The Spectral Theorem

An implementation in SageMath

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Quick reminder on "special" matrices

- For a matrix A over a field $K \in \{\mathbb{R}, \mathbb{C}\}$, we introduce the notation $A^* := (\overline{A})^\intercal$, i.e. the complex conjugate transpose of A.

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- Of course, over $K = \mathbb{R}$, this reduces to $A^* = A^\intercal$

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  (b) orthogonal over $\mathbb{R}$ / unitary over $\mathbb{C}$
  $\iff A^* \cdot A = A \cdot A^* = \mathbb{I}$
  (c) self-adjoint (symmetric over $\mathbb{R}$ / Hermitian over $\mathbb{C}$)
  $\iff A^* = A$
- Note that both orthogonal / unitary and self-adjoint already imply normal.

The Spectral Theorem over $\mathbb{C}$

Let $n \in \mathbb{N}$. For $A \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ a matrix, it holds:

(a) A normal $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{C}^n$ consisting of the eigenvectors of A. Furthermore, then there exist $P, D \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ such that:

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- P is unitary and of the form $P = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}$.
- D is diagonal and of the form

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

with $\lambda_1, \dots, \lambda_n$ the eigenvalues of A.

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$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

  with $\lambda_1, \dots, \lambda_n$ the eigenvalues of A.
- $P \cdot D \cdot P^* = A$ and $P^* \cdot A \cdot P = D$.

The Spectral Theorem over $\mathbb{C}$

(b) A unitary $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{C}^n$ such that (a) holds and for all eigenvalues λ of A, we have $|\lambda| = 1$.

The Spectral Theorem over $\mathbb{C}$

(b) A unitary $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{C}^n$ such that (a) holds and for all eigenvalues λ of A, we have $|\lambda| = 1$.

(c) A self-adjoint (Hermitian) $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{C}^n$ such that (a) holds and all eigenvalues λ of A are real.

The Spectral Theorem over $\mathbb{R}$

Let $n \in \mathbb{N}$. For $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ a matrix, it holds:

(a) A normal $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$ such that there exist $P, D \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that:

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- P is orthogonal and of the form

$$P = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}.$$

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(a) A normal $\iff \exists$ ONB $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ such that there exist $P, D \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that:

- P is orthogonal and of the form
$$P = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}.$$
- $\exists s, t \in \mathbb{N}$ such that D is of the form
$$D = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_s & & & \\ & & & A_1 & & \\ & & & & \ddots & \\ & & & & & A_t \end{pmatrix}$$

The Spectral Theorem over $\mathbb{R}$

with $\lambda_1, ..., \lambda_s \in \mathbb{R}$ the (real) eigenvalues of A and $\forall i = 1, \dots, t$:

$$A_i = \begin{pmatrix} \mu_i & -\nu_i \\ \nu_i & \mu_i \end{pmatrix}$$

with $\mu_i, \nu_i \in \mathbb{R}$.

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with $\mu_i, \nu_i \in \mathbb{R}$.

- $P \cdot D \cdot P^\intercal = A$ and $P^\intercal \cdot A \cdot P = D$.

The Spectral Theorem over $\mathbb{R}$

(b) A orthogonal $\iff$ $\exists$ ONB $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ such that (a) holds and $\forall i = 1, \dots, s : \lambda_i \in \{-1, 1\}$ and

$$A_i = \begin{pmatrix} \cos(\phi_i) & \sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{pmatrix}$$

with $\phi_i \in (0, \pi)$ $\forall i = 1, \dots, t$.

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with $\phi_i \in (0, \pi)$ $\forall i = 1, \ldots, t$.

(c) A self-adjoint (symmetric) $\iff \exists$ ONB $(b_1, \ldots, b_n)$ of $\mathbb{R}^n$ such that (a) holds and $s = n$, i.e. $(b_1, \ldots, b_n)$ consists of the eigenvectors of A and D is a diagonal matrix.

Implementation in SageMath

- Now, we are going to see the implentation of the Spectral Theorem in SageMath.

Bibliography

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