GROUP-19

MONTHLY ASSIGNMENT

MAT 120

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Contents

1. Heart Attack

(a)

We know,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\ dy = \mid J \mid dr\ d\theta$$

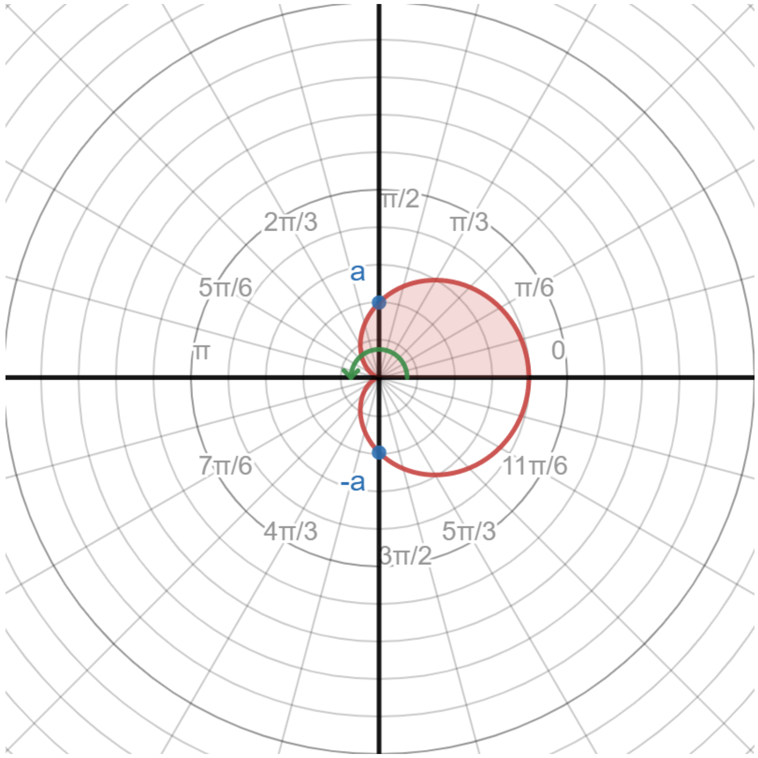
Now,

$$\begin{aligned}
\mid J \mid &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2(\theta) - (-r\sin^2(\theta)) \\
&= r\cos^2(\theta) + r\sin^2(\theta) \\
&= r\{\cos^2(\theta) + \sin^2(\theta)\} \\
\mid J \mid &= r \\
\therefore dx\ dy &= r\ dr\ d\theta
\end{aligned}$$

$$\begin{aligned}
\iint_R xy\ dx\ dy &= \iint_R (r\cos\theta)(r\sin\theta)\ r\ dr\ d\theta \\
&= \iint_R r^3 \sin\theta\cos\theta\ dr\ d\theta \\
\iint_R y\ dx\ dy &= \iint_R r\sin\theta\ r\ dr\ d\theta \\
&= \iint_R r^2\ \sin\theta\ dr\ d\theta \\
\therefore \frac{\iint_R xy\ dx\ dy}{\iint_R y\ dx\ dy} &= \frac{\iint_R r^3 \sin\theta\cos\theta\ dr\ d\theta}{\iint_R r^2\ \sin\theta\ dr\ d\theta}
\end{aligned}$$

The region 'R' is revolved around x-axis, so the axis of symmetry is the x-axis. Therefore, the center of gravity will be on the axis of symmetry (x-axis). Thus, y and z must be zero.

(b)



Here, x-axis is the symmetry for the given shape. Using symmetry, we can find the integral for the shaded region and then multiply it by 2. For shaded region, Range:

$$0 \ \leq \theta \leq \pi$$

$$0 \leq r \leq a(1 + \cos\theta)$$

$$\begin{aligned} \overline{X} &= \frac{\iint_R r^3 \sin\theta \cos\theta \ dr \ d\theta}{\iint_R r^2 \ \sin\theta \ dr \ d\theta} \\ &= \frac{\int_0^\pi \int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta \ dr \ d\theta}{\int_0^\pi \int_0^{a(1+\cos\theta)} r^2 \ \sin\theta \ dr \ d\theta} \end{aligned}$$

Considering for one side of the symmetry (0 to π).

$$\begin{aligned}
\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\ dr\ d\theta &= \int_0^{\pi}\left[\frac{r^4}{4}\sin\theta\cos\theta\right]_0^{a(1+\cos\theta)} d\theta \\
&= \frac{1}{4}\int_0^{\pi} a^4(1+\cos\theta)^4\sin\theta\cos\theta\ d\theta \\
&= \frac{a^4}{4}\int_0^{\pi}(1+\cos\theta)^4\sin\theta\cos\theta\ d\theta
\end{aligned}$$

Let,

$$\begin{aligned}
u &= \cos\theta \\
\therefore du &= -\sin(\theta)d\theta \\
\int(1+\cos\theta)^4\sin\theta\cos\theta\ d\theta &= \int(1+u)^4u(-du) \\
&= -\int u(1+u)^4du \\
Let, \\
s &= 1+u \\
\therefore ds &= du \\
-\int u(1+u)^4du &= -\int(s-1)s^4ds \\
&= \int s^4-s^5ds \\
&= \frac{s^5}{5}-\frac{s^6}{6}+C \\
&= \frac{(1+u)^5}{5}-\frac{(1+u)^6}{6}+C \\
\therefore\int(1+\cos\theta)^4\sin\theta\cos\theta\ d\theta &= \frac{(1+\cos\theta)^5}{5}-\frac{(1+\cos\theta)^6}{6}+C
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{a^4}{4}\int_0^{\pi}(1+\cos\theta)^4\sin\theta\cos\theta\ d\theta &= \frac{a^4}{4}\left[\frac{(1+\cos\theta)^5}{5}-\frac{(1+\cos\theta)^6}{6}\right]_0^{\pi}\\
&= \frac{a^4}{4}\left\{\frac{32}{5}-\frac{32}{3}\right\}\\
\therefore \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\ dr\ d\theta &= -\frac{16}{15}a^4
\end{aligned}$$

Again,

$$\begin{aligned}
\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\ \sin\theta\ dr\ d\theta &= \int_0^{\pi}\left[\frac{r^3}{3}\sin\theta\right]_0^{a(1+\cos\theta)} d\theta\\
&= \frac{1}{3}\int_0^{\pi} a^3(1+\cos\theta)^3\sin\theta\ d\theta\\
&= \frac{a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\ d\theta
\end{aligned}$$

Let,

$$\begin{aligned}
u &= \cos\theta\\
\therefore du &= -\sin(\theta)d\theta\\
\int(1+\cos\theta)^3\sin\theta\ d\theta &= \int(1+u)^3\ (-du)\\
&= -\int(1+u)^3du\\
&Let,\\
s &= 1+u\\
\therefore ds &= du\\
-\int(1+u)^3du &= -\int s^3ds\\
&= -\frac{s^4}{4}+C\\
&= -\frac{(1+u)^4}{4}+C\\
\therefore \int(1+\cos\theta)^3\sin\theta\ d\theta &= -\frac{(1+\cos\theta)^4}{4}+C
\end{aligned}$$

