On Solving the Maximum Flow Problem with Conflict Constraints

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Abstract—The Maximum Flow Problem with Conflict Constraints is a generalization that adds conflict constraints to a classical optimization problem on networks used to model several real-world applications. In the last few years several approaches, both heuristic and exact, have been proposed to attack the problem. In this paper we consider a mixed integer linear program and solve it with an open-source solver. Computational results on the benchmark instances commonly used in the literature of the problem are reported. All the 160 instances benchmark instances normally used in the literature are solved to optimality for the first time, with 28 instances closed for the first time. Moreover, in the process, 6 improvements to the best-known heuristic solutions are also found.

Index Terms—maximum flow, conflict constraints, exact solutions, heuristic solutions

I. Introduction

The study of maximum flow problems in network optimization has a long history starting in the 1950s [1], with research driven by real applications in the transportation and communication fields. In this paper we consider the Maximum Flow Problem with Conflicts (MFPC), which consists of retrieving a maximum flow from a given source to a given sink in a given network, while complying with classical capacity constraints of the connections, but also preventing positive flow from being contemporarily carried out by the arcs of any conflicting pair. Notice that the latter restriction makes the problem considerably harder than the original version.

The MFPC was first studied in [2], where the authors considered both positive disjunctive constraints (at least one of two arcs has to be selected)and negative disjunctive constraints (at most one of two arcs can be selected), showing the NP-hardness of all the resulting variants. Later, the use of negative disjunctive constraints to model conflicts between arc pairs in order to model real-world applications, was studied in [3], where some mixed-integer linear programming formulations and exact methods for the problem such as a Benders Decomposition, a Branch & Bound and a Russian Doll Search were provided. More recently, the authors of [4] proposed a heuristic approach for the problem called Kernousel, obtained by merging two metaheuristic approaches, one based on Carousel Greedy [5] and the other on Kernel Search [6].

Applications of the MFPC can be found in the routing of messages on telecommunication networks, where conflicts might be used to avoid the use of links of contractually-incompatible providers [7], in the optimization of personnel scheduling [8], in transportation, where conflicts constraints might be used to avoid paying alternative (cumulative) tolls while solving routing applications [7]. Other more specific applications can be found in the image segmentation domain [9].

In this paper a mixed integer linear programming model for the problem is considered and attacked by the open-source solver CP-SAT, part of the Google OR-Tools [10] suite for optimization. The experimental results on the benchmark instances previously adopted in the literature improve the state-of-the-art, with several improved heuristic solutions retrieved in very short time, and all the instances solved to proven optimality for the first time. This achievement is not unexpected, being in line with some recent results obtained on other combinatorial optimization problems [11], [12], [13].

The paper is organized as follows. In Section II the Maximum Flow Problem with Conflict Constraints is formally introduced and defined, while in Section III the mixed integer linear programming model adopted is discussed. Section IV is devoted to the description of the benchmark instances previously adopted in the literature and to the presentation of the experimental results obtained by the new model we propose. Finally, conclusions about the work presented are drawn in Section V.

II. Problem Description

The Maximum Flow Problem with additional Conflict Constraints can be defined on a directed graph $G = (V, A)$, with V denoting the set of nodes and A the set of arcs. The node set includes a source node s and a sink node t, while each arc $(i, j) \in A$ is associated with a capacity $u_{ij} \in \mathbb{N}^+$, indicating the maximum amount of flow that can be sent from node i to node j through the direct connection between them.Moreover, two arcs $(i, j), (k, l) \in A$ are said to be *in conflict* if at most one of them can have positive flow in a feasible solution. The set of arcs in conflict with the arc $(i, j) \in A$ is defined by the set $\delta(i, j)$. The objective of the MFPC is to identify the maximum flow from s to t that satisfies the capacity and

the conflict constraints, and to route such a flow through the network.

A small example of an MFPC instance (taken from [4]) and an optimal solution for the instance are provided in Figure 1.

III. A MIXED INTEGER LINEAR PROGRAMMING MODEL

In this section a model for the MFPC [3] is described. A variable $f_{ij} \in \mathbb{Z}_0^+$ is introduced for each $(i,j) \in A$, and represents the flow carried by the arc in the solution under investigation. In order to model conflicts, a second set of variables is introduced. For each $(i,j) \in A$, a binary variable x_{ij} is considered, and takes value 1 if $f_{ij} > 0$, 0 otherwise. Finally, a variable $z \in \mathbb{Z}_0^+$ is introduced to capture the total flow from s to t passing through the network in the solution under investigation. The resulting model is as follows.

$$\max \quad z \tag{1}$$

$$s.t. \quad \sum_{(j,i)\in A} f_{ji} - \sum_{(i,j)\in A} f_{ij} = \begin{cases} -z & \text{if } i = s \\ +z & \text{if } i = r \\ 0 & \text{otherwise} \end{cases} \quad i \in V \tag{2}$$

$$f_{ij} \leq u_{ij}x_{ij} \qquad\qquad (i,j) \in A \tag{3}$$

$$x_{ij} + x_{kl} \leq 1 \qquad (i,j) \in A, (k,l) \in \delta_{ij} \tag{4}$$

$$f_{ij} \geq 0 \qquad\qquad (i,j) \in A \tag{5}$$

$$x_{ij} \in \{0,1\} \qquad\qquad (i,j) \in A \tag{6}$$

The objective function (1) maximizes the total flow from s to t. Constraints (2) deal with flow conservation and set z at the same time. Notice that these constraints are the core, together with (5), of the classic formulation for the Maximum Flow Problem. Inequalities (3) regulate the activation of x variables as soon as the respective f variables carry some positive flow. Constraints (4) model conflicts and forbid conflicting edges to be active at the same time. Finally the domains of the variables are specified in constraints (5) and (6).

