663 Final Project: LDA Collapsed Gibbs Sampling

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1. Abstract

We approach Latent Dirichlet Allocation (LDA), a generative hierachical Bayesian model, via collapsed Gibbs sampling on text corpora in order to make inferences on latent variable parameters. In this hierarchy, documents are represented as random mixtures over latent topics, and topics are characterized by a distribution over words. We are interested in obtaining samples from the posterior latent topic marginals in order to estimate the distribution of words in a particular topic, and the distribution of topics witin a document. We describe the theoretical work presented by Griffiths and Steyvers, and recreate their algorithm with increased speed ups via ***

2. Background

We are primarily using “Finding Scientific Topics” (Griffiths and Styevers) [2] to understand the theoretical basis of the collapsed Gibbs sampler, but we also reference “Latent Dirichlet Allocation” (Blei et. al.)[1] and “Fast Collapsed Gibbs Sampling for Latent Dirichlet Allocation” (Porteous et. al.)[3]. Please note that the notation within this section matches that of Griffiths et. al. In general, the concept of collapsed Gibbs sampling is to introduce latent variables, integrate out their underlying parameters, and ultimately provide an easy and cost efficient way to update the full conditionals. We can then utilize these samples to obtain estimates of the latent parameters of interest. By collapsing, or integrating out the parameters of the latent variables, the algorithm significantly cuts down the number of steps within the markov chain. Moroever, the updating procedure involves a fairly cost-effecient summing operation.

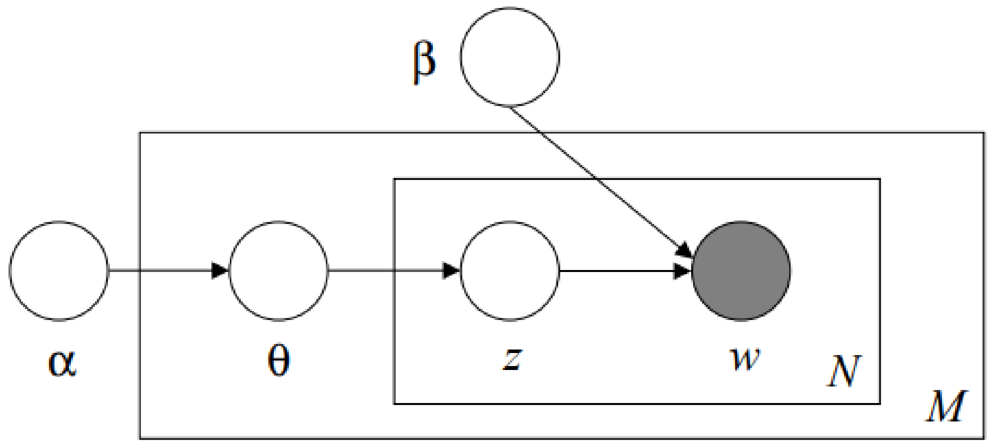
To begin, we are primarily interested in garnering samples from:

$$p(\mathbf{z}|\mathbf{w}) = \frac{p(\mathbf{z},\mathbf{w})}{\Sigma_{\mathbf{z}} p(\mathbf{z},\mathbf{w})}$$

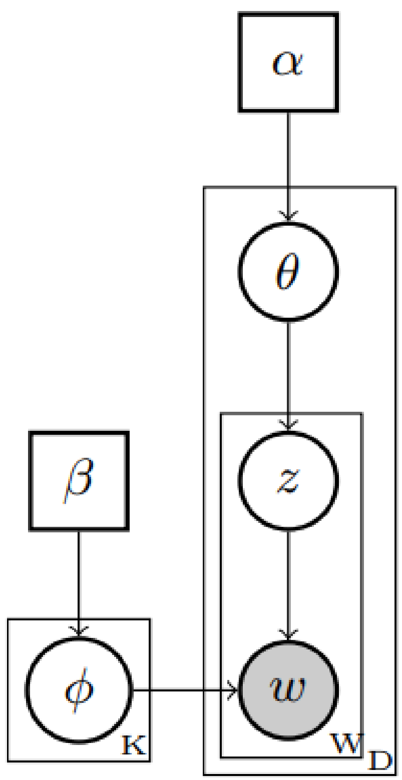
Where $p(\mathbf{z}|\mathbf{w})$ represents the posterior distribution over the assigments of words to topics. However, the denominator cannot be factorized and this quantity cannot be computed directly. The approach by Griffiths et. al. is to convert this to an MCMC set up on top of a slightly altered LDA setting first provided by Blei et. al.

We discuss the hierarchical Bayesian model and the use of conjugacy before examining the results provided by Griffiths and Steyvers. We also attempt to fill in holes not originally provided by the authors.

Consider the graphical model representation of LDA from Blei et. al.:



We contrast this to the graphical LDA model used by Griffiths et. al. (and Porteous et. al.):



Where we noticably see that the two constructions differ by ϕ over K topics.

