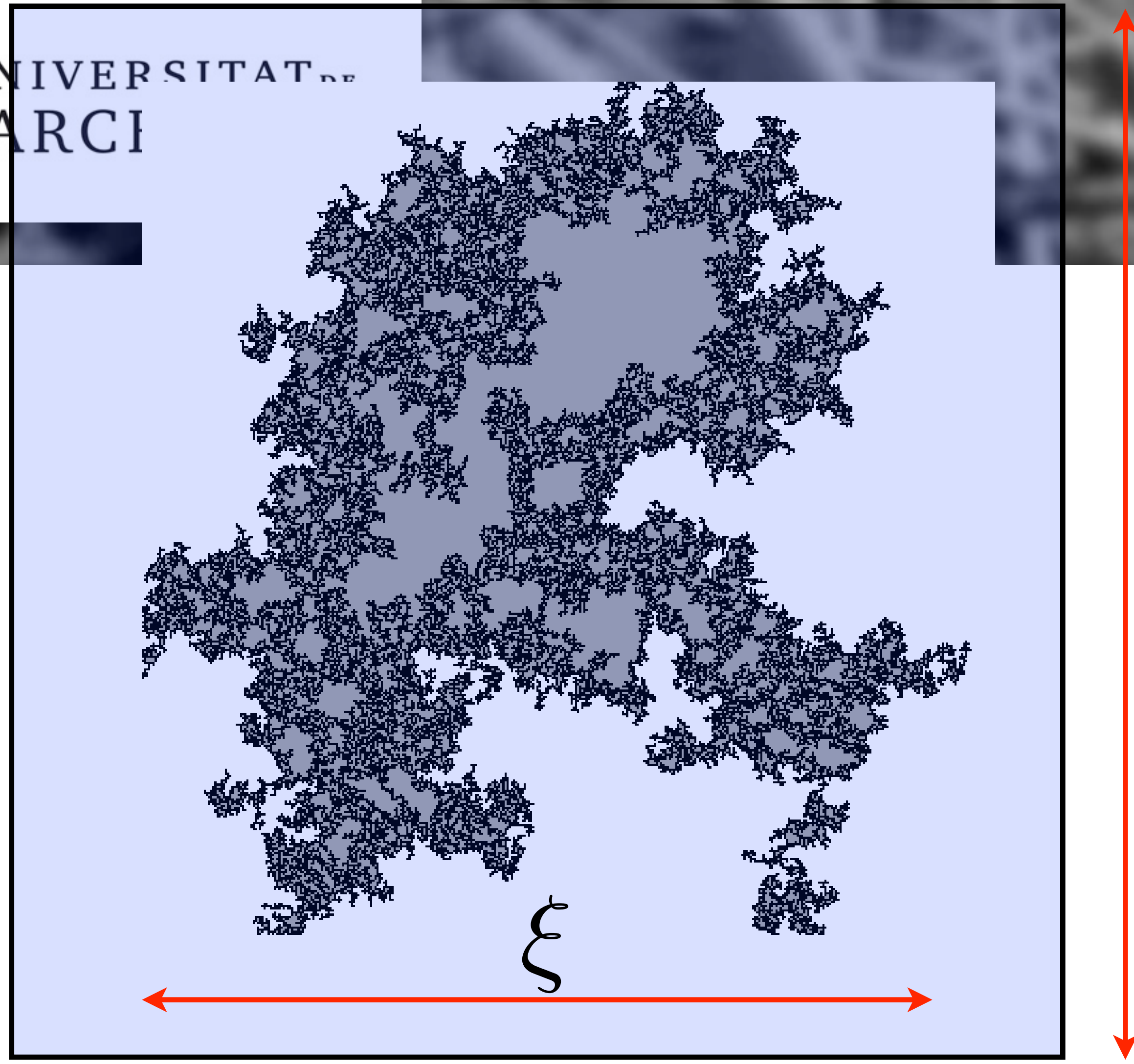

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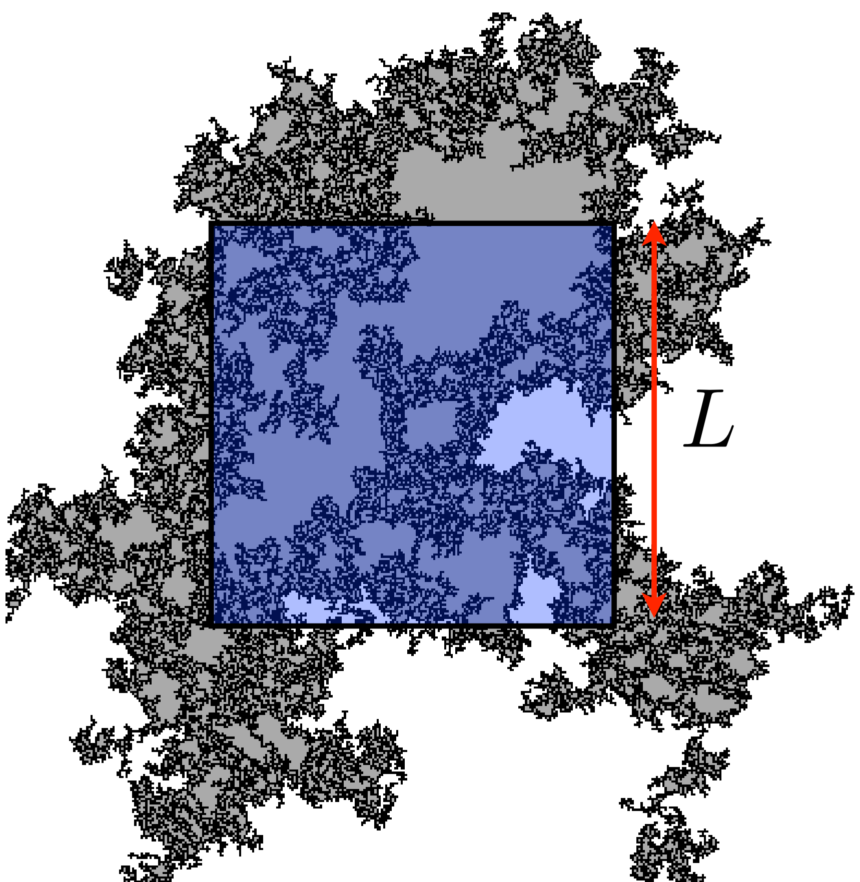

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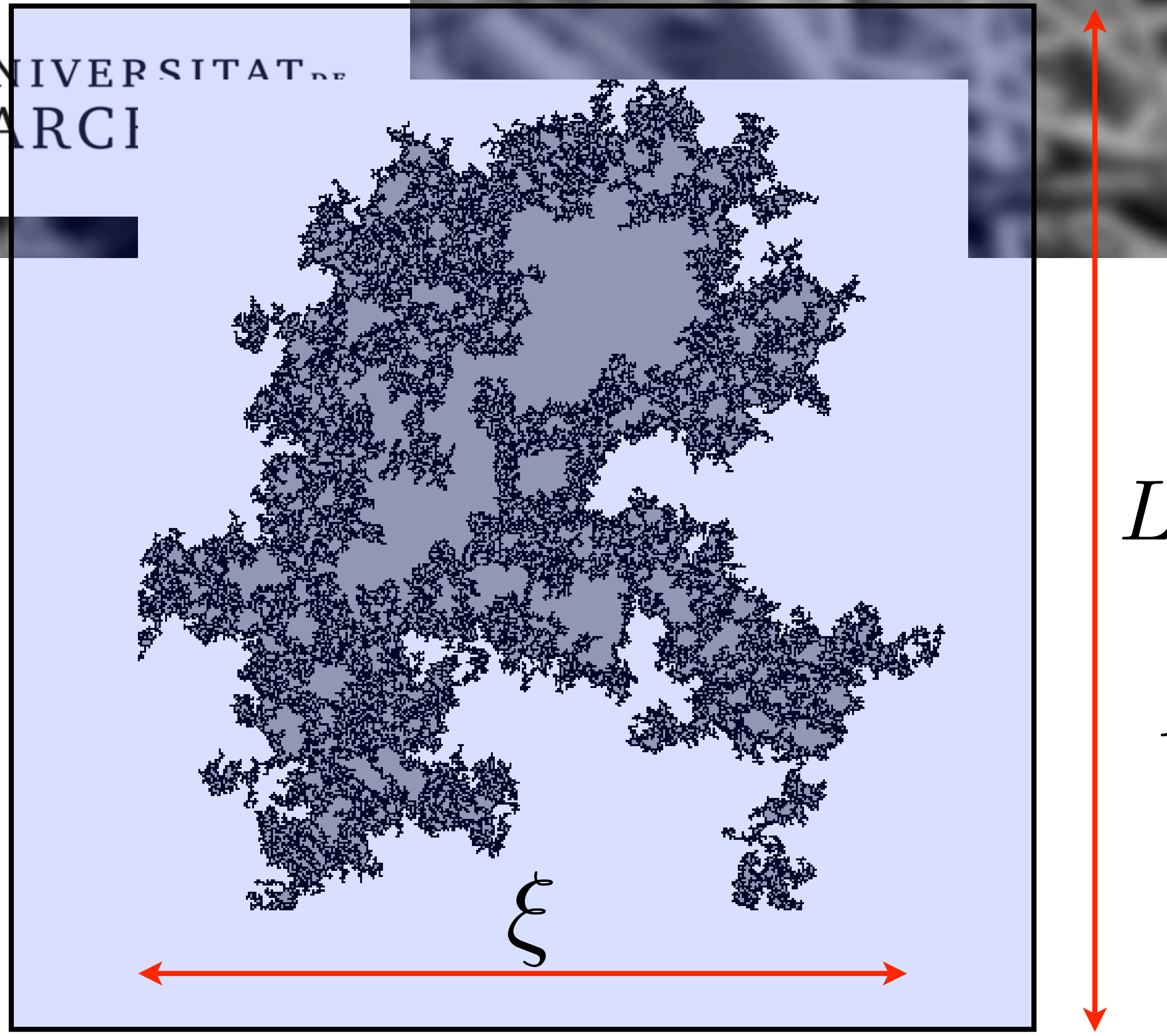

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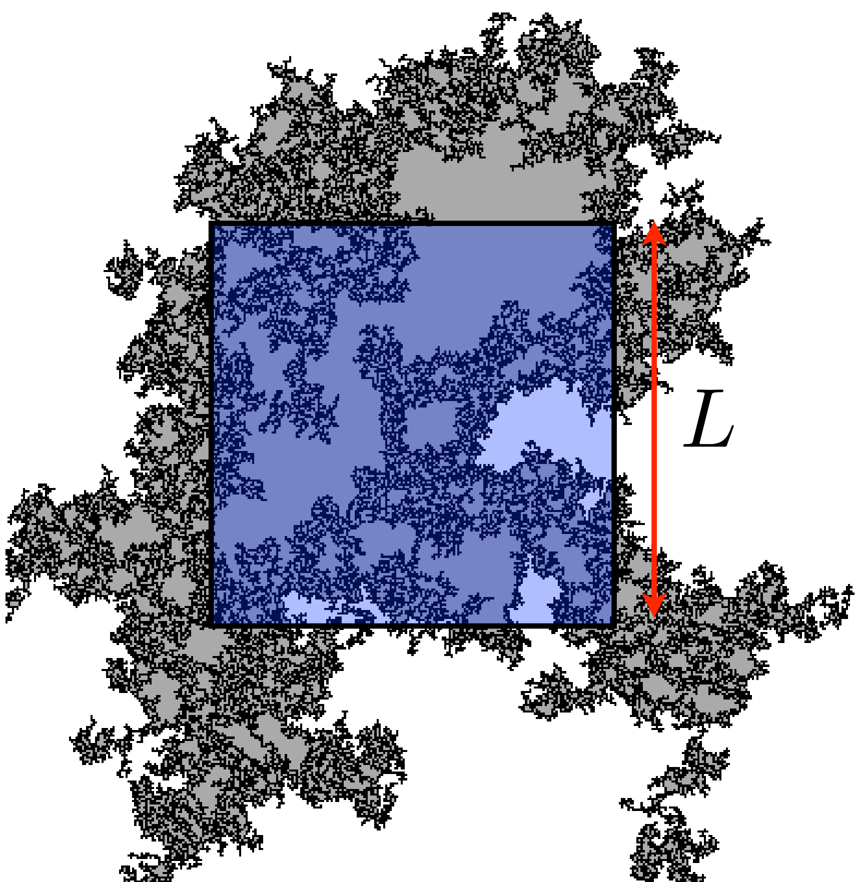

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$$\boxed{d_f = d - \frac{\beta}{\nu}}$$

hyperscaling relation

In summary, given the exponents $\{\tau, \sigma, d\}$

$$P_\infty \sim (p - p_c)^\beta \qquad \beta = \frac{\tau - 2}{\sigma}$$

$$\langle s \rangle \sim |p - p_c|^{-\gamma} \qquad \gamma = \frac{3 - \tau}{\sigma}$$

$$\xi \sim |p - p_c|^{-\nu} \qquad \nu = \frac{\tau - 1}{\sigma d}$$

$$d_f = \frac{d}{\tau - 1}$$

$$\begin{cases} M \sim P_\infty L^d & \text{if} \quad L > \xi \\ M \sim L^{d_f} & \text{if} \quad L < \xi \end{cases}$$

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FSS for the order parameter

$$P_\infty(p, L) = L^{-\frac{\beta}{\nu}} F\left[|p-p_c|L^{\frac{1}{\nu}}\right]$$

$$F(x) \sim \begin{cases} x^\beta & x \gg 1 \\ \text{cte} & x \ll 1 \end{cases}$$

FSS for the average cluster size

$$\langle s(p,L)\rangle = L^{\frac{\gamma}{\nu}} G\left[|p-p_c|L^{\frac{1}{\nu}}\right]$$

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This is the best way to measure from simulations the critical exponents

(Bond) percolation on networks

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Degree distribution $P(k)$

Probability that a randomly chosen node has degree k

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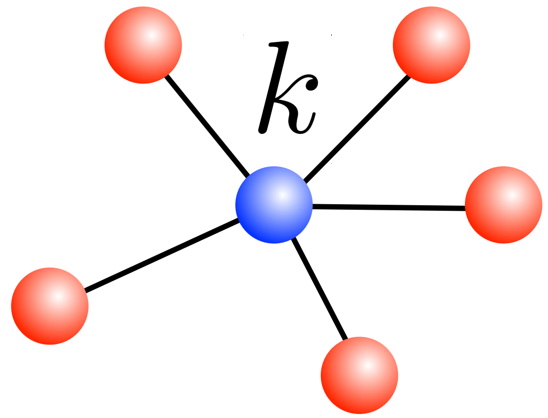

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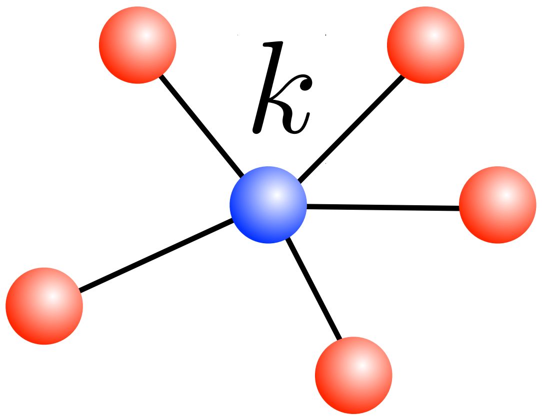

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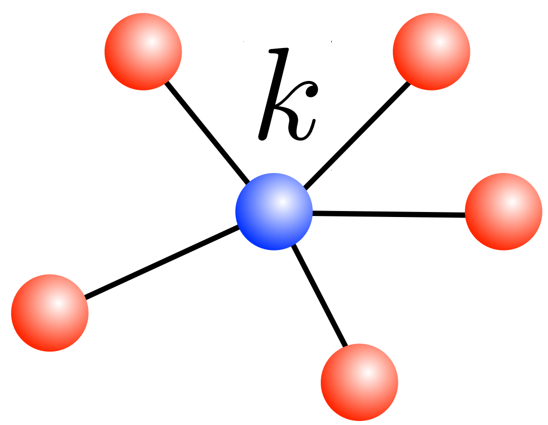

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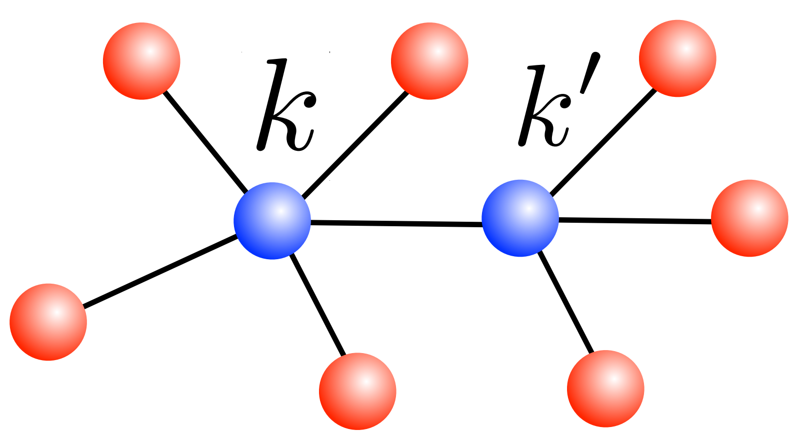

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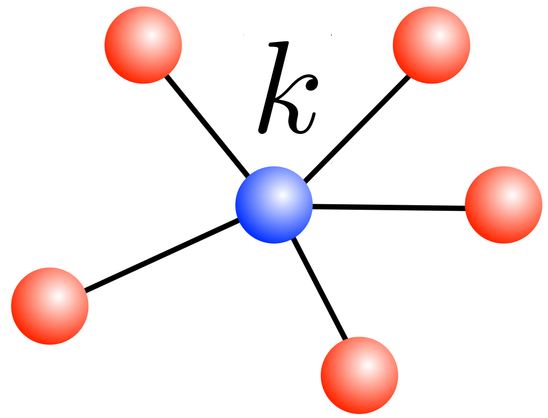

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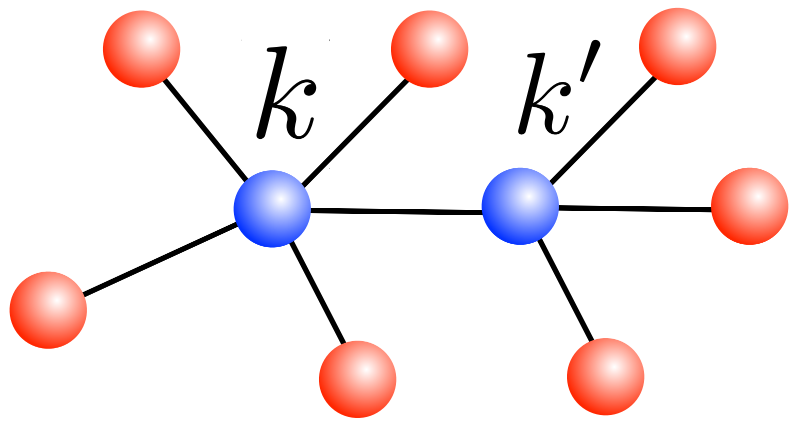

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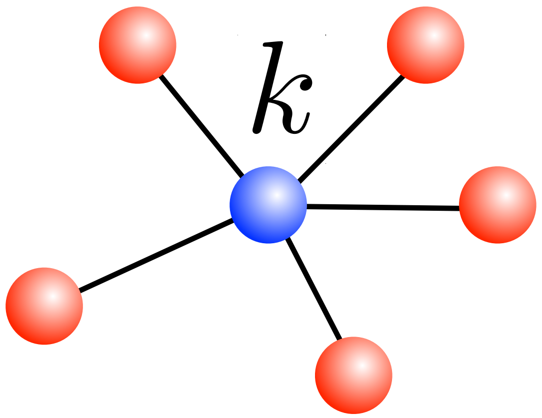

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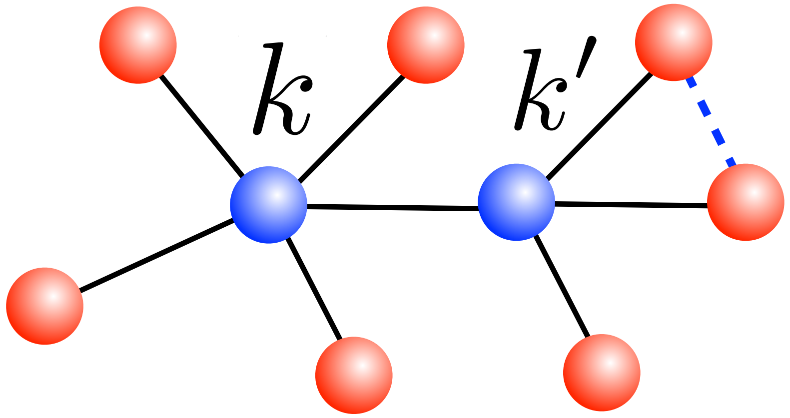

Universitat de Barcelona

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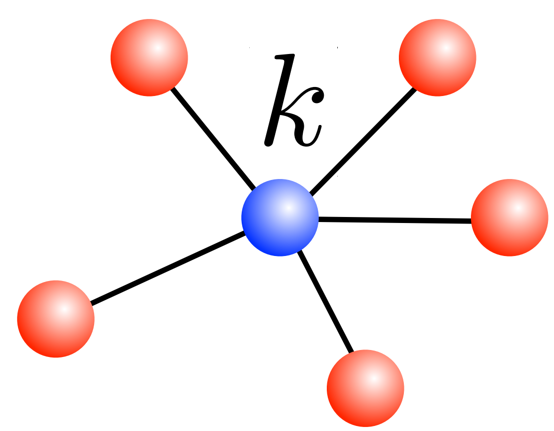

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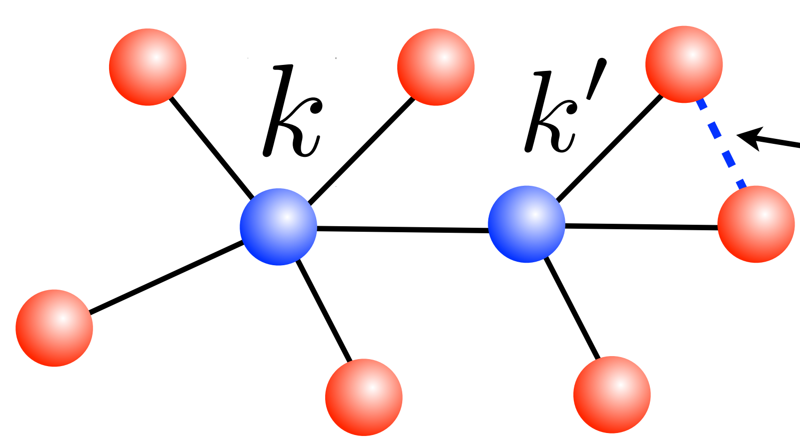

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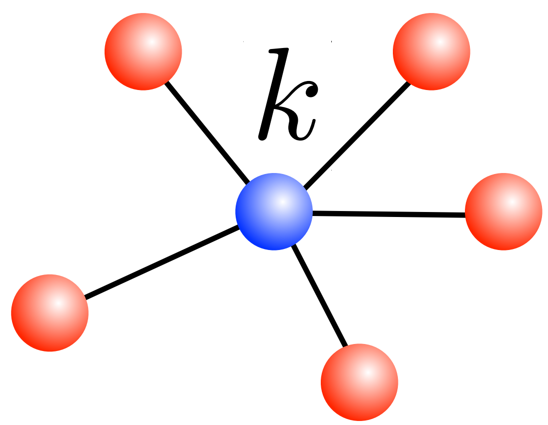

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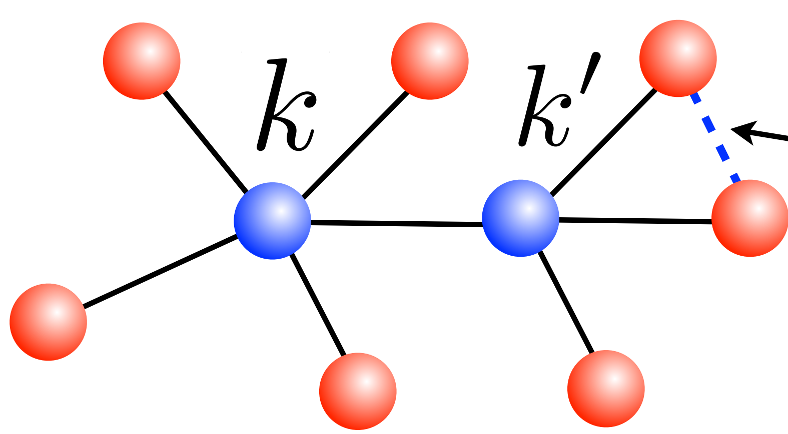
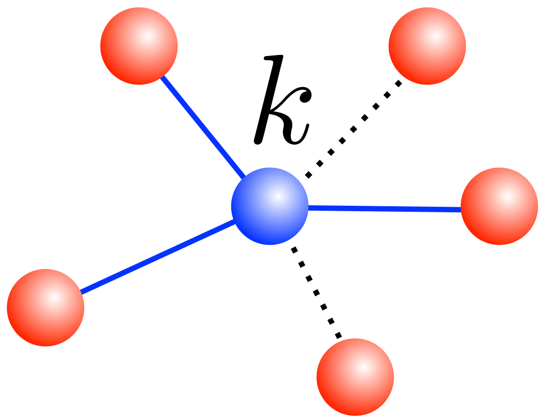


probability that this link exists $\sim \mathcal{O}\left(\frac{1}{N}\right)$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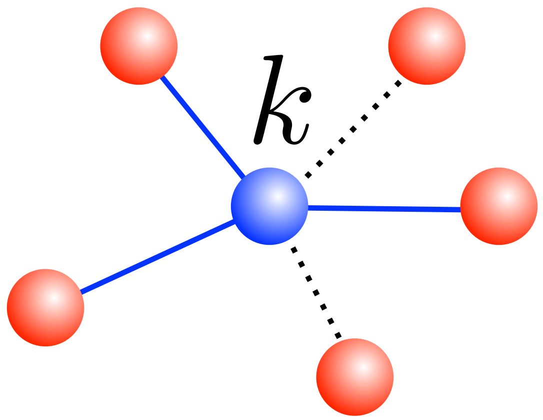

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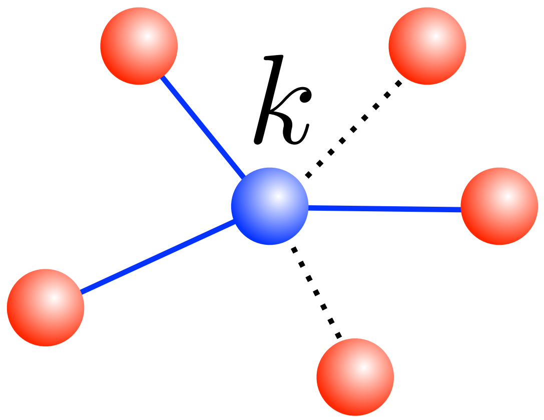

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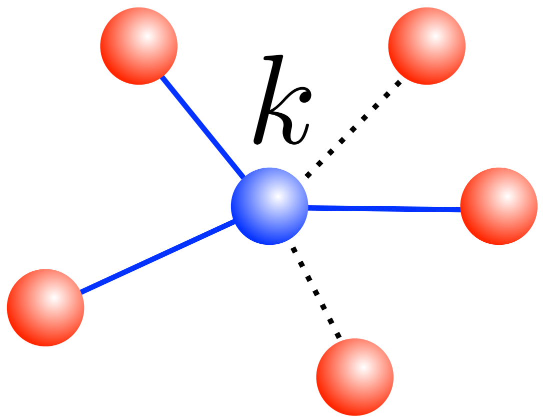

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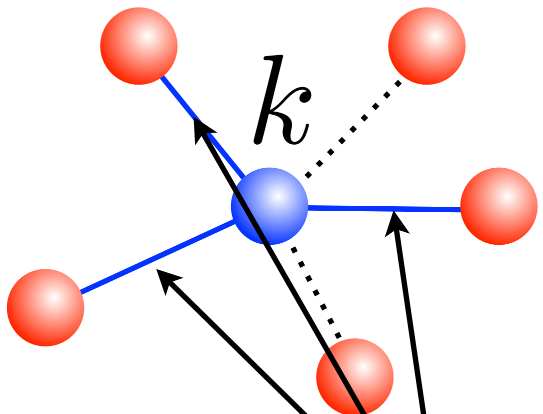

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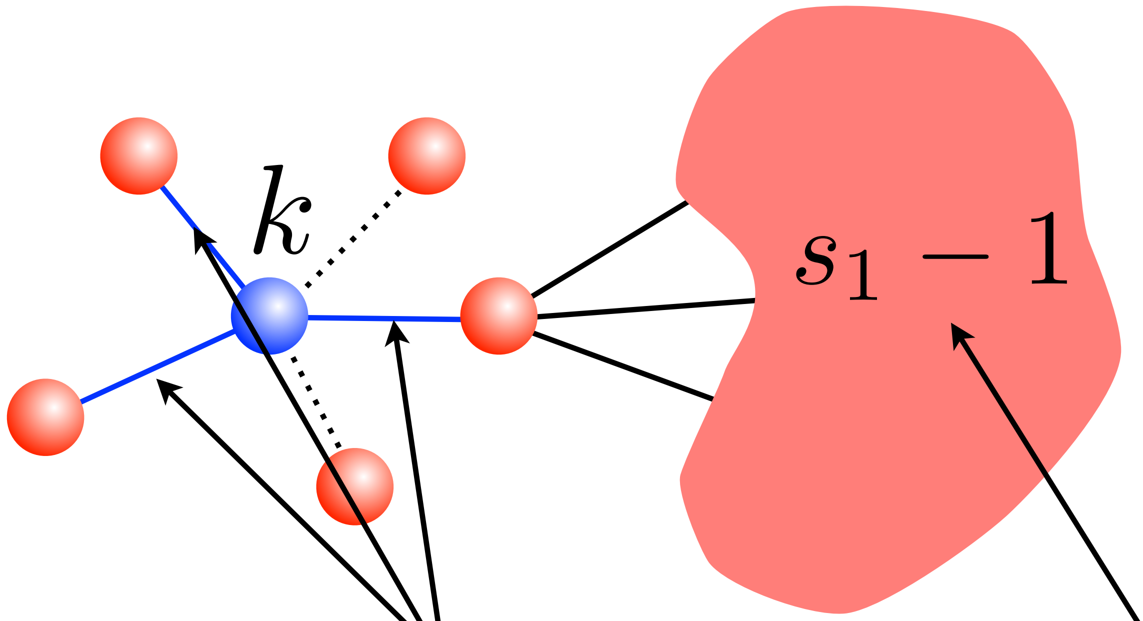

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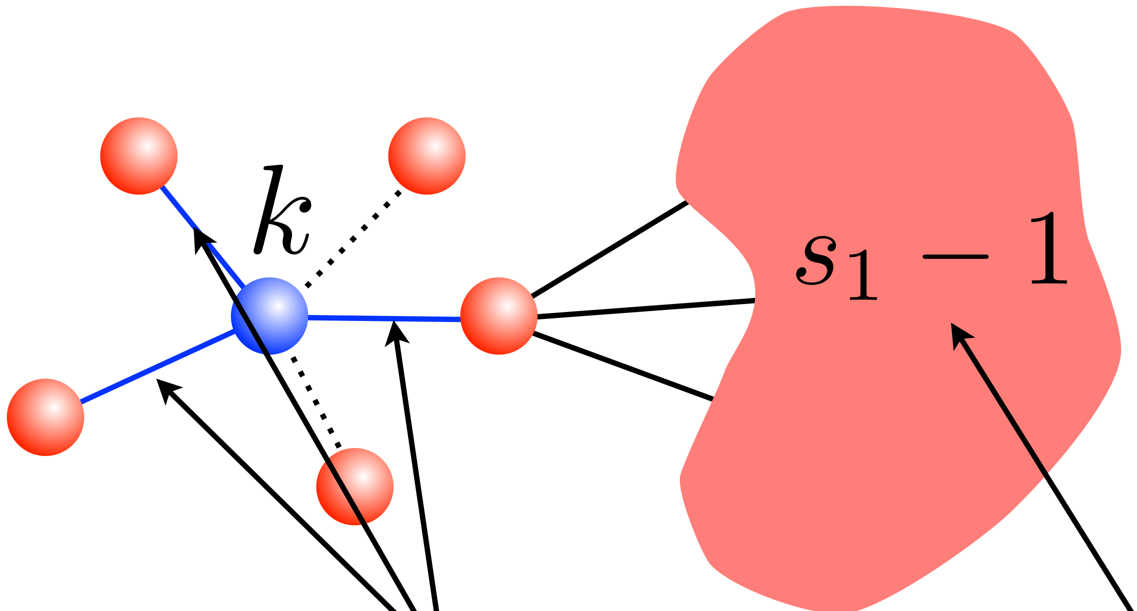

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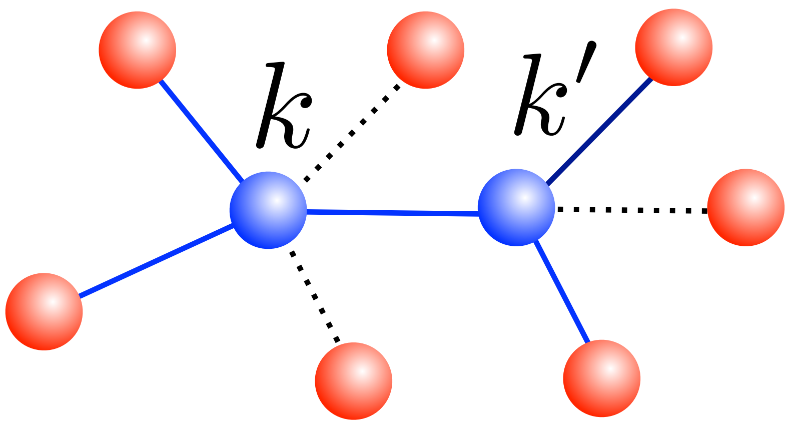
$$\hat{P}(z|k) = z \left[1 - p + p\hat{G}(z|k) \right]^k$$


k
k'

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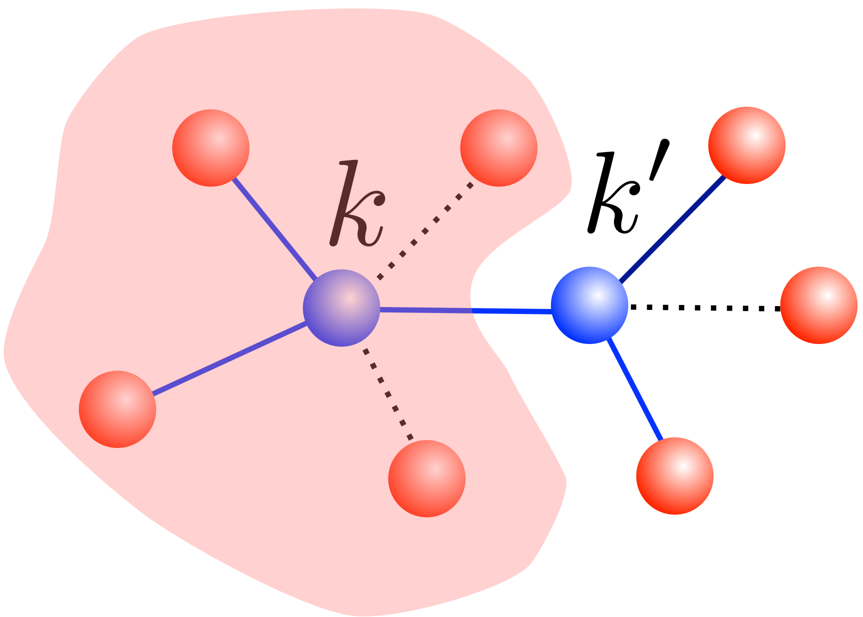

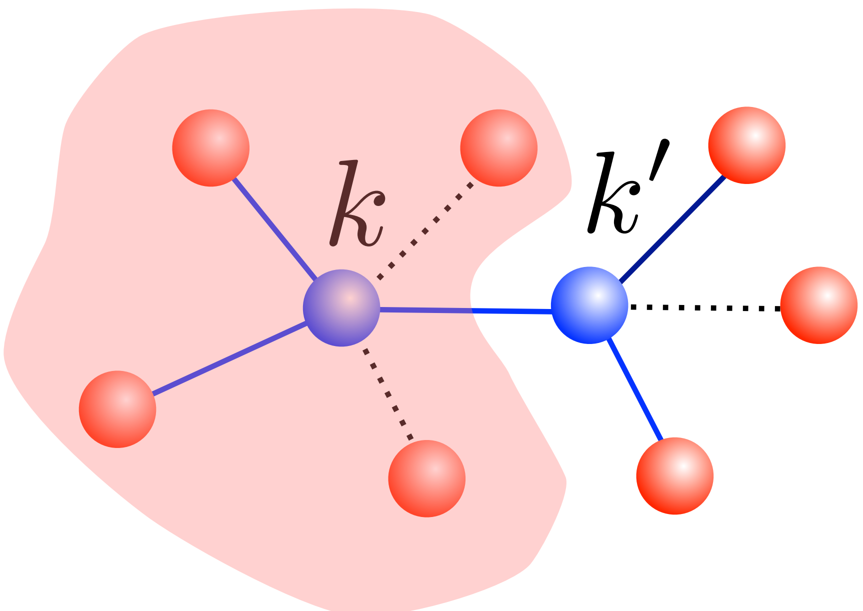

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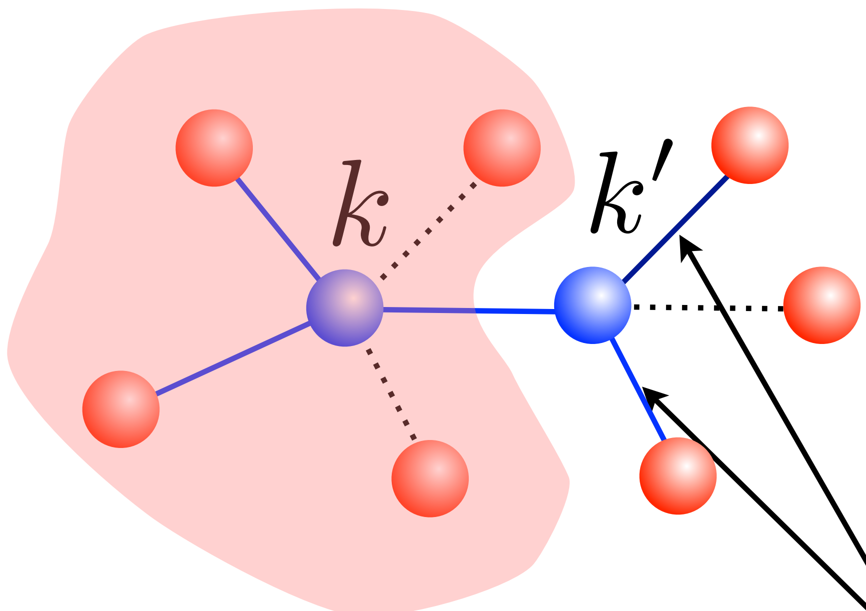

