Basix

30 January 2021 10:52 AM

Sequence

A collection S of a real number is said to be a sequence of numbers if corresponding to every positive integer n, there exist a unique element S.

Monotonic sequence

Monotonic Increasing sequence

$a1 \leq a2 \leq a3 \ldots ..$

Monotonic decreasing sequence

$a1 \geq a2 \geq a3 \ldots ..$

Convergent, divergent, oscillating sequence

Convergent

A sequence S = $\{a_n\}$ is convergent if $\lim_{n\to\infty} a_n \ is\ finite$

Divergent

A sequence S = $\{a_n\}$ is divergent if $\lim_{n\to\infty} a_n \ is\ infinite$

Oscillating

A sequence which is neither convergent nor divergent

Theorems

31 January 2021 02:48 PM

- A convergent sequence determines its limit uniquely
- Every convergent sequence is bounded
- A monotonic increasing sequence which is bounded above is convergent and converges to its exact upper bound or supremum.
- A monotonic decreasing sequence which is bounded below is convergent and converges to its exact lower bound.
- A monotonic increasing sequence diverges to +inf. If it is not bounded above.
- A monotonic decreasing sequence diverges to -inf. If it is not bounded below.

Infinite series

31 January 2021 03:20 PM

If $\{u_n\}$ is a sequence of real numbers, then the experesion $\sum_{n=1}^{\infty} u_n$
Is called an infinite series.

Convergent series : if the sequence forming the series is convergent then the series is also convergent.

divergent series : if the sequence forming the series is divergent then the series is also divergent.

Oscillatory series : if the sequence forming the series is Oscillatory then the series is also Oscillatory.

Absolute convergence

An infinite series $\sum_{n=1}^{\infty} u_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} | u_n | \; is\ convergent$

Conditionally convergent

If the given series is convergent but not absolutely convergent then it is called conditionally convergent.

Geometric series

Converges if common ratio belongs to (-1,1)
Diverges if Cr >= 1
Oscillatory if CR <= -1

P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

convergent if p>1
Divergent if p<= 1

Power series

31 January 2021 07:18 PM

Power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Power series in x a_n are independent coefficients
Radius of convergence is the value of x for which |x - c|<1

- GP is also a power series
- The powers series may only converge for certain interval called interval of convergence
- If two power series are convergent then their sum , difference product also converges.
- The quotient of two power series not term by term but combined sum up to a certain n is also a power series provided
  - Both numerator and denominator converges in the same interval
  - Denominator does not vanish
  - $$\frac{a_0 + a_1x + a_2x^2 + \ldots + a_nx^n}{b_0 + b_1x + b_2x^2 + \ldots + b_nx^n} = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$
    Since $\sum a_nx^n = \sum b_nx^n . \sum c_nx^n$,
    $a_0 = b_0c_0$, $c_1 = c_0b_1 + c_1b_0$, etc
    whence $c_0, c_1, c_2, \ldots$ can be calculated
- If two power series has same interval of convergence and represented by same term function then the power series are identical.

Series test

31 January 2021 05:09 PM

- By defination

  $\lim_{n\to\infty} a_n$ *is finite* $\to$ *convergent*

  $\lim_{n\to\infty} a_n$ *is infinite* $\to$ *divergent*

  Else oscillating
- Divergence test

$$\lim_{n\to\infty} a_n \text{ does not exist, or}$$

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

- Integral test

$$f \text{ continuous, positive, decreasing on } [1, \infty)$$

$$\text{such that } a_n = f(n)$$

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

  this does *not* mean that the sum of the series is that same as the value of the integral.