Seen in a different light, the complete probability model by Griffiths et. al. is:

$$\begin{aligned} w_i | z_i, \phi^{(z_i)} &\sim \mathit{Discrete}(\phi^{(z_i)}) \\ \phi &\sim \mathit{Dirichlet}(\beta) \\ z_i | \theta^{(d_i)} &\sim \mathit{Discrete}(\theta^{(d_i)}) \\ \theta &\sim \mathit{Dirichlet}(\alpha) \end{aligned}$$

We can also express the *Discrete* as *Categorical* or *Multinomial* with only one trial. Also note that while α and β could be vector specified, here we assume one value for $\boldsymbol{\alpha} = \mathbf{1}\alpha$ and similarly $\beta = \mathbf{1}\beta$, i.e. we have symmetric Dirichlet distributions where α and β are fixed hyperparameters and each vector component $\boldsymbol{\alpha}_i$ represents the equivalent prior weight placed on each topic k in a document, whereas each β_i represents the equivalent prior weight on each word in a topic. The approach by Griffiths et. al. was to integrate out the latent variables ϕ and $\boldsymbol{\theta}$. However, they leave much of the calculations to the reader, of which we attempt to spell out below. I.e. we can note that:

$$p(\mathbf{z}|\mathbf{w}) = \frac{p(\mathbf{z},\mathbf{w})}{\Sigma_{\mathbf{z}} p(\mathbf{z},\mathbf{w})}$$

where

$$p(\mathbf{w},\mathbf{z}) = p(\mathbf{w}|\mathbf{z})p(\mathbf{z})$$

where

$$p(\mathbf{z},\mathbf{w},\phi) = p(\mathbf{w}|\mathbf{z},\phi)p(\mathbf{z}|\phi)p(\phi)$$

where integrating ϕ out of $p(\mathbf{w}|\mathbf{z},\phi)p(\phi)$ is equivalent to $p(\mathbf{w}|\mathbf{z})$. However, we know via Multinomial-Dirichlet conjugacy, that $p(\phi|\mathbf{w},\mathbf{z})\ \alpha\ \frac{p(\mathbf{w}|\mathbf{z},\phi)p(\phi)}{p(\mathbf{w},\mathbf{z})}\ \alpha\ p(\mathbf{w}|\mathbf{z},\phi)p(\phi)$ and we can then note that the right most term has the kernel of an updated Dirichlet distribution $\phi \sim Dir(\beta + \boldsymbol{\Sigma}\mathbf{w_i})$, and consequently combine this information with the fact that it is a probability distribution that must integrate to one and as such must take the normalizing constant of this updated Dirichlet form, or stated differently, we can conclude $p(\phi|\mathbf{w},\mathbf{z}) = Dir(\beta + \Sigma\mathbf{w}_i)$. By integrating ϕ we have equivalently (using slight of hand in the second integral):

$$p(\mathbf{w}|\mathbf{z}) = \int_\phi p(\mathbf{w}|\mathbf{z},\phi)p(\phi)d\phi\ \alpha\ \int_\phi Dir(B + \Sigma\mathbf{w}_i)d\phi = \int_\phi \frac{\Gamma(\Sigma(\beta_i + \Sigma\mathbf{w_i}))}{\Pi\,\Gamma(\beta_i + \Sigma\mathbf{w_i})} \Pi\phi_i^{(\beta_i + \Sigma\mathbf{w_i})-1} d\phi$$

where $p(\mathbf{w}|\mathbf{z},\phi)p(\phi)\ \alpha\ Dir(\beta + \Sigma\mathbf{w}_i)$ but then since the left hand side is a probability distribution, then we know that the proportionality constant is precisely the normalizing factor of the updated Dirichlet to which $p(\mathbf{w}|\mathbf{z},\phi)p(\phi)$ is proportional, and as such the chained equation above can replace the proportionaly sign with an equality. or

$$p(\mathbf{w}|\mathbf{z}) = \int_\phi p(\mathbf{w}|\mathbf{z},\phi)p(\phi)d\phi = \int_\phi Dir(B + \Sigma\mathbf{w}_i)d\phi = \int_\phi \frac{\Gamma(\Sigma(\beta_i + \Sigma\mathbf{w_i}))}{\Pi\,\Gamma(\beta_i + \Sigma\mathbf{w_i})} \Pi\phi_i^{(\beta_i + \Sigma\mathbf{w_i})-1} d\phi$$

Griffiths et. al. does not provide the above steps, rather the authors provide the final form (below) with ϕ integrated out, i.e.

$$P(\mathbf{w} \mid \mathbf{z}) = \left(\frac{\Gamma(W\beta)}{\Gamma(\beta)^W}\right)^T \prod_{j=1}^{T} \frac{\prod_w \Gamma\left(n_j^{(w)} + \beta\right)}{\Gamma\left(n_j^{(\cdot)} + W\beta\right)},$$

A number of steps seem to missing which we attempt to uncover below by reverse engineering from the results to find forms that might be of interest.