IV. COMPUTATIONAL EXPERIMENTS

The benchmark instances adopted in the literature are introduced in Section IV-A, while a comparison of the results achieved in the present work with those of the methods previously appeared in the literature, is presented in Section IV-B.

A. Benchmark Instances

The computational experiments were conducted on the collection of benchmark instances originally proposed in Şuvak et al. [3]. By construction, each of these instances is constructed to have at least one feasible solution with a non-zero flow, and the characteristics of each instance depend on four parameters:

- n, the number of nodes of the graph. Parameter n takes value from the set $\{40, 50, 60, 70, 80\}$;
- p. the arc density, with $p = mn(n-1)$, where m denotes the number of arcs of the graph. Parameter p takes value from the set $\{0.3, 0.4, 0.5, 0.6\}$;
- d, the conflict density, $d = 2wm(m-1)$, with w denoting the number of conflicting arc pairs. Parameter d takes value from the set $\{0.3, 0.4, 0.5, 0.6\}$;

- I, defining the arc capacity range. In setting $I = 1$ the capacities are take from the interval $[10, 15]$, in setting $I = 2$ the capacities are take from the interval $[15, 20]$.

The detailed characteristics of the 160 instances obtained by combining the different possible values of the parameters can be found [4].

B. Experimental Results

The model discussed in Section III has been solved with the Google OR-Tools CP-SAT solver [10] version 9.12. The experiments have been run on a computer equipped with an Intel Core i7 12700F CPU. The results are compared with those of the methods previously appeared in the literature. Namely, the exact methods discussed in [3] (Benders Decomposition, *BD*, Branch & Bound, *B&B*; Russian Doll Search, *RDS*) and the three metaheuristics presented in [4] (Carousel Greedy, *CG*; Kernel Search, *KS*; Kernousel, *KO*). The computation times of all the metaheuristic methods were obtained on a computer equipped with an Intel Core i7-3770 CPU, while the results of the exact methods were obtained on a normalized time of 2840.51 seconds on the same machine [4]. Therefore, in order to have a fair comparison, we present our computation times normalized to the same processor, according to the ratio based on the Whetstone benchmark provided at http://gene.disi.unitn.it/test/cpu_list.php.

The results are presented in Table I where for each instance (*ID*) the best-known (*BK*) values for lower (*LB*) and upper (*UB*) bounds before the present study are reported. For each exact method the percentage gap (*Gap*) from the best known lower and upper bounds obtained on several runs of maximum 2840.51 seconds are presented. The gap for the lower bounds is calculated as $100 \cdot \frac{BK_{LB}-LB}{BK_{LB}}$, where BK_{LB} is the best-known lower bound, and LB is the value provided by the method under investigation. Analogously, the gap for the upper bound is obtained as $100 \cdot \frac{UB-BK_{UB}}{BK_{UB}}$. For each of the metaheuristic approaches we report the percentage gap for the upper bound (*Gap UB*) and the computation time (*Sec*). Finally, for the new approach discussed in this paper (CP-SAT) we report the optimal solution retrieved (*Opt*), the time required to the retrieve the optimal solution (*Sec heu*) – useful for a comparison with the metaheuristics methods – and the optimality-proof time (*Sec opt*). Improved bounds and newly closed instances by CP-SAT are highlighted.

The results of Table I show that CP-SAT is able to improve the state-of-the-art results on the benchmark instances commonly adopted in the literature. In detail, all the instances are closed, with optimality proven for the first time for 28 of the 160 instances, with substantial improvements on the upper bounds. In 6 cases the value of the previously best-known heuristic solutions is improved, in some cases significantly. In terms of computation times, exact solutions are provided on average in about 200 seconds, with only two instances requiring more than 1 hour, and none above 2 hours. These computation times are already of the same order of magnitude as some of the heuristic techniques considered. When the time to retrieve the optimal solution, but without proving optimality
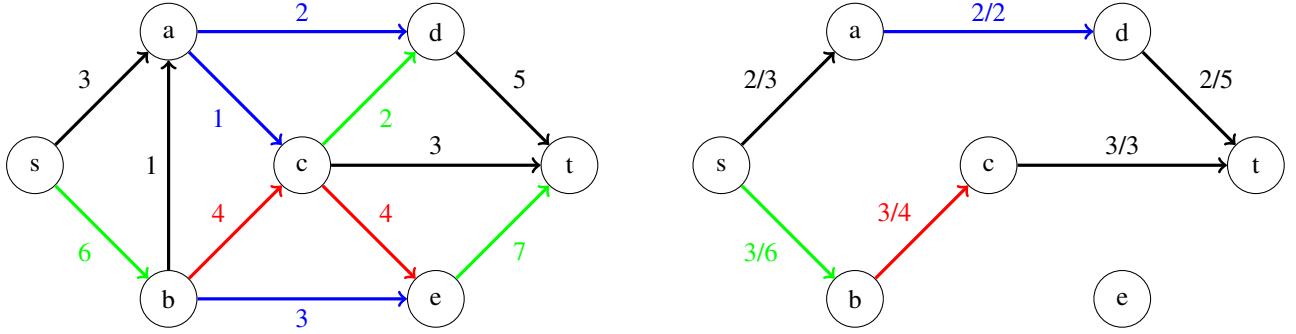
Fig. 1: On the left an example of a small MFPC instance is shown, where the arc capacities are placed by the arcs. Arcs in black are not affected by conflicts while any two arcs sharing the same red, blue or green color are in conflict. On the right a feasible solution carrying the maximal possible flow from s to t without violating any conflict is shown.

