MA0301 Elementary discrete mathematics
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Solutions — exercise 5

$\boxed{2}$ Let $Y := \{1, 2, 3, 4, \ldots, 600\}$. Use the inclusion-exclusion principle to find the number of positive integers Y that are not divisible by 3, 5 or 7.

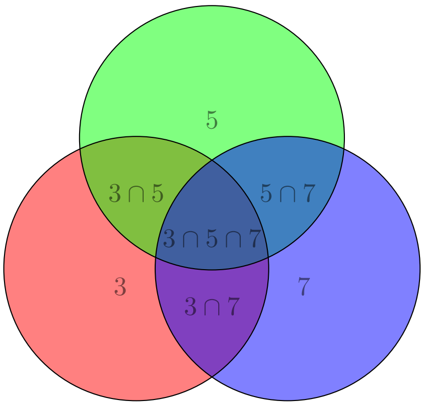

Figure 1

While not relevant for the exam, the Venn diagram displayed in figure 1 can shed some light on why the inclusion-exclusion principle works. We are interested in finding cardinality[1] of the union $3 \cup 5 \cup 7$, where the numbers are now interpreted as sets e.g. $3 = \{3n \mid n \in Y\}$ is the subset of Y consisting of all numbers divisible by 3 and so forth. The following argument might be a bit hard to digest, so try to read it while carefully studying figure 1.

Simply adding the number of elements in each set will give us an over-estimate. As an example 15 is counted twice, once in the set with numbers divisible by 3, and once in the set consisting of numbers divisible by 5. Thus $|3 \cup 5 \cup 7| \neq |3| + |5| + |7|$. To remedy this we can remove every union of two integers: $|3 \cap 5| + |3 \cap 7| + |3 \cap 7|$. However, this creates another problem. Take $105 = 3 \cdot 5 \cdot 7$. This number is included three times as it lies in 3, 5 and 7, alas it is also removed three times since it lies in $3 \cap 5$, $3 \cap 7$ and $5 \cap 7$. Thus, we need to include back every number that is a multiple of 3, 5 and 7. This gives us our final expression,

$$|3 \cup 5 \cup 7| = |3| + |5| + |7| - |3 \cap 5| - |3 \cap 7| - |5 \cap 7| + |3 \cap 5 \cap 7|,$$

[1]In mathematics, the cardinality of a set A is a measure of the "number of elements of the set", $|A|$.

which is precisely the inclusion-exclusion principle. The number of integers that are a multiple of k beneath n are precisely $\lfloor n/k \rfloor$. Every number divisible by 3, 5 or 7 is thus

$$\begin{aligned}|3 \cup 5 \cup 7| &= |3| + |5| + |7| - |3 \cap 5| - |3 \cap 7| - |5 \cap 7| + |3 \cap 5 \cap 7| \\ &= \lfloor \frac{600}{3} \rfloor + \lfloor \frac{600}{5} \rfloor + \lfloor \frac{600}{7} \rfloor - \lfloor \frac{600}{3 \cdot 5} \rfloor - \lfloor \frac{600}{3 \cdot 7} \rfloor - \lfloor \frac{600}{5 \cdot 7} \rfloor + \lfloor \frac{600}{3 \cdot 5 \cdot 7} \rfloor \\ &= 200 + 120 + 85 - 40 - 28 - 17 + 5 = 325\,.\end{aligned}$$

As such, there are $|Y| - |3 \cup 5 \cup 7| = 275$ numbers not divisible by 3, 5 or 7 amongst the first 600 integers.

$\boxed{3}$ Use the principle of induction to show that for all natural numbers n,

$$4 \sum_{i=1}^{n} i(i+2)(i+4) = n(n+1)(n+4)(n+5).$$

Base case: For $n = 1$

$$LHS = 4 \sum_{i=1}^{1} i(i+2)(i+4) = 4 \cdot 1 \cdot 3 \cdot 5 = 60\,, \quad RHS = 1(1+1)(1+4)(1+5) = 60\,,$$

so the hypothesis holds for the base case.

inductive step: Assume that for all k we have

$$4 \sum_{i=1}^{k} i(i+2)(i+4) = k(k+1)(k+4)(k+5).$$

We wish to prove that this implies that the hypothesis holds for $k+1$.

$$\begin{aligned}RHS &= (k+1)(k+2)(k+5)(k+6) \\ LHS &= 4 \sum_{i=1}^{k+1} i(i+2)(i+4) \\ &= 4(k+1)(k+3)(k+5) + 4 \sum_{i=1}^{k} i(i+2)(i+4) \\ &= 4(k+1)(k+3)(k+5) + k(k+1)(k+4)(k+5) \\ &= [4(k+3) + k(k+4)](k+1)(k+5) \\ &= (k+1)(k+2)(k+5)(k+6).\end{aligned}$$

As $LHS = RHS$ the rest follows by induction. The steps performed in the calculation above where respectively: (1)

$$\sum_{i=1}^{n+1} a_i = a_1 + a_2 + \cdots + a_n + a_{n+1} = (a_1 + a_2 + \cdots a_n) + a_{n+1} = a_{n+1} + \sum_{i=1}^{n} a_i.$$

(2) use the induction hypothesis. (3) We have $4(k+3) + k(k+4) = k^2 + 8k + 12 = (k+6)(k+2)$ either by the quadratic formula, or inspection (as $6 + 2 = 8$ and $6 \cdot 2 = 12$).