$$
\begin{aligned}
P(w \mid z) =& \left(\frac{\Gamma(W\beta)}{\Gamma(\beta)^W}\right)^{\top} \prod_{j=1}^{T} \frac{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)}{\Gamma\left(n_j^{(.)} + W\beta\right)} \\
&= \int_\phi \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi \\
&= \underbrace{\prod_{j=1}^{T} \frac{\Pi_w \Gamma\left(n_j^{(w)} + \beta\right)}{\Gamma\left(n_j^{(.)} + W\beta\right)} \cdot \prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)}}_{\text{multiply by 1.}} \int_\phi \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi \\
&= \prod_{j=1}^{T} \frac{\Pi_w \Gamma\left(n_j^{(w)} + \beta\right)}{\Gamma\left(n_j^{(.)} + W\beta\right)} \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})} \int_\phi \prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi
\end{aligned}
$$

and if $\left(\frac{\Gamma(W\beta)}{\Gamma(\beta)^W}\right)^{\top} = \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})}$ then this would imply that $\int_\phi \prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi$ integrates to 1, but this seems potentially reasonable since $\prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)}$ seems to resemble a product that would yield $\frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})}$ i.e. we would be integrating over the support of the updated $\phi \sim Dir(\beta + \mathbf{\Sigma w_i})$. However in the interest of not speculating, we can again "multiply by 1".

$$
\begin{aligned}
P(w \mid z) &= \prod_{j=1}^{T} \frac{\Pi_w \Gamma\left(n_j^{(w)} + \beta\right)}{\Gamma\left(n_j^{(.)} + W\beta\right)} \cdot \prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)} \int_\phi \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi \\
&= \prod_{j=1}^{T} \frac{\Pi_w \Gamma\left(n_j^{(w)} + \beta\right)}{\Gamma\left(n_j^{(.)} + W\beta\right)} \left(\frac{\Gamma(W\beta)}{\Gamma(\beta)^W}\right)^{\top} \int_\phi \left(\frac{\Gamma(W\beta)}{\Gamma(\beta)^W}\right)^{-T} \prod_{j=1}^{T} \frac{\Gamma\left(n_j^{(.)} + W\beta\right)}{\Pi_w \Gamma\left(n_j^{(\omega)} + \beta\right)} \frac{\Gamma(\Sigma(\beta_i + \Sigma \mathbf{w_i}))}{\Pi\, \Gamma(\beta_i + \Sigma \mathbf{w_i})} \Pi \phi_i^{(\beta_i + \Sigma \mathbf{w_i}) - 1} d\phi
\end{aligned}
$$

implying that everything in the integral integrates to one. We do not find very much intuition to what we've found, but felt an attempt to fill in the holes not presented within the literature mentioned might be useful, and could be worth returning to at a later point.

Note that $n_j^{(.)}$ is the number of times word w was assigned to topic j in vector of assignments $\mathbf{z}$ above.

Similarly, we note that $p(\mathbf{z})$ is the same as integrating θ out of the joint $p(\mathbf{z}, \theta) = p(\mathbf{z}|\theta)p(\theta)$, but just as before we have a Multinomial-Dirichlet conjugacy and we can procede in the same manner as before to obtain

$$
P(\mathbf{z}) = \left(\frac{\Gamma(T\alpha)}{\Gamma(\alpha)^T}\right)^{D} \prod_{d=1}^{D} \frac{\prod_j \Gamma\left(n_j^{(d)} + \alpha\right)}{\Gamma\left(n^{(d)} + T\alpha\right)}
$$

The authors then jump to provide the form of the full conditional for $p(z_i|\mathbf{z_{-i}}, \mathbf{w})$ such that

$$
P(z_i = j \mid \mathbf{z}_{-i}, \mathbf{w}) \propto \frac{n_{-i,j}^{(w_i)} + \beta}{n_{-i,j}^{(\cdot)} + W\beta} \frac{n_{-i,j}^{(d_i)} + \alpha}{n_{-i,n}^{(d_i)} + T\alpha}
$$

Where at first glance we can notice a blend of the forms from $P(\mathbf{w}|\mathbf{z})$ and $P(\mathbf{z})$ on a univariate-like scale as we might expect since the full conditional is proportionate to the joint, and the joint is equal to the product of $P(\mathbf{w}|\mathbf{z})P(\mathbf{z})$. Given all of the $\mathbf{z_{-i}}$ are constants, we note that many terms / counts can be cancelled out through the proportionate sign, however, we see information from the words still included. Of course, this would make sense as we are thinking of the words as a likelihood function within our Bayesian machinery. Moreover, we note that the jth marginal of a Dirichlet distribution is a Beta distribution, and in this vein our jth topic full conditional might be seen as proportionate to the product of two Beta distributions (where again, we have cancelled out terms through the proportionate sign). Note that $n_{-i}^{(\cdot)}$ is the count that does not include current assignment ot z_i. The authors note that this is actually a relatively effecient calculation since we just have to sum a relatively small number of nonzero counts.

