Lagrange Polynomials

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Definitions 1

Basis

Let V be a vector space over a field K and $v_1, \ldots, v_n \in V$. $\{v_1, \ldots, v_n\}$ are a basis of V iff $\forall v \in V : v = \sum_{i=1}^{n} \alpha_i v_i$ and $\{v_1, \ldots, v_n\}$ are linearly independent. With $\forall i \in 1 \ldots n : \alpha_i \in K$. [1]

Lagrange Polynomials

Let $\{x_0, \ldots, x_n\}$ be a set of values with $x_i \neq x_j$ if $i \neq j$. For this set of values, we can define Lagrange polynomials: $\{\ell_0(x), \ldots, \ell_n(x)\}$ with

$$\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}.$$ [2]

Definitions 2

Dimension

The cardinality of a basis. The maximal cardinality of a set of vectors that can be linearly independent. [1]

P_n

P_n is a vector space of polynomials defined as $P_n = \{p \in R[X] : \deg(p) \leq n\}$ with $n \in \mathbb{N}_0$ where $R[X]$ is the ring of polynomials.[2]

Proofs 1

Theorem 1

$\ell_j(x_i) = \delta_{ij}$

Proof.

Let $x_i \in \{x_0, \dots, x_n\}$ and $j \neq i$ then we conclude:

$$\ell_j(x_i) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^{i-1} \frac{x_i - x_k}{x_j - x_k} \cdot 0 \cdot \prod_{\substack{k=i+1 \\ k \neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = 0$$

$$\ell_j(x_j) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x_j - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^{n} 1 = 1$$

$\square$

Proofs 2

Theorem 2

Monomials from degree 0 to n make up the basis of the vector space of polynomials of degree $x \leq n$, also called P_n, with $n \in \mathbb{N}_0$.

Proof.

Because monomials have different degrees, and hence can't cancel each other out. Therefore:

$$0 = \sum_{i=0}^{n} \alpha_i x^i \Leftrightarrow \alpha_i = 0 \; i \in 0 \dots n$$

Using monomials, we can also construct any polynomial up to the degree of n because

$$\forall p \in P_n : p = \sum_{i=0}^{n} \alpha_i x^i$$

because of how polynomials are defined. Hence, the monomials form a basis of P_n. □

Because we require $n+1$ monomials, we can say that $\dim(P_n) = n+1$.

Proofs 3

Theorem

For the set of values $\{x_0, \ldots x_n\}$*, the corresponding Lagrange polynomials* $\{\ell_0, \ldots \ell_n\}$ *will be linearly independent in* P_n*.*

Proofs 4

Proof.

- Proof that $\ell_j \in P_n$:
  $\ell_j = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}$ hence the polynomial is defined by some constant $\prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{1}{x_j - x_k}$ and n linear factors $(x - x_k)$ each multiplication of a linear factor with another raises the degree by one $\Rightarrow \deg(\ell_j(x)) = n \Rightarrow \ell_j \in P_n$
- Proof of linear independence:
  Let's assume $\{\ell_0, \dots \ell_n\}$ are linearly dependent and use the δ_{ij} property of ℓ_j

$$\exists j \in \{1, \dots, n\} : \ell_j(x) = \sum_{\substack{i=0 \\ i\neq j}}^{n} \alpha_i \ell_i(x) \Rightarrow \ell_j(x_j) = \sum {}_{i\neq j}^{i=0^n} \alpha_i \ell_i(x_j)$$

$$\Rightarrow 1 = \ell_j(x_j) = \sum {}_{i\neq j}^{i=0^n} \alpha_i \underbrace{\ell_i(x_j)}_{=0} = 0 \lightning$$

$\Rightarrow$ Lagrange polynomials are linearly independent

$\square$

Proofs 5

Theorem

Let V be a vector space, $M = \{v_1, \ldots, v_n\}$, $\#M = \dim(V)$, $M \subset V$ and $\{v_1, \ldots, v_n\}$ are linearly independent, then M is a basis of V.

Proof.

Let there be one more vector in V that is linearly independent of vectors in M, then $\dim(V) \geq n+1$ due to how the dimension is defined, which is a contradiction.
We now know that $\forall u \in V$ $\{v_1, \ldots, v_n, u\}$ will be linearly dependent.

$$\Rightarrow \forall u \in V \exists \alpha_i \in K \forall i \in 1, \ldots, n : u = \sum_{i=1}^{n} \alpha_i v_i$$

And, because vectors in M are linearly independent, M must be the basis of V, because of how the basis is defined. □

Proofs 6

Theorem

Lagrange polynomials $\{\ell_0, \ldots, \ell_n\}$ *construct a basis of* P_n *for* $n \in \mathbb{N}_0 : n < \infty$.

Proof.

We know that $\#\{\ell_0, \ldots, \ell_n\} = n + 1 = \dim(P_n)$. The Lagrange polynomials are linearly independent and $\{\ell_0, \ldots, \ell_n\} \subset P_n$ and according to our previous theorem, $\{\ell_0, \ldots, \ell_n\}$ is a basis of P_n. $\square$

Sources

[1] Brown William A. (1991), Matricies and vector spaces, New York M. Dekker, ISBN 978-0-8247-8419-5 page 107

[2] Rannacher, R. 2017. Numerik 0: Einführung in die Numerische Mathematik. Heidelberg University Publishing. Page 24