$\boxed{7}$ Use the laws of set theory to show for arbitrary sets A, B, C that:

a) If $(A \cup B) \subseteq (A \cap B)$ then $A = B$.

To prove that $A = B$ we show that $A \subseteq B$ and $B \subseteq A$ when the conditions holds

$$A \subseteq A \cup B \subseteq A \cap B \subseteq B$$
$$B \subseteq A \cup B \subseteq A \cap B \subseteq A,$$

and we are done.

b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

So we are to prove De Morgan's first law, which suddenly is different in set notation compared to logic... Again we prove both inclusions.

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Assume that $x \in \overline{A \cap B}$. Then $x \neq A \cap B$, so either $x \notin A$ or $x \notin B$. Which, is the same as to say that $x \in \overline{A}$ or $x \in \overline{B}$. In either case $x \in \overline{A} \cup \overline{B}$.

$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$: Let $x \in \overline{A} \cup \overline{B}$. Then we know that either $x \notin A$ or $x \notin B$. Then it is not in the intersection either $x \notin A \cap B$, which is to say $x \in \overline{A \cap B}$ and we are done.

c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

As before we prove both inclusions

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$: Assume that $x \in A \cap (B \cup C)$. This implies that $x \in A$ and $x \in B \cup C$. The last expression is the same as saying that $x \in B$ or $x \in C$. If $x \in B$ then $x \in A \cap B$ (as $x \in A$), otherwise if $x \in C$ then $x \in A \cap C$. As x is in either of those we at least have x in the union $x \in (A \cap B) \cup (A \cap C)$.

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$: Assume now that $x \in (A \cap B) \cup (A \cap C)$, as such $x \in (A \cap B)$ or $x \in (A \cap C)$. In the first case $x \in A$ and $x \in B$, in the latter $x \in A$ and $x \in C$. Regardless, $x \in A$. This leaves us with two cases either $x \in B$ or $x \in C$. In either case, x has to lie in the union $x \in B \cup C$. This proves the inclusion $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ and we are done.

Section 4.1

$\boxed{8}$ Prove each of the following for all $n \geq 1$ by the Principle of Mathematical induction

a) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

Base case: For $n = 1$ we have

$$LHS = (2 \cdot 1 - 1)^2 = 1\,, \qquad RHS = \frac{1(2\cdot 1 - 1)(2 \cdot 1 + 1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1,$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statement holds for $n = k$, in other words

$$\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

Need to show that this implies that the statement holds for $n = k+1$

$$\begin{aligned}
RHS &= \frac{(k+1)(2k+1)(2k+3)}{3} \\
LHS &= \sum_{i=1}^{k+1} (2i-1)^2 \\
&= (2k+1)^2 + \sum_{i=1}^{k} (2i-1)^2 \\
&= (2k+1)^2 + \frac{k(2k-1)(2k+1)}{3} \\
&= \frac{(2k+1)[3(2k+1) + k(2k-1)])}{3} \\
&= \frac{(k+1)(2k+1)(2k+3)}{3}\,.
\end{aligned}$$

As $3(2k+1) + k(2k-1) = 2k^2 + 5k + 3 = (k+1)(2k+3)$ either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

b) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$.

Base case: For $n = 1$ we have,

$$LHS = 1 \cdot (2 \cdot 1 + 1) = 3\,, \qquad RHS = \frac{1(1+1)(2 \cdot 1 + 7)}{6} = \frac{1 \cdot 2 \cdot 9}{6} = 3,$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statement holds for $n = k$, that is

$$\sum_{i=1}^{k} i(i+2) = \frac{k(k+1)(2k+7)}{6}.$$

Wish to show that this implies that the statement holds for $n = k + 1$.

$$\begin{aligned}
RHS &= \frac{(k+1)(k+2)(2k+9)}{6} \\
LHS &= \sum_{i=1}^{k+1} i(i+2) \\
&= (k+1)(k+3) + \sum_{i=1}^{k} i(i+2) \\
&= (k+1)(k+3) + \frac{k(k+1)(2k+7)}{6} \\
&= \frac{(k+1)[(6(k+3) + k(2k+7))]}{6} \\
&= \frac{(k+1)(k+2)(2k+9)}{6}.
\end{aligned}$$