Moreover, the full conditional has an intuitive nature such that the left term can be interpreted as the probability of w_i under topic j, and the right term can be interpreted as the probability of topic j in document d_i. Of course, once our chain has converged, we can take our samples to be independent samples of the posterior topic marginals.

Lastly, Griffiths et. al. then discuss the estimation of θ_j and ϕ_j. The authors note their estimates

$$\hat{\phi}_j^{(w)} = \frac{n_j^{(w)}+\beta}{n_j^{(\cdot)}+W\beta}$$
$$\hat{\theta}_j^{(d)} = \frac{n_j^{(d)}+\alpha}{n^{(d)}+T\alpha}$$

correspond to the predictive distributions over new words w and new topics z, conditioned on $\mathbf{w}$ and $\mathbf{z}$, but leaves it for the reader to infer how. It's not precisely clear what they mean, but we postulate they could be describing finding the mean of the posterior predictive draws. If so, we illustrate the idea noting that $p(w_{+1}|\mathbf{w})$ is the same as integrating $\mathbf{z}$ out from $p(w_{+1}, \mathbf{z}|\mathbf{w})$ which is the same as integrating out $\mathbf{z}$ from $p(w_{+1}|\mathbf{z}, \mathbf{w})p(\mathbf{z}|\mathbf{w})$. Since we have the samples from $p(\mathbf{z}|\mathbf{w})$, and since we can presumably sample from w_{+1} now that we are conditioning on $\mathbf{z}$, we can apply Monte Carlo sampling to approximate the posterior predictive distribution for w_{+1}. From here we can take the average of these samples to represent the estimate of the posterior predictive mean as $\hat{\phi}$. Of course, the estimates provided above are with respect to the jth $\hat{\phi}_j$ which requires essentially the same procedure. Note that we can follow the same method to obtain the estimate $\hat{\theta}_j$.

Backtracking in time, Blei et. al. resorted to variational bayes to overcome intractability issues, whereas others have resorted to expectation propogation. We still note that Blei et. al. discuss a key component surrounding the concept of exchangeability when calculating $p(\mathbf{w}, \mathbf{z})$. These authors claim that since the topics are infinitely exchangeable within a document in LDA, then by de Finetti's representation theorem, the joint distribution of the topics conditioned on θ are then *independent and identically distributed*, such that since

$$p(z_1, \ldots, z_N) = p(z_{\pi(1)}, \ldots, z_{\pi(N)})$$

then

$$p(\mathbf{w}, \mathbf{z}) = \int p(\theta)\left(\prod_{n=1}^{N} p(z_n \mid \theta)p(w_n \mid z_n)\right)d\theta$$

In other words, why it is possible to express the joint as a product in the manner presented above. This notion of exchangeability and di Finetti's theorem was not mentioned within Griffiths, but we thought it important to discuss as it was a critical tool in developing the LDA model within Blei et. al. from which Griffiths et. al. built upon.

Clearly, the approach in Griffiths et. al. provides a huge speed up as we can integrate out the parameters of the latent variables and simply sum a relatively small number of nonzero counts in our quest to obtain posterior samples. However, one of the disadvantages to this algorithm is the number of full conditionals that need to be sampled. Following the work from Griffiths et. al. we note that Porteous et. al. have developed a way to "reduce the inner-loop" or reducing the number of K topics to sample from to a much smaller number of full conditionals. The argument is based on the notion that "for any particular word and document, the sampling distributions of interst are frequently skewed such that most of the probability mass is concentrated on a small fraction of the total number of topics K." The authors provide an alogorithm "Fast LDA", which takes advantage of this concentrated probability notion mentioned above.

We draw slightly on the works of Porteous et. al., noting that the authors claim that this concentration occurs “for most real data sets after several iterations of the Gibbs sampler”, and note their experiment with the New York Times Corpus, and the very different PubMed Corpus. Despite the very different nature of the Corpora, the findings were quite similar as illustrated below