(column *Sec heu*) is considered, the time required is aligned with (and often shorter than) that of the other heuristics, notwithstanding the higher quality of the solutions provided. When compared with the previous exact methods, the results are clearly improved.

V. CONCLUSIONS

A mixed integer linear formulation for the Maximum Flow Problem with Conflict Constraints is considered, and solved via an open-source solver.

The results of an experimental campaign, run on the benchmark instances adopted in the previous literature, are reported. The results indicate that the proposed approach is able to improve 6 best-known results and to find for the first time a proven optimal solution for all the 160 instances considered in the previous literature, with 28 instances closed for the first time.

REFERENCES

[1] J. L. Ford and D. Fulkerson, "Maximal flow through a network," *Canadian Journal of Mathematics*, vol. 8, pp. 399–404, 1956.

[2] U. Pferschy and J. Schauer, "The maximum flow problem with disjunctive constraints," *Journal of Combinatorial Optimization*, vol. 26, no. 1, pp. 109–119, 2013.

[3] Z. Şuvak, I. K. Altınel, and N. Aras, "Exact solution algorithms for the maximum flow problem with additional conflict constraints," *European Journal of Operational Research*, vol. 287, no. 2, pp. 410–437, 2020.

[4] F. Carrabs, R. Cerulli, R. Mansini, D. Serra, and C. Sorgente, "Hybridizing carousel greedy and kernel search: A new approach for the maximum flow problem with conflict constraints," *European Journal of Operational Research*, 2025.

[5] C. Cerrone, R. Cerulli, and B. Golden, "Carousel greedy: A generalized greedy algorithm with applications in optimization," *Computers and Operations Research*, vol. 85, pp. 97–112, 2017.

[6] E. Angelelli, R. Mansini, and M. G. Speranza, "Kernel search: A new heuristic framework for portfolio selection," *Computational Optimization and Applications*, vol. 51, no. 1, pp. 345–361, 2012.

[7] A. Schrijver, "On the history of the transportation and maximum flow problems," *Math. Program.*, vol. 91, pp. 437–445, 2002.

[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Eds., *Introduction to Algorithms*, 3rd ed. MIT Press, 2009.

[9] J. Kleinberg and E. Tardos, *Algorithm Design*. Addison Wesley, 2006.

[10] L. Perron and F. Didier, "Google OR-Tools - CP-SAT," Google, 2025, https://developers.google.com/optimization/cp/cp_solver/ [Accessed: 2025-03-18].

[11] R. Montemanni and M. Dell'Amico, "Solving the parallel drone scheduling traveling salesman problem via constraint programming," *Algorithms*, vol. 16, no. 1, p. 40, 2023.

[12] R. Montemanni, M. Dell'Amico, and A. Corsini, "Parallel drone scheduling vehicle routing problems with collective drones," *Computers & Operations Research*, vol. 163, p. 106514, 2024.

[13] R. Montemanni, "Solving a home healthcare routing and scheduling problem with real-world features," in *Proceedings of the 9th International Conference on Machine Learning and Soft Computing*. Springer, to appear, 2025.

TABLE I: Computational results.

