Mathematic Eng.

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CONTENTS

Chapter 1

LINEAR PROGRAMMING PROBLEM

1.1 The main problem of linear programming

The main problem of linear programming is the minimization or the maximization of a linear function subject to linear constraints. The function whose greatest or least value is being sought is called an objective function and the collection of the values of the variables at which the greatest or the least value is attained defines the so-called optimal plan. Any other collection of values complying with the restrictions defines the feasible plan (solution).

Assume that the constraints are given as a consistent system of m linear inequalities in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... +_{1n} x_n \geq b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \geq b_{2,} \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \geq b_{m.} \end{cases}$$

it is required to choose among the nonnegative

solutions of the system a solution as a result of
which the linear (objective) function

$$L = c_1x_1 + c_2x_2 + ... + c_nx_n + c_o$$

assumes the greatest (the least) value or, as they say, it is required to maximize (minimize) the linear form L.

We shall show now how to solve this problem by a geometrical method for which purpose we shall consider a consistent system of linear inequalities in two and three variables. Suppose we are given, in addition, the linear function

$$L = c_1x_1 + c_2x_2 + c_0.$$

Let us choose from among the set of points $(x_1; x_2)$ belonging to the domain of solutions of the consistent system of inequalities such points which import to the given linear function the least (the greatest) value. For each point of the plane the function L assumes a fixed value $L = L_1$. The set of all such points is the straight line

$$c_1x_1 + c_2x_2 + c_0 = L_1$$

perpendicular to the vector $C(c_1; c_2)$ which starts at the origin if we move this line parallel to itself in the positive direction of the vector C, then the linear function

$$L = c_1x_1 + c_2x_2 + c_0$$

will increase, and if we move the line in the opposite direction, it will decrease suppose that when the line L moves in the positive direction of vector C, it first comes across the polygon of the solutions at its vertex and then, being in the position L_1, the line L becomes supporting, and on this line the function L assumes the least value. In its further movement in the same (positive) direction, the straight line L will pass through another vertex of the polygon of solutions, leaving the domain of solutions, and will also become a supporting line L_2 ; on that line, the function L assumes the greatest value among all the values attained on the polygon of solutions.

Thus, minimization and maximization of the linear function

$$L = c_1x_1 + c_2x_2 + c_0$$

on the polygon of solutions are attained at points of intersection of the polygon and the lines of support perpendicular to the vector $C(c_1; c_2)$. The line of support may have either one point in common with the polygon of solutions (the vertex of the polygon), or an infinite set of points (this set being a side of the polygon).

By analogy with the aforesaid, the linear function of three variables

$$L = c_1x_1 + c_2x_2 + c_3x_3 + c_0$$

assumes a constant value on the plane perpendicular to the vector $C\,(c_1; c_2; c_3)$. The least and the greatest value of the function on the polyhedron of solutions are attained at the points of intersection of the polyhedron and the planes of support perpendicular to the vector $C\,(c_1; c_2; c_3)$. A plane of support may have either one point in common with the polyhedron of solutions (its vertex), or an infinite set of points (the set being an edge or a face of the polyhedron).

Example 1 *Maximize the linear form*

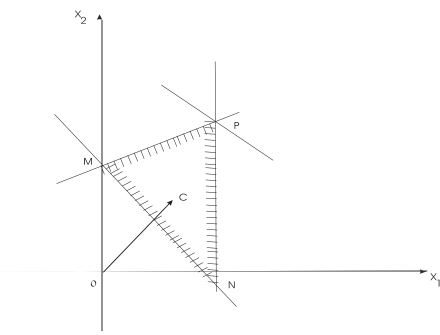
$$L = 2x_1 + 2x_2$$

subject to the constraints

$$\begin{aligned} 3x_1 - 2x_2 &\geq -6, \\ 3x_1 + x_2 &\geq 3, \\ x_1 \quad &\leq 3 \end{aligned}$$

Solution: Replacing the inequality signs of strict equalities, we construct the domain of solutions from the equations of the straight lines

$$\begin{aligned} 3x_1 - 2x_2 + 6 &= 0, \\ 3x_1 + x_2 - 3 &= 0, \\ x_1 \quad &= 3 \end{aligned}$$



The domain of solutions of the inequalities is the triangle MNP. Mow we construct the vector $C(2;2)$. Then, when leaving the triangle of solutions, the line of support passes through the point $P(3;15/2)$ and, therefore, at the point P the linear function

$$L = 2x_1 + 2x_2$$

assumes the greatest value, that is, is maximized, and $L_{\max} = 2.3 + 2.(15/2) = 21$.

Example 2 *Minimize the linear function*

$$L = 12x_1 + 4x_2$$

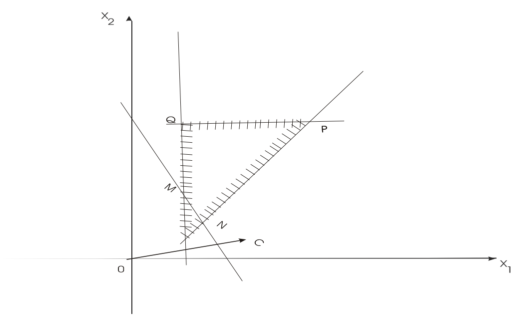
subject to the constraints

$$\begin{aligned} x_1 + x_2 &\geq 2, \\ x_1 \qquad &\geq 1/2, \\ x_2 &\leq 4, \\ x_1 - x_2 &\leq 0 \end{aligned}$$

Solution: Replacing the inequality signs by the signs of strict equalities, we construct the domain of solutions bounded by the straight lines

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_1 \qquad &= 1/2, \\ x_2 &= 4, \\ x_1 - x_2 &= 0. \end{aligned}$$

The domain of solutions is the polygon $MNPQ$



Next we construct the vector $C(12;4)$. The line of support passes through the point $M(1/2;3/2)$, which is the first point of intersection of the polygon of solutions and the line L on the way of the movement of that line in the positive direction of the vector C. At the point M the linear function

$$L = 12x_1 + 4x_2$$

assumes the least value

$$L_{\min} = 12.(1/2) + 4.(3/2) = 12.$$

Example 3 *Consider the problem*

Maximize

$$f(x) = x_1 + x_2$$

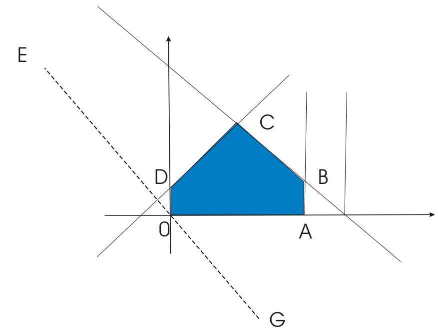
Subject to

$$\begin{aligned} -2x_1 + x_2 &\leq 1 \\ x_1 \qquad &\leq 2 \\ x_1 + x_2 &\leq 3 \end{aligned}$$

and

$$x_1 \geq 0 \qquad x_2 \geq 0$$

Solution:



Here the polygon $OABCD$ represents the set of feasible solutions and the objective function has the same slope as the third constraint. The line Eg, $x_1 + x_2 = 0$ is parallel to the face BC of the feasible region. Hence on shifting Eg parallel to itself it will coincide with BC. Consequently any point on the bounding lineBC will be optimal solution with the same maximum value for the objective function. In this case we have an infinite number of optimal solutions the two points B and C represents optimal extreme solutions.

$$\begin{aligned} B &= (x_1 = 2, \quad x_2 = 1) \\ C &= (x_1 = 1, \quad x_2 = 2) \end{aligned}$$

and all the convex liner combination of these two points, *i.e.* all the points lying on the line BC between B and C represent optimal solution. The optimal value of the objective function will be $f(x) = 3$.

1. maximize
$$f(x) = 5x_1 + 3x_2$$
subject to
$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \end{aligned}$$
and
$$x_1 \geq 0,\ x_2 \geq 0$$

2. Minimize
$$f(x) = 3x_1 - x_2$$
$$\begin{aligned} x_1 + x_2 &\geq 3 \\ x_1 - x_2 &\leq 1 \\ x_2 &\leq 2 \end{aligned}$$
and
$$x_1 \geq 0 \quad x_2 \geq 0$$

3. Minimize
$$f(x) = 3x_1 + x_2$$
Subject to
$$\begin{aligned} x_1 - x_2 &\leq 1 \\ x_1 - x_2 &\geq 3 \end{aligned}$$

and

$$x_1 \geq 0, \;\; x_2 \geq 0$$

1.2 The Simplex Method

The simplex method is a technique developed to solve linear programming problems. Applying this method leads to one of the following cases:

1. a finite optimal solution is found.

2. an infinite optimal solution is positively identified.

3. the problem has no feasible solution.

The simplex method is an algebraic iterative procedure which will solve exactly any linear programming problem in a finite number of steps, or give an indication that there is an unbounded solution.

The simplex method is a procedure for moving step by step from a given extreme point to an optimal extreme point. At each step, it is possible to move only to what intuitively are adjacent extreme points. The simplex method moves a long an edge of the region of feasible solutions form one extreme point to an adjacent one. Of all the adjacent extreme points, the one chosen is that which gives the greatest increase (or greatest decrease) in the objective function.

1.2.1 The simplex method in a tabular form:

Consider the following linear programming problem:-

Example 4 *Determine x_1 and x_2 that maximize the objective function*

$$f(x) = c_1x_1 + c_2x_2$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 &\leq b_2 \\ a_{31}x_1 + a_{32}x_2 &\leq b_3 \end{aligned}$$

and $\qquad x_1 \geq 0 \qquad x_2 \geq 0$

Solution: The algebraic solution for this problem can be arranged in a tabular form as will be shown :

The original simplex tableau :

After converting the structural constraints into equations the problem can be arranged in a tableau as follows :

basic variables	x_1	x_2	x_3	x_4	x_5	constants
x_3	a_{11}	a_{12}	1	0	0	b_1
x_4	a_{21}	a_{22}	0	1	0	b_2
x_5	a_{31}	a_{32}	0	0	1	b_3
$-f(x)$	c_1	c_2	0	0	0	0

we can read equations (**??**) from this tableau by leaving the first column, then

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + x_3 \qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \quad x_4 \quad &= b_2 \\
a_{31}x_1 + a_{32}x_2 + \qquad x_5 &= b_3
\end{aligned}$$

which are the same as equations (**??**).
On the other hand, starting with the first column, we can read the rows as

$$\begin{aligned}
x_3 &= b_1 - a_{11}x_1 - a_{12}x_2 \\
x_4 &= b_2 - a_{21}x_1 - a_{22}x_2 \\
x_5 &= b_3 - a_{31}x_1 - a_{32}x_2
\end{aligned}$$

Also the last row is

$$\begin{aligned}
-f(x) &= 0 - c_1x_1 - c_2x_2 \\
f(x) &= c_1x_1 + c_2x_2
\end{aligned}$$

From the first and last columns we can read the initial basic feasible solution, i.e.

$$x_3 = b_1,\ x_4 = b_2,\ \ x_5 = b_3$$

(basic variables)

$$x_1 = 0 \quad x_2 = 0$$

(non-basic variables)

and

$$-f(x) = 0 \quad i.e. \ \ f(x) = 0$$

To test the optimally of this initial basic feasible solution, we look at the coefficients in the last row of the tableau, i.e. c_1 and c_2. If both are $-ve$ the solution is optimal. If at least one them is positive, the solution is not optimal c_1 and c_2 are both positive, then the solution is not optimal.

The transformed simplex tableau:

Examine c_1, c_2 in the last row and suppose that c_1 is greater than $c_2 (i.e. c_1 > c_2)$ then x_1 will be basic variable and its column is called the pivot column.

Then divide the constants b_1, b_2 and b_3 given in the last column by the corresponding elements a_{11}, a_{21} and a_{31} in the pivot column. Suppose that the ratio b_2/a_{21} is the least, then x_4 will be the non-basic variables and its row is called the pivot row.

The element a_{21} at the intersection of the pivot column and pivot row is called the pivot element.

This solution has x_3, x_1, x_5 as basic variables and x_4, x_2 as non-basic variables. Now the next tableau is formed by transforming the elements in the original simplex tableau as follows:

1. The pivot row is transformed by dividing all its elements by the pivot element.

2. The pivot column is transformed by replacing all its elements by zeros except the pivot element which becomes 1.

3. Other elements which are neither in the pivot column nor the pivot row are transformed as follows :

Let a_{12} be such as element and a_{11}be the element lying in the same row and the pivot column.

		pivot column	
	a_{11}		a_{12}
pivot row	a_{21}		a_{22}

a_{22} be the element lying in the same column and the pivot row. and a_{21} is the pivot element.

Then a_{12} is transformed to

$$a_{12} - \frac{a_{11}a_{22}}{a_{21}},$$

which is the same as a'_{12}. Thus the transformed simplex tableau becomes

basic variables	x_1	x_2	x_3	x_4	x_5	constants
x_3	0	a'_{12}	1	a'_{14}	0	b'_1
x_1	1	a'_{22}	0	a'_{24}	0	b'_2
x_5	0	a'_{32}	0	a'_{34}	1	b'_3
$-f(x)$	0	c'_2	0	c'_4	0	$-c$

This solution is :

$$x_3 = b'_1, x_1 = b'_2 \quad , \; x_5 = b'_3$$

basic variables

$$x_2 = 0 \;\; ; \;\; x_4 = 0$$

non-basic variables .

and

$$f(x) = c.$$

To test the optimally, we examine the coefficients $(c'_2 \& c'_4)$ in the last row. If both are negative the solution is optimal. If at least one is positive, the solution is not optimal and the same rules are repeated till we reach the optimal solution.

Now, we can summarize the rules for the simplex method in tabular form:

The decision rules:

1. Testing optimally: Examine the coefficients of the objective function in the last row of the tableau. If the problem is to maximize and they are negative or zeros, the solution is optimal. If the problem is to minimize and they are positive or zeros, the solution is optimal

2. The basic variables: If the problem is to maximize, then the non-basic variable associated with the largest positive coefficient in the last row is the basic variable.

3. The non-basic variables: Divide the constants in the last column by the corresponding elements in the pivot column. The basic variable associated with the least of these ratios is the non-basic variable.

The transformation rules:

1. The pivot row is transformed by dividing all elements by pivot element.

2. The pivot column is transformed by replacing its elements by zeros except the pivot element which becomes 1.

		pivot column		
	b		a	
	d		c	pivot row

3. The remaining elements are transformed by applying the following rule:

$$a' = a - bc/d$$

Example 5 *Find* $x_1 \geq 0, \quad x_2 \geq 0$ *that maximize*

$$f(x) = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 - 2x_2 &\leq 12 \\ x_1 - 5x_2 &\leq 2 \\ -x_1 + 2x_2 &\leq 4 \end{aligned}$$

Solution: The first step is

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_3	3	-2	1	0	0	12
x_4	1	-5	0	1	0	2
x_5	-1	2	0	0	1	4
$-f(x)$	2	1	0	0	0	0

The second step

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_3	0	13	1	-3	0	6
x_1	1	-5	0	1	0	2
x_5	0	-3	0	1	1	6
$-f(x)$	0	11	0	-2	0	-4

The third step

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_2	0	1	$1/13$	$-3/13$	0	$6/13$
x_1	1	0	$5/13$	$-2/13$	0	$56/13$
x_5	0	0	$3/13$	$4/13$	1	$96/13$
$-f(x)$	2	0	$-11/13$	$7/13$	0	$-118/13$

The forth step is

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_2	0	1	$1/4$	0	$3/4$	6
x_1	1	0	$1/2$	0	$1/2$	8
x_4	0	0	$3/4$	1	$13/4$	24
$-f(x)$	0	0	$-5/4$	0	$-7/4$	-22

The optimal solution is

$$x_1 = 8, \qquad x_2 = 6, \quad x_4 = 24$$

basic variables

$$x_3 = 0, \qquad x_5 = 0$$

non-basic non-basic variables

and

$$f(x) = 22$$

Exercise 6 *maximize*

$$f(x) = 3x_1 + 5x_2$$

Subject to

$$\begin{aligned} x_1 \qquad & \leq 4 \\ x_2 & \leq 6 \\ 3x_1 + 2x_2 & \leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

1.2.2 The canonical form:

As we have seen, the linear programming problem is to find values of $x_1, x_2, .., x_n$ which satisfy the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1.1}$$

and

$$x_j \geq 0 \quad , \quad j = 1, 2, ..., \quad n$$

and minimize the objective function

$$z = c_1 x_1 + c_2 x_2 + ...c_n x_n \tag{1.2}$$

where a_{ij}, b_i and c_j $(i = 1, ..., m;\ j = 1, ..., n)$ are constants and z denotes the objective function. we shall assume that equations (1.1) are linearly independent.

Let us assume that we have found, some how, an extreme point of the convex set, $i.e$ we have found a feasible basic. Without loss of generality we can consider $x_1, x_2, ..., x_m$ to be the basic variables. Then we know that the $m \times m$ system.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2m}x_m &= b_2 \\
\vdots \qquad\qquad\qquad &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mm}x_m &= b_m
\end{aligned}$$

posses a unique solution which consists of nonnegative numbers. Now let us re-arrange the original system (1.1) in the following form. Divide the first equation by $a_{11}(a_{11} \neq 0)$ and eliminate x_1 from the other equations and from the objective form by algebraic manipulation, using a_{11} position as a pivot. This is done by subtracting from each equation i the new equation multiplied by a_{i1}. Then we obtain

$$
\begin{aligned}
x_1 + \overline{a}_{12}x_2 + ... + \overline{a}_{1n}x_n &= \overline{b}_1 \\
\overline{a}_{22}x_2 + ... + \overline{a}_{2n}x_n &= \overline{b}_2 \\
\vdots \qquad\qquad &= \vdots \\
\overline{a}_{m2}x_2 + ... + \overline{a}_{nn}x_n &= \overline{b}_m \\
\overline{c}_2x_2 + ... + \overline{c}_nx_n &= z - \overline{z}
\end{aligned}
$$

where $\overline{a}_{ij}, \overline{b}_i, \overline{c}_j$ and $\overline{z}$ are constants. If $a_{11} = 0$, pick an equation that has a non-zero coefficient of $x_1$and let i index of this equation be 1. Now divide the second equation of this new arrangement by $a_{22}(\overline{a}_{22} \neq 0)$ and eliminate $x_2$from the other equations and from the objective form using $\overline{a}_{22}$ position as a pivot. Repeat this for $x_3, x_4, ..., x_m, i.e.$ for all the other basic variables. The result is the following system:

$$
\begin{aligned}
x_1 \qquad\quad + a'_{1m+1}x_{m+1} + ... \qquad + a'_{1n}x_n &= b'_1 \\
x_2 \quad + a'_{2m+1}x_{m+1} + ... \qquad + a'_{2n}x_n &= b'_2 \\
\vdots \qquad\qquad\qquad &= \vdots \\
x_m + a'_{mn+1}x_{m+1} + ... \qquad + a'_{mn}x_n &= b'_m \\
c'_{m+1}x_{m+1} + ... \qquad + c'_nx_n &= z - z_0
\end{aligned} \tag{1.3}
$$

where a'_{ij}, b'_i , c'_j are the modified values of a_{ij}, b_i, c_j and z_0 are constants. The system (1.3) is completely equivalent to the original system (1.1). The system is called the canonical form for the basis $x_1, x_2, ..., x_m$.The coefficients c'_j are called the relative cost coefficient. The coefficients of a basic variable in the canonical form constitute a unit vector.

