Norwegian University of Science and Technology
Department of Mathematics

MA0301 Elementary discrete mathematics
Spring 2018

Solutions — exercise 10

Section 1.1 & 1.2

11 Three small towns, designated by A, B, C are interconnected by a system of two-way roads, as shown in Fig. 4

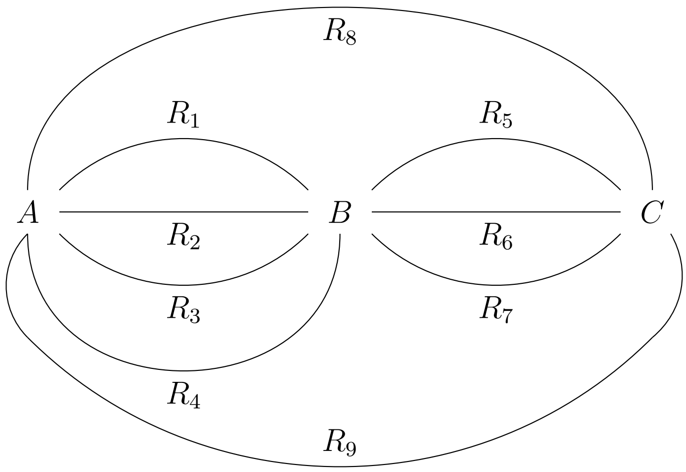


Figure 1

a) In how many ways can Linda travel from town A to town C?

To reach C we can either go directly along R_8 or R_9, this gives 2 paths, or travel through B. In the last case there are 4 paths to B, and for each of these there are 3 paths from B to C. In total

$$\#\text{ paths from } A \text{ to } C: \quad 2 + 4 \cdot 3 = 14.$$

b) How many different round strips can Linda travel from town A to town C and back to town A?

From part **b)** we know that there are 14 paths from A to C, and since every path is a two-way path, this means there are 14 paths from C to A. Thus, for each of the 14 paths we have taken to reach C we can choose 14 different paths to return.

$$\#\text{ paths from } A \text{ to } C \text{ and back to } C: \quad 14^2 = 196.$$

c) How many of the round trips in part **b)** are such that the return trip (from town C to town A) is atleast partially different from the route Linda takes from town A to town C?

There are 14 paths from A to C, thus once we have reached C there are 13 paths that we have *not* taken. Thus, to obtain a partially different path, we just have to pick one of the 13 paths not yet taken.

$$\#\text{ round trips } A \to C \to A \text{ with different return path:} \quad 14 \cdot 13 = 182.$$

$\boxed{24}$ Show that for all integers $n, r \geq 0$, if $n + 1 \geq r$ then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r),$$

where $P(n, r) = n!/(n-r)!$ denotes the number of permutations.

Using that $n! = n(n-1)!$ (Example: $5! = 5 \cdot 4!$) gives us the relation directly

$$P(n+1, r) = \frac{(n+1)!}{(n+1-r)!} = \frac{n+1}{n+1-r} \cdot \frac{n!}{(n-r)!} = \frac{n+1}{n+1-r} P(n, r)$$

$\boxed{25}$ Find the values(s) of n in each of the following:

(a) $P(n, 2) = 90$

(b) $P(n, 3) = 3P(n, 2)$

(c) $2P(n, 2) + 50 = P(2n, 2)$

Note that for all $n \in \mathbb{N}$ we have

$$P(n, 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) \tag{1}$$

As such we are looking for two integers, n, $n-1$ such that $n \cdot (n-1) = 90$. By inspection we see that $90 = 10 \cdot 9 = 10(10-1)$, so $n = 10$. We also have that $90 = (-9)(-10) = -9(-9-1)$ so another solution is $n = -9$, however I do not think we are counting negative solutions when dealing with permutations.

