

عنوان البحث

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**Abstract**

Maximize the profit, Minimize the cost. These all I could say about Linear Programming. As all companies are seeking to improve their products but with constrains. These improvements should accomplish goal which is to make the most of their equipment, materials and their stuff employees. They are also looking after achieving an annual income that is to be off course more than the previous year.

**Introduction**

The need to get the ‘best’ out of a system is a very strong motivation in much of

engineering. A typical problem may be to obtain the maximum amount of product or to

minimize the cost of a process or to find a configuration that gives maximum strength.

Sometimes what is ‘best’ is easy to define, but frequently the problem is not so clear

cut, and a lot of thought is required to reach an appropriate function to optimize. In most

cases there are very severe and natural constraints operating: the problem may be one

of maximizing the amount of product, subject to the supply of materials; or it may be

minimizing the cost of production, with constraints due to safety standards. Indeed,

much of modern optimization is concerned with constraints and how to deal with them.

We will discuss two methods that could be used to solve this kind of problems. First one is the graphical method in which the solution is attained by constructing the domain of the solution from the inequalities after changing them into equalities. The second method is the simplex method which is an iterative procedure and a technique developed to solve linear programming problems. Then we will touch the surface of the dual problems.

**Research Project Contents**

The main problem of linear programming

1.1 The main problem of Linear programming is the minimization and the maximization of a linear function subject to linear constraints. The function to be maximized or minimized is called an objective function and the collection of the value of the variables at which the greatest or the least value is attained defines the what is called optimal plan. Any other collection of vines complying with the restrictions defines the feasible plan.

The constrains looks something like the system of linear inequalities below:

………………………

The objective function my look like the linear function below:

and it is required to maximize or minimize the linear form L.

This problem can be solved by the graphical method following the steps below:

**Step 1: Formulate the LP (Linear programming) problem.**

**Step 2: Construct a graph and plot the constraint lines.**

**Step 3: Determine the valid side of each constraint line.**

**Step 4: Identify the feasible solution region.**

**Step 5: Plot the objective function on the graph.**

**Step 6: Find the optimum point.**

Example 1: A health-conscious family wants to have a very well controlled vitamin C-rich mixed fruit-breakfast which is a good source of dietary fibre as well; in the form of 5 fruit servings per day. They choose apples and bananas as their target fruits, which can be purchased from an online vendor in bulk at a reasonable price.

Bananas cost 30 rupees per dozen (6 servings) and apples cost 80 rupees per kg (8 servings). Given: 1 banana contains 8.8 mg of Vitamin C and 100-125 g of apples i.e. 1 serving contains 5.2 mg of Vitamin C.

Every person of the family would like to have at least 20 mg of Vitamin C daily but would like to keep the intake under 60 mg. How much fruit servings would the family have to consume on a daily basis per person to minimize their cost?

**Answer:** We begin stepwise with the formulation of the problem first. The constraint variables – ‘x’ = number of banana servings taken and ‘y’ = number of servings of apples taken. Let us find out the objective function now.

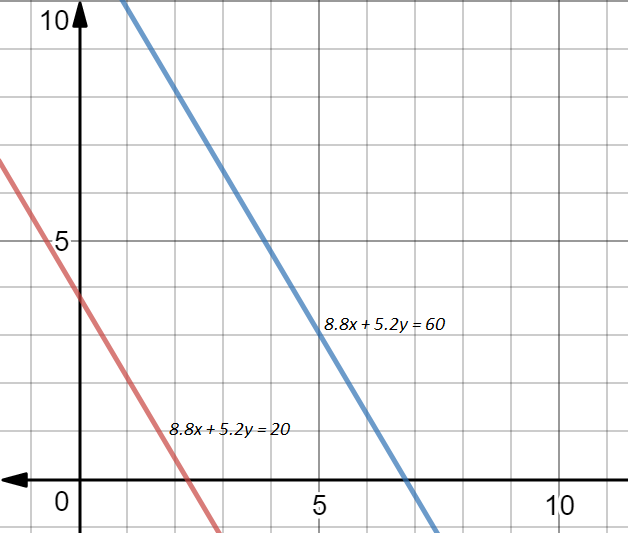
* Cost of a banana serving = 30/6 rupees = 5 rupees. Thus, the cost of ‘x’ banana servings = 5x rupees
* Cost of an apple serving = 80/8 rupees = 10 rupees. Thus, the cost of ‘y’ apple servings = 10y rupees

