Department of Mathematics and Natural Sciences

MAT 120

MONTHLY ASSIGNMENT

SPRING 2021

The submission is due May 9th "09/05/2021". *Solve all problems.* ***Please write your Name, ID and Section on the first page of the assignment answer script (this includes the entire team) - you have to do this for both handwritten or LaTeX submission.***

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You can only submit a PDF file - image or doc files won't be accepted. Before submitting the PDF, please rename the PDF file in the format - SET_ID_SECTION.

Answer the questions by yourself.Plagiarism will lead to an F grade in the course. **Total marks is "200".**It will be converted to 20 and if you do your *entire* work using LaTeX you will get a bonus 50 marks, which will be converted to 5. So highest marks you can get out of 20 is 25 provided you do everything correct and you submit your assignment in time

1. **Heart Attack**

(a) Let a closed region $\mathbf{R}$ be revolved about the $x - axis$. Also let (x, y) be any arbitrary point in the region. Then the coordinate of the center of gravity of the solid generated by revolving the region $\mathbf{R}$ is given by

$$(\bar{x}, \bar{y}, \bar{z}) = (\frac{\iint_{\mathbf{R}} xy\, dx\, dy}{\iint_{\mathbf{R}} y\, dx\, dy}, 0, 0) \tag{1}$$

Explain why $\bar{y}$ and $\bar{z}$ must be zero. Use the formula (1) given in Cartesian coordinates to prove that, if $\mathbf{R}$ is the region bounded by the polar curve $r = f(\theta)$, then formula transforms into:

$$(\bar{x}, \bar{y}, \bar{z}) = (\frac{\iint_{\mathbf{R}} r^3 \sin\theta \cos\theta\, dr\, d\theta}{\iint_{\mathbf{R}} r^2 \sin\theta\, dr\, d\theta}, 0, 0) \tag{2}$$

[5]

(b) Using the formula given in (2), find the center of gravity of the solid generated by revolving the region bounded by the polar curve

$$r = a(1 + \cos\theta) \tag{3}$$

about its axis of symmetry.

The name of this particular curve is *cardioid* (originating from a Greek word meaning *heart*). You may need to experiment with the equation of the cardioid given in (3) to determine the limits of the integrals. Try sketching the curve to have a better idea. Including the figure is not necessary.

[10]

(c) You have learnt how to find the area of a surface generated by revolving a curve about the coordinate axes. Try to understand the derivation of the formula you use to find the area from any reference book. Using that experience, derive the formula for the volume of the solid generated by revolving a region $\mathbf{R}$ bounded by a polar curve $r = f(\theta)$ about the $x - axis$:

$$V = 2\pi \iint_{\mathbf{R}} r^2 \sin\theta\, dr\, d\theta \tag{4}$$

[10]

(d) Using the formula in (4), find the volume of the solid generated by revolving the cardioid in (3) about its axis of symmetry. You need to carefully determine the limits of the integrals and be cautious about any symmetry available in the shape.

[10]

(e) You know that the formula for arc length of a curve $y = f(x)$ is given by

$$L = \int_p^q \sqrt{1 + [f'(x)]^2} dx \tag{5}$$

Now you have to prove that, for a polar curve $r = f(\theta)$, the formula takes the form:

$$L = \int_a^b \sqrt{r^2 + [\frac{dr}{d\theta}]^2} d\theta \tag{6}$$

You should start from formula (5).

[5]

(f) Use formula (6) to find the perimeter of the cardioid (3). As always, be careful while you determine the limits of the integration.

[10]

2. **Particle in a Cylinder**

A particle exists in three dimensions and is trapped inside a solid S. The cross section of the cylinder C on the xy plane is the region bounded between $r = \cos(\theta)$ and $r = \sin(\theta)$ in the **first** quadrant. All the points in the solid S exists inside the cylinder C bounded between the planes $z = -y$ and $z = y$.

Before we can get into the particle dynamics, we need to introduce the notion of "inner product." You are already familiar with the dot product of two vectors. Dot products give a sense of how much the two vectors are aligned with each other. In layman's terms, a function can be thought of as a vector with infinite components. Lets put this in perspective, a $3d$ vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ has three components a_x, a_y, and a_z. Now, lets consider a function f defined by the map $f : x \mapsto 7x$, where $x \in [-3, 2]$; then the *"xth"* component of the *"vector"* f is just $f(x)$, or equivalently $7x$. Since x can take the value of any real number between -3 and 2 and that there are infinite real numbers between any two numbers, then it follows that the *"vector"* f has infinite components. In the case of the dot product, we have to take a summation, and in the case of an inner product, we have to take an *integral*. The definition of the inner product of two functions g and h (denoted by $\langle g|h\rangle$) over a volume S is defined as