Now,

$$\frac{a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta = \frac{a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$
$$= \frac{a^3}{3}(-4)$$
$$\therefore \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\ \sin\theta\ dr\ d\theta = -\frac{4}{3}a^3$$

for the entire rotation we multiply with 2.

$$\overline{X} = \frac{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\ dr\ d\theta}{\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\ \sin\theta\ dr\ d\theta}\times 2$$
$$= \frac{-\frac{16}{15}a^4}{-\frac{4}{3}a^3}\times 2$$
$$\therefore \overline{X} = \frac{8}{5}a$$

$\therefore$ Center of gravity $= \left(\frac{8}{5}a,\ 0,\ 0\right)$

(c)

$$\text{Surface Area} = \iint_R f(x)\ dx\ dy$$

Given,

$$\begin{aligned} V &= 2\pi \iint_R f(x)\ dx\ dy \\ &= 2\pi \iint_R r\sin\theta\ dx\ dy\ ;\ [\because y = f(x) = r\sin\theta] \end{aligned}$$

$$We\ know,$$

$$dx\ dy\ = \mid J \mid\ dr\ d\theta$$

Now,

$$\begin{aligned} \mid J \mid &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2(\theta) - (-r\sin^2(\theta)) \\ &= r\cos^2(\theta) + r\sin^2(\theta) \\ &= r\{\cos^2(\theta) + \sin^2(\theta)\} \\ \mid J \mid &= r \end{aligned}$$

$$\therefore dx\ dy = r\ dr\ d\theta$$

$$
\begin{aligned}
V &= 2\pi \iint_R r \sin\theta \; dx \; dy \\
&= 2\pi \iint_R r \sin\theta \; r \; dr \; d\theta \\
\therefore V &= 2\pi \iint_R r^2 \sin\theta \; dr \; d\theta
\end{aligned}
$$

(d)

$$
\begin{aligned}
V &= 2\pi \int \int_R r^2 \sin\theta dr d\theta \\
&= 2\pi \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1+\cos\theta)} r^2 \sin\theta dr d\theta \\
&= 2\pi \int_0^\pi \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta d\theta \\
&= \frac{2\pi}{3} \int_0^\pi a^3(1+\cos\theta)^3 \sin\theta d\theta
\end{aligned}
$$

Let,

$$
\begin{aligned}
u &= 1+\cos\theta \\
\therefore\ \sin\theta\ d\theta &= -du
\end{aligned}
$$

θ	0	π
u	2	0

$$
\begin{aligned}
\therefore V &= \frac{2\pi}{3} \int_2^0 a^3 u^3 (-du) \\
&= \frac{-2\pi a^3}{3} \int_2^0 u^3 du \\
&= \frac{-2\pi a^3}{3} \left[\frac{u^4}{4}\right]_2^0 \\
&= \frac{-2\pi a^3}{3} \left[0 - \frac{16}{4}\right] \\
&= \frac{8\pi a^3}{3}
\end{aligned}
$$

(e)

$$\begin{aligned}
L &= \int_p^q \sqrt{1+[f'(x)]^2}dx \\
&= \int_p^q \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= \int_p^q \sqrt{\frac{dx^2+dy^2}{dx^2}}dx \\
&= \int_p^q \sqrt{(dx^2)\left(\frac{dx^2+dy^2}{dx^2}\right)} \\
&= \int \sqrt{dx^2+dy^2}
\end{aligned}$$

Now,

$$\begin{aligned}
dL &= \sqrt{dx^2+dy^2} \\
\Longrightarrow dL^2 &= dx^2+dy^2 \\
\Longrightarrow \frac{dL^2}{d\theta^2} &= \frac{dx^2}{d\theta^2}+\frac{dy^2}{d\theta^2} \\
\Longrightarrow dL^2 &= \left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]d\theta^2 \\
\Longrightarrow dL &= \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta \\
& \quad \text{[integrating both sides]} \\
\therefore L &= \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta
\end{aligned}$$

Now,

$$
\begin{aligned}
x &= r(\theta)\cos\theta \\
y &= r(\theta)\sin\theta \\
\therefore \left(\frac{dx}{d\theta}\right)^2 &= \left(-r\sin\theta + \cos\theta\frac{dr}{d\theta}\right)^2 \\
\therefore \left(\frac{dy}{d\theta}\right)^2 &= \left(r\cos\theta + \sin\theta\frac{dr}{d\theta}\right)^2 \\
\therefore dL^2 &= \Bigg[\cos^2\theta\left(\frac{dr}{d\theta}\right)^2 - 2r\sin\theta\cos\theta\left(\frac{dr}{d\theta}\right) + r^2\sin^2\theta + \\
&\qquad r^2\cos^2\theta + 2r\sin\theta\cos\theta\left(\frac{dr}{d\theta}\right) + \sin^2\theta\left(\frac{dr}{d\theta}\right)^2\Bigg]d\theta^2 \\
\implies dL^2 &= \left[r^2(\sin^2\theta + \cos^2\theta) + \left(\frac{dr}{d\theta}\right)^2(\sin^2\theta + \cos^2\theta)\right]d\theta^2 \\
\implies dL^2 &= \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]d\theta^2 \\
\therefore dL &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \\
&\quad \text{[integrating both sides]} \\
\therefore L &= \int_a^b \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\,d\theta
\end{aligned}
$$

[proved]

(f)

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$r = a(1 + \cos\theta)$$
$$= a + a\cos\theta$$
$$\therefore \frac{dr}{d\theta} = -a\sin\theta$$
$$\therefore \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$$

$\therefore$ Perimeter:

$$= 2\int_0^\pi \sqrt{[a^2(1+\cos\theta)^2] + a^2\sin\theta} d\theta$$
$$= 2\int_0^\pi \sqrt{[a^2(1+2\cos\theta+\cos^2\theta)] + a^2\sin\theta} d\theta$$
$$= 2\int_0^\pi \sqrt{a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)} d\theta$$
$$= 2a\int_0^\pi \sqrt{2+2\cos\theta} d\theta$$
$$= 2a\int_0^\pi \sqrt{2(1+\cos\theta)} d\theta$$
$$= 2a\int_0^\pi \sqrt{2\left[2\cos^2\left(\frac{\theta}{2}\right)\right]} d\theta$$
$$= 4a\int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta$$
$$= 4a\left[\frac{\sin(\theta/2)}{1/2}\right]_0^\pi$$
$$= 8a(\sin(\pi/2) - \sin 0)$$
$$= 8a$$

2. Particle in a Cylinder

(a)

$$< \psi|\psi >= 1$$

The probability of finding the particle outside S is 0 and the probability of finding the particle inside S is non-zero; hence the summation of all probabilities (total probability of finding the particle inside S) is 1.

