MA0301 Elementary discrete mathematics
Spring 2018

Solutions — exercise 8

Section 2.2

$\boxed{13}$ Verify that

$$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)],$$

for primitive statements p, q and r.

To save some space in the truth diagram we denote LHS $= [(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ and RHS $= [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$. As LHS $=$ RHS in table 1 we are done.

Table 1: Truth diagram for problem $\boxed{13}$

p	q	r	$p \leftrightarrow q$	$q \leftrightarrow r$	$r \leftrightarrow p$	$p \rightarrow q$	$q \rightarrow r$	$r \rightarrow p$	LHS	RHS
F	F	F	T	T	T	T	T	T	T	T
F	F	T	T	F	F	T	T	F	F	F
F	T	F	F	F	T	T	F	T	F	F
F	T	T	F	T	F	T	T	F	F	F
T	F	F	F	T	F	F	T	T	F	F
T	F	T	F	F	T	F	T	T	F	F
T	T	F	T	F	F	T	F	T	F	F
T	T	T	T	T	T	T	T	T	T	T

$\boxed{14}$ For primitive statements p, q,

a) verify that $p \rightarrow [q \rightarrow (p \wedge q)]$ is a tautology. (NOT PART OF THE EXERCISE)

b) verify that $(p \vee q) \rightarrow [q \rightarrow q]$ is a tautology by using the result from part **a)** along with the substitution rules and laws of logic.

As $q \rightarrow q$ is a tautology in itself, we have $(p \vee q) \rightarrow T_0$. Which is only false if $(p \vee q)$ is False and T_0 is True. However, T_0 is always True, as such $(p \vee q) \rightarrow [q \rightarrow q]$ is always True, and thus a tautology.

However, this proof does not use part **a)**. Let us remedy this with a second proof.

Steps	**Reasons**
$T_0 \Leftrightarrow p \to [q \to (p \wedge q)]$	Part **a)**
$\Leftrightarrow (p \vee q) \to [q \to ((p \vee q) \wedge q)]$	Substitution rule $p \to (p \vee q)$
$\Leftrightarrow (p \vee q) \to [q \to q]$	Absorption Laws $q \wedge (q \vee p) \Leftrightarrow q$

c) is $[p \vee q] \to [q \to (p \wedge q)]$ a tautology?

No. Let q be True and p False. Then $p \vee q$ is True. However $q \to (p \wedge q)$ is False as $p \wedge q$ is False and q is True. As such $[p \vee q]$ does not always imply $q \to (p \wedge q)$. This can also be seen from table 2.

Table 2: Truth table for $[p \vee q] \to [q \to (p \wedge q)]$ from problem $\boxed{14}$ part **a)**

p	q	$p \vee q$	$p \wedge q$	$q \to (p \wedge q)$	$[p \vee q] \to [q \to (p \wedge q)]$
F	F	F	F	T	T
F	T	T	F	F	F
T	F	T	F	T	T
T	T	T	T	T	T

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$\boxed{19}$ **c)** For $k \in \mathbb{Z}^+$ verify that $k^3 = \binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3}$.

By using the definition of the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$ a straight forward computation yields

$$\begin{aligned}
&\binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3}\\
=&\frac{k!}{3!(k-3)!} + 4\frac{(k+1)!}{3!(k-2)!} + \frac{(k+2)!}{3!(k-1)!}\\
=&\frac{k(k-1)(k-2)(k-3)!}{3!(k-3)!} + 4\frac{(k+1)k(k-1)(k-2)!}{3!(k-2)!} + \frac{(k+2)(k+1)k(k-1)!}{3!(k-1)!}\\
=&\frac{k}{3!}\Big[(k-1)(k-2) + 4(k+1)(k-1) + (k+2)(k+1)\Big]\\
=&\frac{k}{3!}\Big[(k^2-3k+2) + (4k^2-4) + (k^2+3k+2)\Big]\\
=&\frac{k}{3!}\Big[6k^2\Big] = k^3
\end{aligned}$$

which is what we wanted to show.

d) Use part **c)** to show that

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4} = \frac{n^2(n+1)^2}{4}$$

This problem is trivial once we have shown the Hockey-stick identity

$$\sum_{t=0}^{n} \binom{t}{k} = \sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}. \tag{1}$$