1.2.3 Improving the basis:

The basic feasible solution is given immediately from the canonical form by giving zero values to the non-basic variables, we get the solution

$$\begin{aligned} x_i &= 0 \qquad i = m+1, m+2, ..., n \\ x_i &= b'_1 \qquad i = 1, ..., m \end{aligned} \tag{1.4}$$

and the value of the objective function is

$$z = z_0$$

As the basic is feasible we must have $b'_i \geq 0$ for $i = 1, 2, ..., m$. Now we want to move to an adjacent extreme point whose z value is less than z_0. Looking at the c'_j coefficients of the canonical form (1.3), if one of them is negative, say $c'_s < 0$. This means that if we give to the non-basic variable x_s some positive value *i.e.*. we change it to a basic variable, the value of z will decrease because z is given by $z = z_0 + c'_s x_s$. As x_s is increased, we move away from the original extreme point. We move along an edge of the set, constantly decreasing z as x_s gets bigger and bigger.

Now, let us rewrite the canonical form (1.1) by moving the entire x_s column to the right

$$\begin{aligned}
x_1 \qquad\qquad + a'_{1m+1}x_{m+1} + ... + a'_{1n}x_n &= b'_1 - a'_{1s}x_s \\
x_2 \qquad + a'_{2m+1}x_{m+1} + ... + a'_{2n}x_n &= b'_2 - a'_{2s}x_s \\
\vdots \qquad\qquad\qquad &= \vdots \\
x_m + a'_{mm+1}x_{m+1} + ... + a'_{mn}x_n &= b'_m - a'_{ms}x_s \\
c'_{m+1}x_{m+1} \quad + ... + c'_n x_n &= z - c'_s x_s - z_0
\end{aligned} \tag{1.5}$$

We see from (1.5) that the basic variables and the objective function have the following functional dependence on x_s :

$$\begin{aligned}
x_{11} &= b'_1 - a'_{1s}x_s \qquad\qquad i = 1, 2, ..., m \\
z &= z_0 \quad + c'_s x_s
\end{aligned}$$

$c'_s < 0$, so that z decreases with increasing x_s.

The a'_{1s} can have any sign and finite value. Hence, as x_s increases, some of the variables will increase $(if \quad a'_{is} < 0)$ some will not change at all $(if \quad a'_{1s} = 0)$, and some will decrease $(if \quad a'_{is} > 0)$. The decreasing variables should not become negative, we must have

$$b'_1 - a'_1 x_s \geq 0$$

$$x_s \leq \frac{b'_1}{a'_{1s}} a'_{1s} > 0 \tag{1.6}$$

This imposes upper bounds on x_s for different i.

If x_s is to satisfy(1.6) for all i, then x_s cannot exceed the smallest of these upper bounds. Let $i = r$ be the value of i for which the ratio $\frac{b'_i}{a'_{is}}$ is smallest (for $a'_{is} > 0$) Then

$$\max \quad x_s = \min_{a'_{is} > 0} \frac{b'i}{a'is} = \frac{b'_r}{a'_{rs}} \tag{1.7}$$

If we let x_s takes on its maximum value as given by (1.6), then x_r is reduced to zero. This means that we have removed x_r form the basic and replaced it by x_s.

1.2.4 The simplex method in the compact form:

Looking at the previous example 5, which be solved by the simplex method in a tabular form, we notice that a unit matrix always present in each tableau, it correspond to the columns of the basic variables of each iteration. However, a unit matrix does not contain numerical informations which need be recorded. We can remove it and know that it always exists. We then will have columns for non-basic variable only. The tableau for the kth iteration will have the form:

$basic$	non-basic $x_{m+1} \ldots\ldots x_s \ldots\ldots x_n$	$values$
x_1	$a'_{1m+1} \ldots\ldots a'_{1s} \ldots\ldots a'_{1n}$	b'_1
x_2	$a'_{2m+1} \ldots\ldots a'_{2s} \ldots\ldots a'_{2n}$	b'_2
x_r	$a'_{rm+1} \ldots\ldots a'_{rs} \ldots\ldots a'_{rn}$	b'_r
x_m	$a'_{mm+1} \ldots\ldots a'_{ms} \ldots\ldots a'_{mn}$	b'_m
$-z$	$c'_{mm+1} \ldots\ldots c'_s \ldots\ldots c'_n$	$-z_0$

The tableau for the $k+1\underline{th}$ iteration will be :

$basic$	non-basic variables x_{m+1} $\cdots$ x_r $\cdots$ x_n	$values$
x_1	$a'_{1m+1} - a'_{1s}a'^{*}_{rm+1} .... - a'_{1s}/a'_{rs} .... a'_{in} - a'_{1s}a'^{*}_{rm}$	$b'_1 - a'_{1s}b'^{*}_r$
x_2	$a'_{2m+1} - a'_{2s}a'^{*}_{rm+1} .... - a'_{2s}/a'_{rs} .... a'_{2n} - a'_{2s}a'^{*}_{rn}$	$b'_2 - a'_{2s}b'^{*}_r$
$\vdots$		$\vdots$
x_s	$a'_{rm+1}/a'_{rs} = a'^{*}_{rm+1} .... 1/a'_{rs} ..... a'_{rn}/a'_{rs} = a'^{*}_{rn}$	$b'_r/a'_{rs} = b'^{*}_r$
$\vdots$		$\vdots$
x_m	$a'_{mm+1} - a'_{ms}a'^{*}_{rm+1} ... a'_{ms}/a'_{rs} ... a'_{mn} - a'_{ms}a'^{*}_{rn}$	$b'_m - a'_{ms}b'^{*}_r$
$-z$	$c'_{m+1} - c'_s\ a'^{*}_{rm+1} ..... - c'_s/a'_{rs} .... c'_n - c'_s a'^{*}_{rn}$	$-z_o - c'_s b'^{*}_r$

The rules for the transformation tableau from iteration to the next one are as follows:

1. The pivot element is replaced by its reciprocal.
2. The pivot row is divided by the pivot element.
3. The pivot column is divided by the negative of the pivot element.
4. Every other element is reduced by the quantity: element in the same row and pivot column multiplied by element in the same column and pivot row and divided by the pivot element.
5. The headings of the pivot row and pivot column are interchanged.

To illustrate this method let us consider the following example:

$$\begin{aligned} x_1 + 2x_2 &\leq 10 \\ x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\leq 2 \\ x_1 - 2x_2 &\leq 1 \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 &\rightarrow \max \\ \text{and } x_1, x_2 &\geq 0, \end{aligned}$$

Solution: Reversing the signs of the c_j to convert the problem to a minimization one. Then the calculations are shown in the following compact tableau:

$non - basic$ variables / $basic$	x_1	x_2	$values$
x_3	1	2	10
x_4	1	1	6
x_5	1	-1	2
x_6	$\boxed{1}$	-2	1
$-z$	-2	1	0

initial tableau

basic \ *non − basic* variables	x_6	x_2	*values*
x_3	-1	4	9
x_4	-1	3	5
x_5	-1	$\boxed{1}$	1
x_1	$+1$	-2	1
$-z$	2	-5	2

1 is iteration .

basic \ non-basic variables	x_6	x_5	*values*
x_3	3	-4	5
x_4	$\boxed{2}$	-3	2
x_2	-1	1	1
x_1	-1	2	3
$-z$	-3	5	7

2 nd iteration .

basic \ non-basic variables	x_4	x_5	*value*
x_3	$-3/2$	$1/2$	2
x_6	$1/2$	$-3/2$	1
x_2	$1/2$	$-1/2$	2
x_1	$1/2$	$1/2$	4
$-z$	$3/2$	$1/2$	10

3 rd iteration optimal tableau

from the optimal tableau we can see that the optimal solution is :

$$x_1 = 4 \qquad\qquad x_2 = 2$$

$$z_{\min} = -10$$

and

$$z_{\max} = 10$$

1.2.5 The two-phase Simplex method

Solution of Linear programming problems with structural constraints of the type $\geq$:

Consider the following Linear programming problem:

Determine x_1 and x_2 that maximize (or minimize) the objective function:

$$f(x) = c_1x_1 + c_2x_2 \tag{1.8}$$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 &\geq b_2 \\ a_{31}x_1 + a_{32}x_2 &\geq b_3 \end{aligned} \tag{1.9}$$

and

$$x_1 \geq 0, x_2 \geq 0 \tag{1.10}$$

In this case we convert the structural constraints into equations by subtracting non-negative slack variables x_3, x_4 and x_5. The system (1.9) becomes:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 - x_3 \qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \qquad - x_4 \qquad &= b_2 \\ a_{31}x_1 + a_{32}x_2 \qquad\qquad - x_5 &= b_3 \end{aligned} \tag{1.11}$$

If we let $x_1 = 0, x_2 = 0$ we get the initial basic solution $x_3 = -b_1,\ x_2 = -b_2,\ x_5 = -b_3$. But this solution does not satisfy the non-negative constraints and therefore it is not feasible.

In order to start with an initial basic feasible solution we assume the system by the non-negative variables x_6, x_7 and x_8, which are called artificial variables. Thus we obtain the following:-

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 - x_3 \qquad + x_6 \qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 \qquad\quad - x_4 + x_7 \qquad &= b_2 \\ a_{31}x_1 + a_{32}x_2 \qquad\qquad\quad - x_5 + x_8 &= b_3 \end{aligned} \tag{1.12}$$

and

$$x_j \geq 0 \qquad j = 1, 2, \cdots, 8.$$

If we let $x_1 = 0, x_2 = 0, \cdots, x_5 = 0$, then we get $x_6 = b_1,\ x_7 = b_2, x_8 = b_3$, which form a basic feasible solution for the set of constraints (1.12). But this is not a solution for the original constraints (1.11). So it can be used as a starting solution for the simplex iterative procedure. Therefore to arrive at a basic feasible solution and at the same time satisfies the original constraints, we should reduce the artificial variables to zero. To do this, let

$$w = x_6 + x_7 + x_8 \tag{1.13}$$

Since each of x_6, x_7, x_8 is non-negative, then the smallest value that w can take is zero, ($i.e.$ $\min w = 0$) at the same time when w reaches zero, each of the artificial variables has to be zero. Thus the first part of the solution is to find the values of $x_1, x_2, \cdots, x_5$ which minimize w. This is called phase I.

Now, since x_6, x_7, x_8 are selected as basic variables, then w should be expressed in terms of the non-basic variables by adding the equations in (1.12) and subtracting from (1.13) . We get

$$x_1 \sum_{i=1}^{3} a_{i1} + x_2 \sum_{i=1}^{3} a_{i2} - x_3 - x_4 - x_5 + w = \sum_{i=1}^{3} b_i$$

Therefore

$$w = d_1 x_1 + d_2 x_2 + x_3 + x_4 + x_5 + w_0 \tag{1.14}$$

where

$$d_1 = \sum_{i=1}^{3} a_{i1}, d_2 = \sum_{i=1}^{3} a_{i2}$$

and

$$w_0 = \sum_{i=1}^{3} b_i$$

and then the problem is to minimize (1.14) subject to

$$
\begin{array}{lcl}
c_1x_1 + c_2x_2 \qquad\qquad\qquad - f(x) & = & 0 \\
a_{11}x_1 + a_{12}x_2 - x_3 \qquad + x_6 & = & b_1 \\
a_{21}x_1 + a_{22}x_2 \qquad\qquad - x_4 + x_7 & = & b_2 \\
a_{31}x_1 + a_{32}x_2 \qquad\qquad\quad - x_5 + x_8 & = & b_3
\end{array}
$$

and

$$x_j \geq 0, \qquad j = 1, 2, \cdots, 8.$$

Basic var.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i
x_6	a_{11}	a_{12}	-1	0	0	1	0	0	b_1
x_7	a_{21}	a_{22}	0	-1	0	0	1	0	b_2
x_8	a_{31}	a_{32}	0	0	-1	0	0	1	b_3
$-f(x)$	c_1	c_2	0	0	0	0	0	0	0
$-w$	d_1	d_2	1	1	1	0	0	0	$-w_0$

Then we apply the rules of the simplex method until we get the minimum value of w. if $\min(w) > 0$, then there is no feasible solution for the original problem. On the other hand if $\min(w) = 0$, then the problem has a solution. And in the last tableau the artificial variables and the elements in the last row (*i.e.* for w) become all zeros except for the columns with headings x_6, x_7 and x_8 where the elements are ones. This is called phase I. The second part of the solution is to eliminate from the final transformed simplex tableau (which is obtained in phase I) the last row and the columns for the artificial variables. Then we apply the rules of the simplex method to minimize (or maximize) the modified objective function $f(x)$. This is called phase II.

Example 7 *Determine x_1 and x_2 that minimize the objective function*

$$f(x) = 3x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 &\geq 40 \\ 3x_1 + 2x_2 &\geq 50 \end{aligned}$$

and

$$x_1, x_2 \geq 0$$

Solution: By subtracting the slack variables x_3, x_4 and adding the artificial variables x_5, x_6, the structural constraints become

$$\begin{aligned} 2x_1 + 4x_2 - x_3 + x_5 \quad &= 40 \\ 3x_1 + 2x_2 \quad - x_4 + x_6 &= 50 \end{aligned}$$

where

$$x_j \geq 0, \quad j = 1, 2, \cdots, 6$$

let

$$x_5 + x_6 = w$$

Adding the structural constraints and subtracting from the w form, we get

$$5x_1 + 6x_2 - x_3 - x_4 + w = 90$$

So, the new problem is

$$\min w = 90 - 5x_1 - 6x_2 + x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + 4x_2 - x_3 + x_5 \quad &= 40 \\ 3x_1 + 2x_2 \quad - x_4 + x_6 &= 50 \\ 3x_1 + x_2 \quad\quad - f(x) &= 0 \end{aligned}$$

This can be written in a tabular form as follows:

Basic varb.	x_1	x_2	x_3	x_4	x_5	x_6	const.
x_5	2	4	-1	0	1	0	40
x_6	3	2	0	-1	0	1	50
$-f(x)$	3	1	0	0	0	0	0
$-w$	-5	-6	1	1	0	0	-90

Applying the simplex method for minimizing w, we get the following successive transformed tableau.

Basic varb.	x_1	x_2	x_3	x_4	x_5	x_6	const.
x_5	2	4	-1	0	1	0	40
x_6	3	2	0	-1	0	1	50
$-f(x)$	3	1	0	0	0	0	0
$-w$	-5	-6	1	1	0	0	-90
x_2	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	10
x_6	2	0	$\frac{1}{2}$	-1	$-\frac{1}{2}$	1	30
$-f(x)$	$\frac{5}{2}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	-10
$-w$	-2	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	-30
x_2	0	1	$-\frac{3}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{5}{2}$
x_1	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	15
$-f(x)$	0	0	$-\frac{3}{8}$	$\frac{5}{4}$	$\frac{3}{8}$	$-\frac{5}{4}$	$-\frac{95}{2}$
$-w$	0	0	0	0	1	1	0

Now $\min(w) = 0$, which means that $x_5 = x_6$ and the problem has a feasible solution. A basic feasible solution for the original problem is

$$\begin{aligned} x_1 &= 15, x_2 = \frac{5}{2}, \\ x_3 &= x_4 = 0 \end{aligned}$$

This ends phase I.

Phase II starts by reducing the last tableau to the ordinary tableau for minimizing $f(x)$. Thus the initial tableau for the original problem is:

Basic varb.	x_1	x_2	x_3	x_4	const.
x_2	0	1	$-\frac{3}{8}$	$\frac{1}{4}$	$\frac{5}{2}$
x_1	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	15
$-f(x)$	0	0	$-\frac{3}{8}$	$\frac{5}{4}$	$-\frac{95}{2}$

This tableau means that

$$\begin{aligned} x_3 &= x_4 = 0, \\ x_1 &= 15, x_2 = \frac{5}{2} \end{aligned}$$

is a basic feasible solution for the original problem.

Applying the simplex method for minimizing $f(x)$, then the transformed simplex tableau is

Basic varb.	x_1	x_2	x_3	x_4	const.
x_2	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	25
x_3	4	0	1	-2	60
$-f(x)$	$\frac{3}{2}$	0	0	$\frac{1}{2}$	-25

The coefficient in the last row are non-negative.

Therefore optimal solution is reached. This solution is

$$\begin{aligned} x_1 &= x_4 = 0 && \text{non-basic variables} \\ x_2 &= 25, x_3 = 60 && \text{basic variables} \end{aligned}$$

and

$$f(x) = 25$$

Solution of Linear programming problems having mixed types of constraints:

This means that the constraints of the types $\leq, \geq, =$ with the assumption that the constant coefficient b_i are non-negative. So we will consider the following example.

Example 8 *Determine x_1 and x_2 that minimize*

$$f(x) = 2x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 8 \\ 6x_1 + 4x_2 &\geq 12 \\ x_1 + 4x_2 &= 20 \end{aligned}$$

and

$$x_1, x_2 \geq 0$$

Solution: By adding slack and artificial variables, then the constraints are converted into equations and the problem becomes:

To minimize

$$f(x) = 2x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &= 8 \\ 6x_1 + 4x_2 - x_4 + x_5 \quad &= 12 \\ x_1 + 4x_2 \qquad\qquad + x_6 &= 20 \end{aligned}$$

let

$$x_5 + x_6 = w$$

then

$$-7x_1 - 8x_2 + x_4 = w - 32$$

and

$$x_j \geq 0, \quad j = 1, 2, \cdots, 6$$

The successive tables for solving phase I are:

Basic varb.	x_1	x_2	x_3	x_4	x_5	x_6	const.
x_3	1	1	1	0	0	0	8
x_5	6	4	0	-1	1	0	12
x_6	1	4	0	0	0	1	20
$-f(x)$	2	4	0	0	0	0	0
$-w$	-7	-8	0	1	0	0	-32
x_3	$-\frac{1}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	5
x_2	$\frac{3}{2}$	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	3
x_6	-5	0	0	1	-1	1	8
$-f(x)$	64	0	0	1	-1	0	-12
$-w$	5	0	0	-1	2	0	-8

Basic varb.	x_1	x_2	x_3	x_4	x_5	x_6	const.
x_3	$\frac{3}{4}$	0	1	0	0	$\frac{1}{4}$	3
x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	5
x_4	-5	0	0	1	-1	1	6
$-f(x)$	1	0	0	0	0	-1	-20
$-w$	0	0	0	0	1	1	0

Since $\min(w) = 0$, then the problem has a basic feasible solution and the final simplex tableau is reduced to the ordinary tableau for minimizing $f(x)$:

Basic varb.	x_1	x_2	x_3	x_4	const.
x_3	$\frac{3}{4}$	0	1	0	3
x_2	$\frac{1}{4}$	1	0	0	5
x_4	-5	0	0	1	6
$-f(x)$	1	0	0	0	-20

The coefficients in the last row are non-negative and accordingly the solution is optimal. This solution is

$$x_1 = 0, x_2 = 5$$

and

$$f(x) = 20$$

1.2.6 Degeneracy:

In determining the variable to be removed from the basis, we find the minimum of the ratios $\frac{b'_i}{a'_{is}}$ $(a'_{is} \geq 0)$ and if this minimum occurs for $i = r$, then the basic variable which appears on the rth row is the one that is dropped. This is fine as lon as a unique minimum exists. What if a tie occurs for two or more values of i? which variable should be dropped from the basic set ? Assume we drop the one with the lowest index. In obtaining the new canonical form, it will then turn out that, not only will the dropped variable be reduced to zero, but also all the other variables which tied with it; but they are still in the basis. A basis which contains variables at zero is called degenerate, and the basic variables at zero are called degenerate variables.