(b)

The cylindrical coordinate system will be a good choice to evaluate the integrals because it is easier to determine the limits for the given bounded region and equation. It is also easier to convert the equation to a simpler form ($\sqrt{x^2+y^2} = r$) using this coordinate system.

$$< g|h > = \iiint_S g(x,y,z)^* h(x,y,z)\ dV$$

For,

$$< \psi|\psi > = \iiint_S \psi(x,y,z)^* \psi(x,y,z)\ dV$$

Since, $\psi(x,y,z)$ is real and complex conjugate of a real number is also real, we can write-

$$\begin{aligned} < \psi|\psi > &= \iiint_S (\psi(x,y,z))^2\ dV \\ &= \iiint_S \frac{A^2}{\sqrt{x^2+y^2}}\ dV \end{aligned}$$

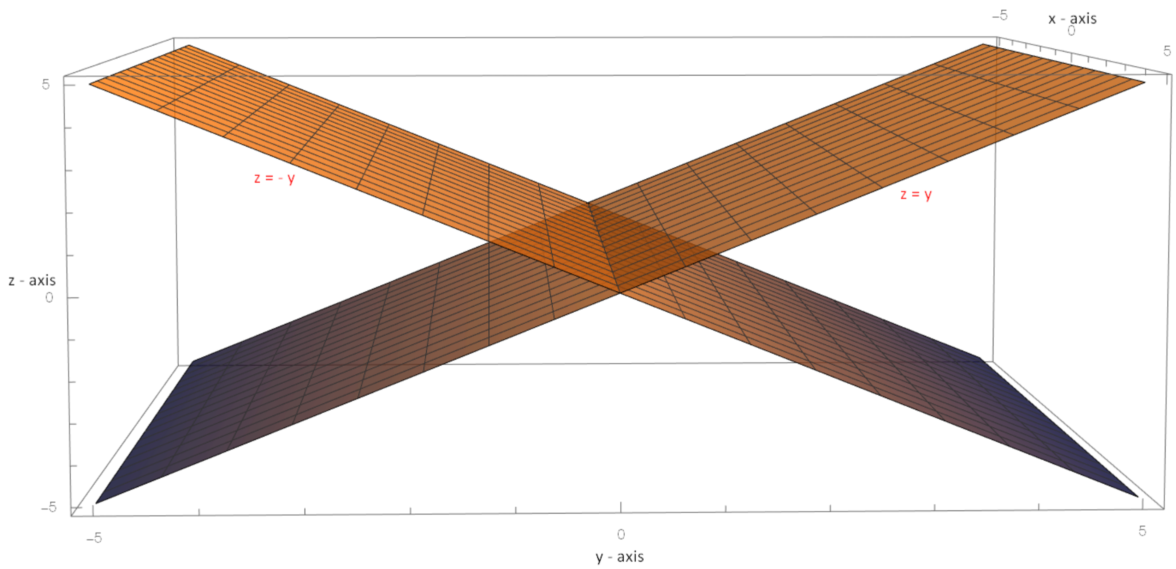


Figure 1: $z = y$ and $z = -y$ planes

We know: $-y \leq z \leq y$.

So in terms of cylindrical coordinate system, $-r\sin\theta \leq z \leq r\sin\theta$.

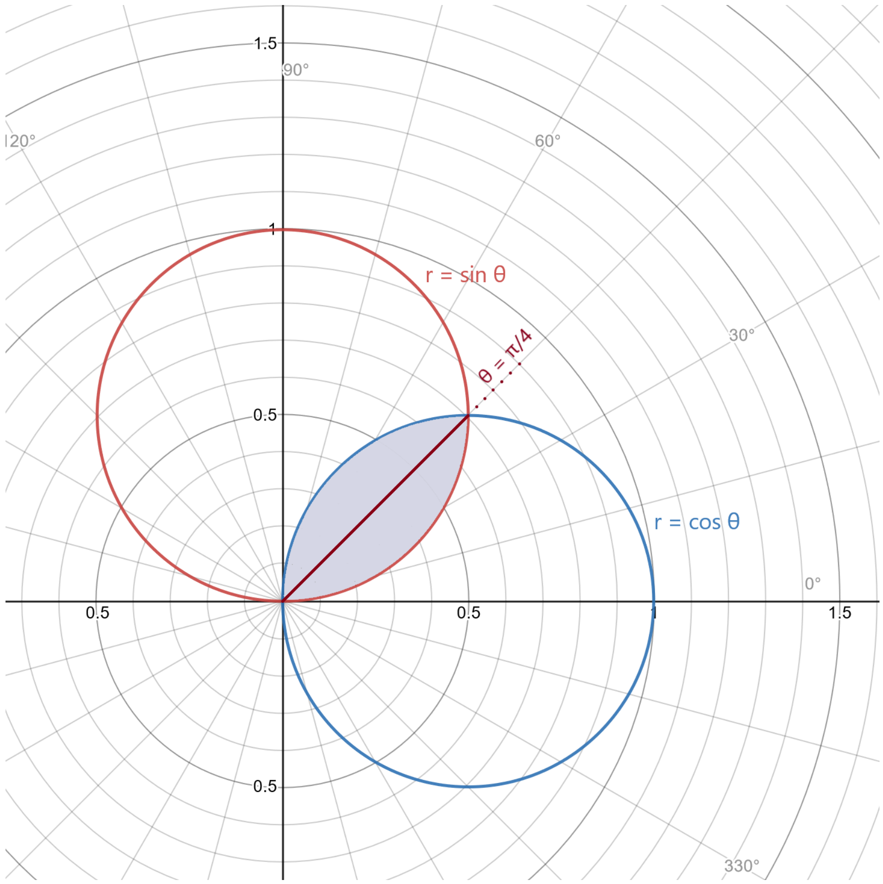


Figure 2: xy plane

Using the graph above,

$$\sin\theta = \cos\theta$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}$$

So, the limits are:

$$-r\sin\theta \leq z \leq r\sin\theta$$

$$0 \leq \theta \leq \frac{\pi}{4} \; and \; \frac{\pi}{4} < \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq \sin\theta \; and \; 0 < r \leq \cos\theta$$

Finally, putting the limits into the equation and substituting the value of $\sqrt{x^2+y^2} = r$.

$$
\begin{aligned}
<\psi|\psi> &= \iiint_S \frac{A^2}{r}\, dV \\
<\psi|\psi> &= \int_0^{\frac{\pi}{4}} \int_0^{\sin\theta} \int_{-r\sin\theta}^{r\sin\theta} \frac{A^2}{r}\, dV + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_{-r\sin\theta}^{r\sin\theta} \frac{A^2}{r}\, dV
\end{aligned}
$$

(c)

$$
\begin{aligned}
dV &= |\frac{\partial(x,y,z)}{\partial(z,r,\theta)}|\, dz\, dr\, d\theta \\
&= \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{vmatrix} dz\, dr\, d\theta \\
&= \begin{vmatrix} 0 & \cos\theta & -r\sin\theta \\ 0 & \sin\theta & r\cos\theta \\ 1 & 0 & 0 \end{vmatrix} dz\, dr\, d\theta \\
&= (r\cdot\cos^2\theta + r\cdot\sin^2\theta)\, dz\, dr\, d\theta \\
&= r\cdot(\cos^2\theta + \sin^2\theta)\, dz\, dr\, d\theta \\
&= r\ dz\, dr\, d\theta
\end{aligned}
$$

(d)

We know, $<\psi|\psi>=1$.

