

GROUP - 19

MONTHLY

# ASSIGNMENT

MAT 120

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## 1. Heart Attack

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(a)

We know,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dx \, dy = |J| \, dr \, d\theta$$

Now,

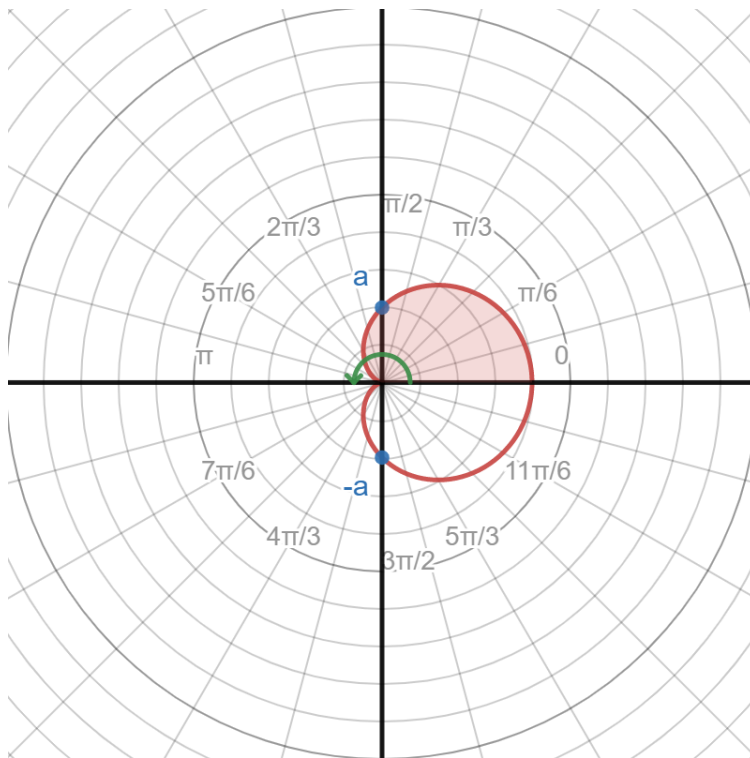
$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2(\theta) - (-r \sin^2(\theta)) \\ &= r \cos^2(\theta) + r \sin^2(\theta) \\ &= r \{\cos^2(\theta) + \sin^2(\theta)\} \\ |J| &= r \end{aligned}$$

$$\therefore dx \, dy = r \, dr \, d\theta$$

$$\begin{aligned} \iint_R xy \, dx \, dy &= \iint_R (r \cos \theta)(r \sin \theta) r \, dr \, d\theta \\ &= \iint_R r^3 \sin \theta \cos \theta \, dr \, d\theta \\ \iint_R y \, dx \, dy &= \iint_R r \sin \theta r \, dr \, d\theta \\ &= \iint_R r^2 \sin \theta \, dr \, d\theta \\ \therefore \frac{\iint_R xy \, dx \, dy}{\iint_R y \, dx \, dy} &= \frac{\iint_R r^3 \sin \theta \cos \theta \, dr \, d\theta}{\iint_R r^2 \sin \theta \, dr \, d\theta} \end{aligned}$$

The region 'R' is revolved around x-axis, so the axis of symmetry is the x-axis. Therefore, the center of gravity will be on the axis of symmetry (x-axis). Thus, y and z must be zero.

(b)



Here, x-axis is the symmetry for the given shape. Using symmetry, we can find the integral for the shaded region and then multiply it by 2. For shaded region, Range:

$$0 \leq \theta \leq \pi$$

$$0 \leq r \leq a(1 + \cos \theta)$$

$$\begin{aligned} \overline{X} &= \frac{\iint_R r^3 \sin \theta \cos \theta \, dr \, d\theta}{\iint_R r^2 \sin \theta \, dr \, d\theta} \\ &= \frac{\int_0^\pi \int_0^{a(1+\cos \theta)} r^3 \sin \theta \cos \theta \, dr \, d\theta}{\int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \sin \theta \, dr \, d\theta} \end{aligned}$$

Considering for one side of the symmetry (0 to  $\pi$ ).

$$\begin{aligned}
 \int_0^\pi \int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta \, dr \, d\theta &= \int_0^\pi \left[ \frac{r^4}{4} \sin\theta \cos\theta \right]_0^{a(1+\cos\theta)} d\theta \\
 &= \frac{1}{4} \int_0^\pi a^4 (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta \\
 &= \frac{a^4}{4} \int_0^\pi (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta
 \end{aligned}$$

Let,

$$u = \cos\theta$$

$$\therefore du = -\sin(\theta)d\theta$$

$$\begin{aligned}
 \int (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta &= \int (1+u)^4 u (-du) \\
 &= - \int u(1+u)^4 du
 \end{aligned}$$

Let,

$$s = 1 + u$$

$$\therefore ds = du$$

$$- \int u(1+u)^4 du = - \int (s-1)s^4 ds$$

$$= \int s^4 - s^5 ds$$

$$= \frac{s^5}{5} - \frac{s^6}{6} + C$$

$$= \frac{(1+u)^5}{5} - \frac{(1+u)^6}{6} + C$$

$$\therefore \int (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta = \frac{(1+\cos\theta)^5}{5} - \frac{(1+\cos\theta)^6}{6} + C$$

Now,

$$\begin{aligned}\frac{a^4}{4} \int_0^\pi (1 + \cos \theta)^4 \sin \theta \cos \theta \, d\theta &= \frac{a^4}{4} \left[ \frac{(1 + \cos \theta)^5}{5} - \frac{(1 + \cos \theta)^6}{6} \right]_0^\pi \\ &= \frac{a^4}{4} \left\{ \frac{32}{5} - \frac{32}{3} \right\} \\ \therefore \int_0^\pi \int_0^{a(1+\cos \theta)} r^3 \sin \theta \cos \theta \, dr \, d\theta &= -\frac{16}{15} a^4\end{aligned}$$

Again,

$$\begin{aligned}\int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \sin \theta \, dr \, d\theta &= \int_0^\pi \left[ \frac{r^3}{3} \sin \theta \right]_0^{a(1+\cos \theta)} d\theta \\ &= \frac{1}{3} \int_0^\pi a^3 (1 + \cos \theta)^3 \sin \theta \, d\theta \\ &= \frac{a^3}{3} \int_0^\pi (1 + \cos \theta)^3 \sin \theta \, d\theta\end{aligned}$$

Let,

$$\begin{aligned}u &= \cos \theta \\ \therefore du &= -\sin(\theta) d\theta \\ \int (1 + \cos \theta)^3 \sin \theta \, d\theta &= \int (1 + u)^3 (-du) \\ &= - \int (1 + u)^3 du\end{aligned}$$

Let,

$$\begin{aligned}s &= 1 + u \\ \therefore ds &= du \\ - \int (1 + u)^3 du &= - \int s^3 ds \\ &= -\frac{s^4}{4} + C \\ &= -\frac{(1 + u)^4}{4} + C \\ \therefore \int (1 + \cos \theta)^3 \sin \theta \, d\theta &= -\frac{(1 + \cos \theta)^4}{4} + C\end{aligned}$$