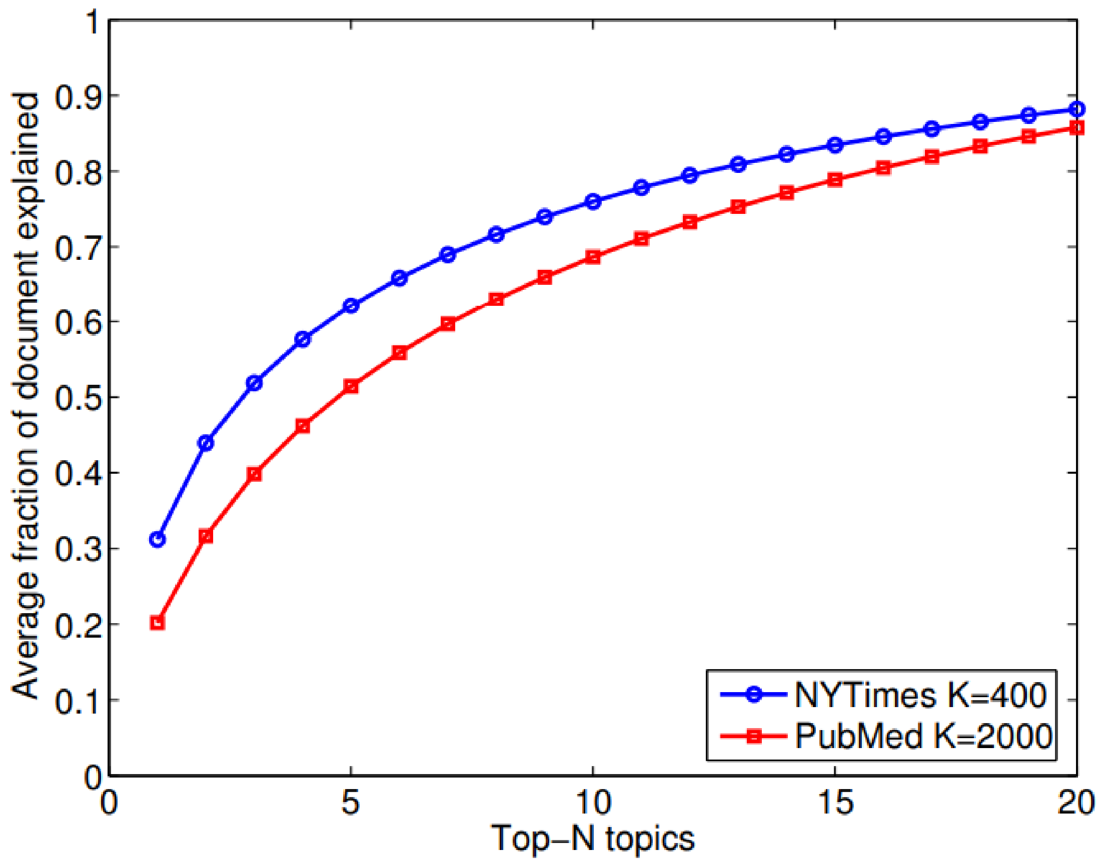


Figure 4: Average fraction of a document explained by top-20 topics, for NYTimes (K=400 topics) and PubMed (K=2000 topics). We see that, on average, the top-20 topics in any document account for approximately 90% of the words in the document.

We note that while our model draws slightly on Porteous et. al., we are manually setting our topic number which may not be the most advantageous aspect of our model. One might look into the BNP approaches that allow for allow for the number of topics to grow. Moreover, we still have to sample each topic whereas the Porteous et. al. model is drastically faster since it does not have to do this (and peforms even better with even larger K).

For Griffiths et. al. the purpose of the algorithm was centered on the title of the paper. That is, they were focused on “Finding Scientific Topics”. Griffiths et. al. used Bayesian model selection to establish the number of topics (where as we choose 20), and they illustrated that the extracted topics from a corpora of abstracts from PNAS was capable of providing meaningful structure in the data consistent with the class designations provided by the authors of the articles. This is quite useful for experts as a “first-order approximation to the kind of knowledge avialable” within a corpora of documents. In other words, it can help scientists decide whether or not to invest valuable time investigating a particular corpora.

In a similar vein, we might also like to apply the same approach to single papers. Knowing in advance what topics largely explain a particular topic can also save time.

Known possible applications also include identifying “hot topics” as Griffiths and Steyvers point out by examining dynamics and tagging abstracts to illustrate semantic content.

These are concepts that could help anyone hoping to better expend their valuable research time. In this vein, whatever the topic area might be, it might be wise to begin the research process by first running an LDA model on a few relevant corpora before digging in one's specified research.

Moreover, population genetics, machine learning, and social networks are just a few of the many applications for which LDA can be applied.

3. Algorithm

3.1. High Level Description

The goal is to generate samples from posterior topic marginals through collapsed Gibbs sampling in order to then find estimates of $\hat{\phi}_{wk}$ and $\hat{\theta}_{kj}$ where the former is the probability of word w occuring in topic k and the later is the probability of topic k occuring in the jth document. Noting that $\hat{\phi}_{\mathbf{k}}$ represents the estimate for the distribution of words in topic k and $\hat{\phi}_{\mathbf{d}}$ represents the estimate for the distribution of topics in document d. We attempted to elaborate on the commentary of predictive distributions very briefly stated in Griffiths et. al., whereas no elaboration was provided in Porteous et. al. That said, the forms provide a gateway into their underlying intuition as we are weighting in the prior α and β respectively in combination with the counts of interest to update the underlying vector components of the parameters characterizing the multinomial distributions generating words and topics.

In general, the first major notion is to underscore the idea of reinforcing a certain "stickyness" for similar topics within the same document - i.e. to place a higher probability on a new word having an assignment to a topic already popular within the same document.

The second idea is to encourage a "stickyness" for words belonging to the same topic - i.e. words often assigned to the same topic should have more probability of being assigned to said topic (for new instances of the same word).