So we can write,

$$<\psi|\psi>=\int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}\int_{-r\sin\theta}^{r\sin\theta}\frac{A^2}{r}\ dV+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-r\sin\theta}^{r\sin\theta}\frac{A^2}{r}\ dV=1$$

Now, substituting the value of dV using the Jacobian determinant and solving:

$$\begin{aligned}
<\psi|\psi> &= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}\int_{-r\sin\theta}^{r\sin\theta}\frac{A^2}{r}\ \cdot r\ dz\,dr\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-r\sin\theta}^{r\sin\theta}\frac{A^2}{r}\ \cdot r\ dz\,dr\,d\theta=1\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}\int_{-r\sin\theta}^{r\sin\theta}A^2\ dz\,dr\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-r\sin\theta}^{r\sin\theta}A^2\ dz\,dr\,d\theta=1\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}[A^2z]_{-r\sin\theta}^{r\sin\theta}\ dr\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}[A^2z]_{-r\sin\theta}^{r\sin\theta}\ dr\,d\theta=1\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}2A^2r\sin\ \theta\ dr\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}2A^2r\sin\ \theta\ dr\,d\theta=1\\
&= \int_0^{\frac{\pi}{4}}[A^2r^2\sin\ \theta]_0^{\sin\theta}\ d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[A^2r^2\sin\ \theta]_0^{\cos\theta}\ d\theta=1\\
&= \int_0^{\frac{\pi}{4}}A^2\sin^3\theta\ d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}A^2\cos^2\theta\sin\ \theta\ d\theta=1\\
&= A^2\int_0^{\frac{\pi}{4}}\sin^3\theta\ d\theta+A^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\sin\ \theta\ d\theta=1
\end{aligned}$$

First,

$$\int\sin^3\theta\ d\theta\ =\int(1-\cos^2\theta)\sin\ \theta\ d\theta$$

Let $u=\cos\,\theta$ and $\frac{du}{d\theta}=-\sin\,\theta$.

$$\int-1+u^2\ du=[-u+\frac{u^3}{3}]=[-\cos\ \theta+\frac{\cos^3\theta}{3}]$$

Likewise for,

$$\int\cos^2\theta\sin\ \theta\ d\theta$$

Let $u=\cos\,\theta$ and $\frac{du}{d\theta}=-\sin\,\theta$.

$$\int-u^2\ du=[-\frac{u^3}{3}]=[-\frac{\cos^3\theta}{3}]$$

So putting the limits into these integrals,

$$
\begin{aligned}
&A^2[-\cos\,\theta + \frac{\cos^3\theta}{3}]_0^{\frac{\pi}{4}} + A^2[-\frac{\cos^3\theta}{3}]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 1\\
&A^2[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} - (-1 + \frac{1}{3})] + A^2[0 - (-\frac{\sqrt{2}}{12})] = 1\\
&\frac{2-\sqrt{2}}{3}A^2 = 1\\
&A^2 = \frac{6+3\sqrt{2}}{2}\\
&A = (\frac{6+3\sqrt{2}}{2})^{\frac{1}{2}}
\end{aligned}
$$

(e)

We know,

$$< \psi|\hat{r}\psi > = \iiint_S \psi(x,y,z)^*\ \hat{r}\psi(x,y,z)\ dV$$

Since, $\hat{r}\psi = r\psi$,

$$
\begin{aligned}
< \psi|\hat{r}\psi > &= \iiint_S (\psi(x,y,z))^2 \cdot r\ dV\\
&= \iiint_S \frac{A^2}{r} \cdot r\ dV\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}\int_{-r\sin\theta}^{r\sin\theta} A^2\ \cdot r\ dz\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-r\sin\theta}^{r\sin\theta} A^2\ \cdot r\ dz\,dr\,d\theta\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} [A^2rz]_{-r\sin\theta}^{r\sin\theta}\ dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} [A^2rz]_{-r\sin\theta}^{r\sin\theta}\ dr\,d\theta\\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} 2A^2r^2\sin\,\theta\ dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} 2A^2r^2\sin\,\theta\ dr\,d\theta\\
&= \int_0^{\frac{\pi}{4}} [\frac{2A^2r^3\sin\,\theta}{3}]_0^{\sin\theta}\ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\frac{2A^2r^3\sin\,\theta}{3}]_0^{\cos\theta}\ d\theta\\
&= \int_0^{\frac{\pi}{4}} \frac{2A^2\sin^4\theta}{3}\ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2A^2\cos^3\theta\sin\,\theta}{3}\ d\theta\\
&= \frac{2A^2}{3}\int_0^{\frac{\pi}{4}} \sin^4\theta\ d\theta + \frac{2A^2}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta\sin\,\theta\ d\theta
\end{aligned}
$$

For,

$$\int_0^{\frac{\pi}{4}} \sin^4\theta \; d\theta = \int_0^{\frac{\pi}{4}} \sin^3\theta \sin\,\theta \; d\theta$$

Applying integration by parts such that, $f(x) = \sin^3\theta$ and $g'(x) = \sin\,\theta$:

$$\int_0^{\frac{\pi}{4}} \sin^3\theta \sin\,\theta \; d\theta \;\; = \;\; [-\sin^3\theta \cos\,\theta - \int -3\sin^2 \cos^2\theta \; d\theta]_0^{\frac{\pi}{4}}$$

Using trigonometric identities we know:

$$3\;\sin^2\theta\;\cos^2\theta = 3\left(\frac{\sin\,2\theta}{2}\right)^2 = \frac{3}{8}(1 - \cos\,4\theta)$$

So, substituting this value:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sin^3\theta \sin\,\theta \; d\theta &= [-\sin^3\theta\cos\,\theta - \frac{3}{8}\int (1 - \cos\,4\theta)]_0^{\frac{\pi}{4}} \\
&= [-\sin^3\theta\cos\,\theta - \frac{3}{8}(\theta - \frac{\sin\,4\theta}{4})]_0^{\frac{\pi}{4}} \\
&= \frac{3\pi - 8}{32}
\end{aligned}$$

Now for,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta \sin\,\theta \; d\theta$$

Let $u = \cos\,\theta$ and $\frac{du}{d\theta} = -\sin\,\theta$.

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3\theta \sin\,\theta \; d\theta &= \int -u^3 du \\
&= [-\frac{u^4}{4}] \\
&= [\frac{-\cos^4\theta}{4}]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{1}{16}
\end{aligned}$$

Putting these values into the original equation of $< \psi|\hat{r}\psi >$

$$\begin{aligned}
< \psi|\hat{r}\psi > &= \frac{2A^2}{3}\cdot\frac{3\pi - 8}{32} + \frac{2A^2}{3}\cdot\frac{1}{16} \\
&= \frac{3\pi - 8}{48}A^2 + \frac{1}{24}A^2 \\
&= \frac{3\pi - 6}{48}A^2
\end{aligned}$$

Finally putting the value of $A^2 = \frac{6+3\sqrt{2}}{2}$ (known from 2c)

$$
\begin{aligned}
< \psi|\hat{r}\psi > &= \frac{3\pi - 6}{48} \cdot \frac{6+3\sqrt{2}}{2} \\
&= \frac{18\pi + 9\pi\sqrt{2} - 36 - 18\sqrt{2}}{96}
\end{aligned}
$$

(f)