Example 9 *Determine x_1 and x_2 to maximize*

$$f(x) = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \qquad\quad &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 6 \end{aligned}$$

and

$$x_1, x_2 \geq 0$$

Solution: By adding the positive slack variables. the problem becomes:

To maximize

$$f(x) = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \qquad\quad + x_3 \qquad\quad &= 4 \\ x_2 \quad\ + x_4 \quad &= 6 \\ 3x_1 + 2x_2 \qquad\quad + x_5 &= 6 \end{aligned}$$

and

$$x_j \geq 0, \quad j = 1, 2, \cdots, 5$$

The original simplex tableau is given as follows:

Basic varb.	x_1	x_2	x_3	x_4	x_5	const.
x_3	1	0	1	0	0	4
x_4	0	1	0	1	0	6
x_5	3	2	0	0	1	12
$-f(x)$	3	5	0	0	0	0

By applying the rules of the simplex method, then the first transformed tableau is given as follows:

Basic varb.	x_1	x_2	x_3	x_4	x_5	const.
x_3	1	0	1	0	0	4
x_2	0	1	0	1	0	6
x_5	3	0	0	-2	1	0
$-f(x)$	3	0	0	-5	0	-30

We find that the basic variable x_5 has a zero value. Such a solution where one or more of the basic variables are zeros, is called a degenerate solution. Continuing the simplex method, then the next transformed simplex tableau is given by:

Basic varb.	x_1	x_2	x_3	x_4	x_5	const.
x_3	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	4
x_2	0	1	0	1	0	6
x_1	1	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0
$-f(x)$	0	0	0	-3	-1	-30

The optimal solution is

$$\begin{aligned} x_1 &= 0, x_2 = 6, x_3 = 4, \\ x_4 &= x_5 = 0 \end{aligned}$$

and the maximum value of the objective function is

$$f(x) = 30$$

which is the same as in the preceding iteration.

1.3 Duality

1.3.1 Definition of the dual problem:

We shall show that to every linear-programming problem there corresponds another linear programming problem which is called the dual of the original problem. We shall denote the original problem as the primal. The optimal solution of the primal solution of the dual are intimately connected. If the optimal solution to one is known then the optimal solution to the other is readily available. This fact is important because situations can arise where the dual is easier to solve than the primal.

Consider the following problem, which we shall denote as the primal.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\geq b_2 \\ \vdots \qquad\qquad &\geq \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\geq b_m \end{aligned} \tag{1.15}$$

$$\begin{aligned} z &= c_1x_1 + c_2x_2 + ... + c_nx_n \to \min \\ x_j &\geq 0 \end{aligned}$$

where the signs of all parameters are arbitrary. Let us now form a now problem which is obtained by transposing the rows and columns of (1.15), including the right-hand side and the objective function, reversing the inequalities, and maximizing instead of minimizing. Then we obtain.

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m &\leq c_2 \\ \vdots \qquad\qquad &\leq \vdots \\ a_{1n}y_1 + a_{2n}y_2 + ...a_{mn}y_m &\leq c_n \end{aligned} \tag{1.16}$$

$$\begin{aligned} v &= b_1y_1 + b_2y_{22} + ... + b_my_m \text{—}\max \\ y_i &\geq 0 \end{aligned}$$

By definition, (1.16) is the dual of (1.15), and $y_1, y_2, ...y_m$ are called the dual variables.

Let us consider now the linear programming problem in the standard form of equality constraints as the primal problem.

Find the values for $x_1, ..., x_n$ that minimize

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

subject to the m equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\qquad &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + almnx_n &= b_m
\end{aligned}$$

and

$$x_j \geq 0 \qquad , \ j = 1, ..., n.$$

The dual to this problem is:

Find the values $y_1, ..., y_m$ which maximize the objective function

$$v = b_1y_1 + b_2y_2 + ... + b_my_m$$

subject to the inequalities

$$\begin{aligned}
a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m &\leq c_2 \\
\vdots \qquad\qquad\qquad &\leq \vdots \\
a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m &\leq c_n
\end{aligned}$$

The variables y_j are not restricted to be non-negative.

Generally to each constraint of the primal problem there correspond z variable of the dual problem. If the constraint is inequality then the corresponding variables is restricted to be nonnegative while the variables that correspond to equation are not restricted in sign.

Example 10 *Write the dual to the following problem*

Maximize

$$z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad\quad &\leq 4 \\ x_2 &\leq 6 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

The dual problem can be written as:

Minimize

$$V = 4y_1 + 6y_2 + 18y_3$$

subject to

$$\begin{aligned} y_1 \quad\quad + 3y_3 &\geq 3 \\ y_2 + 2y_3 &\geq 5 \end{aligned}$$

and

$$y_1 \geq 0, \quad y_2 \geq 0$$

Example 11 *The primal problem is:*

minimize

$$z = x_1 + x_2 + x_3$$

subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_1 - 2x_2 \quad\quad\;\; &\leq 3 \\ 2x_2 \quad - x_3 &\geq 4 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0,$$

x_3 is unrestricted in sign.

The second constraint can be written

$$-x_1 + 2x_2 \geq -3$$

The dual problem is

maximize

$$V = 5y_1 - 3y_2 + 4y_3$$

subject to

$$\begin{aligned} y_1 - y_2 \quad\quad\quad &\leq 1 \\ -3y_1 + 2y_2 + 2y_3 &\leq 1 \\ 4y_1 \quad\quad\quad - y_3 &= 1 \end{aligned}$$

and y_1 is unrestricted in sign and

$$y_2 \geq 0 \;\; , y_3 \geq 0.$$

1.3.2 Duality Theorems

In the following theorems, we are going to use the standard forms for the primal and the dual problems given as:

I. The primal problem

To minimize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\geq b_i \qquad i = 1, 2, ..., m \\ \text{and} \qquad x_j &\geq 0 \qquad j = 1, 2, ..., n \end{aligned}$$

II. The dual problem

To maximize

$$z' = \sum_{j=1}^{m} b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\leq c_j \qquad j = 1, 2, ..., n \\ \text{and} \qquad y_i &\geq 0 \qquad i = 1, 2, ..., m \end{aligned}$$

Theorem 12 *The dual of the dual is the primal*

This theorem is obvious and implies a completely symmetrical relationship between the primal and the dual problems.

Theorem 13 *For any feasible solution, the values of the objective function for the primal problem is always greater than or equal to the value of the objective function for the dual problem .*

In other words, if $x_1, x_2, ..., x_n$ & $y_1, y_2, ..., y_m$ are feasible solution to the primal and the dual problems respectively, then

$$z = \sum_{j=1}^{n} c_j x_j \geq z' = \sum_{i=1}^{m} b_i y_i$$

Proof. Consider the primal and the dual problems as given by I &II.
From the dual problem

$$c_j \geq \sum_{i=1}^{m} a_{ij} y_i \tag{1.17}$$

Since $x_j \geq 0$, then we can multiply both sides of (1.17) by x_j and then sum over j, we get

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \quad a_{ij} y_i \right) x_j \tag{1.18}$$

Interchanging the summation signs of the right hand side of (1.18), we have

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \quad a_{ij} x_j \right) y_i \tag{1.19}$$

But from the primal problem

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$

Then substituting in (1.19) we get

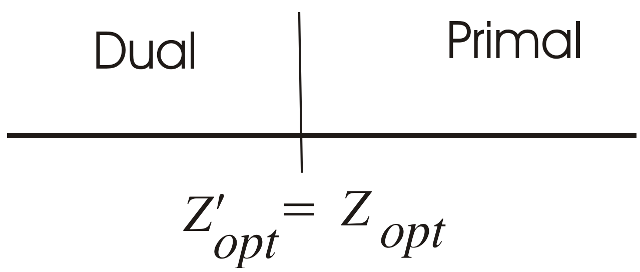
$$\begin{aligned} \sum_{j=1}^{n} c_j x_j &\geq \sum_{i=1}^{m} b_i y_i \\ i.e.\ \ z &\geq z'. \end{aligned}$$

which completes the proof. ■

Theorem 14 *If there exist finite feasible solutions for both the primal and the dual problems, then the values of the objective functions corresponding to their optimal solutions are equal.*

A constructive proof of this theorem is established by means of the simplex algorithm. From theorems 13 and 14 we have the following informations about the possible ranges of the values of the objective functions for the feasible solutions of the primal and the dual problems.

This can be illustrated by the following diagram:



This can be used in the following way:

1. If we know a feasible solution of the dual problem then we have a lower bound for the optimal value (minimal) of the objective function of the primal problem

$$z_{opt} \geq z'^0$$

2. If we know a feasible solution of the primal problem then we have an upper bound for the optimal value (maximum) of the objective function of the dual problem.

$$z'_{opt} \leq z^0$$

3. If an optimal solution of either problems is known then the value of the objective function of the other problem is also known.

$$z'_{opt} = z_{opt}.$$

Theorem 15 *If the primal (dual) problem has feasible solutions and the dual (primal) problem has no feasible solution, then the primal (dual) problem has unbounded solutions.*

Chapter 2

GRADIENT, DIVERGENCE AND CURL

2.1 Vector functions of several scalar variables

It is assumed that the reader has studied the real- valued functions of More than one variables in ordinary (scalar) calculus. For example, $\phi(x, y, z)$ is a function of three scalar variables x, y, z. Such functions will be referred to as "scalar functions of several scalar variables." We now extend this definition as follows:

If to each ordered triple (x, y, z) of scalars, there corresponds a vector $f(x, y, z)$, then $f(x, y, z)$ is said to be vector function of three scalar variables x, y, z.

The temperature in a room and the velocity of a moving particle are respectively the examples of scalar and vector functions.

Remark 16 *We can also talk about functions of more than three variables in a similar manner.*

Remark 17 *We can decompose $f(x, y, z)$ as*

$$f(x, y, z) = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k$$

where

$$f_1(x,y,z),\ \ f_2(x,y,z),\ \ f_3(x,y,z)$$

are three scalar functions of three scalar variables x, y, z.

Remark 18 *The definitions of limit and continuity of $f(x,y,z)$are analogous to those for $f(t)$.*

Remark 19 *If $\varphi(x,y,z)$ and $f(x,y,z)$ be defined at each point (x,y,z) in a certain region of space, then these are said to be scalar point function and vector point function respectively.*

2.2 Partial derivatives (or differential coefficient).

It $f(x,y,z)$ be a vector function of three scalar variables, then

$$\underline{\lim}_{\delta x \to 0} \frac{f(x+\delta x,y,z)}{\delta x}$$

If it exists, is called the partial derivative of f with respect to x, and is denoted by $\dfrac{\partial f}{\partial x}$.

Similarly $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$and may be defined. Further, Since the derivatives of $f(x,y,z)$ may be again the functions of x, y, z we may also have second order partial derivatives.

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\ \ ,\ \ \frac{\partial f}{\partial z}, \cdots$$

and other higher order derivatives.

Remark 20 *The reader will note that the definitions of concepts given in Sections 2.1 and 2.3 are analogous to those in ordinary (scalar) calculus and therefore their properties are also analogous.*

2.3 The vector differential operator ∇.

The vector operator ∇ (read as del or nabla) is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\ i + \frac{\partial}{\partial y}\ j + \ \frac{\partial}{\partial z}\ k \equiv \sum i \frac{\partial}{\partial x}$$

Remark 21 *The symbolic components*

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}$$

can be treated as three components of ∇ *and hence* ∇ *may considered as a symbolic vector itself. It serves as a vector differential operator, when it operates upon a function distributively.*

2.4 The operator $\vec{a}\cdot\ \nabla$, a being any vector.

The operator a. ∇ is defined by the equation

$$\vec{a}\cdot\ \nabla \equiv \vec{a}\cdot i \frac{\partial}{\partial x}\ + \vec{a}\cdot j \frac{\partial}{\partial y} + \vec{a}\cdot\ k\ \frac{\partial}{\partial z}\ \equiv \sum \vec{a}\cdot i \frac{\partial}{\partial x}$$

$$\therefore \left(\vec{a}\cdot\nabla\right)\ f \equiv \vec{a}\cdot i \frac{\partial f}{\partial x}\ + \vec{a}\cdot j \frac{\partial f}{\partial y} + \vec{a}\cdot\ k\ \frac{\partial f}{\partial z}\ \equiv \sum \left(\vec{a}\cdot i \frac{\partial}{\partial x}\right)\ f$$

and

$$\left(\vec{a}\cdot\nabla\right)\quad \phi\equiv\vec{a}\cdot i\frac{\partial\phi}{\partial x}\ +\vec{a}\cdot j\frac{\partial\phi}{\partial y}+\vec{a}\cdot\ k\ \frac{\partial\phi}{\partial z}\ \equiv\sum\left(\vec{a}\cdot i\frac{\partial}{\partial x}\right)\ \phi$$

If $\vec{a}\ = a_1 i + a_2 j + a_3 k$, then $\vec{a}\cdot\nabla \equiv a_1\frac{\partial}{\partial x}\ + a_2\frac{\partial}{\partial y} + a_3\ \frac{\partial}{\partial z}$

$\therefore \left(\vec{a}\cdot\nabla\right) f \equiv a_1\frac{\partial f}{\partial x}\ + a_2\frac{\partial f}{\partial y} + a_3\ \frac{\partial f}{\partial z}$

and

$$\left(\vec{a}\cdot\nabla\right)\phi \equiv a_1\frac{\partial\phi}{\partial x}\ + a_2\frac{\partial\phi}{\partial y}+a_3\ \frac{\partial\phi}{\partial z}$$

Remark 22

$$\left(\vec{a}\cdot\nabla\right)\phi = \vec{a}\cdot\nabla\phi$$

follows easily.

Remark 23 *Since* ∇ *and* f *are both vectors,* ∇f *is not defined. Hence* $a.\nabla f$ *means* $(a.\nabla)f$ *only.*

Remark 24

$$\left(\vec{a}\cdot\nabla\right) f \neq f\left(\vec{a}\cdot\nabla\right),$$

because the L.H.S. is a vector function, while the R.H.S. is a vector operator only.

2.5 Gradient of a scalar point function.

If $\phi(x,y,z)$ be a continuously differential scalar point function then the gradient of ϕ, written as grad ϕ or $\nabla\phi$, is defined as

$$\operatorname{grad}\phi = \nabla\phi \equiv\ i\frac{\partial\phi}{\partial x}+\ j\frac{\partial\phi}{\partial y}+\ k\frac{\partial\phi}{\partial z}\ \equiv\sum i\frac{\partial\phi}{\partial x}$$

Remark 25 *Gradient of a scalar point function $\phi(x, y, z)$ is a vector whose components are*

$$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \text{ and } \frac{\partial \phi}{\partial z}$$

Remark 26 *It is easily seen that the necessary and sufficient condition for a scalar point function ϕ to be constant is that $\nabla\phi = 0$.*

Remark 27 *If c is a constant, then we see that*

$$\begin{aligned} \nabla(c\phi) &= i\frac{\partial(c\phi)}{\partial x} + j\frac{\partial(c\phi)}{\partial y} + k\frac{\partial(c\phi)}{\partial z} \\ &= c\left[i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \right] \\ &= c\nabla(\phi) \end{aligned}$$

2.6 Divergence of a vector point function.

If $f(x, y, z)$ be a continuously differentiable vector point function, then the divergence of f, written as div f or $\nabla.f$, is defined as

$$\operatorname{div} f = \nabla \cdot f \equiv i\cdot\frac{\partial f}{\partial x} + j\cdot\frac{\partial f}{\partial y} + k\cdot\frac{\partial f}{\partial z} \equiv \sum i\cdot\frac{\partial f}{\partial x}$$

Remark 28 *Divergence of a vector point function is a scalar quantity. It may be treated as a scalar product of the operator ∇ with the vector f. However, it should be noted carefully that*

$$f \cdot \nabla \neq \nabla \cdot f$$

Remark 29

If $\overrightarrow{f} = f_1 i + f_2 j + f_3 k$, then

$$\begin{aligned} \operatorname{div} f &= \nabla \cdot f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (f_1 i + f_2 j + f_3 k) \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \end{aligned}$$

Remark 30 *A vector f is said to be solenoidal if* div $f = 0$.

Remark 31 *If c is a constant, then $\nabla \cdot (cf) = c\ \nabla \cdot f$.*

2.7 Curl of a vector point function.

If $f(x, y, z)$ be a continuously differentiable vector point function, then the curl of f, written as curl f or $\nabla x f$, is defined as

$$\operatorname{curl} f = \nabla \times f \equiv i\times\frac{\partial f}{\partial x} + j\times\frac{\partial f}{\partial y} + k\times\frac{\partial f}{\partial z} \equiv \sum i\times\frac{\partial f}{\partial x}$$

Remark 32 *Curl of a vector point function is a vector quantity. It may be treated as a vector product of the operator ∇ with the vector f. However, it should be noted carefully that*

$$f \times \nabla \neq \nabla \times f$$

Remark 33 *If*

$$f = f_1 i + f_2 j + f_3 k$$

then,

$$
\begin{aligned}
\operatorname{curl} f &= \nabla \times f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (f_1 i + f_2 j + k) \\
&= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}
\end{aligned}
$$

Note that in the expansion of the above determinant the operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

must precede f_1, f_2, f_3.

Remark 34 *A vector* f *is said to be irrotational if* $\operatorname{curl} \ f = 0$.

Remark 35 *If* c *is a constant, then*

$$\nabla \times (cf) = c\nabla \times f$$

Remark 36 $\operatorname{curl} \ f$ *is also called rotation of* f *and is written as* $rot \ f$.

2.8 Laplace's (or Laplacian) operator ∇^2.

Laplace's operator is denoted and defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \sum \frac{\partial^2}{\partial x^2}$$

$$\therefore \nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \equiv \sum \frac{\partial^2 f}{\partial x^2}$$

and

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \sum \frac{\partial^2 \phi}{\partial x^2}$$

The equation $\nabla^2 \phi = 0$ is called Laplace's equation. Any of its solution is known is known as an Harmonic function.

