عنوان البحث

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القسم / الشعبة	:عام
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اسم المقرر	*Engineering Mathematics 4*:
كود المقرر	--------:
اسم الطالب	:عبد الرحمن عماد الدين محمد الشافعي
الرقم القومي	30004301700253:
البريد الأكاديمي	: ab.al.shafei@el-eng.menofia.edu.eg
اشراف	: أ.د./ وداد على الجنايني : أ.د./ رمضان الشنواني

Abstract

Maximize the profit, Minimize the cost. These all I could say about Linear Programming. As all companies are seeking to improve their products but with constrains. These improvements should accomplish goal which is to make the most of their equipment, materials and their stuff employees. They are also looking after achieving an annual income that is to be off course more than the previous year.

Introduction

The need to get the 'best' out of a system is a very strong motivation in much of engineering. A typical problem may be to obtain the maximum amount of product or to minimize the cost of a process or to find a configuration that gives maximum strength. Sometimes what is 'best' is easy to define, but frequently the problem is not so clear cut, and a lot of thought is required to reach an appropriate function to optimize. In most cases there are very severe and natural constraints operating: the problem may be one of maximizing the amount of product, subject to the supply of materials; or it may be minimizing the cost of production, with constraints due to safety standards. Indeed, much of modern optimization is concerned with constraints and how to deal with them. We will discuss two methods that could be used to solve this kind of problems. First one is the graphical method in which the solution is attained by constructing the domain of the solution from the inequalities after changing them into equalities. The second method is the simplex method which is an iterative procedure and a technique developed to solve linear programming problems. Then we will touch the surface of the dual problems.

Research Project Contents

The main problem of linear programming

1.1 The main problem of Linear programming is the minimization and the maximization of a linear function subject to linear constraints. The function to be maximized or minimized is called an objective function and the collection of the value of the variables at which the greatest or the least value is attained defines the what is called optimal plan. Any other collection of vines complying with the restrictions defines the feasible plan. The constrains looks something like the system of linear inequalities below:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}\, x_n \geq b_1,$$

$$a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n \geq b_{2,}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}\, x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m.$$

The objective function my look like the linear function below:

$$L = c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_o$$

and it is required to maximize or minimize the linear form L.

This problem can be solved by the graphical method following the steps below:

Step 1: Formulate the LP (Linear programming) problem.

Step 2: Construct a graph and plot the constraint lines.

Step 3: Determine the valid side of each constraint line.

Step 4: Identify the feasible solution region.

Step 5: Plot the objective function on the graph.

Step 6: Find the optimum point.

Example 1: A health-conscious family wants to have a very well controlled vitamin C-rich mixed fruit-breakfast which is a good source of dietary fibre as well; in the form of 5 fruit servings per day. They choose apples and bananas as their target fruits, which can be purchased from an online vendor in bulk at a reasonable price.

Bananas cost 30 rupees per dozen (6 servings) and apples cost 80 rupees per kg (8 servings). Given: 1 banana contains 8.8 mg of Vitamin C and 100-125 g of apples i.e. 1 serving contains 5.2 mg of Vitamin C.

Every person of the family would like to have at least 20 mg of Vitamin C daily but would like to keep the intake under 60 mg. How much fruit servings would the family have to consume on a daily basis per person to minimize their cost?

Answer: We begin stepwise with the formulation of the problem first. The constraint variables – ‘x’ = number of banana servings taken and ‘y’ = number of servings of apples taken. Let us find out the objective function now.

- Cost of a banana serving = 30/6 rupees = 5 rupees. Thus, the cost of ‘x’ banana servings = 5x rupees
- Cost of an apple serving = 80/8 rupees = 10 rupees. Thus, the cost of ‘y’ apple servings = 10y rupees