In sum, the overal goal was getting samples from the posterior marginals via collapsed Gibbs sampling, and in the process utilize counts to estimate the parameters of interest. We spell out the specifics below.

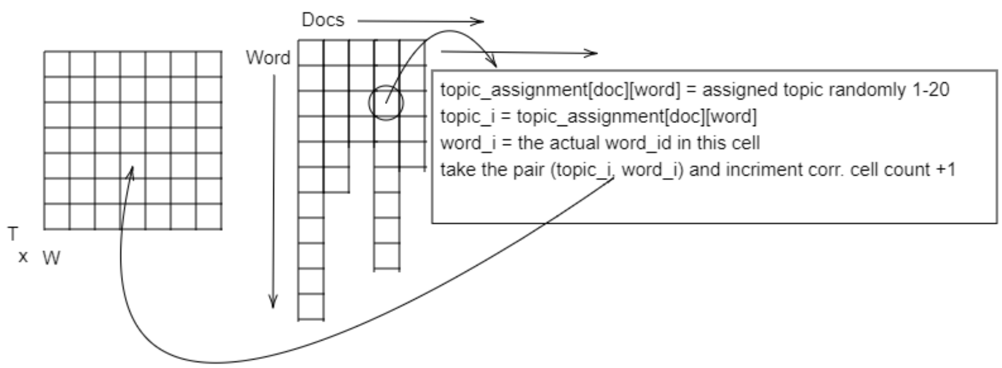
3.2. Detailed Description

Before moving onto the sampler, we should discuss *** (tokenizing etc...)

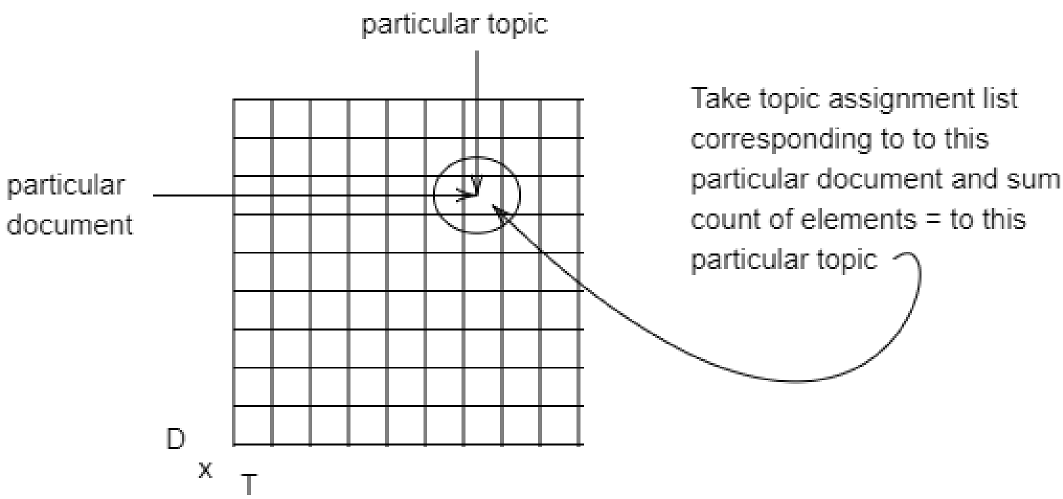
Now we discuss the algorithm. Before running the sampler, we want to note the structure of the "word-topic count matrix" and the "document-topic count matrix".

- T is total topics
- W is total words

Here is a diagram of how the $\mathbf{word-topic}$ count matrix works (left image)



Whereas the **document − topic** matrix functions as:



and the algorithmic details are such that:

Algorithm 1: Setting up

```
Result: pre-Gibbs preparation
initialization;
- clean data (lower case, remove punctuation etc.)
- docs ← list of lists: sublists correspond to document-specific words
- assign an id to each word:
vocabulary ← empty set
i ← 0
for d in documents do
    for word in document do
        if word in vocabulary then
            continue
        end
        vocabulary[word] ← i
        i ← i + 1
    end
end
docs id ← replace all words with their id's
ta ← list of lists: sublists correspond document word-topic assign.
W ← number of words in vocabulary
D ← number of documents
T ← number of topics = 20 **change to K**
- Create word − topic count matrix:
wt ← T by W empty matrix
for document in D do
    for word in document do
        ta[document][word] ← random choice 1-20
        ti = temp topic id ← ta[document][word]
        wi = temp word id ← docs id[document][word]
        wt[ti, wi] ← +1
    end
end
- Create document − topic count matrix:
dt ← D by T empty matrix
for document in D do
    for topic in T do
        dt[document,word] ← Σ#{elements of ta[document] == t}
    end
end
```