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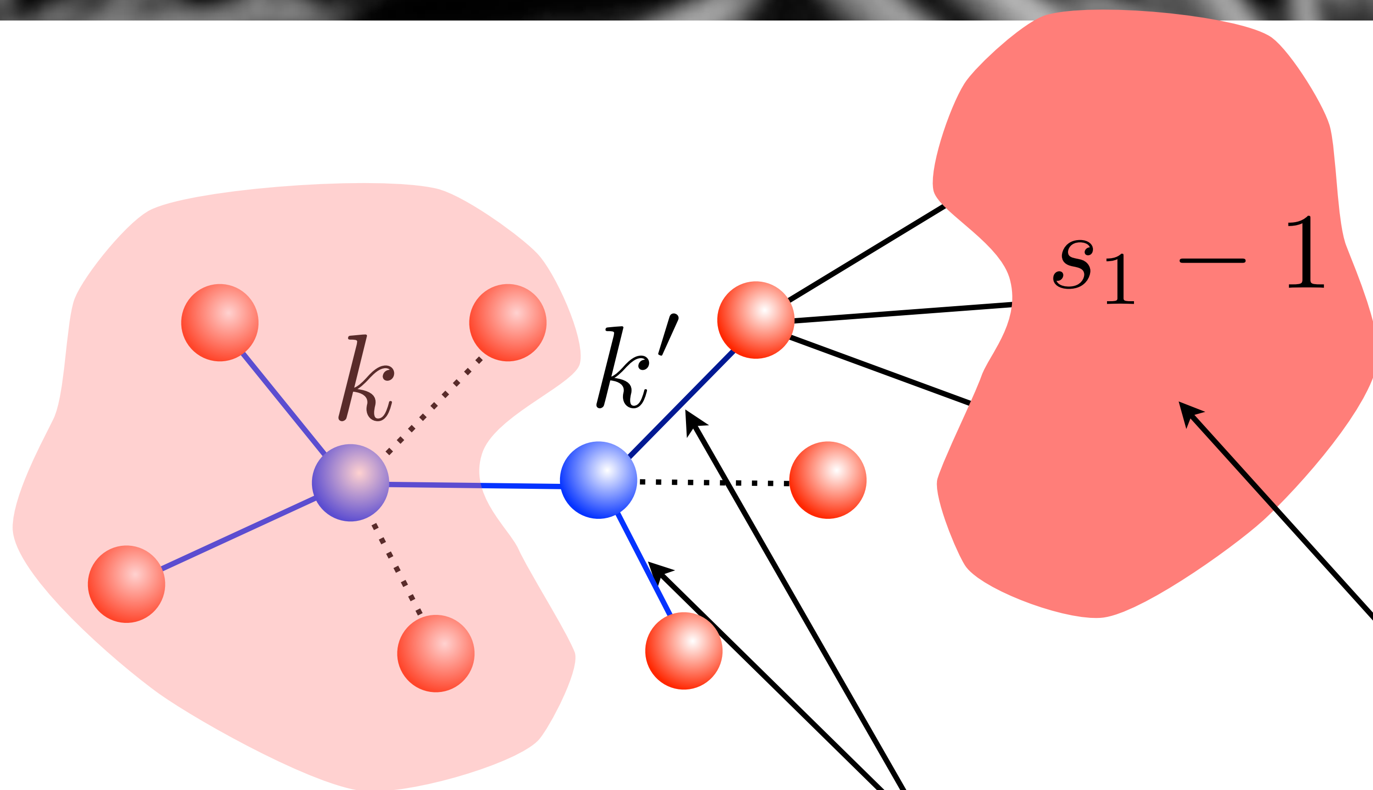

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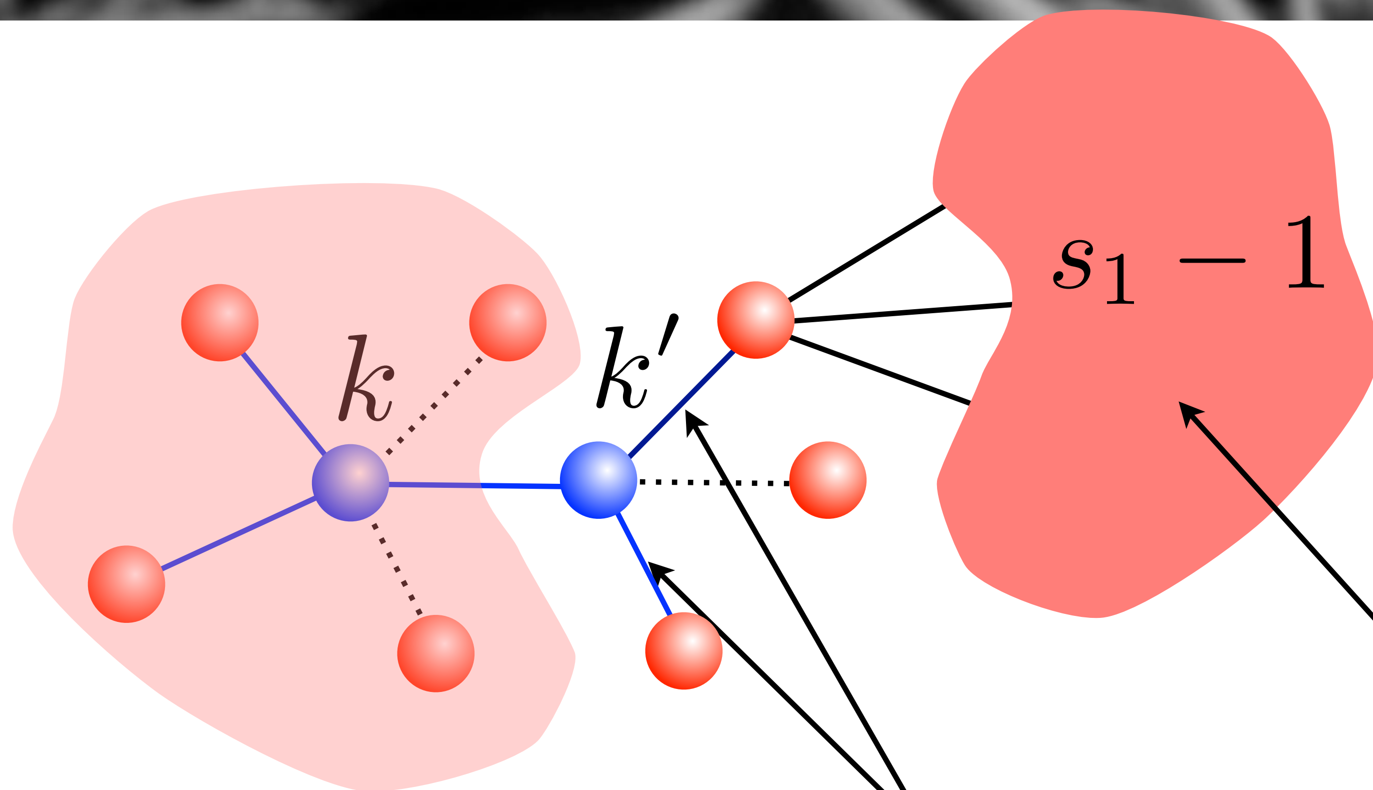

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$$\hat{G}(z|k) = z \sum_{k'} P(k'|k) \left[1 - p + p\hat{G}(z|k') \right]^{k'-1}$$

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Order parameter

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M. Boguñá and R. Pastor-Satorras, Phys. Rev. E 66, 047104 (2002)
M. Boguñá, R. Pastor-Satorras, and A. Vespignani Phys. Rev. Lett 90, 028701 (2003)
Y. Moreno and A. Vázquez, Phys. Rev. E 67, 015101(R) (2003)

For uncorrelated random graphs

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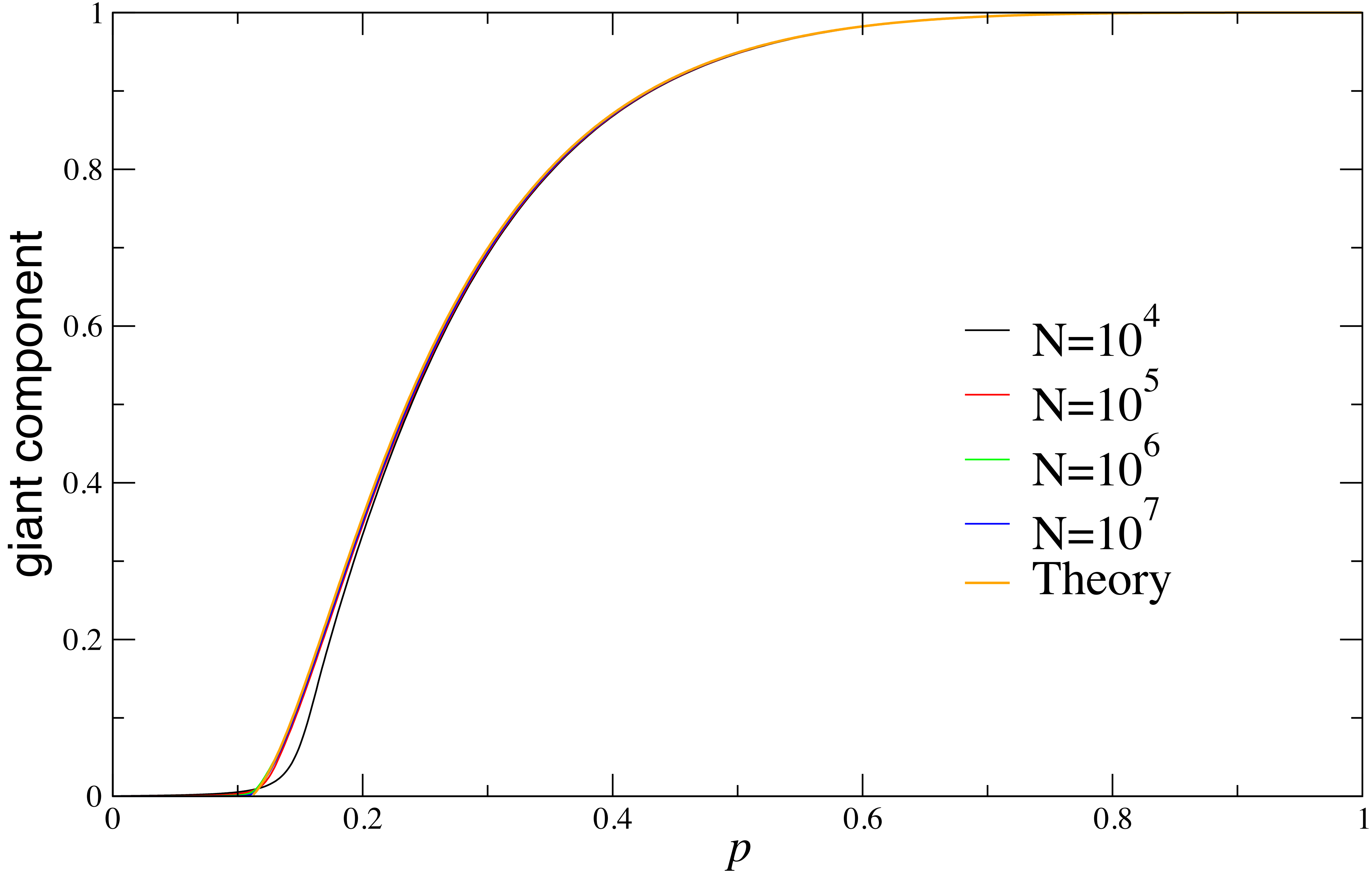
$$\boxed{\begin{array}{ll} y = 1 - \sum_{k'} \frac{k' P(k')}{\langle k \rangle} \left[1 - py\right]^{k'-1} & \\ & p_c = \frac{\langle k \rangle}{\langle k(k-1) \rangle} \\ P_\infty = 1 - \sum_k P(k) \left[1 - py\right]^k & \end{array}}$$

Power law degree distribution with γ=3.5



What about finite size scaling?

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Erdos-Renyi graph
with average degree 5

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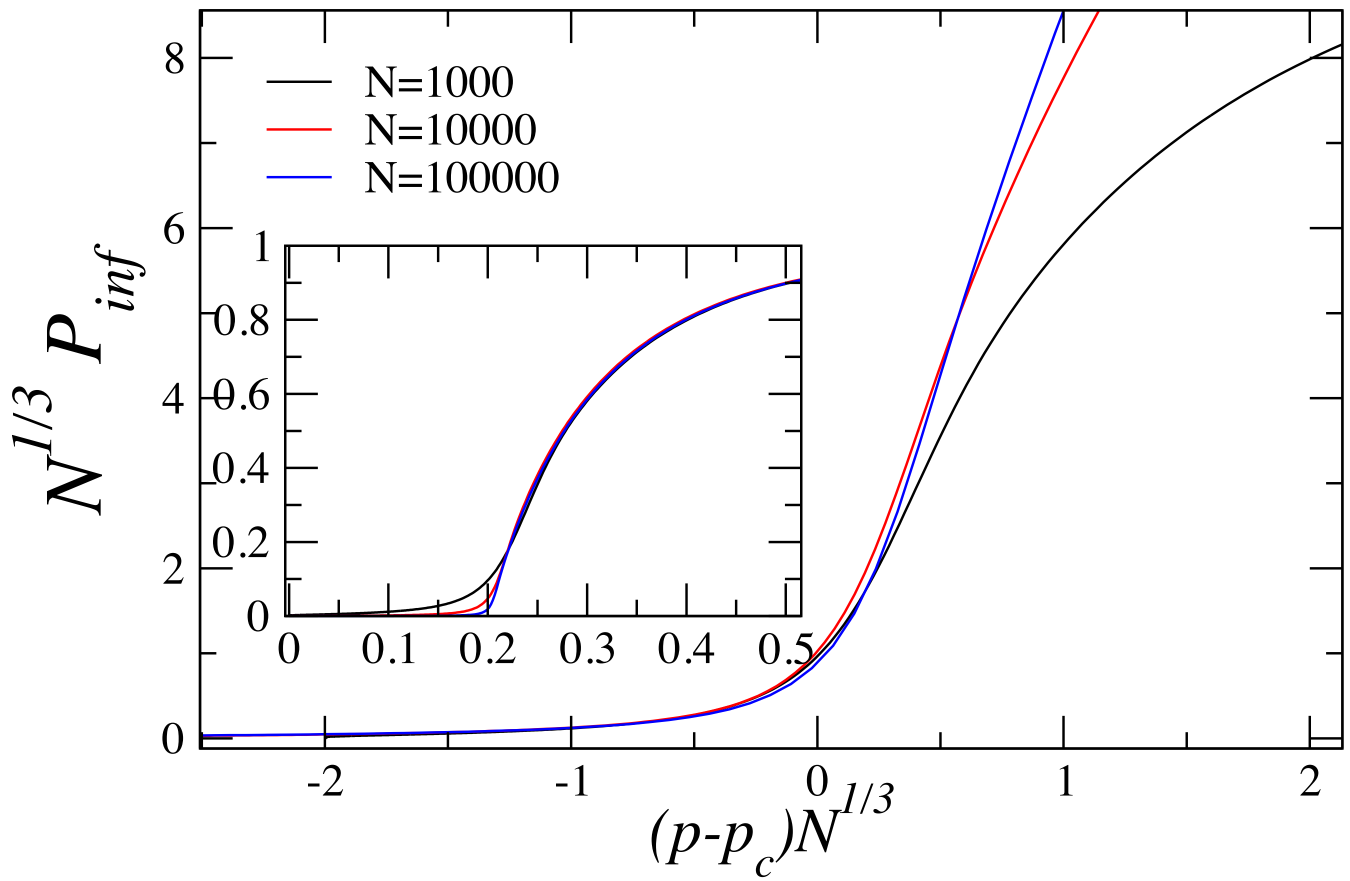

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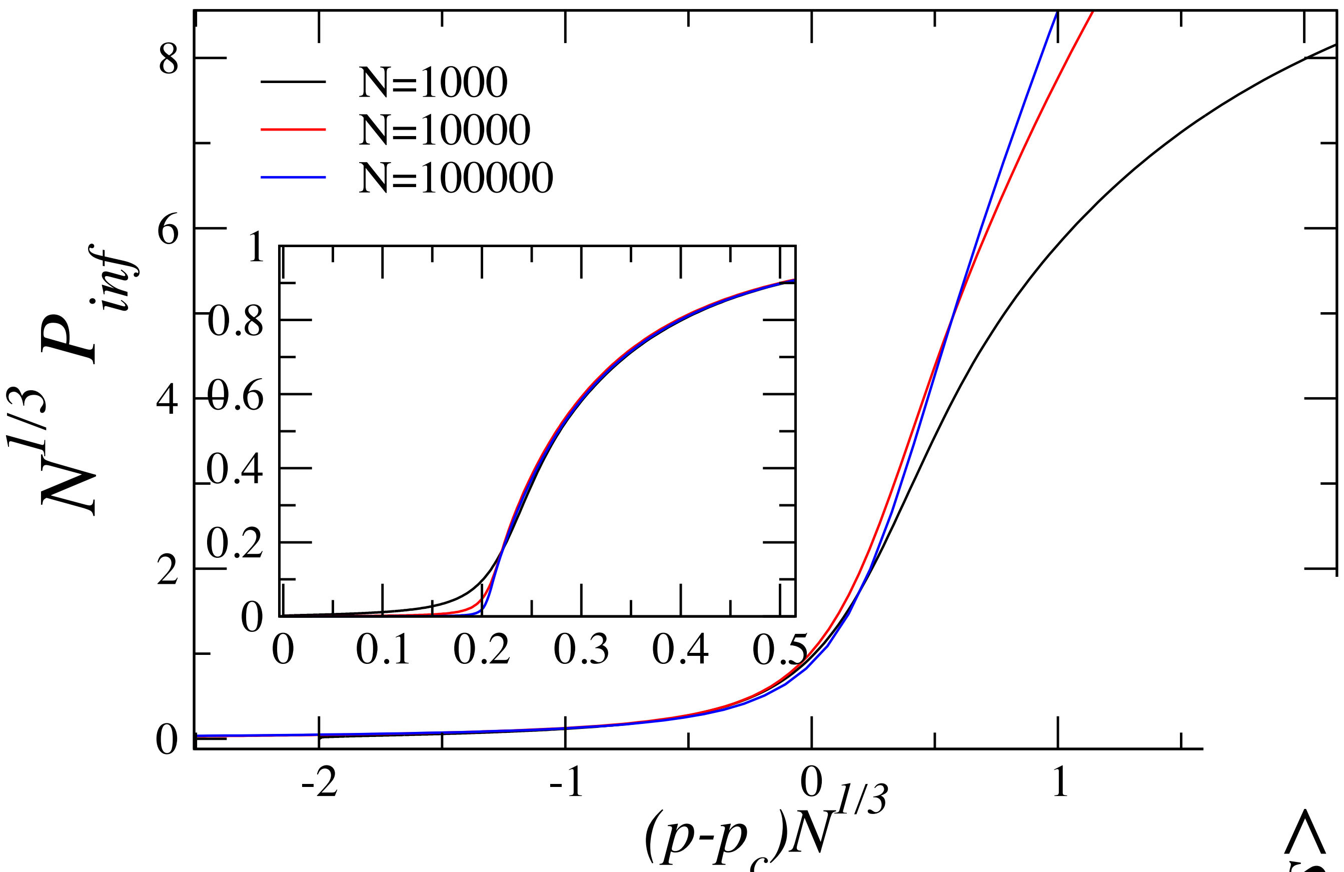

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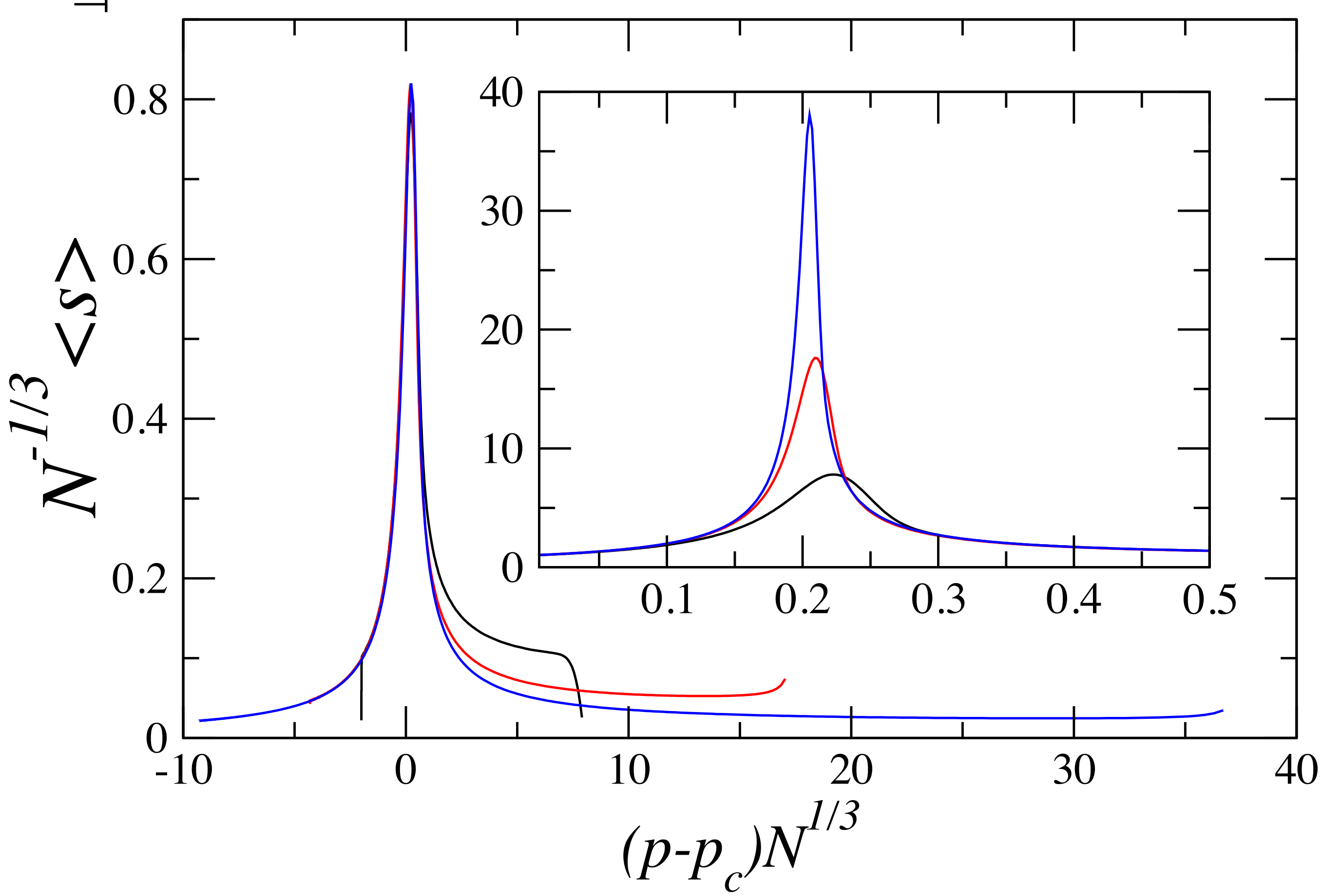

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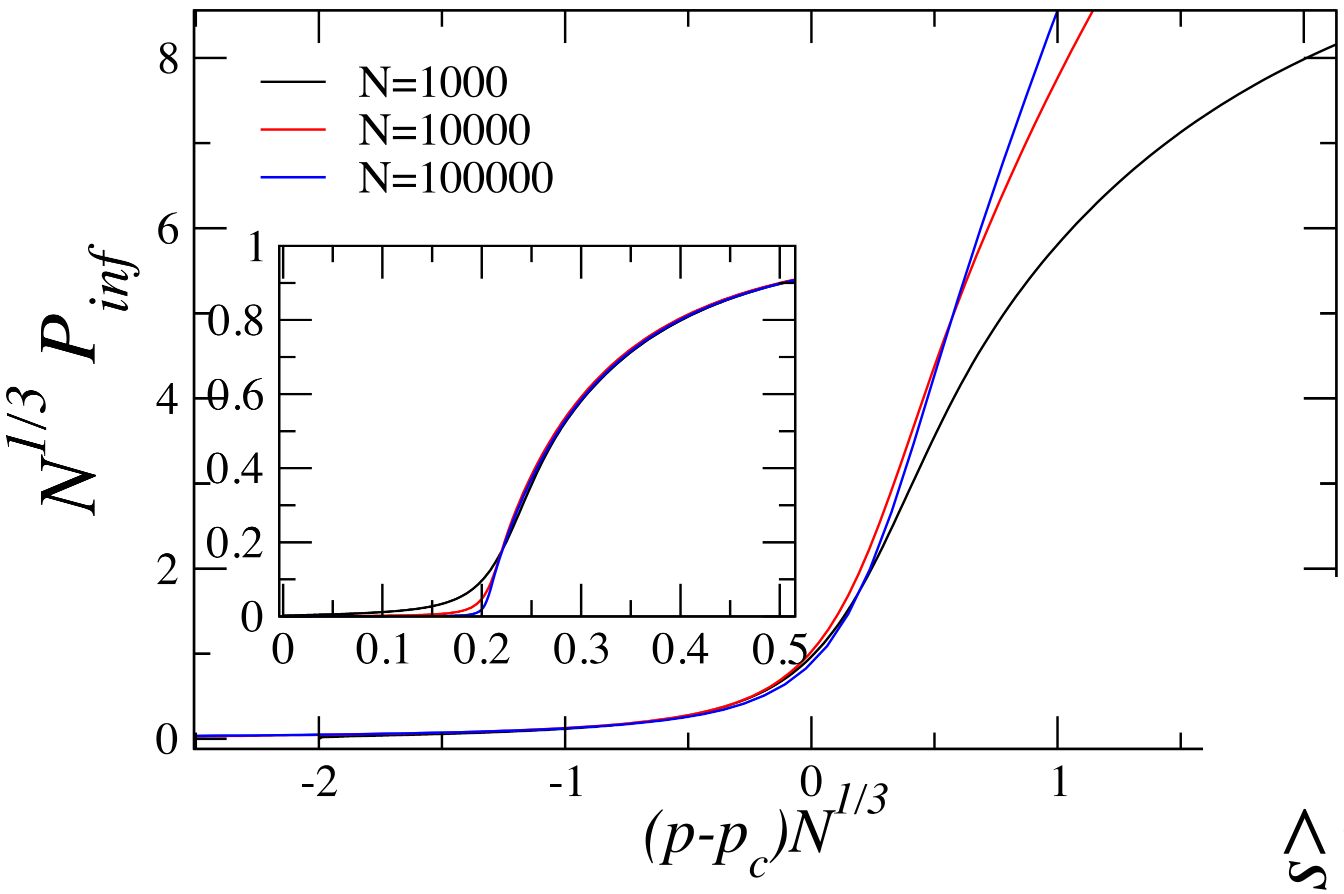

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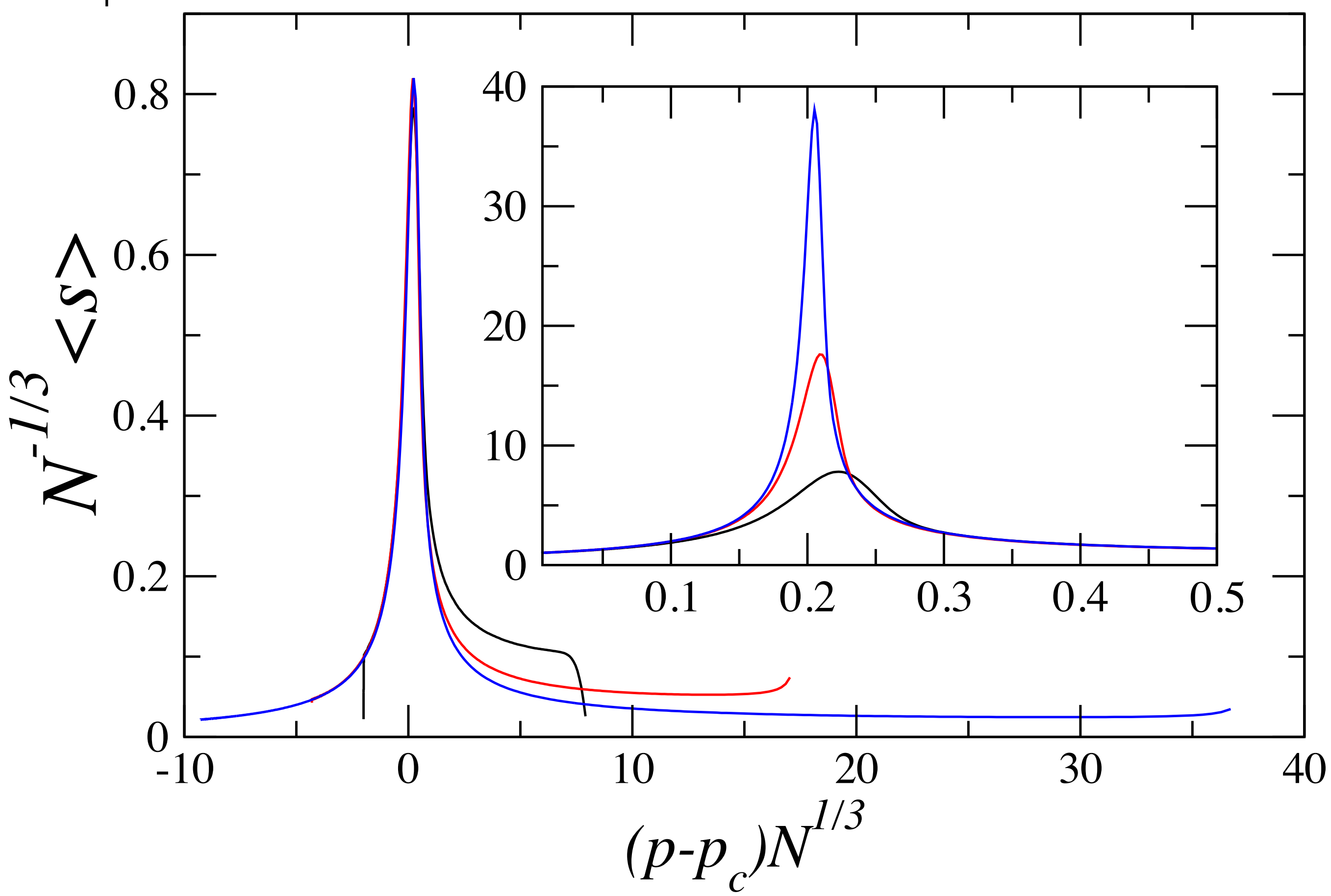
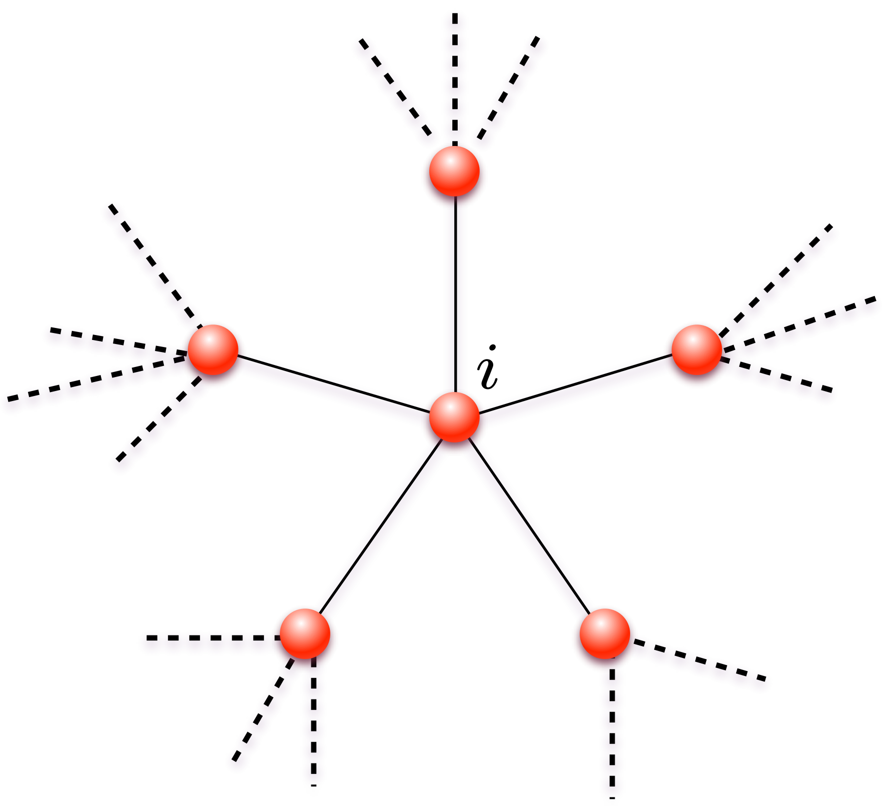


Unfortunately, it doesn't work so well for scale-free networks

Percolation on clustered networks



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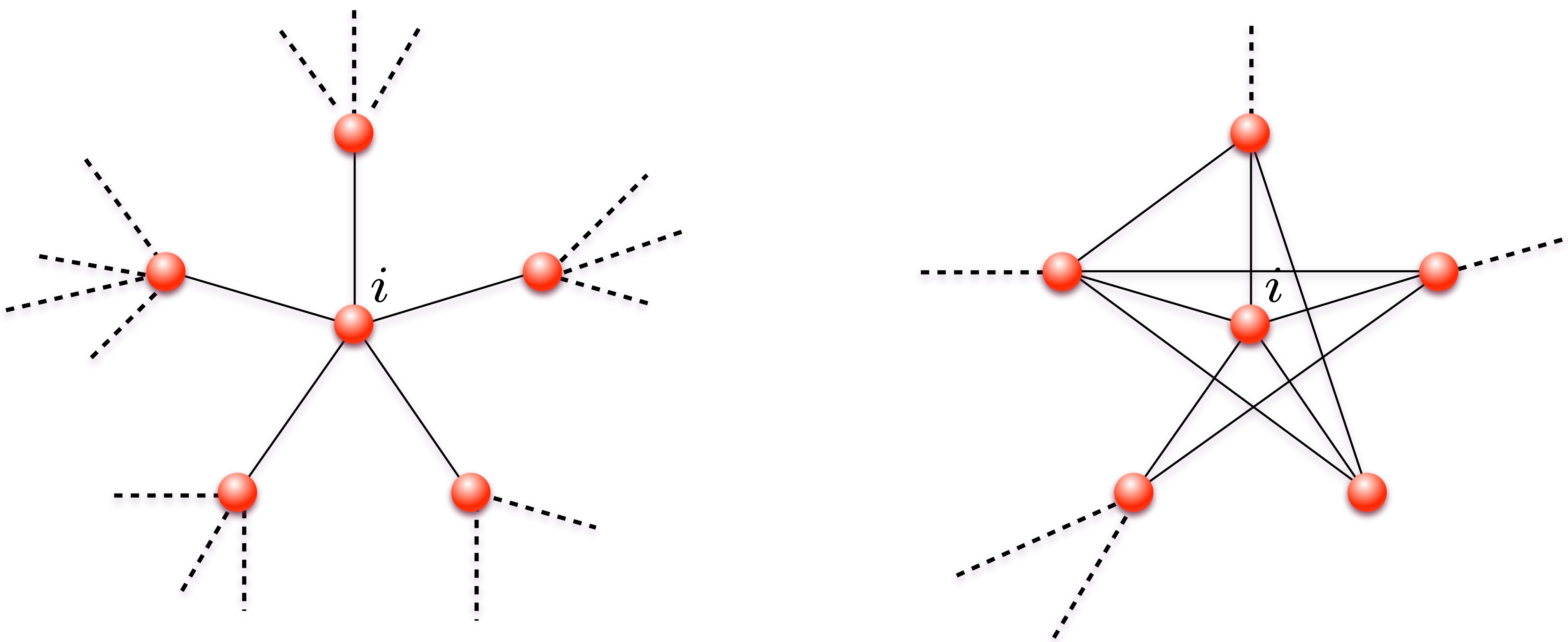
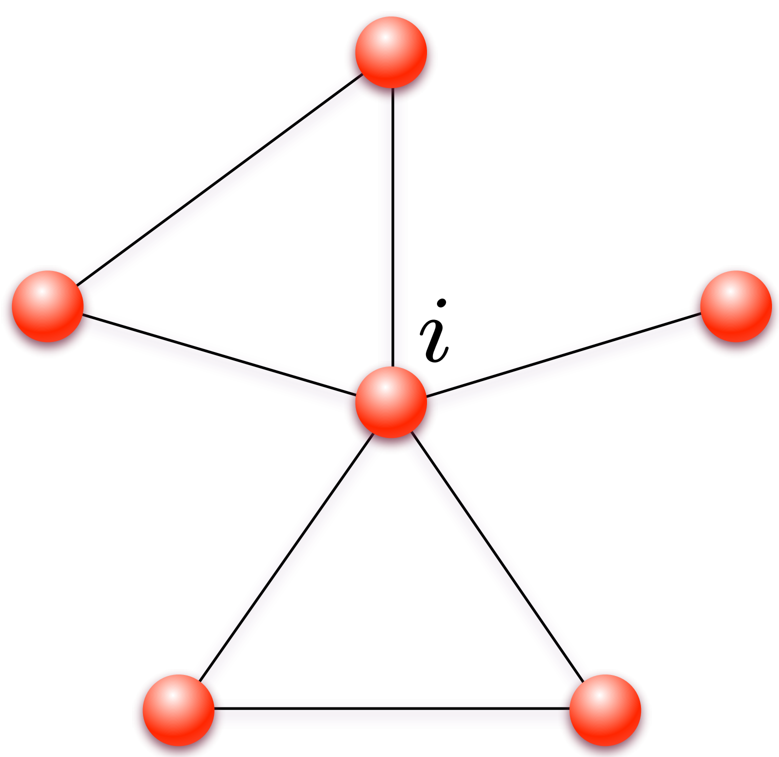


Weak vs. strong clustering

Weak vs. strong clustering

Weak clustering:
Triangles are disjoint



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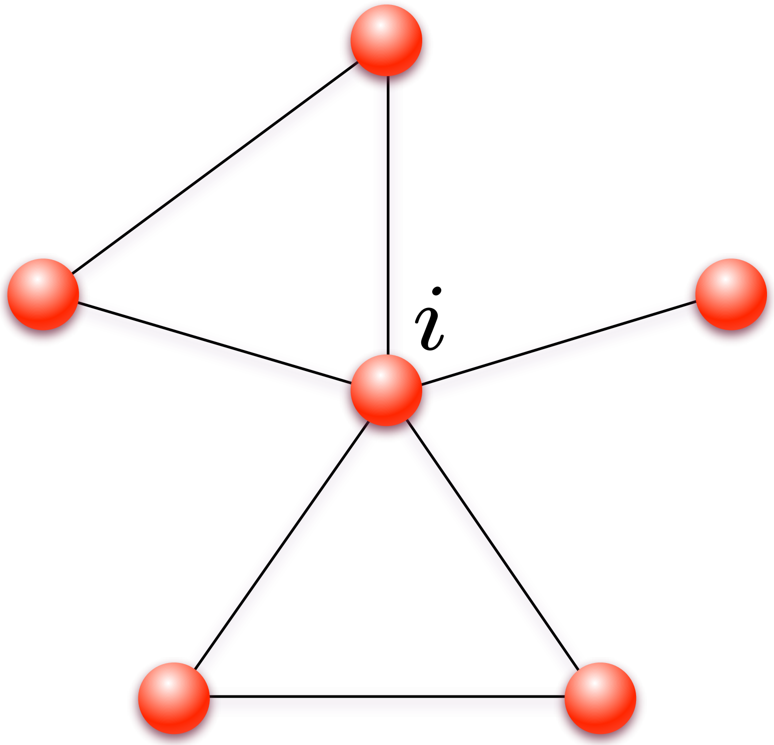
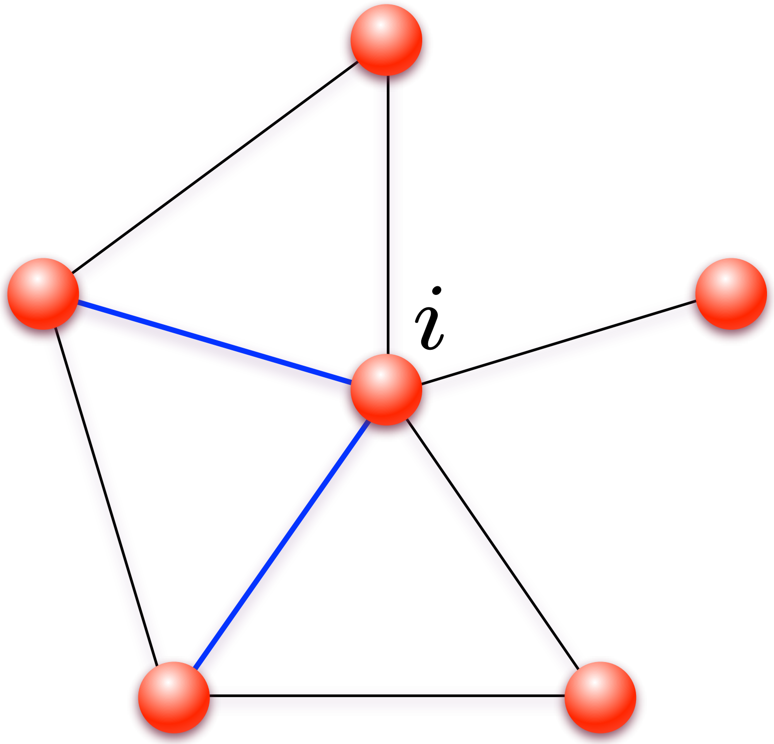


Strong clustering: Triangles coalesce into some links



Weakly clustered networks

$$\boxed{p_c = \frac{1}{\Lambda_m}}$$

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average number of triangles going through the edge *k-k'*

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One can prove that $\Lambda_m^{clust} \leq \Lambda_m^{unclust}$

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Weak clustering hinders percolation

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$$\Lambda_m^{clust} \to \infty \text{ when } \langle k^2 \rangle \to \infty$$

Weak clustering hinders percolation

What about strong clustering?

