# Lagrange Polynomials

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# Definitions 1

## Basis

let $V$ be a vectorspace above a field $K$ and $v_1 \dots v_n \in V$ $\{v_1 \dots v_n\}$ are a Basis of $V$ iff $\forall v \in V : v = \sum_{i=1}^{n} \alpha v_i$ and $\{v_1 \dots v_n\}$ are linearly indipendant. With $\forall i \in 1 \dots n : \alpha_i \in K$

## lagrange poylnomials

let $\{x_0 \dots x_n\}$ be a set of values with $x_i \neq x_j$ if $i \neq j$. For this set of values we can define lagrange polynomials: $\{\ell_0(x) \dots \ell_n(x)\}$ with $\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}$

# Definitions 2

## Dimension

the cardinality of a basis. The maximal cardinality of a set, of vectors, that can be linearly indipendant.

## $P_n$

$P_n$ is a vectorspace of polynomials defined as $P_n = \{p \in R[X] : deg(p) \leq n\}$ with $n \in \mathbb{N}_0$ where $R[X]$ is the ring of polynomials

# Proofs 1

**theorem 1**

$\ell_j(x_i) = \delta_{ij}$

**Proof.**

let $x_i \in \{x_0 \dots x_n\}$ and $j \neq i$ then we conclde:

$$\ell_j(x_i) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = \prod_{\substack{k=0\\k\neq j}}^{i-1} \frac{x_i - x_k}{x_j - x_k} \cdot 0 \cdot \prod_{\substack{k=i+1\\k\neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = 0$$

$$\ell_j(x_j) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{x_j - x_k}{x_j - x_k} = \prod_{\substack{k=0\\k\neq j}}^{n} 1 = 1$$

□

# Proofs 2

**theorem 2**

monoms from degree 0 to $n$ make up the basis of the vectorspace of Polynomials of degree $x \leq n$, also called $P_n$, with $n \in \mathbb{N}_0$

**Proof.**

Because monoms have different degrees, and hence can't cancel eachother out. Therefore:

$$0 = \sum_{i=0}^{n} \alpha_i x^i \Leftrightarrow \alpha_i = 0 \; i \in 0 \ldots n$$

using monoms we can also construct any polynomial up to the degree of n becuase

$$\forall p \in P_n : p = \sum_{i=0}^{n} \alpha_i x^i$$

because of how polynomials are definied. Hence the monoms form a basis of $P_n$ $\square$

because we require $n + 1$ monoms we can say, that $dim(P_n) = n + 1$

# Proofs 3

**Theorem**

*for the set of values* $\{x_0, \dots x_n\}$ *the according lagrange polynomials* $\{\ell_0, \dots \ell_n\}$ *will be linearly indipendant in* $P_n$

**Proof.**

- proof that $\ell_j \in P_n$:
  $\ell_j = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}$ hence the polynomial is defined by some constant $\prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{1}{x_j - x_k}$ and $n$ linear factors $(x - x_k)$ each multiplication of a linear factor with another raises the degree by one $\Rightarrow deg(\ell_j(x)) = n \Rightarrow \ell_j \in P_n$
- proof linear indipendance:
  lets assume $\{\ell_0, \dots \ell_n\}$ are linearly dipendant. and use the $\delta_{ij}$ propertly of $\ell_j$

$$\exists j \in \{1, \dots, n\} : \ell_j(x) = \sum_{\substack{i=0 \\ i\neq j}}^{n} \alpha_i \ell_i(x) \Rightarrow \ell_j(x_j) = \sum_{\substack{i=0 \\ i\neq j}}^{n} \alpha_i \ell_i(x_j)$$

$$\Rightarrow 1 = \ell_j(x_j) = \sum_{\substack{i=0 \\ i\neq j}}^{n} \alpha_i \underbrace{\ell_i(x_j)}_{=0} = 0 \lightning$$

$\Rightarrow$ lagrange polynomials are linearly independant

$\square$

# Proofs 6

**Theorem**

*Let $V$ be a vectorspace, $\#\{v_1, \ldots, v_n\} = dim(V)$ $\{v_1, \ldots, v_n\} \subset V$ and $\{v_1, \ldots, v_n\}$ are linearly indipendant, then $\{v_1, \ldots, v_n\}$ is a basis von $V$*

**Proof.**

$\square$

# Proofs 6

**Theorem**

*Lagrange polynomials construct a basis of $P_n$ lagrange polynomials $\{\ell_0, \ldots, \ell_n\}$ construct a basis of $P_n$ for a $n \in \mathbb{N}_0 : n < \infty$*

**Proof.**

we know that $\#\{\ell_0, \ldots, \ell_n\} = n + 1 = dim(P_n)$. The Lagrange polynomials are linearly indipendant and $\{\ell_0, \ldots, \ell_n\} \subset P_n$ and according to one of our previous theorems $\{\ell_0, \ldots, \ell_n\}$ is a basis of $P_n$ $\square$