The Cantor-Schröder-Bernstein Theorem

A formal proof in Lean4

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Table of contents

Theorem and appreciation

Theorem (Cantor-Schröder-Bernstein)

Let α and β be two sets. If there exist injective functions $f : \alpha \to \beta$ and $g : \beta \to \alpha$, then there exists a bijective function $h : \alpha \to \beta$.

Example (7)

Let $4 = \{0, 1, 2, 3\}$, $A := 4^{\mathbb{N}}$ and $B := \mathcal{P}(\mathbb{N})$. Here is a bijection:

$$h(x) := \{2n | q(x, n) \equiv 1\} \cup \{2n + 1 | p(x, n) \equiv 1\}$$

where p, q are defined as

$$\begin{aligned} p(x, n) &:= (x(n) \equiv 1(mod4)) \vee (x(n) \equiv 3(mod4)), \\ q(x, n) &:= (x(n) \equiv 2(mod4)) \vee (x(n) \equiv 3(mod4)). \end{aligned}$$

If we want to use the theorem, here are the injections:

$$f : A \to B, x \mapsto \{p_n^{x(n)} | n \in \mathbb{N}, p_n \text{ is prime}\}$$

$$g : B \to A, S \mapsto (\lambda n \to 1, \text{if } n \in S;\ 0,\ \text{else}).$$

Proof

One way to prove the theorem is to construct h [1].

Some observations:

- We have f and g^{-1} from α to β.
- When their co-domains are restricted, both f, g are bijective.
- We need to partition α so that we can make use of f and g^{-1}.

[1] Another possible approach is discussed on the "Remarks" section.

Proof

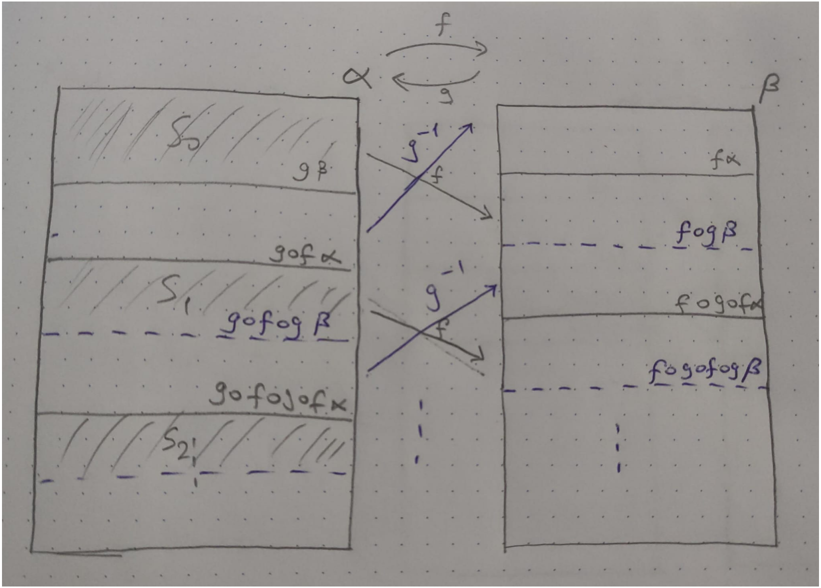

Figure: Partition of α

Proof

Define the sets S_n for $n \in \mathbb{N}$ recursively as follows:

$$S_0 = \alpha \setminus g(\beta),$$
$$S_(n+1) = g(f(S_n)).$$

Let S $:= \bigcup_{n=0}^{\infty} S_n$.

Define the function $h : \alpha \to \beta$ as follows:

$$h\ x := \begin{cases} f\ x & \text{,if } x \in \mathrm{S}, \\ g^{-1}\ x & \text{,if } x \notin \mathrm{S}. \end{cases}$$

Proof

Now, we need to verify that h is bijective:

- Injectivity: Suppose $h\ x = h\ y$. We need to show $x = y$.
- Surjectivity: For every $y \in \beta$, we need to find an $x \in \alpha$ such that $h\ x = y$.
  - $g\ y \in S$
  - $g\ y \notin S$

Axioms

We can see the axioms used in our theorem by typing "#print axioms schroeder-bernstein".

1. Classical.choice
2. propext
3. Quot.sound

   From 1, 2 and "Quot"; we get "funext":

4. funext

   1, 2 and 4 implies LEM:

5. LEM

Some remarks

- On the axiom of choice
  - Cantor's proof is a consequence of "the linear order of cardinal numbers" which was later shown by Hartogs to be equivalent to the AC [3].
  - The proof we use here assumes the AC (in the parts related with "invFun")
  - There is a proof (from König) not relying on the AC [3].
- On the law of excluded middle
  - $\forall p$: Prop, p v $\neg p$
  - LEM can be derived from Classical.choice (+ propext + funext) and is not explicitly shown among the axioms [4].
  - No constructive proof: CSB implies LEM [5].
- A similar theorem (CSB-Escardó) for Types [6].

Sources

1. For the code and the proof: https://leanprover-community.github.io/mathematics_in_lean/
2. For theorem finding: https://leanprover-community.github.io/mathlib4_docs/
3. https://en.wikipedia.org/wiki/Schröder-Bernstein_theorem
4. https://lean-lang.org/theorem_proving_in_lean4/axioms_and_computation.html
5. Chad E Brown, Cécilia Pradic. Cantor-Bernstein implies Excluded Middle. 2023. ffhal-02103517v3f
6. https://unimath.github.io/agda-unimath/foundation.cantor-schroder-bernstein-escardo.html
7. Ünlü, Özgün. Bilkent University Math 123 Midterm-2, 2023.
8. Christian Merten (Heidelberg Uni.) & Elif Üsküplü (Indiana Uni.)