Using equation (1) we have

$$P(n, 2) = P(n, 3) \;\Leftrightarrow\; n(n-1) = n(n-1)(n-2) \;\Leftrightarrow\; n(n-1)(3-n) = 0$$

So $n = 0$, $n = 1$ or $n = 3$.In the last step we simply factored out the common term,

$$n(n-1)\cdot 1 - n(n-1)\cdot(n-2) = n(n-1)[1-(n-2)] = n(n-1)(3-n).$$

Again we use the idea from equation (1) to rewrite the equation

$$2P(n,2) + 50 = P(2n,2) \;\Leftrightarrow\; 2n(n-1) + 50 = 2n(2n-1) \Leftrightarrow\; 2n^2 = 50$$

Since $50 = 2\cdot 5^2$ we see by inspection that the solutions to the equation above is $n = \pm 5$. Where again I am quite sure that we are only interested in the positive solution.

$\boxed{27}$ **b)** How many distinct paths are there from $(1,0,5)$ to $(8,1,7)$ in Euclidian three-space if each more is one of the following types?

(H): $(x,y,z) \to (x+1,y,z)$:
(V): $(x,y,z) \to (x,y+1,z)$:
(A): $(x,y,z) \to (x,y,z+1)$

Each path consists of 2 H's, 1 V and 7 A's. There are

$$\frac{10!}{2!\cdot 1!\cdot 7!} = 360$$

ways to arrange these 10 letters, and this is the number of paths.

c) Generalize the results in part **b)**.

If a, b, and c are any real numbers and m, n, p are non-negative integers, then the number of paths from (a,b,c) to $(a+m, b+n, c\ p)$ is

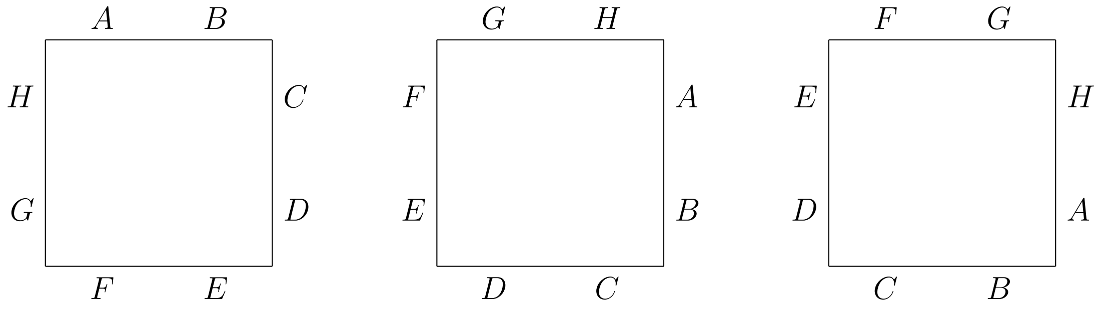
$$\frac{(m+n+p)!}{m!\cdot n!\cdot p!}$$

$\boxed{36}$ **a)** In how many ways can eight people, denoted A, B, …, H be seated about the square table shown in figure 2. Where figures 2a and 2b are considered the same but are distinct from figure 2c?

The exclusion of arrangements that can be obtained from rotation comes to the same as the extra condition that A is seated e.g. at the upper side. This because in any case there is exactly one rotation that brings him there. Then there are 2 possibilities for A. Now B has 7 possible seats to choose from, C has 6 possible seats to choose from and so forth. In total we have

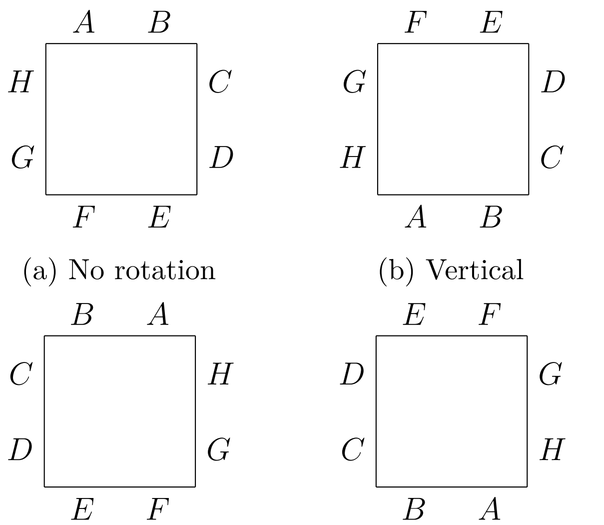
$$\#\text{ Distinct table seatings for 8 people: } 2\cdot 7! = 7\cdot 6\cdots 2\cdot 2 = 10080$$