ID	BK		Previous Exact Methods						Previous Heuristic Methods						New Results				
			BD [3] Gap		B&B [3] Gap		RDS [3] Gap		CG [4]		KS [4]		KO [4]				CP-SAT		Gap
	LB	UB	LB	UB	LB	UB	LB	UB	Gap UB	Sec	Gap UB	Sec	Gap UB	Sec	Opt	Sec heu	Sec opt	LB	UB
1	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	8.0	0.0	4.6	0.0	13.1	37	1.2	1.6	0.0	0.0
2	49	49	0.0	0.0	0.0	0.0	0.0	0.0	24.5	5.7	0.0	3.7	0.0	9.4	49	1.3	1.6	0.0	0.0
3	24	24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0	6.3	0.0	7.6	24	1.7	1.9	0.0	0.0
4	33	33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.6	0.0	4.7	0.0	9.6	33	1.6	1.9	0.0	0.0
5	12	12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.5	0.0	12.1	0.0	7.4	12	2.1	2.8	0.0	0.0
6	34	34	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.5	0.0	4.9	0.0	8.4	34	2.1	2.7	0.0	0.0
7	13	13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.0	0.0	6.7	0.0	8.4	13	2.3	2.8	0.0	0.0
8	15	15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.7	0.0	5.5	0.0	7.7	15	2.3	3.0	0.0	0.0
9	35	35	0.0	0.0	0.0	0.0	0.0	0.0	2.9	6.6	2.9	6.8	0.0	12.7	35	2.2	2.7	0.0	0.0
10	51	51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.0	0.0	19.6	0.0	17.0	51	2.3	7.2	0.0	0.0
11	26	26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.6	0.0	10.6	0.0	14.3	26	3.0	3.9	0.0	0.0
12	50	50	0.0	0.0	0.0	0.0	0.0	0.0	22.0	4.8	0.0	14.8	0.0	14.4	50	3.0	3.8	0.0	0.0
13	13	13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.5	0.0	12.6	0.0	12.9	13	4.0	5.2	0.0	0.0
14	19	19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.3	0.0	10.0	0.0	11.7	19	3.9	4.1	0.0	0.0
15	25	25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.3	0.0	9.7	0.0	14.5	25	4.7	6.2	0.0	0.0
16	19	19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.2	0.0	11.3	0.0	13.5	19	4.9	6.5	0.0	0.0
17	43	43	0.0	0.0	0.0	0.0	0.0	0.0	20.9	13.5	18.6	37.5	11.6	55.7	43	3.8	11.5	0.0	0.0
18	55	55	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.5	0.0	8.2	0.0	19.9	55	3.8	7.7	0.0	0.0
19	33	33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	9.7	3.0	13.2	0.0	23.4	33	5.2	6.7	0.0	0.0
20	35	35	0.0	0.0	0.0	0.0	0.0	0.0	5.7	14.0	5.7	11.3	5.7	26.6	35	5.2	6.2	0.0	0.0
21	27	27	0.0	0.0	0.0	0.0	0.0	0.0	3.7	6.4	0.0	15.8	0.0	21.7	27	6.4	8.4	0.0	0.0
22	34	34	0.0	0.0	0.0	0.0	0.0	0.0	5.9	6.2	0.0	14.5	0.0	19.9	34	6.5	7.7	0.0	0.0
23	15	15	0.0	0.0	0.0	0.0	0.0	0.0	13.3	4.0	6.7	14.0	0.0	17.3	15	7.1	8.9	0.0	0.0
24	34	34	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.6	0.0	13.8	0.0	17.0	34	7.0	8.8	0.0	0.0
25	45	45	0.0	0.0	0.0	0.0	0.0	0.0	8.9	15.3	0.0	14.6	0.0	34.0	45	5.7	19.5	0.0	0.0
26	80	80	0.0	0.0	0.0	0.0	0.0	0.0	13.8	16.9	11.3	26.0	0.0	38.0	80	5.8	14.9	0.0	0.0
27	30	30	0.0	0.0	0.0	0.0	0.0	0.0	16.7	4.7	20.0	38.0	16.7	43.0	30	7.5	16.0	0.0	0.0
28	48	48	0.0	0.0	0.0	0.0	0.0	0.0	31.3	16.8	0.0	17.6	0.0	35.1	48	7.4	13.0	0.0	0.0
29	26	26	0.0	0.0	0.0	0.0	0.0	0.0	7.7	7.6	0.0	17.7	0.0	25.3	26	8.6	10.9	0.0	0.0
30	36	36	0.0	0.0	0.0	0.0	0.0	0.0	2.8	5.5	0.0	20.5	0.0	26.9	36	8.3	10.6	0.0	0.0
31	24	24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.5	0.0	22.6	0.0	27.8	24	9.9	13.1	0.0	0.0
32	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	7.0	0.0	19.2	0.0	26.2	37	9.8	11.9	0.0	0.0
33	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.9	5.4	10.1	0.0	20.5	37	3.2	4.6	0.0	0.0
34	48	48	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.1	2.1	10.7	0.0	21.9	48	3.2	4.5	0.0	0.0
35	24	24	0.0	0.0	0.0	0.0	0.0	0.0	16.7	2.6	8.3	10.0	8.3	13.0	24	4.5	5.2	0.0	0.0
36	33	33	0.0	0.0	0.0	0.0	0.0	0.0	45.5	3.1	0.0	11.1	0.0	15.9	33	4.6	5.8	0.0	0.0