$$\text{Total Cost } C = 5x + 10y$$

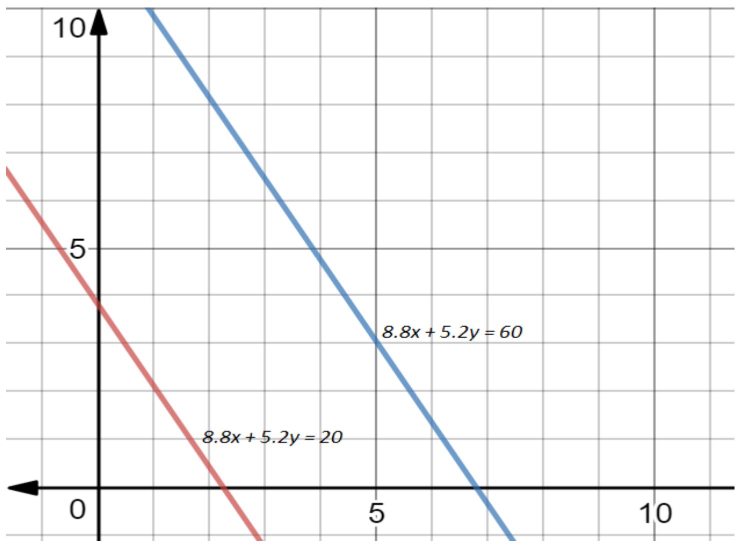
Constraints: $x \geq 0$, $y \geq 0$ (non-negative number of servings)

Total Vitamin C intake:

$$8.8x + 5.2y \geq 20 \qquad (1)$$

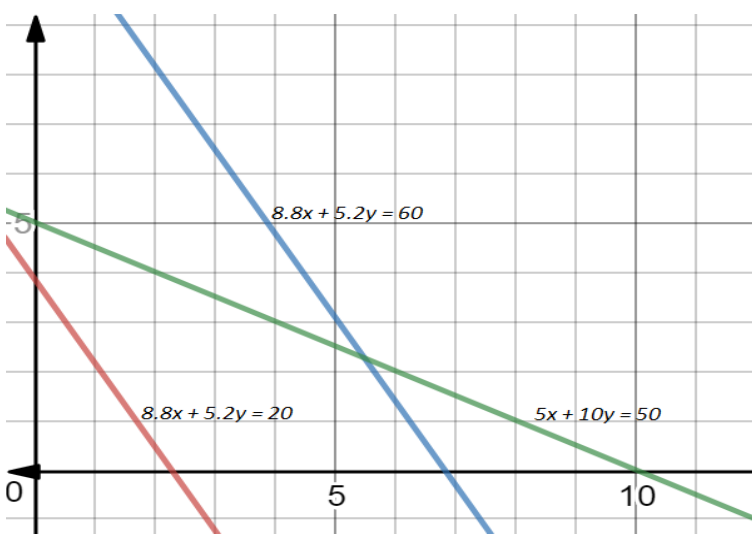
$$8.8x + 5.2y \leq 60 \qquad (2)$$

Now let us plot a graph with the constraint equations-



To check for the validity of the equations, put x=0, y=0 in (1). Clearly, it doesn't satisfy the inequality. Therefore, we must choose the side opposite to the origin as our valid region. Similarly, the side towards origin is the valid region for equation (2)

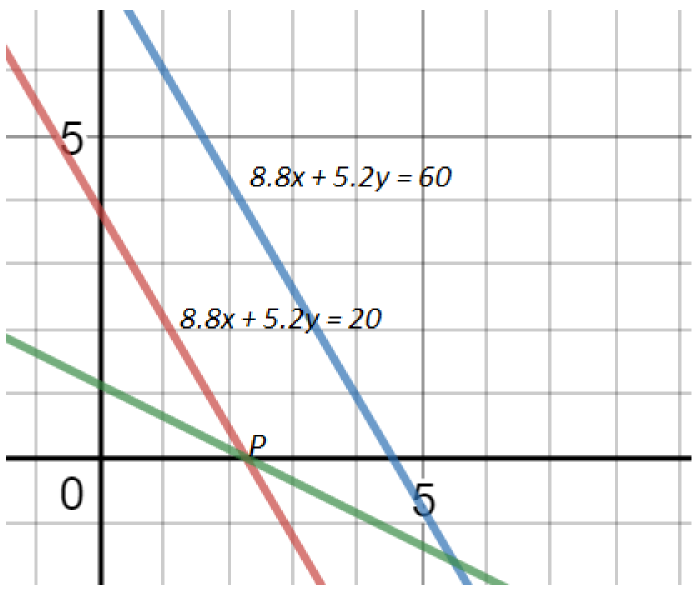
Feasible Region: As per the analysis above, the feasible region for this problem would be the one in between the red and blue lines in the graph! For the direction of the objective function; let us plot 5x+10y = 50.