Theorem 37 *To show that*

$$\text{div} \quad \text{grad}\, \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Proof:

$$\begin{aligned} \text{div} \quad \text{grad}\, \phi = \text{div} \quad (\nabla \phi) &= \nabla \cdot \nabla \phi \\ &= \nabla \cdot \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi \end{aligned}$$

Remark 38 *From the above theorem, we get a new second order differential operator, via div grad, which can be written as*

$$\text{div} \quad \text{grad}\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2$$

Remark 39 *We write $\nabla \cdot \nabla$ as ∇^2 in keeping with the notion for a scalar, product, namely $a \cdot a = a^2$.*

2.9 Vector identities involving Diff.Op.

Let $\phi(x, y, z)$ and $\psi(x, y, z)$ be continuously differentiable scalar point functions and let $f(x, y, z)$ and $g(x, y, z)$ be continuously differentiable vector point functions. Then we have the following vector identities:

(i) $\text{grad}\,(\phi + \psi) = \text{grad}\,\phi + \text{grad}\,\psi$ or $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

(ii) $\text{div}\,(f + g) = \text{div}\,f + \text{div}\,g$ or $\nabla \cdot (f + g) = \nabla \cdot f + \nabla \cdot g$

(iii) $\text{curl}\,(f + g) = \text{curl}\,f + \text{curl}\,g$ or $\nabla \times (f + g) = \nabla \times f + \nabla \times g$

(iv) $\text{grad}\,(\phi\psi) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi$ or $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(v) $\text{grad}\left(\dfrac{\phi}{\psi}\right) = \dfrac{\psi\,\text{grad}\,\phi - \phi\,\text{grad}\,\psi}{\psi^2}$

(vi) $\text{div}\,(\phi f) = \phi\,\text{div}\,f + f \cdot \text{grad}\,\phi$ or $\nabla \cdot (\phi f) = \phi(\nabla \cdot f) + f \cdot \nabla\phi$

(vii) $\text{curl}\,(\phi f) = \phi\,\text{curl}\,f + \text{grad}\,\phi \times f$ or $\nabla \times (\phi f) = \phi(\nabla \times f) + \nabla\phi \times f$

$(viii)$ $\text{div}\,(f \times g) = g \cdot \text{curl}\,f - f \cdot \text{curl}\,g$ or $\nabla \cdot (f \times g) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g)$

(ix) $\text{curl}\,(f \times g) = f\,\text{div}\,f - g\,\text{div}\,f - (g \cdot \nabla)\,f \quad - (f \cdot \nabla)\,g.$

(x) $\text{grad}\,(f \cdot g) = f \times \text{curl}\,g + g \times \text{curl}\,f + (f \cdot \nabla)\,g + (g \cdot \nabla)\,f$

$$\frac{1}{2}\,\text{grad}\,f^2 = f \times \text{curl}\,f + (f \cdot \nabla)\,f$$

(xi) $\operatorname{curl}\operatorname{grad}\phi = 0$ $i.e.$ $\nabla \times (\nabla\phi) = 0.$

(xii) $\operatorname{div}\operatorname{curl} f = 0$ $i.e.$ $\nabla \cdot (\nabla \times f) = 0.$

$(xiii)$ $\operatorname{curl}\operatorname{curl} f = \operatorname{grad}\operatorname{div} f - \nabla^2 f$ $i.e.$ $\nabla \times (\nabla \times f) = \nabla (\nabla \cdot f) - \nabla^2 f$.

(xiv) $\operatorname{grad}\operatorname{div} f = \operatorname{curl}\operatorname{curl} f + \nabla^2 f.$

Proof: The proofs of these identities are based on the following definitions:

$$\begin{aligned} \text{grad } \phi &= \sum i\frac{\partial\phi}{\partial x}, \quad \text{div } f = \sum i\cdot\frac{\partial f}{\partial x}, \\ \text{curl } f &= \sum i\times\frac{\partial f}{\partial x}, \end{aligned}$$

(i)

$$\begin{aligned} \operatorname{grad}(\phi+\psi) &= \sum i\frac{\partial(\phi+\psi)}{\partial x} \\ &= \sum i\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial x}\right) = \sum i\frac{\partial\phi}{\partial x}+\sum i\frac{\partial\psi}{\partial x} \\ &= \operatorname{grad}\phi + \operatorname{grad}\psi \end{aligned}$$

(ii)

$$\begin{aligned} \text{div } (f+g) &= \sum i\cdot\frac{\partial(f+g)}{\partial x} \\ &= \sum i\cdot\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right) = \sum i\cdot\frac{\partial f}{\partial x}+\sum i\cdot\frac{\partial g}{\partial x} \\ &= \operatorname{div} f + \operatorname{div} g \end{aligned}$$

(iii)

$$\begin{aligned}
\operatorname{curl}(f+g) &= \sum i\times\frac{\partial(f+g)}{\partial x} \\
&= \sum i\times\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)=\sum i\times\frac{\partial f}{\partial x}+\sum i\times\frac{\partial g}{\partial x} \\
&= \operatorname{curl} f+\operatorname{curl} g.
\end{aligned}$$

(iv)

$$\begin{aligned}
\operatorname{grad}\quad(\phi\psi) &= \sum i\frac{\partial(\phi\psi)}{\partial x}=\sum i\left(\phi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\phi}{\partial x}\right) \\
&= \phi\sum i\frac{\partial\psi}{\partial x}+\psi\sum i\frac{\partial\phi}{\partial x} \\
&= \phi\operatorname{grad}\psi+\psi\operatorname{grad}\phi
\end{aligned}$$

(v)

$$\begin{aligned}
\operatorname{grad}\left(\frac{\phi}{\psi}\right) &= \sum i\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right)=\sum i\frac{\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}}{\psi^2} \\
&= \frac{\psi\sum i\frac{\partial\phi}{\partial x}-\phi\sum i\frac{\partial\psi}{\partial x}}{\psi^2} \\
&= \frac{\psi\operatorname{grad}\phi-\phi\operatorname{grad}\psi}{\psi^2}
\end{aligned}$$

(vi)

$$\begin{aligned}
\operatorname{div} \quad (\phi f) &= \sum i \cdot \frac{\partial}{\partial x} (\phi f) \\
&= \sum i \cdot \left(\frac{\partial \phi}{\partial x} f + \phi \frac{\partial f}{\partial x} \right) = \sum i \frac{\partial \phi}{\partial x} \cdot f + \sum \phi i \cdot \frac{\partial f}{\partial x} \\
&= f \cdot \sum i \frac{\partial \phi}{\partial x} + \phi \sum i \cdot \frac{\partial f}{\partial x} \\
&= f \cdot \operatorname{grad} \phi + \phi \operatorname{div} f
\end{aligned}$$

(vii)

$$\begin{aligned}
\operatorname{curl} \quad (\phi f) &= \sum i \times \frac{\partial}{\partial x} (\phi f) \\
&= \sum i \times \left(\frac{\partial \phi}{\partial x} f + \phi \frac{\partial f}{\partial x} \right) = \sum i \frac{\partial \phi}{\partial x} \times f + \sum \phi i \times \frac{\partial f}{\partial x} \\
&= \left(\sum i \frac{\partial \phi}{\partial x} \right) \times f + \phi \sum i \times \frac{\partial f}{\partial x} \\
&= \operatorname{grad} \phi \times f + \phi \operatorname{curl} f
\end{aligned}$$

$(viii)$

$$\begin{aligned}
\operatorname{div} \quad (f \times g) &= \sum i \cdot \frac{\partial}{\partial x} (f \times g) = \sum i \cdot \left(\frac{\partial f}{\partial x} \times g + f \times \frac{\partial g}{\partial x} \right) \\
&= \sum \left\{ i \cdot \left(\frac{\partial f}{\partial x} \times g \right) \right\} + \sum \left\{ i \cdot \left(f \times \frac{\partial g}{\partial x} \right) \right\} \\
&= \sum \left\{ i \times \left(\frac{\partial f}{\partial x} \cdot g \right) \right\} - \sum \left\{ \left(i \times \frac{\partial g}{\partial x} \right) \cdot f \right\}
\end{aligned}$$

Using the properties of scalar triple products, namely,

$$\begin{aligned} a\cdot(b\times c) &= (a\times b)\cdot c, \\ a\cdot(b\times c) &= -a\cdot(c\times b) = -(a\times c)\cdot b \end{aligned}$$

$$\begin{aligned} \therefore\ \text{div}\quad (f\times g) &= g\cdot\sum i\times\frac{\partial f}{\partial x} - f\cdot\sum i\times\frac{\partial g}{\partial x} \\ &= g\cdot\text{curl}\, f - f\cdot\text{curl}\, g \end{aligned}$$

(ix)

$$\begin{aligned} \text{curl}\quad (f\times g) &= \sum i\times\frac{\partial}{\partial x}(f\times g) = \sum i\times\left(\frac{\partial f}{\partial x}\times g + f\times\frac{\partial g}{\partial x}\right) \\ &= \sum\left\{i\times\left(\frac{\partial f}{\partial x}\times g\right)\right\} + \sum\left\{i\times\left(f\times\frac{\partial g}{\partial x}\right)\right\} \\ &= \sum\left\{(i\cdot g)\frac{\partial f}{\partial x} - \left(i\cdot\frac{\partial f}{\partial x}\right)g\right\} + \sum\left\{\left(i\cdot\frac{\partial g}{\partial x}\right)f - (i\cdot f)\frac{\partial g}{\partial x}\right\} \end{aligned}$$

$$\begin{aligned} &= \sum(g\cdot i)\frac{\partial f}{\partial x} - g\sum i\cdot\frac{\partial f}{\partial x} + f\sum i\cdot\frac{\partial g}{\partial x} - \sum(f\cdot i)\frac{\partial g}{\partial x} \\ &= (g\cdot\nabla)f - g\,\text{div}\, f + f\,\text{div}\, g - (f\cdot\nabla)g \end{aligned}$$

Using definition of divergence and section 2.4 we find that

$$\begin{aligned} \text{curl}\quad (f\times g) &= f\quad\text{div}\quad g - g\quad\text{div}\quad f \\ &\quad + (g\cdot\nabla)f - (f\cdot\nabla)g \end{aligned}$$

(x)

$$
\begin{aligned}
(f \cdot g) &= \sum i \frac{\partial}{\partial x}(f \cdot g) \qquad (2.1)\\
&= \sum i \left(\frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x}\right)\\
&= \sum i \left(g \cdot \frac{\partial f}{\partial x}\right) + \sum i \left(f \cdot \frac{\partial g}{\partial x}\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
g \times \operatorname{curl} f = g \times \sum i \times \frac{\partial f}{\partial x} &= \sum g \times \left(i \times \frac{\partial f}{\partial x}\right)\\
&= \sum \left[\left(g \cdot \frac{\partial f}{\partial x}\right) i - (g \cdot i) \frac{\partial f}{\partial x}\right]
\end{aligned}
$$

$$g \times \operatorname{curl} f = \sum i \left(g \cdot \frac{\partial f}{\partial x}\right) - \sum (g \cdot i) \frac{\partial f}{\partial x} \qquad (2.2)$$

$$\therefore \sum i \left(g \cdot \frac{\partial f}{\partial x}\right) = g \times \operatorname{curl} f + (g \cdot \nabla) f$$

[Rewriting and using section 2.4]
Similarly,

$$\therefore \sum i \left(f \cdot \frac{\partial g}{\partial x}\right) = f \times \operatorname{curl} g + (f \cdot \nabla) g \qquad (2.3)$$

Putting the values from (2.2) and (2.3) in (2.1), we have

$$
\begin{aligned}
\operatorname{grad}(f \cdot g) &= g \times \operatorname{curl} f + (g \cdot \nabla) f\\
&\quad + f \times \operatorname{curl} g + (f \cdot \nabla) g
\end{aligned}
$$

Deduction. Putting $g = f$ and noting $f.f = f^2$, we get

$$\text{grad}\, f^2 = 2f \times \text{curl}\, f + 2\,(f \cdot \nabla)\, f$$

or

$$\frac{1}{2}\text{grad}\, f^2 = f \times \text{curl}\, f + (f \cdot \nabla)\, f$$

(xi)

$$\text{curl}\,\text{grad}\,\phi = \text{curl}\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

Using Remark 33, section 2.7

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) i - \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) j - \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) k = 0$$

because

$$\frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y} \qquad etc.$$

(xii) Let $f = (f_1 i + f_2 j + f_3 k)$

Then, by Remark 33, section 2.7,

$$\text{curl}\, f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k$$

Then, using Remark 29, section 2.6, we have

$$\begin{aligned}
\operatorname{div}\operatorname{curl} f &= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0
\end{aligned}$$

because

$$\frac{\partial^2 f_3}{\partial x \partial y} = \frac{\partial^2 f_3}{\partial y \partial x}$$

$(xiii)$ Let $f = (f_1 i + f_2 j + k)$

Then, we have

$$\operatorname{curl} f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k$$

Then, using Remark 29, section 2.6 in determinate form, we get

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} f &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] i + \text{two similar terms} \\
&= \left[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_3}{\partial z \partial x}\right] i + \text{two similar terms}
\end{aligned}$$

$$= \left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y \partial x} + \frac{\partial^2 f_3}{\partial z \partial x} - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right)\right] i + \text{two similar terms}$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f\right] i + \text{two similar terms}$$

$$\begin{aligned}
&= \left[\frac{\partial}{\partial x}\operatorname{div} f - \nabla^2 f_1\right] i + \left[\frac{\partial}{\partial y}\operatorname{div} f - \nabla^2 f_2\right] j + \left[\frac{\partial}{\partial z}\operatorname{div} f - \nabla^2 f_3\right] k \\
&= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\operatorname{div} f - \nabla^2 \left(f_1 i + f_2 j + k\right) \\
&= \operatorname{grad}\operatorname{div} f - \nabla^2 f.
\end{aligned}$$

(XIV) Re-writing the identity XIII, we have

$$\operatorname{grad}\operatorname{div} f = \operatorname{curl}\operatorname{curl} f + \nabla^2 f.$$

2.10 List of some useful results.

Following results will be frequently used in solving various problems of this chapter.

(i) For the position vector r of a point (x, y, z) in space, we have

$$r = xi + yj + zk = \sum xi \quad \text{so that} \frac{\partial r}{\partial x} = i, \frac{\partial r}{\partial y} = j, \frac{\partial r}{\partial z} = k$$

Again $r = |\overrightarrow{r}| = \sqrt{x^2 + y^2 + z^2}$ so that $r^2 = x^2 + y^2 + z^2$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(ii) If f be any vector in space, then

$$f=(f\cdot i)\,i+(f\cdot j)\,j+(f\cdot k)\,k=\sum (f\cdot i)\,i$$

(iii)

$$\sum (i\cdot i)=(i\cdot i)+(j\cdot j)+(k\cdot k)=1+1+1=3$$

In particular, div $c=0$ and curl $c=0$, where c is a constant vector. Again, grad $c=0$ if c is a scalar.

Example 40 *If* $\phi(x,y,z)=x\ \sin\ y+y\ \cos\ z$, *find* grad ϕ *at* $(0,0,0)$.

Solution:

$$\begin{aligned}\operatorname{grad}\phi &= i\frac{\partial\phi}{\partial x}+\ j\frac{\partial\phi}{\partial y}+\ k\frac{\partial\phi}{\partial z}\\ &= i\,(\sin y)+j\,(x\cos y+\cos z)+k\,(-y\sin z)\end{aligned}$$

grad ϕ at $(0,0,0)=j$, putting $x=0,y=0,z=0$.

Example 41 *Prove that* div $\overrightarrow{r}=3$ *and* curl $\overrightarrow{r}=0$.

Solution:

$$\begin{aligned}\operatorname{div}\ \overrightarrow{r} &= \sum i\cdot\frac{\partial\,\overrightarrow{r}}{\partial x}=\sum (i\cdot i)=i\cdot i+j\cdot j+k\cdot k=3\\ \operatorname{curl}\ \overrightarrow{r} &= \sum i\times\frac{\partial\,\overrightarrow{r}}{\partial x}=\sum (i\times i)=i\times i+j\times j+k\times k=0\end{aligned}$$

Example 42 *Determine the constant a so that the vector $f = (x+3y)i + (y-2z)j + (x+az)k$ is solenoidal.*

Solution:

$$\operatorname{div} f = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 2+a$$

Since f is solenoidal, div $f = 0$. then $2+a=0$ or $a=-2$.

Example 43 *Show that the vector $f = (\sin y + z)i + (x\cos y - z)j + (x-y)k$ is irrotational.*

Solution:

$$\operatorname{curl} f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}$$

$$\begin{aligned} &= \left[\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right] i \\ &\quad - \left[\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial z}(\sin y + z)\right] j \\ &\quad + \left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right] k \\ &= 0 \end{aligned}$$

Since curl $f = 0$, by definition f is irrotational.