There is a fundamental difference between strong and weak clustering

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Weak clustering

$$\bar{c}(k) < \frac{1}{k-1}, \forall k$$

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$$\boxed{\bar{c}(k) < \frac{1}{k-1}, \forall k}$$

$$\Downarrow$$

$$m_{ij} < 1$$

What about strong clustering?

There is a fundamental difference between strong and weak clustering

Weak clustering	Strong clustering
$$\bar{c}(k) < \frac{1}{k-1}, \forall k$$	$$\bar{c}(k) > \frac{1}{k-1}, \forall k \in \mathcal{D}$$

$$\Downarrow$$

$$m_{ij} < 1$$

What about strong clustering?

There is a fundamental difference between strong and weak clustering

Weak clustering

$$\boxed{\bar{c}(k) < \frac{1}{k-1}, \forall k}$$

$$\Downarrow$$

$$m_{ij} < 1$$

Strong clustering

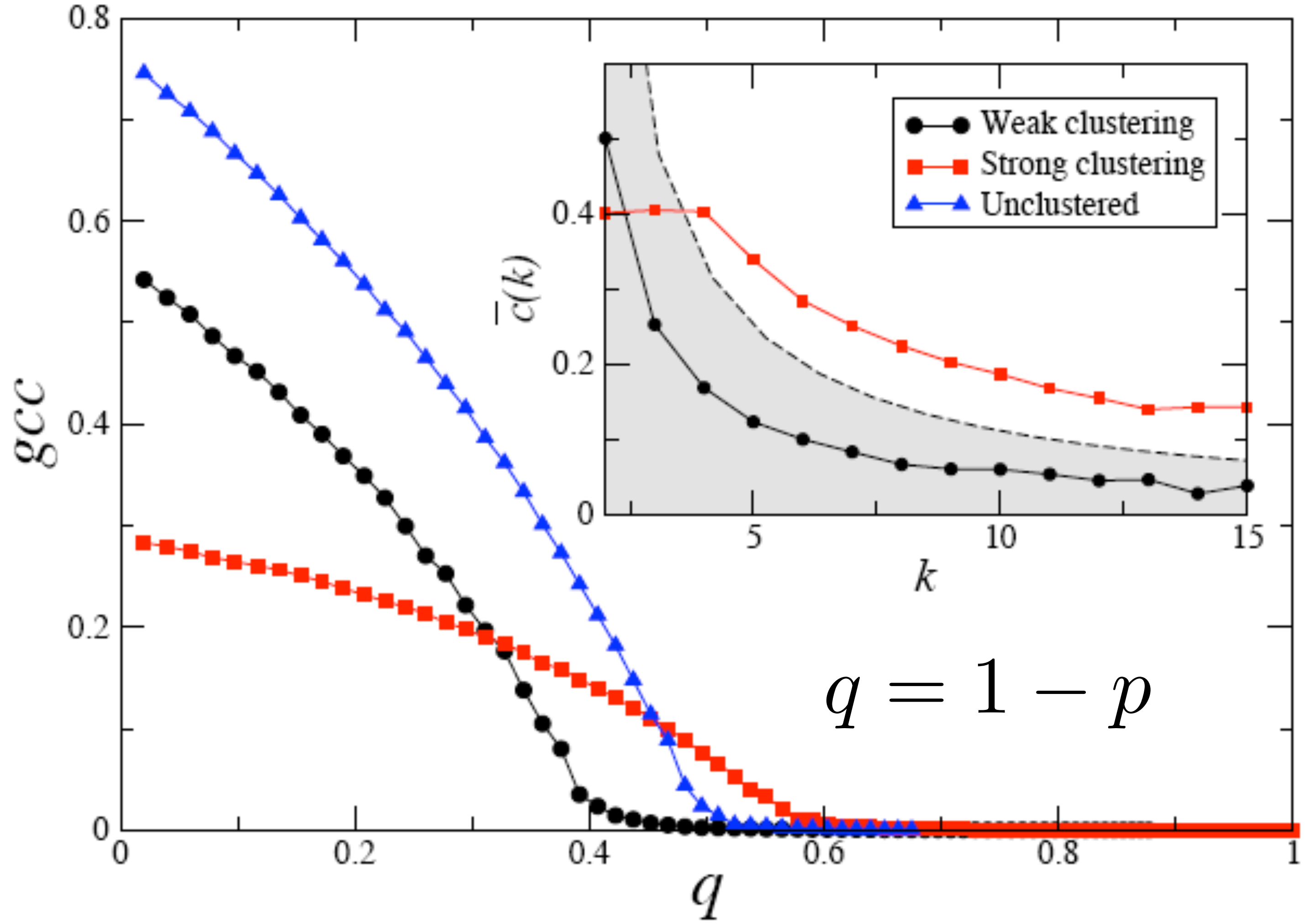
$$\boxed{\bar{c}(k) > \frac{1}{k-1}, \forall k \in \mathcal{D}}$$

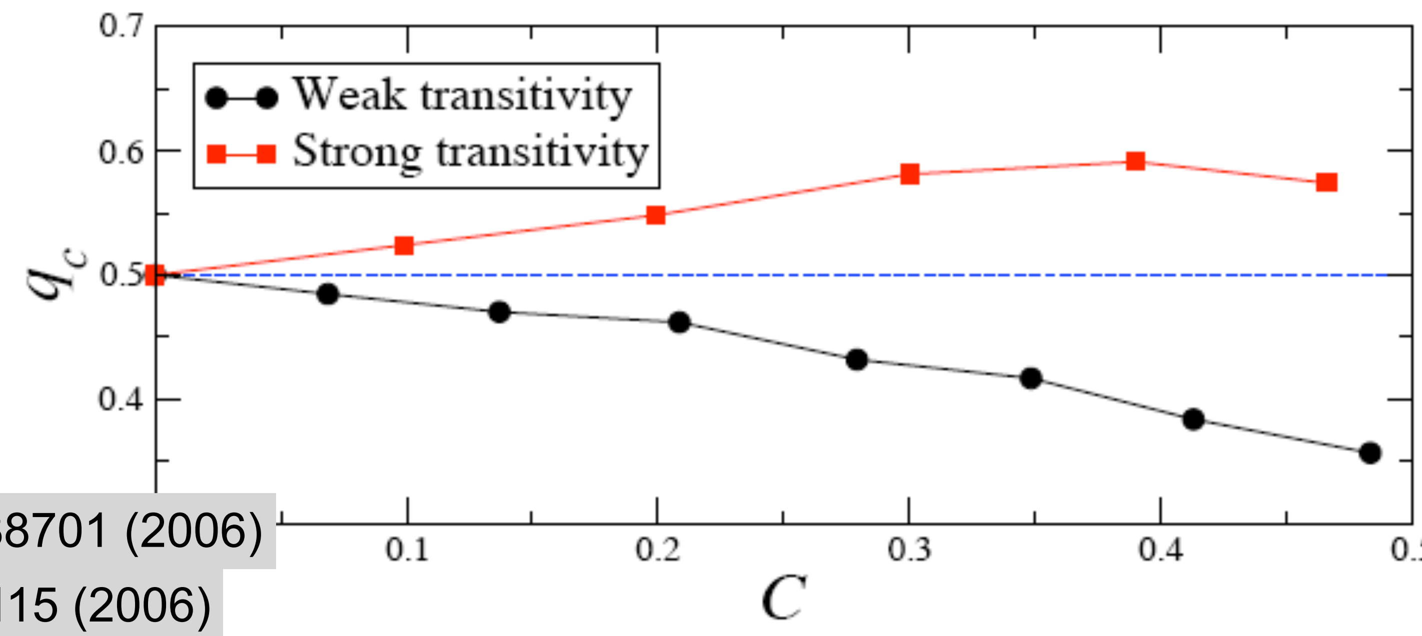
$$\Downarrow$$

$$m_{ij} > 1$$

Universitat de Barcelona

Exponential degree distribution





M. A. Serrano, and M. Boguñá, Phys. Rev. Lett. 97, 088701 (2006)

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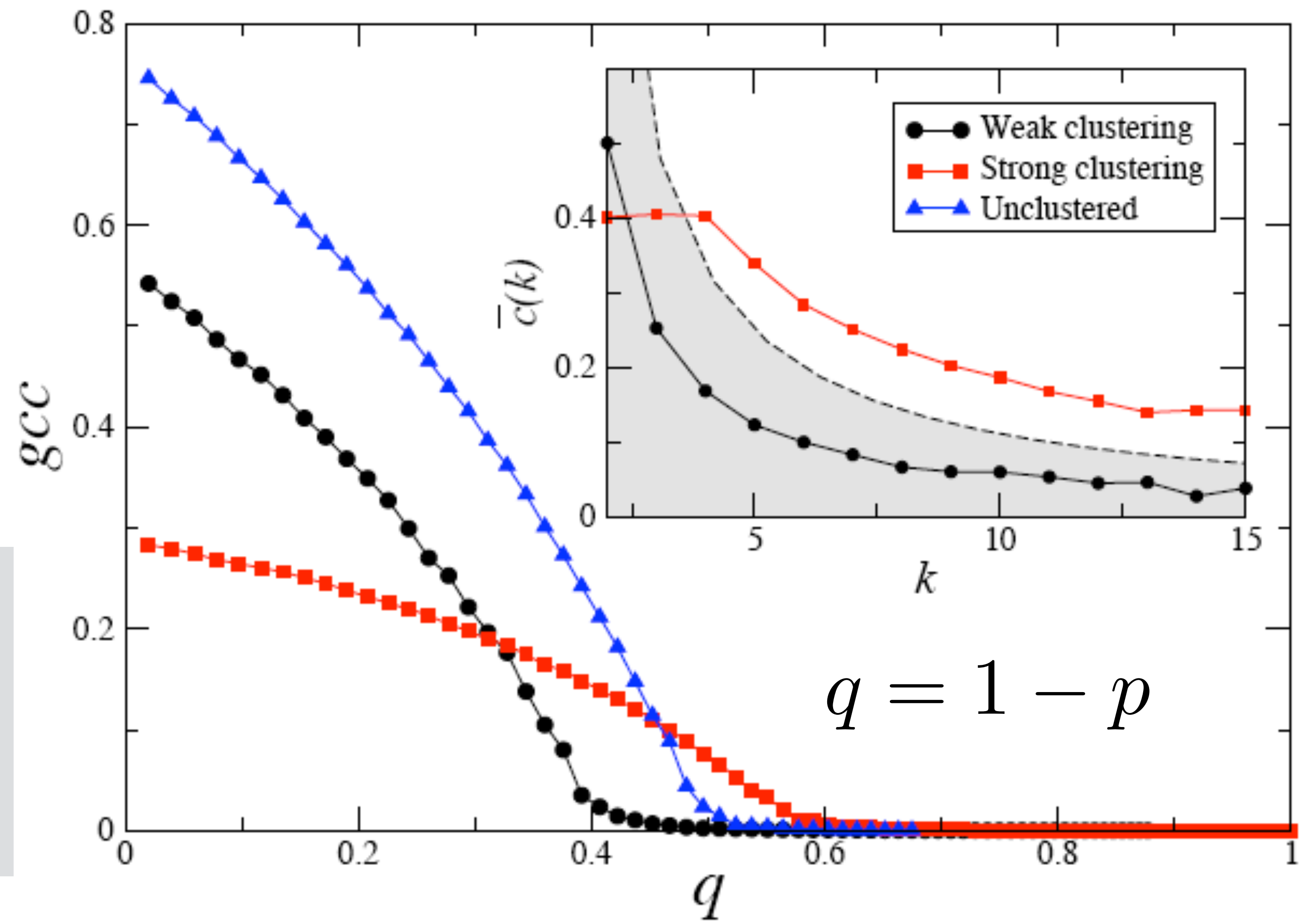
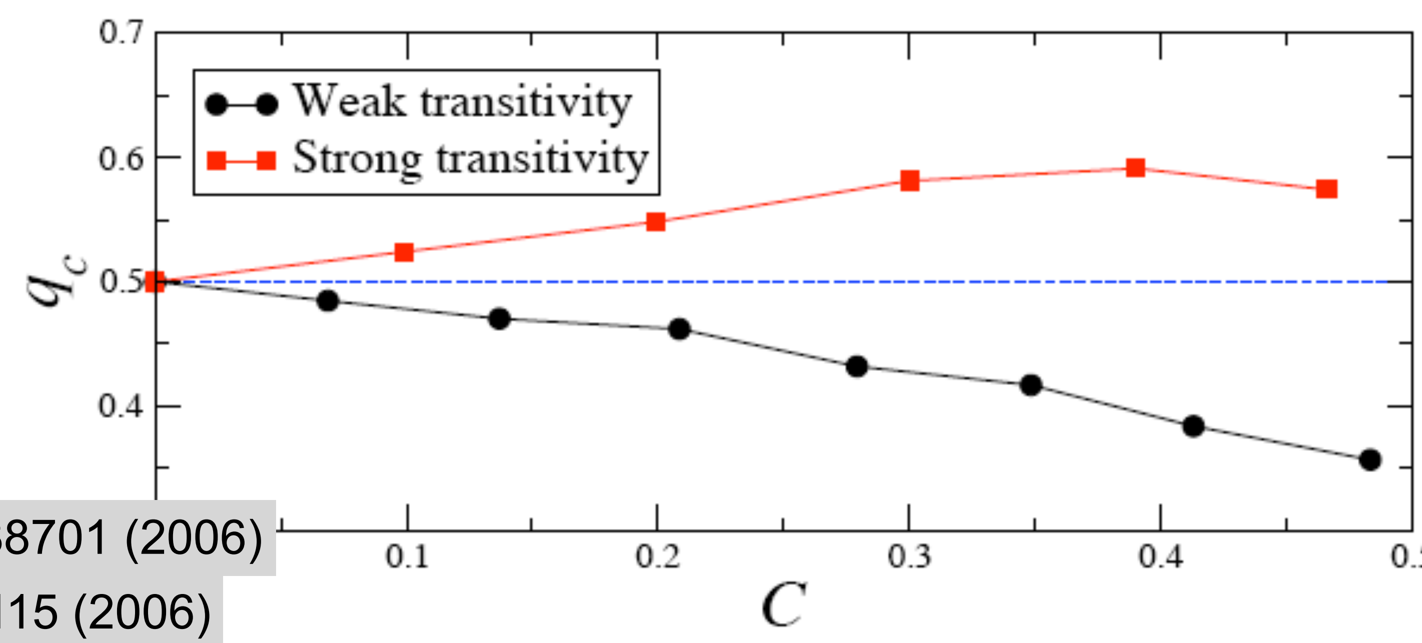
Exponential degree distribution



Strong clustering favors the onset of the giant component



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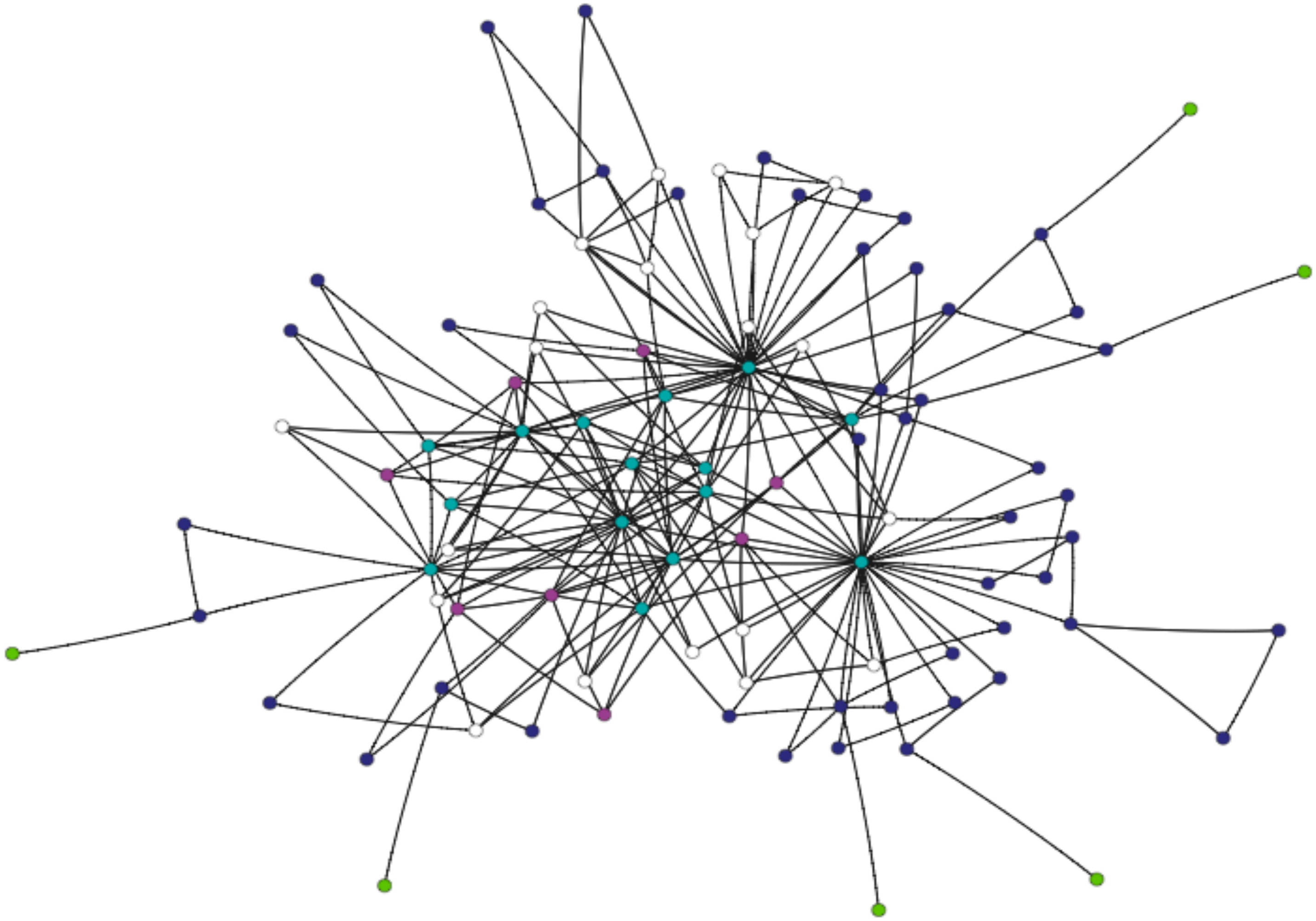
k-core decomposition

k-core

Maximal subgraph with vertices having *k* or more connections within the subgraph



P. Colomer-de-Simón and M. Boguñá, Phys. Rev. X 4, 041020 (2014)

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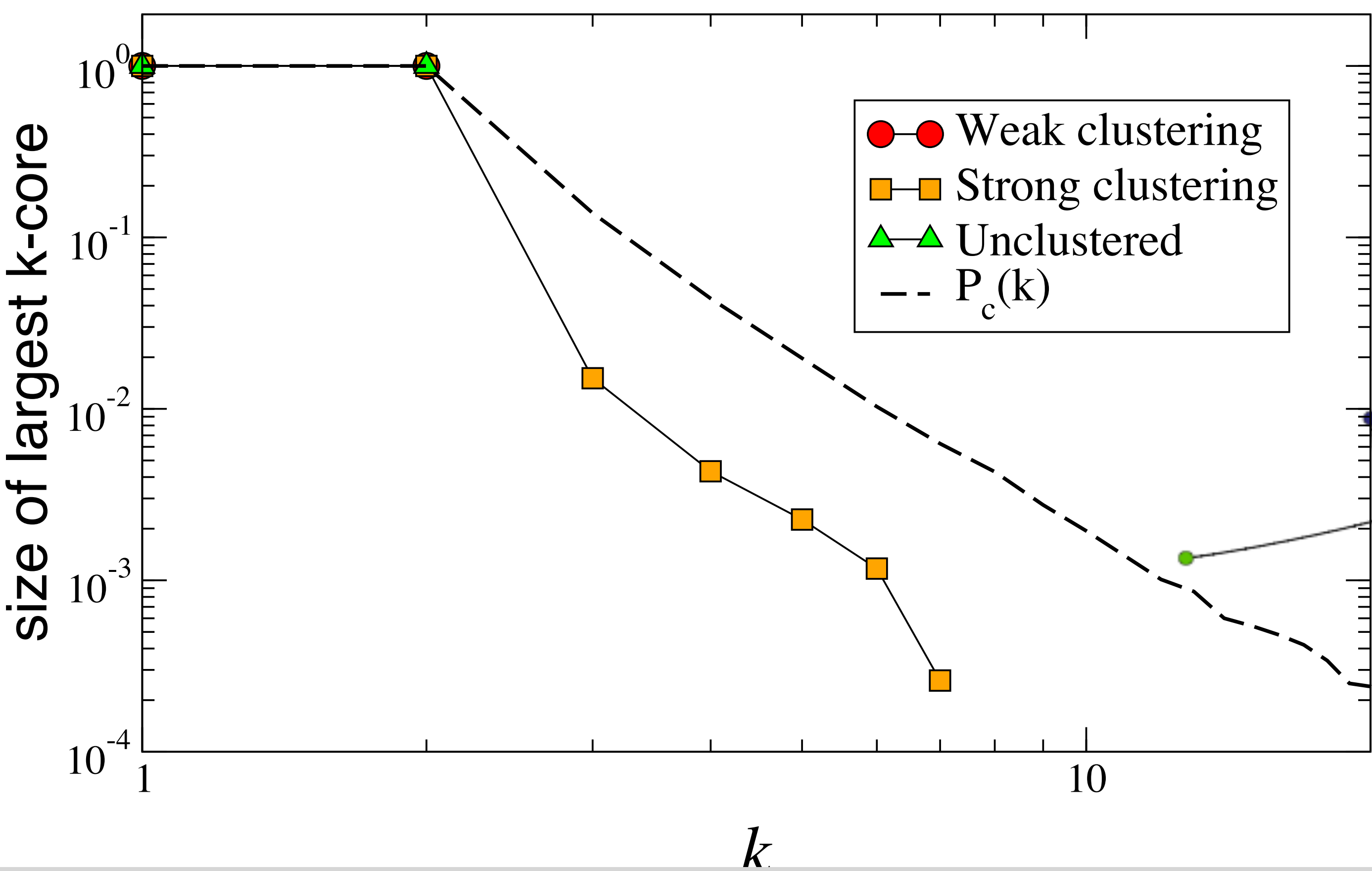
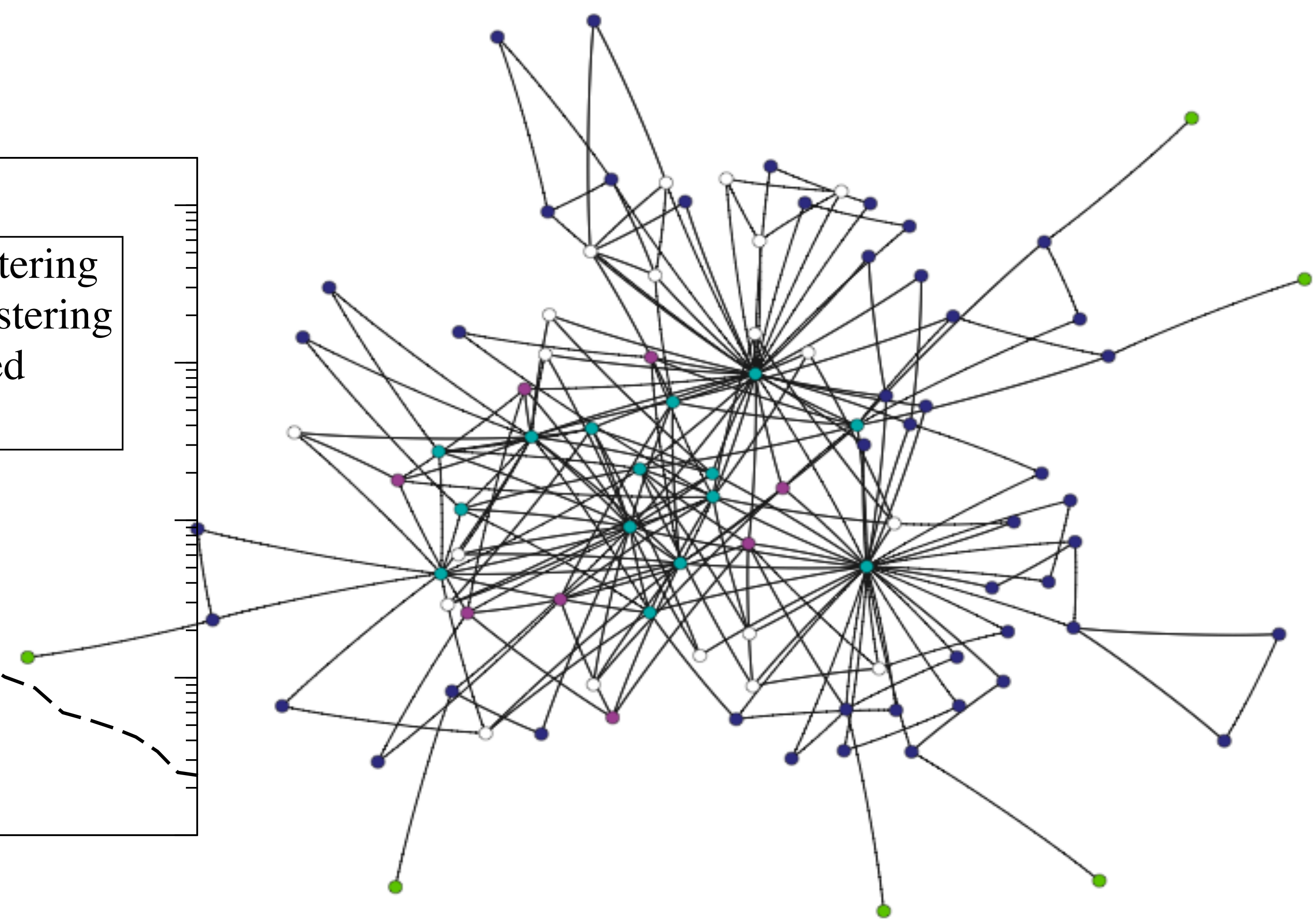
Homogeneous network





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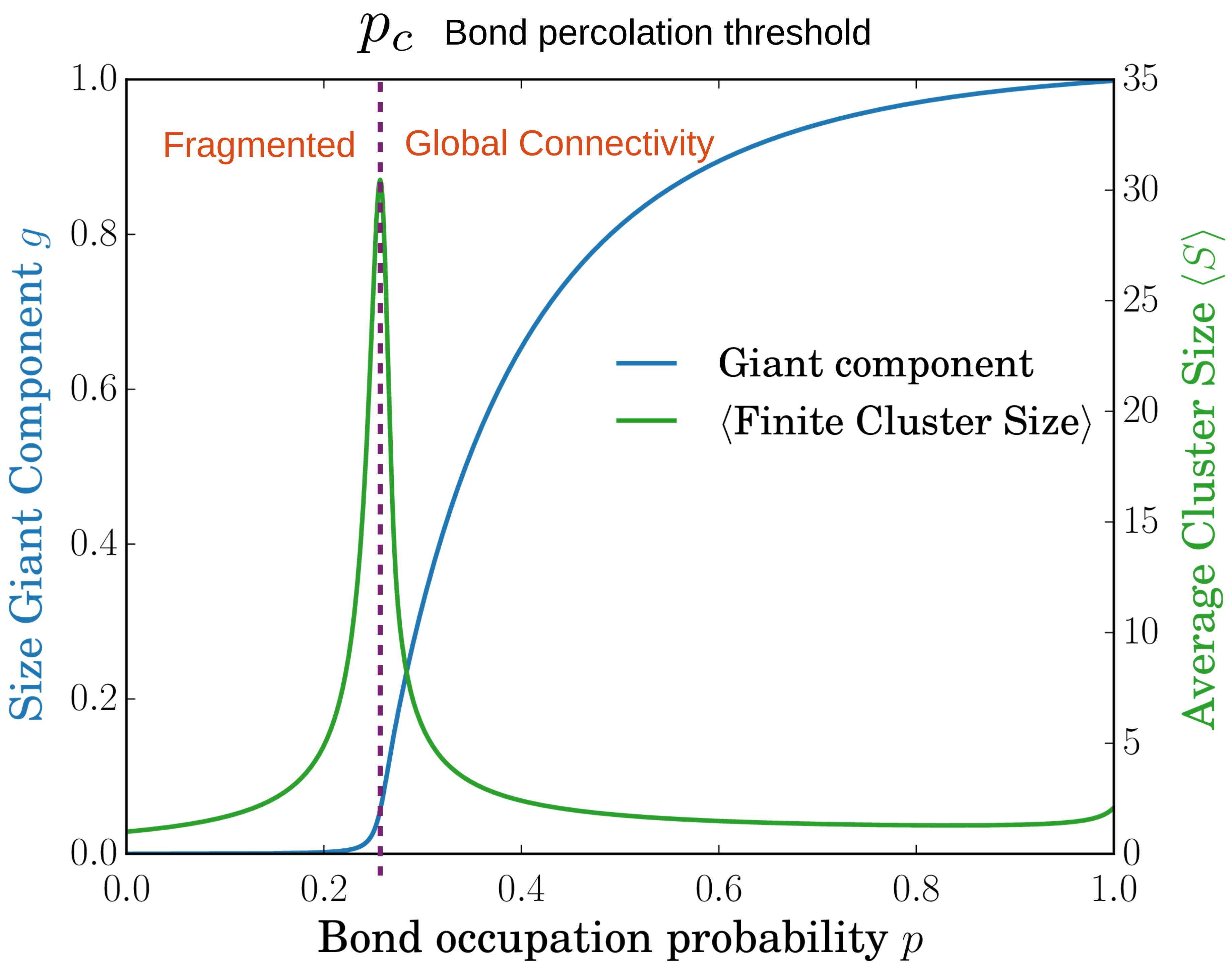
Bond Percolation



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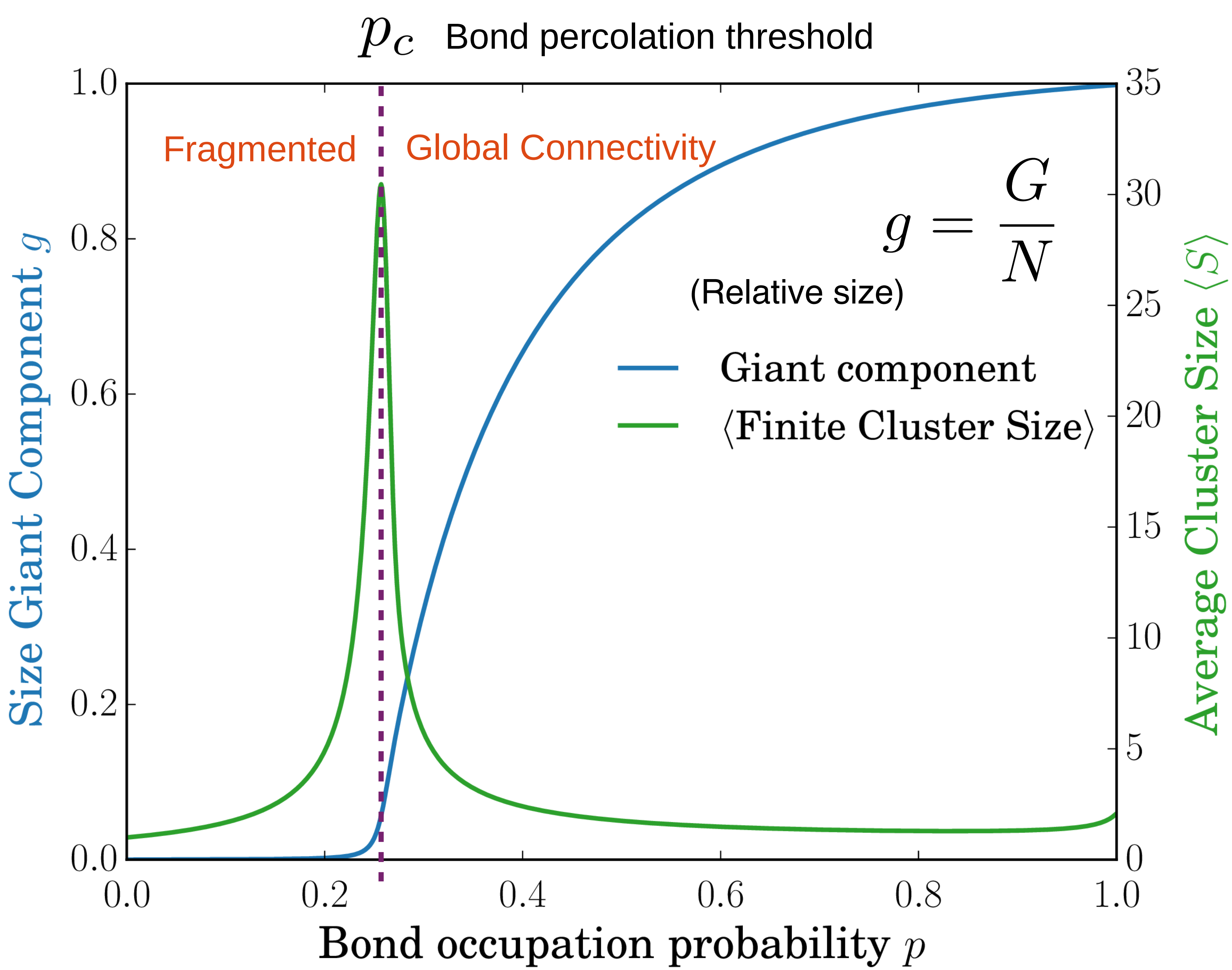