Now,

$$\begin{aligned}\frac{a^3}{3} \int_0^\pi (1 + \cos \theta)^3 \sin \theta \, d\theta &= \frac{a^3}{3} \left[ -\frac{(1 + \cos \theta)^4}{4} \right]_0^\pi \\ &= \frac{a^3}{3}(-4) \\ \therefore \int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \sin \theta \, dr \, d\theta &= -\frac{4}{3}a^3\end{aligned}$$

for the entire rotation we multiply with 2.

$$\begin{aligned}\bar{X} &= \frac{\int_0^\pi \int_0^{a(1+\cos \theta)} r^3 \sin \theta \cos \theta \, dr \, d\theta}{\int_0^\pi \int_0^{a(1+\cos \theta)} r^2 \sin \theta \, dr \, d\theta} \times 2 \\ &= \frac{-\frac{16}{15}a^4}{-\frac{4}{3}a^3} \times 2 \\ \therefore \bar{X} &= \frac{8}{5}a\end{aligned}$$

$$\therefore \text{Center of gravity} = \left( \frac{8}{5}a, 0, 0 \right)$$



(c)

$$\text{Surface Area} = \iint_R f(x) \, dx \, dy$$

Given,

$$\begin{aligned} V &= 2\pi \iint_R f(x) \, dx \, dy \\ &= 2\pi \iint_R r \sin \theta \, dx \, dy ; [\because y = f(x) = r \sin \theta] \end{aligned}$$

*We know,*

$$dx \, dy = |J| \, dr \, d\theta$$

Now,

$$\begin{aligned} |J| &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2(\theta) - (-r \sin^2(\theta)) \\ &= r \cos^2(\theta) + r \sin^2(\theta) \\ &= r \{\cos^2(\theta) + \sin^2(\theta)\} \\ |J| &= r \end{aligned}$$

$$\therefore dx \, dy = r \, dr \, d\theta$$

$$\begin{aligned} V &= 2\pi \iint_R r \sin \theta \, dx \, dy \\ &= 2\pi \iint_R r \sin \theta \, r \, dr \, d\theta \\ \therefore V &= 2\pi \iint_R r^2 \sin \theta \, dr \, d\theta \end{aligned}$$

(d)

$$\begin{aligned}
V &= 2\pi \int \int_R r^2 \sin \theta dr d\theta \\
&= 2\pi \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1+\cos \theta)} r^2 \sin \theta dr d\theta \\
&= 2\pi \int_0^\pi \left[ \frac{r^3}{3} \right]_0^{a(1+\cos \theta)} \sin \theta d\theta \\
&= \frac{2\pi}{3} \int_0^\pi a^3 (1 + \cos \theta)^3 \sin \theta d\theta
\end{aligned}$$

Let,

$$u = 1 + \cos \theta$$

$$\therefore \sin \theta d\theta = -du$$

$\theta$	0	$\pi$
$u$	2	0

$$\begin{aligned}
\therefore V &= \frac{2\pi}{3} \int_2^0 a^3 u^3 (-du) \\
&= \frac{-2\pi a^3}{3} \int_2^0 u^3 du \\
&= \frac{-2\pi a^3}{3} \left[ \frac{u^4}{4} \right]_2^0 \\
&= \frac{-2\pi a^3}{3} \left[ 0 - \frac{16}{4} \right] \\
&= \frac{8\pi a^3}{3}
\end{aligned}$$

(e)

$$\begin{aligned}
L &= \int_p^q \sqrt{1 + [f'(x)]^2} dx \\
&= \int_p^q \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
&= \int_p^q \sqrt{\frac{dx^2 + dy^2}{dx^2}} dx \\
&= \int_p^q \sqrt{(dx^2) \left(\frac{dx^2 + dy^2}{dx^2}\right)} \\
&= \int \sqrt{dx^2 + dy^2}
\end{aligned}$$

Now,

$$\begin{aligned}
dL &= \sqrt{dx^2 + dy^2} \\
\Rightarrow dL^2 &= dx^2 + dy^2 \\
\Rightarrow \frac{dL^2}{d\theta^2} &= \frac{dx^2}{d\theta^2} + \frac{dy^2}{d\theta^2} \\
\Rightarrow dL^2 &= \left[ \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \right] d\theta^2 \\
\Rightarrow dL &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
&\quad \text{[integrating both sides]} \\
\therefore L &= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta
\end{aligned}$$

Now,

$$\begin{aligned}
 x &= r(\theta) \cos \theta \\
 y &= r(\theta) \sin \theta \\
 \therefore \left( \frac{dx}{d\theta} \right)^2 &= \left( -r \sin \theta + \cos \theta \frac{dr}{d\theta} \right)^2 \\
 \therefore \left( \frac{dy}{d\theta} \right)^2 &= \left( r \cos \theta + \sin \theta \frac{dr}{d\theta} \right)^2 \\
 \therefore dL^2 &= \left[ \cos^2 \theta \left( \frac{dr}{d\theta} \right)^2 - 2r \sin \theta \cos \theta \left( \frac{dr}{d\theta} \right) + r^2 \sin^2 \theta + \right. \\
 &\quad \left. r^2 \cos^2 \theta + 2r \sin \theta \cos \theta \left( \frac{dr}{d\theta} \right) + \sin^2 \theta \left( \frac{dr}{d\theta} \right)^2 \right] d\theta^2 \\
 \implies dL^2 &= \left[ r^2 (\sin^2 \theta + \cos^2 \theta) + \left( \frac{dr}{d\theta} \right)^2 (\sin^2 \theta + \cos^2 \theta) \right] d\theta^2 \\
 \implies dL^2 &= \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right] d\theta^2 \\
 \therefore dL &= \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta \\
 &\quad \text{[integrating both sides]} \\
 \therefore L &= \int_a^b \sqrt{r^2 + \left[ \frac{dr}{d\theta} \right]^2} d\theta
 \end{aligned}$$

[proved]

(f)

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$r = a(1 + \cos \theta)$$

$$= a + a \cos \theta$$

$$\therefore \frac{dr}{d\theta} = -a \sin \theta$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin^2 \theta$$

$\therefore$  Perimeter:

$$\begin{aligned} &= 2 \int_0^\pi \sqrt{[a^2(1 + \cos \theta)^2] + a^2 \sin^2 \theta} d\theta \\ &= 2 \int_0^\pi \sqrt{[a^2(1 + 2 \cos \theta + \cos^2 \theta)] + a^2 \sin^2 \theta} d\theta \\ &= 2 \int_0^\pi \sqrt{a^2(1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)} d\theta \\ &= 2a \int_0^\pi \sqrt{2 + 2 \cos \theta} d\theta \\ &= 2a \int_0^\pi \sqrt{2(1 + \cos \theta)} d\theta \\ &= 2a \int_0^\pi \sqrt{2 \left[ 2 \cos^2 \left( \frac{\theta}{2} \right) \right]} d\theta \\ &= 4a \int_0^\pi \cos \left( \frac{\theta}{2} \right) d\theta \\ &= 4a \left[ \frac{\sin(\theta/2)}{1/2} \right]_0^\pi \\ &= 8a(\sin(\pi/2) - \sin 0) \\ &= 8a \end{aligned}$$

## 2. Particle in a Cylinder

---

(a)

$$\langle \psi | \psi \rangle = 1$$

The probability of finding the particle outside S is 0 and the probability of finding the particle inside S is non-zero; hence the summation of all probabilities (total probability of finding the particle inside S) is 1.