Another way to obtain this answer is as follows: Ignoring symmetries and rotations there are 8! ways to place 8 persons around the table. If we reflect the table around the x-axis, or y-axis we obtain the same table placement. These are shown in figure 3. So for every table placement, we have 4 identitcal positions. Thus, $8!/4 = 8\cdot 7!/4 = 2\cdot 7!$ as before.



(a) (b) (c)

Figure 2



(a) No rotation (b) Vertical (c) Horizontal (d) Horizontal & vertical

Figure 3: The 4 rotation symmetries

b) If two of the eight people, say A and B, do not get along well, how many different seattings are possible with A and B not sitting next to each other?

We begin as before by placing A. To avoid symmetries A is once again placed on the upper half of the table, and as such have 2 seats to choose from. Now as B does not want to sit next to A he only has 5 seats to choose from, while C still has 6. Continuing we get that the total number of arrangements such that A does not sit beside B are

$$\#\text{ Arrangements such that } A \text{ does not sit beside } B\text{:} \quad 2 \cdot 5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 7200$$

Section 1.3

16 Determine the value of each of the following summations,

c) $\sum_{i=0}^{10} 1 + (-1)^i = (1-1) + (1+1) + \cdots + (1+1) = 6 \cdot 2 = 12$

Notice in general that for every *odd* $i = 2k+1$ then $1 + (-1)^{2k+1} = 1 - 1 = 0$, and similarly for *even* $i = 2k$ then $1 + (-1)^{2k} = 2$. Let $n \in \mathbb{N}$ then in general we have

$$\begin{aligned}\sum_{i=0}^{2n} 1 + (-1)^i &= (1-1) + (1+1) + \cdots + (1+1) \\ &= \sum_{i=0}^{n} (1 + (-1)^{2k-1}) + (1 + (-1)^{2k}) = \sum_{i=0}^{n} (1-1) + (1+1) = 2(n+1)\end{aligned}$$

d) $\sum_{k=n}^{2n} (-1)^k$ where n is an odd positive integer

Notice that when k is odd we have $(-1)^k = -1$ and $(-1)^k = 1$ when k is even. Thus, every other term cancel each other out. Since n is odd we will always sum an even number of terms, so the sum is zero.

$$\begin{aligned}\sum_{k=n}^{2n} (-1)^k &= (-1)^n + (-1)^{n+1} + \cdots + (-1)^{2n-1} + (-1)^{2n} \\ &= (-1+1) + (-1+1) + \cdots + (-1+1) = 0\end{aligned}$$

the first term $(-1)^n$ is negative because n is *odd*.

e) $\sum_{i=1}^{6} i(-1)^i = 1(-1)^1 + 2(-1)^2 + \cdots + 6(-1)^6 = -1 + 2 - 3 + 4 - 5 + 6 = 3$

Let $n \in \mathbb{N}$ then in general we have

$$\begin{aligned}\sum_{i=1}^{2n} i(-1)^i =& (-1+2) + (-3+4) + \cdots + \Big[(2k-1)(-1)^{2k-1} + 2k(-1)^{2k}\Big] \\ =& \sum_{k=1}^{n} \Big[(2k-1)(-1)^{2k-1} + 2k(-1)^{2k}\Big] = \sum_{k=1}^{n} \Big[(2k-1)(-1) + 2k\Big] = \sum_{k=1}^{n} 1 = n\end{aligned}$$