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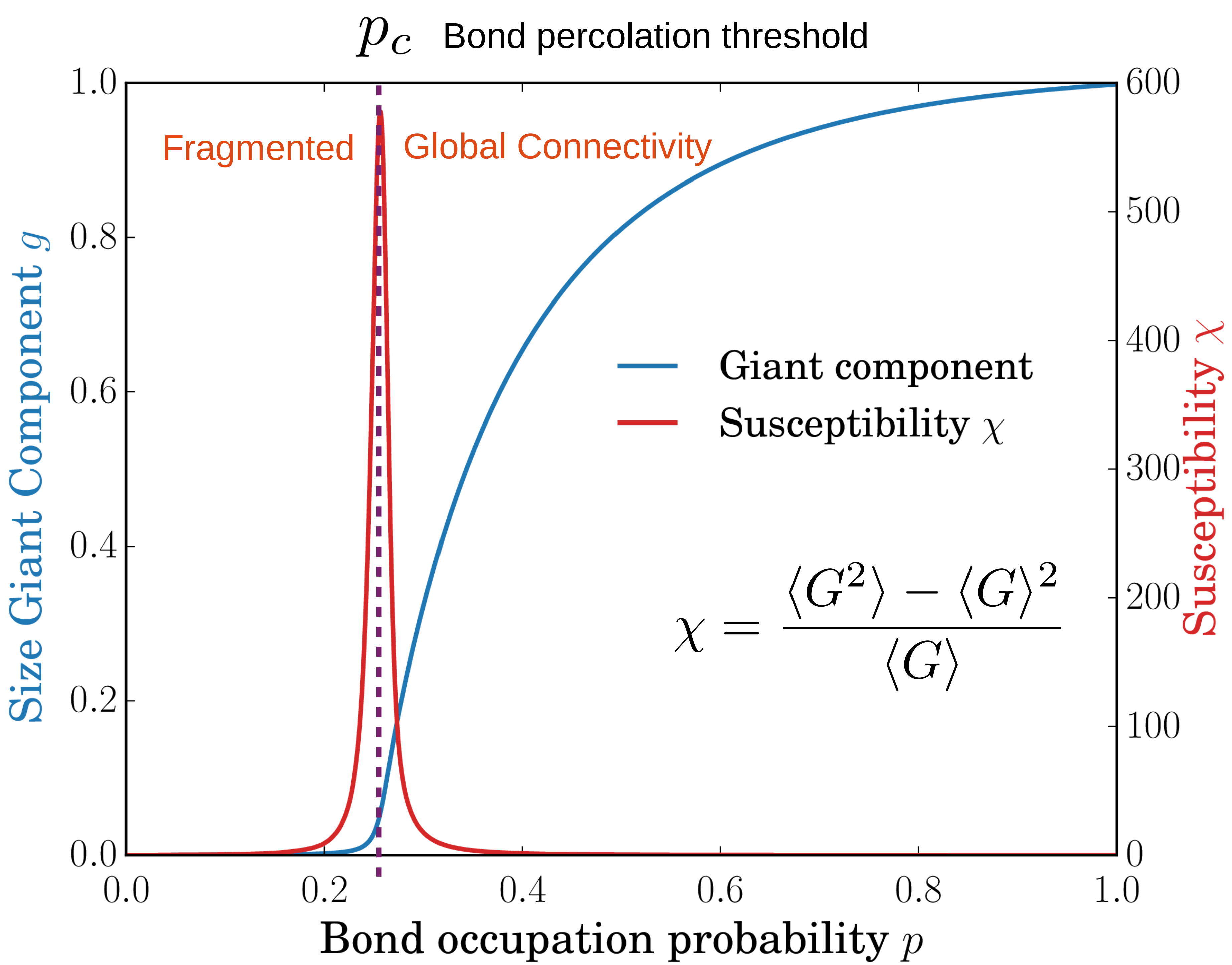
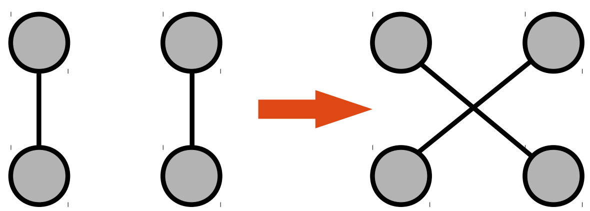


Maximally random model

- Rewiring



$\rightarrow$ Fix $P(k)$

GitHub *https://github.com/polcolomer/RandNetGen*

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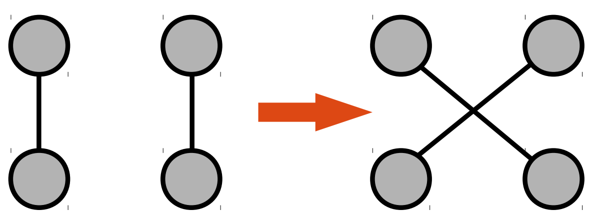


$\rightarrow$ Fix $P(k)$

- Annealed Metropolis-Hastings

$$H = \sum_k |\bar{c}^*(k) - \bar{c}(k)| \rightarrow \text{Fix} \quad \bar{c}(k)$$

current clustering

target clustering

GitHub *https://github.com/polcolomer/RandNetGen* exponential random graph

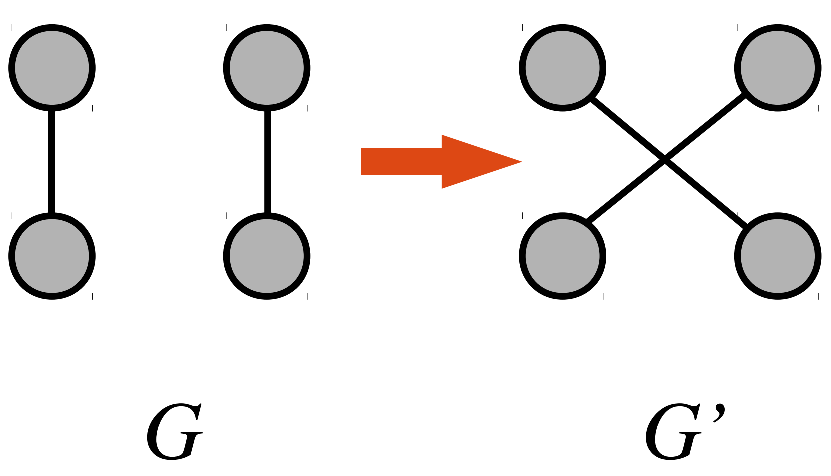
$$P(G) \propto e^{-\beta H(G)}$$

$\beta \longrightarrow$ inverse of the temperature

$$H(G) = \sum_k |\bar{c}^*(k) - \bar{c}(k)|$$

in the limit $\beta \to \infty$ the system will approach the state of minimum energy and so clustering will approach the target clustering

Metropolis-Hastings algorithm



with probability

$$\text{Prob} = \min\left(1, e^{\beta[H(G)-H(G')]}\right)$$

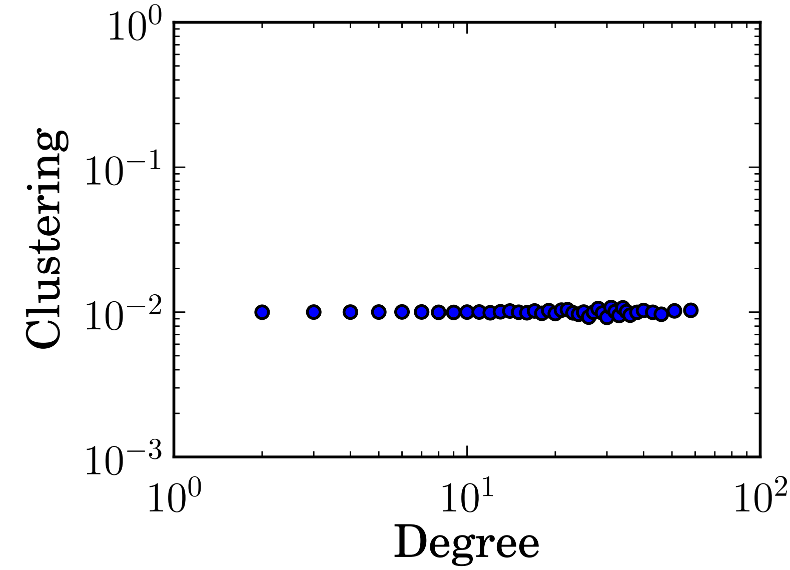
accept the change. Otherwise, keep the graph G unchanged

Our algorithm

1. Fix the degree sequence and close a network with any procedure
2. Set $\beta = 0$ and run Metropolis-Hastings with at least 200E rewiring attempts
3. Set $\beta = \beta_0 = 50$ and run Metropolis-Hastings with at least 200E rewiring attempts
4. Increase the value of β by 10% and go to step #3
5. At each iteration, measure the fraction of proposed rewiring events with $\Delta H > 0$

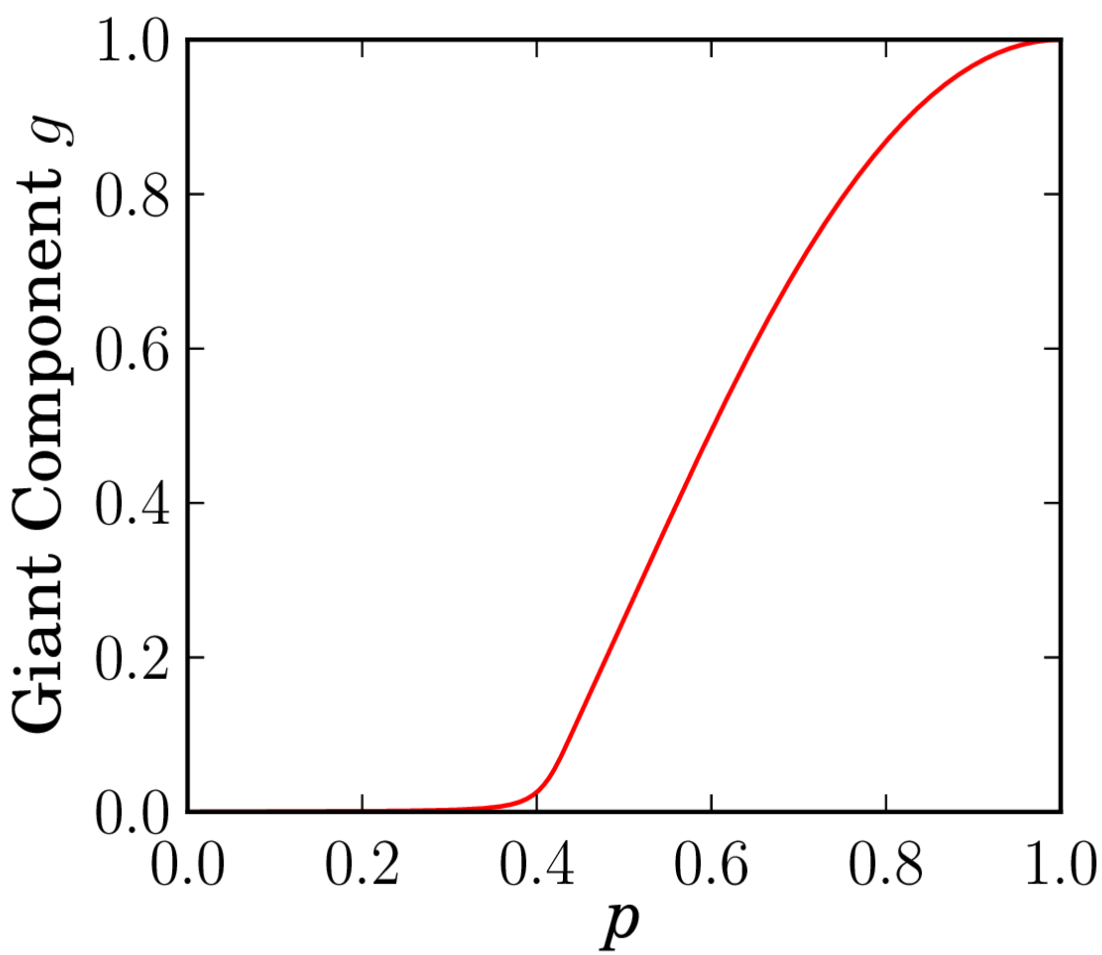
   that are accepted. When this fraction is smaller than 5×10^{-5} stop the algorithm

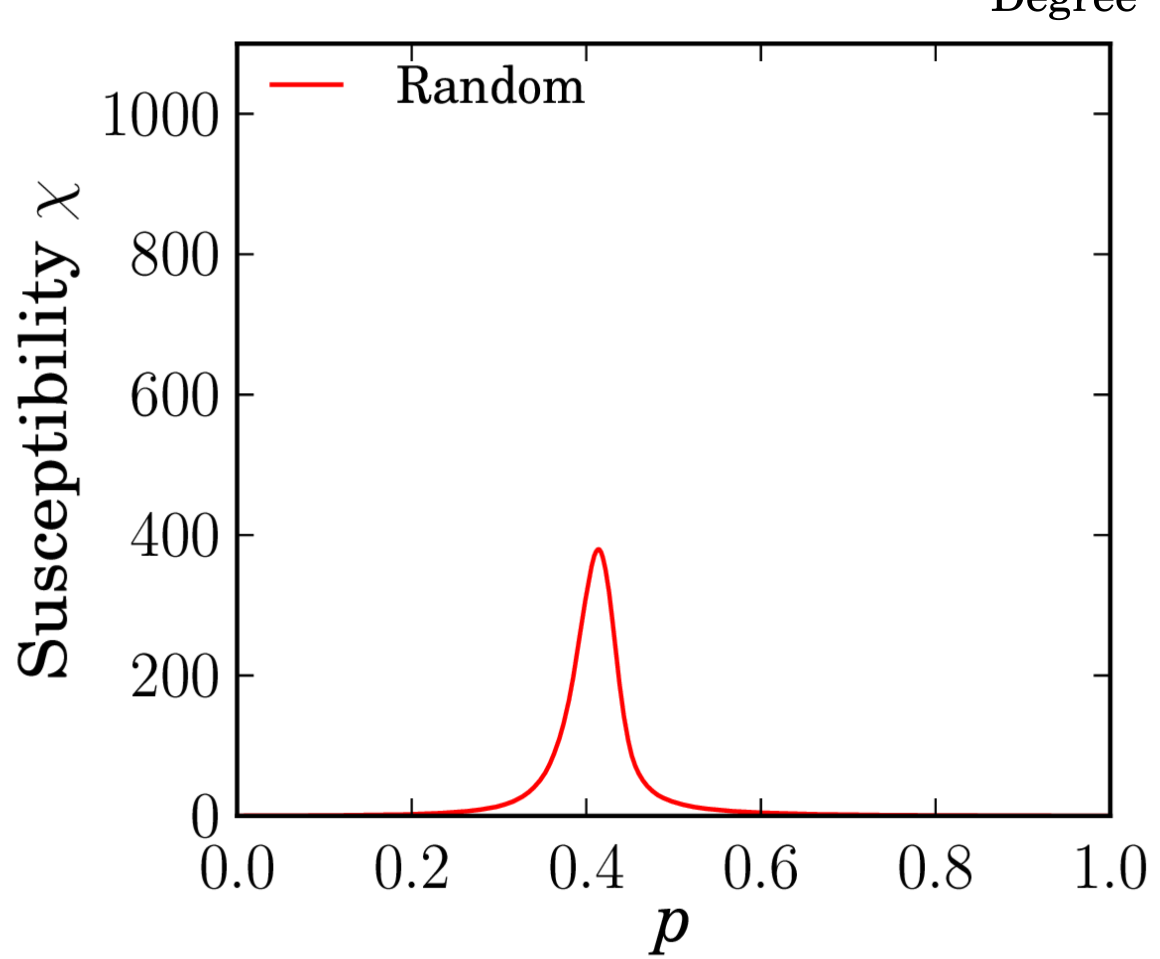
Effect of clustering



$$P(k) \sim k^{-4}$$

Random





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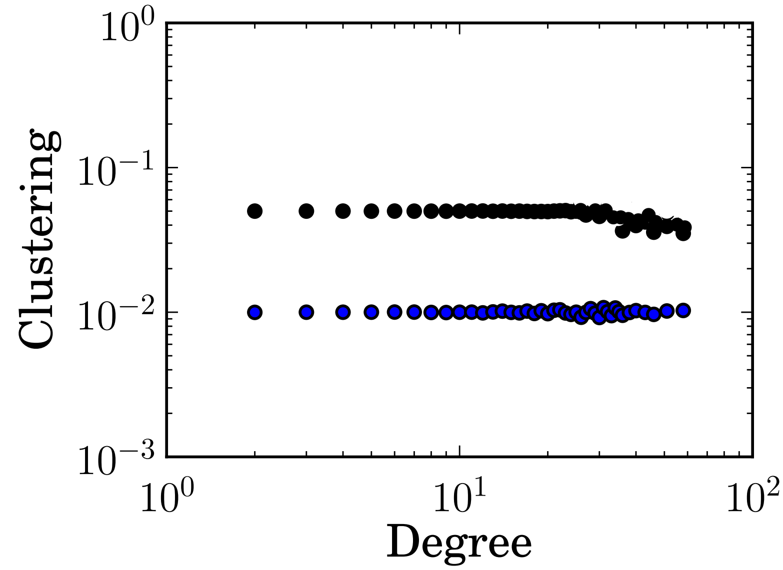
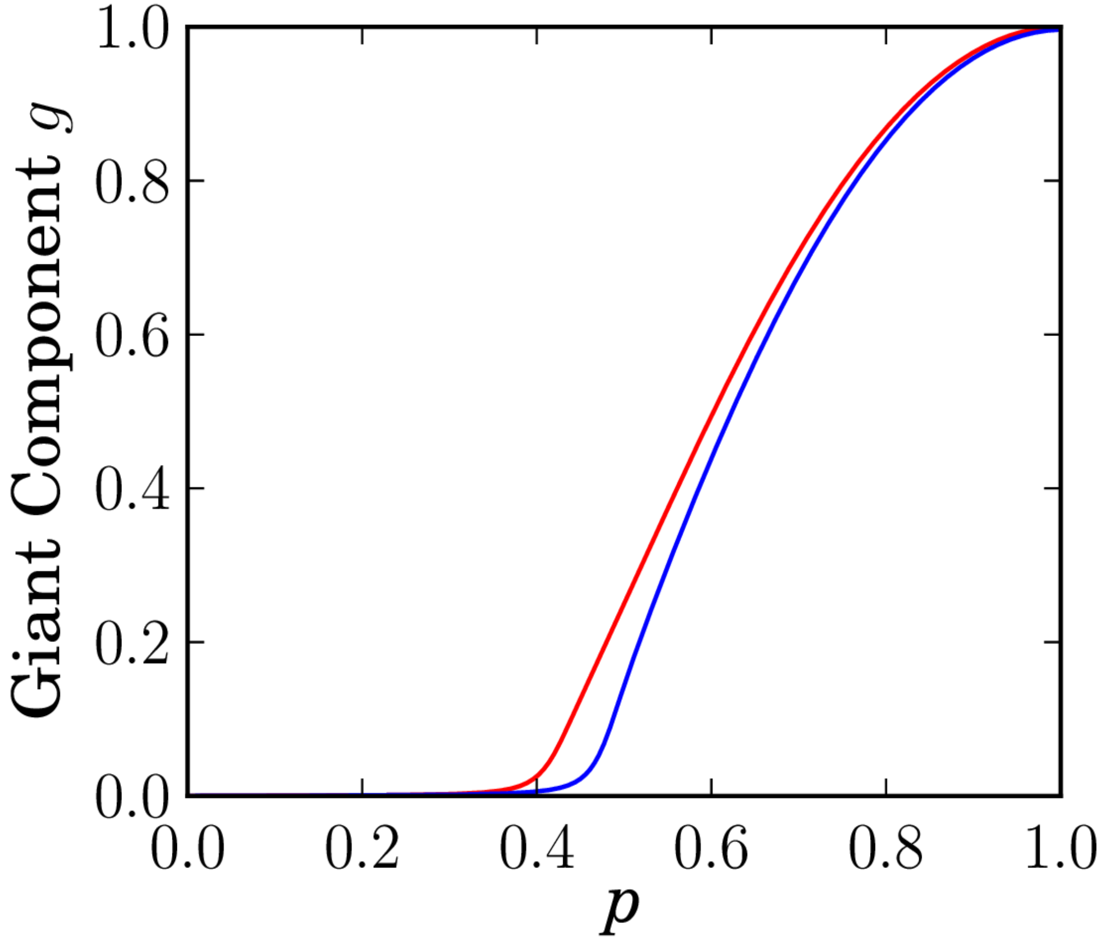
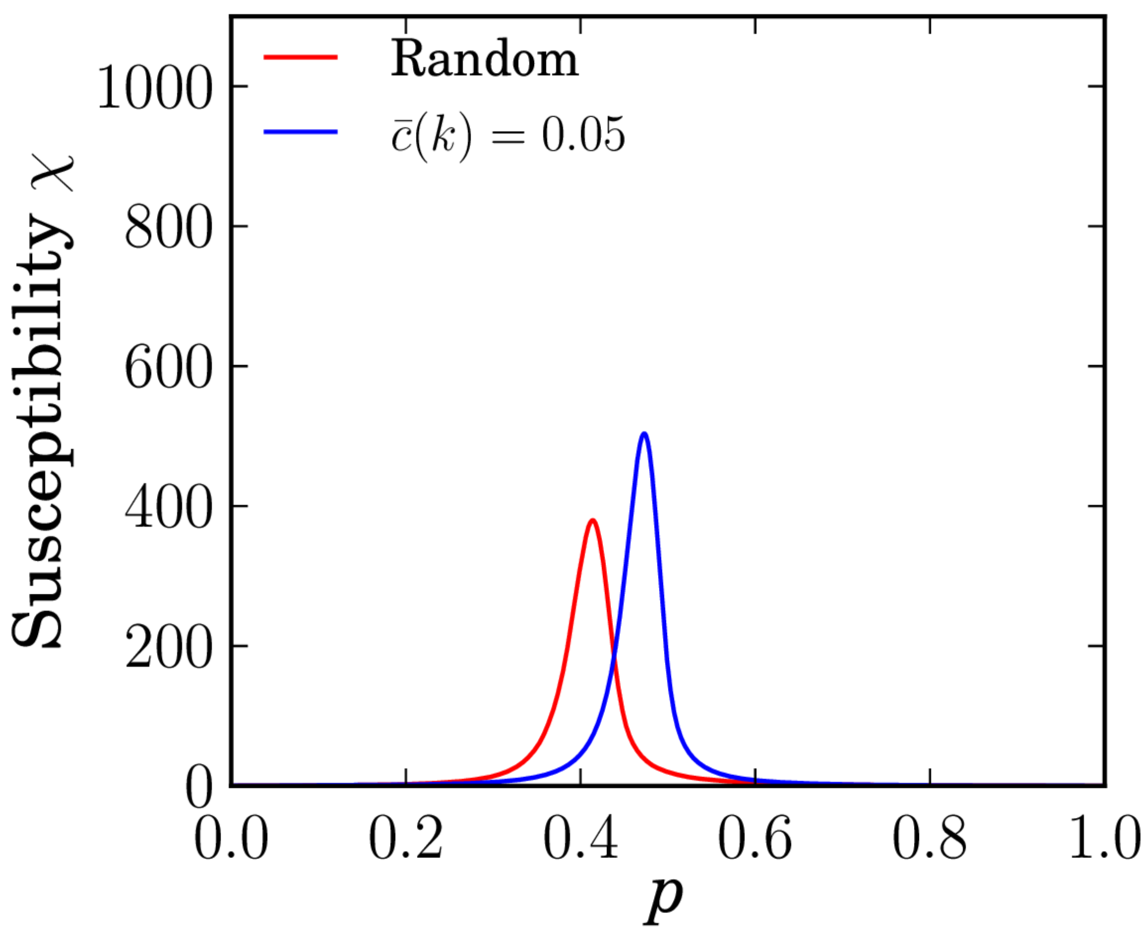


$$P(k) \sim k^{-4} \qquad \bar{c}(k) = 0.05$$





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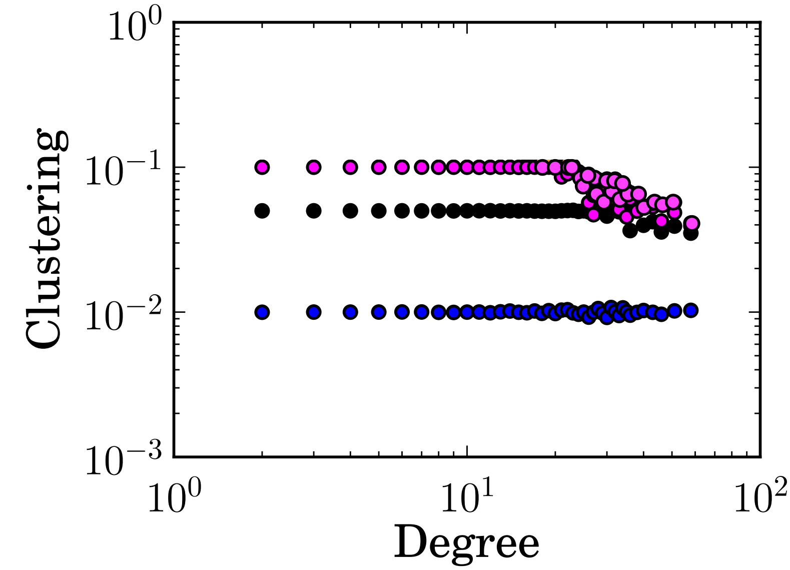
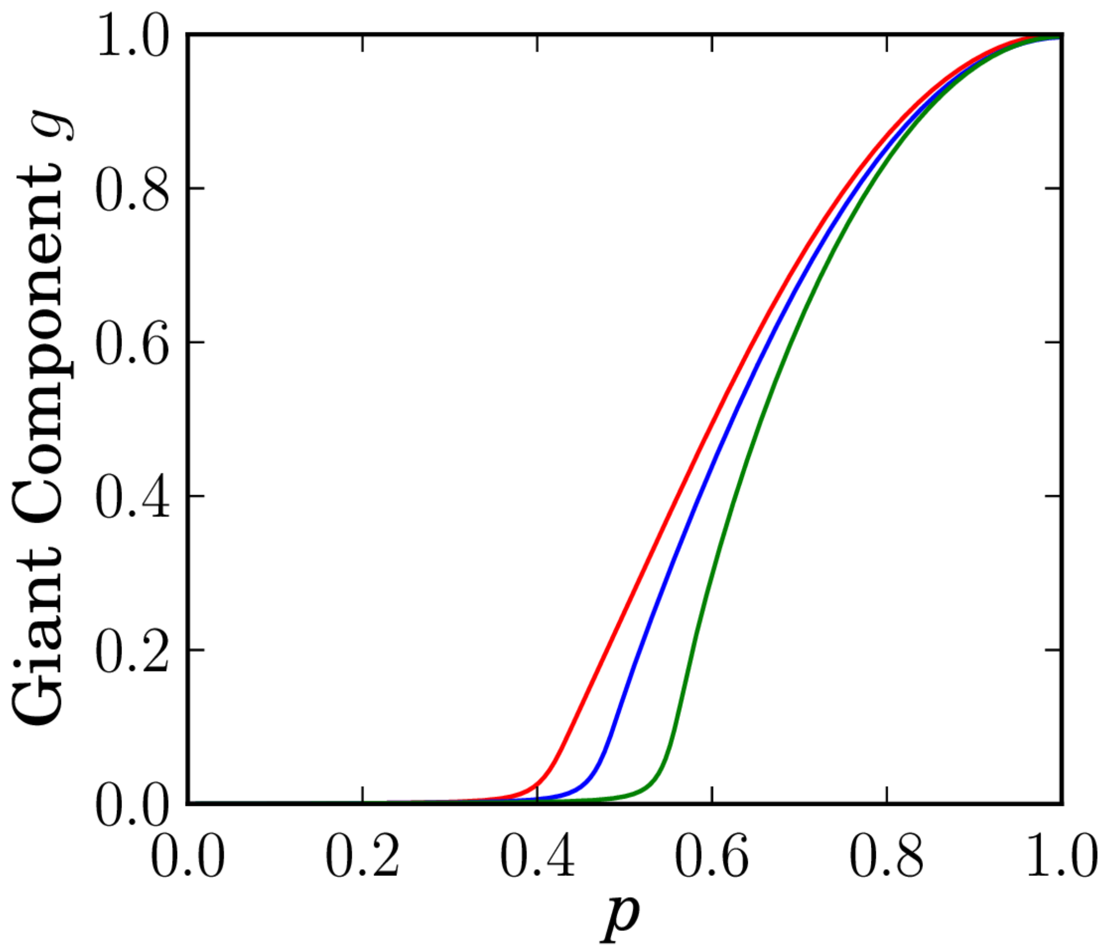
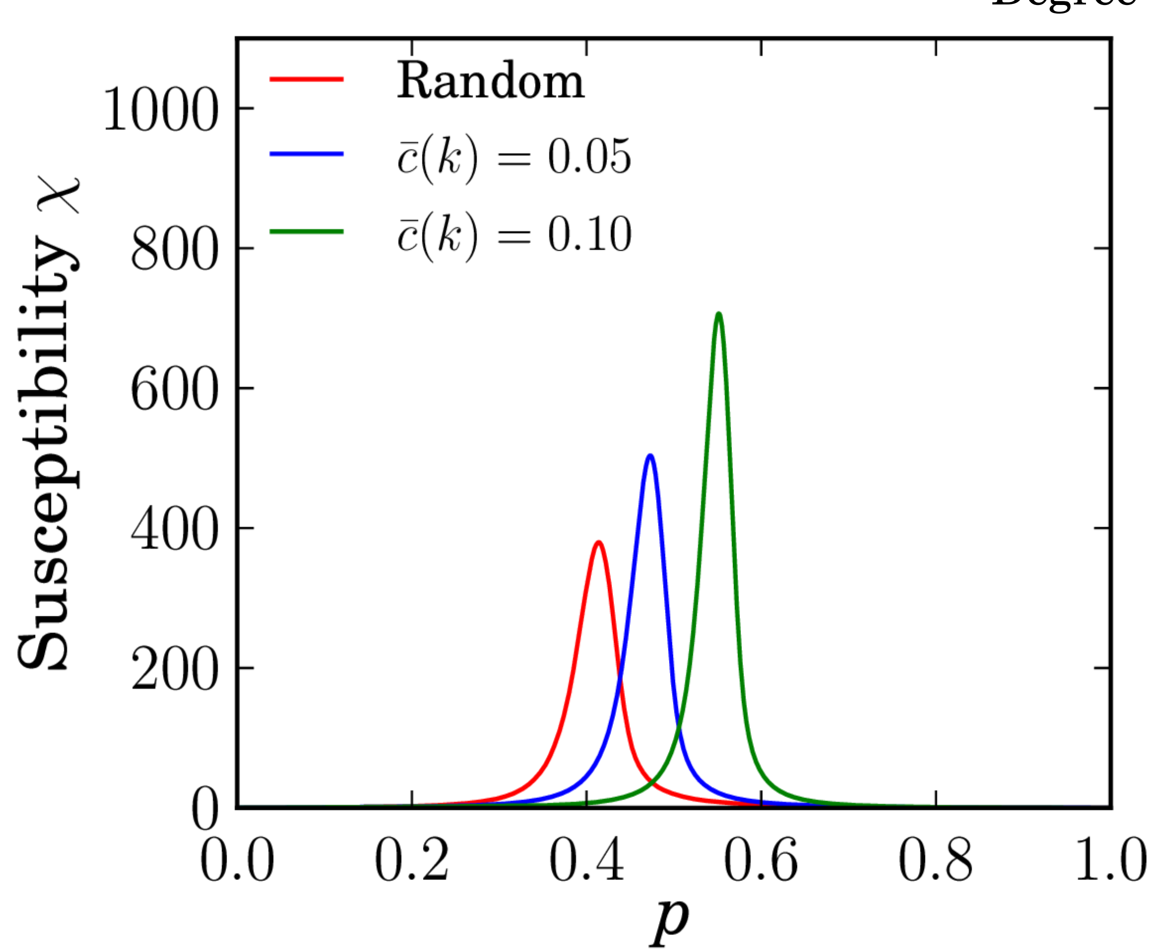
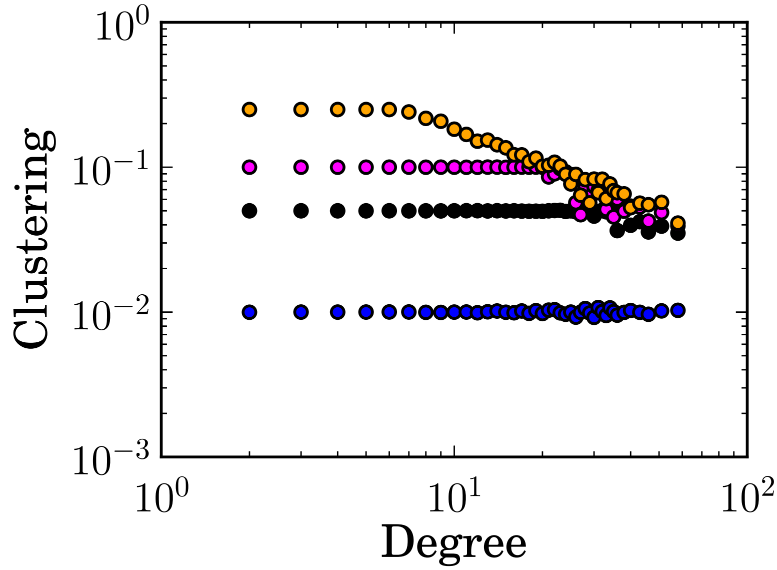