(b)

The cylindrical coordinate system will be a good choice to evaluate the integrals because it is easier to determine the limits for the given bounded region and equation. It is also easier to convert the equation to a simpler form ( $\sqrt{x^2 + y^2} = r$ ) using this coordinate system.

$$\langle g | h \rangle = \iiint_S g(x, y, z)^* h(x, y, z) dV$$

For,

$$\langle \psi | \psi \rangle = \iiint_S \psi(x, y, z)^* \psi(x, y, z) dV$$

Since,  $\psi(x, y, z)$  is real and complex conjugate of a real number is also real, we can write-

$$\begin{aligned} \langle \psi | \psi \rangle &= \iiint_S (\psi(x, y, z))^2 dV \\ &= \iiint_S \frac{A^2}{\sqrt{x^2 + y^2}} dV \end{aligned}$$

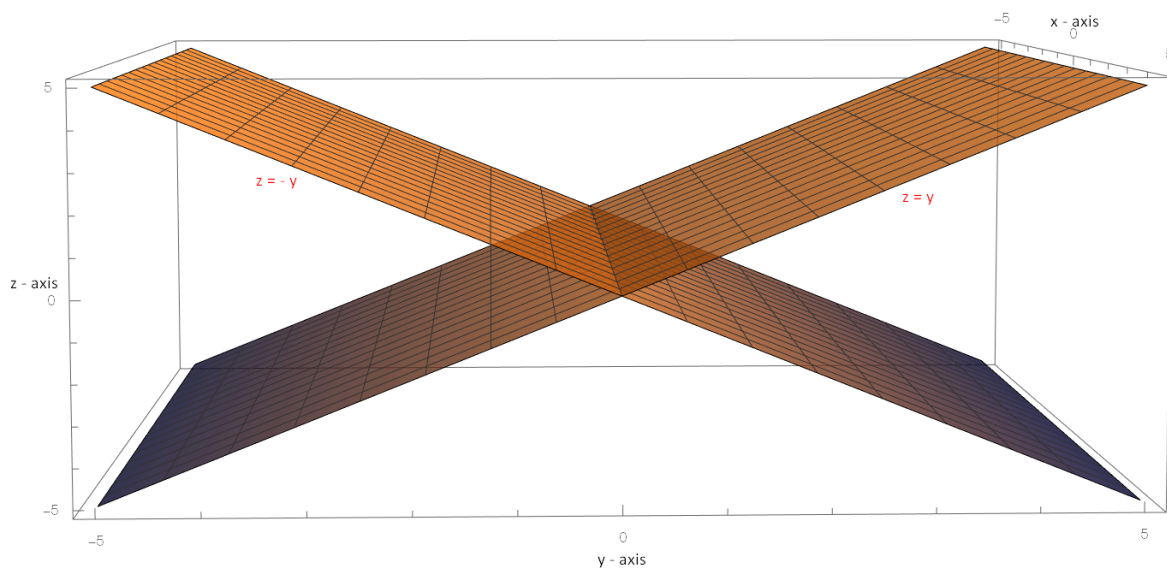
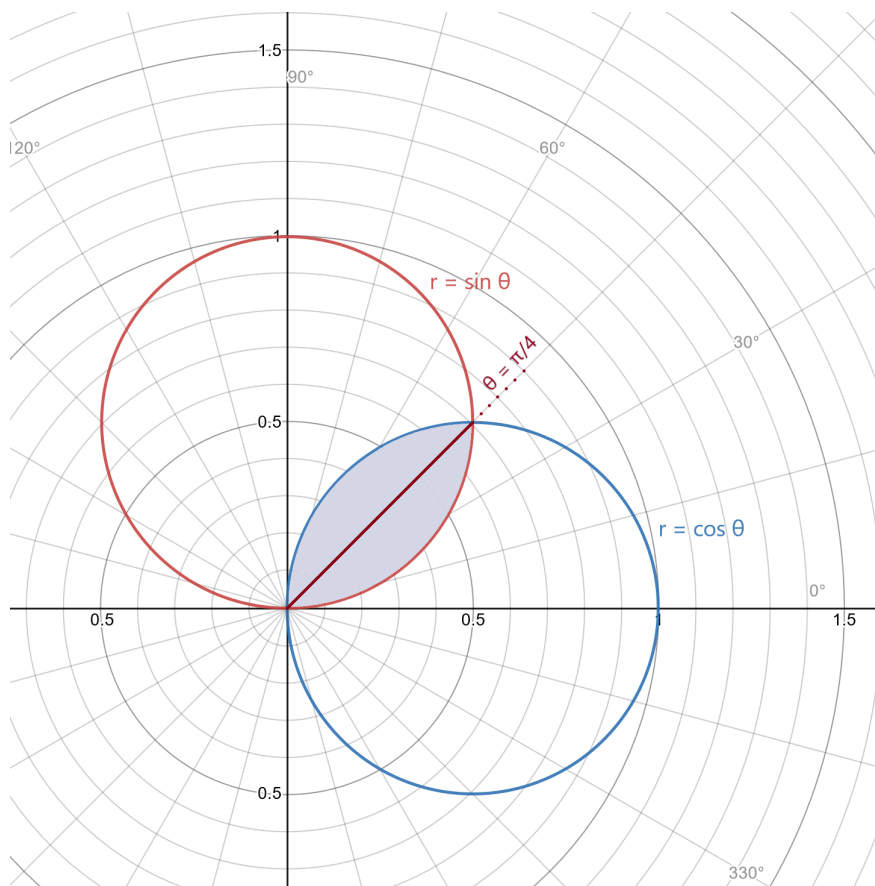


Figure 1:  $z = y$  and  $z = -y$  planes

We know:  $-y \leq z \leq y$ .

So in terms of cylindrical coordinate system,  $-r \sin \theta \leq z \leq r \sin \theta$ .



Figure 2:  $xy$  plane

Using the graph above,

$$\sin \theta = \cos \theta$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}$$

So, the limits are:

$$-r \sin \theta \leq z \leq r \sin \theta$$

$$0 \leq \theta \leq \frac{\pi}{4} \text{ and } \frac{\pi}{4} < \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq \sin \theta \text{ and } 0 < r \leq \cos \theta$$

Finally, putting the limits into the equation and substituting the value of  $\sqrt{x^2 + y^2} = r$ .

$$\begin{aligned}
 \langle \psi | \psi \rangle &= \iiint_S \frac{A^2}{r} dV \\
 \langle \psi | \psi \rangle &= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} dV + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} dV
 \end{aligned}$$

(c)

$$dV = \left| \frac{\partial(x, y, z)}{\partial(z, r, \theta)} \right| dz dr d\theta$$

$$= \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} dz dr d\theta$$

$$\begin{aligned}
 &= \begin{vmatrix} 0 & \cos \theta & -r \sin \theta \\ 0 & \sin \theta & r \cos \theta \\ 1 & 0 & 0 \end{vmatrix} dz dr d\theta \\
 &= (r \cdot \cos^2 \theta + r \cdot \sin^2 \theta) dz dr d\theta \\
 &= r \cdot (\cos^2 \theta + \sin^2 \theta) dz dr d\theta \\
 &= r dz dr d\theta
 \end{aligned}$$

(d)

We know,  $\langle \psi | \psi \rangle = 1$ .