If $\hat{A}(kf) = k\hat{A}f$ and $\hat{r}\psi = r\psi$,

Then,

$$\hat{r}^2\psi = \hat{r}(\hat{r}\psi) = \hat{r}(r\psi) = r^2\psi$$

(g)

$$
\begin{aligned}
<\psi|\hat{r}^2\psi> &= \iiint_S \psi(x,y,z)^* \ \hat{r}^2\psi(x,y,z) \ dV \\
&= \iiint_S \frac{A^2}{r}\cdot r^2 \ dV \\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta}\int_{-r\sin\theta}^{r\sin\theta} A^2 \ \cdot r^2 \ dz\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{-r\sin\theta}^{r\sin\theta} A^2 \ \cdot r^2 \ dz\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} [A^2r^2z]_{-r\sin\theta}^{r\sin\theta} \ dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} [A^2r^2z]_{-r\sin\theta}^{r\sin\theta} \ dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}}\int_0^{\sin\theta} 2A^2r^3\sin\ \theta \ dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} 2A^2r^3\sin\ \theta \ dr\,d\theta \\
&= \int_0^{\frac{\pi}{4}} [\frac{A^2r^4\sin\ \theta}{2}]_0^{\sin\theta} \ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\frac{A^2r^4\sin\ \theta}{2}]_0^{\cos\theta} \ d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{A^2\sin^4\theta\sin\ \theta}{2} \ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{A^2\cos^4\theta\sin\ \theta}{2} \ d\theta \\
&= \frac{A^2}{2}\int_0^{\frac{\pi}{4}} \sin^4\theta\sin\ \theta \ d\theta + \frac{A^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\sin\ \theta \ d\theta \\
&= \frac{A^2}{2}\int_0^{\frac{\pi}{4}} (1-\cos^2\theta)^2\theta\sin\ \theta \ d\theta + \frac{A^2}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\sin\ \theta \ d\theta
\end{aligned}
$$

Let $u = \cos\ \theta$ and $\frac{du}{d\theta} = -\sin\ \theta$.

$$
\begin{aligned}
<\psi|\hat{r}^2\psi> &= \frac{A^2}{2}\int -1+2u^2-u^4 \ du + \frac{A^2}{2}\int -u^4 \ du \\
&= \frac{A^2}{2}[-u+\frac{2u^3}{3}-\frac{u^5}{5}] + \frac{A^2}{2}[\frac{-u^5}{5}] \\
&= \frac{A^2}{2}[-\cos\ \theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5}]_0^{\frac{\pi}{4}} + \frac{A^2}{2}[\frac{-\cos^5\theta}{5}]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{A^2}{2}(\frac{32\sqrt{2}-43}{60\sqrt{2}}) + \frac{A^2}{2}(\frac{1}{20\sqrt{2}}) \\
&= A^2(\frac{32\sqrt{2}-43}{120\sqrt{2}}) + A^2(\frac{1}{40\sqrt{2}}) \\
&= (\frac{4\sqrt{2}-5}{15\sqrt{2}})A^2
\end{aligned}
$$

Finally substituting the value of $A^2 = \frac{6+3\sqrt{2}}{2}$ (known from 2c),

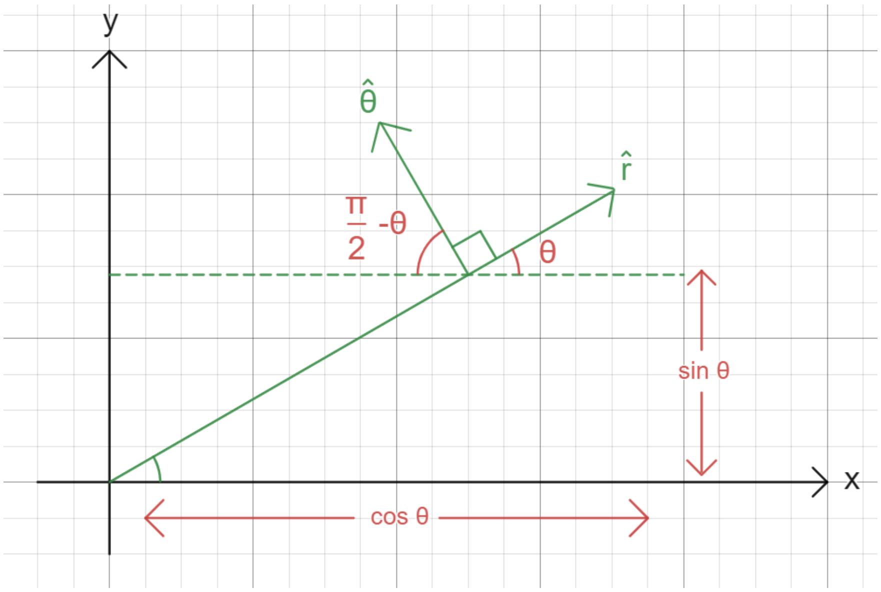
$$
\begin{aligned}
< \psi|\hat{r}^2\psi > &= \frac{4\sqrt{2}-5}{15\sqrt{2}} \cdot \frac{6+3\sqrt{2}}{2} \\
&= \frac{3-\sqrt{2}}{10}
\end{aligned}
$$

(h)

$$
\begin{aligned}
\sigma_r^2 &= < \psi|\hat{r}^2\psi > - < \psi|\hat{r}\psi >^2 \\
\sigma_r^2 &= \frac{3-\sqrt{2}}{10} - (\frac{18\pi + 9\pi\sqrt{2} - 36 - 18\sqrt{2}}{96})^2 \\
\sigma_r^2 &= 0.0250587 \\
\sigma_r &= 0.1583
\end{aligned}
$$

3. Rotating Tube

(a)



$$
\begin{aligned}
\hat{r} &= x\hat{i} + y\hat{j} \\
&= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \\
&= r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})
\end{aligned}
$$

Also we know,

$$
\begin{aligned}
\vec{r} &= r\,\hat{r} \qquad\qquad [\text{where } \hat{r} \text{ is the unit vector}] \\
or,\ r\,\hat{r} &= r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \\
\therefore\ \hat{r} &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}
\end{aligned}
$$

From the figure,

$$
\begin{aligned}
\hat{\theta} &= -\cos\left(\frac{\pi}{2} - \theta\right)\hat{i} + \sin\left(\frac{\pi}{2} - \theta\right)\hat{j} \\
&= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}
\end{aligned}
$$

According to question,

$$
\begin{aligned}
v &= \frac{dr}{dt}\,\hat{r} + r\,\frac{d\theta}{dt}\,\hat{\theta} \\
or,\; v &= r'(t)\,\hat{r} + r\,\theta'(t)\,\hat{\theta} \\
or,\; \frac{dv}{dt} &= \frac{d}{dt}\Big(r'(t)\,\hat{r} + r\,\theta'(t)\,\hat{\theta}\Big) \\
or,\; f_r &= r''(t)\,\hat{r} + r'\,\frac{d\hat{r}}{dt} + r'\theta'(t)\,\hat{\theta} + r\,\theta''(t)\,\hat{\theta} + r\,\theta'(t)\,\frac{d\hat{\theta}}{dt} \qquad (1)
\end{aligned}
$$

now,

$$
\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}\Big(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\Big) \\
&= \theta'(t)\Big(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\Big) \\
&= \theta'(t)\,\hat{\theta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}\Big(-\sin\theta\,\hat{i} + \cos\theta\hat{j}\Big) \\
&= -\theta'(t)\cos\theta\,\hat{i} - \theta'(t)\sin\theta\,\hat{j} \\
&= -\theta'(t)\,[\cos\theta + \sin\theta] \\
&= -\theta'(t)\,\hat{r}
\end{aligned}
$$