As $6(k+3) + k(2k+7) = 2k^2 + 13k + 18 = (k+2)(2k+9)$ either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

c) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

While the question *clearly* states that the proof has to be done by induction, I can not help myself to show the following proof as the series is telescoping

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Base case: For $n = 1$ we have

$$LHS = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}, \qquad RHS = \frac{1}{1+1} = \frac{1}{2},$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statement holds for $n = k$, that is

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Wish to show that this implies that the statement holds for $n = k + 1$

$$\begin{aligned}
RHS &= \frac{k+1}{k+2} \\
LHS &= \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \\
&= \frac{1}{(k+1)(k+2)} + \sum_{i=1}^{k} \frac{1}{k(k+1)} \\
&= \frac{1}{(k+1)(k+2)} + \frac{k}{k+1} \\
&= \frac{1+k(k+2)}{(k+2)(k+1)} = \frac{(1+k)^2}{(k+2)(k+1)} = \frac{1+k}{k+2}
\end{aligned}$$

As $1 + k(k + 2) = k^2 + 2k + 1^2 = (k + 1)^2$ either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

$\boxed{12}$ **a)** Prove that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$ where $i \in \mathbb{C}$ and $i^2 = -1$.

By De Moivre's laws we have

$$(\cos\theta + i\sin\theta)^2 = (e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i\sin\theta$$

However, this might be considered cheating as we are supposed to prove this theorem later. Expanding the square gives the same results

$$\begin{aligned}
(\cos\theta + i\sin\theta)^2 &= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta \\
&= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta) = \cos(2\theta) + i\sin(2\theta)
\end{aligned}$$

b) Using induction, prove that for all $n \in \mathbb{Z}^+$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

Base case: As $n \in \mathbb{Z}^+$, the base case is $n = 0$,

$$RHS = (\cos\theta + i\sin\theta)^0 = 1\,, \qquad LHS = \cos(0\cdot\theta) + i\sin(0\cdot\theta) = 1.$$

as $\cos 0 = 1$ and $\sin 0 = 0$ so the hypothesis holds for the base case. (Step **a)** shows that the hypothesis holds for $n = 1$ as well.)

Inductive step: Assume that the statement holds for $n = k$ that is

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta,$$

want to prove that this implies that the statement holds for $n = k+1$. Now

$$\begin{aligned}
RHS &= \cos(k+1)\theta + i\sin(k+1)\theta \\
LHS &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\
&= (\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta
\end{aligned}$$

since $RHS = LHS$ the statement follows by induction.

c) Verify that $1 + i = \sqrt{2}(\cos 45° + i\sin 45°)$, and compute $(1+i)^{100}$.

As $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, the first part is trivial

$$\sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$$

and so $(1+i)^{100} = (\sqrt{2})^{100}(\cos\pi/4 + i\sin\pi/4)^{100} = 2^{50}(\cos 25\pi + i\sin 25\pi) = -2^{50}$

$\boxed{16}$ **a)** For $n = 3$ let $X_3 = 1, 2, 3$. Now consider the sum

$$s_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} = \sum_{\emptyset \neq A \subseteq X_3} \frac{1}{p_A},$$

where p_A denotes the product of all elements in a non-empty subset A of X_3. Note that the sum is taken over all the non-empty subsets of X_3. Evaluate this sum

Finding the common denominator a straight forward computation yields

$$\begin{aligned}
s_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}\right) + \left(\frac{1}{3} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3}\right) \\
&= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}\right) + \left(1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}\right)\frac{1}{3} = 2 + (1+2)\frac{1}{3} = 3
\end{aligned}$$

b) Repeat the calculation in step **a)** for s_2 and s_4.

By "accident" s_2 has already been computed

$$s_2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2} = 1 + \frac{1}{2} + \frac{1}{2} = 2$$

s_4 is done in the same matter, albeit a bit more work has to be done

$$\begin{aligned}
s_4 &= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 4}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 4}\\
&+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 4}+\frac{1}{1\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{1\cdot 2\cdot 3\cdot 4}\\
&=\Big(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{1\cdot 2\cdot 3}\Big)\\
&+\Big[\frac{1}{4}+\frac{1}{1\cdot 4}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 4}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{1\cdot 3\cdot 4}+\frac{1}{1\cdot 2\cdot 4}+\frac{1}{1\cdot 2\cdot 3\cdot 4}\Big]\\
&=\Big(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{1\cdot 2\cdot 3}\Big)\\
&+\Big[1+\Big(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{2\cdot 3}+\frac{1}{1\cdot 3}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}\Big)\Big]\cdot\frac{1}{4}\\
&= s_3+[1+s_3]\frac{1}{4}=3+(1+3)\frac{1}{4}=4
\end{aligned}$$

c) Conjecture a general result suggested by the calculations from steps **a)** and **b)**. Prove your conjecture using the Principle of Mathematical induction.