Now take a ruler and place it on the straight line of the objective function. Start sliding it from the left end of the graph. What do we want here? We want the minimum value of the cost i.e. the minimum value of the optimum function C. Thus, we should slide the ruler in such a way that a point is reached, which:

1) lies in the feasible region
2) is closer to the origin as compared to the other points

This would be our Optimum Point. I've marked it as P in the graph. It is the one which you will get at the extreme right side of the feasible region here. I've also shown the position in which your ruler needs to be to get this point by the line in green.



Now we must calculate the coordinates of this point. To do this, just solve the simultaneous pair of linear equations:

$$y = 0$$
$$8.8x + 5.2y = 20$$

We'll get the coordinates of 'P' as (2.27, 0). This implies that the family must consume 2.27 bananas and 0 apples to minimize their cost and function according to their diet plan.

The Simplex Method
The simplex method is an algebraic iterative procedure which will solve exactly any linear programming problem in a finite number of steps or give an indication that there is an unbounded solution.

Applying this method leads to one of the following cases:
1. a finite optimal solution is found.
2. an infinite optimal solution is positively identified.
3. the problem has no feasible solution.

The simplex method is a procedure for moving step by step from a given extreme point to an optimal extreme point. At each step, it is possible to move only to what intuitively are adjacent extreme points.

1 The simplex method in a tabular form:
Consider the following linear programming problem: -

Example 4 Determine x1 and x2 that maximize the objective function

$$f\ (x) = c_1x_1 + c_2x_2$$

Subject to

$$a_{11}x_1 + a_{12}x_2 \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 \geq b_2$$
$$a_{31}x_1 + a_{32}x_2 \geq b_3$$
$$and\ \ x_1 \geq 0\ ,\ x_2 \geq 0$$

Solution: The algebraic solution for this problem can be arranged in a tabular form as will be shown:
The original simplex tableau:
After converting the structural constraints into equations, the problem can be arranged in a tableau as follows:

basic variables	x_1	x_2	x_3	x_4	x_5	constants
x_3	a_{11}	a_{12}	1	0	0	b_1
x_4	a_{21}	a_{22}	0	1	0	b_2
x_5	a_{31}	a_{32}	0	0	1	b_3
$-f(x)$	c_1	c_2	0	0	0	0

we can read equations from this tableau by leaving the first column, then

$$a_{11}x_1 + a_{12}x_2 + x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + x_4 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + x_5 = b_3$$

which are the same as equations.
On the other hand, starting with the first column, we can read the rows as

$$x_3 = b_1 - a_{11}x_1 - a_{12}x_2$$
$$x_4 = b_2 - a_{21}x_1 - a_{22}x_2$$
$$x_5 = b_3 - a_{31}x_1 - a_{32}x_2$$

Also, the last row is

$$-f(x) = 0 - c_1x_1 - c_2x_2$$
$$f(x) = c_1x_1 + c_2x_2$$

From the first and last columns we can read the initial basic feasible solution, i.e.

$$x_3 = b_1;\ x_4 = b_2;\ x_5 = b_3$$

(basic variables)