37	21	21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.3	0.0	11.1	0.0	16.5	21	5.9	7.8	0.0	0.0
38	33	33	0.0	0.0	0.0	0.0	0.0	0.0	9.1	5.6	0.0	12.8	0.0	19.2	33	5.8	7.5	0.0	0.0
39	13	13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.0	0.0	12.0	0.0	14.7	13	6.3	8.1	0.0	0.0
40	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.3	0.0	10.5	0.0	15.3	37	6.4	8.2	0.0	0.0
41	37	37	0.0	0.0	0.0	0.0	0.0	0.0	2.7	9.6	24.3	31.0	2.7	31.5	37	6.4	15.0	0.0	0.0
42	53	53	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.6	5.7	20.5	0.0	25.6	53	6.4	9.3	0.0	0.0
43	35	35	0.0	0.0	0.0	0.0	0.0	0.0	2.9	8.5	28.6	23.6	0.0	34.0	35	8.0	14.3	0.0	0.0
44	38	38	0.0	0.0	0.0	0.0	0.0	0.0	2.6	5.0	2.6	18.7	0.0	24.9	38	7.8	9.9	0.0	0.0
45	21	21	0.0	0.0	0.0	0.0	0.0	0.0	42.9	5.2	33.3	26.2	0.0	32.6	21	9.6	11.8	0.0	0.0
46	18	18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	6.3	5.6	16.6	0.0	25.4	18	9.5	12.0	0.0	0.0
47	24	24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	11.1	0.0	22.9	0.0	32.1	24	11.1	13.1	0.0	0.0
48	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	10.1	0.0	25.1	0.0	36.8	37	11.0	14.9	0.0	0.0
49	40	40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	21.5	2.5	35.3	0.0	83.7	40	9.4	24.5	0.0	0.0
50	57	57	0.0	0.0	3.5	15.8	0.0	0.0	3.5	16.2	3.5	35.9	3.5	49.1	57	9.4	22.8	0.0	0.0
51	40	40	0.0	0.0	0.0	0.0	0.0	0.0	5.0	13.6	0.0	29.5	0.0	41.5	40	13.4	23.4	0.0	0.0
52	53	53	0.0	0.0	0.0	0.0	0.0	0.0	0.0	6.5	0.0	27.9	0.0	37.1	53	12.6	24.4	0.0	0.0
53	25	25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	17.8	0.0	29.6	0.0	46.9	25	16.9	26.8	0.0	0.0
54	38	38	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.8	0.0	28.4	0.0	33.2	38	16.5	23.2	0.0	0.0
55	25	25	0.0	0.0	0.0	0.0	0.0	0.0	56.0	5.0	0.0	31.1	0.0	34.5	25	20.5	25.4	0.0	0.0
56	18	18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	5.3	0.0	28.1	0.0	34.6	18	20.2	24.5	0.0	0.0
57	49	67	33.3	37.6	27.3	42.9	30.3	681.6	2.0	34.1	2.0	71.2	0.0	192.3	**49**	14.1	235.5	0.0	**-27.3**
58	73	73	0.0	0.0	0.0	42.5	2.7	575.3	0.0	23.6	0.0	89.6	0.0	153.3	73	13.9	54.0	0.0	0.0
59	38	38	0.0	0.0	0.0	0.0	0.0	0.0	10.5	13.8	2.6	40.3	0.0	67.5	38	20.4	35.5	0.0	0.0
60	51	51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	22.8	3.9	56.1	0.0	96.1	51	20.6	43.5	0.0	0.0
61	39	39	0.0	0.0	0.0	0.0	0.0	0.0	2.6	13.4	2.6	39.6	2.6	53.5	39	27.7	35.1	0.0	0.0
62	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	18.0	0.0	41.9	0.0	61.3	37	27.7	40.3	0.0	0.0
63	26	26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	15.0	11.5	49.4	0.0	75.0	26	34.7	40.8	0.0	0.0
64	33	33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	12.8	0.0	53.8	0.0	66.1	33	35.0	42.6	0.0	0.0
65	36	36	0.0	0.0	0.0	22.2	0.0	0.0	5.6	11.5	27.8	19.8	0.0	32.7	36	7.3	20.5	0.0	0.0
66	53	53	0.0	0.0	0.0	0.0	0.0	0.0	5.7	11.7	5.7	16.0	0.0	29.7	53	7.4	16.2	0.0	0.0
67	22	22	0.0	0.0	0.0	0.0	0.0	0.0	4.5	5.0	0.0	16.7	0.0	24.1	22	8.9	11.1	0.0	0.0
68	34	34	0.0	0.0	0.0	0.0	0.0	0.0	2.9	17.0	5.9	14.3	0.0	33.4	34	9.1	11.0	0.0	0.0
69	27	27	0.0	0.0	0.0	0.0	0.0	0.0	0.0	11.8	0.0	25.5	0.0	38.8	27	11.4	13.7	0.0	0.0
70	33	33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	6.3	0.0	21.1	0.0	32.0	33	11.6	13.8	0.0	0.0
71	12	12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	5.1	0.0	23.8	0.0	32.0	12	13.9	16.3	0.0	0.0
72	19	19	0.0	0.0	0.0	0.0	0.0	0.0	10.5	10.8	5.3	26.3	5.3	41.9	19	13.7	16.5	0.0	0.0
73	39	40	2.5	2.6	2.5	28.2	2.5	446.2	0.0	22.5	10.3	56.6	0.0	108.1	**39**	13.0	36.4	0.0	**-2.5**
74	64	64	0.0	0.0	0.0	4.7	0.0	0.0	0.0	20.4	15.6	40.1	0.0	56.5	64	13.0	32.2	0.0	0.0
75	26	26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	24.8	0.0	27.2	0.0	55.9	26	18.3	23.2	0.0	0.0