Example 44 *If a and b be constant vectors, show that (i)* grad $(\overrightarrow{r}.a) = a$
(ii) grad $[\overrightarrow{r}ab] = a \times b$

(iii) div $(\overrightarrow{r} \times a) = 0$ (iv) curl $(\overrightarrow{r} \times a) = -2a$

Solution: (i)

$$\begin{aligned}
\text{grad } (\overrightarrow{r}.\overrightarrow{a}) &= \sum i \frac{\partial}{\partial x}(\overrightarrow{r}.\overrightarrow{a}) = \sum i \left(\frac{\partial \overrightarrow{r}}{\partial x} \cdot \overrightarrow{a} + \overrightarrow{r} \cdot \frac{\partial \overrightarrow{a}}{\partial x} \right) \\
&= \sum i(i.\overrightarrow{a}) = \overrightarrow{a}
\end{aligned}$$

(ii) grad $[rab]$=grad $[r.(a \times b)] = a \times b$, by part (i)

(iii)

$$\begin{aligned}
\text{div } (\overrightarrow{r} \times \overrightarrow{a}) &= \sum i \cdot \frac{\partial}{\partial x}(\overrightarrow{r} \times \overrightarrow{a}) = \sum i \cdot \left(\frac{\partial \overrightarrow{r}}{\partial x} \times \overrightarrow{a} + \overrightarrow{r} \times \frac{\partial \overrightarrow{a}}{\partial x} \right) \\
&= \sum i \cdot \left(\frac{\partial \overrightarrow{r}}{\partial x} \times \overrightarrow{a} \right) = \sum i \cdot (i \times \overrightarrow{a}) \\
&= \sum (i \times i) \cdot \overrightarrow{a} \qquad \text{as } a \cdot (b \times c) = (a \times b) \cdot c \\
&= (i \times i) \cdot \overrightarrow{a} + (j \times j) \cdot \overrightarrow{a} + (k \times k) \cdot \overrightarrow{a} = 0
\end{aligned}$$

(vi)

$$\begin{aligned}
\text{curl } (\overrightarrow{r}\times\overrightarrow{a}) &= \sum i\times\frac{\partial}{\partial x}(\overrightarrow{r}\times\overrightarrow{a}) = \sum i\times\left(\frac{\partial\overrightarrow{r}}{\partial x}\times\overrightarrow{a}+\overrightarrow{r}\times\frac{\partial\overrightarrow{a}}{\partial x}\right) \\
&= \sum i\times\left(\frac{\partial\overrightarrow{r}}{\partial x}\times\overrightarrow{a}\right) = \sum i\times(i\times\overrightarrow{a}) \\
&= \sum\left[\left(i\cdot\overrightarrow{a}\right)i-(i\cdot i)\cdot\overrightarrow{a}\right] \\
&= \sum\left(i\cdot\overrightarrow{a}\right)i-\sum(i\cdot i)\cdot\overrightarrow{a} \\
&= \overrightarrow{a}-[(i\cdot i)\cdot\overrightarrow{a}+(j\cdot j)\cdot\overrightarrow{a}+(k\cdot k)] \\
&= \overrightarrow{a}-3\overrightarrow{a}=-2\overrightarrow{a}
\end{aligned}$$

Example 45 *If* $r=|\overrightarrow{r}|$ *where* $r=xi+yj+zk$ *prove that*

(i) $\nabla\log|\overrightarrow{r}|=\dfrac{\overrightarrow{r}}{r}$ $\qquad$ (ii) $\nabla r^n=nr^{n-2}\overrightarrow{r}$

(iii) $\nabla f(r)=f'(r)\nabla r$ $\qquad$ (iv) $\nabla\left(\dfrac{1}{r}\right)=-\dfrac{\overrightarrow{r}}{r^3}$

Solution: (i)

$$\begin{aligned}
\nabla\log|\overrightarrow{r}| &= \nabla\log r=\sum i\frac{\partial\log r}{\partial x} \\
&= \sum i\frac{1}{r}\frac{\partial r}{\partial x}=\sum i\frac{1}{r}\frac{x}{r} \\
&= \frac{1}{r^2}(xi+yj+zk)=\frac{\overrightarrow{r}}{r}
\end{aligned}$$

(ii)

$$\begin{aligned}\nabla r^n &= \sum i \frac{\partial r^n}{\partial x} = \sum i n r^{n-1} \frac{\partial r}{\partial x} \\ &= \sum i n r^{n-1} \frac{x}{r} = n r^{n-2} \sum ix \\ = nr & ^{n-2} (xi + yj + zk) = nr^{n-2} \overrightarrow{r}\end{aligned}$$

(iii)

$$\begin{aligned}\nabla f(r) &= \sum i \frac{\partial}{\partial x} f(r) = \sum i f'(r) \frac{\partial r}{\partial x} \\ = f'(r) & \quad \sum i \frac{\partial r}{\partial x} = f'(r) \nabla r\end{aligned}$$

(iv)

$$\begin{aligned}\nabla \left(\frac{1}{r}\right) &= \sum i \frac{\partial}{\partial x} \left(\frac{1}{r}\right) = \sum i \left(\frac{-1}{r^2} \frac{\partial r}{\partial x}\right) \\ &= -\sum i \frac{1}{r^2} \frac{x}{r} = -\frac{1}{r^3} \sum ix \\ &= -\frac{\overrightarrow{r}}{r^3}\end{aligned}$$

Example 46 (i) *Show that* $\operatorname{div} \quad (r^3 \overrightarrow{r}) = (n+3) r^n$, *where* $r = |\overrightarrow{r}|$ *and* $\overrightarrow{r} = xi + yj + zk$. *If* $r^n \overrightarrow{r}$ *is solenoidal, then prove that* $n + 3 = 0$. *Prove that*

$$\nabla^2 r^n \overrightarrow{r} = n(n+3) r^{n-2} \overrightarrow{r}.$$

Solution:

$$\text{div} \quad (r^n \overrightarrow{r}) = r^n \quad \text{div} \quad r \quad + \quad r \cdot \quad \text{grad} \quad r^n,$$

by identity VI, section 2.9

$$\begin{aligned} &= 3r^n + \overrightarrow{r} \cdot (nr^{n-2} \overrightarrow{r}) \\ &= 3r^n + nr^{n-2} (\overrightarrow{r} \cdot \overrightarrow{r}) = (3+n)\, r^n \end{aligned}$$

Again, $r^n \overrightarrow{r}$ is stomodeal $\Rightarrow$ div $(r^n \overrightarrow{r}) = 0 \Rightarrow n + 3 = 0$
(ii)

$$\begin{aligned} \nabla^2 r^n \overrightarrow{r} &= \text{grad}\,\text{div}\,(r^n \overrightarrow{r}) = \text{grad}\,[(3+n)\, r^n] \quad \text{by part } (i) \\ &= (3+n)\,\text{grad}\, r^n = (3+n)\, nr^{n-2} \overrightarrow{r} \\ &= n\,(n+3)\, r^{n-2} \overrightarrow{r}. \end{aligned}$$

Example 47 (i) *Show that* div $(r^3 \overrightarrow{r}) = (n+3)\, r^n$, *where* $r = |\overrightarrow{r}|$ *and* $\overrightarrow{r} = xi + yj + zk$. *If* $r^n \overrightarrow{r}$ *is solenoidal, then prove that* $n + 3 = 0$. *Prove that*

$$\nabla^2 r^n \overrightarrow{r} = n\,(n+3)\, r^{n-2} \overrightarrow{r}.$$

Solution:

$$\text{div} \quad (r^n \overrightarrow{r}) = r^n \quad \text{div} \quad r \quad + \quad r \cdot \quad \text{grad} \quad r^n,$$

by identity VI, section 2.9

$$
\begin{aligned}
&= 3r^n + \overrightarrow{r} \cdot \left(n r^{n-2} \overrightarrow{r}\right) \\
&= 3r^n + n r^{n-2} \left(\overrightarrow{r} \cdot \overrightarrow{r}\right) = (3+n)\, r^n
\end{aligned}
$$

Again, $r^n \overrightarrow{r}$ is stomodeal $\Rightarrow$ div $(r^n \overrightarrow{r}) = 0 \Rightarrow n + 3 = 0$

(ii)

$$
\begin{aligned}
\nabla^2 r^n \overrightarrow{r} &= \operatorname{grad} \operatorname{div} \left(r^n \overrightarrow{r}\right) = \operatorname{grad}\left[(3+n)\, r^n\right] \quad \text{by part } (i) \\
&= (3+n) \operatorname{grad} r^n = (3+n)\, n r^{n-2} \overrightarrow{r} \\
&= n\,(n+3)\, r^{n-2} \overrightarrow{r}.
\end{aligned}
$$

Example 48 *A vector f is both solenoidal as well as irrotational. Show that f can be expressed as the gradient of a scalar function ϕ, where ϕ satisfies the Laplace's equation $\nabla^2 \phi = 0$.*

Solution: Since f is irrotational, curl $f = 0$. Again we know that (identity XI, section 2.9) curl grad $f = 0$. So it follows that $f =$ grad f. Further, since f is solenoidal, div $f = 0$ or div grad $f = 0$ or $\nabla^2 \phi = 0$.

Example 49 *If $\overrightarrow{r}$ be position vector and a, b be constant vectors, prove:*

$$
(i) \qquad \nabla\left(\overrightarrow{b} \cdot \nabla \frac{1}{r}\right) = -\frac{b}{r^3} + \frac{3\left(\overrightarrow{b} \cdot \overrightarrow{r}\right) \overrightarrow{r}}{r^5}
$$

$$
(ii) \qquad \overrightarrow{a} \cdot \nabla\left(b \cdot \nabla \frac{1}{r}\right) = \frac{3\left(\overrightarrow{a} \cdot \overrightarrow{r}\right)\left(\overrightarrow{b} \cdot \overrightarrow{r}\right)}{r^5} - \frac{\left(\overrightarrow{a} \cdot \overrightarrow{b}\right)}{r^3}
$$

Solution: Using example 45 (iv),

$$\nabla\frac{1}{r}=-\frac{\overrightarrow{r}}{r^3}$$

Thus, we get

$$\nabla\left(\overrightarrow{b}\cdot\nabla\frac{1}{r}\right) = \nabla\left(\overrightarrow{b}\cdot\frac{-\overrightarrow{r}}{r^3}\right)$$

$$=-\nabla\left(\frac{\overrightarrow{b}\cdot\overrightarrow{r}}{r^3}\right)=-\sum i\frac{\partial}{\partial x}\left(\frac{\overrightarrow{b}\cdot\overrightarrow{r}}{r^3}\right)$$

$$=-\sum i\left[\frac{\partial}{\partial x}\left(\frac{1}{r^3}\frac{\partial}{\partial x}\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)+\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\frac{\partial r^{-3}}{\partial x}\right)\right]$$

$$=-\sum i\left[\left(\frac{1}{r^3}\frac{\partial}{\partial x}\left(\frac{\partial\overrightarrow{b}}{\partial x}\cdot\overrightarrow{r}+\overrightarrow{b}\cdot\frac{\partial\overrightarrow{r}}{\partial x}\right)+\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\left(-3r^{-4}\right)\frac{\partial\overrightarrow{r}}{\partial x}\right)\right]$$

$$=-\sum i\left[\left(\frac{1}{r^3}\frac{\partial}{\partial x}\left(0+\overrightarrow{b}\cdot i\right)+\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\left(-\frac{3}{r^4}\right)\frac{x}{r}\right)\right]$$

$$\begin{aligned} &= -\frac{1}{r^3}\sum\left(\overrightarrow{b}\cdot i\right)i+\frac{3}{r^5}\sum xi \\ &= -\frac{1}{r^3}\overrightarrow{b}+\frac{3}{r^5}\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\overrightarrow{r}\end{aligned}$$

$$(ii\quad)\ \overrightarrow{a}\cdot\nabla\left(b\cdot\nabla\frac{1}{r}\right) = \overrightarrow{a}\cdot\left[-\frac{1}{r^3}\overrightarrow{b}+\frac{3}{r^5}\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\overrightarrow{r}\right]$$

$$= \frac{3\left(\overrightarrow{a}\cdot\overrightarrow{r}\right)\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)}{r^5} - \frac{\left(\overrightarrow{a}\cdot\overrightarrow{b}\right)}{r^3}$$

Example 50 *(i) If A and B are irrotational, prove that $A \times B$ is solenoidal.*

(ii) Prove that $f(r)\overrightarrow{r}$ is irrotational.

Solution: Since A and B are irrotational, curl $A =$ curl $B = 0$.

then,

$$\begin{aligned} \text{div }(A \times B) = B\cdot\text{curl }A - A\cdot\text{curl }B \\ &= B\cdot 0 - A\cdot 0 \\ &= 0. \end{aligned}$$

Since div $(A \times B) = 0$, if follows that $A \times B$ is solenoidal.

(ii) Using identity VII, section 2.9, we have

$$\text{curl }[f(r)\overrightarrow{r}] = \text{grad }f(r)\times\overrightarrow{r} + f(r)\,\text{curl }\overrightarrow{r}. \tag{2.4}$$

From example 45 (i) and example 41 (iii),

$$\text{grad }\ f(r) = f^{'}(r)\nabla(r) \text{ and curl }\ \overrightarrow{r} = 0.$$

Also

$$\begin{aligned}\nabla r &= \sum i \frac{\partial r}{\partial x} = \sum i \frac{x}{r} \\ &= \frac{1}{r} \sum ix = \frac{\overrightarrow{r}}{r}\end{aligned}$$

then, from equation (2.6),

$$\text{curl} \quad [f(r)\overrightarrow{r}] = \frac{\overrightarrow{r}}{r} \times \overrightarrow{r} = 0$$

Since curl $[f(r)\overrightarrow{r}] = 0$, it follows that $f(r)\overrightarrow{r}$ is irrotational.

Example 51 *Given that* $\rho F = \nabla P$ *where* ρ, P, F *are point functions, prove that* $F \cdot \text{curl} \ \ F = 0$.

Solution: From given relation we have $F = \left(\frac{1}{\rho}\right) \nabla P$ so that curl $F =$ curl $\left[\left(\frac{1}{\rho}\right) \nabla P\right]$.

$$\begin{aligned}\text{curl} \ \ F = \text{curl} \quad &\left[\left(\frac{1}{\rho}\right) \nabla P\right] = \frac{1}{\rho} \text{curl} \, \nabla P + \text{grad} \left(\frac{1}{\rho}\right) \times \nabla P \\ = \quad &\text{grad} \left(\frac{1}{\rho}\right) \times \nabla P = \nabla \left(\frac{1}{\rho}\right) \times \nabla P\end{aligned}$$

$$\begin{aligned}\therefore F \cdot \text{curl} \, F &= \frac{1}{\rho} \nabla P \cdot \left[\nabla \left(\frac{1}{\rho}\right) \times \nabla P\right] \\ &= \frac{1}{\rho} \left[\nabla P, \nabla \left(\frac{1}{\rho}\right), \nabla P\right] = 0,\end{aligned}$$

since the value of scalar triple product vanishes when it has two equal vectors.

Example 52 *(i) If A and B are irrotational, prove that AxB is solenoidal.*

(ii) Prove that $f(r)\overrightarrow{r}$ is irrotational.

Solution: Since A and B are irrotational, curl $A=$ curl $B=0$.

then,

$$\text{div}\quad (A\times B)=B\cdot\text{curl}\quad A-A\cdot\text{curl}\quad B.$$

$$=B\cdot 0-A\cdot 0=0$$

Since div $(A\times B)=0$, if follows that $A\times B$ is solenoidal.

(ii) Using identity VII, section 2.9, we have

$$\text{curl}\ [f(r)\overrightarrow{r}]=\text{grad}\ f(r)\times\overrightarrow{r}+f(r)\,\text{curl}\ \overrightarrow{r}. \tag{2.5}$$

From example 45 (i) and example 41 (iii) ,

$$\text{grad}\ f(r)=f^{'}(r)\nabla(r)\quad\text{and}\quad\text{curl}\ \overrightarrow{r}=0.$$

Also

$$\nabla r=\sum i\frac{\partial\overrightarrow{r}}{\partial x}=\sum i\frac{x}{r}=\frac{1}{r}\sum ix=\frac{1}{r}\overrightarrow{r}$$

then, from equation (2.6),

$$\text{curl}\ \overrightarrow{r}=\text{curl}\left[f^{'}(r)\overrightarrow{r}\right]=\frac{1}{r}\overrightarrow{r}\times\overrightarrow{r}=0$$

Since curl $[f(r)\overrightarrow{r}]=0$, it follows that $f(r)\overrightarrow{r}$ is irrotational.

Example 53 *If* $A=\frac{x}{r}i+\frac{y}{r}j+\frac{z}{r}k=\frac{1}{u^2}$, *where* $r=x^2+y^2+z^2$ *find the value of* curl grad u.

Solution:

$$\text{curl} \quad \text{grad} \quad u = \text{curl}\left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \frac{-x}{(x^2+y^2+z^2)^{3/2}} & \frac{-y}{(x^2+y^2+z^2)^{3/2}} & \frac{-z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix} \text{ since } u = \left(x^2+y^2+z^2\right)^{3/2}$$

$$= i\left[\frac{\partial}{\partial y}\left\{-z\left(x^2+y^2+z^2\right)^{-3/2}\right\} - \frac{\partial}{\partial z}\left\{-y\left(x^2+y^2+z^2\right)^{-3/2}\right\}\right]$$

$$-j\left[\frac{\partial}{\partial x}\left\{-z\left(x^2+y^2+z^2\right)^{-3/2}\right\} - \frac{\partial}{\partial z}\left\{-x\left(x^2+y^2+z^2\right)^{-3/2}\right\}\right]$$

$$+ \quad k\left[\frac{\partial}{\partial x}\left\{-y\left(x^2+y^2+z^2\right)^{-3/2}\right\} - \frac{\partial}{\partial y}\left\{-x\left(x^2+y^2+z^2\right)^{-3/2}\right\}\right]$$

$$= i\left[z\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2y - y\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2z\right]$$

$$- \ j\left[z\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2x - x\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2z\right]$$

$$+ \quad k\left[y\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2x - x\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot 2y\right]$$

$$= \quad 0 \text{ as required}$$

Example 54 *A vector function f is the product of a scalar function and the gradient of a scalar function. Show that $f.\operatorname{curl}\ f = 0$.*

Solution: Let $f = \psi\,\text{grad}\ \ \varphi$, where ψ and φ are scalar functions. Then

$$\begin{aligned}\text{curl}\ \ f &= \text{curl}\,(\psi\,\text{grad}\ \ \varphi)\\ &= \psi\,\text{curl}\ \ \text{grad}\ \ \varphi + (\text{grad}\ \ \psi)\times(\text{grad}\ \ \varphi)\\ &= (\text{grad}\ \ \psi)\times(\text{grad}\ \ \varphi)\end{aligned}$$

[since curl grad $\varphi = 0$] ,then,

$$\begin{aligned}f\cdot\text{curl}\ \ f &= (\psi\,\text{grad}\ \ \varphi)\cdot(\text{grad}\ \ \psi)\times(\text{grad}\ \ \varphi)\\ &= \psi\,[\text{grad}\ \ \varphi \qquad \text{grad}\ \ \psi \qquad \text{grad}\ \ \varphi] = 0.\end{aligned}$$

Exercise

1. If $f = \left(2x^2y - x^4\right)i + \left(e^{xy} - y\sin x\right)j + \left(x^2\cos\right)k$, verify that $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$.
2. If $f = \left(2x^2yz\right)i + \left(2xz^3\right)j + \left(xz^2\right)k$, and $\overrightarrow{s} = (2z)\,i + (y)\,j - \left(x^2\right)k$ find $\frac{\partial^2 f}{\partial x\partial y}\left(\overrightarrow{r}\times\overrightarrow{s}\right)$ at the point $(1, 0, -2)$.

   (a) If $\phi(x, y, z) = 3x^2y - y^3z^2$, find grad φ at the point $(1, -2, -1)$

   (b) If $\phi(x, y, z) = x^2y - y^3x + z^2$, find grad φ at the point $(1, 1, 1)$

   (a) If $F = xy^2i + 2x^2yzj - 3yz^3k$ find div F at the point $(1, -1, 1)$.