- Comparison test

$$a_n, b_n > 0 \text{ for all } n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

- Alternating series test (lebnitz theorem)

$$a_n = (-1)^{n+1} \cdot b_n, b_n > 0 \text{ for all } n, b_n \text{ decreasing}$$

$$\lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

- Absolute convergence test

$$\sum_{n=1}^{\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

- Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Ratio test fails if the above limit is 1

- Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Root test fails if the above limit is 1

Taylor series

31 January 2021 07:30 PM

The Taylor series of a function f(x) that in infinitely differentiable at a real or complex no, a is the following power series

$$f(x) = f(x) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \quad at\ x = a$$

If we put a=0 above we get a Maclaurin series

Limit & Continuity

01 February 2021 11:12 AM

Limit

In function of one variable for a limit to exist we only have to check it from 2 directions that is the right hand limit and left hand limit if these 2 limits are equal then the function is said to have a limit at a point p.

for a function of 2 variables z=f(x, Y) we graph it as a surface and so there may be infinite number of directions to approach a particular point (a ,b).

In such a situation we approach the limit in the following ways

- $\epsilon\ \delta$ defination

  Statement : if a *number l* is such that for any positive number ϵ *we can find some positive no.* $\delta(dpending\ on\ \epsilon)$

  Such that | f(x, y) - l | $< \epsilon$ *whenever* $|x-a| < \delta$ *and* $|y-b| < \delta$

  Then we write

  $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = l}$

- To verify whether the limit exist or not

  Statement : if their exist two functions y = f1(x) and y = f2(x)

  Such that $\lim_{x\to a} f1(x) = b = \lim_{x\to a} f2(x)$

  For which $\lim_{x\to a} f\big(x, f1(x)\big) \neq \lim_{x\to a} f\big(x, f2(x)\big)$

  Then the $\lim_{(x,y)\to(a,b)} f(x,y)$ *does not exist*

Repeated limits

$\lim_{x\to a} f(x,y)$ *will give a finction in y say* $f(y)$

then $\lim_{y\to b} f(y)$ *gives a certain value*

The limit $\lim_{y\to b} \lim_{x\to a} f(x,y)$ is called repeated limit

$\lim_{y\to b} \lim_{x\to a} f(x,y)$ *may not be equal to*

$\lim_{x\to a} \lim_{y\to b} f(x,y)$ *may not be equal to* $\lim_{(x,y)\to(a,b)} f(x,y)$

If $\lim_{y\to b} \lim_{x\to a} f(x,y) \neq \lim_{x\to a} \lim_{y\to b} f(x,y)$

This $\lim_{(x,y)\to(a,b)} f(x,y)$ *DNE*

Points

Existence of partial derivative does not ensure continuity of a function of more than one variable

Chain rule

01 February 2021 04:53 PM

- $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$

- If f =f(u) and u=f(x, y)
  - $\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}$
  - $\frac{\partial f}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y}$

- *If f=f(u, v) and u=f(x, y) and v=f(x, y)*
  - $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$
  - $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$

- *If f=f(u, v) and u=f(x) and v=f(x)*
  - $\frac{df}{dx} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}$

Total differentiable

01 February 2021 05:15 PM

Let z = f(x, y) be a function of two independent variables x and y. The function f(x, y) is said to be differentiable or total differentiable at (x, y) if $f_x \text{ and } f_y$ exist at (x, y) and if its increment $\Delta f \text{ or } \Delta z \text{ can be expressed as}$

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

The total differential df can be expressed as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Second order total differential is just

$$d^2 f = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 f$$

Basix

01 February 2021 06:09 PM

Defination : an equation involving x, y and $\frac{dy}{dx}$ is called a differential equation

Formation and solution : mathematically differential equation can be formed by eliminating a number of independent parameters.

Lets $f(x, y, C1, C2, C3 \ldots, Cn) = 0$ is given where it is said that c1...Cn are all independent conctants.

Then we will differentiate the equation n times w.r.t x and get n equation and use them to eliminate and get the required D - equation.

The D - equations represents a family of curves.

The rest of the discussions will be on the solution of D equations

Order and degree:

- Order - the order of the highest differential coefficient
  - The no. Of independent parameters
  - $\frac{d^n y}{dx^n} \rightarrow \textit{the highest value of the n appearing in equation}$
- Degree: the highest power of the highest order differential coefficient
  - In other words consider the equation to be a polynomial in highest order term the degree of that polynomial.