(Continued on next page)

TABLE I: Computational results (continued).

| ID | BK | | Previous Exact Methods | | | | | | Previous Heuristic Methods | | | | | | New Results | | | | |
| | | | BD [3] Gap | | B&B [3] Gap | | RDS [3] Gap | | CG [4] | | KS [4] | | KO [4] | | Opt | CP-SAT | | Gap | |
	LB	UB	LB	UB	LB	UB	LB	UB	Gap UB	Sec	Gap UB	Sec	Gap UB	Sec		Sec heu	Sec opt	LB	UB
76	38	38	0.0	0.0	0.0	0.0	0.0	0.0	2.6	11.9	2.6	39.6	2.6	53.0	38	18.7	32.5	0.0	0.0
77	29	29	0.0	0.0	0.0	0.0	0.0	0.0	0.0	24.2	3.4	45.9	0.0	77.0	29	26.2	31.8	0.0	0.0
78	37	37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	19.0	0.0	42.6	0.0	61.9	37	25.4	34.4	0.0	0.0
79	14	14	0.0	0.0	0.0	0.0	0.0	0.0	14.3	7.9	0.0	42.7	0.0	61.7	14	30.7	37.3	0.0	0.0
80	37	37	0.0	0.0	0.0	0.0	0.0	0.0	8.1	7.1	0.0	36.6	0.0	51.2	37	30.5	36.9	0.0	0.0
81	53	61	34.4	37.0	13.1	15.1	34.4	607.5	0.0	51.0	22.6	113.6	0.0	214.7	**53**	24.2	69.9	0.0	**-13.1**
82	68	84	38.1	45.1	19.0	23.5	36.9	538.2	5.9	29.2	1.5	76.3	0.0	172.0	**68**	24.3	69.5	0.0	**-19.0**
83	40	40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	15.0	10.0	89.4	0.0	118.9	40	34.8	60.6	0.0	0.0
84	39	39	0.0	0.0	0.0	17.9	0.0	0.0	0.0	16.3	0.0	64.9	0.0	75.2	39	35.0	54.6	0.0	0.0
85	26	26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	21.3	0.0	78.6	0.0	85.3	26	46.7	65.4	0.0	0.0
86	37	37	0.0	0.0	0.0	0.0	0.0	0.0	5.4	14.6	0.0	60.3	2.7	71.7	37	47.0	58.3	0.0	0.0
87	25	25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	18.8	0.0	77.1	0.0	86.7	25	59.9	70.9	0.0	0.0
88	19	19	0.0	0.0	0.0	0.0	0.0	0.0	5.3	9.5	0.0	59.8	0.0	66.1	19	59.9	71.6	0.0	0.0
89	55	84	36.9	60.0	36.9	52.7	41.7	732.7	1.8	38.6	9.1	179.8	0.0	317.9	**57**	38.6	232.4	**-3.6**	**-32.1**
90	68	103	50.5	101.0	48.5	51.5	47.6	704.4	20.6	38.8	23.5	224.5	0.0	319.7	**70**	38.1	521.7	**-2.9**	**-32.0**
91	41	41	0.0	0.0	0.0	4.9	0.0	0.0	0.0	56.5	0.0	98.2	0.0	170.3	41	58.3	119.7	0.0	0.0
92	53	53	0.0	0.0	0.0	30.2	0.0	588.7	0.0	32.4	0.0	96.2	0.0	142.4	53	58.4	134.3	0.0	0.0
93	27	27	0.0	0.0	0.0	0.0	0.0	0.0	3.7	11.6	0.0	81.5	0.0	103.8	27	79.0	104.2	0.0	0.0
94	53	53	0.0	0.0	0.0	0.0	0.0	0.0	0.0	39.3	0.0	81.5	0.0	122.7	53	79.3	100.6	0.0	0.0
95	24	24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	23.3	0.0	83.4	0.0	107.2	24	89.2	106.8	0.0	0.0
96	33	33	0.0	0.0	0.0	0.0	0.0	0.0	3.0	12.0	0.0	84.9	0.0	95.4	33	88.4	107.1	0.0	0.0
97	39	39	0.0	0.0	0.0	71.8	5.1	387.2	7.7	32.1	0.0	37.7	0.0	96.6	39	13.4	30.6	0.0	0.0
98	47	47	25.5	17.0	19.1	10.6	0.0	0.0	25.5	15.1	27.7	35.8	0.0	82.1	47	13.4	33.1	0.0	0.0
99	23	23	0.0	0.0	0.0	0.0	0.0	0.0	4.3	20.0	13.0	29.7	4.3	52.1	23	19.6	24.3	0.0	0.0
100	36	36	0.0	0.0	0.0	0.0	0.0	0.0	0.0	14.2	0.0	31.5	0.0	45.8	36	20.1	25.2	0.0	0.0
101	25	25	0.0	0.0	0.0	0.0	0.0	0.0	52.0	6.9	0.0	35.1	0.0	42.2	25	26.4	32.2	0.0	0.0
102	19	19	0.0	0.0	0.0	0.0	0.0	0.0	15.8	6.3	0.0	42.4	0.0	49.9	19	27.6	34.9	0.0	0.0
103	14	14	0.0	0.0	0.0	0.0	0.0	0.0	7.1	5.6	0.0	41.3	0.0	53.3	14	32.2	38.9	0.0	0.0
104	20	20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	6.1	0.0	37.9	0.0	44.5	20	33.7	39.4	0.0	0.0
105	56	76	40.8	65.2	36.8	35.7	36.8	558.9	14.3	50.2	19.6	107.1	0.0	275.1	**56**	31.3	198.1	0.0	**-26.3**
106	65	65	20.0	28.2	0.0	3.1	0.0	0.0	23.1	20.5	44.6	44.0	18.5	103.9	65	29.9	55.8	0.0	0.0
107	31	31	0.0	0.0	0.0	12.9	0.0	0.0	0.0	20.1	0.0	80.9	0.0	82.8	31	43.9	64.7	0.0	0.0
108	36	36	0.0	0.0	0.0	0.0	0.0	0.0	0.0	27.3	2.8	51.1	0.0	84.1	36	43.9	61.1	0.0	0.0
109	26	26	0.0	0.0	0.0	0.0	0.0	0.0	7.7	22.6	0.0	70.5	0.0	110.8	26	61.9	78.1	0.0	0.0
110	33	33	0.0	0.0	0.0	0.0	0.0	0.0	3.0	18.3	3.0	71.1	3.0	98.1	33	61.6	79.8	0.0	0.0
111	25	25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	15.1	0.0	73.6	0.0	90.3	25	74.0	87.2	0.0	0.0