   (b) If $F = x^2zi - 2y^2z^2j + 2xy^2zk$ find div F at the point $(1, -1, 1)$

3. Show that $\nabla \cdot \nabla\varphi = \nabla^2\varphi$ where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

4. Show that

$$\text{div}\,\{\overrightarrow{a} \times (\overrightarrow{r} \times \overrightarrow{a})\} = 2a^2$$

where $\overrightarrow{a}$ is constant vector,

$$\overrightarrow{r} = xi + yj + zk \quad \text{and} \quad |\overrightarrow{a}| = a$$

5. If $\overrightarrow{r}$ be position vector and a, b be constant vectors, prove:

$$(i) \qquad \nabla\left(\overrightarrow{b}\cdot\nabla\frac{1}{r}\right) = -\frac{b}{r^3} + \frac{3\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)\overrightarrow{r}}{r^5}$$

$$(ii) \qquad \overrightarrow{a}\cdot\nabla\left(b\cdot\nabla\frac{1}{r}\right) = \frac{3\left(\overrightarrow{a}\cdot\overrightarrow{r}\right)\left(\overrightarrow{b}\cdot\overrightarrow{r}\right)}{r^5} - \frac{\left(\overrightarrow{a}\cdot\overrightarrow{b}\right)}{r^3}$$

6. Given that $\rho F = \nabla P$ where ρ, P, F are point functions, prove that $F\cdot\text{curl}\;F = 0$.

7. If $A = \frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k$,where

$$\begin{aligned} r^2 &= x^2+y^2+z^2 \\ &= \frac{1}{u^2}, \end{aligned}$$

find the value of curl grad u.

Example 55 *A vector f is both solenoidal as well as irrotational. Show that f can be expressed as the gradient of a scalar function ϕ, where ϕ satisfies the laplace's equation $\nabla^2\phi = 0$.*

Solution: Since f is irrotational, curl $f = 0$. Again we know that (identity XI, section 2.9) curl grad $f = 0$. So it follows that $f =$ grad f. Further, since f is solenoidal, div $f = 0$ or div grad $f = 0$ or $\nabla^2\phi = 0$.

Example 56 *(i) If A and B are irrotational, prove that $A \times B$ is solenoidal.*

(ii) Prove that $f(r)\overrightarrow{r}$ is irrotational.
Solution: Since A and B are irrotational, curl $A =$ curl $B = 0$.
then,

$$\begin{aligned} \text{div } (A \times B) = B \cdot \text{curl } A - A \cdot \text{curl } B \\ &= B \cdot 0 - A \cdot 0 \\ &= 0. \end{aligned}$$

Since div $(A \times B) = 0$, if follows that $A \times B$ is solenoidal.
(ii) Using identity VII, section 2.9, we have

$$\text{curl } [f(r)\overrightarrow{r}] = \text{grad } f(r) \times \overrightarrow{r} + f(r)\,\text{curl } \overrightarrow{r}. \tag{2.6}$$

From example 45 (i) and example 41 (iii),

$$\text{grad } f(r) = f^{'}(r)\nabla(r) \text{ and curl } \overrightarrow{r} = 0.$$

Also

$$\begin{aligned}\nabla r &= \sum i\frac{\partial r}{\partial x}=\sum i\frac{x}{r} \\ &= \frac{1}{r}\sum ix=\frac{\overrightarrow{r}}{r}\end{aligned}$$

then, from equation (2.6),

$$\text{curl }\left[f(r)\overrightarrow{r}\right]=\frac{\overrightarrow{r}}{r}\times\overrightarrow{r}=0$$

Since curl $[f(r)\overrightarrow{r}]=0$, it follows that $f(r)\overrightarrow{r}$ is irrotational.

2.11 Green's theorem in the plane

(Relation between plane surface integral and line integral).

If R is a closed region in the $xy-$plane bounded by a simple closed curve C and if $\phi(x,y)$ and $\psi(x,y)$ are continuous functions having continuous partial derivatives in R, then

$$\oint_c \psi dx+\phi dy=\iint_R\left(\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial y}\right)dxdy$$

where C is traversed in the positive (anti-clockwise) direction.

2.12 Stokes' Theorem

(Relation between surface and line integrals).

If F is any continuously differentiable vector point function and S is a surface bounded by a curve C, then

$$\oint_c \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_s \text{curl}\, \overrightarrow{F} \cdot \overrightarrow{n}\, ds$$

where the unit normal vector n at any point of S is drawn in the direction in which a right handed screw would move when rotated in the sense of description of C.

Another form. The line integral of the tangential component of a vector F taken round a closed curve C is equal to the surface integral of the normal component of the curl F taken over any surface S having C as its boundary.

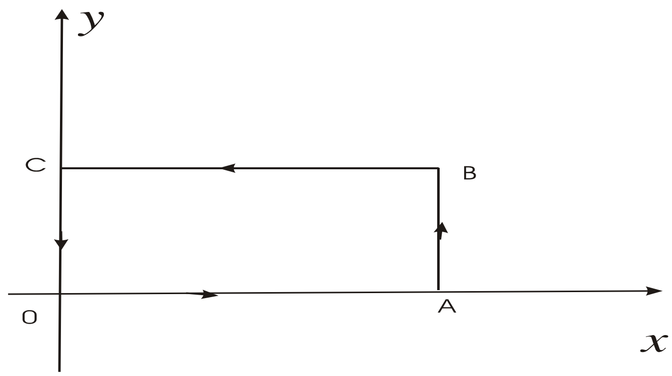
2.13 Gauss' Divergence Theorem.

(Relation between surface and volume integrals).

If F is a continuously differentiable vector point function and V is the volume bounded by a closed surface S, then

$$\iint_S F \cdot \overrightarrow{ds} = \iiint_v \text{div}\, F dV$$

Another from. The normal surface integral of a continuously differentiable vector point function F over a closed surface S enclosing a volume V, is equal to the volume integral of the divergence of F taken over the volume V.



2.14 Illustrative Examples.

Example 57 *Evaluate by Green's theorem in the plane*

$$\oint_c \left(x^3 - \cosh y\right) dx + (y + \sin x)\, dy,$$

where C *is the rectangle with vertices* $(0,0)\,,(\pi,0)\,,\ (\pi,1)\,,(0,1)\,.$

Solution: The path of integration C consists of the straight lines OA, AB, BC and CO where the coordinates of A, B, C are $(\pi,0)\,,(\pi,1)$ and $(0,1)$ and respectively. Let R be the plane region bounded by C.

Using Green's theorem in plane, we have

$$\oint_c \left(x^3 - \cosh y\right) dx + (y + \sin x)\, dy$$

$$= \iint_R \left(\frac{\partial (y + \sin x)}{\partial x} - \frac{\partial (x^3 - \cosh y)}{\partial y} \right) dxdy$$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\, dxdy = \int_{x=0}^{\pi} [y \cos x + \sinh y]_0^1 \, dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\, dx = [\sin x + x \cosh 1 - x]_0^{\pi}$$

$$\pi \cosh 1 - \pi = \pi (\cosh 1 - 1)\,.$$

Example 58 *(a) Verify Green's theorem in the plane for*

$$\oint_c \left(x^2 + y^2\right) dx - 2xydy,$$

where C is the rectangle bounded by

$$y = 0, x = 0, y = b, x = a$$

(b) Verify Stokes' theorem for

$$F = \left(x^2 + y^2\right) i - 2xyj$$

taken round the rectangle bounded by

$$y = 0, x = a, y = b, x = 0$$

(c) Verify Stokes' theorem for the vector field defined by

$$F = \left(x^2 + y^2\right) i - 2xyj$$

in the rectangular region in the $xy-$*plane bounded by the lines*

$$y = 0, x = a, y = b, x = 0$$

Solution: (a) For figure, refer 57. Here coordinates of A, B, C are $(a, 0), (a, b)$, $(0, b)$ respectively. Let R be the region bounded by

$$y = 0, x = 0, y = b, x = a$$

Then, by Green's theorem in plane, we must have

$$\oint_c \left(x^2 + y^2\right) dx - 2xydy$$

$$= \iint_R \left(\frac{\partial\left(-2xy\right)}{\partial x} - \frac{\partial\left(x^2 + y^2\right)}{\partial y} \right) dxdy \tag{2.7}$$

$L.H.S. = -2ab^2$ see previous chapter.

$$R.H.S. = \int_{x=0}^{a} \int_{y=0}^{1} (\quad - 4y)\, dxdy = -4 \int_0^a \left[\frac{y^2}{2} \right]_0^b dx$$

$$= -2b^2 \int_0^a dx = -2ab^2$$

Hence the Green's theorem (2.7) is verified.

(b) Since the given surface is plane rectangle in $xy-$plane, by Stokes' theorem in plane, we get (2.7). Now verify the theorem as in part (a).

Second Method. By Stokes' theorem, we must have

$$\oint_c F \cdot d\overrightarrow{r} = \iint_S \operatorname{curl} F \cdot \overrightarrow{n}\, ds \tag{2.8}$$

$L.H.S. = -2ab^2$ see previous chapter.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = -4yk$$

Here n =unit vector $\perp$ to $xy-$plane $= k$

$$\therefore R.H.S. = \iint_R (-4yk) \cdot k\,dxdy \quad \text{since } S = R$$

$$= \int_{x=0}^{a} \int_{y=0}^{1} (\; - 4y)\, dxdy = -2ab^2 \qquad \text{as before}$$

then, (2.8) is true. Hence the theorem is verified.

(c) Proceed as in part (b).

Example 59 *If C be a closed curve, then show that*

$$(i) \quad \oint_c \overrightarrow{r} \cdot d\overrightarrow{r} = 0 \quad (ii) \qquad \oint_c \phi \nabla\phi \cdot d\overrightarrow{r} = 0$$

Solution: (i) Using Stokes' theorem, we have

$$\oint_c \overrightarrow{r} \cdot d\overrightarrow{r} = \iint_S \text{curl}\, \overrightarrow{r} \cdot \overrightarrow{n}\, ds = 0 \qquad \text{since curl}\, \overrightarrow{r} = 0$$

(ii) Using Stokes' theorem, to vector function $\phi\nabla\phi$, we get

$$\begin{aligned} \oint_c (\phi\nabla\phi) \cdot d\overrightarrow{r} &= \iint_S \text{curl}\,(\phi\nabla\phi) \cdot \overrightarrow{n}\, ds \\ &= \iint_S [\phi\, \text{curl}\, \nabla\phi + \nabla\phi \times \nabla\phi] \cdot \overrightarrow{n}\, ds = 0 \end{aligned}$$

Since $\text{curl}\, \nabla\phi = 0$, and $\nabla\phi \times \nabla\phi = 0$.

Example 60 *Apply Gauss' Divergence theorem to show that*

$$(i) \quad \iint_S \overrightarrow{r} \cdot \overrightarrow{n}\, ds = 3V \quad (ii) \quad \iint_S \nabla r^2 \cdot \overrightarrow{ds} = 6V$$

where V is the volume of the space enclosed by the surface S.

Proof: (i) Applying Gauss' divergence theorem, we get

$$\begin{aligned} \iint_S \overrightarrow{r} \cdot \overrightarrow{n}\, ds &= \iiint_V \text{div}\, \overrightarrow{r}\, dV \\ &= 3\iiint_V dV = 3V \end{aligned}$$

as required.

(ii) Left as an exercise.

Example 61 *(a) If* $F = axi + byj + czk, \quad a, b, c$ *are constants, show that*

$$\iint_S F \cdot \overrightarrow{n}\, ds = (4\pi/3)(a+b+c),$$

where S *is the surface of a unit sphere.*

(b) If $F = axi + byj + czk, \quad a, b, c$ are constants, show that

$$\iint_S F \cdot \overrightarrow{n}\, ds = (4\pi/3)(a+b+c),$$

S being the surface of the sphere

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$$

(c) If S is any closed surface enclosing a volume V and

$$F = xi + 2yj + 3zk,$$

prove that

$$\iint_S F \cdot \overrightarrow{n}\, ds = ds = 6V$$

Solution:

$$\iint_s F \cdot n \; dS = \iiint_v \operatorname{div} \quad F \; dV, \text{ by Gauss' divergence theorem}$$

$$\iiint\limits_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$\begin{aligned} &= (a+b+c)\iiint\limits_V dV = (a+b+c)\,V \\ &= (a+b+c)\times(4\pi/3)\,(1)^3 = (4\pi/3)\,(a+b+c) \end{aligned}$$

(b) and (c) Do yourself as in part (a).

Example 62 *(a) Verify divergence theorem for*

$$F = \left(x^2 + yz\right) i + \left(y^2 - zx\right) j + \left(z^2 + xy\right) k$$

taken over the rectangular parallelepiped

$$0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le c$$

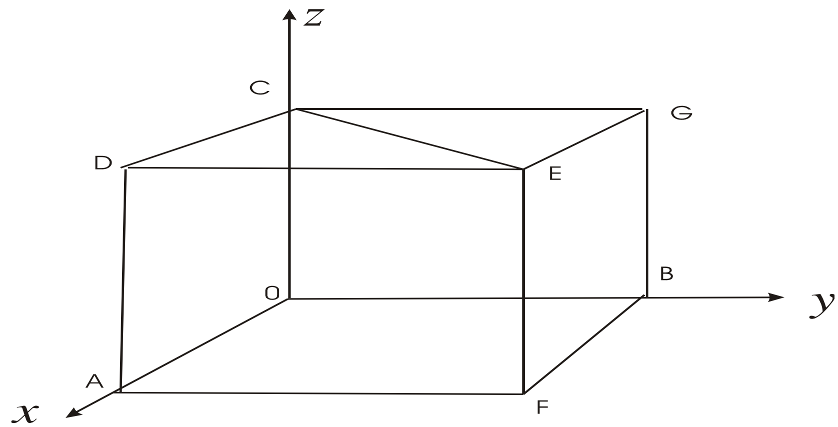
(b) Verify the divergence theorem for the function

$$F = x^2 i + zj + yzk$$

over the unit cube.

(c) Verify the divergence theorem for

$$F = 4xzi + y^2 j + yzk$$



taken over the cube bounded by

$$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$$

Solution: (a) Let OX, OY, OZ, be the coordinate axes and let i, j, k be unit vectors along these axes respectively. Let $S_1, S_2, S_3, S_4, S_5, S_6$, be the faces $DEFA, CGBO$, $EGBF, DCOA, DEGC$ and $OAFB$ respectively of the given rectangular parallelepiped as shown in the figure.

Let S denote the entire surface of the parallelepiped and let V be the volume enclosed by S. Then by Gauss' divergence theorem we must have

$$\iint_S F \cdot \overrightarrow{n}\, ds = \iiint_V \operatorname{div} F dV \tag{2.9}$$

$$\begin{aligned} \because \quad \operatorname{div} F &= \frac{\partial}{\partial x}\left(x^2+yz\right)+\frac{\partial}{\partial y}\left(y^2-zx\right)+\frac{\partial}{\partial z}\left(z^2-xy\right) \\ &= 2\left(x+y+z\right) \end{aligned}$$

Then, R.H.S. of (2.9)

$$2\iiint\limits_V (x+y+z)\,dxdydz = 2\int\limits_{x=0}^{a}\int\limits_{y=0}^{b}\int\limits_{z=0}^{c1} (x+y+z)\,dxdydz$$

$$= 2\int\limits_{x=0}^{a}\int\limits_{y=0}^{b} \left[x+y+\frac{z^2}{2}\right]_0^c dxdy$$

$$= 2\int\limits_{x=0}^{a} \left[cxy+\frac{1}{2}cy^2+\frac{1}{2}c^2y\right]_0^b dx$$

$$= 2\int\limits_{x=0}^{a} \left[cxb+\frac{1}{2}cb^2+\frac{1}{2}c^2b\right] dx = 2\left[cxb+\frac{1}{2}cb^2+\frac{1}{2}c^2b\right]_0^a$$

$$cba^2+cb^2a+c^2ba \tag{2.10}$$

The surface integral on $L.H.S.$ of (2.9) is the contribution due to the six faces of the parallelepiped.

Then $L.H.S.$ of (2.9)

$$\iint\limits_{S_1} F\cdot\overrightarrow{n}\,ds+\iint\limits_{S_2} F\cdot\overrightarrow{n}\,ds+\iint\limits_{S_3} F\cdot\overrightarrow{n}\,ds$$

$$\iint\limits_{S_4} F\cdot\overrightarrow{n}\,ds+\iint\limits_{S_5} F\cdot\overrightarrow{n}\,ds+\iint\limits_{S_6} F\cdot\overrightarrow{n}\,ds \tag{2.11}$$

For the face S_1 i.e. $DEFA, x = a$ and $n = i$.

$$\therefore \iint\limits_{S_1} F\cdot\overrightarrow{n}\,ds = \iint\limits_{S_1} F\cdot i ds = \iint\limits_{S_1} \left(x^2 - yz\right) dydz$$

$$= \int\limits_{z=0}^{c}\int\limits_{y=0}^{b} \left(a^2 - yz\right) dydz \qquad \text{since } x = a \text{ on } S_1$$

$$= \int\limits_{z=0}^{c} \left[a^2 y - \frac{1}{2}y^2 z\right]_0^b dz = \int\limits_{z=0}^{c} \left(a^2 b - \frac{1}{2}b^2 z\right) dz$$

$$= \left[a^2 bz - \frac{1}{4}b^2 z^2\right]_0^c = a^2 bc - \frac{1}{4}b^2 c^2$$

For the face S_2 i.e. $CGBO, x = 0$ and $n = -i$ (Note)

$$\therefore \iint\limits_{S2} F\cdot\overrightarrow{n}\,ds = \iint\limits_{S_2} F\cdot(-i)\,ds = -\iint\limits_{S_2} \left(x^2 - yz\right) dydz$$

$$= -\int\limits_{z=0}^{c}\int\limits_{y=0}^{b} (-yz)\,dydz \qquad \text{since } x = 0 \text{ on } S_2$$

$$= \int\limits_{z=0}^{c} z\left[\frac{1}{2}y^2\right]_0^b dz = \int\limits_{z=0}^{c} \frac{1}{2}b^2\left[\frac{1}{2}z^2\right]_0^c dz = \frac{1}{4}b^2 c^2$$

For the face S_3 i.e. $EGBF, y = b$ and, $n = j$

$$\therefore \iint_{S_3} F \cdot \overrightarrow{n}\, ds = \iint_{S_3} F \cdot j ds = \iint_{S_3} \left(y^2 - zx\right) ds$$

$$= \int_0^a \int_0^c \left(b^2 - yz\right) dz dx \qquad \text{since } y = b \text{ on } S_3$$

$$= \int_0^a \left[b^2 z - \frac{1}{2} z^2 x\right]_0^c dx = \int_0^a \left(b^2 c - \frac{1}{2} c^2 x\right) dx$$

$$= \left[b^2 cx - \frac{1}{4} c^2 x^2\right]_0^a = b^2 ca - \frac{1}{4} a^2 c^2$$

For the face S4 i.e. $DEGC, z = c, n = k.$

$$\therefore \iint_{S_4} F \cdot \overrightarrow{n}\, ds = \iint_{S_4} F \cdot (-j)\, ds = -\iint_{S4} \left(y^2 - zx\right) dz dx$$

$$= \int_0^a \int_0^c (-yz)\, dz dx \qquad \text{since } y = 0 \text{ on } S_4$$

$$= \int_0^a x \left[\frac{1}{2} z^2\right]_0^c dx = \frac{1}{2} c^2 \int_0^a x dx = \frac{1}{4} c^2 a^2$$

For the face S_5 i.e. $DEGC, z = c, n = k.$

$$\therefore \iint_{S_5} F \cdot \overrightarrow{n}\, ds = \iint_{S_5} F \cdot j ds = \iint_{S_5} \left(z^2 - xy\right) ds$$

$$= \int_0^a \int_0^b \left(c^2 - yz\right) dxdy \qquad \text{since } z = c \text{ on } S_5$$

$$= \int_0^a \left[c^2 y - \frac{1}{2} y^2 x\right]_0^b dx = \int_0^a \left(c^2 b - \frac{1}{2} b^2 x\right) dx$$

$$= \left[c^2 bx - \frac{1}{4} b^2 x^2\right]_0^a = c^2 ba - \frac{1}{4} a^2 b^2$$

For the face S_6 i.e. $OAFB, z = 0, n = -k$.

$$\therefore \iint_{S_6} F \cdot \overrightarrow{n}\, ds = \iint_{S_6} F \cdot (-k)\, ds = -\iint_{S_6} \left(z^2 - xy\right) ds$$

$$= \int_0^a \int_0^b (yz)\, dxdy \quad \text{since } z = c \text{ on } S_6$$

$$= \int_0^a \left[\frac{1}{2} y^2 x\right]_0^b dx = \frac{1}{2} b^2 \int_0^a x dx = \frac{1}{4} b^2 a^2$$

Using the above results in (2.11), we find that $L.H.S.$ of (2.9)

$$\left(a^2 bc - \frac{1}{4} b^2 c^2\right) + \frac{1}{4} b^2 c^2 + \left(b^2 ca - \frac{1}{4} a^2 c^2\right) + \frac{1}{4} a^2 c^2$$
$$+ \left(c^2 ba - \frac{1}{4} a^2 b^2\right) + \frac{1}{4} a^2 b^2$$

$$a^2 bc + b^2 ca + c^2 ba \tag{2.12}$$

From (2.10) and (2.12), we see that (2.9) is true. Hence the theorem is verified .