First order first degree

01 February 2021 06:30 PM

- Exact
- Linear & burnoulli

Exact

01 February 2021 06:35 PM

A D - equation of the form *M(x, y)dx + N(x, y)dy = 0*
Is exact or total if LHS is exact differential of some function in x & y
i.e. *du = M.dy +N.dx*

Then the solution becomes $\mathrm{u(x, y) + c = 0}$

Working method:

Write the equation in the following form
M(x, y)dx + N(x, y)dy = 0

Check if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If yes then the LHS is exact deferential

The solution is

$$\int_{y-constant} Mdx + \int_{x=0} Ndy = C$$

Some times the LHS is not exact differential

In such cases multiply both side with terms such that it becomes one
This term is called integrating factor

List of some exact differentials

To be completed

Linear

01 February 2021 06:54 PM

A D- equation is called linear if the unknown variable and its derivative occurs only in first degree.

$$\frac{dy}{dx} + P(x).y = Q(x)$$

If p(x) and Q(x) are functions of x only or are constant then the equation is called linear first order equation.

Solution:

$$\frac{dy}{dx} + P(x).y = Q(x)$$

$$multiplying\ both\ side\ by\ e^{\int Pdx}\quad this\ is\ integrating\ factor$$

$$e^{\int Pdx}\ \frac{dy}{dx} + e^{\int Pdx}\ P(x).y = e^{\int Pdx}\ Q(x)$$

$$\frac{d}{dx}\left(y.e^{\int Pdx}\right) = e^{\int Pdx}\ Q(x)$$

$$y.e^{\int Pdx} = \int e^{\int Pdx}\ Q(x)\ \mathrm{dx}$$

Bernoulli's equation

Equations of the form

$$f'(y)\frac{dy}{dx} + P(x).f(y) = Q(x)$$

Is not a linear first order equation but can be converted

Let t = f(y)

$$\frac{dt}{dx} = f'(y)\frac{dy}{dx}$$

Hence the equation changes to

$$\frac{dt}{dx} + P(x).t = Q(x)$$

Now this is a linear first order equation

This sort of equation is called Bernoulli's equation

A typical example is

$$\frac{dy}{dx} + P(x).y = Q(x)y^n$$

Dividing by y^n

$$y^{-n}\frac{dy}{dx} + P(x).y^{1-n} = Q(x)$$

Which is a Bernoulli's equation

First order higher degree

01 February 2021 07:15 PM

A general first order higher degree equation can be written as

f(x, y, P) = 0

Where $P = \frac{dy}{dx}$

A general form can be written as

$$p^n + f_1(x, y)p^{n-1} + f_2(x, y)p^{n-2} + f_3(x, y)p^{n-3} + \cdots + f_n(x, y) = 0$$

This type of equation can be solved by four methods

- Solvable for P
- Solvable for y
- Solvable for x
- Clairaut's equation

Solvable for P

01 February 2021 07:15 PM

The LHS of the general form can be written as n linear forms

$(\mathrm{P} - f_1(x, y))(\mathrm{P} - f_2(x, y))(\mathrm{P} - f_3(x, y)) \dots (\mathrm{P} - f_n(x, y)) = 0$

Solve each linear factors

$(\mathrm{P} - f_1(x, y)) \Rightarrow \phi_1(x, y, c) = 0$

$(\mathrm{P} - f_2(x, y)) \Rightarrow \phi_2(x, y, c) = 0$

.

.

.

