Lagrange Polynomials

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16 July 2024

Definitions 1

Basis

let V be a vectorspace above a field K and $v_1 \dots v_n \in V$ $\{v_1 \dots v_n\}$ are a Basis of V iff $\forall v \in V : v = \sum_{i=1}^{n} \alpha v_i$ and $\{v_1 \dots v_n\}$ are linearly indipendant. With $\forall i \in 1 \dots n : \alpha_i \in K$

lagrange poylnomials

let $\{x_0 \dots x_n\}$ be a set of values with $x_i \neq x_j$ if $i \neq j$. For this set of values we can define lagrange polynomials: $\{\ell_0(x) \dots \ell_n(x)\}$ with $\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}$

Definitions 2

Dimension

the cardinality of a basis. The maximal cardinality of a set, of vectors, that can be linearly indipendant.

P_n

P_n is a vectorspace of polynomials defined as $P_n = \{p \in R[X] : deg(p) \leq n\}$ with $n \in \mathbb{N}_0$ where $R[X]$ is the ring of polynomials

Proofs 1

theorem 1

$\ell_j(x_i) = \delta_{ij}$

Proof.

let $x_i \in \{x_0 \dots x_n\}$ and $j \neq i$ then we conclde:

$$\ell_j(x_i) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = \prod_{\substack{k=0\\k\neq j}}^{i-1} \frac{x_i - x_k}{x_j - x_k} \cdot 0 \cdot \prod_{\substack{k=i+1\\k\neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = 0$$

$$\ell_j(x_j) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{x_j - x_k}{x_j - x_k} = \prod_{\substack{k=0\\k\neq j}}^{n} 1 = 1$$

□

Proofs 2

theorem 2

monoms from degree 0 to n make up the basis of the vectorspace of Polynomials of degree $x \leq n$, also called P_n, with $n \in \mathbb{N}_0$

Proof.

Because monoms have different degrees, and hence can't cancel eachother out. Therefore:

$$0 = \sum_{i=0}^{n} \alpha_i x^i \Leftrightarrow \alpha_i = 0 \ i \in 0 \ldots n$$

using monoms we can also construct any polynomial up to the degree of n becuase

$$\forall p \in P_n : p = \sum_{i=0}^{n} \alpha_i x^i$$

because of how polynomials are definied. Hence the monoms form a basis of P_n $\square$

because we require $n+1$ monoms we can say, that $dim(P_n) = n+1$

Proofs 3

Theorem

for the set of values $\{x_0, \dots x_n\}$ *the according lagrange polynomials* $\{\ell_0, \dots \ell_n\}$ *will be linearly indipendant in* P_n

Proofs 4

Proof.

- proof that $\ell_j \in P_n$:
  $\ell_j = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$ hence the polynomial is defined by some constant $\prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{1}{x_j - x_k}$ and n linear factors $(x - x_k)$ each multiplication of a linear factor with another raises the degree by one $\Rightarrow deg(\ell_j(x)) = n \Rightarrow \ell_j \in P_n$
- proof linear indipendance:
  lets assume $\{\ell_0, \ldots \ell_n\}$ are linearly dipendant. and use the δ_{ij} propertly of ℓ_j

$$\exists j \in \{1, \ldots, n\} : \ell_j(x) = \sum_{\substack{i=0 \\ i \neq j}}^{n} \alpha_i \ell_i(x) \Rightarrow \ell_j(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^{n} \alpha_i \ell_i(x_j)$$

$$\Rightarrow 1 = \ell_j(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^{n} \alpha_i \underbrace{\ell_i(x_j)}_{=0} = 0 \lightning$$

$\Rightarrow$ lagrange polynomials are linearly independant

$\square$

Theorem

Let V be a vectorspace,$M = \{v_1, \ldots, v_n\}$ $\#M = dim(V)$ $M \subset V$ and $\{v_1, \ldots, v_n\}$ are linearly indipendant, then M is a basis von V

Proof.

let there be one more vector in V that is linearly independant to vectors in M then $dim(V) \geq n + 1$ due to how the dimension is defined. Which is a contradiction.
we now know that $\forall u \in V$ $\{v_1, \ldots, v_n, u\}$ will be linearly dipendant.

$$\Rightarrow \forall u \in V \exists \alpha_i \in K \forall i \in 1, \ldots, n : u = \sum_{i=1}^{n} \alpha_i v_i$$

and, because vecotrs in M are linearly indipendant M must be the basis of V, because of how the basis is defined $\square$

Theorem

Lagrange polynomials construct a basis of P_n lagrange polynomials $\{\ell_0, \ldots, \ell_n\}$ construct a basis of P_n for a $n \in \mathbb{N}_0 : n < \infty$

Proof.

we know that $\#\{\ell_0, \ldots, \ell_n\} = n + 1 = dim(P_n)$. The Lagrange polynomials are linearly indipendant and $\{\ell_0, \ldots, \ell_n\} \subset P_n$ and according to our previous theorem $\{\ell_0, \ldots, \ell_n\}$ is a basis of P_n $\square$