if we substitute in (3.1) we get,

$$
\begin{aligned}
f_r &= r''(t)\,\hat{r} + r'\,\frac{d\hat{r}}{dt} + r'\theta'(t)\,\hat{\theta} + r\,\theta''(t)\,\hat{\theta} + r\,\theta'(t)\,\frac{d\hat{\theta}}{dt} \\
f_r &= r''(t)\,\hat{r} + r'\,\theta'(t)\,\hat{\theta} + r'(t)\,\theta'(t)\,\hat{\theta} + r\,\theta''(t)\,\hat{\theta} + r\,\theta'(t)\,[-\theta'(t)\,\hat{r}]
\end{aligned}
$$

considering r components,

$$
\begin{aligned}
f_r &= r''(t) + r\,\theta'(t) \times [-\theta'(t)] \\
&= r''(t) - r\,\{(\theta'(t)\}^2
\end{aligned}
$$

$$
\therefore f_r = \frac{d\,r^2}{dt} - r\left(\frac{d\theta}{dt}\right)^2
$$

[proved]

(b)

If we consider the particles F or Force's components. There will be two components one is $mg\cos\theta$ and the other one is $mg\sin\theta$. The only Force is active during rotation is $mg\sin\theta$. Because the force is directly proportional to sine of the angular displacement. So we get,

$$F = -mg\sin\omega t$$

We also know, $F = ma$ or $F = m \cdot f_r$. If we substitute,

$$\begin{aligned}
m \cdot f_r &= -mg\sin\omega t \\
\cancel{m} \cdot f_r &= -\cancel{m}\, g\sin\omega t \\
f_r &= -g\sin\omega t \\
\frac{d\,r^2}{dt} - r\left(\frac{d\theta}{dt}\right)^2 &= -g\sin\omega t && \text{[from (a)]} \\
\frac{d\,r^2}{dt} - r\omega^2 &= -g\sin\omega t && \left[\because \omega = \frac{d\theta}{dt}\right]
\end{aligned}$$

[showed]

(c)

Given,

$$\frac{d^2r}{dt^2} - r\omega^2 = -g\sin\omega t$$

C.F:

$$r'' - r\omega^2 = 0$$

let,

$$\begin{aligned} r &= e^{kt} \\ r' &= k\,e^{kt} \\ r'' &= k^2\,e^{kt} \end{aligned}$$

now,

$$\begin{aligned} k^2\,e^{kt} - e^{kt}\omega^2 &= 0 \\ or,\ e^{kt}(k^2-\omega^2) &= 0 \\ or,\ k^2-\omega^2 &= 0 \\ \therefore\ k &= \pm\,\omega \end{aligned}$$

$$C.F = C_1\,e^{\omega t} + C_2\,e^{-\omega t}$$

P.I:

we can write the equation as,

$$(D^2-\omega^2)r = -g\sin\omega t$$

$$\begin{aligned} \text{P.I} &= \frac{1}{D^2-\omega^2}\times(-g\sin\omega t) \\ &= -g\left[\frac{1}{D^2-\omega^2}\times\sin\omega t\right] \\ &= -g\left[\frac{1}{-\omega^2-\omega^2}\times\sin\omega t\right] \\ &\qquad [\text{substituting } -\omega^2 \text{ instead of } D^2] \\ &= -g\times\frac{\sin\omega t}{-2\,\omega^2} \\ &= \frac{g\sin\omega t}{2\,\omega^2} \\ &= \frac{g}{2\,\omega^2}\ \sin\omega t \end{aligned}$$

$$\begin{aligned} \therefore r(t) &= C.F + P.I \\ &= C_1\,e^{\omega t} + C_2\,e^{-\omega t} + \frac{g}{2\,\omega^2}\ \sin\omega t \end{aligned}$$

(d)

From (c) we got,

$$r(t) = C_1\, e^{\omega t} + C_2\, e^{-\omega t} + \frac{g}{2\,\omega^2}\ \sin\omega t$$

From hyperbolic trigonometric function we know that,

$$\begin{aligned}\cosh(a) + \sinh(a) &= e^{a}\\ \cosh(a) - \sinh(a) &= e^{-a}\end{aligned}$$

Now,

$$\begin{aligned} r(t) &= C_1\,\{\cosh\omega t + \sinh\omega t\} + C_2\{\cosh\omega t - \sinh\omega t\} + \frac{g}{2\,\omega^2}\ \sin\omega t \\ &= C_1\ \cosh\omega t + C_1\ \sinh\omega t + C_2\ \cosh\omega t - C_2\ \sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t \\ &= (C_1 + C_2)\cosh\omega t + (C_1 - C_2)\sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t \end{aligned}$$

let,

$$\begin{aligned} C_1 + C_2 &= L \\ C_1 - C_2 &= M \end{aligned}$$

$$\therefore r(t) = L\ \cosh\omega t + M\ \sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t$$

(e)

From the question we got that when $t = 0$ radius was a.

So,

$$\begin{aligned}
r(t) &= L\ \cosh\omega t + M\ \sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t \\
or,\ a &= L\ \cosh(\omega\times 0) + M\ \sinh(\omega\times 0) + \frac{g}{2\,\omega^2}\ \sin(\omega\times 0) \\
or,\ a &= L + 0 + 0 \qquad \qquad \left[\begin{array}{l} \because\ \cosh(0) = 1 \\ \quad \sinh(0) = 0 \\ \quad \sin(0) = 0 \end{array}\right] \\
\therefore L &= a
\end{aligned}$$

after differentiating,

$$\begin{aligned}
r(t) &= L\ \cosh\omega t + M\ \sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t \\
or,\ \frac{dr}{dt} &= \frac{d}{dt}\left[a\ \cosh\omega t + M\ \sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t\right] \\
or,\ v &= a\,\omega\ \sinh\omega t + M\,\omega\ \cosh\omega t + \frac{g}{2\,\omega}\ \cos\omega t
\end{aligned}$$

when $t = 0$,

$$\begin{aligned}
v &= 0 + M\,\omega + \frac{g}{2\,\omega} \\
or,\ M\,\omega &= v - \frac{g}{2\,\omega} \\
or,\ M\,\omega &= \frac{2\cdot\omega\cdot v - g}{2\,\omega} \\
or,\ M &= \frac{2\cdot\omega\cdot v - g}{2\,\omega^2} \\
\therefore\ M &= \frac{v}{\omega} - \frac{g}{2\,\omega^2}
\end{aligned}$$

So,

$$r(t) = a\cosh\omega t + \left(\frac{v}{\omega} - \frac{g}{2\,\omega^2}\right)\sinh\omega t + \frac{g}{2\,\omega^2}\ \sin\omega t$$

[showed]

4. Charged Quantam Particles

(a)

In equation (15),

$$
\begin{aligned}
\psi(r,\theta,\phi) &= F(r)G(\theta,\phi)\\
\implies \frac{\partial\psi}{\partial r}(r,\theta,\phi) &= G(\theta,\phi)\frac{d}{dr}(F(r))\\
\implies \frac{\partial\psi}{\partial\theta}(r,\theta,\phi) &= F(r)\frac{\partial}{\partial\theta}(G(\theta,\phi))\\
\implies \frac{\partial^2\psi}{\partial\phi^2}(r,\theta,\phi) &= F(r)\frac{\partial^2}{\partial\phi^2}(G(\theta,\phi))
\end{aligned}
$$