$$P(k) \sim k^{-4} \qquad \bar{c}(k) = 0.10$$

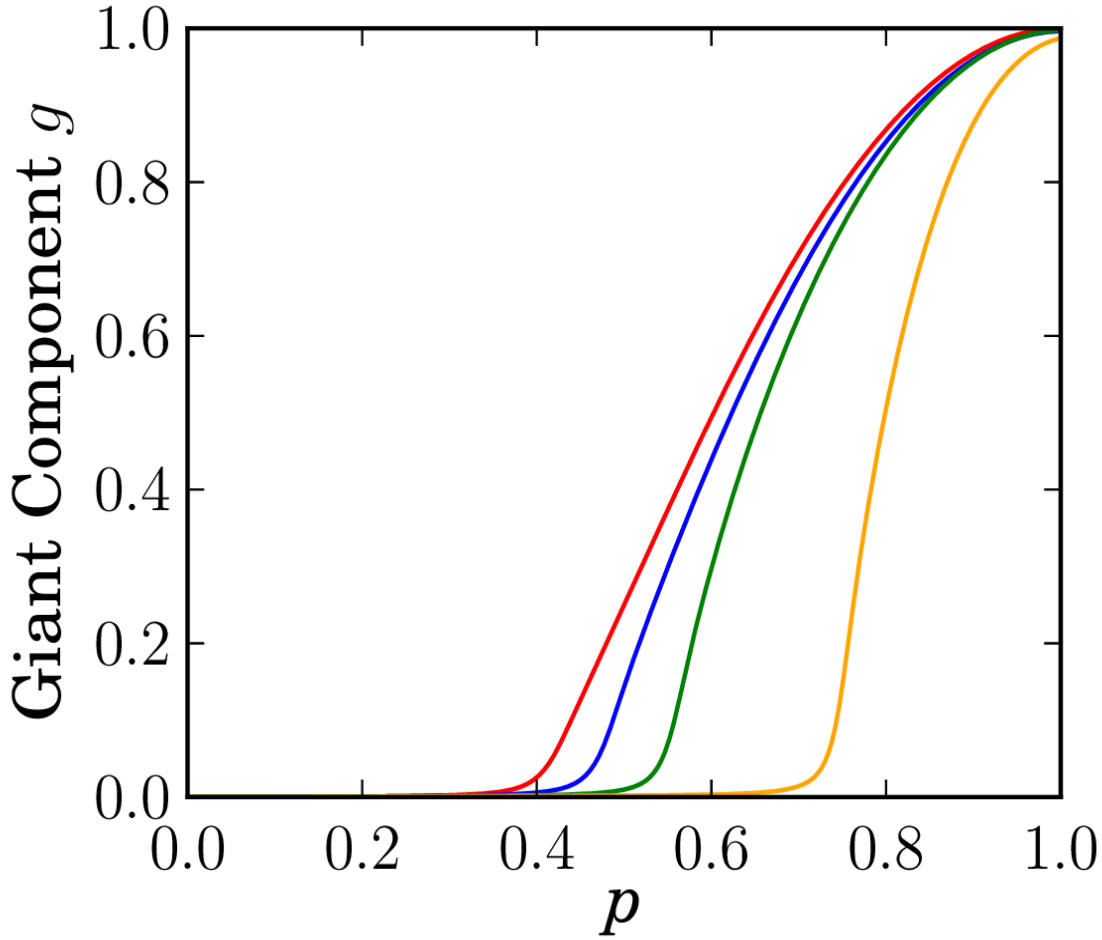


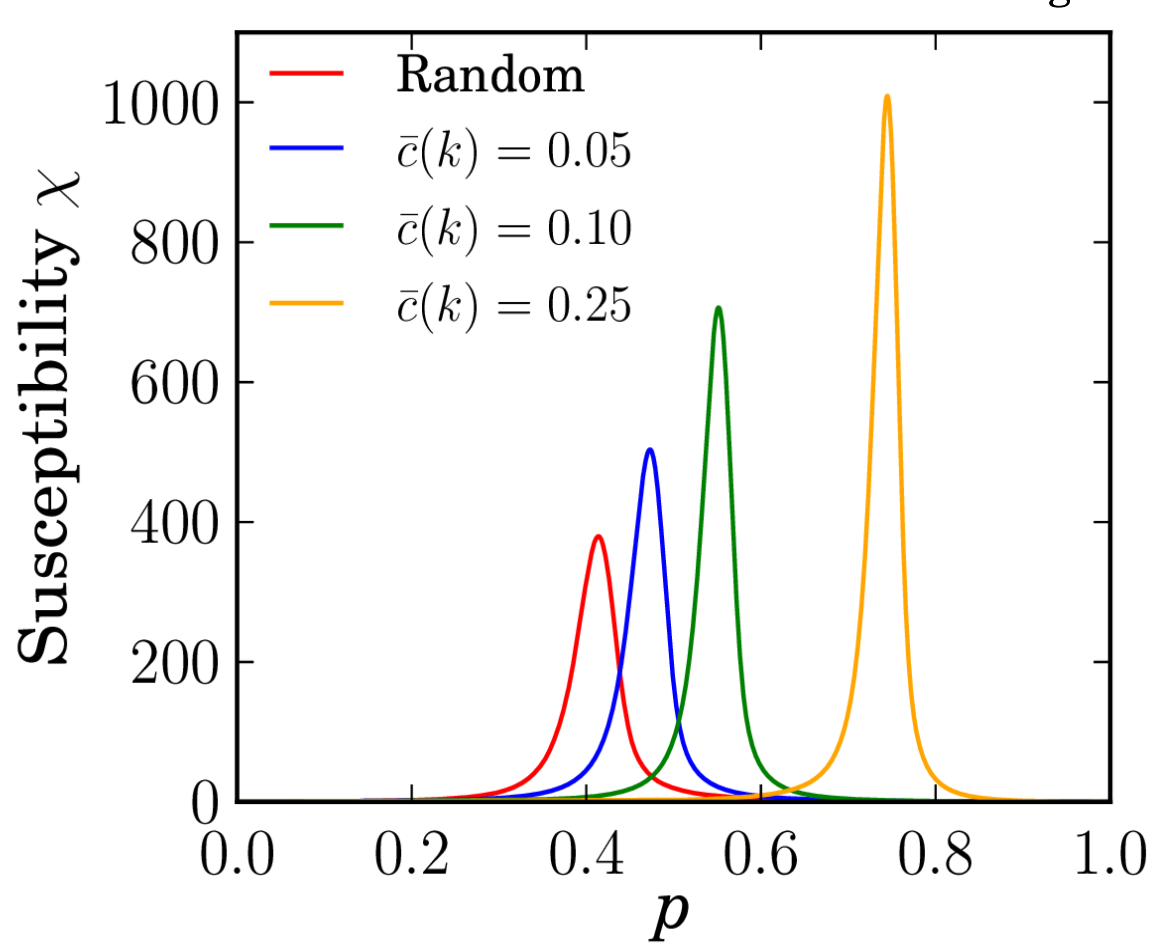


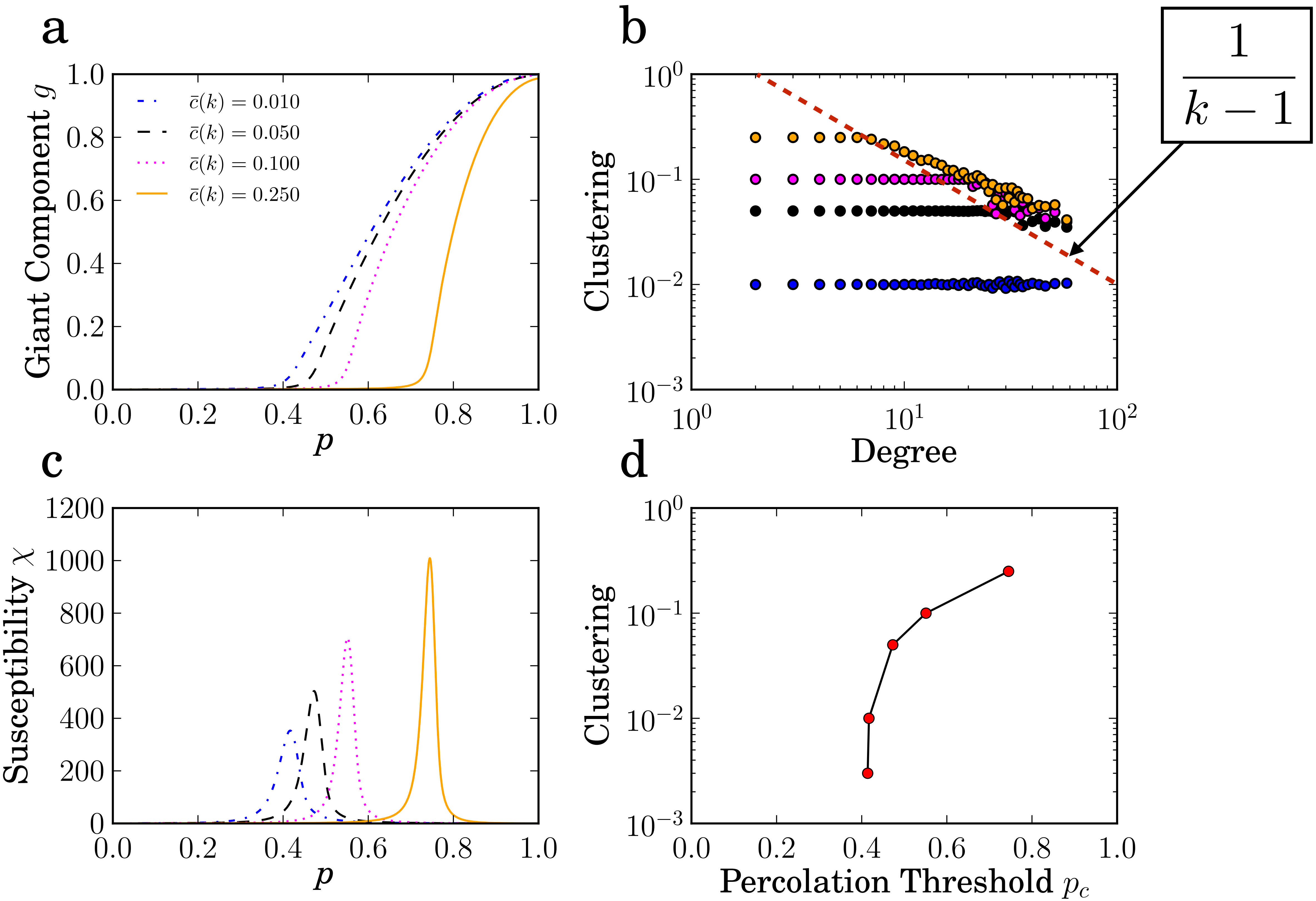
Effect of clustering



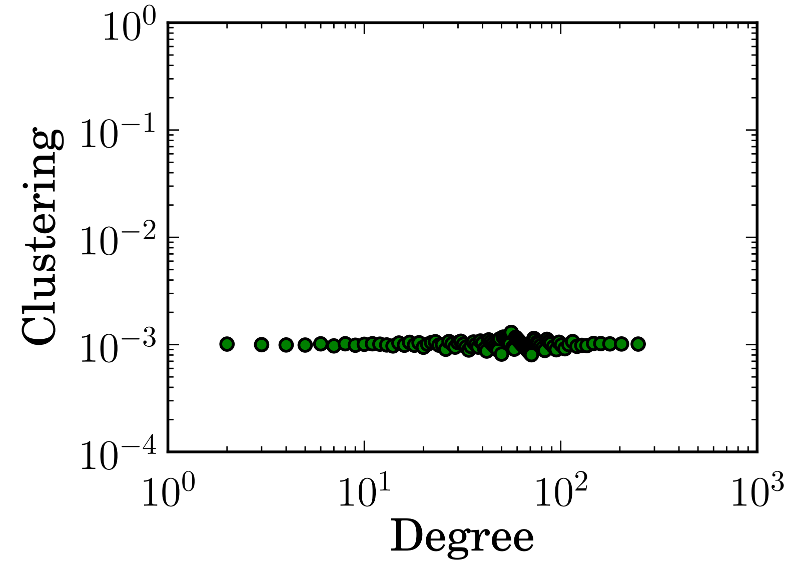
$$P(k) \sim k^{-4} \qquad \bar{c}(k) = 0.25$$





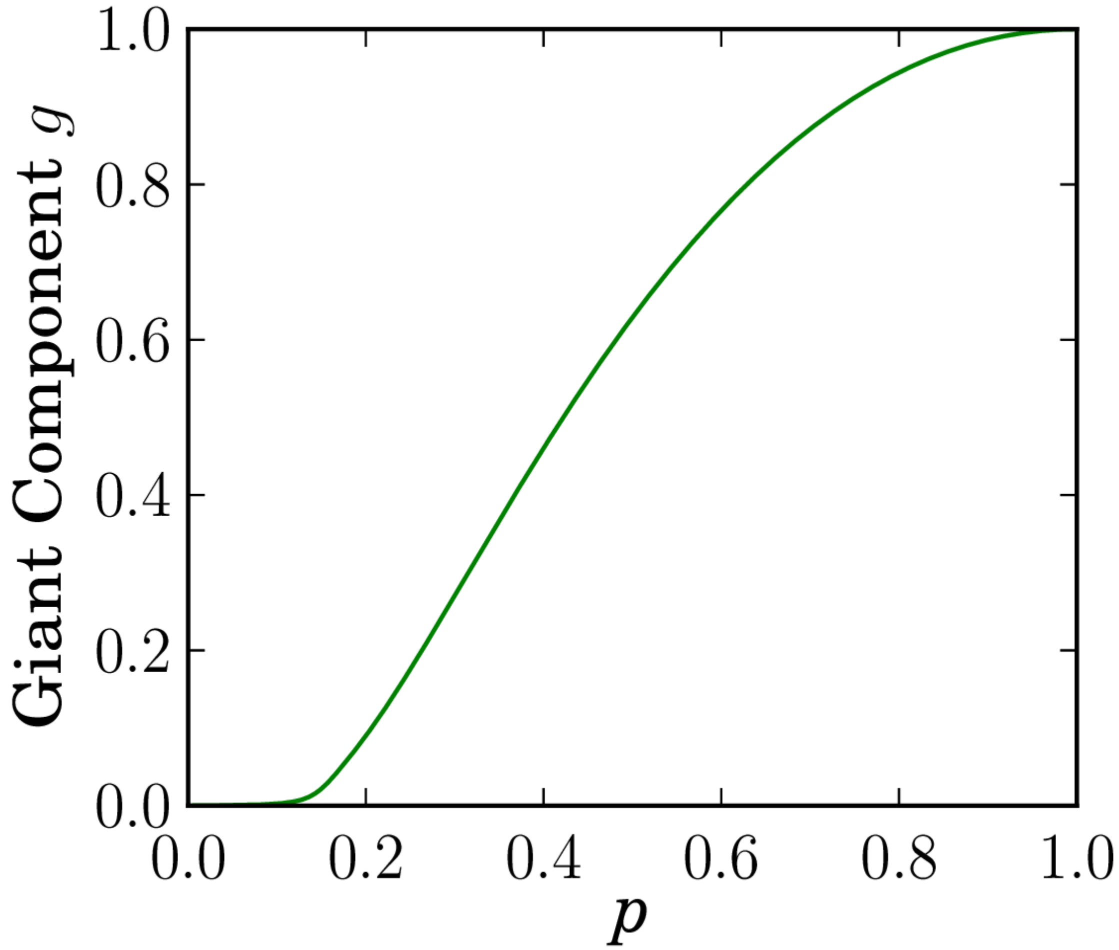

UNIVERSITAT DE BARCELONA
a
b
$\frac{1}{k-1}$
Giant Component g
$\bar{c}(k) = 0.010$
$\bar{c}(k) = 0.050$
$\bar{c}(k) = 0.100$
$\bar{c}(k) = 0.250$
p
Clustering
Degree
c
d
Susceptibility χ
p
Clustering
Percolation Threshold p_c

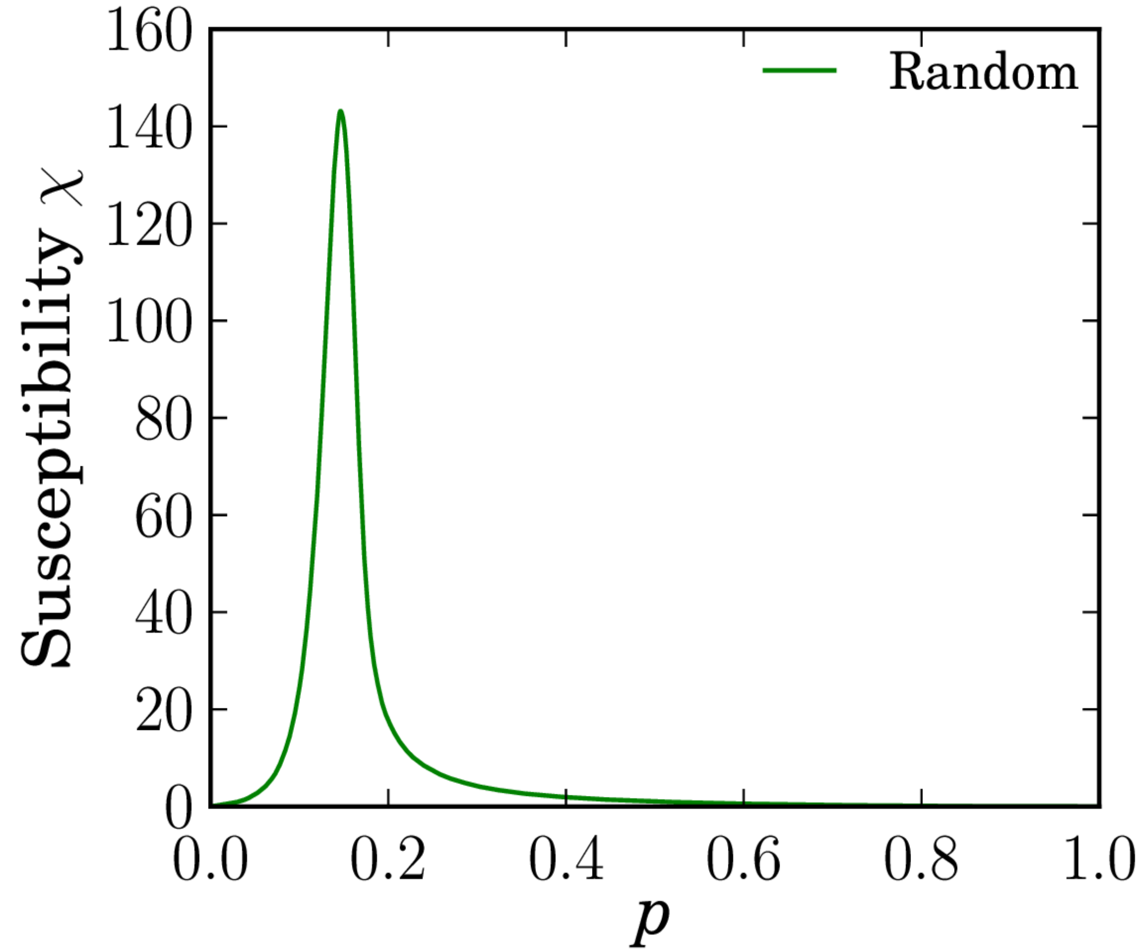
Effect of clustering



$$P(k) \sim k^{-3.1}$$

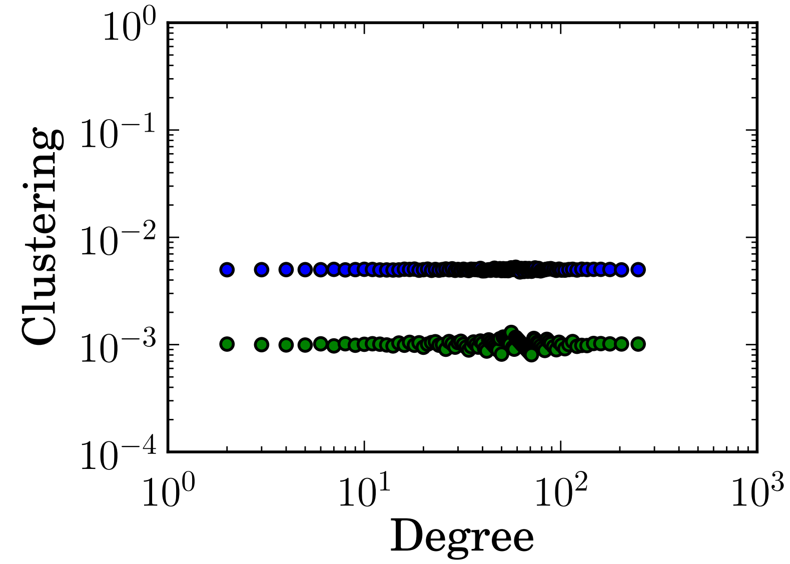
Random

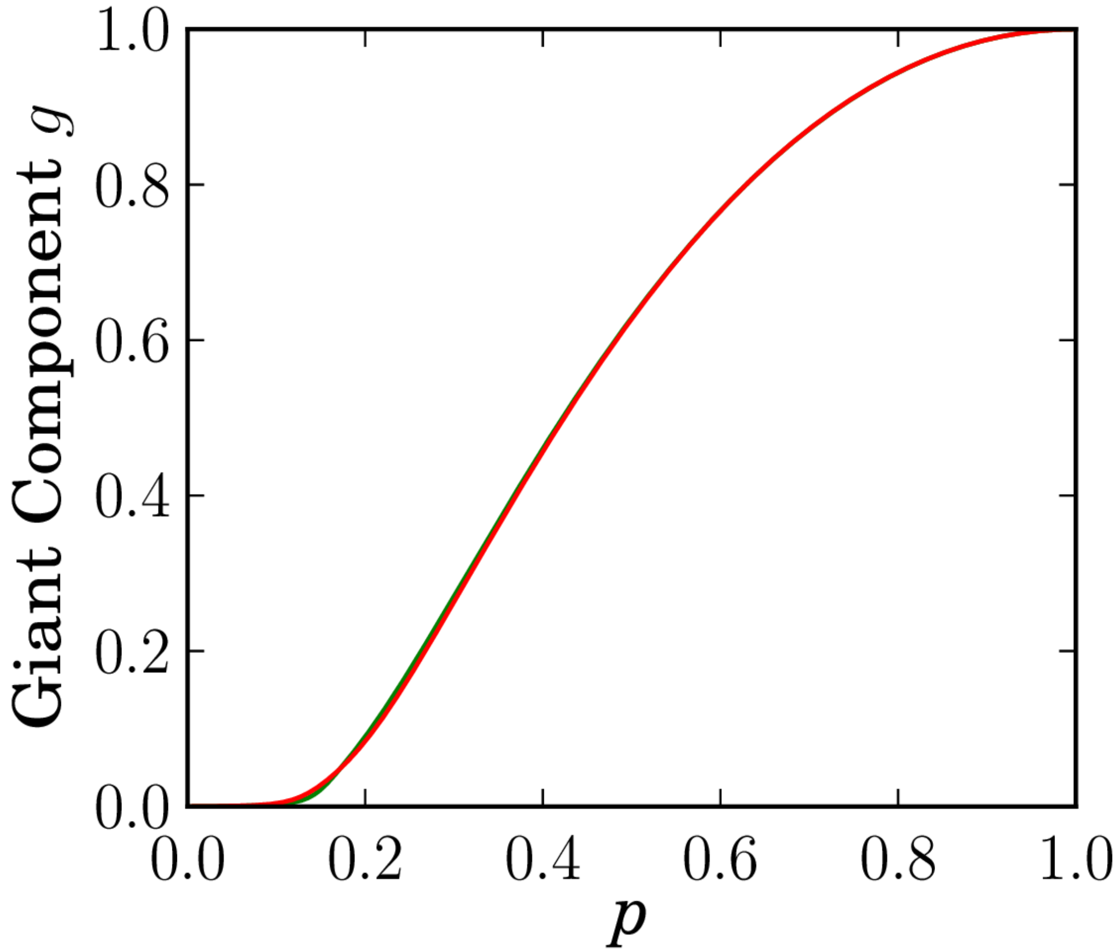


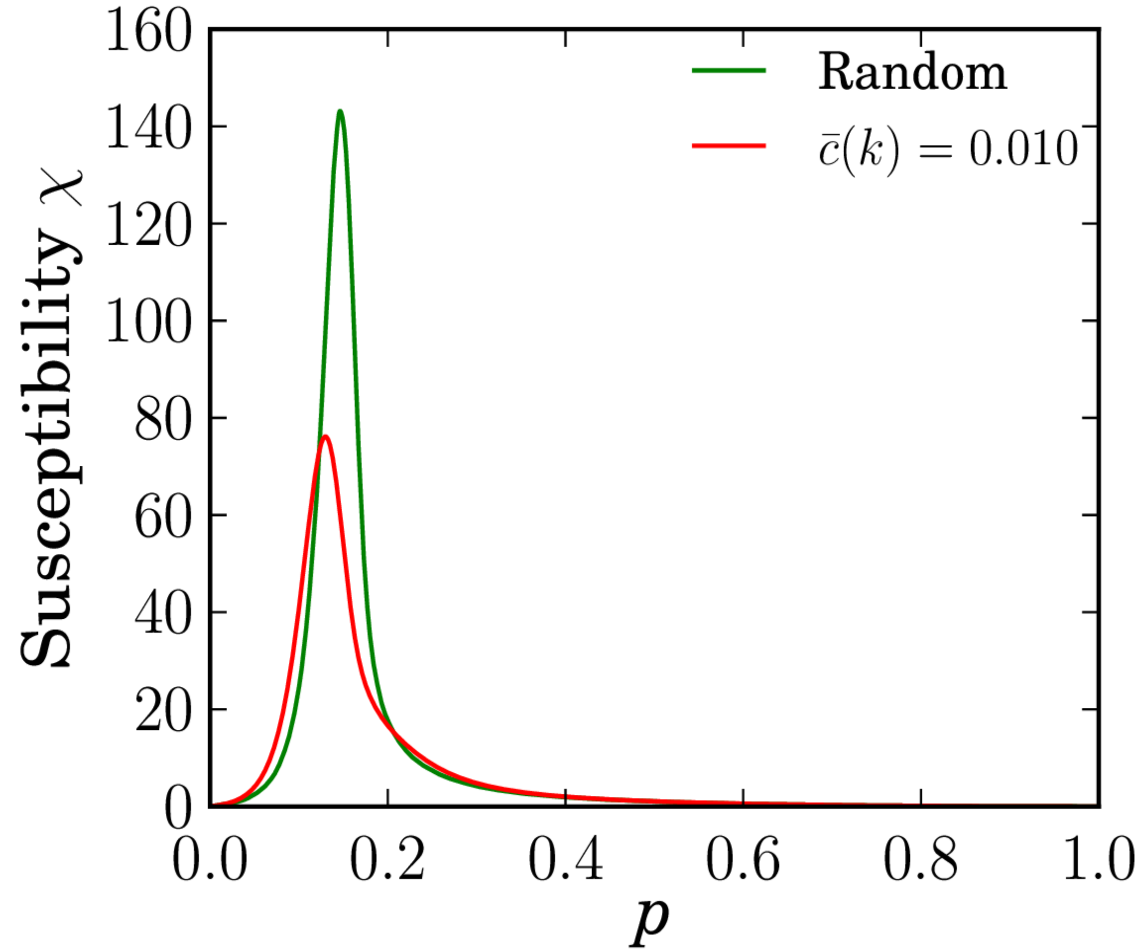


Effect of clustering

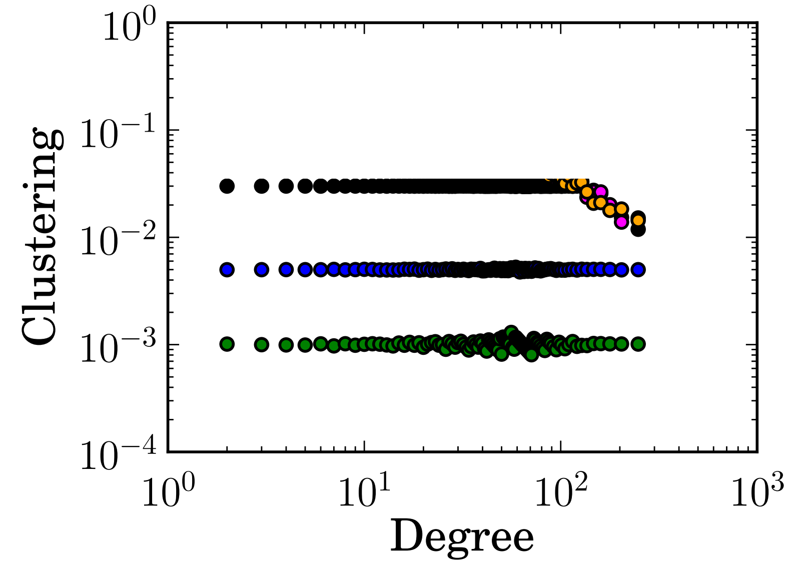
$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.01$$