$$\langle g|h\rangle = \iiint_S g(x, y, z)^* \, h(x, y, z) \; dV, \tag{7}$$

where $g(x, y, z)^*$ is the complex conjugation of $g(x, y, z)$.
The probability of finding the particle anywhere outside the solid S is 0 and the probability of finding the particle inside the solid S is non zero. The **probability density function** (PDF in short) which describes this fact mathematically is

$$\text{PDF} = \psi(x, y, z)^* \, \psi(x, y, z) \,, \tag{8}$$

where ψ is

$$\psi(x, y, z) = \begin{cases} \frac{A}{(x^2+y^2)^{\frac{1}{4}}} & (x, y, z) \text{ lies within } S \\ 0, & \text{otherwise,} \end{cases}$$

where A is a positive constant.

(a) Write down the value of $\langle\psi|\psi\rangle$, explain your reasoning in one sentence. (Your answer should be a number, do not evaluate any integral)

[2]

(b) Now write down the integral(s) with the limits in order to evaluate $\langle\psi|\psi\rangle$ using equation (7) **and** explain why cylindrical coordinates will be a good choice to evaluate the integral (Do not evaluate the integral yet)

[10]

(c) Find the volume element in cylindrical coordinates with the help of the Jacobian determinant

$$dV = \left|\frac{\partial (x, y, z)}{\partial (z, r, \theta)}\right| dz\ dr\ d\theta$$

[4]

(d) Find the value of A

[10]

When dealing with probabilities, we are often interested in calculating the expectation values of certain quantities. Before we move into that, we need to understand what are operators. Operators act on functions and change them. Lets define the following operator, $\hat{r}$ which acts on a function ψ and produces r (the number which represents axial distance) times ψ. Mathematically,

$$\hat{r}\psi = r\psi \tag{9}$$

To find the perpendicular distance from the z axis (the axial distance r) that we *expect* the particle to be found at, we need to find the *expectation* value of r. Given that, the expectation value of r is the inner product $\langle\psi|\hat{r}\psi\rangle$

(e) Evaluate $\langle\psi|\hat{r}\psi\rangle$ with the help of equation (9)

[10]

An operator can act on a function twice; we represent this by a square on the operator. Numbers can be taken outside of the operator when it acts on a function (i.e. if $\hat{A}$ is an operator acting on a function kf, where k is a number, then $\hat{A}(kf) = k\hat{A}f$)

(f) Evaluate $\hat{r}^2\psi$ using equation (9)

[2]

(g) Given that, the expectation value of r^2 is the inner product $\langle\psi|\hat{r}^2\psi\rangle$, find the expectation value of r^2 [10]

(h) We can finally calculate the uncertainty in the axial distance using the relation

$$\sigma_r^2 = \langle\psi|\hat{r}^2\psi\rangle - \langle\psi|\hat{r}\psi\rangle^2\,,$$

where σ_r is the uncertainty in r.
Find the uncertainty in the axial distance of the particle's location. [2]

3. **Rotating Tube**

(a) In a Cartesian plane, if the position vector of a moving particle is given by $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, then the x and y components of velocity is given by $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively. On the other hand, the polar components, i.e., the r and θ components of velocity is given by $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$. Using these, prove that the r component of acceleration is

$$f_r = \frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2 \tag{10}$$

[10]

(b) A smooth straight thin tube revolves with uniform angular velocity ω in a vertical plane about one end which is fixed. A particle inside the tube is sliding along the tube with a constant velocity v. At time $t = 0$, the tube was horizontal and the particle was at a distance a from the fixed end. Show that the motion of the particle can be described by the differential equation

$$\frac{d^2r}{dt^2} - r\omega^2 = -g\sin\omega t \tag{11}$$

[Hint: Use formula (10)]

[7]

(c) Find the complementary function (CF) and the particular integral (PI) of the differential equation (11).

[15]

(d) Hence or otherwise, show that the general solution can be written as

$$r(t) = L\cosh wt + M\sinh wt + \frac{g}{2\omega^2}\sin\omega t \tag{12}$$

[8]

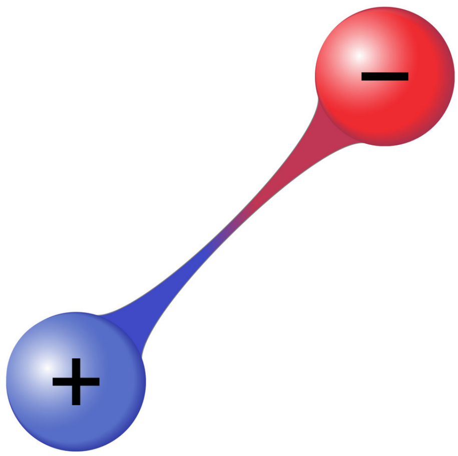
(e) Identify the initial conditions from the description in part (b) and use them to find L and M. Hence show that

$$r(t) = a\cosh wt + (\frac{v}{\omega} - \frac{g}{2\omega^2})\sinh wt + \frac{g}{2\omega^2}\sin\omega t \tag{13}$$