Algorithm 2: Collapsed Gibbs Sampling

Result: full conditional topic samples
initialization;
$\alpha = 1$ $\beta = .001$ iterations = ?
RUN SAMPLER : iterate *1000(?)* times through the chain:
for *i in iterations* **do**
  for *document in D* **do**
    for *word in document* **do**
      initialize topic assignment to token w:
      t0 $\leftarrow$ ta[d][w]
      token id:*
      wid $\leftarrow$ docs id[document][word]
      remove token w (when sampling for token w):*
      dt[document, t0] $\leftarrow$ dt[document,t0] - 1
      wt[t0,wid] $\leftarrow$ wt[t0,wid] - 1
      $p_{dt} \leftarrow \frac{(dt[document,:]+\alpha)}{\Sigma dt[document,:]+T*\alpha}$
      $p_{wt} \leftarrow \frac{(wt[:,wid]+\beta)}{wt.sum(axis=1)+W*\beta}$
      - where:
      unnormalized full conditional $p(z)_{ij}^{unnormalized} \leftarrow p_{dt} * p_{wt}$
      - and:
      $p(z)_{ij} \leftarrow \frac{p(z)_{ij}^{unnormalized}}{\Sigma p(z)_{ij}^{unnormalized}} = \frac{n_{-i,j}^{(w_i)}+\beta}{n_{-i,j}^{(\cdot)}+W\beta} \frac{n_{-i,j}^{(d_i)}+\alpha}{n_{-i,n}^{(d_i)}+T\alpha}$
      - draw topic for word n from multinomial using probabilites just calculated:
      t1 $\leftarrow$ np.random.choice(range(T),p $= p(z)_{ij}$
      ta[document][word] $\leftarrow$ t1
      - re-increment with new topic assignment:
      dt[document,t1] $\leftarrow$ dt[document,t1] + 1 wt[t1,wid] $\leftarrow$ wt[t1,wid] + 1
    end
  end
end

Lastly, we can estimate the parameters of interest:

Algorithm 3: Estimation

Result: obtain estimates
$\theta \leftarrow \frac{(dt+\alpha)}{(dt+\alpha).sum(axis=1[:,None]}$
$\phi \leftarrow \frac{wt+\beta}{(wt+\beta).sum(axis=1[:,None]}$
- # words in each topic:
$\phi_{\mathbf{rounded}} \leftarrow round(\phi)$
for *t, topic* **in** *enumerate($\phi_{rounded}$)* **do**
  print("Topic .....???....")
end
- # topics in each document
...

(next page)

Where we ultimately have obtained estimates:

$$\hat{\phi}_j^{(w)} = \frac{n_j^{(w)}+\beta}{n_j^{(\cdot)}+W\beta}$$
$$\hat{\theta}_j^{(d)} = \frac{n_j^{(d)}+\alpha}{n^{(d)}+T\alpha}$$

4. Optimization for Performance

5. Applications to Simulated Data Sets

6. Applications to Real Data Sets

7. Comparative Analysis with Competing Algorithms

8. Discussion / Conclusion

9. References / Bibliography

[1] Blei, D. M., Ng, A. Y. & Jordan, M. I. (2003) J. Machine Learn. Res. 3, 993-1022.

[2] T. L. Griffiths and M. Steyvers. Finding scientific topics. Proc Natl Acad Sci $USA, 101$ Suppl $1 : 5228 - 5235$, April 2004.

[3] I. Porteous, D. Newman, A. Ihler, A. Asuncion, P. Smyth, and M. Welling. Fast collapsed gibbs sampling for latent dirichlet allocation. In KDD '08: Proceeding of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining, pages $569 - -577$, New York, NY, USA, 2008 . ACM. 8

[4] F. Zhao, X. Ren, S. Yang, Q. Han, P. Zhao, and X. Yang, "Latent dirichlet allocation model training with differential privacy," IEEE Transactions on Information Forensics and Security, vol. 16, pp. $1290 - 1305, 2021$.

10. Appendix

In our original plans, we hoped to make this algorithm differentially private. However, limited time and certain issues within the algorithms forced us to abandon this angle for now.

We refrain from a deeper discussion of differential privacy, but note how Zhao et. al.[4] refer to the definition as a randomized mechanism $\mathcal{M} : D \to Y$ such that for any neighboring datasets D, D' that differ by only one record, and for any output $S \subseteq Y$ there is

$$\Pr[\mathcal{M}(D) \in S] \leq \exp(\epsilon) \cdot \Pr\left[\mathcal{M}\left(D'\right) \in S\right]$$

constitutes achieving ϵ-differential privacy, where ϵ controls the privacy level (this is the common definition). The authors then proceed to illustrate how to achieve ϵ-differential privacy wthin a collapsed Gibbs sampler by injecting Laplacian noise on each iteration (also protecting against word count leakage), and by exploiting the fact that collapsed Gibbs sampling has some inherent privacy (helping to protect sampled topics). The full set up is

α, β	hyper-parameters for LDA
$\mathcal{W}$	word space
$\mathcal{K}$	topic space
ϕ_k	topic-word distribution for topic k
ϕ_k^t	probability that word t is generated by topic k
n_m^k	count of words with topic k in document d_m
n_k^t	count of word t with topic k in corpus D
n_t	total word count of $t \in D$
p_k	probability that topic k is sampled
w_r	replaced word
$\mathbf{w}^+$	set of related words
$\mathbf{w}^-$	set of unrelated words
ϵ_i^r	inherent privacy loss on w_r in the i -th iteration
$\epsilon_i^t, \epsilon_i^{t'}$	inherent privacy loss on related words t or t'