So we can write,

$$\langle \psi | \psi \rangle = \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} dV + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} dV = 1$$

Now, substituting the value of dV using the Jacobian determinant and solving:

$$\begin{aligned} \langle \psi | \psi \rangle &= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} \cdot r dz dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} \frac{A^2}{r} \cdot r dz dr d\theta = 1 \\ &= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 dz dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 dz dr d\theta = 1 \\ &= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} [A^2 z]_{-r \sin \theta}^{r \sin \theta} dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} [A^2 z]_{-r \sin \theta}^{r \sin \theta} dr d\theta = 1 \\ &= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} 2A^2 r \sin \theta dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} 2A^2 r \sin \theta dr d\theta = 1 \\ &= \int_0^{\frac{\pi}{4}} [A^2 r^2 \sin \theta]_0^{\sin \theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [A^2 r^2 \sin \theta]_0^{\cos \theta} d\theta = 1 \\ &= \int_0^{\frac{\pi}{4}} A^2 \sin^3 \theta d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} A^2 \cos^2 \theta \sin \theta d\theta = 1 \\ &= A^2 \int_0^{\frac{\pi}{4}} \sin^3 \theta d\theta + A^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 \theta \sin \theta d\theta = 1 \end{aligned}$$

First,

$$\int \sin^3 \theta d\theta = \int (1 - \cos^2 \theta) \sin \theta d\theta$$

Let  $u = \cos \theta$  and  $\frac{du}{d\theta} = -\sin \theta$ .

$$\int -1 + u^2 du = \left[-u + \frac{u^3}{3}\right] = \left[-\cos \theta + \frac{\cos^3 \theta}{3}\right]$$

Likewise for,

$$\int \cos^2 \theta \sin \theta d\theta$$

Let  $u = \cos \theta$  and  $\frac{du}{d\theta} = -\sin \theta$ .

$$\int -u^2 du = \left[-\frac{u^3}{3}\right] = \left[-\frac{\cos^3 \theta}{3}\right]$$

So putting the limits into these integrals,

$$\begin{aligned}
A^2[-\cos \theta + \frac{\cos^3 \theta}{3}]_0^{\frac{\pi}{4}} + A^2[-\frac{\cos^3 \theta}{3}]_{\frac{\pi}{4}}^{\frac{\pi}{2}} &= 1 \\
A^2[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} - (-1 + \frac{1}{3})] + A^2[0 - (-\frac{\sqrt{2}}{12})] &= 1 \\
\frac{2 - \sqrt{2}}{3} A^2 &= 1 \\
A^2 &= \frac{6 + 3\sqrt{2}}{2} \\
A &= (\frac{6 + 3\sqrt{2}}{2})^{\frac{1}{2}}
\end{aligned}$$

(e)

We know,

$$< \psi | \hat{r} \psi > = \iiint_S \psi(x, y, z)^* \hat{r} \psi(x, y, z) dV$$

Since,  $\hat{r}\psi = r\psi$ ,

$$\begin{aligned}
< \psi | \hat{r} \psi > &= \iiint_S (\psi(x, y, z))^2 \cdot r dV \\
&= \iiint_S \frac{A^2}{r} \cdot r dV \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 \cdot r dz dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 \cdot r dz dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} [A^2 r z]_{-r \sin \theta}^{r \sin \theta} dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} [A^2 r z]_{-r \sin \theta}^{r \sin \theta} dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} 2A^2 r^2 \sin \theta dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} 2A^2 r^2 \sin \theta dr d\theta \\
&= \int_0^{\frac{\pi}{4}} [\frac{2A^2 r^3 \sin \theta}{3}]_0^{\sin \theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\frac{2A^2 r^3 \sin \theta}{3}]_0^{\cos \theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{2A^2 \sin^4 \theta}{3} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2A^2 \cos^3 \theta \sin \theta}{3} d\theta \\
&= \frac{2A^2}{3} \int_0^{\frac{\pi}{4}} \sin^4 \theta d\theta + \frac{2A^2}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 \theta \sin \theta d\theta
\end{aligned}$$

For,

$$\int_0^{\frac{\pi}{4}} \sin^4 \theta \, d\theta = \int_0^{\frac{\pi}{4}} \sin^3 \theta \sin \theta \, d\theta$$

Applying integration by parts such that,  $f(x) = \sin^3 \theta$  and  $g'(x) = \sin \theta$  :

$$\int_0^{\frac{\pi}{4}} \sin^3 \theta \sin \theta \, d\theta = [-\sin^3 \theta \cos \theta - \int -3 \sin^2 \cos^2 \theta \, d\theta]_0^{\frac{\pi}{4}}$$

Using trigonometric identities we know:

$$3 \sin^2 \theta \cos^2 \theta = 3 \left( \frac{\sin 2\theta}{2} \right)^2 = \frac{3}{8} (1 - \cos 4\theta)$$

So, substituting this value:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sin^3 \theta \sin \theta \, d\theta &= [-\sin^3 \theta \cos \theta - \frac{3}{8} \int (1 - \cos 4\theta)]_0^{\frac{\pi}{4}} \\ &= [-\sin^3 \theta \cos \theta - \frac{3}{8} (\theta - \frac{\sin 4\theta}{4})]_0^{\frac{\pi}{4}} \\ &= \frac{3\pi - 8}{32} \end{aligned}$$

Now for,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 \theta \sin \theta \, d\theta$$

Let  $u = \cos \theta$  and  $\frac{du}{d\theta} = -\sin \theta$ .

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 \theta \sin \theta \, d\theta &= \int -u^3 du \\ &= \left[ -\frac{u^4}{4} \right] \\ &= \left[ -\frac{\cos^4 \theta}{4} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{16} \end{aligned}$$

Putting these values into the original equation of  $\langle \psi | \hat{r} | \psi \rangle$

$$\begin{aligned} \langle \psi | \hat{r} | \psi \rangle &= \frac{2A^2}{3} \cdot \frac{3\pi - 8}{32} + \frac{2A^2}{3} \cdot \frac{1}{16} \\ &= \frac{3\pi - 8}{48} A^2 + \frac{1}{24} A^2 \\ &= \frac{3\pi - 6}{48} A^2 \end{aligned}$$

Finally putting the value of  $A^2 = \frac{6+3\sqrt{2}}{2}$  (known from 2c)

$$\begin{aligned} \langle \psi | \hat{r} \psi \rangle &= \frac{3\pi - 6}{48} \cdot \frac{6 + 3\sqrt{2}}{2} \\ &= \frac{18\pi + 9\pi\sqrt{2} - 36 - 18\sqrt{2}}{96} \end{aligned}$$

(f)

