Jordan Normal Form and how to find it

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$$A = TDT^{-1} \quad \leftrightarrow \quad D = T^{-1}AT$$

$$A^k = TD^kT^{-1}$$

$$D^k = \begin{bmatrix} (d_{1,1})^k & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (d_{n,n})^k \end{bmatrix}$$

JNF extends diagonalisation concept to nondiagonisable matrixes

$$A = TJT^{-1}$$

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & \\ & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} \lambda_n & 1 & \\ & \lambda_n & 1 \\ & & \lambda_n \end{matrix}} \end{pmatrix}$$

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} \implies J^k = \begin{bmatrix} J_0^k & 0 \\ 0 & J_1^k \end{bmatrix} \implies TJ^kT^{-1} = A^k$$

Camille Jordan stated jordan decomposition theorem in 1870

If $T : V \to V$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of V of the form
$u_1, Tu_1, \ldots, T^{a_1-1}u_1, \ldots, u_k, Tu_k, \ldots, T^{a_k-1}u_k$ where $T^{a_i}u_i = 0$
$1 \leq i \leq k$.

Proof by induction on dimV:

- Assume that $\dim V \geq 1$
- $T^m(V) = \cdots = T(V) = V$ contradict. $\implies$ $T(V)$ properly contained in V
- find $v_1, \ldots, v_l \in T(V)$ so that
  $v_1, Tv_1, \ldots, T^{b_1-1}v_1, \ldots, v_l, Tv_l, \ldots, T^{b_l-1}v_l$
  is a basis for $T(V)$ and $T^{b_i}v_i = 0$ for $1 \leq i \leq l$.
- For $1 \leq i \leq l$ choose $u_i \in V$ such that $Tu_i = v_i$
- extend $T^{b_1-1}v_1, \ldots, T^{b_l-1}v_l$ to basis of $\ker(T)$ by adding vectors $w_1, \ldots, w_m$
- assume $u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m$ is basis for V
- see linear independence by applying T to relation between the vectors
- spans V because of rank-nullity theorem
  $\dim V = (b_1 + 1) + \ldots + (b_l + 1) + m$

Constructing jordan normal form

Find eigenvalues $\det(\lambda I - A) = 0$ their algebraic multiplicity $\alpha(\lambda)$ through characteristic Polynomial $p_A(x) = \det(xI - A)$ Example:

$p_A(x) = (x-1)(x+3)^2(x-2)^4$

$$\begin{pmatrix} \boxed{1} & & & & & & \\ & \boxed{\begin{matrix} -3 & \\ & -3 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{matrix}} \end{pmatrix}$$

$$jordanblock : \begin{array}{|cccc|} 2 & 1? & 0 & 0 \\ 0 & 2 & 1? & 0 \\ 0 & 0 & 2 & 1? \\ 0 & 0 & 0 & 2 \end{array}$$

$$jordanbox : \boxed{\begin{matrix} \lambda_n & 1 & 0 & 0 \\ 0 & \lambda_n & 1 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \lambda_n \end{matrix}}$$

number of boxes in block = geometric multiplicity of λ_n

$$\dim(\ker(A - \lambda_n I))$$

order of blocks in JNF and boxes in block irrelevant

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & \\ & \boxed{2} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} \end{pmatrix}$$

sizes of boxes = sizes of jordan chains: calculate increasing powers until term = algebraic multiplicity:

$$\dim(\ker(A - \lambda_n I)^n)$$

number of Jordan chains with length of at least n =

number of boxes that have at least size n =

$$\dim(\ker(A - \lambda_n I)^n) - \dim(\ker(A - \lambda_n I)^{n-1})$$

Calculating its transformation matrix

$$A = TJT^{-1}$$

- for each eigenvalueλ_n:
- take lowest p for dim$(\ker(A - \lambda_n I)^p) = \alpha(\lambda_n)$
- find $a_p \in \ker(A - \lambda_n I)^p \quad$ but $\quad a_p \notin \ker(A - \lambda_n I)^{p-1}$
- calculate recursively $a_{p-1} := (A - \lambda_n I)a_p$

repeat with $b_p \in \ (A - \lambda_n I)^p$ with p as high as possible such that

$$b_p \notin \text{span}\{\ker(A - \lambda_n I)^{p-1} \cup in\, 4\, calculated\}$$

$$T = (\begin{array}{|c|c|c|c|c|c|c|} \hline \lambda_1(a_1) & ... & \lambda_1(a_p) & \lambda_1(b_1) & ... & \lambda_n(a_1) & ... \\ \hline \end{array})$$

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Quellen:

- Beweis: Mark Wildon
- https://www.youtube.com/watch?v=1vIgCbjToXM
- https://www.youtube.com/watch?v=GVixvieNnyc
- https://www.youtube.com/watch?v=TSdXJw83kyA
- https://en.wikipedia.org/wiki/Jordan_normal_form
- https://www.youtube.com/watch?v=uHW2zThZDEw