$$x_1 = 0 \quad x_2 = 0$$

(non-basic variables)
and

$$-f(x) = 0\ i.e. f(x) = 0$$

To test the optimally of this initial basic feasible solution, we look at the coefficients in the last row of the tableau, i.e. c_1 and c_2. If both are positive, the solution is optimal. If at least one them is positive, the solution is not optimal c_1 and c_2 are both positive, then the solution is not optimal.
The transformed simplex tableau:
Examine c_1, c_2 in the last row and suppose that c_1 is greater than $c_2 (i.e. c_1 > c_2)$ then x_1 will be basic variable and its column is called the pivot column.
Then divide the constants b_1, b_2 and b_3 given in the last column by the corresponding elements a_{11}, a_{21} and a_{31} in the pivot column. Suppose that the ratio $b_2 = a_{21}$ is the least, then x_4 will be the non-basic variables and its row is called the pivot row.
The element a_{21} at the intersection of the pivot column and pivot row is called the pivot element.
This solution has x_3, x_1, x_5 as basic variables and $x_4, x2$ as non-basic variables.
Now the next tableau is formed by transforming the elements in the original simplex tableau as follows:
1. The pivot row is transformed by dividing all its elements by the pivot element.
2. The pivot column is transformed by replacing all its elements by zeros except the pivot element which becomes 1.
3. Other elements which are neither in the pivot column nor the pivot row are transformed as follows:
Let a_{12} be such as element and a_{11}be the element lying in the same row and the pivot column.

		pivot column	
	a_{11}		a_{12}
pivot row	a_{21}		a_{22}

a_{22} be the element lying in the same column and the pivot row. and a_{21} is the pivot element.
Then a_{12} is transformed to

$$a_{12} - \frac{a_{11}a_{22}}{a_{21}},$$

which is the same as a'_{12}. Thus the transformed simplex tableau becomes

$\frac{\text{basic}}{\text{variables}}$	x_1	x_2	x_3	x_4	x_5	constants
x_3	0	a'_{12}	1	a'_{14}	0	b'_1
x_1	1	a'_{22}	0	a'_{24}	0	b'_2
x_5	0	a'_{32}	0	a'_{34}	1	b'_3
$-f(x)$	0	c'_2	0	c'_4	0	$-c$

This solution is:

$$x3 \;=\; b'_1, x_1 \;=\; b'_2 \;, x_5 \;=\; b'_3$$

basic variables

$$x_2 \;=\; 0\,; \;\; x_4 \;=\; 0$$

non-basic variables.
and

$$f(x) \;=\; c.$$

To test the optimally, we examine the coefficients (c'_2&c'_4) in the last row. If both are negative the solution is optimal. If at least one is positive, the solution is not optimal and the same rules are repeated till we reach the optimal solution.
Now, we can summarize the rules for the simplex method in tabular form:
The decision rules:

1. Testing optimally: Examine the coefficients of the objective function in the last row of the tableau. If the problem is to maximize and they are negative or zeros, the solution is optimal. If the problem is to minimize and they are positive or zeros, the solution is optimal

2. The basic variables: If the problem is to maximize, then the non-basic variable associated with the largest positive coefficients in the last row is the basic

variable.
3. The non-basic variables: Divide the constants in the last column by the corresponding elements in the pivot column. The basic variable associated with
the least of these ratios is the non-basic variable.
The transformation rules:
1. The pivot row is transformed by dividing all elements by pivot element.
2. The pivot column is transformed by replacing its elements by zeros except the pivot element which becomes 1.

		pivot column		
	b		a	
	d		c	pivot row

3. The remaining elements are transformed by applying the following rule:

$$a' = a - bc/d$$

Example 5 Find $x_1 \geq 0$; $x_2 \geq 0$ 0 that maximize

$$f(x) = 2x_1 + x_2$$

subject to

$$3x_1 - 2x_2 \leq 12$$
$$x_1 - 5x_2 \leq 2$$
$$-x_1 + 2x_2 \leq 4$$

Solution: The first step is

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_3	3	-2	1	0	0	12
x_4	1	-5	0	1	0	2
x_5	-1	2	0	0	1	4
$-f(x)$	2	1	0	0	0	0

The second step

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_3	0	13	1	-3	0	6
x_1	1	-5	0	1	0	2
x_5	0	-3	0	1	1	6
$-f(x)$	0	11	0	-2	0	-4

The third step

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_2	0	1	$1/13$	$-3/13$	0	$6/13$
x_1	1	0	$5/13$	$-2/13$	0	$56/13$
x_5	0	0	$3/13$	$4/13$	1	$96/13$
$-f(x)$	2	0	$-11/13$	$7/13$	0	$-118/13$

The fourth step

$n.b.v \rightarrow$ $b.v \downarrow$	x_1	x_2	x_3	x_4	x_5	Constants
x_2	0	1	$1/4$	0	$3/4$	6
x_1	1	0	$1/2$	0	$1/2$	8
x_4	0	0	$3/4$	1	$13/4$	24
$-f(x)$	0	0	$-5/4$	0	$-7/4$	-22