If  $\hat{A}(kf) = k\hat{A}f$  and  $\hat{r}\psi = r\psi$ ,

Then,

$$\hat{r}^2\psi = \hat{r}(\hat{r}\psi) = \hat{r}(r\psi) = r^2\psi$$

(g)

$$\begin{aligned}
\langle \psi | \hat{r}^2 | \psi \rangle &= \iiint_S \psi(x, y, z)^* \hat{r}^2 \psi(x, y, z) dV \\
&= \iiint_S \frac{A^2}{r} \cdot r^2 dV \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 \cdot r^2 dz dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_{-r \sin \theta}^{r \sin \theta} A^2 \cdot r^2 dz dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} [A^2 r^2 z]_{-r \sin \theta}^{r \sin \theta} dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} [A^2 r^2 z]_{-r \sin \theta}^{r \sin \theta} dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sin \theta} 2A^2 r^3 \sin \theta dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \theta} 2A^2 r^3 \sin \theta dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \left[ \frac{A^2 r^4 \sin \theta}{2} \right]_0^{\sin \theta} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[ \frac{A^2 r^4 \sin \theta}{2} \right]_0^{\cos \theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{A^2 \sin^4 \theta \sin \theta}{2} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{A^2 \cos^4 \theta \sin \theta}{2} d\theta \\
&= \frac{A^2}{2} \int_0^{\frac{\pi}{4}} \sin^4 \theta \sin \theta d\theta + \frac{A^2}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4 \theta \sin \theta d\theta \\
&= \frac{A^2}{2} \int_0^{\frac{\pi}{4}} (1 - \cos^2 \theta)^2 \theta \sin \theta d\theta + \frac{A^2}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4 \theta \sin \theta d\theta
\end{aligned}$$

Let  $u = \cos \theta$  and  $\frac{du}{d\theta} = -\sin \theta$ .

$$\begin{aligned}
\langle \psi | \hat{r}^2 | \psi \rangle &= \frac{A^2}{2} \int -1 + 2u^2 - u^4 du + \frac{A^2}{2} \int -u^4 du \\
&= \frac{A^2}{2} \left[ -u + \frac{2u^3}{3} - \frac{u^5}{5} \right] + \frac{A^2}{2} \left[ -\frac{u^5}{5} \right] \\
&= \frac{A^2}{2} \left[ -\cos \theta + \frac{2 \cos^3 \theta}{3} - \frac{\cos^5 \theta}{5} \right]_0^{\frac{\pi}{4}} + \frac{A^2}{2} \left[ -\frac{\cos^5 \theta}{5} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{A^2}{2} \left( \frac{32\sqrt{2} - 43}{60\sqrt{2}} \right) + \frac{A^2}{2} \left( \frac{1}{20\sqrt{2}} \right) \\
&= A^2 \left( \frac{32\sqrt{2} - 43}{120\sqrt{2}} \right) + A^2 \left( \frac{1}{40\sqrt{2}} \right) \\
&= \left( \frac{4\sqrt{2} - 5}{15\sqrt{2}} \right) A^2
\end{aligned}$$

Finally substituting the value of  $A^2 = \frac{6+3\sqrt{2}}{2}$  (known from 2c),

$$\begin{aligned}
\langle \psi | \hat{r}^2 \psi \rangle &= \frac{4\sqrt{2} - 5}{15\sqrt{2}} \cdot \frac{6 + 3\sqrt{2}}{2} \\
&= \frac{3 - \sqrt{2}}{10}
\end{aligned}$$

(h)

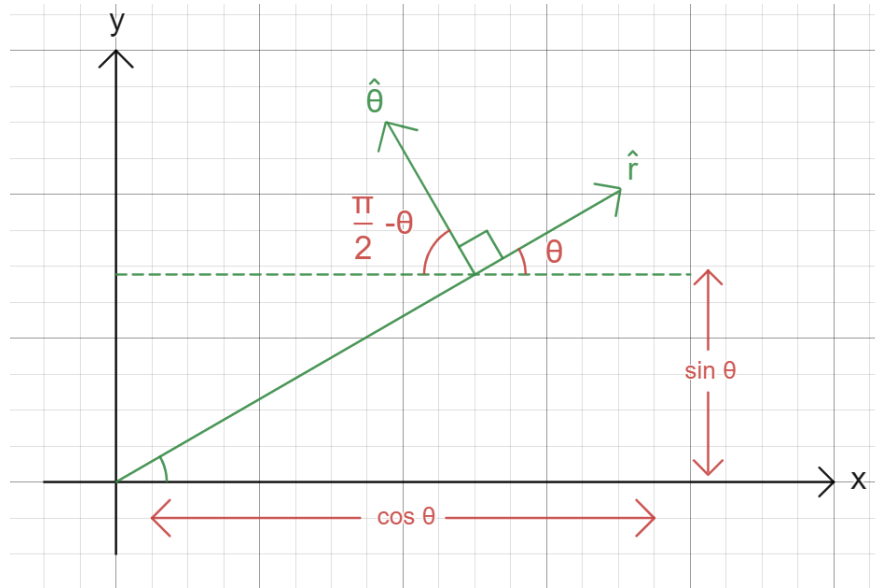
$$\begin{aligned}
\sigma_r^2 &= \langle \psi | \hat{r}^2 \psi \rangle - \langle \psi | \hat{r} \psi \rangle^2 \\
\sigma_r^2 &= \frac{3 - \sqrt{2}}{10} - \left( \frac{18\pi + 9\pi\sqrt{2} - 36 - 18\sqrt{2}}{96} \right)^2 \\
\sigma_r^2 &= 0.0250587 \\
\sigma_r &= 0.1583
\end{aligned}$$



### 3. Rotating Tube

---

(a)



$$\begin{aligned}
 \hat{r} &= x\hat{i} + y\hat{j} \\
 &= r \cos \theta \hat{i} + r \sin \theta \hat{j} \\
 &= r(\cos \theta \hat{i} + \sin \theta \hat{j})
 \end{aligned}$$

Also we know,

$$\begin{aligned}
 \vec{r} &= r \hat{r} \quad [\text{where } \hat{r} \text{ is the unit vector}] \\
 \text{or, } r \hat{r} &= r(\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 \therefore \hat{r} &= \cos \theta \hat{i} + \sin \theta \hat{j}
 \end{aligned}$$

From the figure,

$$\begin{aligned}
 \hat{\theta} &= -\cos \left( \frac{\pi}{2} - \theta \right) \hat{i} + \sin \left( \frac{\pi}{2} - \theta \right) \hat{j} \\
 &= -\sin \theta \hat{i} + \cos \theta \hat{j}
 \end{aligned}$$