$(\mathrm{P} - f_n(x, y)) \Rightarrow \phi_n(x, y, c) = 0$

General solution is

$\phi_1(x, y, c)\phi_2(x, y, c)\phi_3(x, y, c) \dots \phi_n(x, y, c) = 0$

Solvable for y

01 February 2021 07:27 PM

The LHS in general form can be written as

$y = g(x, P)$ ...... A

$\frac{dy}{dx} = F(x, p, \frac{dP}{dx})$

$P = F(x, p, \frac{dP}{dx})$

This equation is linear in P solve it

Let the solution be

$\phi(x, p, C) = 0 \ldots \ldots B$ C is constant

From A and B eliminate P to get the required solution

Solvable for X

01 February 2021 07:27 PM

The LHS in general form can be written as

$x = g(y, P)$ ...... A

$\frac{dx}{dy} = F(y, p, \frac{dP}{dy})$

$\frac{1}{P} = F(y, p, \frac{dP}{dx})$

This equation is linear in P solve it

Let the solution be

$\phi(y, p, C) = 0 \dots \dots B$ C is constant

From A and B eliminate P to get the required solution

Clairaut's equation

01 February 2021 07:38 PM

The general form is expressible as

$y = P.x + f(p)$

Solution is

$y = c.x + f(c)$

Second order linear

03 February 2021 10:44 AM

A second order linear differential equation is of the form

$$\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = X$$

Here p_1 *and* p_2 are constants and X is either a constant or a function of x only

D $= \frac{dy}{dx}$

f(D) = $(D^2 + p_1 D + p_2)$

The equation may be written as follows

f(D)y = X

When X =0 the equation is called homogenous.

Some points

- *All homogenous linear equations allows for superposition*
- *General solution are those solutions that contain some arbitrary constants otherwise they are particular solutions.*
- *if y=u is a general solution of the equation f(d)y=0 and y=v is a particular solution of f(D) y=X then the general solution of the equation f(D)y=X is y = u + v.*
    - *Here u is complementary function C.F and v is called particular integral P.I*

Solution

03 February 2021 11:11 AM

Homogenous equation

Given this equation

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0$$

We assume a trail solution as

$y = e^{mx}$

$Dy = m\, e^{mx}$

$D^2y = m^2\, e^{mx}$

Substituting in the main equation we get

$(m^2 + p_1 m + p_2)\, e^{mx} = 0$

$\Rightarrow m^2 + p_1 m + p_2 = 0$ *... this is the auxilary equation*

Solving this we get two solutions of m let it be m_1 *and* m_2

Cases

- Roots real
  - General Solution $y = c_1\, e^{m_1 x} + c_2 e^{m_2 x}$
- Roots equal
  - General Solution $y = (c_1 + c_2 x)e^{m_2 x}$
- Roots imaginary
  - General Solution $y = e^{RE(m_2)x}(c_1 \sin(\text{img}(m_2)x) + c_2 \cos(\text{img}(m_2)x))$

Non homogenous equation

Y = C.F + P.I

C.F may be found as above

P.I is found as follows

Inverse operator is an operator with following property

$f(D)\{\frac{1}{f(D)}\}X = X$

$\frac{1}{f(D)}$ *is the inverse operator this is P.I*

- $\frac{1}{D}X = \int X dx$
- $\frac{1}{D-a} = e^{ax}\int X e^{-ax} dx$
- *If* $X = e^{ax}$
  - *U=f(a)*
    Repeat until U !=0
    U=f'(a)
  - *P.I* $= \frac{e^{ax}}{U}$

- *Here P.I =* $(x^n)\frac{e^{ax}}{f'''''' \ldots ntimes(a)}$ *if f(a) as well as derivative at a until n-1 times is 0*

Solution is

$\mathrm{Y} = \mathrm{C.F} + \mathrm{P.I}$

$\mathrm{P.I} = \frac{1}{f(D)}$

Case III. Let $X = \sin(ax+b)$ or $\cos(ax+b)$.

In this case we express the operator function $f(D)$ in function of D^2, s $\phi(D^2)$, or in function of D^2 and D, say $\phi(D^2, D)$.