112	18	18	0.0	0.0	0.0	0.0	0.0	0.0	11.1	8.2	0.0	67.8	0.0	78.8	18	72.5	86.5	0.0	0.0
113	41	69	50.7	98.0	44.9	68.3	43.5	914.6	2.4	49.6	0.0	191.0	0.0	500.5	**46**	58.1	400.4	**-12.2**	**-33.3**
114	52	92	47.8	100.0	46.7	76.9	45.7	732.7	0.0	45.0	5.8	124.1	0.0	307.2	**52**	55.7	269.6	0.0	**-43.5**
115	35	35	0.0	0.0	0.0	17.1	0.0	0.0	20.0	16.9	0.0	79.5	0.0	109.3	35	82.8	118.5	0.0	0.0
116	55	81	32.1	79.8	32.1	47.3	56.8	1058.2	1.8	44.5	1.8	100.2	0.0	252.1	**55**	82.7	297.7	0.0	**-32.1**
117	25	25	0.0	0.0	0.0	0.0	0.0	0.0	16.0	21.5	4.0	103.9	4.0	144.5	25	101.2	138.2	0.0	0.0
118	50	50	0.0	0.0	0.0	0.0	0.0	0.0	0.0	11.1	0.0	99.9	0.0	117.0	50	109.4	133.7	0.0	0.0
119	24	24	0.0	0.0	0.0	0.0	0.0	0.0	8.3	12.6	0.0	101.5	0.0	123.5	24	117.6	140.6	0.0	0.0
120	19	19	0.0	0.0	0.0	0.0	0.0	0.0	5.3	10.3	0.0	107.5	0.0	129.6	19	117.0	141.3	0.0	0.0
121	54	92	55.4	125.7	53.3	70.4	45.7	750.0	0.0	48.1	24.1	113.8	0.0	360.2	**56**	90.2	1288.4	**-3.7**	**-39.1**
122	86	145	62.1	105.7	50.3	68.6	57.9	840.7	0.0	77.9	16.3	131.5	0.0	412.1	**86**	91.4	3300.1	0.0	**-40.7**
123	37	60	43.3	105.0	41.7	62.2	41.7	1124.3	8.1	36.4	0.0	115.5	0.0	189.3	**37**	118.2	326.8	0.0	**-38.3**
124	54	66	21.2	49.6	19.7	22.2	18.2	685.2	3.7	39.0	3.7	110.0	3.7	184.8	**54**	118.4	211.1	0.0	**-18.2**
125	32	32	0.0	0.0	0.0	0.0	0.0	0.0	0.0	14.2	0.0	138.9	0.0	170.4	32	136.9	190.6	0.0	0.0
126	53	53	0.0	0.0	0.0	0.0	0.0	0.0	3.8	43.8	13.2	131.5	3.8	176.8	53	142.6	203.9	0.0	0.0
127	25	25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	15.0	0.0	151.4	0.0	183.8	25	159.5	196.1	0.0	0.0
128	36	36	0.0	0.0	0.0	0.0	0.0	0.0	5.6	46.0	0.0	151.8	0.0	203.2	36	160.1	199.8	0.0	0.0
129	38	51	39.2	64.5	25.5	34.2	31.4	660.5	13.2	55.3	13.2	107.6	13.2	183.8	**38**	28.4	81.9	0.0	**-25.5**
130	69	69	0.0	0.0	18.8	30.4	24.6	462.3	0.0	19.8	21.7	56.4	0.0	95.3	69	28.6	66.8	0.0	0.0
131	23	23	0.0	0.0	0.0	8.7	0.0	0.0	4.3	9.7	4.3	49.5	4.3	60.3	23	41.9	60.2	0.0	0.0
132	37	37	0.0	0.0	0.0	0.0	0.0	0.0	8.1	23.3	10.8	65.1	8.1	85.5	37	41.9	59.6	0.0	0.0
133	15	15	0.0	0.0	0.0	0.0	0.0	0.0	20.0	7.8	0.0	63.0	0.0	76.8	15	57.5	70.0	0.0	0.0
134	50	50	0.0	0.0	0.0	0.0	0.0	0.0	0.0	25.9	28.0	66.6	0.0	101.6	50	56.6	68.6	0.0	0.0
135	15	15	0.0	0.0	0.0	0.0	0.0	0.0	20.0	20.0	0.0	62.4	0.0	90.4	15	72.5	85.4	0.0	0.0
136	34	34	0.0	0.0	0.0	0.0	0.0	0.0	5.9	15.9	0.0	67.9	0.0	91.4	34	71.1	83.8	0.0	0.0
137	54	77	30.4	43.6	30.4	57.4	40.7	607.4	0.0	37.6	0.0	127.9	0.0	157.4	**54**	68.6	208.1	0.0	**-30.4**
138	65	90	45.6	58.2	42.2	38.5	44.4	567.7	0.0	38.0	18.5	115.2	0.0	223.3	**65**	65.4	183.5	0.0	**-27.8**
139	36	36	19.4	33.3	19.4	5.6	0.0	0.0	25.0	24.8	19.4	123.5	19.4	141.9	36	89.3	140.3	0.0	0.0
140	50	57	12.3	28.7	12.3	14.0	12.3	676.0	10.0	38.9	28.0	90.8	0.0	148.2	**50**	91.6	147.7	0.0	**-12.3**
141	28	28	0.0	0.0	0.0	0.0	0.0	0.0	17.9	25.0	7.1	99.2	7.1	133.6	28	111.2	140.2	0.0	0.0
142	50	50	0.0	0.0	0.0	0.0	0.0	0.0	0.0	48.2	0.0	108.5	0.0	179.6	50	111.3	149.2	0.0	0.0
143	26	26	0.0	0.0	0.0	0.0	0.0	0.0	3.8	27.2	42.3	119.1	0.0	158.2	26	125.9	149.2	0.0	0.0
144	34	34	0.0	0.0	0.0	0.0	0.0	0.0	5.9	25.7	0.0	113.3	0.0	146.7	34	126.0	152.5	0.0	0.0
145	56	78	48.7	92.4	38.5	39.3	43.6	753.6	10.7	71.2	12.5	248.7	0.0	442.1	**56**	111.5	905.9	0.0	**-28.2**
146	64	112	57.1	127.6	54.5	75.0	58.0	693.8	0.0	73.1	18.8	194.3	0.0	447.3	**64**	109.7	916.8	0.0	**-42.9**
147	39	40	5.0	33.2	5.0	2.6	2.5	620.5	2.6	67.2	28.2	131.1	2.6	234.8	**39**	140.2	186.9	0.0	**-2.5**
148	55	71	46.5	93.7	22.5	29.1	29.6	910.9	7.3	63.2	30.9	137.8	7.3	238.9	**55**	136.4	264.5	0.0	**-22.5**
149	25	25	0.0	16.0	0.0	0.0	0.0	0.0	4.0	58.2	0.0	153.1	0.0	232.4	25	162.5	211.3	0.0	0.0
150	35	35	0.0	0.0	0.0	0.0	0.0	0.0	0.0	48.9	0.0	148.9	0.0	229.1	35	162.9	210.8	0.0	0.0

