Principle Component Analysis – Solved Example

- Given the data in Table, reduce the dimension from 2 to 1 using the Principal Component Analysis (PCA) algorithm.

Feature	Example 1	Example 2	Example 3	Example 4
X_1	4	8	13	7
X_2	11	4	5	14

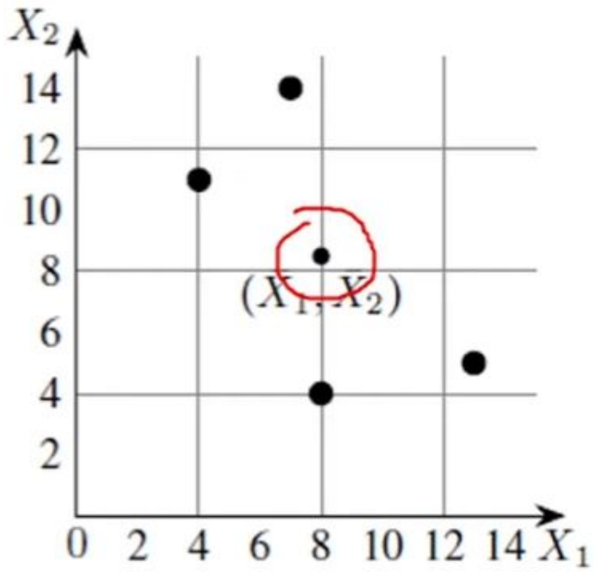
Principle Component Analysis – Solved Example

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

Step 1: Calculate Mean

$$\bar{X}_1 = \frac{1}{4}(4 + 8 + 13 + 7) = 8,$$

$$\bar{X}_2 = \frac{1}{4}(11 + 4 + 5 + 14) = 8.5.$$



Principle Component Analysis – Solved Example

Step 2: Calculation of the covariance matrix.

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$S = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$$

$\overline{X_1} = 8$

$\overline{X_2} = 8.5$

$$\mathrm{Cov}(X_1, X_1) = \frac{1}{N-1}\sum_{k=1}^{N}(X_{1k} - \bar{X}_1)(X_{1k} - \bar{X}_1)$$

$$= \tfrac{1}{3}\left((4-8)^2 + (8-8)^2 + (13-8)^2 + (7-8)^2\right)$$

$$= 14$$

Principle Component Analysis – Solved Example

Step 2: Calculation of the covariance matrix.

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
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$$S = \begin{bmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$\text{Cov}(X_1, X_2) = \frac{1}{N-1} \sum_{k=1}^{N} (X_{1k} - \bar{X}_1)(X_{2k} - \bar{X}_2)$$

$$= \tfrac{1}{3}((4-8)(11-8.5) + (8-8)(4-8.5)$$

$$+ (13-8)(5-8.5) + (7-8)(14-8.5)$$

$$= -11$$

Principle Component Analysis – Solved Example

Step 2: Calculation of the covariance matrix.

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X_1	4	8	13	7
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$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$$

$$\mathrm{Cov}(X_2, X_1) = \mathrm{Cov}(X_1, X_2)$$

$$= -11$$

$$\mathrm{Cov}(X_2, X_2) = \frac{1}{N-1} \sum_{k=1}^{N} (X_{2k} - \bar{X}_2)(X_{2k} - \bar{X}_2)$$

$$= \tfrac{1}{3}\left((11-8.5)^2 + (4-8.5)^2 + (5-8.5)^2 + (14-8.5)^2\right)$$

$$= 23$$

Principle Component Analysis – Solved Example

Step 2: Calculation of the covariance matrix.

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$S = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

Principle Component Analysis – Solved Example

Step 3: Eigenvalues of the covariance matrix

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

The characteristic equation of the covariance matrix is,

$$0 = \det(S - \lambda I)$$

$$= \begin{vmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{vmatrix}$$

$$= (14-\lambda)(23-\lambda) - (-11)\times(-11)$$

$$= \lambda^2 - 37\lambda + 201$$

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}$$

Principle Component Analysis – Solved Example

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

Step 3: Eigenvalues of the covariance matrix

The characteristic equation of the covariance matrix is,

$$0 = \det(S - \lambda I)$$

$$= \begin{vmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{vmatrix}$$

$$= (14-\lambda)(23-\lambda) - (-11) \times (-11)$$

$$= \lambda^2 - 37\lambda + 201$$

$$\lambda = \frac{1}{2}(37 \pm \sqrt{565})$$

$$= 30.3849,\ 6.6151$$

$$= \lambda_1,\ \lambda_2 \quad \text{(say)}$$

$$\overline{X_1} = \mathbf{8}$$

$$\overline{X_2} = \mathbf{8.5}$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (S - \lambda I)\, U$$

$$= \begin{bmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} (14-\lambda)u_1 - 11u_2 \\ -11u_1 + (23-\lambda)u_2 \end{bmatrix}$$

