# Fast Random Rotation Matrices

James Arvo

Program of Computer Graphics

Cornell University

In a previous gem [Arvo 91] I described a method for generating random rotation matrices based on random unit quaternions. As Ken Shoemake points out in his gem [Shoemake 92], that algorithm was flawed in that it did not generate *uniformly distributed* rotation matrices. For a method based on quaternions that corrects this defect, with only slightly more computation, see his algorithm. In this gem I describe a completely different approach to solving the same problem which has the additional benefit of being slightly *faster* than the previous method. The approach is based on the following fact:

*To generate uniformly distributed random rotations of a unit sphere, first perform a random rotation about the vertical axis, then rotate the north pole to a random position.*

The first step of this prescription is trivial. Given a random number, $x_1$, between 0 and 1, the matrix $R$ below does the trick:

$$R = \begin{bmatrix} \cos(2\pi x_1) & \sin(2\pi x_1) & 0 \\ -\sin(2\pi x_1) & \cos(2\pi x_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Here we are assuming that the $z$-axis is the vertical axis, so the "north pole" will be the point $z = (0, 0, 1)$. The second operation is not quite so obvious, but fortunately it can be carried out quite efficiently. Observe that we can take the point $z$ to any other point $p$ on the sphere via a reflection through the plane orthogonal to the line $\overline{zp}$ and containing the origin. Such a reflection is given by the *Householder matrix*

$$H = I - 2vv^{\mathrm{T}} \tag{2}$$

where $v$ is a unit vector parallel to $\overline{zp}$ (See, for instance, [Golub 85]). To turn this into a rotation we need only apply one more reflection (making the determinant positive). A convenient reflection for this purpose is reflection through the origin; that is, scaling by $-1$. Thus, the final rotation matrix can be expressed as the product

$$M = -HR \tag{3}$$

where $R$ is the simple rotation in equation 1. The rotation matrix $M$ will be uniformly distributed within SO(3), the set of all rotations in 3-space, if $H$ takes the north pole to every point on the sphere with equal probability density. This will hold if the image of $z$ under the random reflection is such that both its animuthal angle and the cosine of its polar angle are uniformly distributed. The matrix $H$ in equation 2 will satisfy these requirements if we let

$$v = \begin{bmatrix} \cos(2\pi x_2)\sqrt{x_3} \\ \sin(2\pi x_2)\sqrt{x_3} \\ \sqrt{1 - x_3} \end{bmatrix} \tag{4}$$

where $x_2$ and $x_3$ are two independent uniform random variables in $[0, 1]$. To show this we need only compute $p = Hz$ and verify that $p$ is distributed appropriately. Using the above definition of $v$, we have

$$p = z - 2vv^{\mathrm{T}}z = \begin{bmatrix} -2\cos(2\pi x_2)\sqrt{x_3(1-x_3)} \\ -2\sin(2\pi x_2)\sqrt{x_3(1-x_3)} \\ 2x_3 - 1 \end{bmatrix}. \tag{5}$$

Because the third component of $p$ is the cosine of its polar angle, we see immediately that it is uniformly distributed over $[-1, 1]$, as required. Similarly, from its first two components we see that the azimuthal angle of $p$ is $2\pi x_2$, which is uniformly distributed over $[0, 2\pi]$.

The complete algorithm combining the reflection and simple rotation is shown in Figure 1, and an optimized version in C appears in the appendix. Procedure "random_rotation" requires three uniformly distributed random numbers between 0 and 1. Supplying these values as arguments has several advantages. First, the procedure can be used in conjunction with your favorite pseudo-random number generator, and there are a great many to choose from. Secondly, if we obtain the three random numbers by *stratified* or *jittered* sampling of the unit cube, the resulting rotation matrices will inherit the benefits; namely, less clumping. Finally, if we restrict the range of the random input variables (while maintaining their uniformity), we can generate uniformly distributed perturbations or "wobbles" within given limits.

Figure 2a shows the result of applying 1000 random rotations to a sphere with an arrow painted at one pole. The resulting pattern looks much the same from any vantage point, providing visual confirmation of uniformity. Figure 2b was generated by restricting $x_1$ and $x_3$ to the range $[0, 0.1]$.

# References


[Arvo 91] Arvo, James, "Random Rotation Matrices," *Graphics Gems II*, Academic Press, 1991.

[Golub 85] Golub, Gene H., and Charles F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, MD, 1985.

[Shoemake 92] Shoemake, Ken, "Uniform Random Rotations," *Graphics Gems III*, Academic Press, 1992.