[10]

4. **Charged Quantum Particles**

We have a pair of charged particles with differing masses, m_1 and m_2, differing nature of charges $+q_1$ and $-q_2$, and we want to get some idea about their interaction. When we have a two-body problem, often the problem simplifies if we can turn it into a one-body problem, instead of having two masses, we use a concept known as *equivalent* mass, μ defined as $\mu = \frac{m_1 m_2}{m_1+m_2}$. The two-body problem then becomes equivalent to a one-body problem with mass μ. The following *cartoon diagram* shows the scenerio.



While solving differential equations, we often need the help of operators. In simple terms, the idea of Hamiltonian operator is that we sum the kinetic energy operator with the potential operator. The kinetic operator is momentum operator, $\hat{p}$, squared divided by 2μ. From quantum mechanics $\hat{p} = -i\hbar\nabla$ Then $\hat{p}^2$ becomes $-\hbar\nabla^2$. The potential energy due to two charged particles is given by the operator $\hat{V} = -\frac{q_1 q_2}{4\pi\epsilon_0 r}$ (this results from Coulomb's law). Now, if we let $q_1 q_2 = Q$, the Hamiltonian operator becomes:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Q}{4\pi\epsilon_0 r}$$

The Schrödinger's equation is $\hat{H}\psi = E\psi$ (where E is some constant); we will want to find the solution ψ from this equation. The Laplacian (∇^2) in spherical coordinates is (here $r =$ radial distance, $\theta =$ polar angle and $\phi =$ azimuthal angle

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial^2}{\partial\phi^2}\right).$$

Using the Laplacian from above, we can find expand the Schrödinger's equation ($\hat{H}$ acts on ψ to produce the following, you can think of it as a sort of multiplication)

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]-\frac{Q}{4\pi\epsilon_0 r}\psi=E\psi \tag{14}$$

This problem involves solving the differential equation (14) by converting the partial differential equation into three ordinary differential equations each consisting only one of the independent variables (r, θ and ϕ).

We will assume that $\psi(r,\theta,\phi)$ is separable, that is we can write ψ in the form

$$\psi(r,\theta,\phi)=F(r)\,G(\theta,\phi), \tag{15}$$

where F depends only on r and G depends only on θ and ϕ.

(a) With the help of equation (15) and (14) show that

$$-\frac{\hbar^2}{2\mu}\left[\frac{G}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)+\frac{F}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{F}{r^2\sin^2(\theta)}\left(\frac{\partial^2 G}{\partial\phi^2}\right)\right]-\frac{Q}{4\pi\epsilon_0 r}FG=EFG \tag{16}$$

[4]

(b) Using equation (16) show that

$$\left\{\frac{1}{F}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)+\frac{2\mu r^2}{\hbar^2}\left(\frac{Q}{4\pi\epsilon_0 r}+E\right)\right\}+\frac{1}{G}\left\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\right\}=0 \tag{17}$$

[4]

Note that the part within the curly braces on the left depends only on r and the part on the right depends only on θ and ϕ. Since r, θ and ϕ are completely independent of each other, both parts must differ by the same constant to be equal to zero. We will call this constant $(j(j+1))$, where j is a constant (if j is a constant so is $j(j+1)$). To satisfy equation (17), we must have:

$$\frac{1}{F}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)+\frac{2\mu r^2}{\hbar^2}\left(\frac{Q}{4\pi\epsilon_0 r}+E\right)=j(j+1), \tag{18}$$

and

$$\frac{1}{G}\left\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\right\}=-j(j+1). \tag{19}$$

(c) Show that equation (18) combined with equation (19) in a certain way gives us (17) (this means that equation (18) along with (19) is completely equivalent to equation(17)). [3]

Now we will solve the radial part of the equation

(d) Using equation (18) show that the following is true [4]

$$\frac{2}{r}\frac{dF}{dr} + \frac{d^2F}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(\frac{Q}{4\pi\epsilon_0 r} + E\right) - \frac{j(j+1)}{r^2}\right]F = 0 \tag{20}$$

To solve equation (20), we will first solve the case for *very large* r and call the corresponding F as F_∞.