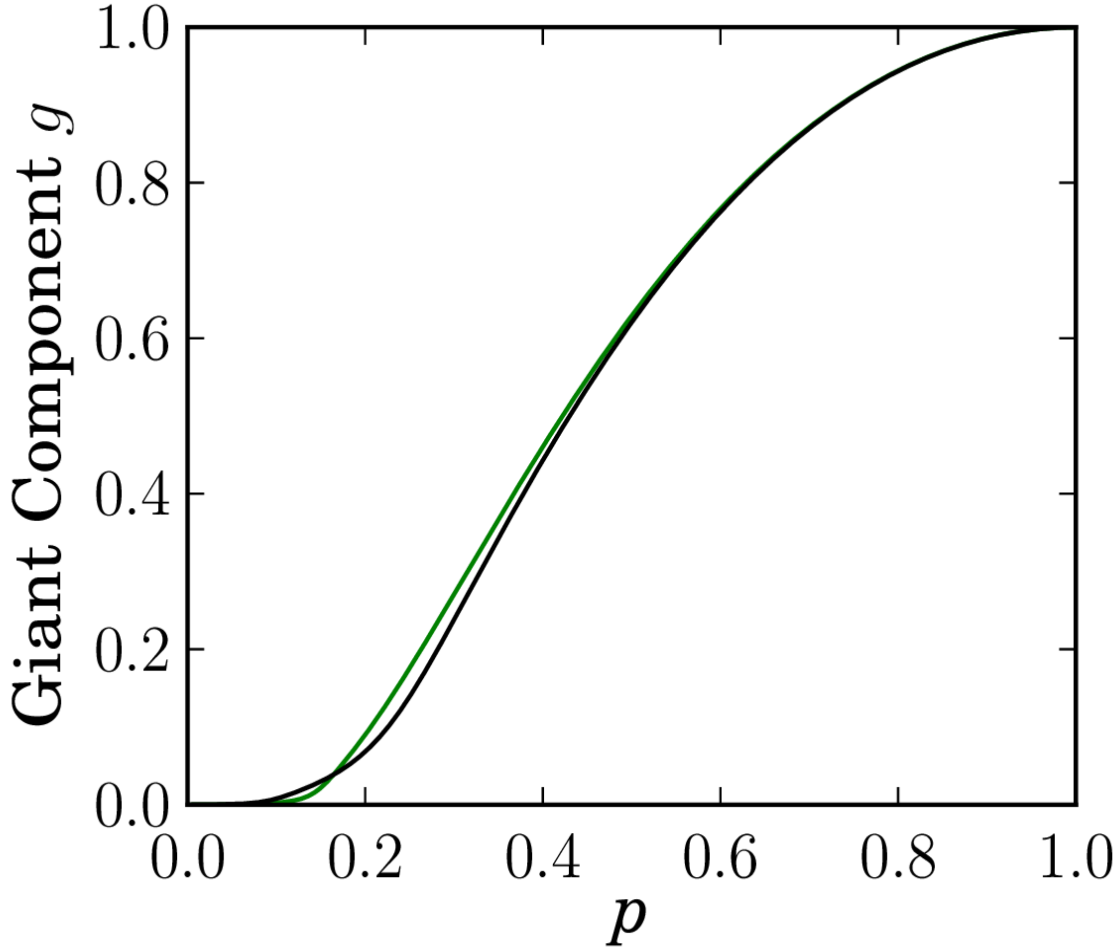


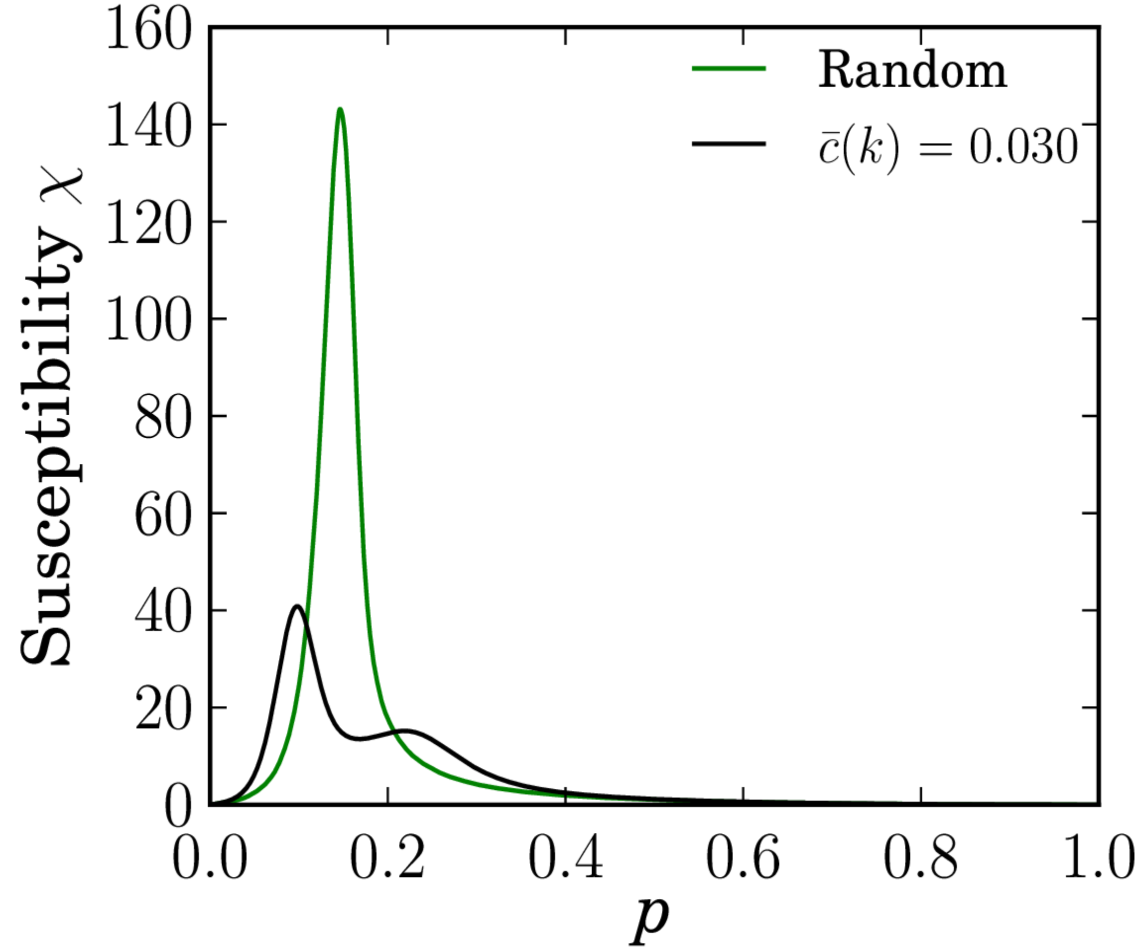


Effect of clustering

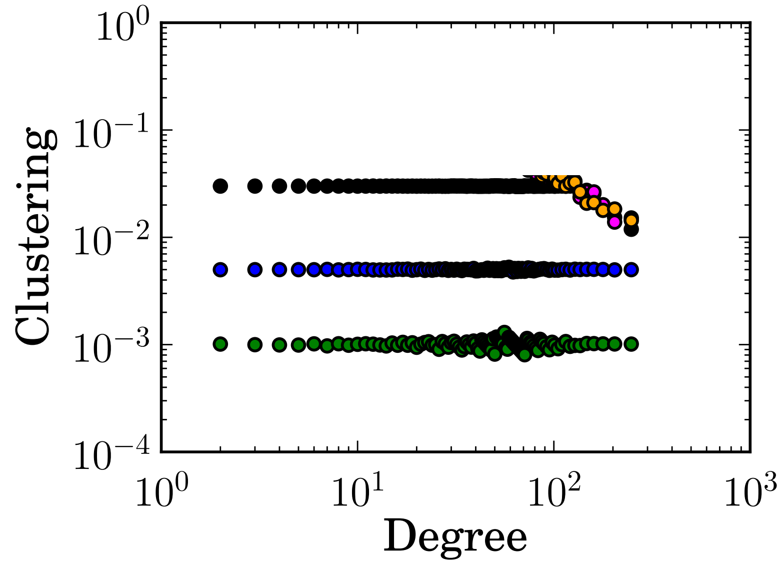


$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.03$$

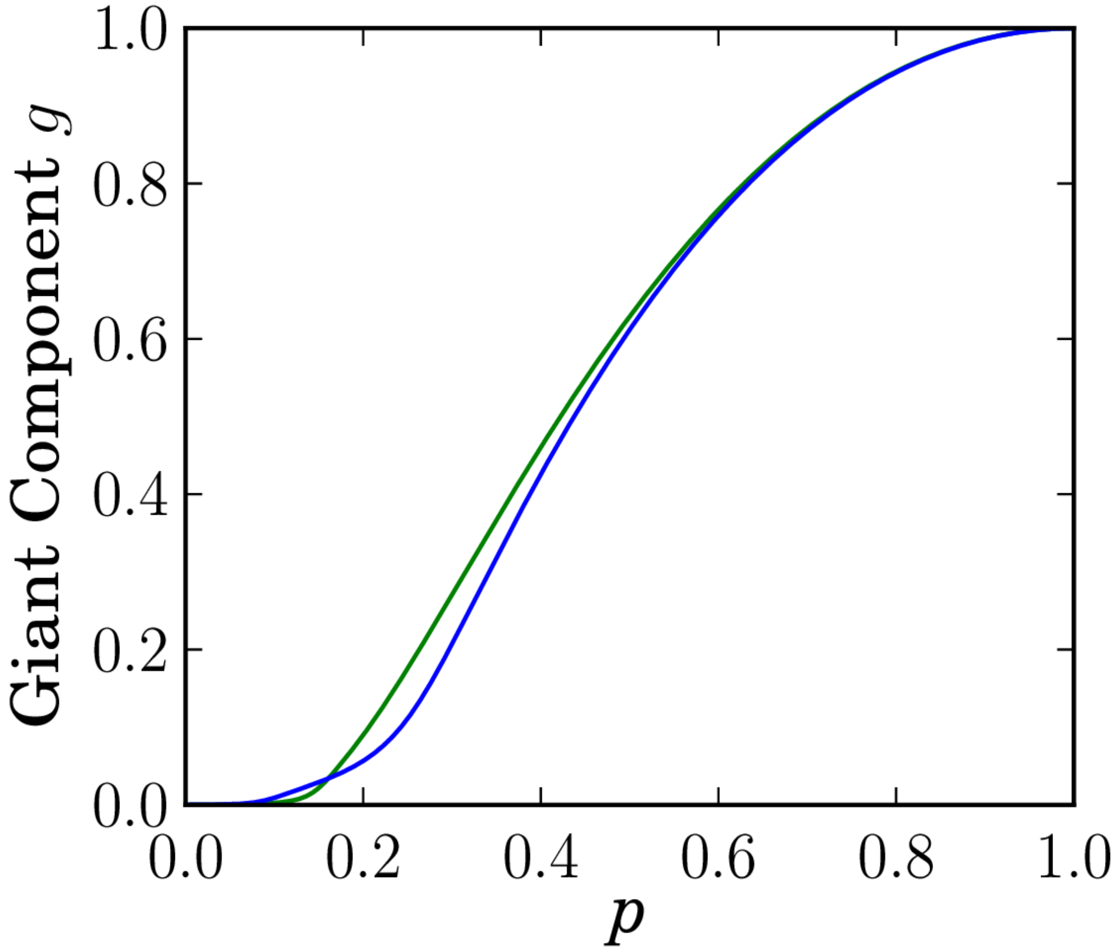


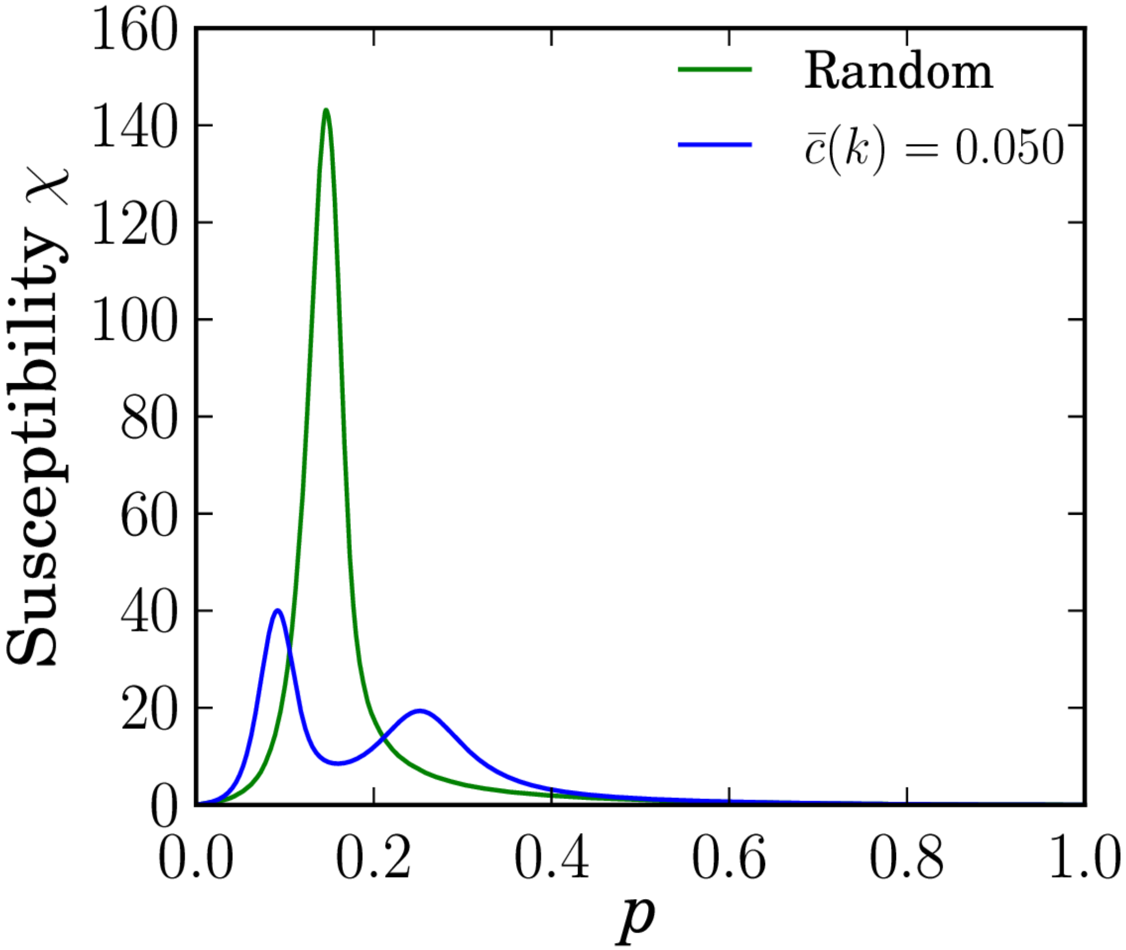


Effect of clustering

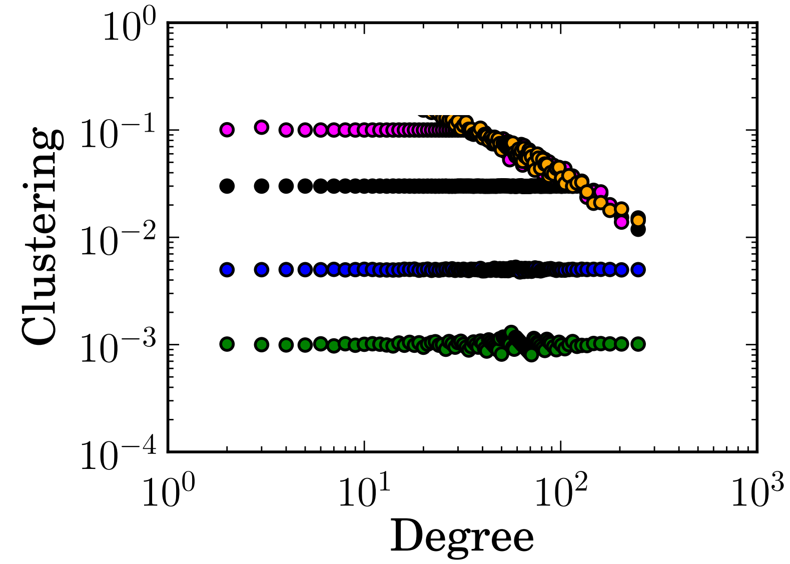


$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.05$$

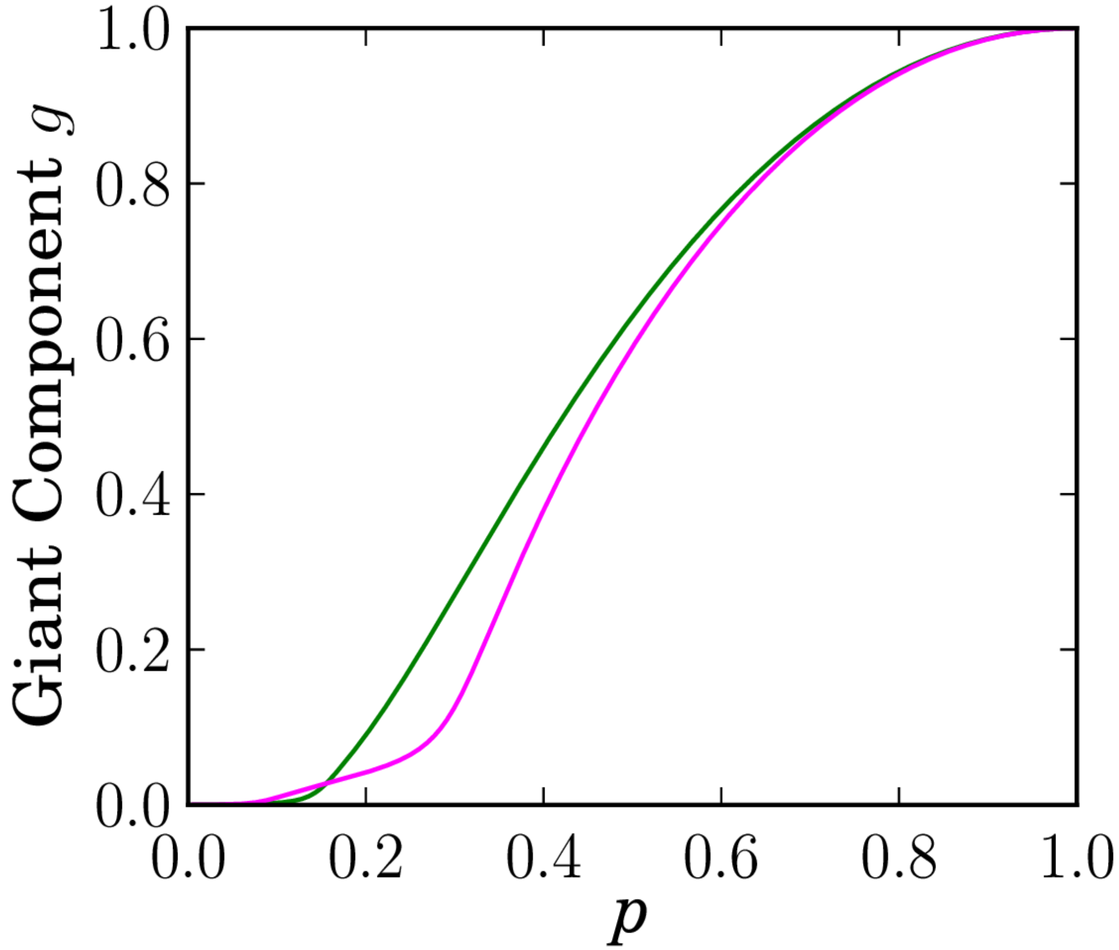


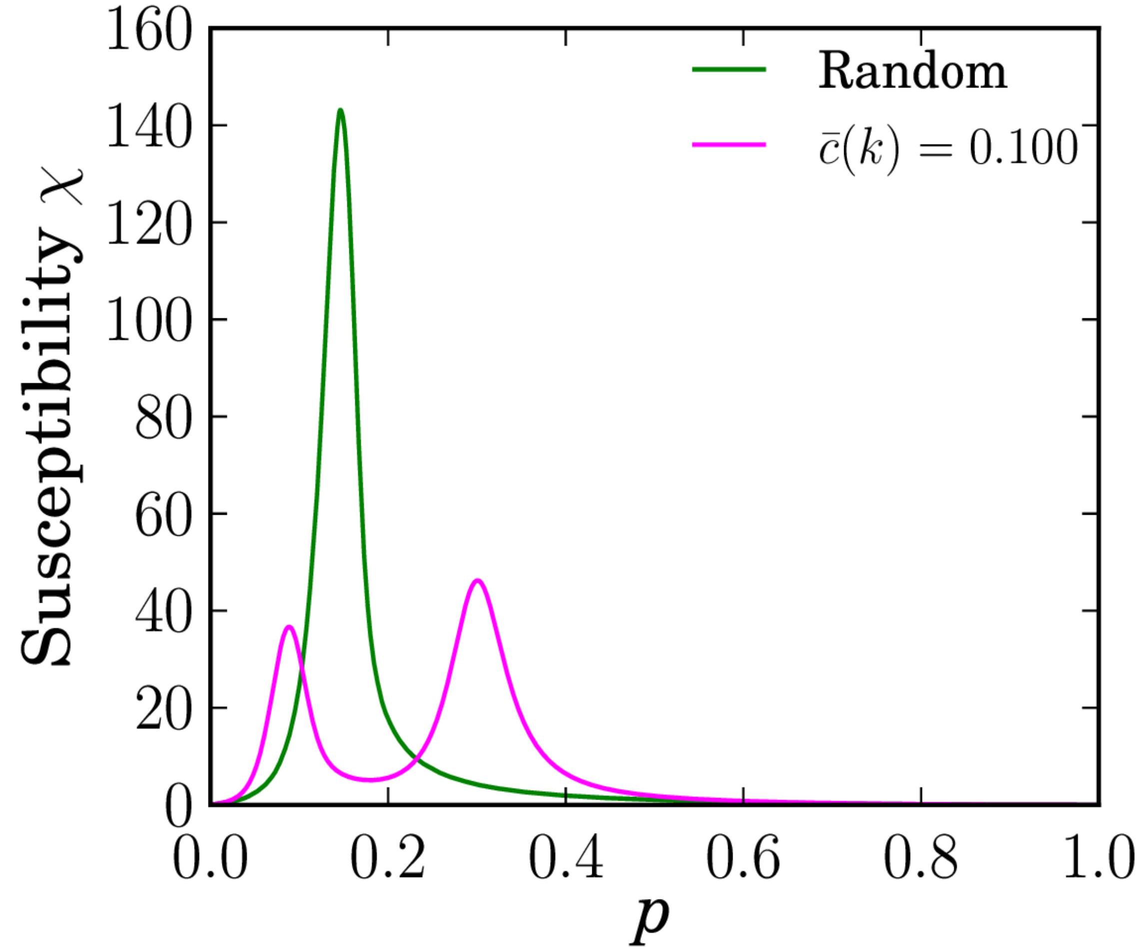


Effect of clustering

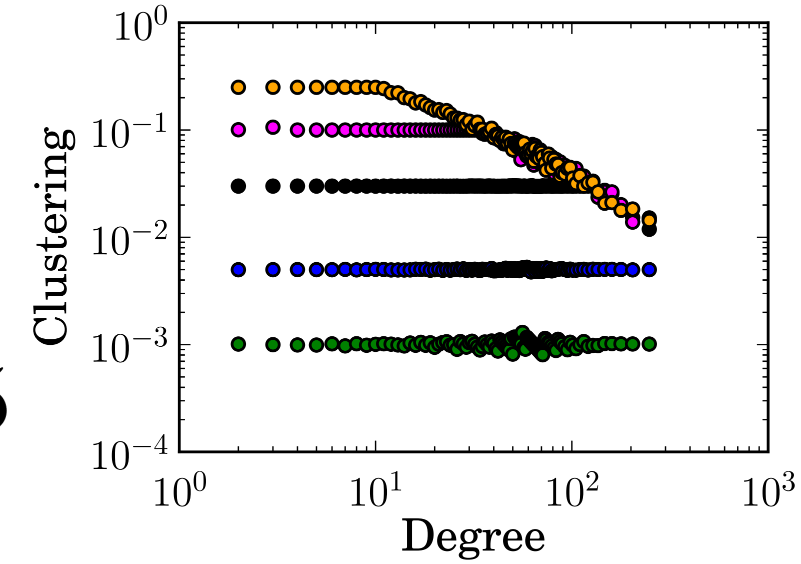


$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.10$$

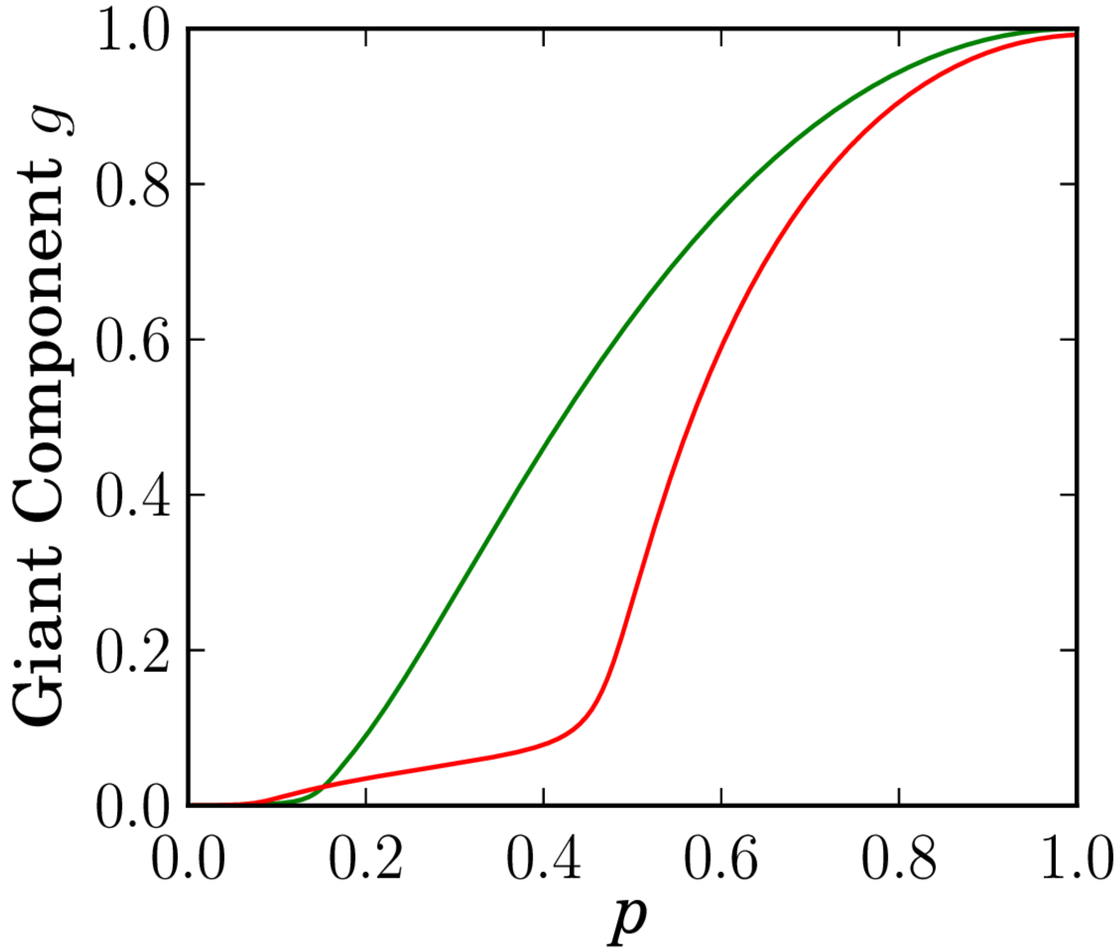


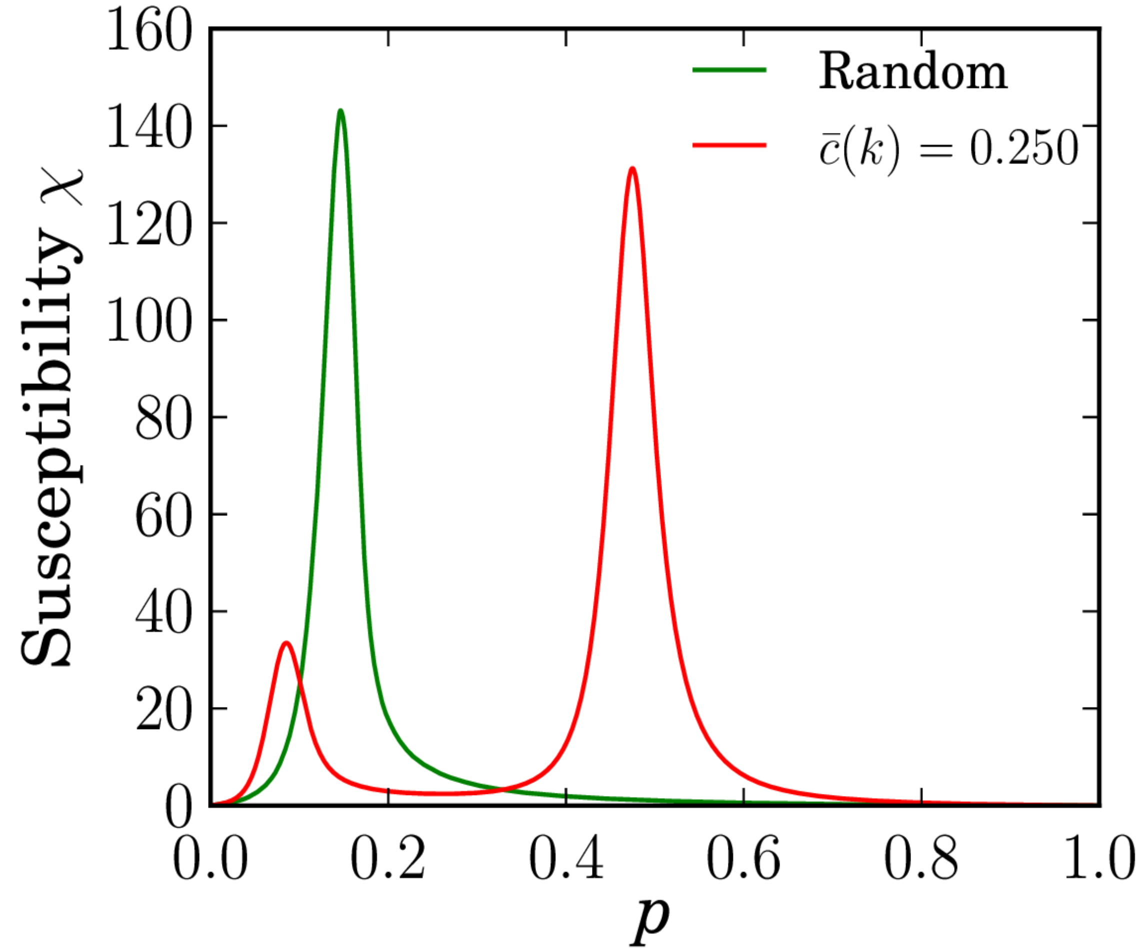


Effect of clustering

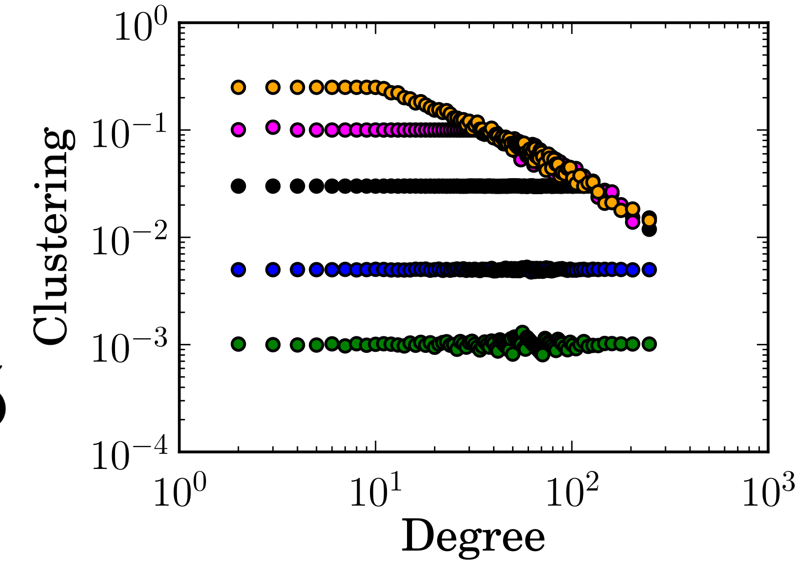


$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.25$$

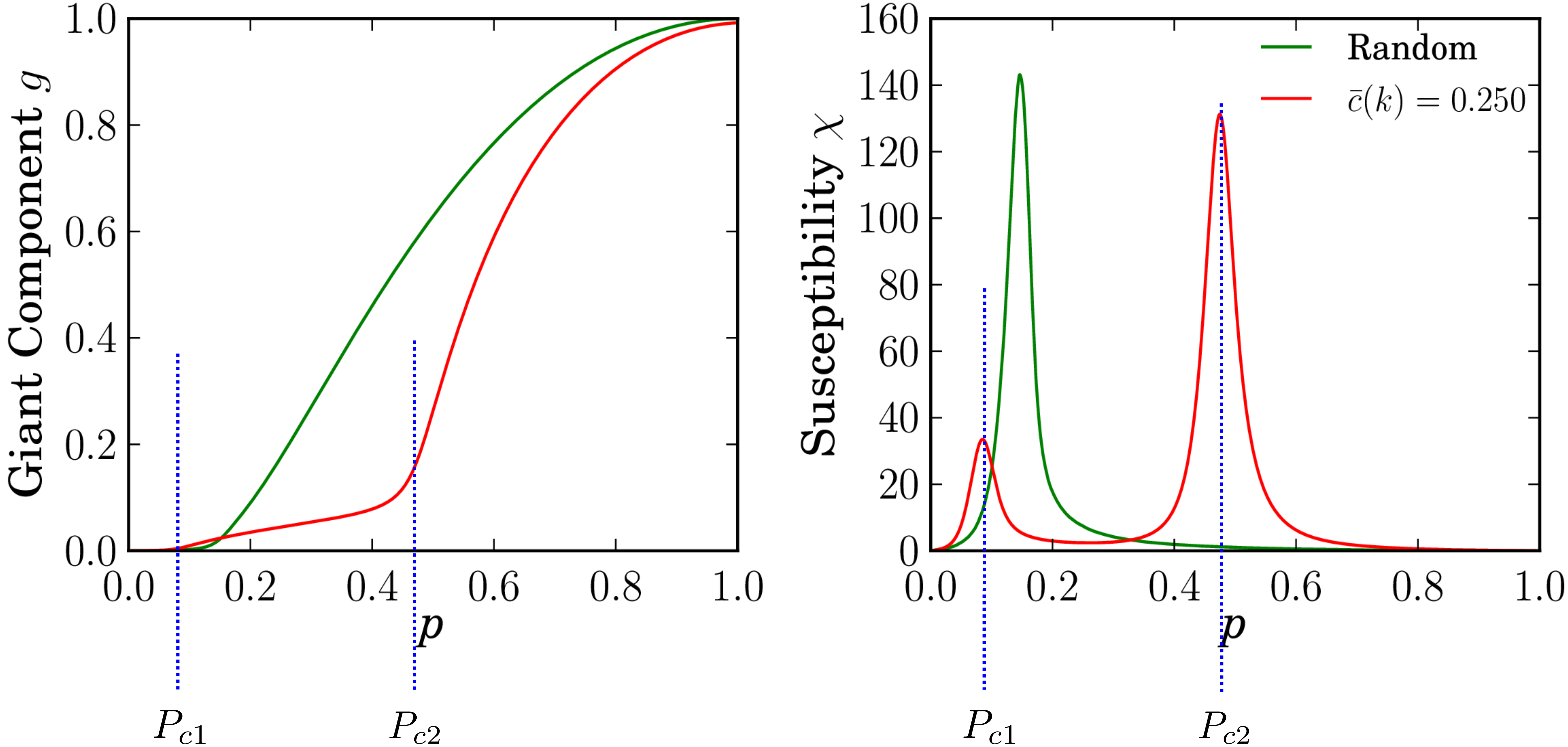


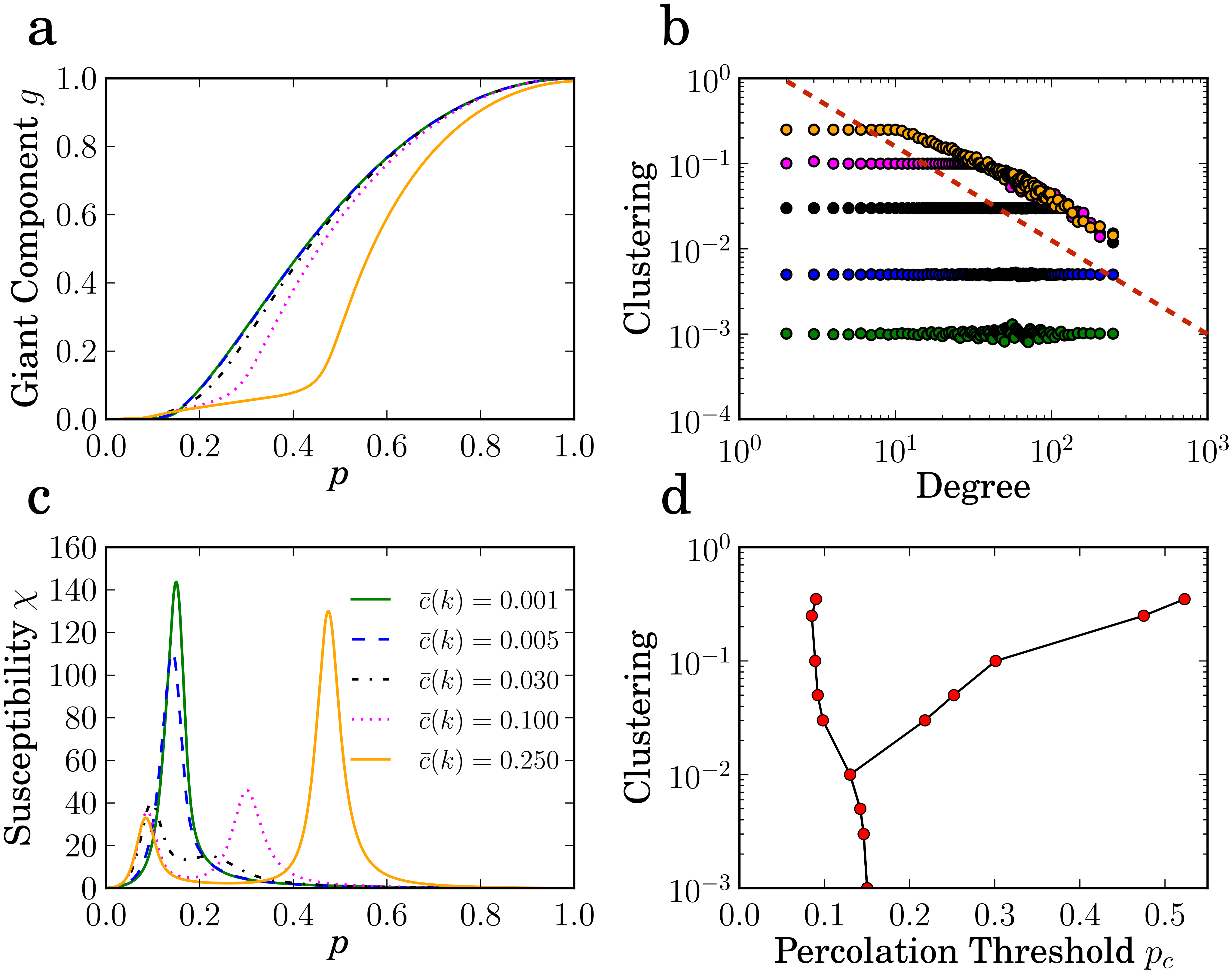


Effect of clustering



$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.25$$




a
Giant Component g
p
b
Clustering
Degree
c
Susceptibility χ
p
$\bar{c}(k) = 0.001$
$\bar{c}(k) = 0.005$
$\bar{c}(k) = 0.030$
$\bar{c}(k) = 0.100$
$\bar{c}(k) = 0.250$
d
Clustering
Percolation Threshold p_c