Then

(i) $\text{P.I.} = \frac{1}{f(D)} sin(ax+b) = \frac{1}{\phi(D^2)} sin(ax+b)$

$$= \frac{sin(ax+b)}{\phi(-a^2)}, \text{ if } \phi(-a^2) \neq 0. \quad \ldots \quad (1$$

D-OPERA... 4-77

(ii) $P.I. = \frac{1}{f(D)} sin(ax+b) = \frac{1}{\phi(D^2, D)} sin(ax+b)$

$$= \frac{1}{\phi(-a^2, D)} sin(ax+b), \text{ if } \phi(-a^2, D) \neq 0. \quad \ldots \quad (17)$$

(iii) $P.I. = \frac{1}{f(D)} sin(ax+b) = \frac{\psi(D)}{\phi(D^2)} sin(ax+b)$

$$= \frac{\psi(D)}{\phi(-a^2)} sin(ax+b), \text{ if } \phi(-a^2) \neq 0. \quad \ldots \quad (18)$$

(iv) $\text{P.I.} = \frac{1}{f(D)} \sin(ax+b) = x\frac{1}{f'(D)} \sin(ax+b), \quad \ldots \quad (19)$

if (i), (ii), (iii) fail

Case IV. Let $X = e^{ax}V$, where V is any function of x.

Then P. I. $= \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$. ... (20)

Proof. Let $U = \frac{1}{f(D+a)} V$

Then $D(e^{ax}U) = e^{ax}DU + ae^{ax}U = e^{ax}(D+a)U$

$$D^2(e^{ax}U) = D\left\{e^{ax}(D+a)U\right\}$$
$$= ae^{ax}(D+a)U + e^{ax}D(D+a)U$$
$$= e^{ax}(D+a)^2U.$$

In this way in general, $D^n(e^{ax}U) = e^{ax}(D+a)^nU$.

Hence $f(D)(e^{ax}U) = e^{ax}f(D+a)^nU$.

$$\therefore \quad f(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\} = e^{ax}V$$

Operating both sides by $\frac{1}{f(D)}$, we get

$$\therefore \quad e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}e^{ax}V.$$

Hence $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.$

Case V. Let $X = xV$, V being any function of x.

$$\text{Then P. I.} = \frac{1}{f(D)}xV = \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V. \quad \ldots$$

Proof. Let $V_1 = \dfrac{1}{f(D)}V$

Then $D(xV_1) = xDV_1 + V_1$

$$D^2(xV_1) = xD^2V_1 + 2DV_1 = xD^2V_1 + \left(\frac{d}{dD}D^2\right)V_1$$

$$\text{Hence } f(D)(xV_1) = xf(D)V_1 + \frac{d}{dD}f(D)V_1$$

$$= xf(D)V_1 + f'(D)V_1.$$

$$\therefore f(D)\left\{x.\frac{1}{f(D)}V\right\} = xV + f'(D).\frac{1}{f(D)}V$$

$$\text{Thus } x\frac{1}{f(D)}V = \frac{1}{f(D)}(xV) + f'(D).\frac{1}{\{f(D)\}^2}V$$

$$\therefore \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - f'(D).\frac{1}{\{f(D)\}^2}V.$$

$$= \left\{x - \frac{1}{f(D)}f'(D)\right\}\frac{1}{f(D)}V.$$

Case VI. If $X = x^n.V$, then

$$\text{P. I.} = \left\{x - \frac{1}{f(D)}f'(D)\right\}^n \frac{1}{f(D)}V$$

where n is a +ve integer.

Last notes

$D^2 + aD + b = 1/b\,(1 + (D^2 + aD)/b)^{-1}$

extra

05 March 2021 07:41 AM

Choice Rules for the Method of Undetermined Coefficients

(a) **Basic Rule.** *If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (4).*

(b) **Modification Rule.** *If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).*

(c) **Sum Rule.** *If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.*

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$. (Verify!)

The method is self-correcting. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Let us illustrate Rules (a)–(c) by the typical Examples 1–3.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

EXAMPLE 1 Application of the Basic Rule (a)

Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5. \tag{5}$$

Solution. *Step 1. General solution of the homogeneous ODE.* The ODE $y'' + y = 0$ has the general solution

$$y_h = A \cos x + B \sin x.$$

Step 2. Solution y_p of the nonhomogeneous ODE. We first try $y_p = Kx^2$. Then $y_p'' = 2K$. By substitution, $2K + Kx^2 = 0.001x^2$. For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides; thus $K = 0.001$ and $2K = 0$, a contradiction.

