Convergence of $\sqrt[n]{n}$ in Lean4

Sophie Weber, David Leeb

16.07.2024

Definitions

Definition (Convergence of a sequence)

A sequence (a_n) converges to x if for every $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ we have $|a_m - x| < \epsilon$.

Lean Code

```
def ConvergesTo (a : ℕ → ℝ) (x : ℝ) : Prop :=
    ∀ ε > 0, ∃ (n : ℕ), ∀ m ≥ n, |a m - x| < ε
```

Lemmas

Lemma (Convergence of a constant sequence)

Let (a_n) be a constant sequence with $a_n = x$ for all n. Then (a_n) converges to x.

Lean Code

```
theorem of_constant (x : ℝ) : ConvergesTo (fun _ ↦ x) x
```

Lemmas

Lemma (Sandwich theorem)

If (a_n), (b_n) and (c_n) are sequences and there exists an $n \in \mathbb{N}$ such that $a_m \leq b_m \leq c_m$ for all $m \geq n$, and both (a_n) and (c_n) converge to x, then (b_n) converges to x.

Lean Code

```
theorem sandwich (a b c : ℕ → ℝ)
(h : ∃ (n : ℕ), ∀ m ≥ n , a m ≤ b m ∧ b m ≤ c m) (x : ℝ)
(ha : ConvergesTo a x) (hc : ConvergesTo c x) :
    ConvergesTo b x
```

Proof

Example

The sequence (a_n) defined by $a_n = \sqrt[n]{n}$ converges to 1.

Proof.

Let (a_n) be the sequence defined by $a_n = \sqrt[n]{n}$.

Remember: Sandwich Theorem
We need to find two sequences (b_n) and (c_n) such that $b_m \leq a_m \leq c_m$ for all $m \geq n$, and both (b_n) and (c_n) converge to 1.

Proof

Proof continued.

Note that for any positive integer $n \geq 1$, we have $1 \leq \sqrt[n]{n}$. This follows from:

$$\begin{aligned} 1 &\leq n \\ \Leftrightarrow 1^n &\leq \sqrt[n]{n}^n \\ \Leftrightarrow 1 &\leq \sqrt[n]{n} \end{aligned}$$

From that we can see, that the sequence (b_n) can be defined by the constant sequence $b_n = 1$ for all n.
From the inequality $1 \leq \sqrt[n]{n}$ we can also derive the equality

$$\sqrt[n]{n} = 1 + d_n$$

for a suitable sequence (d_n) given by: $d_n := \sqrt[n]{n} - 1$

Proof

Proof continued.

To find the second sequence (c_n), we want to show that $\sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n-1}}$.
For $n \geq 2$ this follows from:

$$\begin{aligned} n = \sqrt[n]{n}^n &= (1 + d_n)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \cdot d_n^k \\ &\geq \binom{n}{2} \cdot d_n^2 \\ &= \frac{n!}{(n-2)! \cdot 2!} \cdot d_n^2 \\ &= \frac{n \cdot (n-1)}{2} \cdot d_n^2 \end{aligned}$$

Proof

Proof continued.

What we have so far:

$$n \geq \frac{n \cdot (n-1)}{2} \cdot d_n^2$$

From that we can derive:

$$\Leftrightarrow d_n^2 \leq \frac{2}{n-1}$$

$$\Leftrightarrow d_n \leq \sqrt{\frac{2}{n-1}}$$

Conclusion:

$$\sqrt[n]{n} = 1 + d_n \leq 1 + \sqrt{\frac{2}{n-1}}$$

Proof

Proof continued.

Now we have the sequences (b_n) and (c_n):

$$b_n = 1$$

$$c_n = 1 + \sqrt{\frac{2}{n-1}}$$

They fulfill the condition $b_n \leq a_n \leq c_n$ for all $n \geq 2$:

$$1 \leq \sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n-1}}$$

Proof

Proof continued.

We can easily see, that they both converge to 1:

$$\lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} \left(1 + \sqrt{\frac{2}{n-1}}\right) = 1$$

We can now apply the Sandwich Theorem to conclude that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □

Bibliography

- Forster, O. *Analysis I*. Springer Spektrum, 2015.