The optimal solution is

$$x_1 = 8, \qquad x_2 = 6, \qquad x_4 = 24$$

basic variables

$$x_3 = 0, x5 = 0$$

non-basic variables
and

$$f(x) = 22$$

The simplex method in the compact form:

Looking at the previous example 5, which be solved by the simplex method in a tabular form, we notice that a unit matrix is always present in each tableau, it correspond to the columns of the basic variables of each iteration. However, a unit matrix does not contain numerical information which need be recorded. We can remove it and know that it always exists. We then will have columns for non-basic variable only.

The tableau for the kth iteration will have the form:

$basic$	non-basic $x_{m+1}......x_s........x_n$	$values$
x_1	$a'_{1m+1}......a'_{1s}.......a'_{1n}$	b'_1
x_2	$a'_{2m+1}......a'_{2s}.......a'_{2n}$	b'_2
x_r	$a'_{rm+1}......a'_{rs}.......a'_{rn}$	b'_r
x_m	$a'_{mm+1}.....a'_{ms}.....a'_{mn}$	b'_m
$-z$	$c'_{mm+1}.....c'_s......c'_n$	$-z_0$

The tableau for the k + 1<u>th</u> iteration will be:

$basic$	non-basic variables x_{m+1} $\cdots$ x_r $\cdots$ x_n	$values$
x_1	$a'_{1m+1} - a'_{1s}a'^*_{rm+1}.... - a'_{1s}/a'_{rs}....a'_{in} - a'_{1s}a'^*_{rm}$	$b'_1 - a'_{1s}b'^*_r$
x_2	$a'_{2m+1} - a'_{2s}a'^*_{rm+1}.... - a'_{2s}/a'_{rs}....a'_{2n} - a'_{2s}a'^*_{rn}$	$b'_2 - a'_{2s}b'^*_r$
$\vdots$		$\vdots$
x_s	$a'_{rm+1}/a'_{rs} = a'^*_{rm+1}....1/a'_{rs}.....a'_{rn}/a'_{rs} = a'^*_{rn}$	$b'_r/a'_{rs} = b'^*_r$
$\vdots$		$\vdots$
x_m	$a'_{mm+1} - a'_{ms}a'^*_{rm+1}...a'_{ms}/a'_{rs}...a'_{mn} - a'_{ms}a'^*_{rn}$	$b'_m - a'_{ms}b'^*_r$
$-z$	$c'_{m+1} - c'_s\ a'^*_{rm+1}..... - c'_s/a'_{rs}.......c'_n - c'_sa'^*_{rn}$	$-z_o - c'_sb'^*_r$

The rules for the transformation tableau from iteration to the next one are as follows:

1. The pivot element is replaced by its reciprocal.
2. The pivot row is divided by the pivot element.
3. The pivot column is divided by the negative of the pivot element.
4. Every other element is reduced by the quantity
5. The headings of the pivot row and pivot column

To illustrate this method let us consider the following example:

$$x_1 + 2x_2 \le 10$$
$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1 - 2x_2 \le 1$$
$$2x_1 + x_2 \rightarrow max$$
$$and\ x_1, x2 \ge 0;$$

Solution: Reversing the signs of the c_j to convert the problem to a minimization one. Then the calculations are shown in the following compact tableau:

$basic$ \ $non-basic$ variables	x_1	x_2	$values$
x_3	1	2	10
x_4	1	1	6
x_5	1	-1	2
x_6	$\boxed{1}$	-2	1
$-z$	-2	1	0

initial tableau

$basic$ \ $non-basic$ variables	x_6	x_2	$values$
x_3	-1	4	9
x_4	-1	3	5
x_5	-1	$\boxed{1}$	1
x_1	$+1$	-2	1
$-z$	2	-5	2

1 is iteration .

$basic$ \ non-basic variables	x_6	x_5	$values$
x_3	3	-4	5
x_4	$\boxed{2}$	-3	2
x_2	-1	1	1
x_1	-1	2	3
$-z$	-3	5	7

2 nd iteration .