The second line in Table 2.1 suggests the choice

$$y_p = K_2x^2 + K_1x + K_0. \qquad \text{Then} \qquad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have $K_2 = 0.001$, $K_1 = 0$, $2K_2 + K_0 = 0$. Hence $K_0 = -2K_2 = -0.002$. This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A \cos x + B \sin x + 0.001x^2 - 0.002.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \qquad \text{and} \qquad y'(0) = B = 1.5.$$

Still extra

05 March 2021 07:44 AM

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

(3) $$y'' + p(x)y' + q(x)y = 0$$

on I, and W is the Wronskian of y_1, y_2,

(4) $$W = y_1 y_2' - y_2 y_1' \qquad \text{(see Sec. 2.6).}$$

CAUTION! The solution formula (2) is obtained under the assumption that the ODE is written in standard form, with y'' as the first term as shown in (1). If it starts with $f(x)y''$, divide first by $f(x)$.

Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution. A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$\begin{aligned} y_p &= -\cos x \int \sin x \sec x\, dx + \sin x \int \cos x \sec x\, dx \\ &= \cos x \ln|\cos x| + x \sin x \qquad \text{(Fig. 70)} \end{aligned}$$

Figure 70 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. (Recall from Sec. 2.8 that we have seen $x \sin x$ in connection with resonance, except for notation.) From y_p and the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous ODE we obtain the *answer*

$$y = y_h + y_p = (c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x.$$

Had we included integration constants $-c_1, c_2$ in (2), then (2) would have given the additional $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case. ■

Basix

01 February 2021 09:26 PM

If we have n equation each of same n variables
Say

$$f_1(x_1 x_2 .... x_n), f_2(x_1 x_2 .... x_n), f_3(x_1 x_2 .... x_n), \ldots\ f_n(x_1 x_2 .... x_n)$$

Then the Jacobian of these function J is written as

$$\frac{\partial(f_1, f_2, f_3, \ldots, f_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)} \text{ or } J\left(\frac{f_1, f_2, f_3 \ldots, f_n}{x_1, x_2, x_3, \ldots, x_n}\right)$$

Is the following determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Chain rule

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \frac{\partial(f_1, f_2)}{\partial(z_1, z_2)} \cdot \frac{\partial(z_1, z_2)}{\partial(x_1, x_2)}$$

Basix

01 February 2021 09:41 PM

A function *f(x, y, z,...)* is homogenous function of degree n if
f(tx, ty, tz,....) = t^n *f(x, y, z,...)*

Euler's theorem

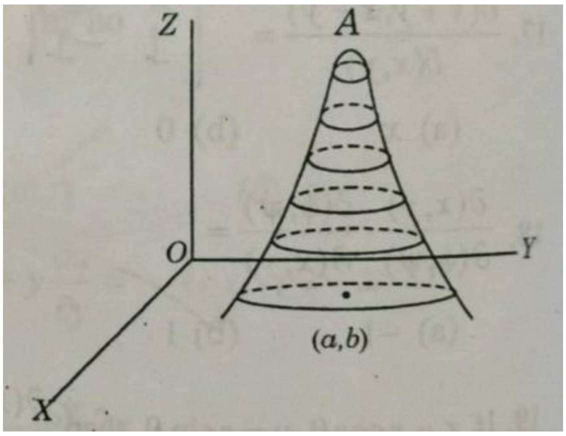
If f(x, y, z) be a homogenous function of degree n than

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)$$

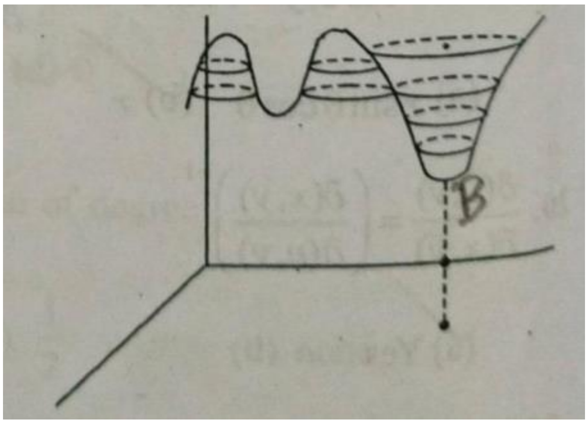
Basix

01 February 2021 09:50 PM

Maxima: a function *f(x, y)* is said to have a local maxima at (a, b) if *f(a, b) ≥ f(x, y) for all points (x, y) in an open circle centered at (a, b).*