Total Cost C = 5x + 10y

Constraints: x ≥ 0, y ≥ 0 (non-negative number of servings)

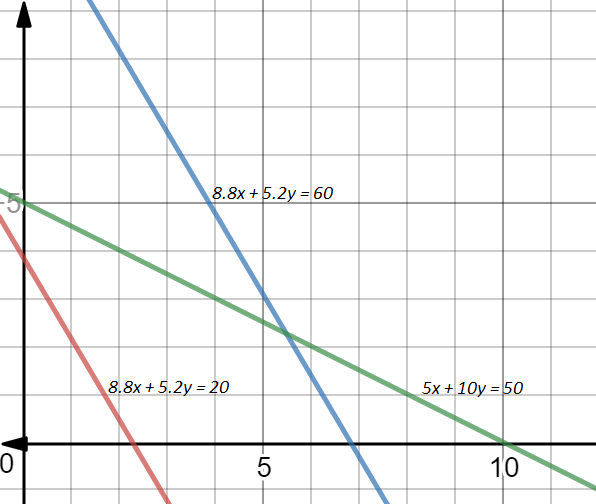
Total Vitamin C intake:  
8.8x + 5.2y ≥ 20        (1)  
8.8x + 5.2y ≤ 60        (2)

Now let us plot a graph with the constraint equations-



To check for the validity of the equations, put x=0, y=0 in (1). Clearly, it doesn’t satisfy the inequality. Therefore, we must choose the side opposite to the origin as our valid region. Similarly, the side towards origin is the valid region for equation (2)

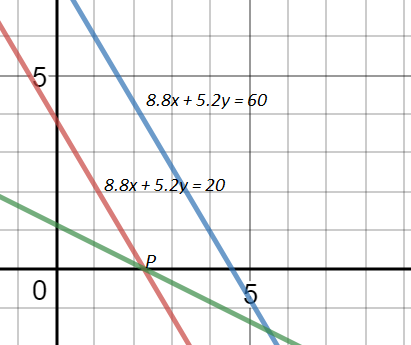
Feasible Region: As per the analysis above, the feasible region for this problem would be the one in between the red and blue [lines](https://www.toppr.com/guides/maths/straight-lines/basics-of-straight-lines/) in the graph! For the [direction](https://www.toppr.com/guides/business-studies/directing/elements-of-directing/) of the objective function; let us plot 5x+10y = 50.



Now take a ruler and place it on the straight line of the objective function. Start sliding it from the left end of the graph. What do we want here? We want the minimum value of the cost i.e. the minimum value of the optimum function C. Thus, we should slide the ruler in such a way that a point is reached, which:

1) lies in the feasible region  
2) is closer to the origin as compared to the other points

This would be our Optimum Point. I’ve marked it as P in the graph. It is the one which you will get at the extreme right side of the feasible region here. I’ve also shown the position in which your ruler needs to be to get this point by the line in green.



Now we must calculate the coordinates of this point. To do this, just solve the simultaneous pair of linear equations:

*y = 0  
8.8x + 5.2y = 20*

We’ll get the coordinates of ‘P’ as (2.27, 0). This implies that the family must consume 2.27 bananas and 0 apples to minimize their cost and function according to their diet plan.

**The Simplex Method**

The simplex method is an algebraic iterative procedure which will solve exactly any linear programming problem in a finite number of steps or give an indication that there is an unbounded solution.

Applying this method leads to one of the following cases:

1. a finite optimal solution is found.

2. an infinite optimal solution is positively identified.

3. the problem has no feasible solution.

The simplex method is a procedure for moving step by step from a given extreme

point to an optimal extreme point. At each step, it is possible to move only to what

intuitively are adjacent extreme points.

**1 The simplex method in a tabular form:**

Consider the following linear programming problem: -

**Example 4** Determine x1 and x2 that maximize the objective function

Subject to

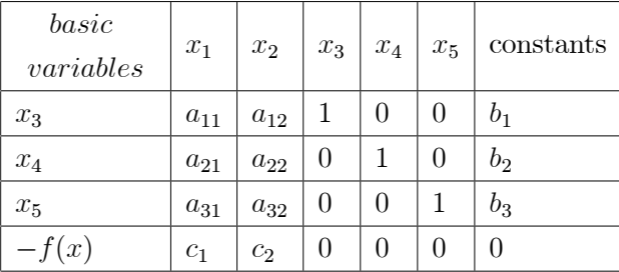
Solution: The algebraic solution for this problem can be arranged in a tabular

form as will be shown:

The original simplex tableau:

After converting the structural constraints into equations, the problem can be

arranged in a tableau as follows:



we can read equations from this tableau by leaving the first column, then

which are the same as equations.

