Habib University - City Campus
Course: CS 412: Algorithms: Design and Analysis
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Examination: Final Exam – Fall 2020 (Online)
Exam Date: Friday, December 11, 2020
Exam Time: 9:00am – 9:00pm

Total Marks: 100 | Duration: 12 hours

Please read the following additional instructions carefully.

- This is a **take home exam**.
- You are **allowed** to use the textbook of the course in addition to the lecture notes/slides provided by the instructor(s) and your own notes that you might have prepared.
- You are **not allowed** to communicate with anyone from the your class or outside for clarification, help, hints, and/or any other support to solve the exam.
- You are **not allowed** to search online for the answers.
- You are **required** to cite all the resources you will use to solve the exam.
- A skeleton LaTeX file is provided separately for your answers.
- You must electronically submit your answers (via LMS) saved in a PDF file created with LaTeX using the provided skeleton file with proper naming convention.
- No other file format is accepted.
- Files submitted with wrong naming will not be graded.
- Multiple submissions are not allowed.

Question 1: [28 points]

For each of the following indicate whether the statement is **True** or **False** [**01 point each**]. Justify your answer [**03 points each**].

(a) An unbiased coin is flipped repeatedly until both heads and tails are obtained. The expected number of times the coin is flipped is 3.

(b) If an undirected graph G with n vertices has k connected components then there are at most $n-k$ edges in G.

(c) Let T be a minimum spanning tree of a graph G. Then for any pair of vertices s and t in G, the shortest path from s to t in G is the shortest path from s to t in T.

(d) Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two *directed graphs* with the same structure (i.e., same vertices and same edges). Let the costs of edges in G_1 be distinct and nonnegative and costs of edges in G_2 be the squares of costs of their corresponding edges in G_1. Then the *shortest paths* in G_1 and G_2 from some vertex s to any other vertex t by DIJKSTRA's algorithm are the same.

(e) Every directed graph is a *directed acyclic graph* (DAG) of its *strongly connected components* (SCC).

(f) There exists an $O(n^2)$ algorithm to generate all the possible bit strings of length n.

(g) The *exact* value returned from the following algorithm MYFUNCTION expressed in terms of n is $n(n+1)/2$.

Algorithm MYFUNCTION

Input: An integer n

Output: The value k

```
k = 0
for i = 0 to n − 1 do
    for j = i downto 0 do
        k = k + 1
return k
```

Question 2: [15 points]

We are given an array $A[1..n]$, which stores a sequence of 0's and then followed by a sequence of 1's.

(a) $\boxed{\text{05 points}}$ Design an $O(\log n)$-time algorithm to find the location of the last 0, i.e., find k such that $A[k] = 0$ and $A[k+1] = 1$.

(b) $\boxed{\text{10 points}}$ Suppose that k is much smaller than n. Is it possible to improve the running time of our algorithm to $O(\log k)$ instead of $O(\log n)$? Justify your answer (i.e., provide an algorithm if your answer is "yes" or a proof if your answer is "no").

Question 3: [10 points]

Let $A[1..n]$ be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an *inversion* of A. Suppose that the elements of A form a *uniform random permutation* i.e., $\Pr\{A[i] > A[j]\} = 1/2$ for all $i < j$. Use indicator random variables to compute the expected number of inversions.

Question 4: [10 points]

Let $A[1..n]$ be an array of *nonnegative integers* and assume there are exactly k 0's in A, note that $0 < k < n$. Consider the following algorithm. Compute the expected number of times the Step **(3)** is executed.

Input: An array $A[1..n]$ of nonnegative integers and an integer k s.t. $0 < k < n$.
Output: An array $B[1..k]$ of indices of all 0's in A.

```
B = [ ]
while |B| ≠ k
    Generate a random number j in {1, ..., n}
    if A[j] = 0
        if j ∉ B then append j to B
return B
```

For example if $A = [0, 5, 2, 0, 8, 9, 0, 3, 1, 7, 0]$ then one possible solution could as $B = [4, 11, 7, 1]$ (the elements in B might not be ordered).

Question 5: [12 points]

Given a rooted tree $T = (V, E)$.

(a) $\boxed{\text{5 points}}$ Design an algorithm that checks whether two vertices $u, v \in V$ are at the same level in T or not.

(b) $\boxed{\text{5 points}}$ What is the time complexity of your algorithm?

(c) $\boxed{\text{2 points}}$ What is the space complexity of your algorithm?

Question 6: [15 points]

For a given directed graph $G = (V, E)$ and weight function $w : E \to \mathbb{R} \cup \{\infty\}$ defined for all edges in G. The *diameter* of the graph G is the length of the *longest shortest path* between any two vertices (u, v).

Use *bottom-up dynamic programming* to solve following questions.

(a) $\boxed{\text{5 points}}$ Express the problem recursively showing that an optimal substructure exists.

(b) $\boxed{\text{5 points}}$ Design an $O(|V|^3)$ algorithm that returns the diameter of the graph G.

(c) $\boxed{\text{5 points}}$ Derive the runtime complexity of the algorithm using asymptotic notation.

Question 7: [10 points]

Find a *longest common subsequence* between the two string sequences:

`CGACATC` and `AGCTC`.