Interesting problem! I conjecture that for every $n \in \mathbb{N}$, $s_n = n$.

Base case: As $n \in \mathbb{N}$ the base case is $n = 1$. Then,

$$LHS = s_1 = \frac{1}{1} = 1\,, \qquad RHS = 1,$$

and so the hypothesis holds for the base case.

Inductive step: We assume that for every k we have

$$s_k = \sum_{\emptyset\neq A\subset X_k}\frac{1}{p_A}$$

wish to prove that this implies that the conjecture holds for s_{k+1}, then

$$s_{k+1} = \sum_{\emptyset\neq A\subset X_{k+1}}\frac{1}{p_A} = \sum_{\emptyset\neq A\subset X_k}\frac{1}{p_B} + \sum_{\{k+1\}\subseteq A\subset X_{k+1}}\frac{1}{p_C} \tag{1}$$

Where the first sum is taken over all non-empty subsets B of X_k and the second sum over all subsets C of X_{k+1} that contain $k+1$. In other words, we do as before and isolate all the terms that contain $k+1$. As seen before we can factor out the $1/(k+1)$ term giving

$$s_{k+1} = s_k + \frac{1}{1+k}(1+s_k) = k + \frac{1}{1+k}(1+k) = 1+k$$

Why the second sum in equation (1) equals $(1+s_k)/(1+k)$ has been shown in excruciating detail in steps **a)** and **b)**. The proof for the conjecture now follows by induction.

17 For $n \in \mathbb{Z}^+$. We define the n'th harmonic number H_n as the sum of the first n reciprocals (of the natural integers), $H_n = \sum_{i=1}^{n} 1/i$.

a) For all $n \in \mathbb{N}$ prove that $1 + \frac{n}{2} \leq H_{2^n}$.

Base case: As $n \in \mathbb{N}$ the base case is $n = 1$. Then

$$LHS = 1 + \frac{1}{2}\,, \qquad RHS = H_{2^{0+1}} = H_2 = 1 + \frac{1}{2}$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statements holds for $n = k$ that is

$$1 + \frac{n}{2} \le H_{2^{n+1}}$$

wish to show that this implies that the inequality holds for $n = k + 1$. Then,

$$\begin{aligned} LHS &= 1 + \frac{k+1}{2} \\ RHS &= H_{2^{k+1}} \\ &= H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k} \\ &\ge H_{2^k} + \frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + \cdots + \frac{1}{2^k+2^k} \\ &= H_{2^k} + \frac{2^k}{2 \cdot 2^k} = H_{2^k} + \frac{1}{2} \ge 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2} \end{aligned}$$

as $LHS = RHS$ the rest follows by induction.

b) Prove that for all $n \in \mathbb{Z}^+$,

$$\sum_{j=1}^{n} jH_j = \left[\frac{n(n+1)}{2}\right] H_{n+1} - \left[\frac{n(n+1)}{4}\right].$$

Base case: As $n \in \mathbb{Z}^+$ the base case is $n = 0$ then

$$RHS = \sum_{j=1}^{0} jH_j = 0, \qquad LHS = \left[\frac{0 \cdot (0+1)}{2}\right] H_{0+1} - \left[\frac{0 \cdot (0+1)}{4}\right] = 0,$$

so the hypothesis holds for the base case.

Inductive step: Assume that

$$\sum_{j=1}^{k} jH_j = \left[\frac{k(k+1)}{2}\right] H_{k+1} - \left[\frac{k(k+1)}{4}\right]$$

holds for every $n = k$. Wish to prove that this implies that the statement holds for $n = k+1$. Now

$$
\begin{aligned}
LHS &= \left[\frac{(k+1)(k+2)}{2}\right] H_{k+2} - \left[\frac{(k+1)(k+2)}{4}\right] \\
RHS &= \sum_{j=1}^{k+1} jH_j = (k+1)H_{k+1} + \sum_{j=1}^{k} jH_j \\
&= (k+1)H_{k+1} + \left[\frac{k(k+1)}{2}\right] H_{k+1} - \left[\frac{k(k+1)}{4}\right] \\
&\frac{(k+1)(k+2)}{2} H_{k+1} - \frac{k(k+1)}{4} \\
&\frac{(k+1)(k+2)}{2} \left[H_{k+2} - \frac{1}{k+2}\right] - \frac{k(k+1)}{4} \\
&\frac{(k+1)(k+2)}{2} H_{k+2} - \frac{2(k+1)}{4} - \frac{k(k+1)}{4} \\
&\frac{(k+1)(k+2)}{2} H_{k+1} - \frac{(k+1)(k+2)}{4}
\end{aligned}
$$

As $LHS = RHS$ the proof follows by induction.