According to question,

$$\begin{aligned}
 v &= \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} \\
 \text{or, } v &= r'(t) \hat{r} + r \theta'(t) \hat{\theta} \\
 \text{or, } \frac{dv}{dt} &= \frac{d}{dt} \left( r'(t) \hat{r} + r \theta'(t) \hat{\theta} \right) \\
 \text{or, } f_r &= r''(t) \hat{r} + r' \frac{d\hat{r}}{dt} + r' \theta'(t) \hat{\theta} + r \theta''(t) \hat{\theta} + r \theta'(t) \frac{d\hat{\theta}}{dt}
 \end{aligned} \tag{1}$$

now,

$$\begin{aligned}
 \frac{d\hat{r}}{dt} &= \frac{d}{dt} (\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= \theta'(t) (-\sin \theta \hat{i} + \cos \theta \hat{j}) \\
 &= \theta'(t) \hat{\theta} \\
 \frac{d\hat{\theta}}{dt} &= \frac{d}{dt} (-\sin \theta \hat{i} + \cos \theta \hat{j}) \\
 &= -\theta'(t) \cos \theta \hat{i} - \theta'(t) \sin \theta \hat{j} \\
 &= -\theta'(t) [\cos \theta \hat{i} + \sin \theta \hat{j}] \\
 &= -\theta'(t) \hat{r}
 \end{aligned}$$

if we substitute in (3.1) we get,

$$\begin{aligned}
 f_r &= r''(t) \hat{r} + r' \frac{d\hat{r}}{dt} + r' \theta'(t) \hat{\theta} + r \theta''(t) \hat{\theta} + r \theta'(t) \frac{d\hat{\theta}}{dt} \\
 f_r &= r''(t) \hat{r} + r' \theta'(t) \hat{\theta} + r' \theta'(t) \hat{\theta} + r \theta''(t) \hat{\theta} + r \theta'(t) [-\theta'(t) \hat{r}]
 \end{aligned}$$

considering  $r$  components,

$$\begin{aligned}
 f_r &= r''(t) + r \theta'(t) \times [-\theta'(t)] \\
 &= r''(t) - r \{(\theta'(t))^2\}
 \end{aligned}$$

$$\therefore f_r = \frac{d r^2}{dt} - r \left( \frac{d\theta}{dt} \right)^2$$

[proved]

(b)

If we consider the particles  $F$  or Force's components. There will be two components one is  $mg \cos \theta$  and the other one is  $mg \sin \theta$ . The only Force is active during rotation is  $mg \sin \theta$ . Because the force is directly proportional to sine of the angular displacement. So we get,

$$F = -mg \sin \omega t$$

We also know,  $F = ma$  or  $F = m \cdot f_r$ . If we substitute,

$$m \cdot f_r = -mg \sin \omega t$$

$$\cancel{m} \cdot f_r = -\cancel{m} g \sin \omega t$$

$$f_r = -g \sin \omega t$$

$$\frac{d r^2}{dt} - r \left( \frac{d\theta}{dt} \right)^2 = -g \sin \omega t \quad [\text{from (a)}]$$

$$\frac{d r^2}{dt} - r \omega^2 = -g \sin \omega t \quad \left[ \because \omega = \frac{d\theta}{dt} \right]$$

[showed]

(c)

Given,

$$\frac{d^2 r}{dt^2} - r\omega^2 = -g \sin \omega t$$

C.F:

$$r'' - r\omega^2 = 0$$

let,

$$r = e^{kt}$$

$$r' = k e^{kt}$$

$$r'' = k^2 e^{kt}$$

now,

$$k^2 e^{kt} - e^{kt}\omega^2 = 0$$

$$\text{or, } e^{kt}(k^2 - \omega^2) = 0$$

$$\text{or, } k^2 - \omega^2 = 0$$

$$\therefore k = \pm \omega$$

$$C.F = C_1 e^{\omega t} + C_2 e^{-\omega t}$$

P.I:

we can write the equation as,

$$(D^2 - \omega^2)r = -g \sin \omega t$$

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2 - \omega^2} \times (-g \sin \omega t) \\ &= -g \left[ \frac{1}{D^2 - \omega^2} \times \sin \omega t \right] \\ &= -g \left[ \frac{1}{-\omega^2 - \omega^2} \times \sin \omega t \right] \\ &\quad [\text{substituting } -\omega^2 \text{ instead of } D^2] \\ &= -g \times \frac{\sin \omega t}{-2\omega^2} \\ &= \frac{g \sin \omega t}{2\omega^2} \\ &= \frac{g}{2\omega^2} \sin \omega t \end{aligned}$$

$$\begin{aligned} \therefore r(t) &= C.F + P.I \\ &= C_1 e^{\omega t} + C_2 e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t \end{aligned}$$

(d)

From (c) we got,

$$r(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t$$

From hyperbolic trigonometric function we know that,

$$\begin{aligned} \cosh(a) + \sinh(a) &= e^a \\ \cosh(a) - \sinh(a) &= e^{-a} \end{aligned}$$

Now,

$$\begin{aligned} r(t) &= C_1 \{\cosh \omega t + \sinh \omega t\} + C_2 \{\cosh \omega t - \sinh \omega t\} + \frac{g}{2\omega^2} \sin \omega t \\ &= C_1 \cosh \omega t + C_1 \sinh \omega t + C_2 \cosh \omega t - C_2 \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \\ &= (C_1 + C_2) \cosh \omega t + (C_1 - C_2) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \end{aligned}$$

let,

$$\begin{aligned} C_1 + C_2 &= L \\ C_1 - C_2 &= M \end{aligned}$$

$$\therefore r(t) = L \cosh \omega t + M \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t$$

(e)

From the question we got that when  $t = 0$  radius was  $a$ .

So,

$$\begin{aligned}
 r(t) &= L \cosh \omega t + M \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \\
 \text{or, } a &= L \cosh (\omega \times 0) + M \sinh (\omega \times 0) + \frac{g}{2\omega^2} \sin (\omega \times 0) \\
 \text{or, } a &= L + 0 + 0 \quad \left[ \begin{array}{l} \because \cosh(0) = 1 \\ \sinh(0) = 0 \\ \sin(0) = 0 \end{array} \right] \\
 \therefore L &= a
 \end{aligned}$$

after differentiating,

$$\begin{aligned}
 r(t) &= L \cosh \omega t + M \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \\
 \text{or, } \frac{dr}{dt} &= \frac{d}{dt} \left[ a \cosh \omega t + M \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \right] \\
 \text{or, } v &= a \omega \sinh \omega t + M \omega \cosh \omega t + \frac{g}{2\omega} \cos \omega t
 \end{aligned}$$

when  $t = 0$ ,

$$\begin{aligned}
 v &= 0 + M \omega + \frac{g}{2\omega} \\
 \text{or, } M \omega &= v - \frac{g}{2\omega} \\
 \text{or, } M \omega &= \frac{2 \cdot \omega \cdot v - g}{2\omega} \\
 \text{or, } M &= \frac{2 \cdot \omega \cdot v - g}{2\omega^2} \\
 \therefore M &= \frac{v}{\omega} - \frac{g}{2\omega^2}
 \end{aligned}$$

So,

$$r(t) = a \cosh \omega t + \left( \frac{v}{\omega} - \frac{g}{2\omega^2} \right) \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t$$

[showed]

## 4. Charged Quantam Particles

---

(a)

In equation (15),

$$\begin{aligned}
 \psi(r, \theta, \phi) &= F(r)G(\theta, \phi) \\
 \Rightarrow \frac{\partial \psi}{\partial r}(r, \theta, \phi) &= G(\theta, \phi) \frac{d}{dr}(F(r)) \\
 \Rightarrow \frac{\partial \psi}{\partial \theta}(r, \theta, \phi) &= F(r) \frac{\partial}{\partial \theta}(G(\theta, \phi)) \\
 \Rightarrow \frac{\partial^2 \psi}{\partial \phi^2}(r, \theta, \phi) &= F(r) \frac{\partial^2}{\partial \phi^2}(G(\theta, \phi))
 \end{aligned}$$