$\boxed{25}$ Determine the coefficient of

a) xyz^2 in $(x+y+z)^4$

b) xyz^2 in $(w+x+y+z)^4$

c) xyz^2 in $(2x-y-z)^4$

d) zyz^{-2} in $(x-2y+3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

For any positive integer m and any nonnegative integer n, the multinomial formula tells us how a sum with m terms expands when raised to an arbitrary power n:

$$(x_1+x_2+\cdots+x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1,k_2,\ldots,k_m} \prod_{t=1}^{m} x_t^{k_t}\,,$$

where

$$\binom{n}{k_1,k_2,\ldots,k_m} = \frac{n!}{k_1!\,k_2!\cdots k_m!}$$

is a multinomial coefficient. The sum is taken over all combinations of nonnegative integer indices k_1 through k_m such that the sum of all k_i is n. That is, for each term in the expansion, the exponents of the x_i must add up to n.

a) xyz^2 in $(x+y+z)^4$ has the coefficient $\displaystyle\binom{4}{1,1,2} = \frac{4!}{1!\cdot 1!\cdot 2!} = 12.$

b) xyz^2 in $(w+x+y+z)^4$ has the coefficient $\displaystyle\binom{4}{0,1,1,2} = \frac{4!}{0!\cdot 1!\cdot 1!\cdot 2!} = 12.$

c) xyz^2 in $(2x-y-z)^4$ has the coefficient $\displaystyle\binom{4}{1,1,2}(2)(-1)(-1)^2 = -\frac{4!\cdot 2}{1!\cdot 1!\cdot 2!} = -24.$

d) zyz^{-2} in $(x-2y+3z^{-1})^4$ has the coefficient $\displaystyle\binom{4}{1,1,2}(-2)(3)^2 = -\frac{4!\cdot 18}{1!\cdot 1!\cdot 2!} = 216.$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$ has the coefficient $\displaystyle\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2 =$ $161\,280.$

$\boxed{33}$ **b)** Given a list $a_0, a_1, a_2, \ldots, a_n$ — of $n+1$ real numbers, where n is a positive integer, determine

$$\sum_{i=1}^{n} a_i - a_{i-1}\,.$$

A series of this form is known as a telescoping series. In mathematics, a *telescoping series* is a series whose partial sums eventually only have a fixed number of terms after cancellation. The cancellation technique, with part of each term cancelling with part of the next term, is known as the *method of differences*.

In particular for the series above

$$\begin{aligned}\sum_{i=1}^{n} a_i - a_{i-1} &= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_{n-1} + a_{n-2}) + (a_n - a_{n-1}) \\ &= -a_0 + (a_1 - a_1) + (a_2 - a_2) + \cdots (a_{n-1} - a_{n-1}) + a_n \\ &= a_n - a_0\end{aligned}$$

c) Determine the value of $\displaystyle\sum_{i=1}^{100} \frac{1}{i+2} - \frac{1}{i+1}$.

This is a telescoping series with $a_i = 1/(i+2)$. Using the result from part **b)** we obtain

$$\sum_{i=1}^{100} \frac{1}{i+2} - \frac{1}{i+1} = \frac{1}{100+2} - \frac{1}{0+2} = -\frac{25}{51}.$$

Section 5.6

$\boxed{8}$ Let $f\colon A \to B$, $g\colon B \to C$. Prove that

(a) if $g \circ f\colon A \to C$ is onto, then g is onto;

(b) if $g \circ f\colon A \to C$ is one-to-one, then f is one-to-one.

(a) Assume that $g \circ f\colon A \to C$ is onto. The function $g \circ f$ from the set A to a set C is onto, if for every element y in the codomain C of f there is at least one element x in the domain A such that $(g \circ f)(x) = y$.

In other words, if $y \in C$, there exists some element $x \in A$ such that $(g \circ f)(x) = y$. Then $g(f(x)) = y$ with $f(x) \in B$, so g is onto.