Using this identity gives us directly

$$\begin{aligned}
\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} \binom{k}{3} + 4\sum_{k=1}^{n} \binom{k+1}{3} + \sum_{k=1}^{n} \binom{k+2}{3} \\
&= \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4} \\
&= \frac{(n+1)n(n-1)(n-2)}{4!} + \frac{(n+2)(n+1)n(n-1)}{4!} + \frac{(n+3)(n+2)(n+1)n}{4!} \\
&= \frac{(n+1)n}{4!}\Big[(n-1)(n-2) + 4(n+2)(n-1) + (n+3)(n+2)\Big] \\
&= \frac{(n+1)n}{4!}\Big[(n^2-3n+2) + (4n^2+4n-8) + (n^2+5n+6)\Big] \\
&= \frac{(n+1)n}{4!}\Big[6n^2+6n\Big] = \left(\frac{n(n+1)}{2}\right)^2,
\end{aligned}$$

which is what we wanted to shown. In the last step we simply used that $6n^2+6n = 6n(n+1)$ and $6/4! = 3 \cdot 2/4 \cdot 3 \cdot 2 = 1/4$. The only thing missing to complete the proof is proving the hockey stick identity from equation (1). In most proofs Pascal's rule is used

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{2}$$

and while this identity can be proven both by direct computation or induction it is perhaps more intuitive to confirm it using Pascal's triangle in table 3.

Table 3: Pascal's triangle, rows 0 through 7. The hockey stick identity confirms, for example: for $n = 4$, $r = 1$: $1 + 2 + 3 = 6$; for $n = 6$, $r = 3$: $1 + 4 + 10 + 20 = 35$.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & \color{red}{1} & & 1 & & & & & & \\
 & & & & & 1 & & \color{red}{2} & & 1 & & & & & \\
 & & & & \color{blue}{1} & & 3 & & \color{red}{3} & & 1 & & & & \\
 & & & 1 & & \color{blue}{4} & & \color{red}{6} & & 4 & & 1 & & & \\
 & & 1 & & 5 & & \color{blue}{10} & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & \color{blue}{20} & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & \color{blue}{35} & & 35 & & 21 & & 7 & & 1
\end{array}$$

The elements in the n'th row is the sum of two elements from the $n-1$'th row. Take the last row in the table as a concrete example, then $1 = 0 + 1$, $7 = 1 + 6$, $21 = 6 + 15$, $35 = 15 + 20$ and so forth.

Base case: Let $n = r$;

$$\sum_{i=r}^{n} \binom{i}{r} = \sum_{i=r}^{r} \binom{i}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{r+1}.$$

Inductive step: Suppose, that for some $k \in \mathbb{N}$, $k \geq r$

$$\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1}.$$

Want to show that this implies that the identity holds for $k+1$

$$\sum_{i=r}^{k+1} \binom{i}{r} = \left(\sum_{i=r}^{k} \binom{i}{r} \right) + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1},$$

and the rest follows by induction. This proves equation (1) and thus concludes the proof.

e) Find $a, b, c, d \in \mathbb{Z}^+$ so that for any $k \in \mathbb{Z}^+$,

$$k^4 = a\binom{k}{4} + b\binom{k+1}{4} + c\binom{k+2}{4} + d\binom{k+3}{4}.$$

Some trial and error gives

$$k^4 = \binom{k}{4} + 11\binom{k+1}{4} + 11\binom{k+2}{4} + \binom{k+3}{4}.$$

This is just a special case of a more general theorem known as Worpitzky's Identity

$$k^n = \sum_{m=0}^{n-1} A(n,m) \binom{k+m}{n} \tag{3}$$

where $A(n,m)$ denotes the Eulerian numbers. One way to define these is by the following recurrence

$$A(n,m) = (m+1)A(n-1,m) + (n-m)A(n-1,m-1) \tag{4}$$

with initial condition $A(0,0) = 1$. While this recurrence can be used to prove equation (3) let's instead give a brief combinatorial proof of this identity.

Let σ be a permutation on n letters. We will call an index $1 \leq i \leq n$ an index of descent if $\sigma(i) > \sigma(i+1)$ or if $i = n$, i.e. a permutation will always end in a descent by our convention. Then our numbers $A(n,k)$ counts the total number of permutations on n letters with precisely k indices of descent Eulerian numbers with slightly shifted indices.

Now we define the notion of a *barred permutation*. A barred permutation on n letters with k bars is a permutation with precisely k bars inserted into the permutation with the restriction that there must be at least one bar inserted between each descent. Note that this means there must always be a bar ending the permutation.

For example, the barred permutations on 3 letters with 2 bars are:

$$\{123||,\ 12|3|,\ 1|23|,\ |123|,\ 13|2|,\ 2|13|,\ 23|1|,\ 3|12|\}.$$

Let $B(n,k)$ denote the number of barred permutations on n letters with k bars. Let us count $B(n,k)$ in two ways.