In equation (14),

$$-\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \left( \frac{\partial^2 \psi}{\partial \phi^2} \right) \right] - \frac{Q}{4\pi\epsilon_0 r} \psi = E\psi :$$

$$\begin{aligned}
 \psi &= FG, \\
 \frac{\partial \psi}{\partial r} &= G \frac{dF}{dr}, \\
 \frac{\partial \psi}{\partial \theta} &= F \frac{\partial G}{\partial \theta} \\
 \frac{\partial^2 \psi}{\partial \phi^2} &= F \frac{\partial^2 G}{\partial \phi^2} \\
 \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) &= \frac{\partial}{\partial r} \left( r^2 G \frac{dF}{dr} \right) \\
 &= G \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) \\
 \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \psi}{\partial \theta} \right) &= \frac{\partial}{\partial \theta} \left( \sin(\theta) F \frac{\partial G}{\partial \theta} \right) \\
 &= F \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right)
 \end{aligned}$$

$\therefore$  equation (14) can be written as,

$$-\frac{\hbar^2}{2\mu} \left[ \frac{G}{r^2} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{F}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{F}{r^2 \sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) \right] - \frac{Q}{4\pi\epsilon_0 r} FG = EFG$$

[shown]

(b)

In equation (16),

$$\begin{aligned} & -\frac{\hbar^2}{2\mu} \left[ \frac{G}{r^2} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{F}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{F}{r^2 \sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) \right] - \frac{Q}{4\pi\epsilon_0 r} FG = EFG \\ \implies & \frac{\hbar^2}{2\mu} \left[ \frac{G}{r^2} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{F}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{F}{r^2 \sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) \right] + \frac{Q}{4\pi\epsilon_0 r} FG + EFG = 0 \\ & \text{[Multiplying both sides by } \frac{2\mu r^2}{FG\hbar^2}] \\ \implies & \frac{r^2}{FG} \left[ \frac{G}{r^2} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{F}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{F}{r^2 \sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) \right] + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) = 0 \\ \implies & \frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{1}{G \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{G \sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) = 0 \\ \implies & \left\{ \frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) \right\} + \frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \left( \frac{\partial^2 G}{\partial \phi^2} \right) \right\} = 0 \\ & \text{[shown]} \end{aligned}$$



(c)

$$\frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) = j(j+1) \quad (18)$$

$$\frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 G}{\partial \phi^2} \right\} = -j(j+1) \quad (19)$$

Here, if we perform the operation, (18) + (19),

$$\begin{aligned} \Rightarrow \frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) + \frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 G}{\partial \phi^2} \right\} &= j(j+1) - \\ &\quad j(j+1) \\ \Rightarrow \left\{ \frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) \right\} + \frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 G}{\partial \phi^2} \right\} &= 0 \end{aligned}$$

[shown]

(d)

$$\frac{1}{F} \frac{d}{dr} \left( r^2 \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) = j(j+1) \quad (18)$$

$$\Rightarrow \frac{1}{F} \left( r^2 \frac{d^2 F}{dr^2} + 2r \frac{dF}{dr} \right) + \frac{2\mu r^2}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) - j(j+1) = 0$$

[multiplying both sides by  $\frac{F}{r^2}$ ]

$$\Rightarrow \frac{2}{r} \frac{dF}{dr} + \frac{d^2 F}{dr^2} + \left[ \frac{2\mu}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) - \frac{j(j+1)}{r^2} \right] F = 0$$

[shown]

(e)

$$\begin{aligned}
\lim_{r \rightarrow \infty} \left\{ \frac{2}{r} \frac{dF}{dr} + \frac{d^2 F}{dr^2} + \left[ \frac{2\mu}{\hbar^2} \left( \frac{Q}{4\pi\epsilon_0 r} + E \right) - \frac{j(j+1)}{r^2} \right] F \right\} &= 0 \\
\Rightarrow 0 \times \frac{dF_\infty}{dr} + \frac{d^2 F_\infty}{dr^2} + \left[ \frac{2\mu}{\hbar^2} (0 + E) - 0 \right] F_\infty &= 0 \\
\Rightarrow \frac{d^2 F_\infty}{dr^2} + \frac{2\mu E}{\hbar^2} F_\infty &= 0 \\
&\text{[shown]}
\end{aligned}$$

(f)

Given,

$$\frac{d^2 F_\infty}{dr^2} - \frac{2\mu E'}{\hbar^2} F_\infty = 0$$

From this, we can write,

$$\begin{aligned}
m^2 - \frac{2\mu E'}{\hbar^2} &= 0 \\
\Rightarrow m &= \pm \frac{\sqrt{2\mu E'}}{\hbar}
\end{aligned}$$

Therefore, the solution should be in the form,

$$\begin{aligned}
F_\infty &= Ae^{m_1 r} + Be^{m_2 r} \\
\therefore F_\infty &= Ae^{-\frac{\sqrt{2\mu E'}}{\hbar} r} + Be^{\frac{\sqrt{2\mu E'}}{\hbar} r}
\end{aligned}$$

[shown]

(g)

Given,

$$F_{\infty} \rightarrow 0$$

$$r \rightarrow \infty$$

$$F_{\infty} = Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r}$$

Now,

$$\text{if } r \rightarrow \infty ; \text{ then } \sqrt{\frac{2\mu E'}{\hbar}}r \rightarrow \infty$$

$$\therefore e^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow 0$$

$$\therefore Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow 0$$

$$\therefore e^{\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow \infty$$

Again,

$$F_{\infty} = Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r}$$

$$Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow 0 \quad [ \because F_{\infty} \rightarrow 0 ]$$

$$Be^{\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow 0 \quad [ \because Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow 0 ]$$

$$\therefore B \rightarrow 0 \quad [ \because e^{\sqrt{\frac{2\mu E'}{\hbar}}r} \rightarrow \infty ]$$

(h)

Given,

$$\begin{aligned} \frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 G}{\partial \phi^2} \right\} &= -j(j+1) \\ G \sin^2(\theta) \frac{1}{G} \left\{ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 G}{\partial \phi^2} \right\} &= -j(j+1) G \sin^2(\theta) \end{aligned} \quad (19)$$

[Multiplying both sides by  $G \sin^2(\theta)$ ]

$$\therefore \sin(\theta) \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{\partial^2 G}{\partial \phi^2} = -j(j+1) G \sin^2(\theta) \quad (24)$$

[showed]

(i)

Given,

$$G(\theta, \phi) = T(\theta)Z(\phi), \quad (25)$$

$$\sin(\theta) \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial G}{\partial \theta} \right) + \frac{\partial^2 G}{\partial \phi^2} = -j(j+1) G \sin^2(\theta) \quad (24)$$

$$\sin(\theta) \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial (TZ)}{\partial \theta} \right) + \frac{\partial^2 (TZ)}{\partial \phi^2} = -j(j+1) (TZ) \sin^2(\theta)$$

$$Z \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) + T \frac{d^2 Z}{d\phi^2} = -j(j+1) (TZ) \sin^2(\theta)$$

$$\frac{1}{T} \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) + \frac{1}{Z} \frac{d^2 Z}{d\phi^2} = -j(j+1) \sin^2(\theta)$$

[Dividing both sides by  $TZ$ ]

$$\therefore \left\{ \frac{1}{T} \left[ \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) \right] + j(j+1) \sin^2(\theta) \right\} + \left\{ \frac{1}{Z} \frac{d^2 Z}{d\phi^2} \right\} = 0 \quad (26)$$