(b) and (c) Proceed as above by taking $a = b = c = 1$.

Exercise

(a) Verify Stokes' theorem for

$$F = (2x - y)\, i - yz^2 j - zy^2 k,$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

(b) Verify Stokes' Theorem for

$$F = (2x - y)\, i - yz^2 j - zy^2 k,$$

where S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ and C is its boundary.

(c) Verify the Gauss' divergence theorem for

$$F = 4xi - 2y^2 j + z^2 k$$

taken over the region bounded by

$$x^2 + y^2 = 4, z = 0 \ \text{ and } z = 3$$

(d) Verify the divergence theorem for the function

$$F = yi + xj + z^2 k$$

over the cylindrical region S bounded by

$$x^2 + y^2 = a^2, z = 0 \ \text{ and } z = h$$

2.15 Orthogonal Curvilinear Coordinates.

Let the rectangular Cartesian coordinates x, y, z of any point P in space be expressed in terms of three independent, single-valued and continuously differentiable scalar point functions as follows:

$$x = x\left(u_1, u_2, u_3\right), y = y\left(u_1, u_2, u_3\right), z = z\left(u_1, u_2, u_3\right) \tag{2.13}$$

Suppose that the Jacobian of x, y, z with respect to u_1, u_2, u_3 does not vanish

$$i.e. \frac{\partial\left(x, y, z\right)}{\partial\left(u_1, u_2, u_3\right)} \neq 0$$

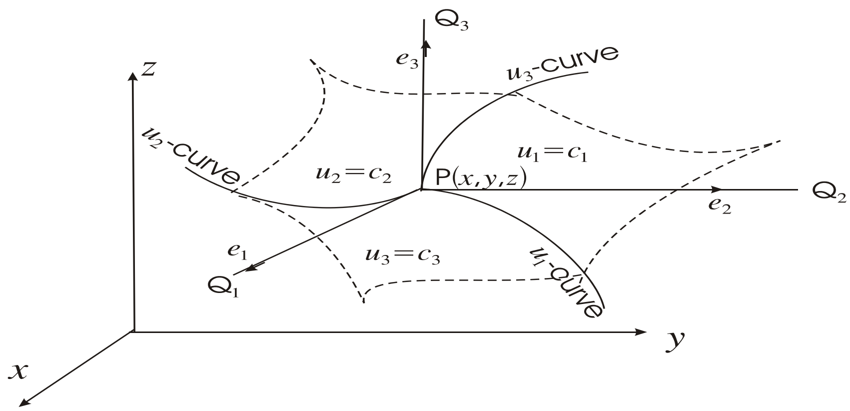
Then the transformation (2.13) can be inverted, i.e.,u_1, u_2, u_3 can be expressed in terms of x, y, z giving

$$u_1 = u_1\left(x, y, z\right), u_2 = u_2\left(x, y, z\right), u_3 = u_3\left(x, y, z\right) \tag{2.14}$$

Due to the conditions imposed on these functions, with each point $P\left(x, y, z\right)$ in space there exists a unique triad of numbers u_1, u_2, u_3 and to each such triad there is a definite point in space.

The set (u_1, u_2, u_3) are called the curvilinear coordinates of P. The set of equations (2.13) and (2.14) define a "transformation of coordinates".

The surfaces $u_1 = c_1, u_2 = c_2, u_3 = c_3$ where c_1, c_2, c_3 are constants, are called coordinate surfaces. The surface on which u_1 is constant is known as u_1-surface.



Similarly, we have u_2-surface and u_3-surface. When taken in pairs these coordinate surfaces intersect each other in curves called coordinate curves: (i) u_1-curve is given by $u_2 = c_2, u_3 = c_3$ (ii) u_2-curve is given by $u_1 = c_1, u_3 = c_3$ (iii) u_3-curve is given by $u_1 = c_1, u_2 = c_2$. The coordinate axes are determined by the tangents PQ_1, PQ_2, PQ_3 to the coordinate curves at the point P. Note carefully that the directions of these coordinate axes depend on the chosen point P of space and consequently the unit vectors associated with them are not necessarily constant.

If at every point $P\,(x, y, z)$, the coordinate axes are mutually perpendicular, then u_1, u_2, u_3 are called orthogonally curvilinear coordinates of P. In the present chapter we shall study such systems only.

Let e_1, e_2, e_3 form a right-handed system of unit vectors tangent to the coordinate curves at P, extending in the directions of increasing u_1, u_2, u_3 respectively.

Then, we have

$$e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1 \tag{2.15}$$

$$e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0 \tag{2.16}$$

$$e_1 \times e_1 = e_2 \times e_2 = e_3 \times e_3 = 0 \tag{2.17}$$

$$e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2 \tag{2.18}$$

We now define three numbers h_1, h_2, h_3 known as "scalar factors" or "material coefficients" as follows:

$$h_1 = \left|\partial r/\partial u_1\right|, h_2 = \left|\partial r/\partial u_2\right|, h_3 = \left|\partial r/\partial u_3\right| \tag{2.19}$$

The position vector $\overrightarrow{r}$ of $p\,(x, y, z)$ is given by

$$\begin{aligned} \overrightarrow{r} &= xi + yj + zk \\ &= x\,(u_1, u_2, u_3)\,i + y\,(u_1, u_2, u_3)\,j + z\,(u_1, u_2, u_3)\,k \\ &= \overrightarrow{r}\,(u_1, u_2, u_3) \end{aligned}$$

Since the equation of u_1-curve is given by $u_2 = c_2, u_3 = c_3$, it follows that a tangent vector to u_1-curve at P is $\partial r/\partial u_1$. Since e_1 is the unit vector in this direction, we have

$$dr/du_1 = \left|\partial r/\partial u_1\right| e_1 = h_1 e_1, \text{ using (2.19)}$$

Thus,

$$\frac{d\overrightarrow{r}}{du_1} = h_1 e_1, \frac{d\overrightarrow{r}}{du_2} = h_2 e_2, \frac{d\overrightarrow{r}}{du_3} = h_3 e_3 \tag{2.20}$$

2.16 Differential of an are length.

Using $r = r\left(u_1, u_{2,} u_3\right)$ and relations (2.20) of Section 2.15, we have

$$d\overrightarrow{r} = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3$$

or

$$d\overrightarrow{r} = h_1 du_1 e_1 + h_2 du_2 e_2 + h_3 du_3 e_3 \tag{2.21}$$

Then the differential of an are length, ds, is given by

$$(ds)^2 = d\overrightarrow{r} \cdot d\overrightarrow{r}$$

$$= (h_1 du_1 e_1 + + h_2 du_2 e_2 + h_3 du_3 e_3) \cdot (h_1 du_1 e_1 + + h_2 du_2 e_2 + h_3 du_3 e_3)$$

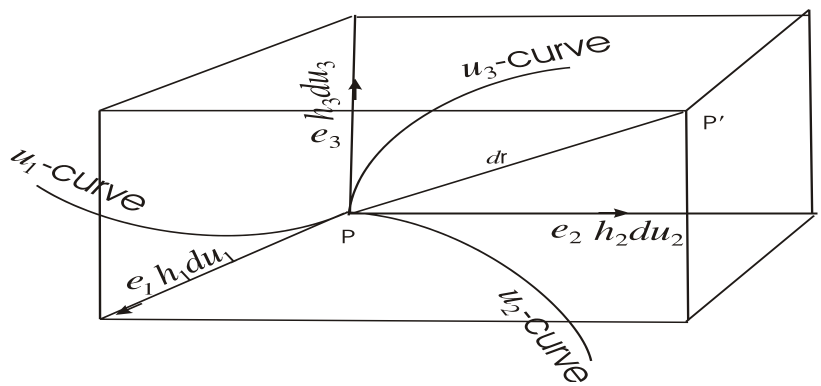
$$(ds)^2 = h_1^2 (du_1)^2 + + h_2^2 (du_2)^2 + h_3^2 (du_3)^2, \tag{2.22}$$

which is known as quadratic differential form.

We are now in a position to give geometrical significance of the coefficients h_1, h_2, h_3. Suppose that an element of $arc \;\; ds$ is directed along u_1-curve so that $du_1 = du_2$. Then the differential of arc length dS_1 along u_2-curve at P is given by $dS_1 = h_1 du_1$ Similarly, dS_2 and dS_3 can be calculated. Thus, we have

$$ds_1 = h_1 du_1, ds_2 = h_2 du_2, ds_3 = h_3 du_3 \tag{2.23}$$

Consider an infinitesimal parallelepiped with one vertex at P as shown in the figure.



Then, from (2.21) we see that (i) lengths of the edges of Parallelepiped are h_1du_1, h_2du_2 and h_3du_3, (ii) The areas of the faces of the Parallelepiped are

$$h_2h_3du_2du_3, h_3h_1du_3du_1 \quad \text{and} \quad h_1h_2du_1du_2 =$$

(iii) volume of the parallelepiped is

$$h_1h_2h_3du_1du_2du_3,$$

2.17 Differential operators O.C.C.

2.17.1 Gradient.

Let $\psi\,(x, y, z)$ be continuously differentiable scalar point function of orthogonal curvilinear coordinates u_1, u_2, u_3. Since grad ψ is a vector, we assume that

$$\operatorname{grad}\psi = \nabla\psi = k_1e_1 + k_2e_2 + k_3e_3, \tag{2.24}$$

Where e_1, e_2, e_3 are unit vectors along the curvilinear coordinate axes, and k_1, k_2, k_3 are unknown scalars, to be determined.

$$\because d\overrightarrow{r} = h_1 du_1 e_1 + h_2 du_2 e_2 + h_3 du_3 e_3 \tag{2.25}$$

and

$$d\psi = \text{grad} \quad \psi \cdot d\overrightarrow{r}$$

$$\therefore d\psi = k_1 h_1 du_1 + k_2 h_2 du_2 + k_3 h_3 du_3 \tag{2.26}$$

But

$$d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3 \tag{2.27}$$

Comparing (2.26) and (2.27), we obtain

$$k_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}, k_2 = \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}, k_3 = \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}$$

then, by (2.24),

$$\text{grad}\,\psi = \nabla\psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1} e_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2} e_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3} e_3, \tag{2.28}$$

which in the required expression. From (2.28), it follows that the components of grad ψ along the vectors are e_1, e_2, e_3 are

$$\left(\frac{1}{h_1}\right)\frac{\partial \psi}{\partial u_1}, \left(\frac{1}{h_2}\right)\frac{\partial \psi}{\partial u_2}, \left(\frac{1}{h_3}\right)\frac{\partial \psi}{\partial u_3}$$

respectively. Rewriting (2.28),

$$\nabla\psi = \left(\frac{e_1}{h_1}\frac{\partial}{\partial u_1} + \frac{e_2}{h_2}\frac{\partial}{\partial u_2} + \frac{e_3}{h_3}\frac{\partial}{\partial u_3}\right)\psi$$

$$\therefore \nabla \equiv \frac{e_1}{h_1}\frac{\partial}{\partial u_1} + \frac{e_2}{h_2}\frac{\partial}{\partial u_2} + \frac{e_3}{h_3}\frac{\partial}{\partial u_3} \tag{2.29}$$

Remark 63 *Replacing ψ by u_1, u_2 and u_3 by turn, (2.28) gives*

$$\nabla u_1 = \frac{e_1}{h_1}, \nabla u_2 = \frac{e_2}{h_2}, \nabla u_3 = \frac{e_3}{h_3} \tag{2.30}$$

Where vectors $\nabla u_1, \nabla u_2, \nabla u_3$ are along normals to the coordinate surfaces $u_1 = c_1, u_2 = c_2$ and $u_3 = c_3$ respectively.

Remark 64 *Using (2.30), relation (2.28) may be rewritten as*

$$\operatorname{grad}\psi = \nabla\psi = \frac{\partial\psi}{\partial u_1}\nabla u_1 + \frac{\partial\psi}{\partial u_2}\nabla u_2 + \frac{\partial\psi}{\partial u_3}\nabla u_3 \tag{2.31}$$

2.17.2 Divergence.

Let $F(u_1, u_2, u_3)$ be continuously differentiable vector point function of orthogonal curvilinear coordinates u_1, u_2, u_3. We assume that

$$F = F_1 e_1 + F_2 e_2 + F_3 e_3 \tag{2.32}$$

where F_1, F_2, F_3 are the components of F along e_1, e_2, e_3 respectively.

Using relation (2.18),then, the equation (2.32) may be rewritten as

$$F = F_1 e_2 \times e_3 + F_2 e_3 \times e_1 + F_3 e_1 \times e_2$$

or

$$F = F_1 h_2 h_3 \nabla u_2 \times \nabla u_3 + F_2 h_3 h_1 \nabla u_3 \times \nabla u_1 + F_3 h_1 h_2 \nabla u_1 \times \nabla u_2$$

$$\operatorname{div} F = \nabla \cdot F = \nabla\cdot\left(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3\right)$$
$$+\nabla\cdot\left(F_2 h_3 h_1 \nabla u_3 \times \nabla u_1\right) + \nabla\cdot\left(F_3 h_1 h_2 \nabla u_1 \times \nabla u_2\right) \tag{2.33}$$

Recall the following identities of chapter ... section...

$$\nabla\cdot(\psi F) = \psi \nabla \cdot F + F \cdot \nabla\psi \tag{2.34}$$

$$\nabla\cdot(F \times G) = \operatorname{curl} F \cdot G - \operatorname{curl} G \cdot F \tag{2.35}$$

$$cur \quad \operatorname{grad} \qquad \psi = 0 \tag{2.36}$$

$$\therefore \nabla\cdot\left(F_1 h_2 h_3 \nabla u_2 \times \nabla u_3\right)$$
$$= F_1 h_2 h_3 \nabla\cdot\left(\nabla u_2 \times \nabla u_3\right) + \left(\nabla u_2 \times \nabla u_3\right) \cdot \nabla\left(F_1 h_2 h_3\right)$$
$$= F_1 h_2 h_3 \left[\operatorname{curl}\operatorname{grad} u_2 \cdot \nabla u_3 - \operatorname{curl}\operatorname{grad} u_3 \cdot \nabla u_2\right]$$
$$+ \left(\nabla u_2 \times \nabla u_3\right) \cdot \nabla\left(F_1 h_2 h_3\right)$$
$$= \left(\nabla u_2 \times \nabla u_3\right) \cdot \nabla\left(F_1 h_2 h_3\right)$$
$$= \left(\nabla u_2 \times \nabla u_3\right) \cdot \left[\frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_1}\nabla u_1 + \frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_2}\nabla u_2 + \frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_3}\nabla u_3\right]$$

$$= \quad \left(\nabla u_2 \times \nabla u_3\right) \cdot \nabla u_1 \frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_1}$$

$$\sin\text{ce } \nabla u_2 \;\times \nabla u_3 \;\cdot \nabla u_2 = 0, \;\; \nabla u_2 \;\times \nabla u_3 \;\cdot \nabla u_3 = 0$$

$$= \frac{1}{h_1 h_2 h_3 \partial u_1}\left(e_2 \times e_3\right) \cdot e_1 \frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_1}$$
$$= \frac{1}{h_1 h_2 h_3}\frac{\partial\left(F_1 h_2 h_3\right)}{\partial u_1} \qquad \text{since } \left(e_2 \times e_3\right) \cdot e_1 = e_1 \cdot e_1 = 1$$

By symmetry,

$$\nabla\cdot(F_2h_3h_1\nabla u_3\times\nabla u_1)=\frac{1}{h_1h_2h_3}\frac{\partial(F_2h_3h_1)}{\partial u_2}$$

and

$$\nabla\cdot(F_3h_1h_2\nabla u_1\times\nabla u_2)=\frac{1}{h_1h_2h_3}\frac{\partial(F_3h_1h_2)}{\partial u_3}$$

Substituting the above values in (2.33), we have

$$\begin{aligned}\operatorname{div}F&=\nabla\cdot F\\ &=\frac{1}{h_1h_2h_3}\left[\frac{\partial(F_1h_2h_3)}{\partial u_1}+\frac{\partial(F_2h_3h_1)}{\partial u_2}+\frac{\partial(F_3h_1h_2)}{\partial u_3}\right]\end{aligned}\tag{2.37}$$

2.17.3 Curl.

Let $F(u_1,u_2,u_3)$ be continuously differentiable vector point function of orthogonal curvilinear coordinates u_1,u_2,u_3. We assume that

$$F=F_1e_1+F_2e_2+F_3e_3\tag{2.38}$$

where F_1,F_2,F_3 are the components of F along e_1,e_2,e_3 respectively. Using (2.30), (2.38) may be re-written as

$$F=h_1F_1\nabla u_1+h_2F_2\nabla u_2+h_3F_3\nabla u_3\tag{2.39}$$

$$\begin{aligned}\operatorname{curl}F=\nabla\times F&=\nabla\times(h_1F_1\nabla u_1)+\nabla\times(h_2F_2\nabla u_2)\\ &\quad+\nabla\times(h_3F_3\nabla u_3)\end{aligned}\tag{2.40}$$