On the other hand, starting with the first column, we can read the rows as

Also, the last row is

From the first and last columns we can read the initial basic feasible solution, i.e.

(basic variables)

(non-basic variables)

and

To test the optimally of this initial basic feasible solution, we look at the coefficients in the last row of the tableau, i.e. and . If both are positive, the solution is

optimal. If at least one them is positive, the solution is not optimal and are

both positive, then the solution is not optimal.

The transformed simplex tableau:

Examine in the last row and suppose that is greater than

then will be basic variable and its column is called the pivot column.

Then divide the constants and given in the last column by the corresponding elements and in the pivot column. Suppose that the ratio

is the least, then will be the non-basic variables and its row is called the

pivot row.

The element at the intersection of the pivot column and pivot row is called

the pivot element.

This solution has as basic variables and as non-basic variables.

Now the next tableau is formed by transforming the elements in the original simplex

tableau as follows:

1. The pivot row is transformed by dividing all its elements by the pivot element.

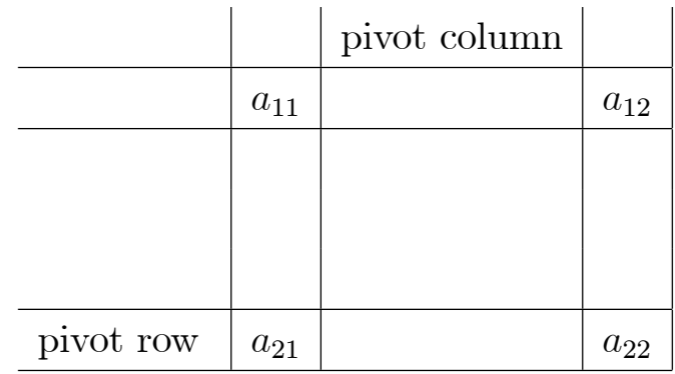
2. The pivot column is transformed by replacing all its elements by zeros except the pivot element which becomes 1.

3. Other elements which are neither in the pivot column nor the pivot row are

transformed as follows:

Let a12 be such as element and a11be the element lying in the same row and the

pivot column.

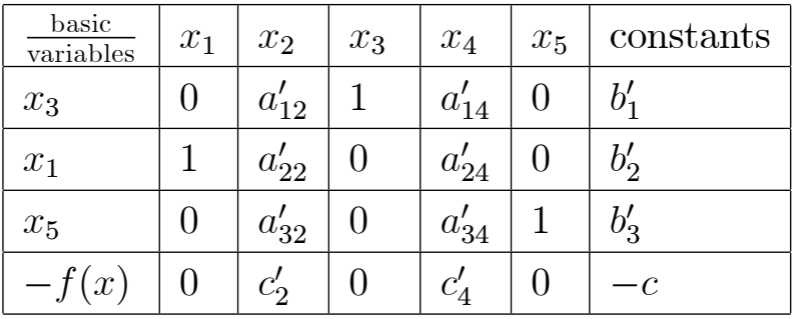


a22 be the element lying in the same column and the pivot row. and is the

pivot element.

Then a12 is transformed to

which is the same as Thus the transformed simplex tableau becomes



This solution is:

basic variables

non-basic variables.

and

To test the optimally, we examine the coefficients in the last row. If both

are negative the solution is optimal. If at least one is positive, the solution is not

optimal and the same rules are repeated till we reach the optimal solution.

Now, we can summarize the rules for the simplex method in tabular form:

The decision rules:

1. Testing optimally: Examine the coefficients of the objective function in the

last row of the tableau. If the problem is to maximize and they are negative

or zeros, the solution is optimal. If the problem is to minimize and they are

positive or zeros, the solution is optimal

2. The basic variables: If the problem is to maximize, then the non-basic variable

associated with the largest positive coefficients in the last row is the basic

variable.