[showed]

(j)

Given,

$$\frac{1}{T} \left[ \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) \right] + j(j+1) \sin^2(\theta) = k^2, \quad (27)$$

$$\frac{1}{Z} \frac{d^2 Z}{d\phi^2} = -k^2 \quad (28)$$

By adding the above two equations, we get,

$$\begin{aligned} & \frac{1}{T} \left[ \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) \right] + j(j+1) \sin^2(\theta) + \frac{1}{Z} \frac{d^2 Z}{d\phi^2} = k^2 - k^2 \\ \therefore & \left\{ \frac{1}{T} \left[ \sin(\theta) \frac{d}{d\theta} \left( \sin(\theta) \frac{dT}{d\theta} \right) \right] + j(j+1) \sin^2(\theta) \right\} + \left\{ \frac{1}{Z} \frac{d^2 Z}{d\phi^2} \right\} = 0 \end{aligned}$$

Since, the first curly brackets is independent of  $\phi$  and  $r$ , therefore we will get a constant and the second curly brackets is independent of  $\theta$  and  $r$ , therefore we will get another constant. These both constants have a equal magnitude  $k^2$  and opposite sign. Therefore the above expression will give 0.

(k)

Given,

$$\frac{d^2 Z}{d\phi^2} + k^2 Z = 0 \quad (28)$$

We Know,

$$\text{if,} \quad ay''(x) + by'(x) + cy(x) = 0 \quad (4.1)$$

We can can write the following equation,

$$am^2 + bm + c = 0 \quad (4.2)$$

And if the discriminant is positive, then the solution of the y of the differential equation,

$$y(x) = Ae^{m_1 x} + Be^{m_2 x} \quad (4.3)$$

Comparing equation (4.1) and equation (28), we get,

$$a = 1$$

$$b = 0$$

$$c = k^2$$

$$m = \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(k^2)}}{2(1)}$$

$$\therefore m = \pm k$$

Putting the value of m in equation (4.3) and substituting  $y(x)$  with  $Z(\phi)$ ,

$$\therefore Z(\phi) = Ae^{k\phi} + Be^{-k\phi} \quad (4.4)$$

(1)

Given,

$$T(\theta) = T_{j,k}(\theta) = n_{j,k} P_j^k(\cos(\theta)) \quad (29)$$

From 4(k), the solution of equation (30),

$$Z(\phi) = Ae^{k\phi} + Be^{-k\phi} \quad (4.4)$$

$$G(\theta, \phi) = T(\theta)Z(\phi) \quad (25)$$

$$\therefore G(\theta, \phi) = n_{j,k} P_j^k(\cos(\theta)) (Ae^{k\phi} + Be^{-k\phi}) \quad (4.5)$$

(m)

Given,

$$F(r) = F_{i,j}(r) = c_{i,j} e^{d_i r} (2d_i r)^j \left[ L_{(i-j-1)}^{(2j+1)}(2d_i r) \right] \quad (23)$$

From 4(l), the solution of  $G(\theta, \phi)$ ,

$$G(\theta, \phi) = n_{j,k} P_j^k(\cos(\theta)) (Ae^{k\phi} + Be^{-k\phi}) \quad (4.5)$$

$$\psi(r, \theta, \phi) = F(r)G(\theta, \phi), \quad (15)$$

$$\therefore \psi(r, \theta, \phi) = \psi_{i,j,k}(r, \theta, \phi) = c_{i,j} e^{d_i r} (2d_i r)^j \left[ L_{(i-j-1)}^{(2j+1)}(2d_i r) \right] n_{j,k} P_j^k(\cos(\theta)) (Ae^{k\phi} + Be^{-k\phi}) \quad (4.6)$$

(n)

Given that, for  $i = 2, j = 1$  and  $k = 0$ , the wave function is

$$\psi_{2,1,0} = Ad_2 r e^{-rd_2} \cos(\theta) \quad (4.7)$$

$$\iiint_{Entire\ Space} |\psi_{i,j,k}|^2 dV = 1 \quad (31)$$

Since,  $V$  is in spherical coordinate system,

$$dV = r^2 \sin(\theta) d\phi d\theta dr$$

Given that, we need to integrate entire space. So, the limits will be,

$$\{(r, \theta, \phi) \in Entire\ Space \mid 0 \leq r \leq \infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\}$$

$$\begin{aligned} \iiint_{Entire\ Space} |\psi_{i,j,k}|^2 dV &= \int_0^\infty \int_0^\pi \int_0^{2\pi} |Ad_2 r e^{-rd_2} \cos(\theta)|^2 r^2 \sin(\theta) d\phi d\theta dr \\ &= A^2 d_2^2 \int_0^\infty \int_0^\pi \int_0^{2\pi} r^4 e^{-2rd_2} \cos^2(\theta) \sin(\theta) d\phi d\theta dr \\ &= A^2 d_2^2 \int_0^\infty \int_0^\pi r^4 e^{-2rd_2} \cos^2(\theta) \sin(\theta) \left( \int_0^{2\pi} d\phi \right) d\theta dr \\ &= A^2 d_2^2 \int_0^\infty \int_0^\pi r^4 e^{-2rd_2} \cos^2(\theta) \sin(\theta) (2\pi) d\theta dr \\ &= 2\pi A^2 d_2^2 \int_0^\infty r^4 e^{-2rd_2} \left( \int_0^\pi \cos^2(\theta) \sin(\theta) d\theta \right) dr \end{aligned}$$

Let,

$$u = \cos(\theta)$$

$$du = -\sin(\theta)$$

$\theta$	0	$\pi$
$u$	1	-1



$$\begin{aligned}
\int_0^\pi \cos^2(\theta) \sin(\theta) \, d\theta &= \int_1^{-1} u^2(-du) \\
&= - \int_1^{-1} u^2 \, du \\
&= - \left[ \frac{u^3}{3} \right]_1^{-1} \\
&= - \left\{ \frac{(-1)^3}{3} - \frac{(1)^3}{3} \right\} \\
&= \frac{2}{3}
\end{aligned}$$

Now,

$$\begin{aligned}
2\pi A^2 d_2^2 \int_0^\infty r^4 e^{-2rd_2} \left( \int_0^\pi \cos^2(\theta) \sin(\theta) \, d\theta \right) dr &= 2\pi A^2 d_2^2 \int_0^\infty r^4 e^{-2rd_2} \frac{2}{3} \, dr \\
&= \frac{4}{3} \pi A^2 d_2^2 \int_0^\infty r^{(5-1)} e^{-(2d_2)r} \, dr \\
&= \frac{4}{3} \pi A^2 d_2^2 \frac{\Gamma(5)}{(2d_2)^5} \\
&\quad \left[ \because \int_0^\infty x^{\alpha-1} e^{-\lambda x} \, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \right] \\
1 &= \frac{4}{3} \pi A^2 d_2^2 \frac{(5-1)!}{32d_2^5} ; \\
&\quad [\text{using eqn (31) and } \Gamma(n) = (n-1)!]
\end{aligned}$$

$$\therefore A = \sqrt{\frac{d_2^3}{\pi}}$$