(e) Explain why can we write the following equation for large r

$$\frac{d^2F_\infty}{dr^2} + \frac{2\mu E}{\hbar^2}F_\infty = 0 \tag{21}$$

[3]

Before we solve for F_∞ we to utilize the fact if $E > 0$ then the particles will behave as if they are *free* (free meaning independent) (there will be too much energy for the particles to stay together if $E > 0$); however we want to interpret a situation where the particles interact with each other, hence we are only interested in $E < 0$. If we let $E' > 0$, then we can write $E = -E'$ (ensuring E is negative because E' is positive) Now we will rewrite equation (21) as

$$\frac{d^2F_\infty}{dr^2} - \frac{2\mu E'}{\hbar^2}F_\infty = 0 \tag{22}$$

Note that μ, E' and $\hbar$ are all positive constants.

(f) Using **auxiliary** equation to solve (22) to show that

$$F_\infty = Ae^{-\frac{\sqrt{2\mu E'}}{\hbar}r} + Be^{\frac{\sqrt{2\mu E'}}{\hbar}r}$$

[4]

Now we need to apply a boundary condition. First we must not allow our solution to contain information about the situation when the particles are free. The particles are free when there is no potential to bound them to each other (i.e. $V = \frac{1}{4\pi\epsilon_0 r} \to 0$). We can see that the potential approaches to 0 when $r \to \infty$. Therefore, the solution must vanish when $r \to \infty$

(g) Use the condition that $F_\infty \to 0$ when $r \to \infty$ to argue that B must approach 0. [3]

To obtain the solution of F for all values of r (including large and small) we will need to rely on a power series $F = F_\infty \sum_{a=0}^{\infty} b_a r^a$, we will *not* explore the analysis of the power series in this exercise. An analysis of the power series will reveals that there are many possible F functions which can be categorized with integers i and j

$$F(r) = F_{i,j}(r) = c_{i,j}e^{-d_i r}(2d_i r)^j\left[L^{(2j+1)}_{(i-j-1)}(2d_i r)\right], \tag{23}$$

where $c_{i,j}$ and d_i are different constants which we specify with i and j; L is a special function called *Laguerre polynomial*. We will utilize the solution for F when we will find the full solution for ψ.

Now we will solve the angular part of the equation. First, we will rewrite equation (19).

(h) Starting from (19), show that

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\frac{\partial^2 G}{\partial\phi^2}=-j(j+1)G\sin^2(\theta) \tag{24}$$

[4]

Now, note that the angular differential equation in equation (24) is still a partial differential equation, to convert it into an ordinary differential equation, we will first separate G as follows:

$$G(\theta,\phi)=T(\theta)\,Z(\phi) \tag{25}$$

(i) Use equation (24) and (25) to show that

$$\left\{\frac{1}{T}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)\right\}+\left\{\frac{1}{Z}\frac{d^2Z}{d\phi^2}\right\}=0 \tag{26}$$

[4]

By the same logical reasoning we used to separate radial and angular part before, we will separate the left and right parts (in curly braces) from equation (26) to obtain the following

$$\frac{1}{T}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)=k^2 \tag{27}$$

and,

$$\frac{1}{Z}\frac{d^2Z}{d\phi^2}=-k^2 \tag{28}$$

(j) Show that equation (27) combined with equation (28) in a certain way gives us (26) (this means that equation (27) along with (28) is completely equivalent to equation(26)). [3]

The solution for equation (27) requires power series analysis, which we will *not* go into; the solution is as follows:

$$T(\theta)=T_{j,k}(\theta)=n_{j,k}P_j^k(\cos(\theta)), \tag{29}$$

here $n_{j,k}$ are constants that can be specified with integers j and k; P_j^k is a special function called *Legendre function*. To solve equation (28) we will rewrite equation (28) as

$$\frac{d^2Z}{d\phi^2}+k^2Z=0 \tag{30}$$

(k) Find $Z(\phi)$ by solving the differential equation with the help of auxiliary equation (note that k^2 is positive). [4]

(l) With the help of equation (25), (29) and the solution of (30) you found in part (k), find $G(\theta,\phi)$. [3]

Now all we need to do is combine all our solutions for each coordinate and obtain our overall solution ψ.

(m) Find $\psi(r,\theta,\phi) = \psi_{i,j,k}(r,\theta,\phi)$ using (15), equation (23) and your solution to part (l). [3]

The solution that you just obtaining contain every possible information we may be able to obtain with regards to the behaviour of the two particle system. As for your last problem, consider that in order to find the constants that we get while solving differential equations, we use certain conditions to find the value of these constants. One of the conditions that our solution ψ must have, is that it will give us 1 if we integrate it's modulus squared with respect to the entire volume. Mathematically,

$$\iiint_{EntireSpace} |\psi_{i,j,k}|^2 dV = 1 \tag{31}$$

Given that, for $i = 2$, $j = 1$ and $k = 0$, the wave function is

$$\psi_{2,1,0} = Ad_2 re^{-rd_2}\cos(\theta),$$

where A and d_2 are positive constants.

(n) Find the value of A in terms of d_2 using equation (31) (Take care with your limits) [4]