Minima: a function *f(x, y)* is said to have a local minima at (a, b) if *f(a, b) ≤ f(x, y) for all points (x, y) in an open circle centered at (a, b).*



Theorem: if a *function f(x, y)* maxima or a minima exist at a point (a, b) where function is defined

And if the partial derivatives W.R.T x and y exist at that point they must be zero.

Conditions for extrema at a point:

- $f_x = 0$ and $f_y = 0 \rightarrow critical\ point$
  - $f_{xx} < 0\ and\ f_{xx}f_{yy}\ -\ {f_{xy}}^2 > 0 \rightarrow maxima$
  - $f_{xx} > 0\ and\ f_{xx}f_{yy}\ -\ {f_{xy}}^2 > 0 \rightarrow minima$
  - $f_{xx}f_{yy}\ -\ {f_{xy}}^2 < 0 \rightarrow neither\ minima\ nor\ maxima\ saddle\ point$
  - $f_{xx}f_{yy}\ -\ {f_{xy}}^2 = 0 \rightarrow inconculsive$

Alternately

- *df*= 0
  - $d^2f < 0\ \rightarrow maxima$
  - $d^2f > 0 \rightarrow minima$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$d^2f = (\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\,)^2\ f$$

Lagrange multiplier

02 February 2021 12:02 PM

Let u = f(x, y, z) be the a function constrained by 2 functions ϕ_1 ϕ_2
Let F be function given by

F(x, y, z) = *f(x, y, z)* + $\lambda_1\ \phi_1(x, y, z) +\ \lambda_2\ \phi_2(x, y, z)$

This $\lambda_1\ \lambda_2$ *are called lagrange multipliers*

Obtain the following equations

$$\frac{\partial F}{\partial x} = 0$$
$$\frac{\partial F}{\partial z} = 0$$
$$\frac{\partial F}{\partial z} = 0$$

Using the above three equation and $\phi_1\ \phi_2$ *we can get* x, y, z *and* $\lambda_1\ \lambda_2$
The value(S) of x, y, z corresponds to critical point.

Whether maxima minima or saddle depends upon $d^2 f$

Basix

02 February 2021 01:46 PM

THE GRADIENT. Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the *gradient* of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\boxed{\nabla\phi\cdot\mathbf{a}}$ and is called the *directional derivative* of ϕ in the direction $\mathbf{a}$. Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\mathbf{a}$.

THE DIVERGENCE. Let $\mathbf{V}(x,y,z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. $\mathbf{V}$ defines a differentiable vector field). Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$, is defined by

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note the analogy with $\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$. Also note that $\nabla\cdot\mathbf{V} \neq \mathbf{V}\cdot\nabla$.

THE CURL. If $\mathbf{V}(x,y,z)$ is a differentiable vector field then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined by

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Direction derivative

02 February 2021 03:59 PM

$\nabla_{\underset{v}{\rightarrow}} f$ called as a directional derivative of f in the direction of $\underset{v}{\rightarrow}$

$\nabla_{\underset{v}{\rightarrow}} f = \underset{v}{\rightarrow} \, . \, \nabla$

Theoreams

05 March 2021 11:08 AM

THE DIVERGENCE THEOREM OF GAUSS states that if V is the volume bounded by a closed surface S and $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S.

STOKES' THEOREM states that if S is an open, two-sided surface bounded by a closed, non-intersecting curve C (simple closed curve) then if $\mathbf{A}$ has continuous derivatives

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called *positive* if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

GREEN'S THEOREM IN THE PLANE. If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$