$basic$ \ non-basic variables	x_4	x_5	$value$
x_3	$-3/2$	$1/2$	2
x_6	$1/2$	$-3/2$	1
x_2	$1/2$	$-1/2$	2
x_1	$1/2$	$1/2$	4
$-z$	$3/2$	$1/2$	10

3 rd iteration optimal tableau

from the optimal tableau we can see that the optimal solution is:

$$x_1 = 4, \quad x_2 = 2$$

$$z_{min} = -10, \quad z_{max} = 10$$

Duality Theorems
In the following theorems, we are going to use the standard forms for the primal and the dual problems given as:
I. The primal problem
To minimize

$$z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, 2, \dots m$$
$$and\ x_j \geq 0 \quad j = 1, 2, \dots n$$

II. The dual problem
To maximize

$$z' = \sum_{j=1}^{m} b_i y\_i$$

subject to

$$\sum_{j=1}^{m} a_{ij} y_i \leq c_j \quad j = 1, 2, \dots n$$
$$and\ y_i \geq 0 \qquad i = 1, 2, \dots m$$

Theorem 12 The dual of the dual is the primal
This theorem is obvious and implies a completely symmetrical relationship between the primal and the dual problems.
Theorem 13 For any feasible solution, the values of the objective function for the primal problem is always greater than or equal to the value of the objective function for the dual problem.
In other words, if $x_1, x_2, \dots\ x_n$ & $y_1, y_2, \dots y_m$ are feasible solution to the primal and the dual problems respectively, then

$$z = \sum_{j=1}^{n} c_j x_j \geq z' = \sum_{i=1}^{m} b_i y_i$$

Proof. Consider the primal and the dual problems as given by I &II.
From the dual problem

$$c_j \geq \sum_{i=1}^{m} a_{ij} y_i \quad (1.17)$$

Since $x_j \geq 0$, then we can multiply both sides of (1.17) by x_j and then sum over j, we get

$$\sum_{j=1}^{n} c_j x_j \; \geq \; \sum_{j=1}^{n} (\sum_{i=1}^{m} (a_{ij} y_i) x_j \qquad (1.18)$$

Interchanging the summation signs of the right hand side of (1.18), we have

$$\sum_{j=1}^{n} c_j x_j \; \geq \; \sum_{j=1}^{n} (\sum_{i=1}^{m} (a_{ij} x_i) y_j \qquad (1.19)$$

But from the primal problem

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_i$$

Then substituting in (1.19) we get

$$\sum_{j=1}^{n} c_j x_j \; \geq \; \sum_{i=1}^{m} b_j y_j$$

$$i.e. z \; \geq \; z'.$$

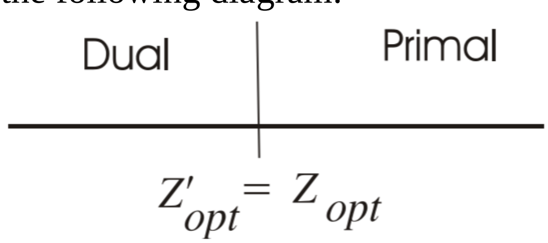
which completes the proof.

Theorem 14 If there exist finite feasible solutions for both the primal and the dual problems, then the values of the objective functions corresponding to their optimal solutions are equal.

A constructive proof of this theorem is established by means of the simplex algorithm.

From theorems 13 and 14 we have the following information about the possible ranges of the values of the objective functions for the feasible solutions of the primal and the dual problems.

This can be illustrated by the following diagram:



This can be used in the following

1. If we know a feasible solution of the dual problem then we have a lower bound for the optimal value (minimal) of the objective function of the primal problem

$$z_{opt} \; \geq \; z'^{0}$$

2. If we know a feasible solution of the primal problem then we have an upper bound for the optimal value (maximum) of the objective function of the dual problem.

$$z'_{opt} \; \geq \; z^{0}$$

3. If an optimal solution of either problems is known then the value of the objective function of the other problem is also known.

$$z'_{opt} = z_{opt}.$$

Theorem 15 If the primal (dual) problem has feasible solutions and the dual (primal) problem has no feasible solution, then the primal (dual) problem has unbounded Solutions

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