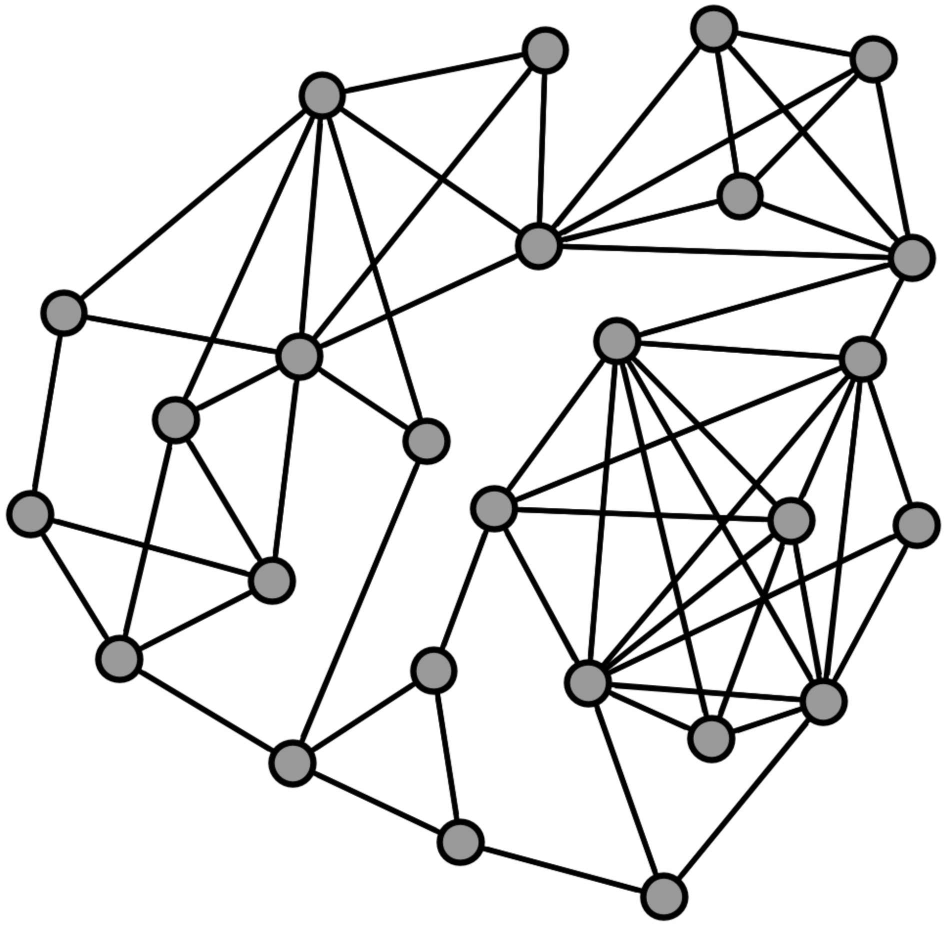
Global organization of clustering

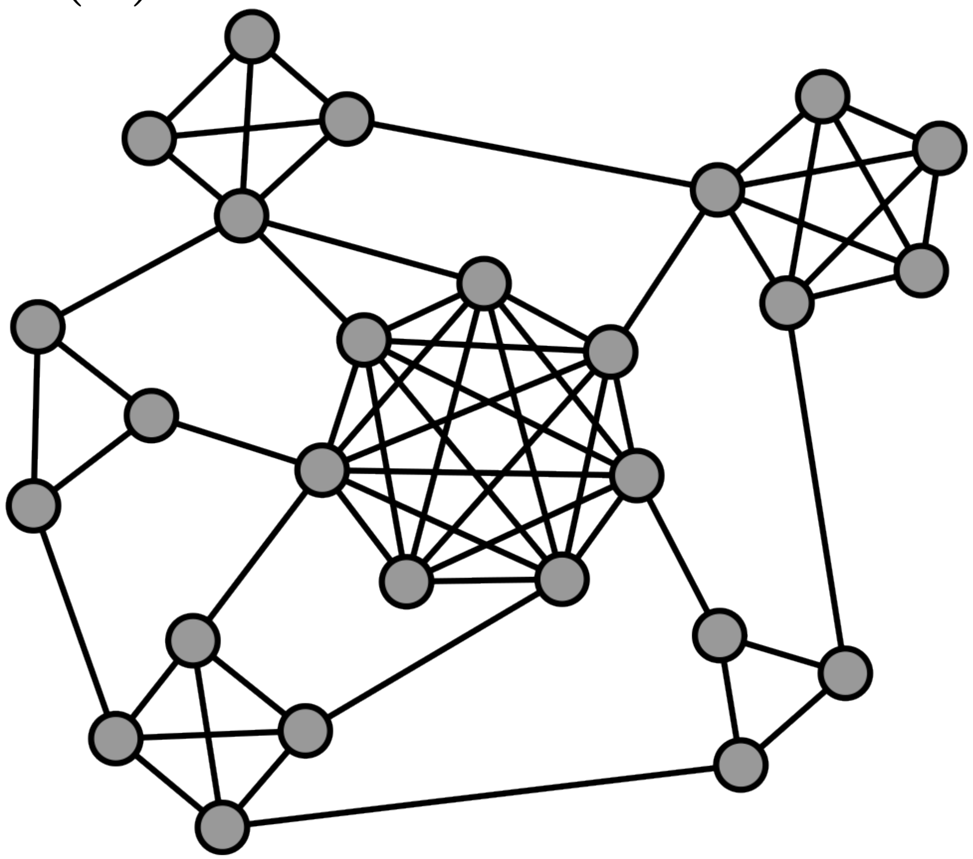
Global organization of clustering

Maximally Random

Maximally Ordered

$P(k)\ \bar{c}(k)$

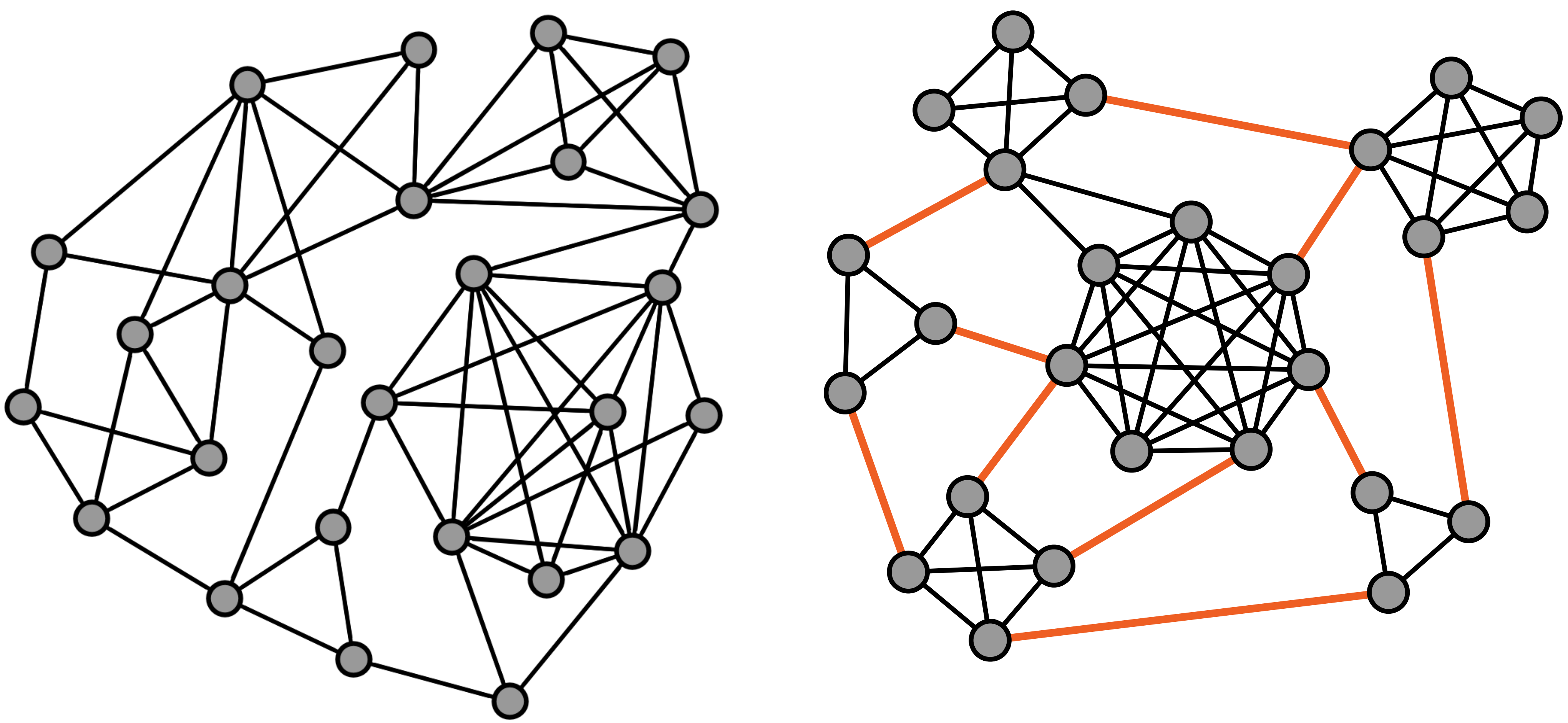




Global organization of clustering

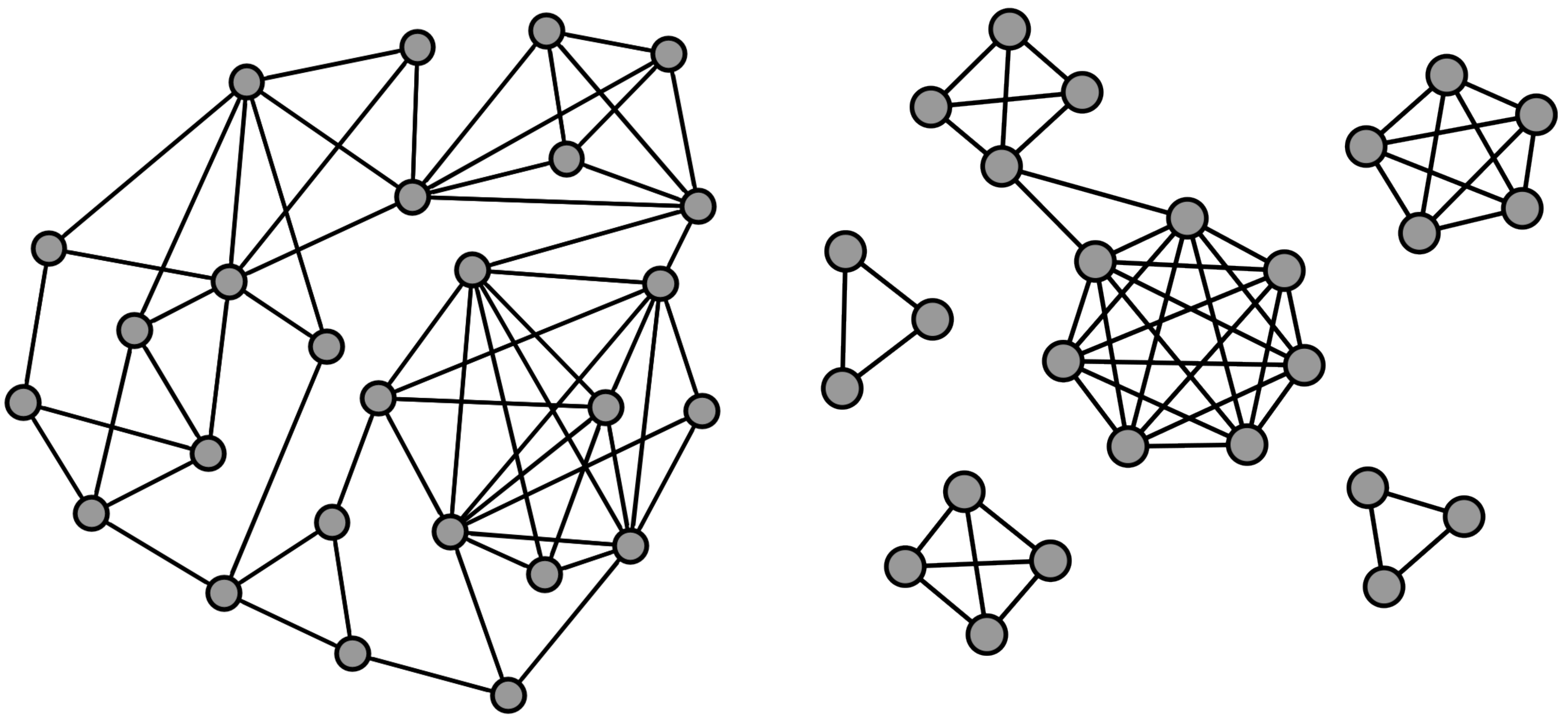
Maximally Random

Maximally Ordered

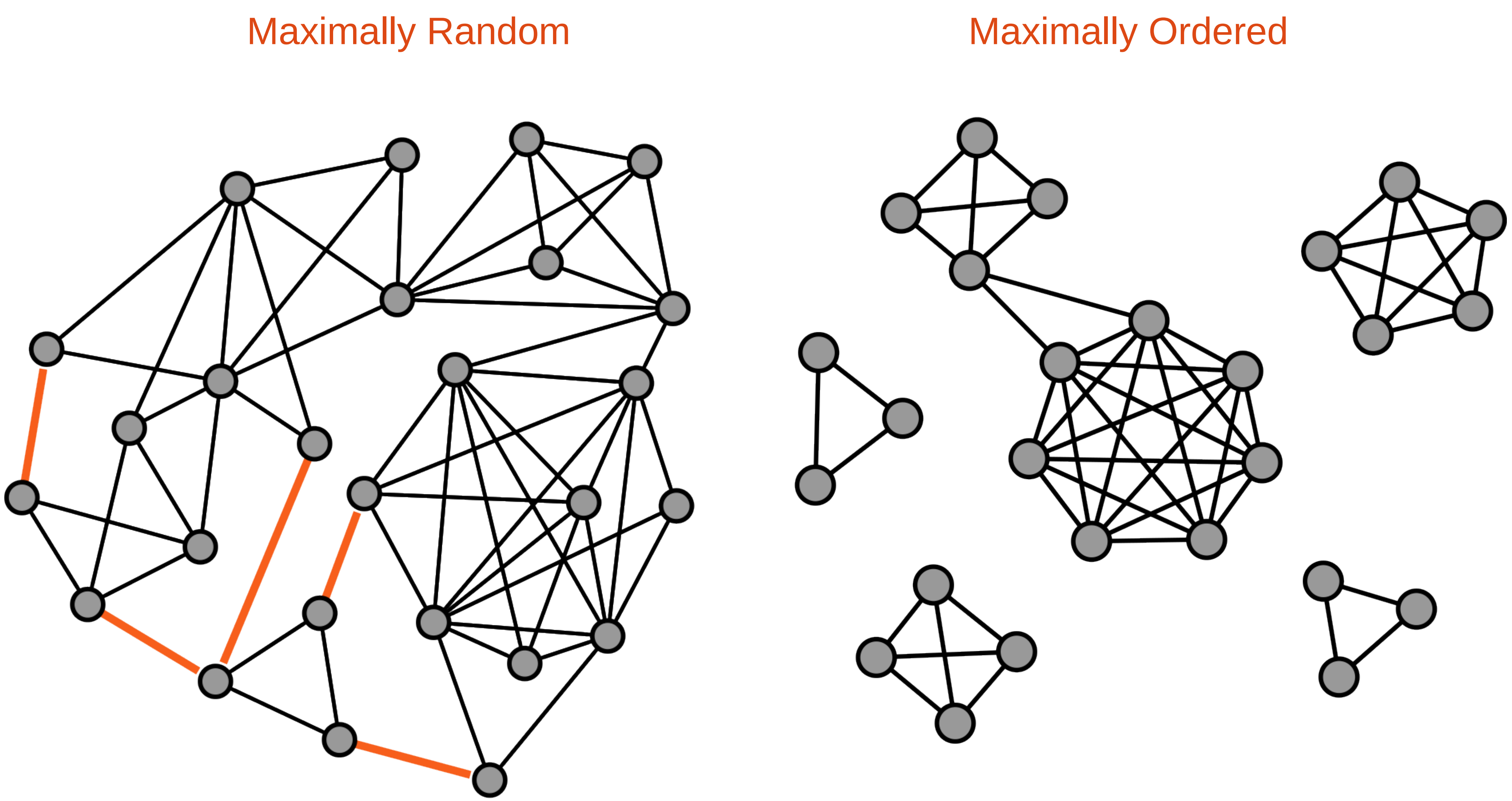


Global organization of clustering

Maximally Random　　　　Maximally Ordered

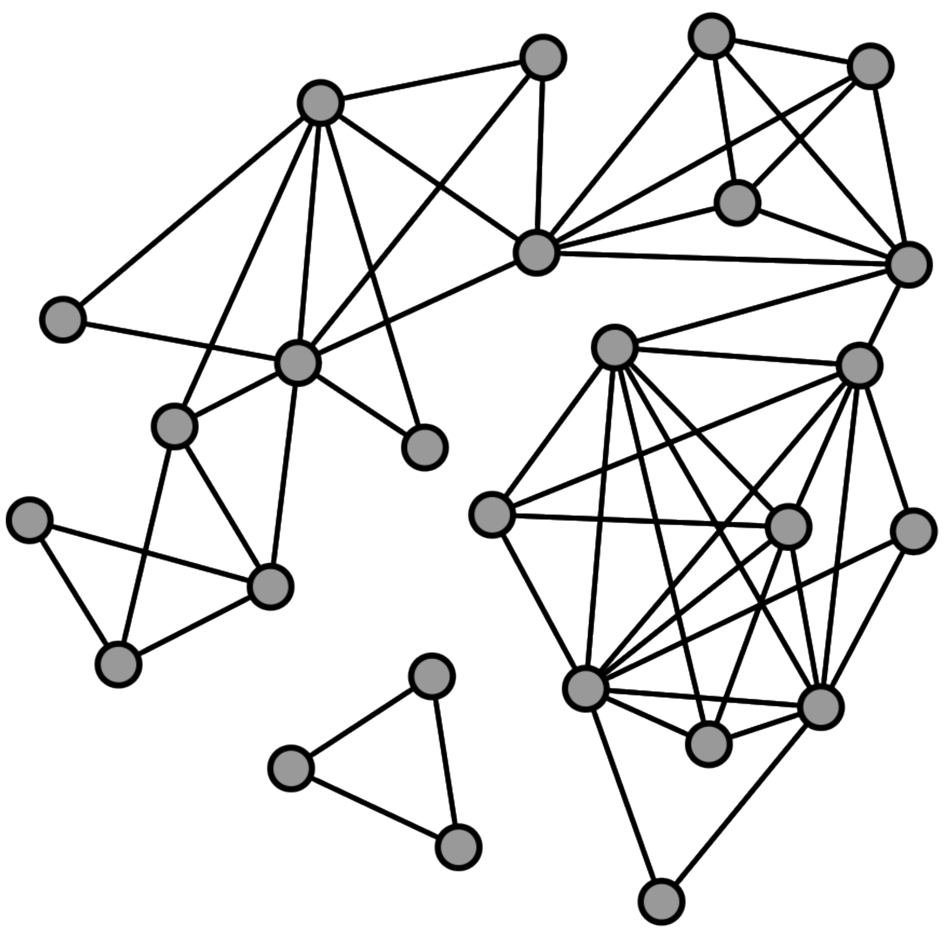


Global organization of clustering

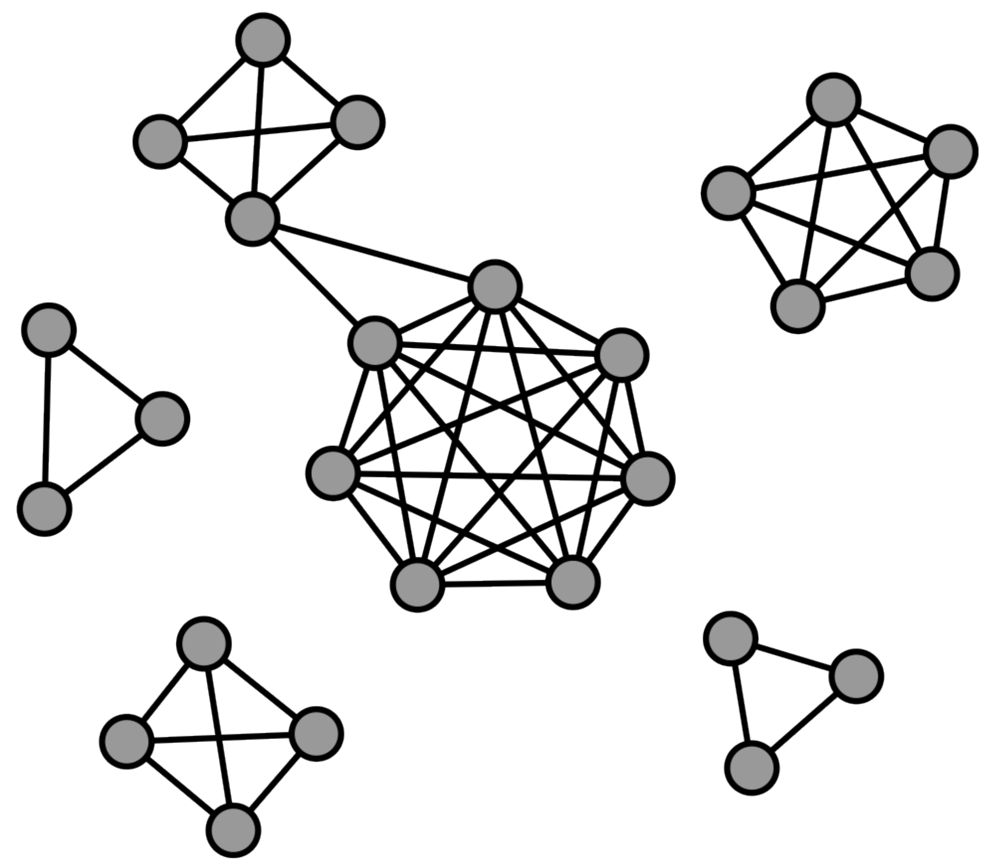


Global organization of clustering

Maximally Random



Maximally Ordered



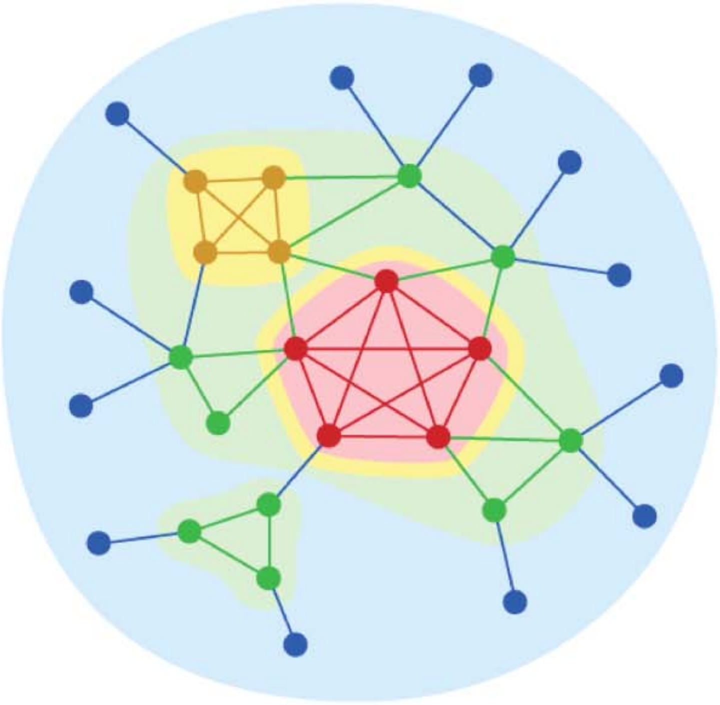
m-cores

m-core: maximal subgraph in which every link has at least multiplicity m with other neighbors in the subgraph

multiplicity m of a link is the number of triangles going through the link

m-cores are obtained in a recursive way

- remove from the original graph all links with multiplicity less than m
- update link multiplicities and remove from the remaining graph all links with multiplicity less than m
- repeat step two until no further removal is possible



P. Colomer-de-Simón, M. A. Serrano, Mariano G. Beiró, J. Ignacio Alvarez-Hamelin, and M. Boguñá, **Deciphering the global organization of clustering in real complex networks**, Scientific Reports 3, 2517 (2013)

http://sourceforge.net/projects/lanet-vi/

LaNet-vi 3.0

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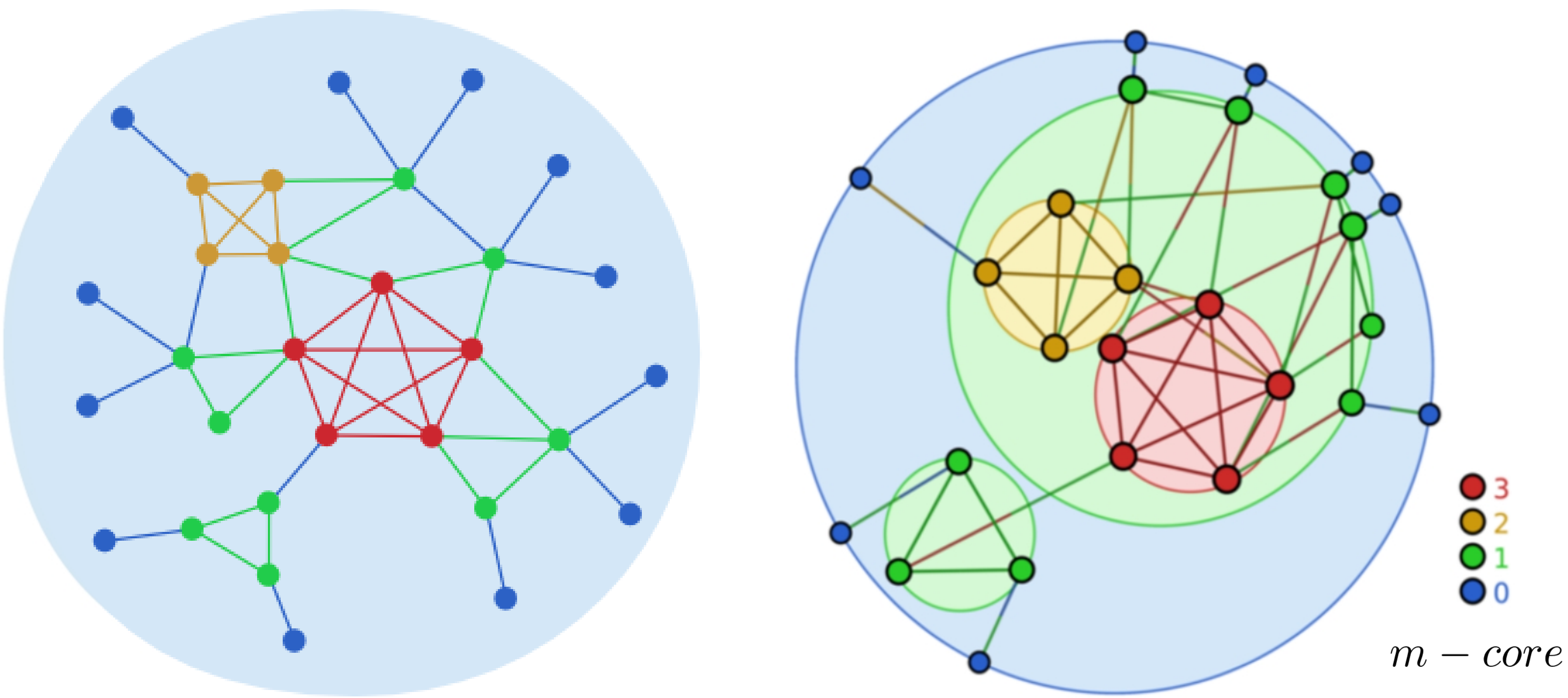
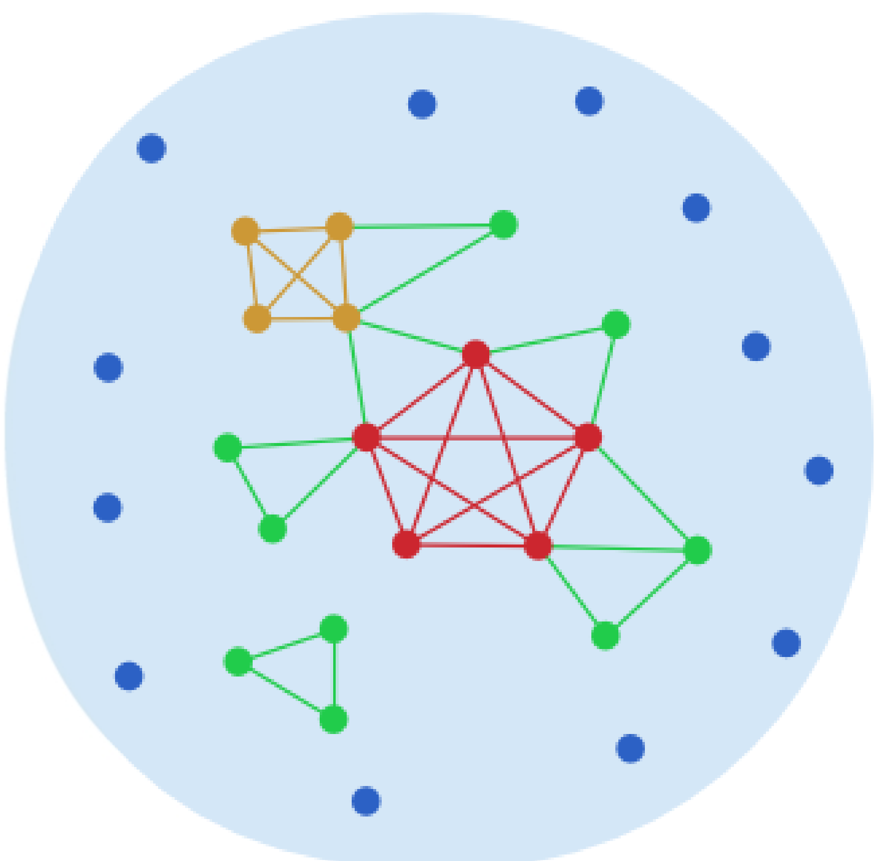
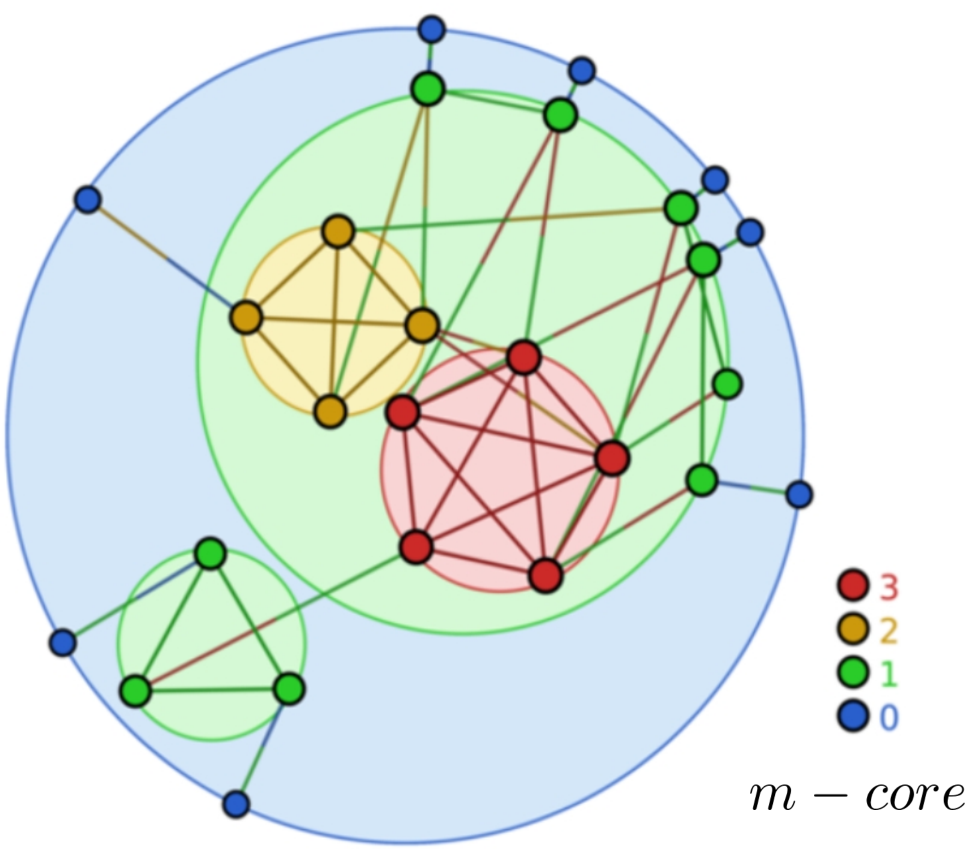


http://sourceforge.net/projects/lanet-vi/

LaNet-vi 3.0





http://sourceforge.net/projects/lanet-vi/

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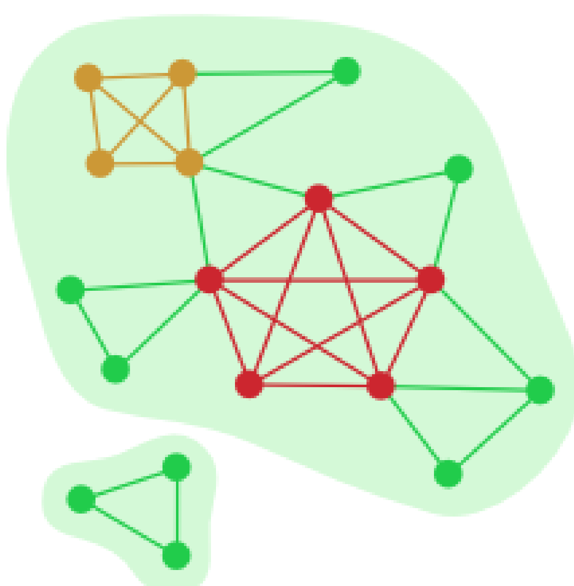
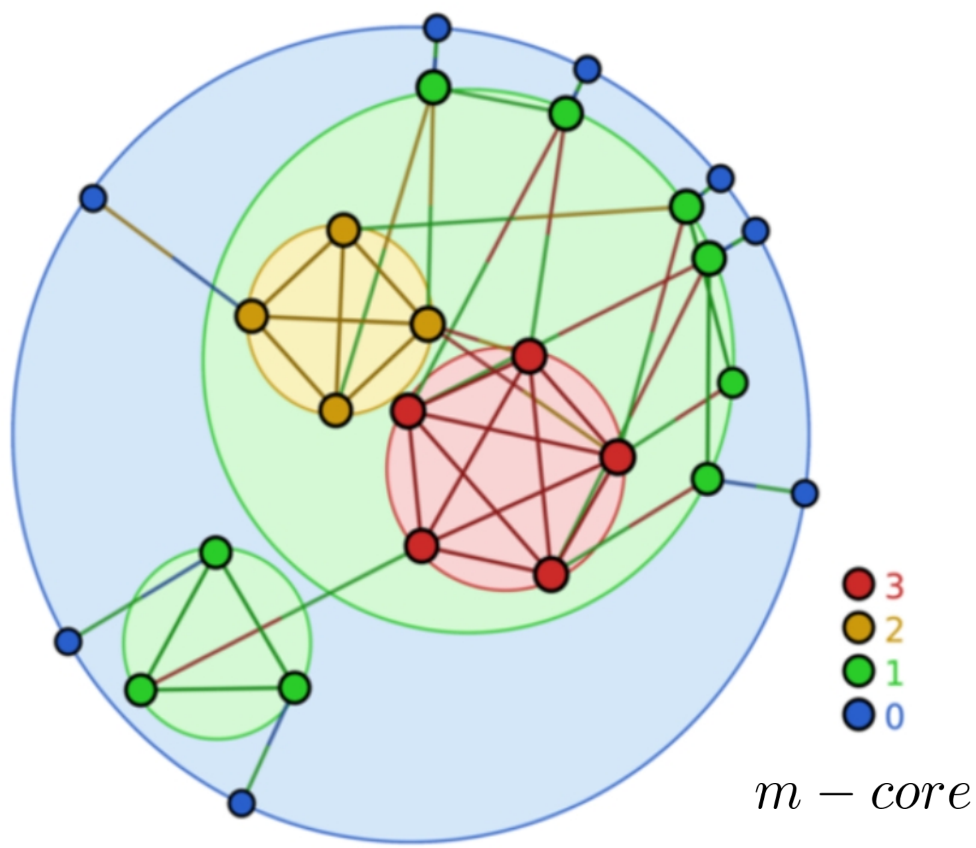
LaNet-vi 3.0





http://sourceforge.net/projects/lanet-vi/

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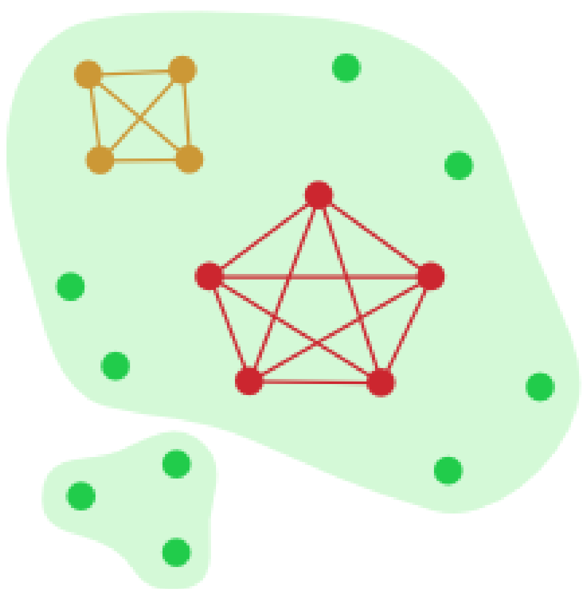
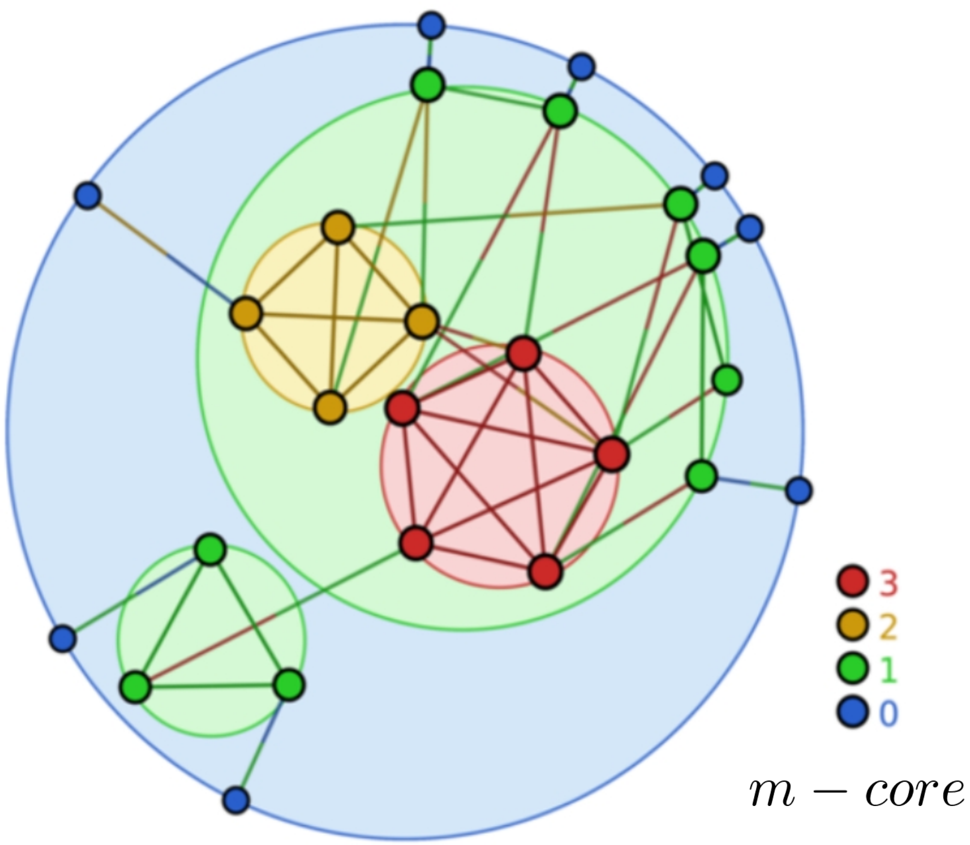
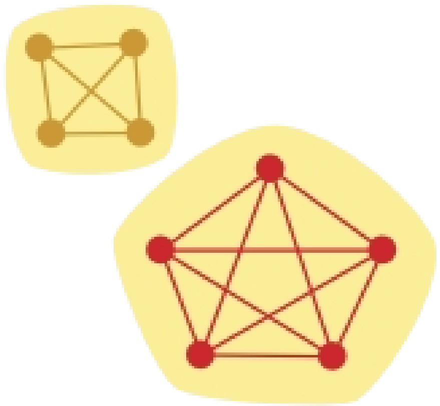
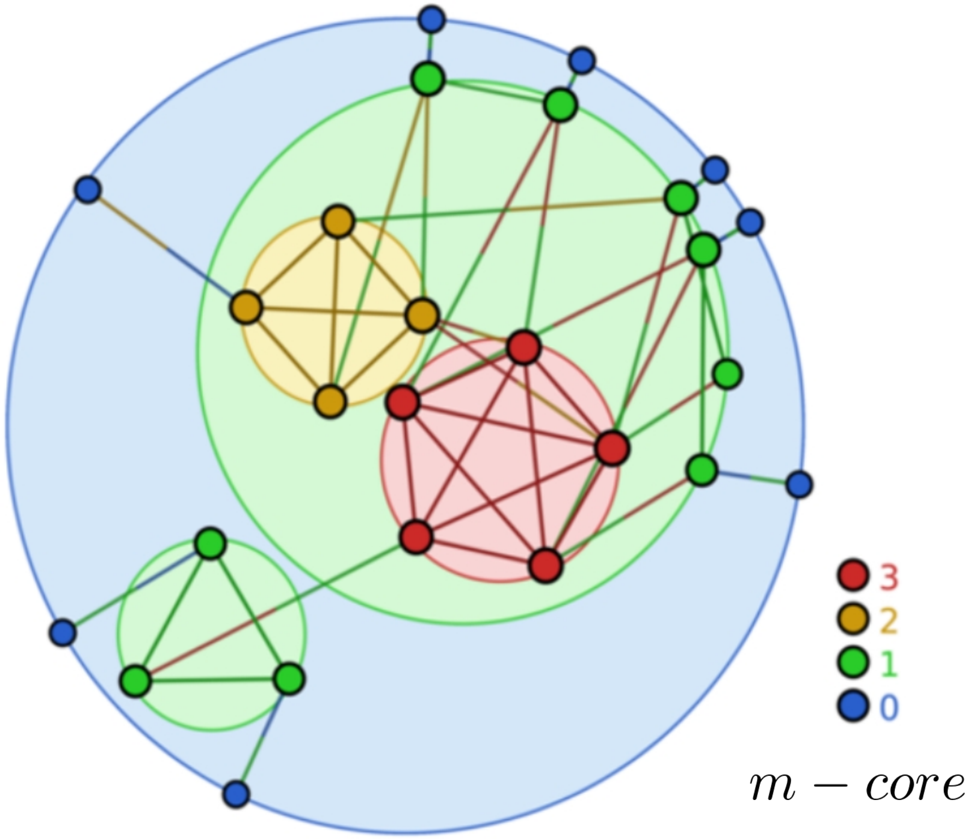




http://sourceforge.net/projects/lanet-vi/

LaNet-vi 3.0





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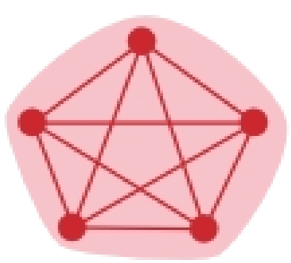

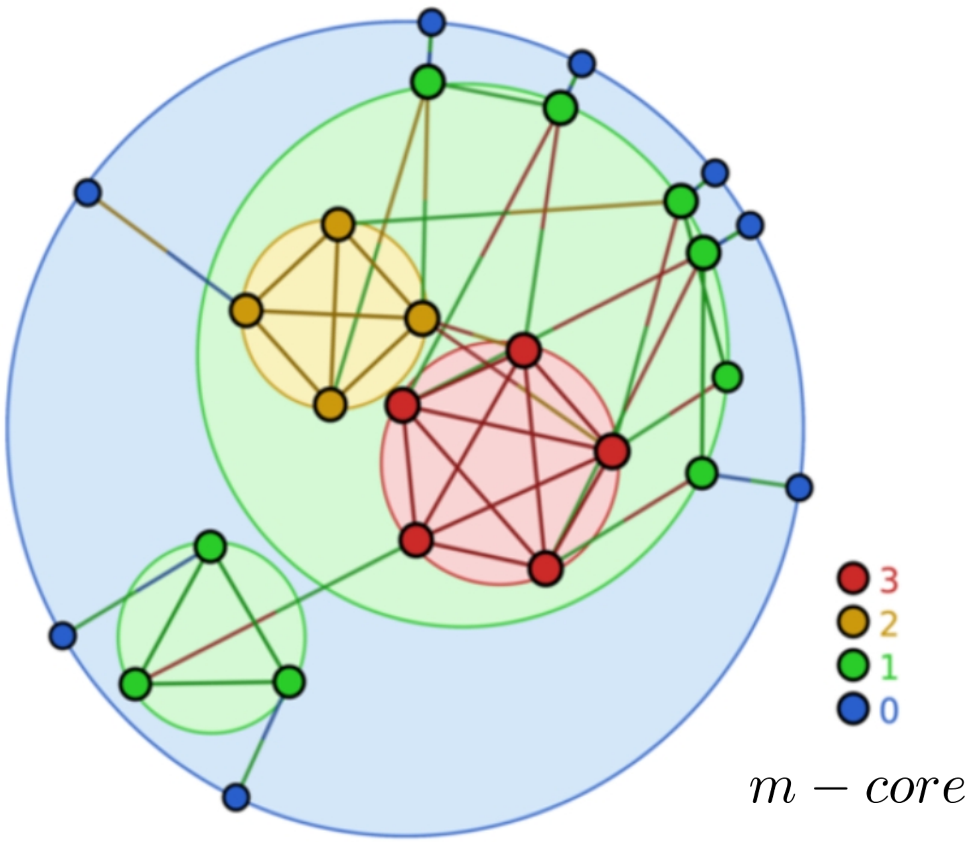

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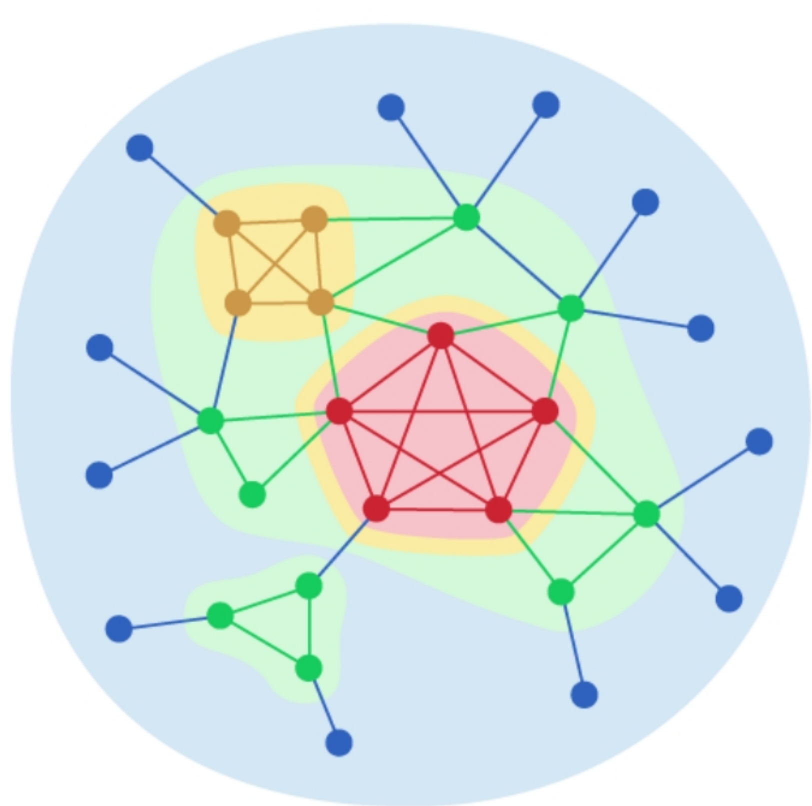
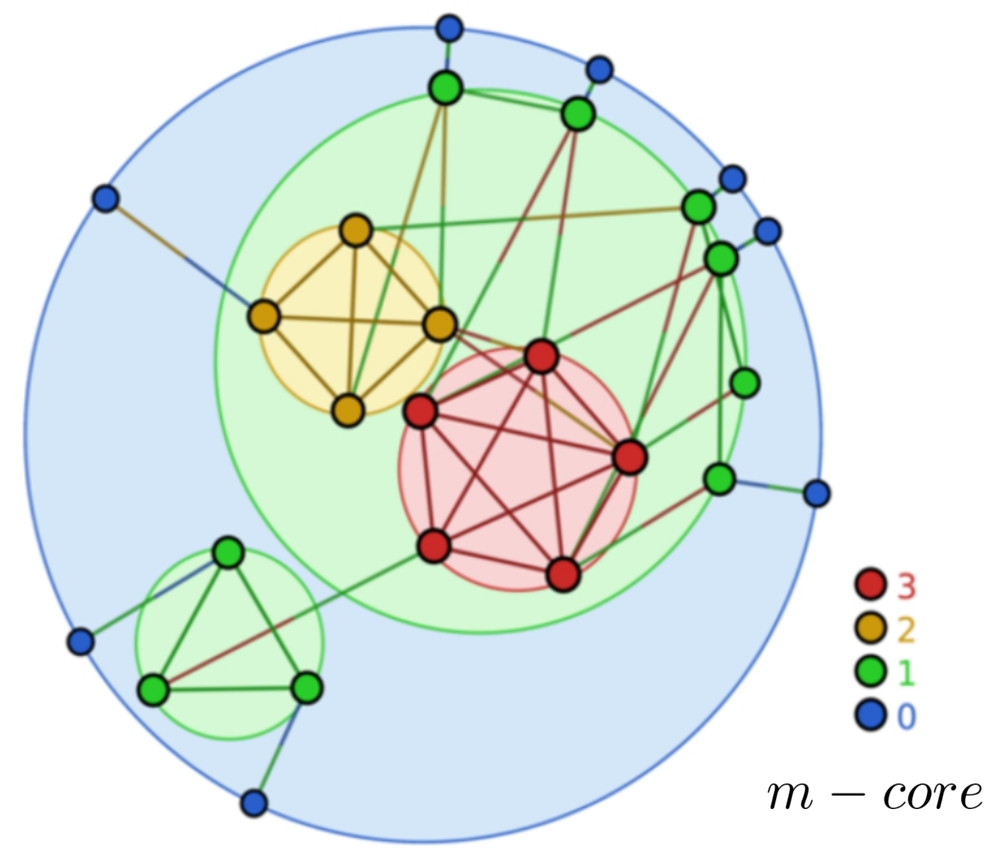
LaNet-vi 3.0