First, note that a barred permutation on n letters with k bars can be obtained from a regular permutation on n letters with $k-i$ descents by placing a bar at each of the $k-i$ indices of descent, and then arbitrarily placing the remaining i bars. The way of placing i bars to separate n objects is $\binom{n+i}{i}$ via stars and bars. Therefore we must have

$$B(n,k) = \sum_{i=0}^{k-1} \binom{n+i}{i} A(n,k-i). \tag{5}$$

Re-indexing the above sum with $j = k - i$, we get

$$B(n,k) = \sum_{j=1}^{k} \binom{n+k-j}{n} A(n,j).$$

On the other hand, we can count $B(n,k)$ directly. Notice that the segment of the permutation between any two bars (if non-empty) is strictly increasing. Therefore the number of barred permutations on n letters with k bars is precisely the number of partitions of the set $\{1,2,\cdots,n\}$ into at most k ordered parts (or equivalently, the number of functions from $\{1,2,\cdots,n\}$ to $\{1,2,\cdots,k\}$). For each element in $\{1,2,\cdots,n\}$, we must choose one of the k partitions it goes into. There are k choices for each of the n elements for a total of k^n such ordered partitions. Therefore we must have

$$B(n,k) = k^n.$$

This establishes the fact that

$$B(n,k) = k^n = \sum_{j=1}^{k} \binom{n+k-j}{n} A(n,j).$$

By re-indexing as done in equation (5) we see that we have proven equation (3) As a last step let us confirm that equation (3) gives us the same answer as trial and error calculation from before:

$$\begin{aligned}\sum_{\ell=0}^{4-1} \binom{\ell+k}{4} A(4,\ell) &= \binom{k}{4}A(4,0) + \binom{k+1}{4}A(4,1) + \binom{k+2}{4}A(4,2) + \binom{k+3}{4}A(4,3) \\ &= \binom{k}{4} + 11\binom{k+1}{4} + 11\binom{k+2}{4} + \binom{k+3}{4} = k^4\end{aligned}$$

Where the calculation of the Eulerian numbers were done by equation (4) and table 4 below. As an example $A(4,1) = (4-1)A(3,0) + (1+1)A(3,1) = 3 \cdot 1 + 2 \cdot 4 = 11$.

Table 4: The Euler triangle displaying the first values of $A(n, m)$.

n/m	0	1	2	3
1	1			
2	1	1		
3	1	4	1	
4	1	11	11	1

Section 5. Suppl

$\boxed{23}$ Given a nonempty set A, let $f\colon A \to A$ and $g\colon A \to A$ where

$$f(a) = g(f(f(a))) \qquad \text{and} \qquad g(a) = f(g(f(a)))$$

for all a in A. Prove that $f = g$.

As done in the book we will simply the notation for the composition of two functions: $f(g(a)) = (f \circ g)(a)$. Further, for all intents and purposes we assume that we always are looking at the point a, and as such write $f(g(a)) = f \circ g$, and similarly denote $f \circ f = f^2$. Then by the definitions

$$f = g \circ f \circ f = g \circ f^2 \qquad \text{and} \qquad g = f \circ g \circ f$$

We have $f = (g) \circ f^2 = (f \circ g \circ f) \circ f^2 = f \circ g \circ f^3$ and $f^2 = f \circ f = f \circ g \circ f \circ f = f \circ g \circ f^2$.

Steps	**Reasons**
$f = g \circ f \circ f$	Definiton of f
$= f \circ g \circ f^3$	Definition of $g = f \circ g \circ f$
$= (f \circ g \circ f^2) \circ f$	
$= f^2 \circ f$	$f \circ g \circ f^2 = f^2$ by the definition of g and f
$= f^2 \circ g \circ f^2$	Definiton of f
$= f \circ (f \circ g \circ f) \circ f$	
$= f \circ g \circ f$	Definiton of g
$= g$	Definiton of g

Thus, $f = g$ which is what we wanted to prove.

$\boxed{27}$ With $A = \{x, y, z\}$, let $f, g\colon A \to A$ be given by $f = \{(x,y),(y,z),(z,x)\}$, $g = \{(x,y),(y,x),(z,z)\}$. Determine each of the following: $f \circ g$, $g \circ f$, f^{-1}, g^{-1}, $(g \circ f)^{-1}$, $f^{-1} \circ g^{-1}$, and $g^{-1} \circ f^{-1}$.

