Lagrange Polynomials

Simon Binder, Mihail Prudnikov

16 July 2024

Intrudu ction

What is the presentaiton about ?

- Lagrange polynomials are used for polynom interpolation:
  you can interpolate any polynomial of degree n exactly knowing only $n+1$ of its points
- very useful in numreical analysis
- We will define lagrange polynomails and show why they form a basis of P_n
- Our References: Rannacher R: introduction to numerics script
  Brown Wiliam A: Matricies and vectorspaces

Definitions 1

Basis

Let V be a vector space over a field K. $B := \{v_1, \ldots, v_n\} \subset V$ is a basis of V if and only if:

- B is linearly independent
- $\forall v \in V : v = \sum_{i=1}^{n} \alpha_i v_i \ \alpha_i \in K : \forall i \in 1, \ldots, n : :$

[1]

Lagrange Polynomials

Let $\{x_0, \ldots, x_n\} \subset \mathbb{R} : \forall i, j \in 0, \ldots, n : x_i \neq x_j$ if $i \neq j$. For this set of values, we can define Lagrange polynomials: $\{\ell_0(x), \ldots, \ell_n(x)\}$ with

$$\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}.$$ [2]

Definitions 2

Dimension

The cardinality of a basis. [1]

Polynomials

A polynomial of a degree $n \in \mathbb{N}$ is defined as:

$$p = \sum_{i=0}^{n} \alpha_i X^i \qquad \alpha_i \in \mathbb{R}$$

with $p \in \mathbb{R}[X]$ where X is the indeterminante

P_n

P_n is a vector space of polynomials of bounded degree, defined as
$P_n = \{p \in R[X] : \deg(p) \leq n\}$ with $n \in \mathbb{N}_0$ where $R[X]$ is the ring of polynomials.[2]

Proofs 1

Theorem 1

$\forall i, j \in 1, \ldots, n : \ell_j(x_i) = \delta_{ij}$

Proof.

Let $x_i, x_j \in \{x_0, \ldots, x_n\}$ and $j \neq i$ then we conclude:

$$\ell_j(x_i) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^{i-1} \frac{x_i - x_k}{x_j - x_k} \cdot 0 \cdot \prod_{\substack{k=i+1 \\ k \neq j}}^{n} \frac{x_i - x_k}{x_j - x_k} = 0$$

$$\ell_j(x_j) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x_j - x_k}{x_j - x_k} = \prod_{\substack{k=0 \\ k \neq j}}^{n} 1 = 1$$

□

Proofs 2

Theorem 2

Monomials of degree 0 to n form a basis of P_n. with $n \in \mathbb{N}_0$.

Proof.

we can evaluate the polynomial at $x = 0$ for the polynomial and all of its derivatives

$$0 = \sum_{i=0}^{n} \alpha_i x^i \Rightarrow 0 = \sum_{i=1}^{n} \alpha_i \cdot 0 + \alpha_0 \Rightarrow \alpha_0 = 0$$

$$\Rightarrow \frac{d}{dx} 0 = 0 = \sum_{i=1}^{n} i \alpha_i x^{i-1} \Rightarrow 0 = \sum_{i=2}^{n} i \alpha_i \cdot 0 + \alpha_1 \Rightarrow \alpha_1 = 0$$

repeat the argument till all of α_i are evaluated $\Rightarrow \forall i \in 0, \ldots, n : \alpha_i = 0 \Rightarrow$ the polynomials are linearly independant

□

Proofs 3

Proof.

Using monomials, we can also construct any polynomial up to the degree of n because

$$\forall p \in P_n : p = \sum_{i=1}^{n} \alpha_i x^i \qquad \alpha_i \in K$$

because of how polynomials are defined. Hence, the monomials form a basis of P_n $\square$

Because we require $n+1$ monomials to span P_n, we can say that $\dim(P_n) = n+1$.

Proofs 4

Theorem

For $\{x_0, \dots x_n\} \subset \mathbb{R}$, as defined above, the corresponding Lagrange polynomials $\{\ell_0, \dots \ell_n\}$ will be linearly independent in P_n.

Proof.

- Proof that $\forall j \in 0 \dots n : \ell_j \in P_n$

$$\begin{aligned}
deg(\ell_j) =& deg\left(\prod_{\substack{k=0\\k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}\right) = deg\left(\prod_{\substack{k=0\\k\neq j}}^{n} (x - x_k)\right) \\
=& \sum_{\substack{k=0\\k\neq j}}^{n} deg(x - x_k) = \sum_{\substack{k=0\\k\neq j}}^{n} 1 = n \\
\Rightarrow & \ell_j \in P_n
\end{aligned}$$

□

Proofs 5

Proof.

- Proof of linear independence:

$$0 = \sum_{i=0}^{n} \alpha_i \ell_i \Rightarrow \sum_{i=0}^{n} \alpha_i \ell_i(x_j) = 0$$
$$\Rightarrow \alpha_j \ell_j(x_j) = 0 \Rightarrow \alpha_j = 0 \text{ because } \ell_j(x_j) = 1$$

because $j \in \{0, \ldots n\}$ was arbitrary but constant we get $\alpha_j = 0 \quad \forall j \in \{0, \ldots n\}$ thus we can conclude linear independance

$\square$

Proofs 6

Theorem

Let V be a vector space, $M = \{v_1, \dots, v_n\}$, $\#M = \dim(V)$, $M \subset V$ and $\{v_1, \dots, v_n\}$ are linearly independent, then M is a basis of V.

Proof.

Let there be one more vector in V that is linearly independent of vectors in M, then $\dim(V) \geq n + 1$ due to how the dimension is defined, which is a contradiction.
We now know that $\forall u \in V$ $\{v_1, \dots, v_n, u\}$ will be linearly dependent.

$$\Rightarrow \forall u \in V : u = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in K \quad i \in 1, \dots, n$$

And, because vectors in M are linearly independent, M must be the basis of V. $\square$

Proofs 7

Theorem

Set of Lagrange polynomials $\{\ell_0, \ldots, \ell_n\}$ *is a basis of* P_n *for* $n \in \mathbb{N}_0$.

Proof.

We know that $\#\{\ell_0, \ldots, \ell_n\} = n + 1 = \dim(P_n)$. The Lagrange polynomials are linearly independent and $\{\ell_0, \ldots, \ell_n\} \subset P_n$. according to our previous theorem, $\{\ell_0, \ldots, \ell_n\}$ is a basis of P_n. $\square$

Sources

[1] Brown William A. (1991), Matricies and vector spaces, New York M. Dekker, ISBN 978-0-8247-8419-5 page 107

[2] Rannacher, R. 2017. Numerik 0: Einführung in die Numerische Mathematik. Heidelberg University Publishing. Page 24