TABLE I: Computational results (continued).

ID	BK		Previous Exact Methods						Previous Heuristic Methods						New Results				
			BD [3] Gap		B&B [3] Gap		RDS [3] Gap		CG [4]		KS [4]		KO [4]		Opt	CP-SAT		Gap	
	LB	UB	LB	UB	LB	UB	LB	UB	Gap UB	Sec	Gap UB	Sec	Gap UB	Sec		Sec heu	Sec opt	LB	UB
151	23	23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	29.3	39.1	180.4	0.0	221.7	23	186.5	232.4	0.0	0.0
152	33	33	0.0	0.0	0.0	0.0	0.0	0.0	42.4	19.5	0.0	177.4	0.0	211.6	33	186.0	231.2	0.0	0.0
153	61	103	60.2	128.7	68.0	68.9	66.0	778.7	0.0	112.4	23.0	401.1	0.0	539.3	**61**	162.2	6203.8	0.0	**-40.8**
154	88	133	48.9	130.5	39.1	51.1	57.9	843.2	3.4	155.5	0.0	314.6	3.4	656.4	**88**	167.6	3916.2	0.0	**-33.8**
155	38	69	52.2	138.5	46.4	81.6	46.4	1339.5	2.6	78.7	0.0	220.6	0.0	375.8	**39**	206.6	2429.4	**-2.6**	**-43.5**
156	53	94	60.6	136.2	59.6	77.4	51.1	1200.0	13.2	100.3	0.0	232.3	0.0	447.0	**53**	201.4	577.4	0.0	**-43.6**
157	38	54	29.6	121.9	29.6	42.1	29.6	1473.7	0.0	57.1	0.0	231.7	0.0	315.1	**38**	266.8	967.4	0.0	**-29.6**
158	48	58	36.2	93.6	17.2	20.8	17.2	1264.6	22.9	54.4	22.9	228.4	22.9	310.8	**53**	272.4	532.2	**-10.4**	**-8.6**
159	26	26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	94.9	3.8	270.4	0.0	387.1	26	302.8	461.7	0.0	0.0
160	36	36	0.0	0.0	0.0	0.0	0.0	0.0	2.8	107.3	0.0	264.8	0.0	392.4	36	302.1	448.4	0.0	0.0
Average			7.4	15.0	6.4	10.0	6.9	154.9	6.1	22.9	5.7	70.2	1.2	112.0		51.9	203.2	-0.2	-4.9