As an example $f(g(x)) = f(y) = z$

$$\begin{aligned}
f \circ g &= \{(x,z),(y,y),(z,x)\} \\
g \circ f &= \{(x,x),(y,z),(z,y)\} \\
f^{-1} &= \{(y,x),(z,y),(x,z)\} \\
g^{-1} = g &= \{(x,y),(y,x),(z,z)\} \\
(g \circ f)^{-1} = g \circ f &= \{(x,x),(y,z),(z,y)\} \\
f^{-1} \circ g^{-1} = (g \circ f)^{-1} &= \{(x,x),(y,z),(z,y)\} \\
g^{-1} \circ f^{-1} &= \{(x,z),(y,y),(z,x)\}
\end{aligned}$$

$\boxed{28}$ **a)** If $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x + 3$, find $f^{-1}(8)$.

We have $f^{-1}(8) = ?$ By taking applying f to both sides and use the definition of the inverse $f(f^{-1}(x)) = x$ we have $8 = f(?)$. Thus, we are asked to find an x such that $f(x) = 8$. Solving yields

$$8 = 5x + 3 \quad \Rightarrow \quad x = (8-3)/5 = 1$$

As such $f^{-1}(8) = 1$. Another equivalent way is to first find the inverse first. Let $y = f(x)$ then $y = 5x + 3$. So $x = (y-3)/5$, or in other words $f^{-1}(y) = (y-3)/5$. Plugging in $f(x) = y = 8$ gives the same as before.

Section 7. Suppl

$\boxed{12}$ The adjacency list representation of a directed graph G is given by the lists in [?]. Construct G from this representation

Table 5: Adjacency list representation

Adjacency List		Index List	
1	2	1	1
2	3	2	4
3	6	3	5
4	3	4	5
5	3	5	8
6	4	6	10
7	5	7	10
8	3	8	10
9	6		

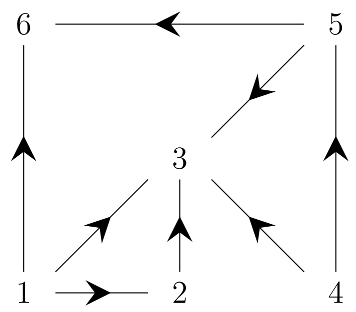

Figure 1: The directed Graph G corresponding to table 5.

$\boxed{16}$ **b)** For all $2 \leq n \leq 35$, show that the Hasse diagram for the set of positive-integer divisors of n looks like one of the nine diagrams in part (a)

For $2 \leq n \leq 35$, n can be written in one of the nine forms:

(i) p: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$

(ii) p^2: $4, 6, 25$

(iii) pq: $6, 10, 14, 15, 21, 22, 26$

(iv) p^3: $8, 27$

(v) p^2q: $12, 20, 28$

(vi) p^4: 16

(vii) p^3q: 24

$(viii)$ pqr: 30

(ix) p^5: 32

where p, q, r denote distinct primes. The Hasse diagrams for these representations are given by the structures in part (a). For $n = 36 = 2^2 \cdot 3^2$, we must introduce a new structure.

17 Let U denote the set of all points in and on the unit square shown in figure 2. That is $U = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Define the relation $\mathscr{R}$ on U by $(a, b)\mathscr{R}(c, d)$ if one of the conditions below holds

1. $(a, b) = (c, d)$
2. $b = d$ and $a = 0$ and $c = 1$
3. $b = d$ and $a = 1$ and $c = 0$.

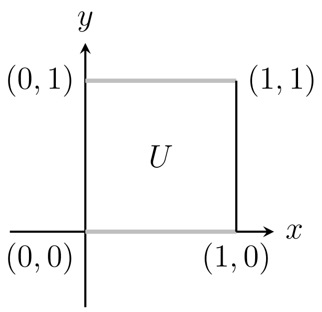

Figure 2

b) List the ordered pairs in the equivalence classes

$$[(0.3, 0.7)], [(0.5, 0)], [(0.4, 1)], [(0, 0.6)], [(1, 0.2)] \tag{6}$$

For $0 \le a \le 1$, $0 \le b \le 1$, how many ordered pairs are in $[(a, b)]$?

In general if $0 < a < 1$ then $[(a, b)] = (a, b)$; otherwise $[(0, b)] = (0, b), (1, b) = [(1, b)]$. The geometric intuition of this is that the highlighted grey parts in figure 2 are "glued" together. This gives the following ordered pairs

$$[(0.3, 0.7)] = \{(0.3, 0.7)\} \qquad [(0.5, 0)] = \{(0.5, 0)\}$$
$$[(0.4, 1)] = \{(0.4, 1)\} \qquad [(0, 0.6)] = \{(0, 0.6), (1, 0.6)\}$$
$$[(1, 0.2)] = \{(0, 0.2), (1, 0.2)\}.$$