Where the algorithm is below

Algorithm 1: HDP-LDA

Input: Document corpus D, Prior parameters α, β, Topic number K, Clipping bound C

Output: Trained topic-word distribution Φ, Privacy loss $\epsilon = T \cdot (\epsilon_L + \epsilon_I)$

// **Initialization**
1 **for** $d_m \in D$ **do**
2   **for** $w = t \in d$ **do**
3   Sample topic: $k \sim Mult(\frac{1}{K} \cdot \mathbf{I}_K)$;
4   Initialize word counts n_k^t and n_m^k;
5   **end**
6 **end**
// **Collapsed Gibbs Sampling**
7 Set $iter = 0$;
8 **while** $iter < T$ **do**
9   Add noise to each n_k^t independently: $n_k^t \leftarrow n_k^t + \eta, \quad \eta \sim Lap(2/\epsilon_L)$;
10   **for** $d \in D$ **do**
11   **for** $w = t \in d$ **do**
12   Clip: $(n_k^t)^{temp} \leftarrow \min\{n_k^t, C\}$;
13   Compute sampling distribution $\mathbf{p}$: $p_k \propto \frac{(n_k^t)^{temp}+\beta}{\sum_{t=1}^{V}(n_k^t+\beta)} \cdot \frac{n_m^k+\alpha}{\sum_{k=1}^{K}(n_m^k+\alpha)}$;
14   Compute inherent privacy loss: $\epsilon_I \leftarrow 2\log(\frac{C}{\beta} + 1)$;
15   Sample topic and update word count n_k^t;
16   **end**
17   **end**
18   $iter \leftarrow iter + 1$;
19 **end**
20 Compute the trained model Φ;

where noticably we see a difference in line 9 where Laplacian noise is added, and line 12 and 14 concerning clipping and inherent privacy loss (on each iteration, summed in total $= \epsilon_I$). Testing utilized defending against *Topic-based Attacks* below.

Algorithm 3: Topic-based Attack Algorithm

Input: The attacked word w_{mn}
Output: The inferred result $\bar{t}$
Set $i = 0$;
while $i < T$ **do**
    Add noise: $(\hat{n_k^t})_i \leftarrow n_k^t + \eta, \quad \eta \sim Lap(2/\epsilon_L)$;
    Record $(\hat{n_k^t})_i$;
    for $w \in D$ **do**
        Sample topic: $k_i \sim \mathbf{P}$;
        if $w = w_{mn}$ **then**
            Record k_i;
        end
    end
    $i \leftarrow i + 1$;
end
Compute $\bar{t} = \arg\max_t P[w_{mn} = t | k_1, ..., k_i, ..., k_T]$ acc. to Equation (24);

Where:

$$
\begin{aligned}
&\Pr(w_{mn} = t \mid k_1, \ldots, k_i, \ldots, k_T) \\
&= \tfrac{\Pr(w_{mn}=t, k_1, \ldots, k_i, \ldots, k_T)}{\Pr(k_1, \ldots, k_i, \ldots, k_T)} \\
&= \tfrac{\prod_i \Pr(w_{mn}=t, k_i)}{\prod_i \Pr(k_i)} = \prod_i \tfrac{\Pr(w_{mn}=t, k_i)}{\Pr(k_i)} \\
&\approx \prod_i \frac{\left(n_{k_i}^{\hat{t}}\right)_i / \sum_{k,t} \left(\hat{n}_k^t\right)_i}{\sum_t \left(n_{k_i}^{\hat{t}}\right)_i / \sum_{k,t} \left(\hat{n_k}^t\right)_i} = \prod_i \frac{\left(n_{k_i}^{\hat{t}}\right)_i}{\sum_t \left(n_{k_i}\right)_i}
\end{aligned}
$$

And while the above may seem relatively reasonable, how the authors arrived at their clipping bound C is slightly unclear. However, they did claim that for for a small number of topics, we can let C be n_t, the total count of word t in D. Moreover, the use of $Lap(2/\epsilon_I)$ is slightly confusing since one of the parameters is implicit, but not explicitly mentioned in the surrounding text. If the assumption is that $\mu = 0$, then it is possible to sample negative values when adding this to the original counts. When we attempted to inject noise as described above into our alogorithm, our resulting probability vector for the topic multinomial had negative entries. We wonder if this was a consequence of the potentially negative laplacian noise added to the counts. To circumvent this, we thought of shifting the Laplace right, taking the absolute value of the noise, drawing extra counts from a Poisson distribution (which also seems more in line with adding noise to a discrete variable), however, everything broke our algorithm in the same manner. Of course, this was a good experience to build some familiarity with techniques capable of altering algorithms to satisfy differential privacy definitions.