QR decomposition using Gram-Schmidt

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Gram-Schmidt

Theorem

Let $\mathbf{V}$ *be a finite dimensional vector space with basis* $\mathbb{B} = \{b_1, b_2, \ldots, b_n\}$. *Define*

$$\mathbf{u}_k = \mathbf{b}_k - \sum_{j=1}^{k-1} \langle \mathbf{b}_k, \mathbf{c}_j \rangle * \mathbf{c}_j, \qquad \mathbf{c}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|} \qquad \textit{for } k = 1, \ldots, n$$

Then: all $\mathbf{u}_k \neq 0$ *and* $\{c_1, c_2, \ldots, c_n\}$ *is orthonormal basis of* $\mathbf{V}$.

Proof by induction.

BS: $\mathbf{u}_1 = \mathbf{b}_1 \neq 0,$ because $b_1, b_2, \ldots, b_n$ are linearly independent

$$\Rightarrow \|\mathbf{u}_1\| \neq 0 \Rightarrow \mathbf{c}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \text{ is well defined and has norm 1}$$

and $\operatorname{span}\{\mathbf{c}_1\} = \operatorname{span}\{\mathbf{b}_1\}$

IH: $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ is an orthonormal basis of $\operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Proof by induction.

IS: $k \mapsto k+1$

Let $\mathbf{u}_{k+1} = \mathbf{b}_{k+1} - \sum_{j=1}^{k} \langle \mathbf{b}_{k+1}, \mathbf{c}_j \rangle \, \mathbf{c}_j$ and $\mathbf{u}_{k+1} \neq 0$, otherwise:

$$\mathbf{b}_{k+1} = \sum_{j=1}^{k} \langle \mathbf{b}_{k+1}, \mathbf{c}_j \rangle \, \mathbf{c}_j \in \operatorname{span}\{\mathbf{c}_1, \dots, \mathbf{c}_k\} = \operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$$

$\Rightarrow \mathbf{b}_1, \dots, \mathbf{b}_{k+1}$ are linearly dependent $\lightning$

$$\Rightarrow \mathbf{c}_{k+1} = \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|}$$

Proof by induction.

Claim: $c_1, \ldots c_{k+1}$ are orthonormal, which follows because:

$$\|\mathbf{c}_j\| = 1 \text{ for } j = 1, \ldots, k+1 \text{ and } \langle \mathbf{c}_j, \mathbf{c}_i \rangle = 0 \text{ for } j \neq i \text{ and } j, i \leq k$$

Let $j \in 1, \ldots, k$:

$$\text{to show } \langle \mathbf{c}_j, \mathbf{c}_{k+1} \rangle = 0, \text{ it's enough to show: } \langle \mathbf{c}_j, \mathbf{u}_{k+1} \rangle = 0$$

Proof by induction.

For that:

$$
\begin{aligned}
\langle \mathbf{c}_j, \mathbf{u}_{k+1} \rangle &= \langle \mathbf{c}_j, \mathbf{b}_{k+1} - \sum_{l=1}^{k} \langle \mathbf{b}_{k+1}, \mathbf{c}_l \rangle * \mathbf{c}_l \rangle \\
&= \langle \mathbf{c}_j, \mathbf{b}_{k+1} \rangle - \sum_{l=1}^{k} \langle \mathbf{b}_{k+1}, \mathbf{c}_l \rangle * \langle \mathbf{c}_j, \mathbf{c}_l \rangle \\
&= \langle \mathbf{c}_j, \mathbf{b}_{k+1} \rangle - \langle \mathbf{b}_{k+1}, \mathbf{c}_j \rangle * 1 \\
&= 0
\end{aligned}
$$

and $\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_{k+1}\} = \operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_{k+1}\}$ $\square$

QR decomposition

Theorem

For $A \in \mathbb{R}^{n\times n}$, $rank(A) = n$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n\times n}$ such that $A = QR$.

Proof.

Let $A = (a_1, \dots, a_n)$ be the column vectors of the matrix A. Since A is regular, the vectors a_i are linearly independent and form a basis of $\mathbb{R}^n$. Let $\{q_1, \dots, q_n\}$ be the orthonormal basis constructed by the Gram-Schmidt process from $\{a_1, \dots, a_n\}$. Then, by $Q = (q_1, \dots, q_n) \in \mathbb{R}^{n\times n}$, an orthogonal matrix is given. The matrix $R := Q^T A$ is regular, and due to the previously proven Gram-Schmidt Theorem, the entries of R satisfy:

$$r_{ij} = \langle q_i, a_j \rangle = 0 \quad \forall j < i.$$

This means: R is an upper triangular matrix. $\square$