Using the identity

$$\nabla\times(\psi F) = \nabla\psi \times F + \psi\,\mathrm{curl}\,F$$

We get

$$\nabla\times(h_1F_1\nabla u_1) = \nabla(h_1F_1)\times\nabla u_1 + h_1F_1\,\mathrm{curl}\,\mathrm{grad}\,u_1$$

$$= \nabla(h_1F_1)\times\nabla u_1$$

$$= \left[\frac{\partial(h_1F_1)}{\partial u_1}\nabla u_1 + \frac{\partial(h_1F_1)}{\partial u_2}\nabla u_2 + \frac{\partial(h_1F_1)}{\partial u_3}\nabla u_3\right]\times\nabla u_1$$

$$= \left[\frac{\partial(h_1F_1)}{\partial u_2}\nabla u_2\times\nabla u_1 + \frac{\partial(h_1F_1)}{\partial u_3}\nabla u_3\times\nabla u_1\right]\times\nabla u_1 \quad \text{since} \quad \nabla u_1\times\nabla u_1 = 0$$

$$= \frac{1}{h_1h_2}\frac{\partial(h_1F_1)}{\partial u_2}e_2\times e_1 + \frac{1}{h_3h_1}\frac{\partial(h_1F_1)}{\partial u_3}e_3\times e_1$$

$$= \frac{-1}{h_1h_2}\frac{\partial(h_1F_1)}{\partial u_2}e_3 + \frac{1}{h_3h_1}\frac{\partial(h_1F_1)}{\partial u_3}e_2$$

Making this and similar substitutions in (2.40), we get

$$\mathrm{curl}\,F = \nabla\times F = \frac{e_2}{h_3h_1}\frac{\partial(h_1F_1)}{\partial u_3} - \frac{e_3}{h_1h_2}\frac{\partial(h_1F_1)}{\partial u_2}$$

$$+\ \frac{e_3}{h_1h_2}\frac{\partial(h_2F_2)}{\partial u_1} - \frac{e_1}{h_2h_3}\frac{\partial(h_2F_2)}{\partial u_3}$$

$$+\ \frac{e_1}{h_2h_3}\frac{\partial(h_3F_3)}{\partial u_2} - \frac{e_2}{h_2h_1}\frac{\partial(h_3F_3)}{\partial u_1}$$

$$= \frac{e_1}{h_2h_3}\left[\frac{\partial(h_3F_3)}{\partial u_2} - \frac{\partial(h_2F_2)}{\partial u_3}\right] + \frac{e_2}{h_3h_1}\left[\frac{\partial(h_1F_1)}{\partial u_3} - \frac{\partial(h_3F_3)}{\partial u_1}\right]$$

$$+\frac{e_3}{h_1h_2}\left[\frac{\partial\left(h_2F_2\right)}{\partial u_1}-\frac{\partial\left(h_1F_1\right)}{\partial u_2}\right] \tag{2.41}$$

$$=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \tag{2.42}$$

2.17.4 Laplacian operator ∇^2.

We have

$$\nabla^2\psi = (\nabla\cdot\nabla\psi)$$

$$= \nabla\cdot\left(\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}e_3\right)$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right] \tag{2.43}$$

2.18 Special Orthogonal coordinate systems.

2.18.1 Cartesian coordinates.

These constitute a trivial case of orthogonal curvilinear coordinate systems. For this case, we have

$$u_1 = x, u_2 = y, u_3 = z, e_1 = i, e_2 = j, e_3 = k, h_1 = h_2 = h_3 = 1 \tag{2.44}$$

Hence formulae (2.28), (2.43), (2.37) and (2.42) of section 2.17 give

1.
$$\operatorname{grad}\psi = \nabla\psi = i\frac{\partial\psi}{\partial x} + j\frac{\partial\psi}{\partial y} + k\frac{\partial\psi}{\partial z} \tag{2.45}$$

2.

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{2.46}$$

3.

$$\nabla \cdot F = i\frac{\partial F_1}{\partial x} + j\frac{\partial F_2}{\partial y} + k\frac{\partial F_3}{\partial z} \tag{2.47}$$

4.

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{2.48}$$

where

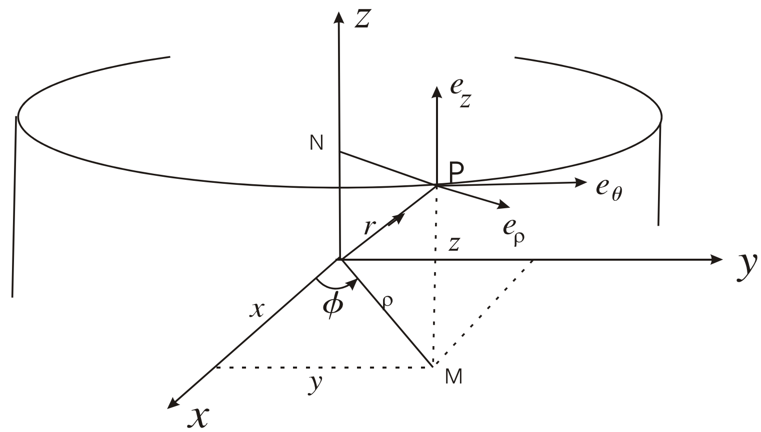
$$F = F_1 i + F_2 j + F_3 k \tag{2.49}$$

F_1, F_2, F_3 being the components of F along i, j, k respectively.

2.18.2 Cylindrical (polar) coordinates.

Let $P\,(x, y, z)$ be any point in space. Let ρ, ϕ, z respectively denote the projection OM of OP on $xy-$plane, the angle which OM makes with $x-$axis and perpendicular PM on $xy-$plane. Then cylindrical coordinates of P are (ρ, ϕ, z) and so here, we have

$$u_1 = \rho, u_2 = \phi, u_3 = z \tag{2.50}$$



Again from the figure, we have

$$x = \rho \cos \phi, y = \sin \phi, z = z \tag{2.51}$$

where

$$\rho \geq 0, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty \tag{2.52}$$

Expressing ρ, ϕ, z in terms of x, y, z , equation (2.51) yields

$$\rho = \sqrt{x^2 + y^2}\ , \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{2.53}$$

The coordinate surfaces are given by

1.
$$\rho = c_1 = \quad \text{i.e.} \quad x^2 + y^2 = c_1^2$$

   cylinders co-axial with $z-$axis.

2.
$$\phi = c_2 \text{ i.e. } \quad y = (\tan c_2)\, x,$$
planes through the $z-$axis.

3. $z = c_3$, *i.e.* planes perpendicular to the $z-$axis.

The point P is the point of intersection of these surfaces. The coordinate curves for ρ, ϕ and z and respectively straight lines perpendicular to the $z-$axis, horizontal circles with centers on the $z-$axis and lines parallel to the $z-$axis.

Let the unit vectors e_1, e_2, e_3 be denoted by e_ρ, e_ϕ, e_z respectively in cylindrical coordinates. Then e_ρ, e_ϕ, e_z extend respectively along NP in the direction of increasing ρ, perpendicular to the plane $ONPM$ in the direction of increasing ϕ and vertically upwards in the direction of increasing z.

Let $\overrightarrow{r}$ be the position vector of P. Then, we have

$$\overrightarrow{r} = xi + yj + zk$$

or

$$\overrightarrow{r} = \rho\cos\phi i + \rho\sin\phi j + zk \tag{2.54}$$

$$\therefore \frac{\partial \overrightarrow{r}}{\partial \rho} = \cos\phi i + \sin\phi j, \frac{\partial \overrightarrow{r}}{\partial \phi} = -\rho\sin\phi i + \rho\cos\phi j, \frac{\partial \overrightarrow{r}}{\partial z} = k \tag{2.55}$$

$$\therefore h_1 = \left|\frac{\partial \overrightarrow{r}}{\partial \rho}\right| = 1, h_2 = \left|\frac{\partial \overrightarrow{r}}{\partial \phi}\right| = \rho, h_3 = \left|\frac{\partial \overrightarrow{r}}{\partial z}\right| = 1 \tag{2.56}$$

Using

$$h_1 e_1 = \frac{\partial \overrightarrow{r}}{\partial u_1}, e_\rho = \cos\phi i + \sin\phi j \quad \text{etc}$$

$$e_\rho = \cos\phi i + \sin\phi j, e_\phi = -\sin\phi i + \cos\phi j, e_z = k, \tag{2.57}$$

From (2.57),

$$e_\rho \cdot e_\phi = e_\phi \cdot e_z = e_z \cdot e_\rho = 0 \tag{2.58}$$

Showing that e_ρ, e_ϕ, e_z are mutually perpendicular and hence cylindrical coordinates are orthogonal curvilinear coordinates. Using formulae (2.28), (2.43), (2.37) and (2.42) of section 2.17, we obtain

$$\operatorname{grad}\psi = \nabla\psi = \frac{\partial\psi}{\partial\rho}e_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}e_\phi + \frac{\partial\psi}{\partial z}e_z \tag{2.59}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{2.60}$$

$$\operatorname{div} F = \nabla\cdot F = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \tag{2.61}$$

$$\operatorname{curl} F = \nabla\times F = \begin{vmatrix} e_\rho & e_\phi & e_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \tag{2.62}$$

where

$$F = F_\rho e_\rho + F_\phi e_\phi + F_z e_z \tag{2.63}$$

F_ρ, F_ϕ, F_z being the components of F along e_ρ, e_ϕ, e_z respectively.

The elementary arc-length in curvilinear coordinates is

$$dS = \sqrt{h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2} \tag{2.64}$$

For cylindrical coordinates, (2.64) reduces to

$$dS = \sqrt{(d\rho)^2 + (d\phi)^2 + (dz)^2} \tag{2.65}$$

The elementary areas in the coordinate planes

$$u_1 u_2, u_2 u_3, u_3 u_1,$$

in curvilinear coordinates are

$$h_1 h_2 du_1 du_2, h_2 h_3 du_2 du_3, h_3 h_1 du_3 du_1$$

Respectively. Hence for cylindrical coordinates the elementary areas in

$$\rho\phi, \phi z, z\rho$$

planes are

$$\rho d\rho d\phi, \rho d\phi dz, dz d\rho$$

respectively.

The elementary volume in curvilinear coordinates is

$$h_1 h_2 h_3 du_1 du_2 du_3$$

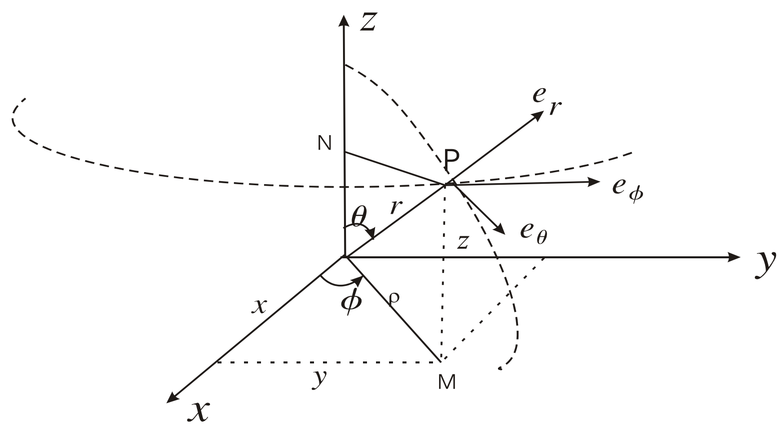
and hence the elementary volume in cylindrical coordinates reduces to

$$\rho d\rho d\phi dz$$

2.18.3 Spherical (polar) coordinates.

Let $P(x, y, z)$ be any point in space. Let r, θ, ϕ respectively denote the distance OP of P from the origin, the angle which OP makes with the $z-$axis and the angle between the projection OM of OP on $xy-$plane and the $x-$axis. Then spherical coordinates of P are (r, θ, ϕ) and so here, we have

$$u_1 = r, u_2 = \theta, u_3 = \phi \tag{2.66}$$



Again, from the figure, we have

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta \tag{2.67}$$

where

$$0 \le r, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi \tag{2.68}$$

Expressing r, θ, ϕ in terms of x, y, z (2.67) yields

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \tan^{-1}\sqrt{(x^2+y^2)}/z, \qquad z = r\cos\theta \tag{2.69}$$

The coordinate surfaces are given by:

1. $$r = c_1 \quad \text{i.e.,} \quad x^2+y^2+z^2 = c_1^2$$

   spheres cocentric with O,

2. $$\theta = c_2 \quad \text{i.e. } x^2+y^2 = \left(\tan^2 c_2\right) z^2 \quad \text{i.e.,}$$

   right circular cones with axis as $z-$axis and vertex at the origin O,

3. $$\phi = c_3, \quad \text{i.e.,} \quad y = (\tan c_3)\, x \quad \text{i.e. },$$

   planes through the $z-$axis.

The point P is the point of intersection of these surfaces.

The coordinate curves for r, θ, ϕ are respectively straight lines passing through the origin, vertical circles with center at the origin and the horizontal circles with center on the $z-$axis.

Let the unit vectors e_1, e_2, e_3 be denoted by e_r, e_θ, e_ϕ respectively in spherical coordinates. Let these unit vectors extend respectively in the directions of r increasing, θ increasing and ϕ increasing.

Let $\overrightarrow{r}$ be the position vector of P. Then, we have

$$\overrightarrow{r} = xi + yj + zk$$

or

$$\overrightarrow{r} = r\sin\theta\left(\cos\phi i + \sin\phi j\right) + r\cos\theta k \tag{2.70}$$

$$\therefore \frac{\partial \overrightarrow{r}}{\partial r} = \sin\theta\left(\cos\phi i + \sin\phi j\right) + \cos\theta k \tag{2.71}$$

$$\frac{\partial \overrightarrow{r}}{\partial \theta} = r\cos\theta\left(\cos\phi i + \sin\phi j\right) - r\sin\theta k \tag{2.72}$$

$$\frac{\partial \overrightarrow{r}}{\partial \phi} = r\sin\theta\left(-\sin\phi i + \cos\phi j\right) \tag{2.73}$$

$$\therefore h_1 = \left|\frac{\partial \overrightarrow{r}}{\partial r}\right| = 1, h_2 = \left|\frac{\partial \overrightarrow{r}}{\partial \theta}\right| = r, h_3 = \left|\frac{\partial \overrightarrow{r}}{\partial \phi}\right| = r\sin\theta \tag{2.74}$$

$$\left.\begin{aligned} e_r &= \sin\theta\left(\cos\phi i + \sin\phi j\right) + \cos\theta k \\ e_\theta &= \cos\theta\left(\cos\phi i + \sin\phi j\right) - \sin\theta k \\ e_\phi &= -\sin\phi i + \cos\phi j \end{aligned}\right\} \tag{2.75}$$

From equation (2.75),

$$e_r \cdot e_\theta = e_\theta \cdot e_\phi = e_\phi \cdot e_r = 0 \tag{2.76}$$

Showing that e_r, e_θ, e_ϕ are mutually perpendicular and hence spherical coordinates are orthogonal curvilinear coordinates. Using formulae (2.28), (2.43), (2.37) and (2.42) of section 2.17, we obtain.

$$\operatorname{grad}\psi = \nabla\psi = \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi \tag{2.77}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{2.78}$$

$$\operatorname{div} F = \nabla\cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ F_\theta)\frac{\partial}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \tag{2.79}$$

$$\operatorname{curl} F = \nabla\times F = \begin{vmatrix} e_r & re_\theta & e_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \tag{2.80}$$

where

$$F = F_r e_r + F_\theta e_\theta + F_\phi e_\phi \tag{2.81}$$

F_r, F_θ, F_ϕ being the components of F along e_r, e_θ, e_ϕ respectively.

The elementary arc-length in curvilinear coordinates is

$$dS = \sqrt{h_1^2(du_1)^2 + h_2^2(du_2)^2 + h_3^2(du_3)^2}$$

Then, for cylindrical coordinates, we have

$$dS = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2} \tag{2.82}$$

The elementary areas in the coordinate planes

$$u_1u_2, u_2u_3, u_3u_1$$

In curvilinear coordinates are

$$h_1h_2du_1du_2, h_2h_3du_2du_3, h_3h_1du_3du_1$$

respectively. Hence for spherical coordinates the elementary areas in

$$r\theta, \theta\phi, \phi r$$

planes are

$$rdrd\theta, r^2\sin\theta d\theta d\phi, r\sin\theta d\phi dr$$

respectively.

The elementary volume in curvilinear coordinates is

$$h_1h_2h_3du_1du_2du_3$$

and hence the elementary volume in spherical coordinate. reduce to $r^2\sin\theta drd\theta d\phi$.

Example 65 *If u_1, u_2, u_3 are orthogonal curvilinear coordinates, show that*

$$\frac{\partial r}{\partial u_1}, \frac{\partial r}{\partial u_2}, \frac{\partial r}{\partial u_3}$$

and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal system of vectors.

Solution: We know that

$$\nabla u_1 = e_1/h_1, \nabla u_2 = e_2/h_2, \nabla u_3 = e_3/h_3$$

$$\begin{aligned}\therefore [\nabla u_1 \nabla u_2 \nabla u_3] &= (\nabla u_1 \times \nabla u_2) \cdot \nabla u_3 \\ &= \left(\frac{1}{h_1}e_1 \times \frac{1}{h_2}e_2\right) \cdot \frac{1}{h_3}e_3 \\ &= \frac{1}{h_1h_2h_3}(e_3 \cdot e_3) = \frac{1}{h_1h_2h_3}\end{aligned}$$

Also,

$$\frac{\partial r}{\partial u_1} = h_1e_1, \frac{\partial r}{\partial u_2} = h_2e_2, \frac{\partial r}{\partial u_3} = h_3e_3$$

Now,

$$\begin{aligned}\frac{\nabla u_2 \times \nabla u_3}{[\nabla u_1 \nabla u_2 \nabla u_3]} &= \frac{(1/h_2)\, e_2 \times (1/h_3)\, e_3}{1/(h_1h_2h_3)} \\ &= h_1e_1 \qquad [\text{since } e_2 \times e_3 = e_1] \\ &= \frac{\partial r}{\partial u_1}\end{aligned}$$

Proceeding similarly we get two more results. Thus, we obtain:

$$\begin{aligned}\frac{\partial r}{\partial u_1} &= \frac{\nabla u_2 \times \nabla u_3}{[\nabla u_1 \nabla u_2 \nabla u_3]}, \qquad \frac{\partial r}{\partial u_2} = \frac{\nabla u_3 \times \nabla u_1}{[\nabla u_1 \nabla u_2 \nabla u_3]}, \\ \frac{\partial r}{\partial u_3} &= \frac{\nabla u_1 \times \nabla u_2}{[\nabla u_1 \nabla u_2 \nabla u_3]}\end{aligned}$$

which shows that $\frac{\partial r}{\partial u_1}, \frac{\partial r}{\partial u_2}, \frac{\partial r}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ form reciprocal system of vectors.

Exercise

Express (i) Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\ \nabla^2 u$ in spherical coordinates if u is independent of ϕ.

(ii) Heat (Diffusion) equation $\frac{\partial u}{\partial t} = k\ \nabla^2 u$ in cylindrical coordinates if u is independent of ϕ.