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

Step 4: Computation of the eigenvectors

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (S - \lambda I)\,U$$

$$= \begin{bmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} (14-\lambda)u_1 - 11u_2 \\ -11u_1 + (23-\lambda)u_2 \end{bmatrix}$$

$$(14-\lambda)u_1 - 11u_2 = 0$$

$$-11u_1 + (23-\lambda)u_2 = 0$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{u_1}{11} = \frac{u_2}{14 - \lambda} = t$$

$$u_1 = 11t, \quad u_2 = (14 - \lambda)t$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{u_1}{11} = \frac{u_2}{14 - \lambda} = t$$

$$u_1 = 11t, \quad u_2 = (14 - \lambda)t$$

$$U_1 = \begin{bmatrix} 11 \\ 14 - \lambda \end{bmatrix}.$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

- To find the first principal components, we need only compute the eigenvector corresponding to the largest eigenvalue.
- In the present example, the largest eigenvalue is λ_1.
- So, we compute the eigenvector corresponding to λ_1.

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

$$U_1 = \begin{bmatrix} 11 \\ 14 - \lambda_1 \end{bmatrix}.$$

- To find a unit eigenvector, we compute the length of U_1 which is given by,

$$\begin{aligned} \|U_1\| &= \sqrt{11^2 + (14 - \lambda_1)^2} \\ &= \sqrt{11^2 + (14 - 30.3849)^2} \\ &= 19.7348 \end{aligned}$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

$$U_1 = \begin{bmatrix} 11 \\ 14 - \lambda_1 \end{bmatrix}.$$

- To find a unit eigenvector, we compute the length of U_1 which is given by,

$$\|U_1\| = \sqrt{11^2 + (14 - \lambda_1)^2}$$
$$= \sqrt{11^2 + (14 - 30.3849)^2}$$
$$= 19.7348$$

$$e_1 = \begin{bmatrix} 11/\|U_1\| \\ (14 - \lambda_1)/\|U_1\| \end{bmatrix}$$
$$= \begin{bmatrix} 11/19.7348 \\ (14 - 30.3849)/19.7348 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

Principle Component Analysis – Solved Example

Step 4: Computation of the eigenvectors

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$U_1 = \begin{bmatrix} 11 \\ 14 - \lambda_1 \end{bmatrix}.$$

- To find a unit eigenvector, we compute the length of U_1 which is given by,

$$\|U_1\| = \sqrt{11^2 + (14 - \lambda_1)^2}$$
$$= \sqrt{11^2 + (14 - 30.3849)^2}$$
$$= 19.7348$$

$$e_1 = \begin{bmatrix} 11/\|U_1\| \\ (14 - \lambda_1)/\|U_1\| \end{bmatrix} = \begin{bmatrix} 11/19.7348 \\ (14 - 30.3849)/19.7348 \end{bmatrix} = \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0.8303 \\ 0.5574 \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

Step 5: Computation of first principal components

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$e_1^T \begin{bmatrix} X_{1k} - \bar{X}_1 \\ X_{2k} - \bar{X}_2 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$e_1^T \begin{bmatrix} X_{1k} - \bar{X}_1 \\ X_{2k} - \bar{X}_2 \end{bmatrix} = \begin{bmatrix} 0.5574 & -0.8303 \end{bmatrix} \begin{bmatrix} X_{11} - \bar{X}_1 \\ X_{21} - \bar{X}_2 \end{bmatrix}$$

$$= 0.5574(X_{11} - \bar{X}_1) - 0.8303(X_{21} - \bar{X}_2)$$

$$= 0.5574(4 - 8) - 0.8303(11 - 8,5)$$

$$= -4.30535$$

$$e_2 = \begin{bmatrix} 0.8303 \\ 0.5574 \end{bmatrix}$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

Step 5: Computation of first principal components

Feature	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14
First Principle Components	-4.3052	3.7361	5.6928	-5.1238

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

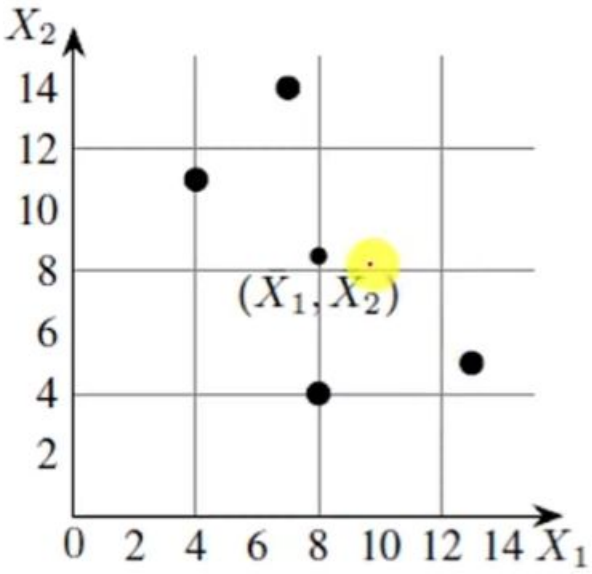
$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