(b) Let $x, y \in A$ if $(g \circ f)(x) = (g \circ f)(y)$ implies that $x = y$ for all pairs x, y then $g \circ f$ is one-to-one. Now,

$$f(x) = f(y) \;\Rightarrow\; g(f(x)) = g(f(y)) \;\Rightarrow\; (g \circ f)(x) = (g \circ f)(y) \;\Rightarrow\; x = y,$$

since $g \circ f$ is one-to-one. Which is what we wanted to prove.

$\boxed{17}$ Let $f, g\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ where for all $x \in \mathbb{Z}^+$, $f(x) = x + 1$ and $g(x) = \max\{1, x - 1\}$, the maximum of 1 and $x - 1$.

a) What is the range of f?

The range of a function f is by definition

$$\{y \mid \text{there exists an } x \text{ in the domain of } f \text{ such that } y = f(x)\}.$$

Since $x \in \mathbb{Z}^+$ and $f(x) = x + 1$, the range of f is $\mathbb{Z}^+/\{-1\} = \{2, 3, 4, \ldots\}$.

b) Is f an onto function?

A function f from a set $\mathbb{Z}^+$ to a set $\mathbb{Z}^+$ is surjective (or onto), or a surjection, if for every element y in the codomain $\mathbb{Z}^+$ of f there is at least one element x in the domain $\mathbb{Z}^+$ of f such that $f(x) = y$.

No. Let $y = 1$, then there is no $x \in \mathbb{Z}$ such that $y = f(x)$.

c) Is the function f one-to-one?

An injective function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. In other words, every element of the function's codomain is the image of at most one element of its domain.

Yes. Let $x, y \in \mathbb{Z}^+$ if $f(x) = f(y)$ implies that $x = y$ for all pairs x, y then f is one-to-one. This is equivalent with the definition above

$$f(x) = f(y) \;\Rightarrow\; x + 1 = y + 1 \;\Rightarrow\; x = y$$

d) What is the range of g?

Another equivalent way of writing the definition of g is as follows:

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ x - 1 & \text{otherwise} \end{cases}$$

Perhaps it is easier to see from this definition that the range of g is $\mathbb{Z}_+$.

e) Is g an onto function?

As the domain and codomain are the same, the function is onto. For every $y \in \mathbb{Z}^+$ we can find a $x \in \mathbb{Z}^+$ such that $y = g(x)$.

f) Is the function g one-to-one?

No. Let $x, y \in \mathbb{Z}^+$ if $f(x) = f(y)$ implies that $x = y$ for all pairs x, y then f is one-to-one. However as $f(1) = f(2) = 1$, f is not one-to-one.

g) Show that $g \circ f = 1_{\mathbb{Z}^+}$.

Using the definition above of g we have that

$$g(f(x)) = \max\{1, f(x) - 1\} = \max\{1, (x + 1) - 1\} = \max\{1, x\} = x,$$

since $x \in \mathbb{Z}^+$. This proves that $g \circ f = 1_{Z^+}$ as wanted.

h) Determine $(f \circ g)(x)$ for $x = 2, 3, 4, 7, 12$ and 25.

We have $(f \circ g)(x) = 1 + \max\{1, x - 1\}$. This gives the following table

Table 1: Shows $f(g(x)) = 1 + \max\{1, x - 1\}$ for various values.

x	2	3	4	7	12	25
$f(g(x))$	2	3	4	7	12	25

i) Do the answers for parts **b)**, **g)** and **h)** contradict the result in Theorem 8?

No. The functions f, g are *not* inverses of each other. The calculations in part **h)** may suggest that $f \circ g = 1_{Z^+}$ since $(f \circ g)(x) = x$ for $x \geq 2$. But we also find that

$$(f \circ g)(1) = f(\max 1, 0) = f(1) = 2,$$

so $(f\circ)(1) \neq 1$, and, consequently, $f \circ g \neq 1_{Z^+}$.

Theorem 8: *A function $f : A \to B$ is invertible if and only if it is one-to-one and onto.*