3. The non-basic variables: Divide the constants in the last column by the corresponding elements in the pivot column. The basic variable associated with

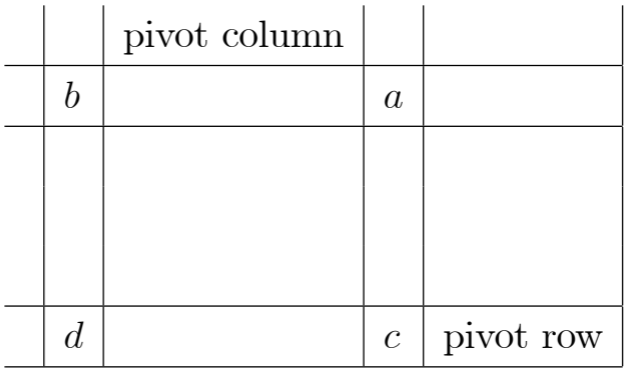
the least of these ratios is the non-basic variable.

The transformation rules:

1. The pivot row is transformed by dividing all elements by pivot element.

2. The pivot column is transformed by replacing its elements by zeros except the

pivot element which becomes 1.

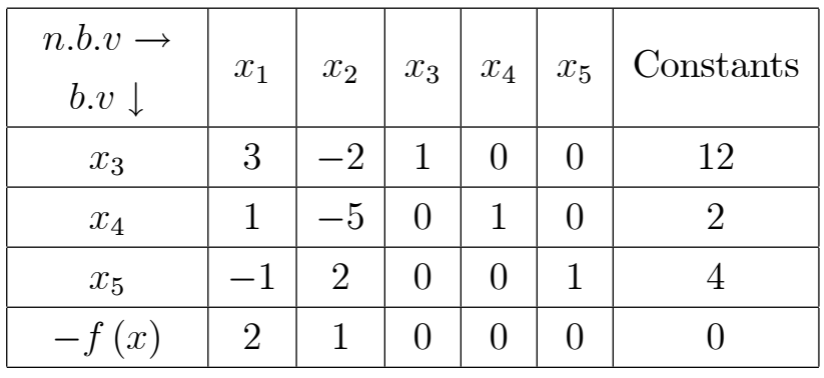


3. The remaining elements are transformed by applying the following rule:

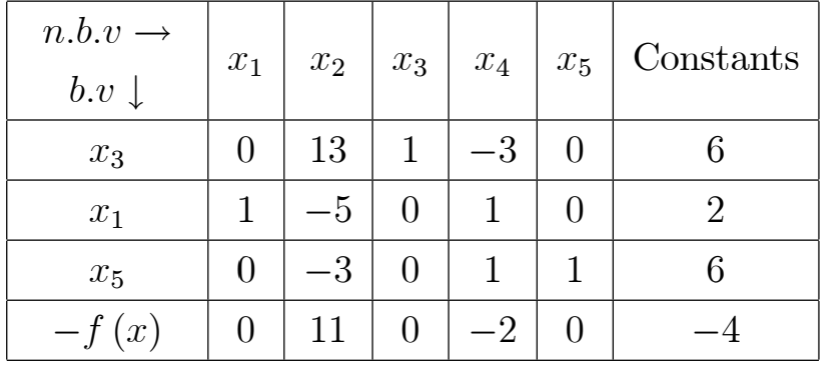
Example 5 Find 0 that maximize

subject to

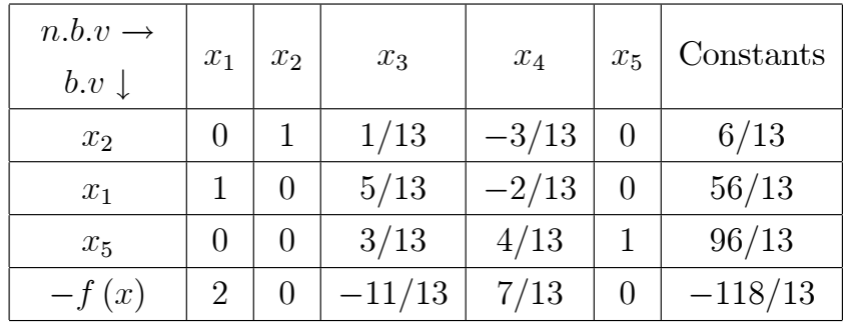
Solution: The first step is



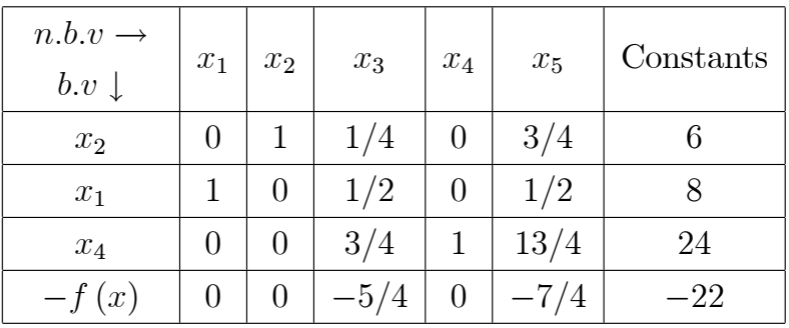
The second step



The third step



The fourth step



The optimal solution is

basic variables

non-basic variables

and