Step 6: Geometrical meaning of first principal components

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14



$$e_1 = \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$e_2 = \begin{bmatrix} 0.8303 \\ 0.5574 \end{bmatrix} \quad S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

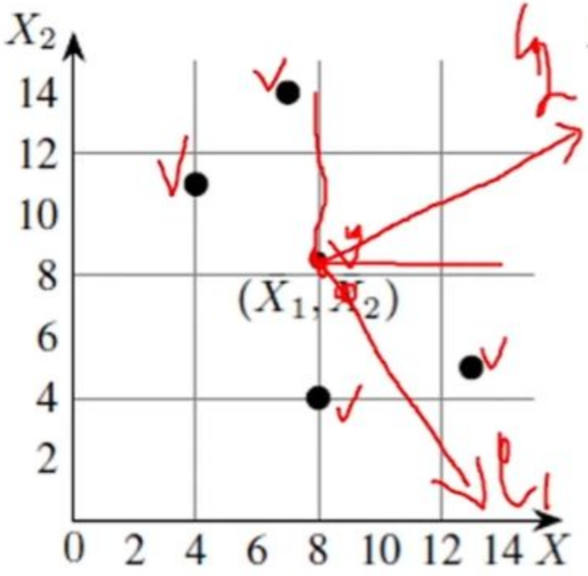
$$\lambda_1 = 30.3849$$

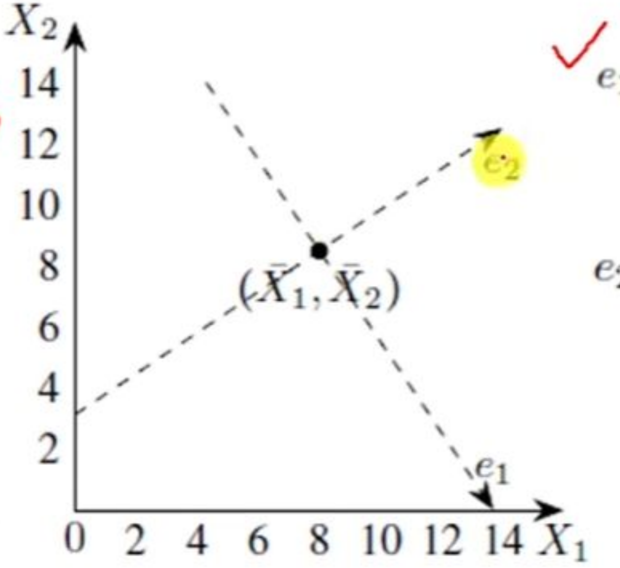
$$\lambda_2 = 6.6151$$

Principle Component Analysis – Solved Example

Step 6: Geometrical meaning of first principal components

F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14





$$e_1 = \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

$$\overline{X_1} = 8$$

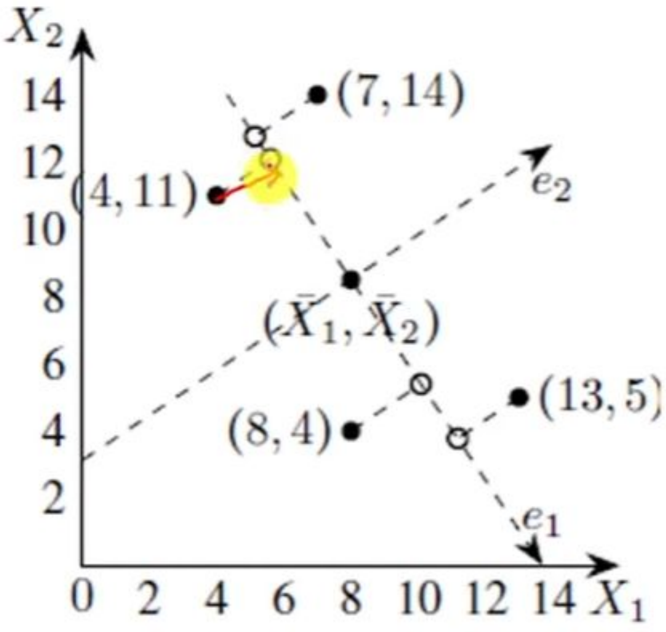
$$\overline{X_2} = 8.5$$

$$e_2 = \begin{bmatrix} 0.8303 \\ 0.5574 \end{bmatrix} \quad S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$

Step 6: Geometrical meaning of first principal components



F	Ex 1	Ex 2	Ex 3	Ex 4
X_1	4	8	13	7
X_2	11	4	5	14

$$e_1 = \begin{bmatrix} 0.5574 \\ -0.8303 \end{bmatrix}$$

$$\overline{X_1} = 8$$

$$\overline{X_2} = 8.5$$

$$e_2 = \begin{bmatrix} 0.8303 \\ 0.5574 \end{bmatrix} \quad S = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

$$\lambda_1 = 30.3849$$

$$\lambda_2 = 6.6151$$