http://sourceforge.net/projects/lanet-vi/

LaNet-vi 3.0

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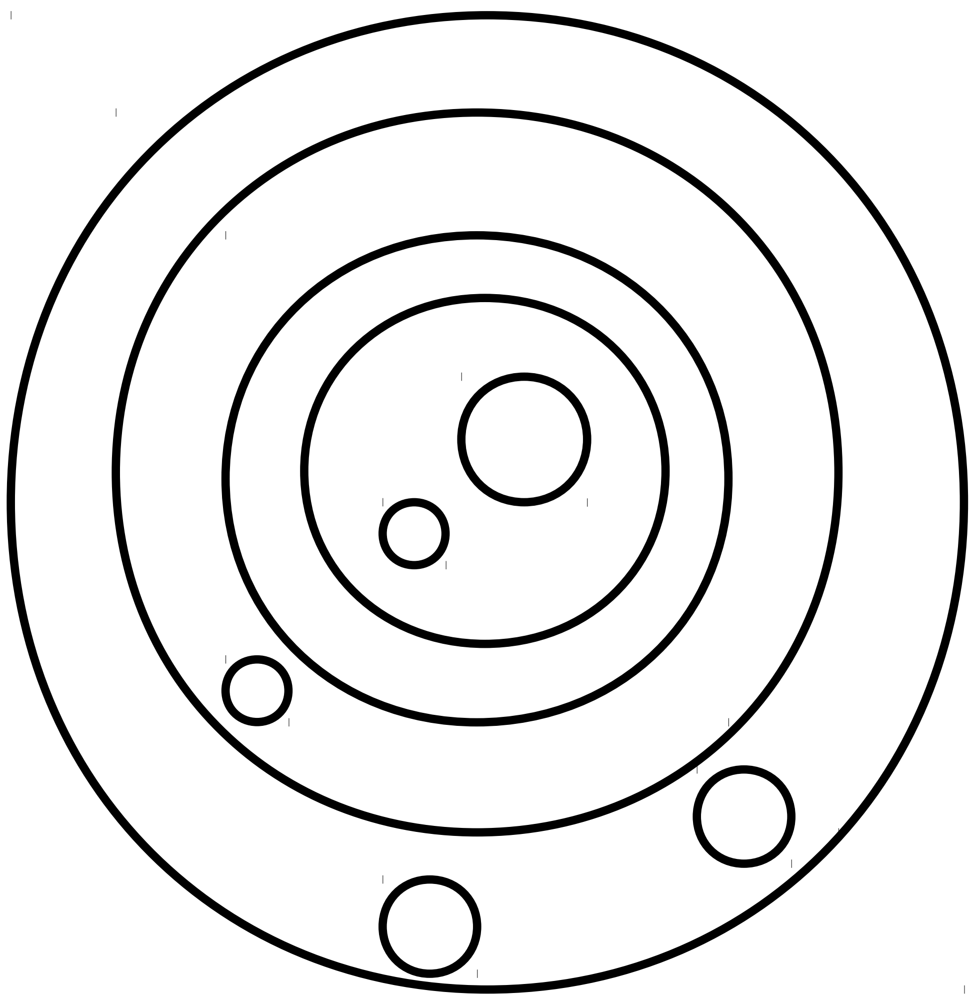
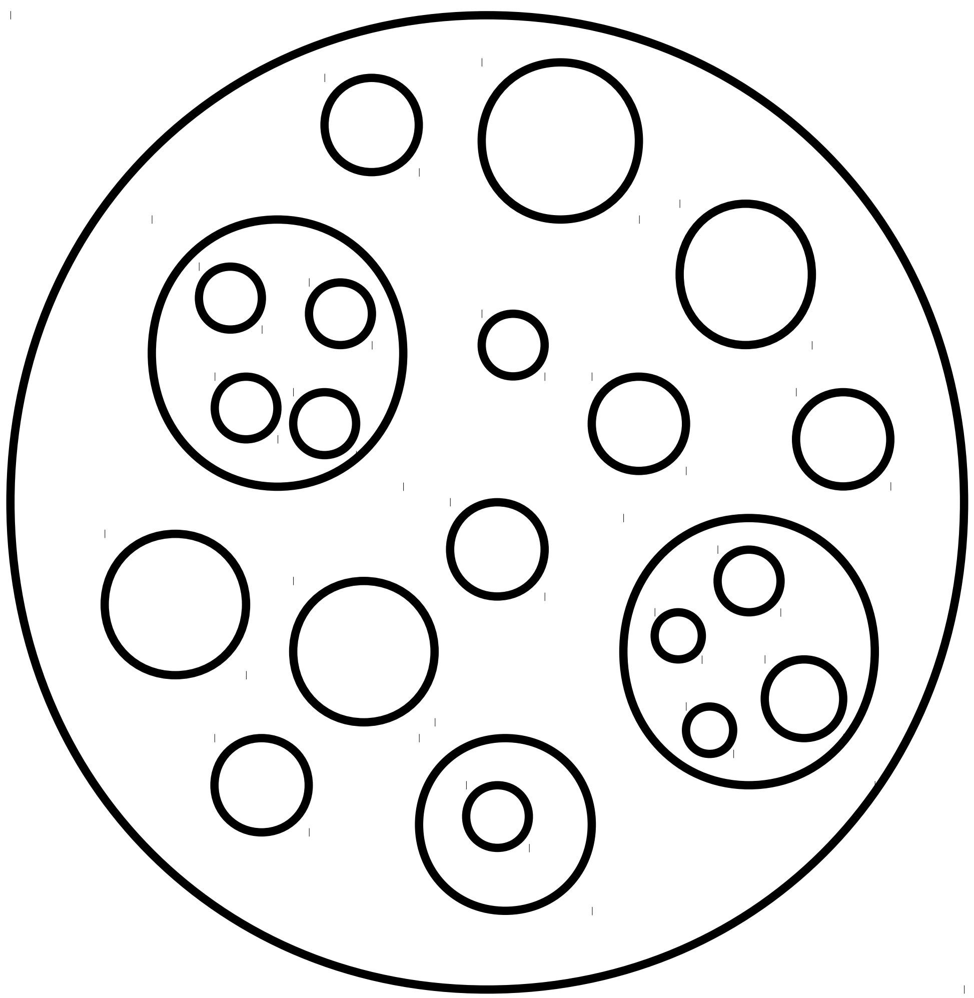
Maximally Random



Maximally Ordered



http://sourceforge.net/projects/lanet-vi/

E. Coli.

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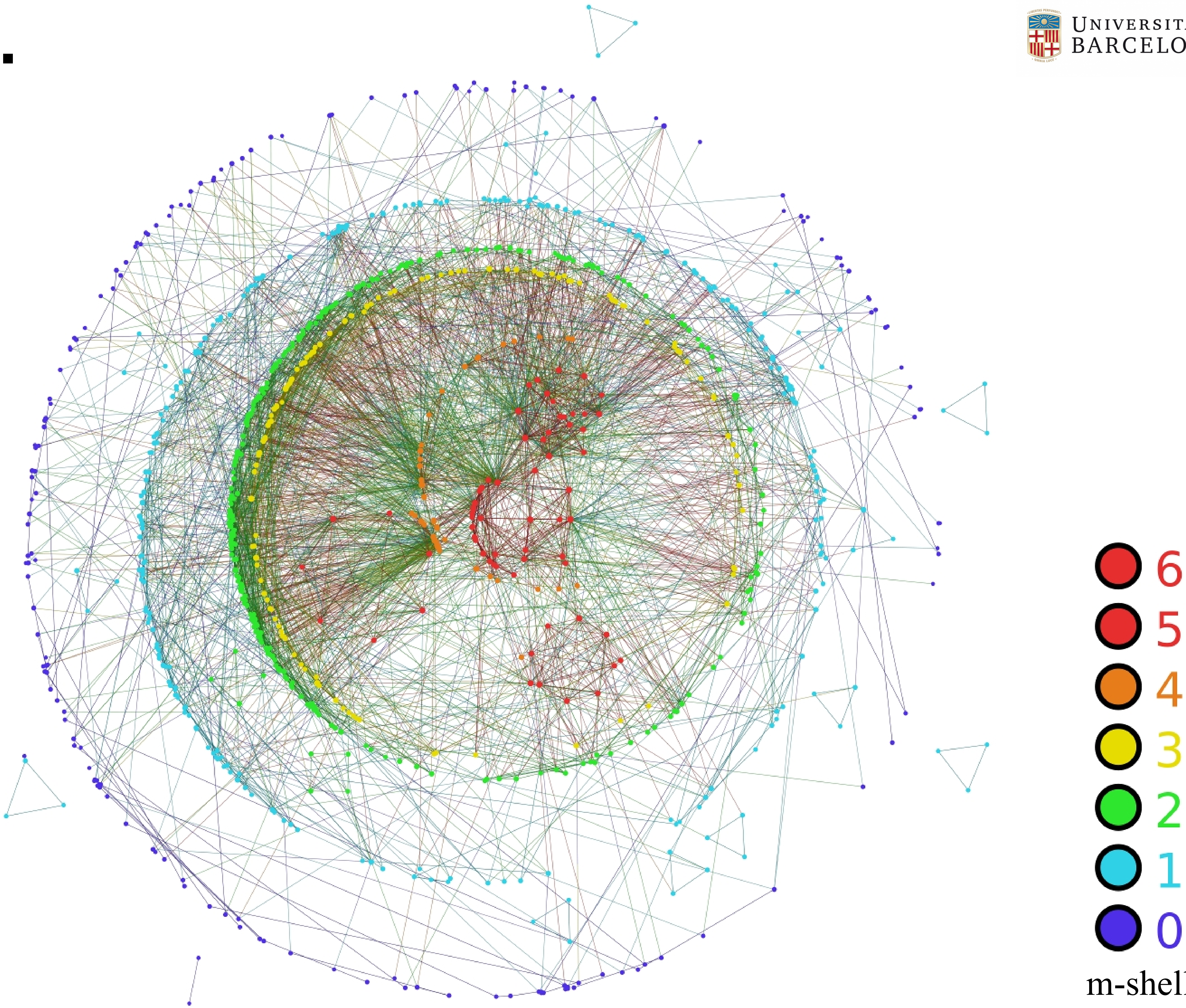
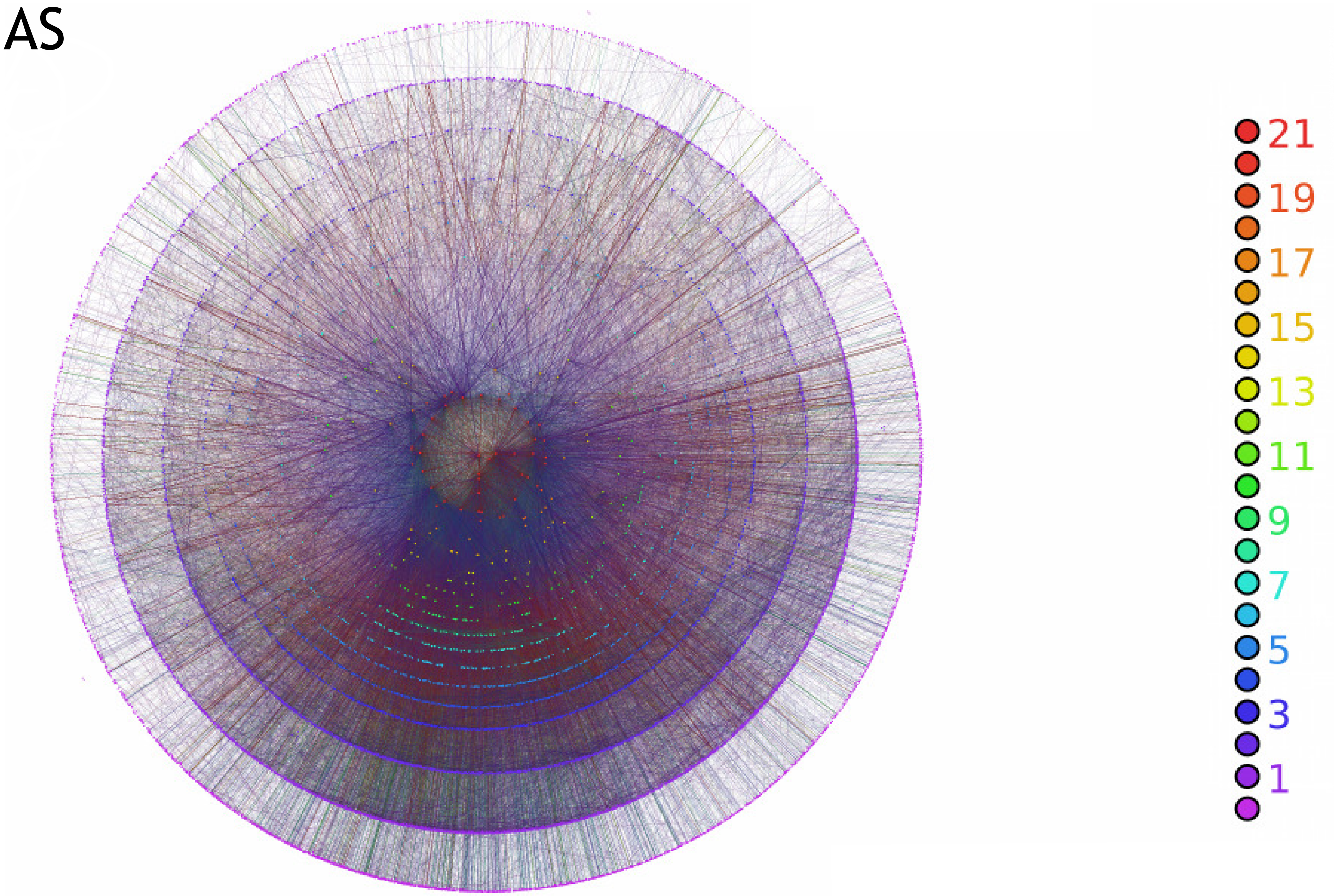


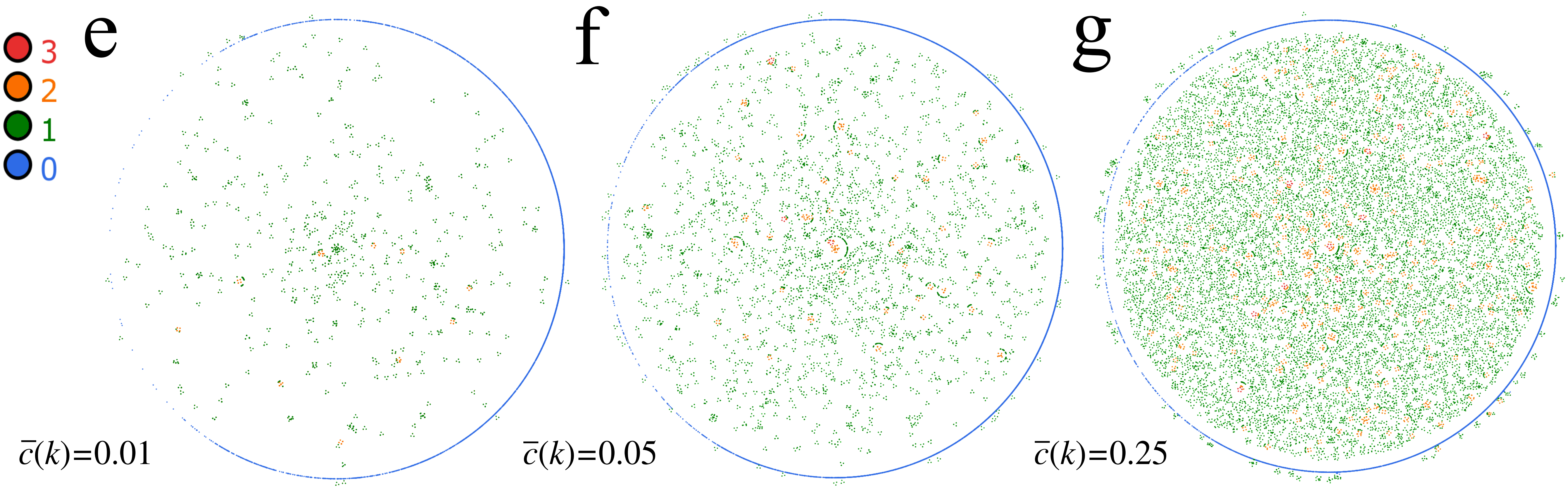
SOURCEFORGE®

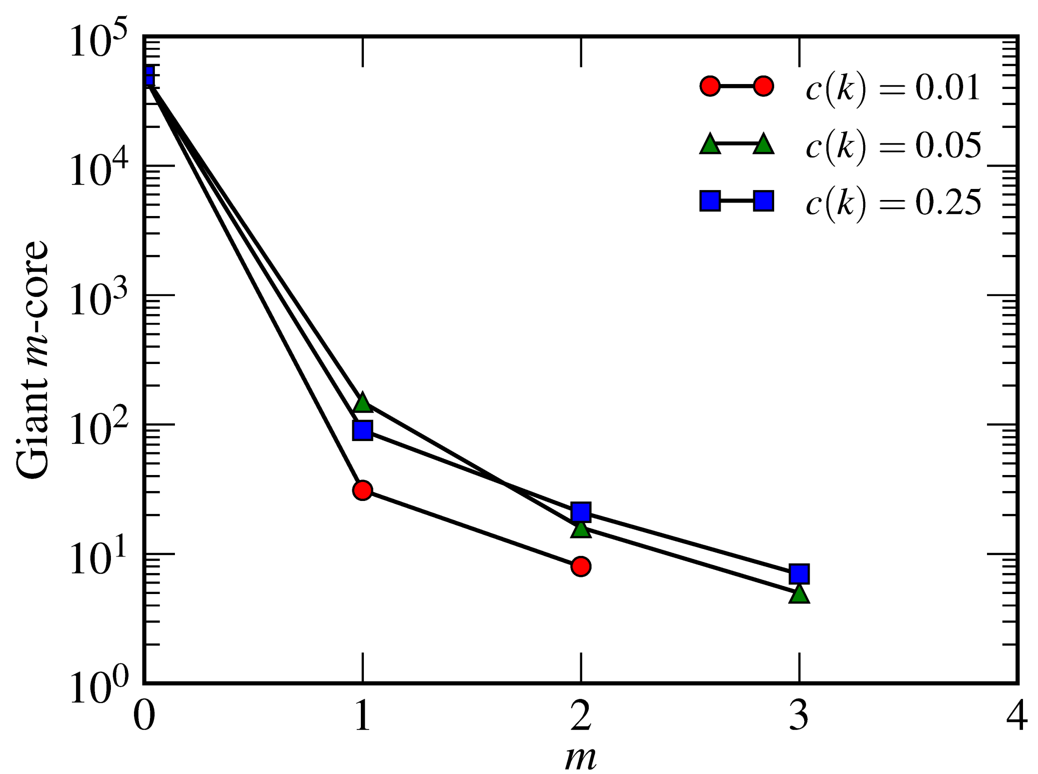
http://sourceforge.net/projects/lanet-vi/

Internet AS



non-heterogeneous networks

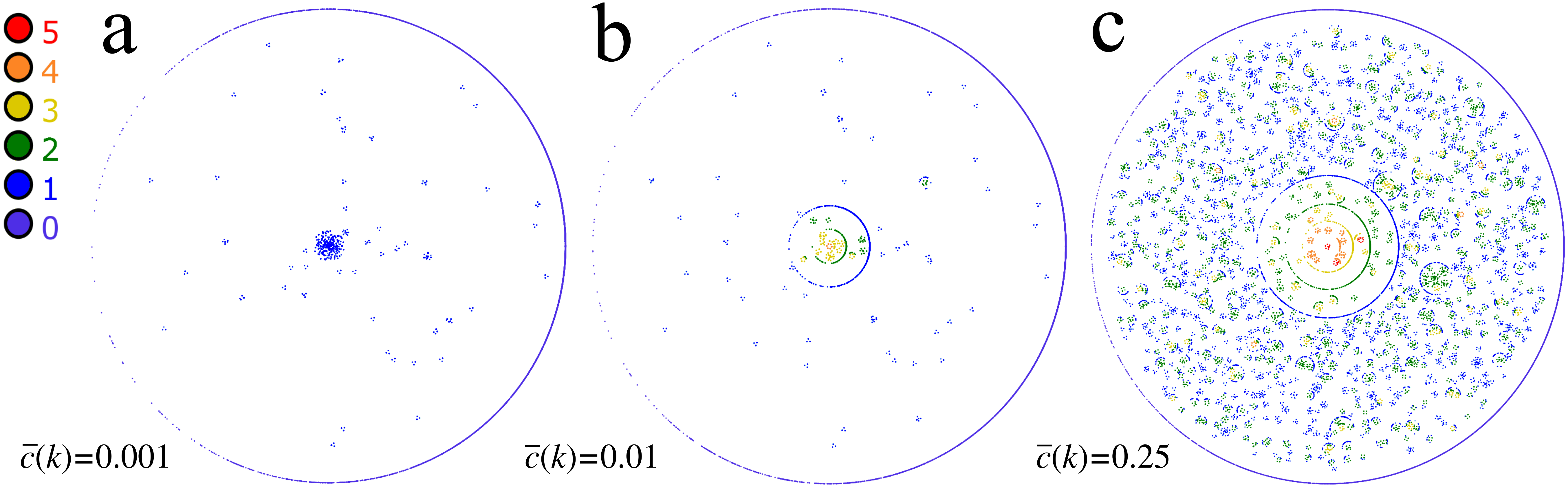


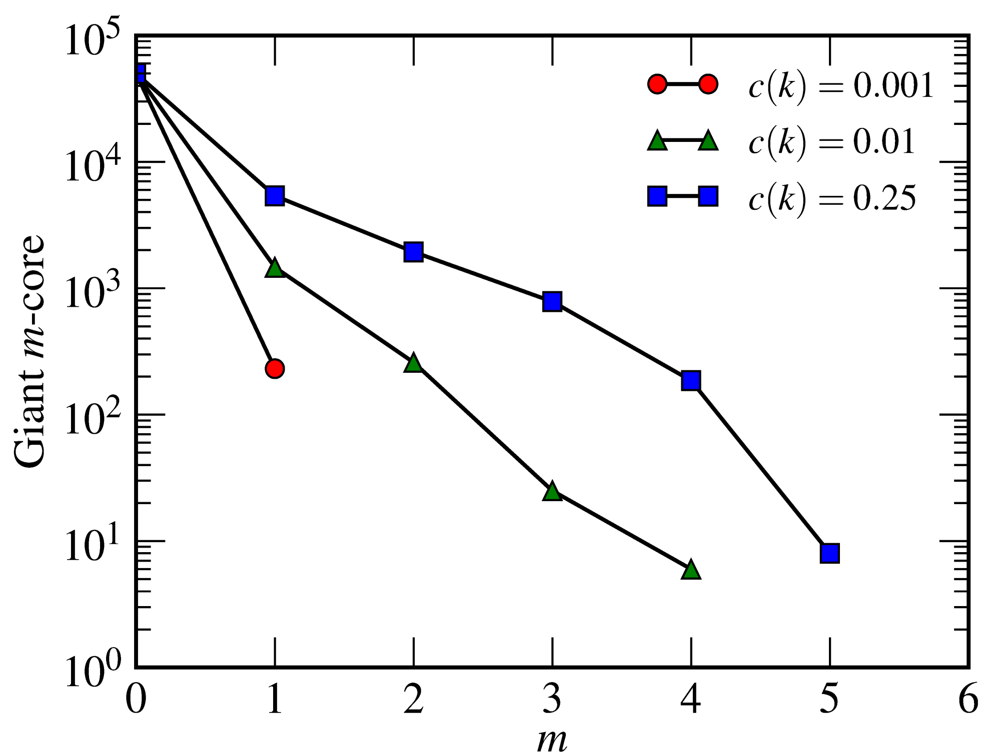


$$P(k) \sim k^{-4}$$

m-core decomposition

heterogeneous networks



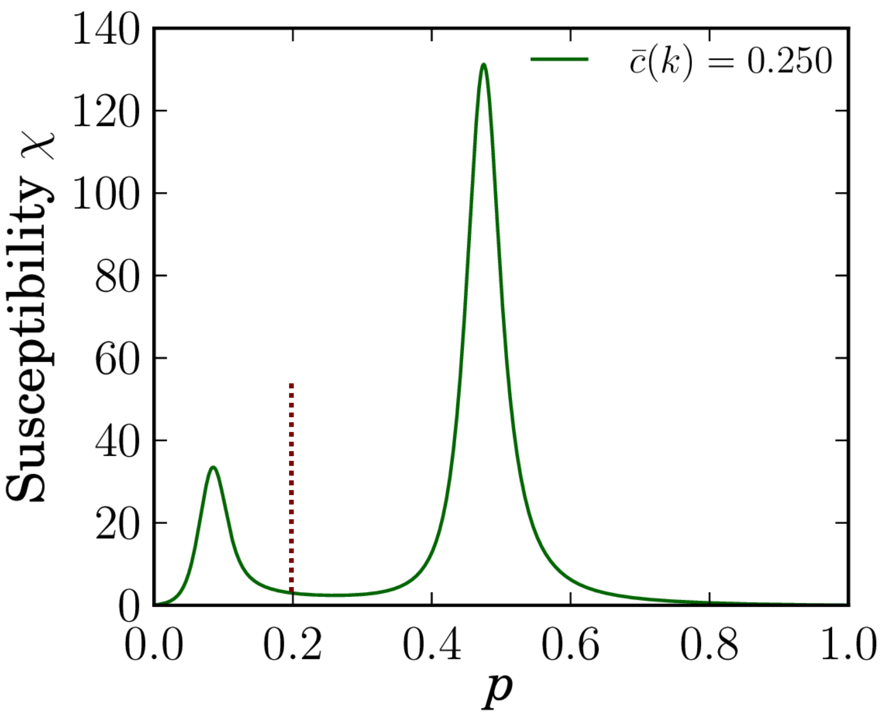


$$P(k) \sim k^{-3.1}$$

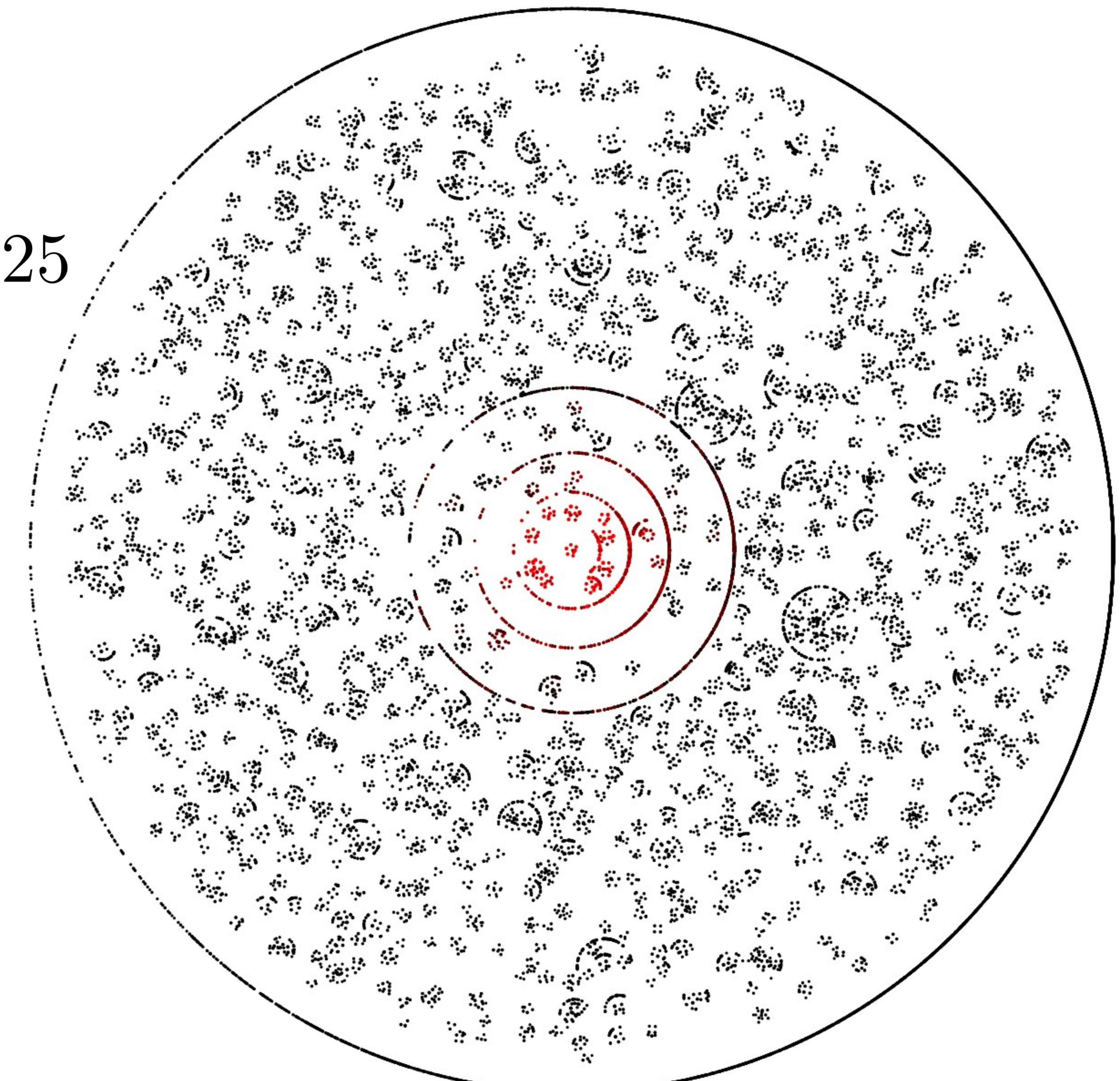
m-core decomposition

Core-Periphery

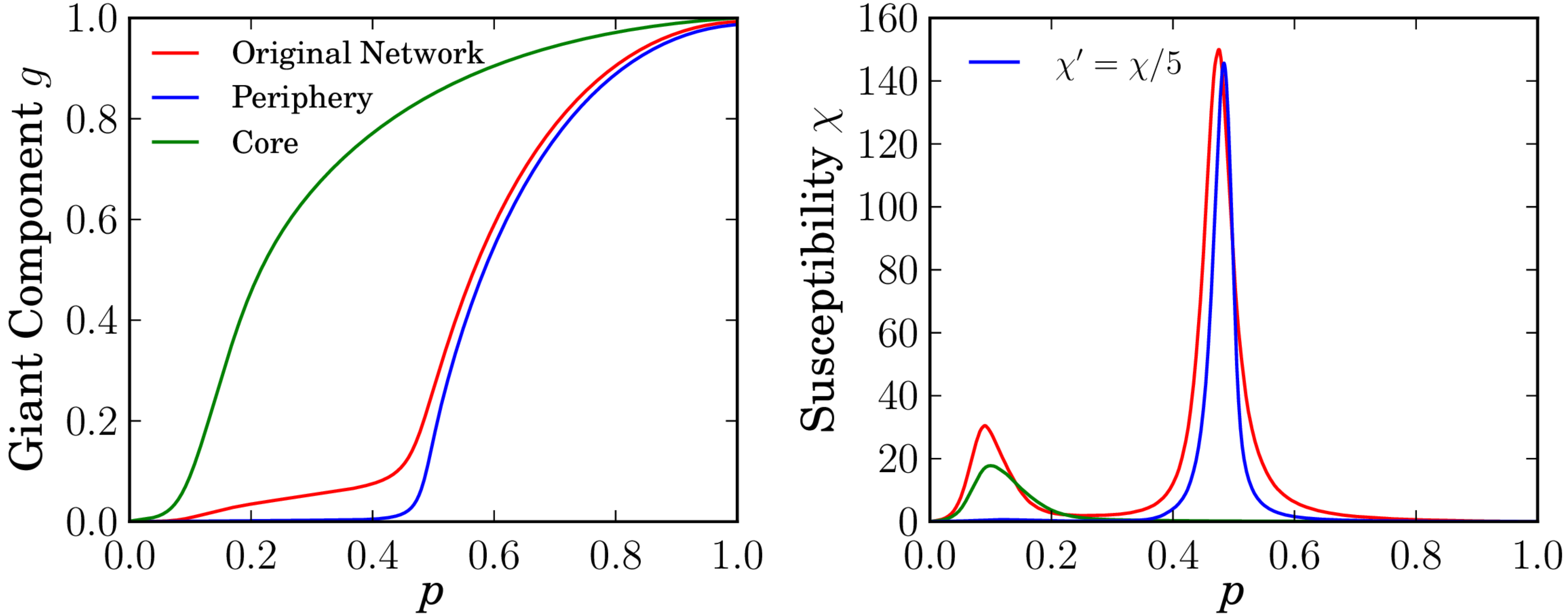
$$P(k) \sim k^{-3.1} \quad \bar{c}(k) = 0.25$$



$$p = 0.2$$

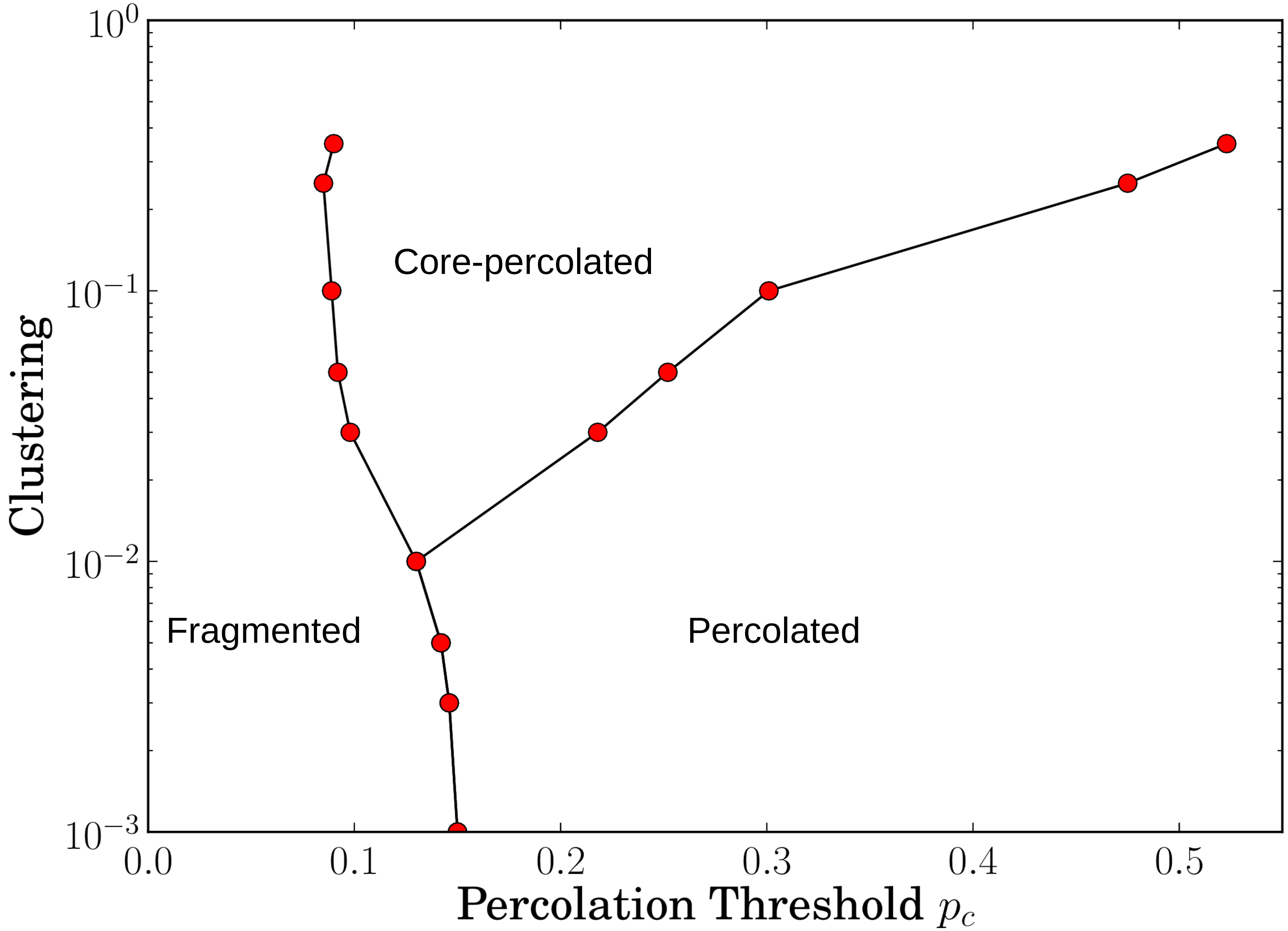


Core-Periphery



The periphery percolates regardless of the core

Percolation phase space



Finite Size Scaling

$$P(k) \sim k^{-3.1} \qquad \bar{c}(k) = 0.25$$