**The simplex method in the compact form:**

Looking at the previous example 5, which be solved by the simplex method in

a tabular form, we notice that a unit matrix is always present in each tableau, it

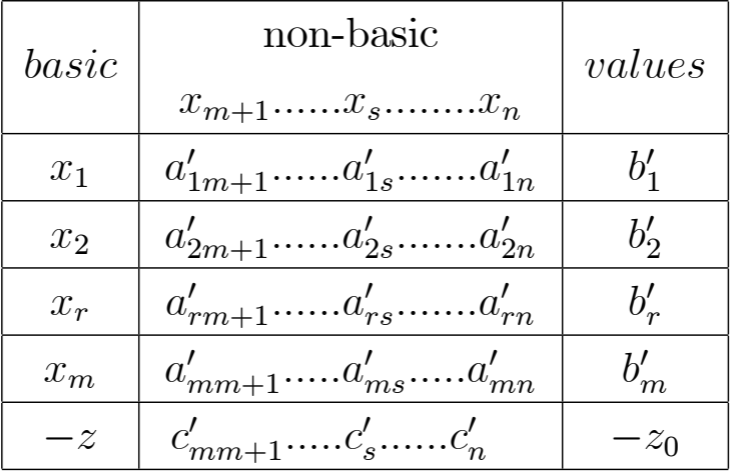
correspond to the columns of the basic variables of each iteration. However, a unit

matrix does not contain numerical information which need be recorded. We can

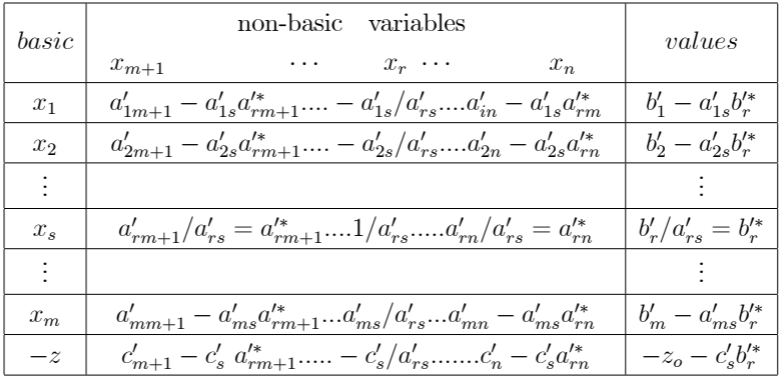
remove it and know that it always exists. We then will have columns for non-basic

variable only.

The tableau for the kth iteration will have the form:



The tableau for the k + 1th iteration will be:



The rules for the transformation tableau from iteration to the next one are as

follows:

1. The pivot element is replaced by its reciprocal.

2. The pivot row is divided by the pivot element.

3. The pivot column is divided by the negative of the pivot element.

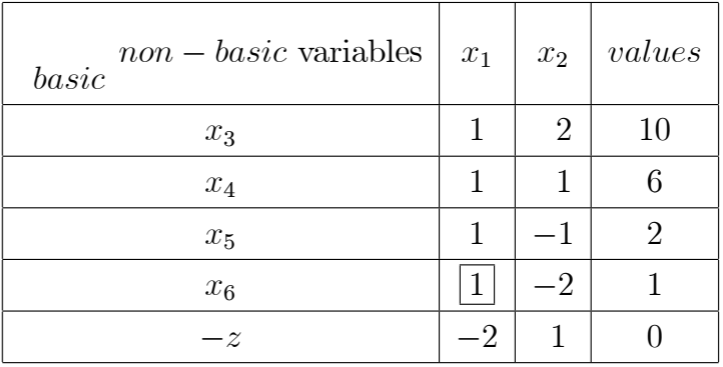
4. Every other element is reduced by the quantity

