

# Lagrange Polynomials

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16 July 2024

# Definitions 1

## Basis

Let  $V$  be a vector space over a field  $K$ .  $B := \{v_1, \dots, v_n\} \subset V$  is a basis of  $V$  if and only if:

- vectors in  $B$  are linearly independent
- $\forall v \in V : \exists \alpha_i \in K : \forall i \in 1, \dots, n : v = \sum_{i=1}^n \alpha_i v_i$

[1]

## Lagrange Polynomials

Let  $\{x_0, \dots, x_n\} \subset \mathbb{R} : \forall i, j \in 0, \dots, n : x_i \neq x_j$  if  $i \neq j$ . For this set of values, we can define Lagrange polynomials:  $\{\ell_0(x), \dots, \ell_n(x)\}$  with

$$\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^n \frac{x - x_i}{x_j - x_i}. \quad [2]$$

## Definitions 2

### Dimension

The cardinality of a basis. The maximal cardinality of a set of vectors that can be linearly independent. [1]

### $P_n$

$P_n$  is a vector space of polynomials defined as  $P_n = \{p \in R[X] : \deg(p) \leq n\}$  with  $n \in \mathbb{N}_0$  where  $R[X]$  is the ring of polynomials.[2]

# Proofs 1

## Theorem 1

$$\ell_j(x_i) = \delta_{ij}$$

## Proof.

Let  $x_i, x_j \in \{x_0, \dots, x_n\}$  and  $j \neq i$  then we conclude:

$$\ell_j(x_i) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x_i - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^{i-1} \frac{x_i - x_k}{x_j - x_k} \cdot 0 \cdot \prod_{\substack{k=i+1 \\ k \neq j}}^n \frac{x_i - x_k}{x_j - x_k} = 0$$

$$\ell_j(x_j) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x_j - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^n 1 = 1$$



# Proofs 2

## Theorem 2

Monomials of degree 0 to  $n$  form a basis of  $P_n$ . with  $n \in \mathbb{N}_0$ .

## Proof.

we can evaluate the polynomial at  $x = 0$  for the polynomial and all of its derivatives

$$\begin{aligned}
 0 &= \sum_{i=0}^n \alpha_i x^i \Rightarrow 0 = \sum_{i=1}^n \alpha_i \cdot 0 + \alpha_0 \Rightarrow \alpha_0 = 0 \\
 \Rightarrow \frac{d}{dx} 0 &= 0 = \sum_{i=1}^n i \alpha_i x^{i-1} \Rightarrow 0 = \sum_{i=2}^n i \alpha_i \cdot 0 + \alpha_1 \Rightarrow \alpha_1 = 0
 \end{aligned}$$

repeat the argument till all of  $\alpha_i$  are evaluated  $\Rightarrow \forall i \in 0, \dots, n : \alpha_i = 0 \Rightarrow$  the polynomials are linearly independent



## Proofs 3

Proof.

Using monomials, we can also construct any polynomial up to the degree of  $n$  because

$$\forall p \in P_n : p = \sum_{i=1}^n \alpha_i x^i$$

because of how polynomials are defined. Hence, the monomials form a basis of  $P_n$



Because we require  $n + 1$  monomials to construct any polynomial in  $P_n$ , we can say that  $\dim(P_n) = n + 1$ .

# Proofs 4

## Theorem

For  $\{x_0, \dots, x_n\} \subset \mathbb{R}$ , the corresponding Lagrange polynomials  $\{\ell_0, \dots, \ell_n\}$  will be linearly independent in  $P_n$ .

## Proof.

- Proof that  $\forall j \in 0 \dots n : \ell_j \in P_n$

$$\begin{aligned} \deg(\ell_j) &= \deg\left(\prod_{\substack{k=0 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}\right) = \deg\left(\prod_{\substack{k=0 \\ k \neq j}}^n (x - x_k)\right) \\ &= \sum_{\substack{k=0 \\ k \neq j}}^n \deg(x - x_k) = \sum_{\substack{k=0 \\ k \neq j}}^n 1 = n \\ &\Rightarrow \ell_j \in P_n \end{aligned}$$



# Proofs 5

## Proof.

- Proof of linear independence:

Let's assume  $\{\ell_0, \dots, \ell_n\}$  is linearly dependent and use the  $\delta_{ij}$  property of  $\ell_j$

$$\exists j \in \{1, \dots, n\} : \ell_j(x) = \sum_{\substack{i=0 \\ i \neq j}}^n \alpha_i \ell_i(x) \Rightarrow \ell_j(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^n \alpha_i \ell_i(x_j)$$

$$\Rightarrow 1 = \ell_j(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^n \alpha_i \underbrace{\ell_i(x_j)}_{=0} = 0 \neq 1$$

$\Rightarrow$  Lagrange polynomials are linearly independent





# Proofs 6

## Theorem

Let  $V$  be a vector space,  $M = \{v_1, \dots, v_n\}$ ,  $\#M = \dim(V)$ ,  $M \subset V$  and  $\{v_1, \dots, v_n\}$  are linearly independent, then  $M$  is a basis of  $V$ .

## Proof.

Let there be one more vector in  $V$  that is linearly independent of vectors in  $M$ , then  $\dim(V) \geq n + 1$  due to how the dimension is defined, which is a contradiction. We now know that  $\forall u \in V \setminus \{v_1, \dots, v_n, u\}$  will be linearly dependent.

$$\Rightarrow \forall u \in V : \forall i \in 1, \dots, n : \exists \alpha_i \in K : u = \sum_{i=1}^n \alpha_i v_i$$

And, because vectors in  $M$  are linearly independent,  $M$  must be the basis of  $V$ , because of how the basis is defined. □

## Proofs 7

## Theorem

*Lagrange polynomials  $\{\ell_0, \dots, \ell_n\}$  are a basis of  $P_n$  for  $n \in \mathbb{N}_0$ .*

## Proof.

We know that  $\#\{\ell_0, \dots, \ell_n\} = n + 1 = \dim(P_n)$ . The Lagrange polynomials are linearly independent and  $\{\ell_0, \dots, \ell_n\} \subset P_n$ . according to our previous theorem,  $\{\ell_0, \dots, \ell_n\}$  is a basis of  $P_n$ . □

# Sources

- [1] Brown William A. (1991), Matricies and vector spaces, New York M. Dekker, ISBN 978-0-8247-8419-5 page 107
- [2] Rannacher, R. 2017. Numerik 0: Einführung in die Numerische Mathematik. Heidelberg University Publishing. Page 24