In equation (14),

$$
-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right]-\frac{Q}{4\pi\epsilon_0 r}\psi = E\psi :
$$

$$
\begin{aligned}
\psi &= FG,\\
\frac{\partial\psi}{\partial r} &= G\frac{dF}{dr},\\
\frac{\partial\psi}{\partial\theta} &= F\frac{\partial G}{\partial\theta}\\
\frac{\partial^2\psi}{\partial\phi^2} &= F\frac{\partial^2 G}{\partial\phi^2}\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) &= \frac{\partial}{\partial r}\left(r^2 G\frac{dF}{dr}\right)\\
&= G\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)\\
\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right) &= \frac{\partial}{\partial\theta}\left(\sin(\theta)F\frac{\partial G}{\partial\theta}\right)\\
&= F\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)
\end{aligned}
$$

$\therefore$ equation (14) can be written as,

$$-\frac{\hbar^2}{2\mu}\Big[\frac{G}{r^2}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{F}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{F}{r^2\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)\Big]-\frac{Q}{4\pi\epsilon_0 r}FG=EFG$$

[shown]

(b)

In equation (16),

$$-\frac{\hbar^2}{2\mu}\Big[\frac{G}{r^2}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{F}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{F}{r^2\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)\Big]-\frac{Q}{4\pi\epsilon_0 r}FG=EFG$$

$$\implies \frac{\hbar^2}{2\mu}\Big[\frac{G}{r^2}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{F}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{F}{r^2\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)\Big]+\frac{Q}{4\pi\epsilon_0 r}FG+EFG=0$$

[Multiplying both sides by $\frac{2\mu r^2}{FG\hbar^2}$]

$$\implies \frac{r^2}{FG}\Big[\frac{G}{r^2}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{F}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{F}{r^2\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)\Big]+\frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r}+E\Big)=0$$

$$\implies \frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{1}{G\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{1}{G\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)+\frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r}+E\Big)=0$$

$$\implies \Big\{\frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big)+\frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r}+E\Big)\Big\}+\frac{1}{G}\Big\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big)+\frac{1}{\sin^2(\theta)}\Big(\frac{\partial^2 G}{\partial\phi^2}\Big)\Big\}=0$$

[shown]

(c)

$$\frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big) + \frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + E\Big) = j(j+1) \tag{18}$$

$$\frac{1}{G}\Big\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\Big\} = -j(j+1) \tag{19}$$

Here, if we perform the operation, (18) + (19),

$$\implies \frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big) + \frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + E\Big) + \frac{1}{G}\Big\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\Big\} = j(j+1) - j(j+1)$$

$$\implies \Big\{\frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big) + \frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + E\Big)\Big\} + \frac{1}{G}\Big\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Big(\sin(\theta)\frac{\partial G}{\partial\theta}\Big) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\Big\} = 0$$

[shown]

(d)

$$\frac{1}{F}\frac{d}{dr}\Big(r^2\frac{dF}{dr}\Big) + \frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + E\Big) = j(j+1) \tag{18}$$

$$\implies \frac{1}{F}\Big(r^2\frac{d^2F}{dr^2} + 2r\frac{dF}{dr}\Big) + \frac{2\mu r^2}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + e\Big) - j(j+1) = 0$$

[multiplying both sides by $\frac{F}{r^2}$]

$$\implies \frac{2}{r}\frac{dF}{dr} + \frac{d^2F}{dr^2} + \left[\frac{2\mu}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r} + E\Big) - \frac{j(j+1)}{r^2}\right]F = 0$$

[shown]

(e)

$$\lim_{r\to\infty}\left\{\frac{2}{r}\frac{dF}{dr}+\frac{d^2F}{dr^2}+\left[\frac{2\mu}{\hbar^2}\Big(\frac{Q}{4\pi\epsilon_0 r}+E\Big)-\frac{j(j+1)}{r^2}\right]F\right\}=0$$
$$\implies 0\times\frac{dF_\infty}{dr}+\frac{d^2F_\infty}{dr^2}+\left[\frac{2\mu}{\hbar^2}\Big(0+E\Big)-0\right]F_\infty=0$$
$$\implies \frac{d^2F_\infty}{dr^2}+\frac{2\mu E}{\hbar^2}F_\infty=0$$

[shown]

(f)

Given,

$$\frac{d^2F_\infty}{dr^2}-\frac{2\mu E'}{\hbar^2}F_\infty=0$$

From this, we can write,

$$m^2-\frac{2\mu E'}{\hbar^2}=0$$
$$\implies m=\pm\frac{\sqrt{2\mu E'}}{\hbar}$$

Therefore, the solution should be in the form,

$$F_\infty=Ae^{m_1 r}+Be^{m_2 r}$$
$$\therefore F_\infty=Ae^{-\frac{\sqrt{2\mu E'}}{\hbar}r}+Be^{\frac{\sqrt{2\mu E'}}{\hbar}r}$$

[shown]

(g)

Given,

$$F_\infty \to 0$$
$$r \to \infty$$
$$F_\infty = Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r}$$

Now,

$$if\ r \to \infty\ ;\ then\ \sqrt{\frac{2\mu E'}{\hbar}}r \to \infty$$
$$\therefore e^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \to 0$$
$$\therefore Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \to 0$$
$$\therefore e^{\sqrt{\frac{2\mu E'}{\hbar}}r} \to \infty$$

Again,

$$F_\infty = Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r}$$
$$Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} + Be^{\sqrt{\frac{2\mu E'}{\hbar}}r} \to 0 \quad [\because F_\infty \to 0]$$
$$Be^{\sqrt{\frac{2\mu E'}{\hbar}}r} \to 0 \quad [\because Ae^{-\sqrt{\frac{2\mu E'}{\hbar}}r} \to 0]$$
$$\therefore B \to 0 \quad [\because e^{\sqrt{\frac{2\mu E'}{\hbar}}r} \to \infty]$$

(h)

Given,

$$\frac{1}{G}\left\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\right\}=-j(j+1) \tag{19}$$

$$G\sin^2(\theta)\frac{1}{G}\left\{\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2 G}{\partial\phi^2}\right\}=-j(j+1)G\sin^2(\theta)$$

[Multiplying both sides by $G\sin^2(\theta)$]

$$\therefore \sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{\partial^2 G}{\partial\phi^2}=-j(j+1)G\sin^2(\theta) \tag{24}$$

[showed]

(i)

Given,

$$G(\theta,\phi)=T(\theta)Z(\phi), \tag{25}$$

$$\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial G}{\partial\theta}\right)+\frac{\partial^2 G}{\partial\phi^2}=-j(j+1)G\sin^2(\theta) \tag{24}$$

$$\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial (TZ)}{\partial\theta}\right)+\frac{\partial^2 (TZ)}{\partial\phi^2}=-j(j+1)(TZ)\sin^2(\theta)$$

$$Z\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)+T\frac{d^2 Z}{d\phi^2}=-j(j+1)(TZ)\sin^2(\theta)$$

$$\frac{1}{T}\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)+\frac{1}{Z}\frac{d^2 Z}{d\phi^2}=-j(j+1)\sin^2(\theta)$$