5. The headings of the pivot row and pivot column

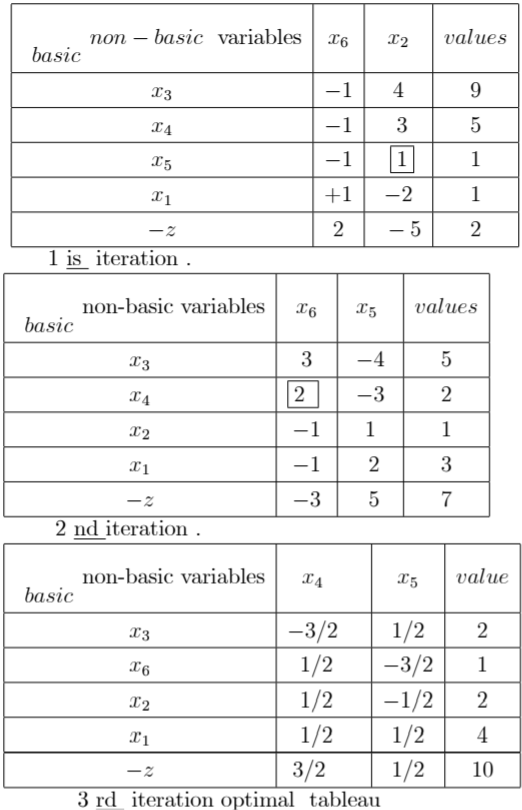
To illustrate this method let us consider the following example:

Solution: Reversing the signs of the to convert the problem to a minimization

one. Then the calculations are shown in the following compact tableau:



initial tableau



from the optimal tableau we can see that the optimal solution is:

**Duality Theorems**

In the following theorems, we are going to use the standard forms for the primal

and the dual problems given as:

I. The primal problem

To minimize

subject to

II. The dual problem

To maximize

subject to

**Theorem 12** The dual of the dual is the primal

This theorem is obvious and implies a completely symmetrical relationship between the primal and the dual problems.

Theorem 13 For any feasible solution, the values of the objective function for the

primal problem is always greater than or equal to the value of the objective function

for the dual problem.

In other words, if are feasible solution to the primal

and the dual problems respectively, then

Proof. Consider the primal and the dual problems as given by I &II.

From the dual problem

Since , then we can multiply both sides of by and then sum over

j, we get

Interchanging the summation signs of the right hand side of (1.18), we have

But from the primal problem

Then substituting in (1.19) we get

which completes the proof.

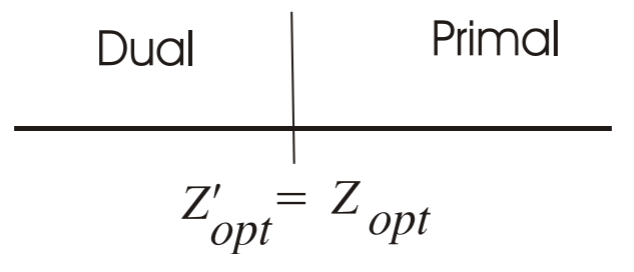
Theorem 14 If there exist finite feasible solutions for both the primal and the dual

problems, then the values of the objective functions corresponding to their optimal solutions are equal.

A constructive proof of this theorem is established by means of the simplex algorithm. From theorems 13 and 14 we have the following information about the possible ranges of the values of the objective functions for the feasible solutions of

the primal and the dual problems.

This can be illustrated by the following diagram:



This can be used in the following

1. If we know a feasible solution of the dual problem then we have a lower bound

for the optimal value (minimal) of the objective function of the primal problem

2. If we know a feasible solution of the primal problem then we have an upper

bound for the optimal value (maximum) of the objective function of the dual

problem.

3. If an optimal solution of either problems is known then the value of the objective function of the other problem is also known.

Theorem 15 If the primal (dual) problem has feasible solutions and the dual (primal) problem has no feasible solution, then the primal (dual) problem has unbounded

Solutions

**References**

1. Spiegel, Murray. Schaum's Outline of Advanced Mathematics for Engineers and Scientists. US: McGraw-Hill, 2009.
2. Strang, G. (2016). Introduction to linear algebra.
3. Anton, H., Rorres, C., & Kaul, A. (2019). Elementary linear algebra: Applications version.
4. Friedberg, S. H., Insel, A. J., & Spence, L. E. (2019). Linear algebra.
5. Karloff, H. (2009). Linear programming. Boston: Birkhäuser.