[Dividing both sides by TZ]

$$\therefore \left\{\frac{1}{T}\left[\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)\right\}+\left\{\frac{1}{Z}\frac{d^2 Z}{d\phi^2}\right\}=0 \tag{26}$$

[showed]

(j)

Given,

$$\frac{1}{T}\left[\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)=k^2, \tag{27}$$

$$\frac{1}{Z}\frac{d^2Z}{d\phi^2}=-k^2 \tag{28}$$

By adding the above two equations, we get,

$$\begin{aligned}\frac{1}{T}\left[\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)+\frac{1}{Z}\frac{d^2Z}{d\phi^2}&=k^2-k^2\\ \therefore\left\{\frac{1}{T}\left[\sin(\theta)\frac{d}{d\theta}\left(\sin(\theta)\frac{dT}{d\theta}\right)\right]+j(j+1)\sin^2(\theta)\right\}+\left\{\frac{1}{Z}\frac{d^2Z}{d\phi^2}\right\}&=0\end{aligned}$$

Since, the first curly brackets is independent of ϕ and r, therefore we will get a constant and the second curly brackets is independent of θ and r, therefore we will get another constant. These both constants have a equal magnitude k^2 and opposite sign. Therefore the above expression will give 0.

(k)

Given,

$$\frac{d^2Z}{d\phi^2} + k^2 Z = 0 \tag{28}$$

We Know,

$$\text{if,} \qquad ay''(x) + by'(x) + cy(x) = 0 \tag{4.1}$$

We can can write the following equation,

$$am^2 + bm + c = 0 \tag{4.2}$$

And if the discriminant is positive, then the solution of the y of the differential equation,

$$y(x) = Ae^{m_1 x} + Be^{m_2 x} \tag{4.3}$$

Comparing equation (4.1) and equation (28), we get,

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= k^2 \\ m &= \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(k^2)}}{2(1)} \\ \therefore m &= \pm k \end{aligned}$$

Putting the value of m in equation (4.3) and substituting $y(x)$ with $Z(\phi)$,

$$\therefore Z(\phi) = Ae^{k\phi} + Be^{-k\phi} \tag{4.4}$$

(l)

Given,

$$T(\theta) \;=\; T_{j,k}(\theta) =\; n_{j,k} P_j^k(\cos(\theta)) \tag{29}$$

From 4(k), the solution of equation (30),

$$Z(\phi) = Ae^{k\phi} + Be^{-k\phi} \tag{4.4}$$

$$G(\theta,\phi) = T(\theta)Z(\phi) \tag{25}$$

$$\therefore\; G(\theta,\phi) = n_{j,k} P_j^k(\cos(\theta)) \left(Ae^{k\phi} + Be^{-k\phi}\right) \tag{4.5}$$

(m)

Given,

$$F(r) \;=\; F_{i,j}(r) =\; c_{i,j} e^{d_i r} (2d_i r)^j \left[L_{(i-j-1)}^{(2j+1)}(2d_i r)\right] \tag{23}$$

From 4(l), the solution of $G(\theta,\phi)$,

$$G(\theta,\phi) = n_{j,k} P_j^k(\cos(\theta)) \left(Ae^{k\phi} + Be^{-k\phi}\right) \tag{4.5}$$

$$\psi(r,\theta,\phi) = F(r)G(\theta,\phi), \tag{15}$$

$$\therefore \psi(r,\theta,\phi) \;=\; \psi_{i,j,k}(r,\theta,\phi) =\; c_{i,j} e^{d_i r} (2d_i r)^j \left[L_{(i-j-1)}^{(2j+1)}(2d_i r)\right] n_{j,k} P_j^k(\cos(\theta)) \left(Ae^{k\phi} + Be^{-k\phi}\right) \tag{4.6}$$

(n)

Given that, for i = 2, j = 1 and k = 0, the wave function is

$$\psi_{2,1,0} = Ad_2re^{-rd_2}\cos(\theta) \tag{4.7}$$

$$\iiint_{Entire\ Space} \mid \psi_{i,j,k} \mid^2 dV = 1 \tag{31}$$

Since, V is in spherical coordinate system,

$$dV = r^2\sin(\theta)\ d\phi\ d\theta\ dr$$

Given that, we need to integrate entire space. So, the limits will be,

$$\{(r, \theta, \phi)\ \epsilon\ Entire\ Space \mid 0 \le r \le \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}$$

$$\begin{aligned}
\iiint_{Entire\ Space} \mid \psi_{i,j,k} \mid^2 dV &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \mid Ad_2re^{-rd_2}\cos(\theta) \mid^2 r^2\sin(\theta)\ d\phi\ d\theta\ dr \\
&= A^2{d_2}^2 \int_0^\infty \int_0^\pi \int_0^{2\pi} r^4e^{-2rd_2}\cos^2(\theta)\sin(\theta)\ d\phi\ d\theta\ dr \\
&= A^2{d_2}^2 \int_0^\infty \int_0^\pi r^4e^{-2rd_2}\cos^2(\theta)\sin(\theta)\left(\int_0^{2\pi} d\phi\right)\ d\theta\ dr \\
&= A^2{d_2}^2 \int_0^\infty \int_0^\pi r^4e^{-2rd_2}\cos^2(\theta)\sin(\theta)\,(2\pi)\ d\theta\ dr \\
&= 2\pi A^2{d_2}^2 \int_0^\infty r^4e^{-2rd_2}\left(\int_0^\pi \cos^2(\theta)\sin(\theta)\ d\theta\right)\ dr
\end{aligned}$$

Let,

$$u = \cos(\theta)$$

$$du = -\sin(\theta)$$

θ	0	π
u	1	-1

$$\begin{aligned}
\int_0^{\pi} \cos^2(\theta)\sin(\theta)\ d\theta &= \int_1^{-1} u^2(-du) \\
&= -\int_1^{-1} u^2\ du \\
&= -\left[\frac{u^3}{3}\right]_1^{-1} \\
&= -\left\{\frac{(-1)^3}{3} - \frac{(1)^3}{3}\right\} \\
&= \frac{2}{3}
\end{aligned}$$

Now,

$$\begin{aligned}
2\pi A^2{d_2}^2 \int_0^{\infty} r^4 e^{-2rd_2} \left(\int_0^{\pi} \cos^2(\theta)\sin(\theta)\ d\theta\right)\ dr &= 2\pi A^2{d_2}^2 \int_0^{\infty} r^4 e^{-2rd_2}\frac{2}{3}\ dr \\
&= \frac{4}{3}\pi A^2{d_2}^2 \int_0^{\infty} r^{(5-1)} e^{-(2d_2)r}\ dr \\
&= \frac{4}{3}\pi A^2{d_2}^2 \frac{\Gamma(5)}{(2d_2)^5} \\
&\qquad \left[\because \int_0^{\infty} x^{\alpha-1}e^{-\lambda x}\ dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}\right] \\
1 &= \frac{4}{3}\pi A^2{d_2}^2 \frac{(5-1)!}{32{d_2}^5}\ ; \\
&\qquad [\text{using eqn (31) and } \Gamma(n) = (n-1)!]
\end{aligned}$$

$$\therefore A = \sqrt{\frac{{d_2}^3}{\pi}}$$