MA0301 Elementary discrete mathematics
Spring 2018

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Department of Mathematics

Solutions — exercise 7

Section 5.2

1 Determine whether or not each of the following relations is a function. If a relation is a function, find its range.

c) $\{(x, y) \mid x, y \in \mathbb{R},\ y = 3x^2 + 1\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$

In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. Thus this relation is a *function* as every input y is related exactly to one output, namely $3x^2 + 1$. The range is $\{7,\ 8,\ 11,\ 16, 23, \ldots\}$ or more succinctly $\{x \in \mathbb{R} \mid 3x^2 + 1\}$.

d) $\{(x, y) \mid x, y \in \mathbb{Q},\ x^2 + y^2 = 1\}$, a relation from $\mathbb{Q}$ to $\mathbb{Q}$

This is *not* a function as both $(0, 1)$ and $(0, -1)$ lie on the unit circle. Thus we have one input $x = 0$ related to *two* outputs $y = 1$, $y = -1$. More generally all the rational points on the unit circle can be described as

$$\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right), \qquad t \in \mathbb{Q}$$

Setting $t = -t$ we have $(\frac{1-t^2}{1+t^2}, -\frac{2t}{1+t^2})$, as such the input $x = \frac{1-t^2}{1+t^2}$ corresponds to two outputs $y = \pm\frac{2t}{1+t^2}$. Showing that there are an *infinite* amount of points prohibiting the circle from being a function. This should be clear by just looking at a picture of the circle as well.

e) $\mathscr{R}$ is a relation from A to B where $|A| = 5$ and $|B| = 6$, and $|\mathscr{R}| = 6$.

Since $6 = |\mathscr{R}| > |A| = 5$, $\mathscr{R}$ cannot be a function. Perhaps this is easier to see with a a concrete example, let $A = \{a, b, c, d, e\}$, and $B = \{1, 2, 3, 4, 5, 6\}$. If $|\mathscr{R}| = 6$, then $\mathscr{R}$ has to contain some element of A *twice*, and thus, it cannot be a function.

For an example if $\mathscr{R} = \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1)\}$, then $\mathscr{R}$ is a function as every element in A is mapped exactly to one value in B, and $|\mathscr{R}| = 5$. However, as every element in A is already used, we can not add another element to $\mathscr{R}$ without mapping an element in A to two different elements in B. We can not be "clever" either and try to add the same element twice, as an intrinsic property of sets are that they consists of only unique elements. Example: $\{1, 1, 1, 2\} = \{1, 2\}$.

$\boxed{3}$ Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$.

a) List five functions from a to b.

We need to make sure to never map a single element in A to two different element in B.

$$\{(1, x), (2, x), (3, x), (4, x)\},$$
$$\{(1, y), (2, y), (3, y), (4, y)\},$$
$$\{(1, z), (2, z), (3, z), (4, z)\},$$
$$\{(1, x), (2, y), (3, z), (4, x)\},$$
$$\{(1, y), (2, z), (3, x), (4, y)\},$$

b) How many functions $f: A \to B$ are there?

Every value in A can be mapped to three different values in B. Thus, there can be in total $|A|^{|B|} = 4^3 = 64$ different functions from A to B.

c) How many functions $f: A \to B$ are one-to-one?

In order for a function to be one-to-one *every* element in A must map to a *different* element in B. As our range is smaller than our domain $|B| < |A|$, there can not exists any functions $f: A \to B$ that are one-to-one.

d) How many functions $g: B \to A$ are there?

Similar as before. Every element in B can now be mapped to 4 different values in A. Thus, there can be in total $|B|^{|A|} = 3^4 = 81$ different functions from B to A.

e) How many functions $g: B \to A$ are one-to-one?

For x we have 4 different choices for values, for y we are now left with 3 choices (as they have to map to unique values), for the last value z there are 2 values left to choose from. In total there are a total of $4 \cdot 3 \cdot 2 = 24$ functions $g: B \to A$ that are one-to-one.

f) How many functions $f: A \to B$ satisfy $f(1) = x$?

The remaining 3 values in A can still be mapped to any of the 3 values in B. Thus, there is a total of $|B|^{|A|-1} = 3^3 = 27$ functions $f: A \to B$ that satisfies $f(1) = x$.

g) How many functions $f: A \to B$ satisfy $f(1) = f(2) = x$?

The remaining 2 values in A can still be mapped to any of the 3 values in B. Thus, there is a total of $|B|^{|A|-2} = 3^2 = 9$ functions $f: A \to B$ that satisfies $f(1) = f(2) = x$.

h) How many functions $f: A \to B$ satisfy $f(1) = x$ and $f(2) = y$?

Whether $f(2)$ is mapped to x or y makes no difference. The remaining 2 values in A can still be mapped to any of the 3 values in B. Thus, there is a total of $|B|^{|A|-2} = 3^2 = 9$ functions $f\colon A \to B$ that satisfies $f(1) = x$ and $f(2) = y$.

$\boxed{5}$ Let $A.B, C \subset \mathbb{R}^2$ where $A = \{(x, y) \mid y = 2x + 1\}$, $B = \{(x, y) \mid y = 3x\}$, and $C = \{(x, y) \mid x - y = 7\}$. Determine each of the following:

c) $\overline{\overline{A} \cup \overline{C}}$

By De Morgans laws and the law of inverses we have

$$\overline{\overline{A} \cup \overline{C}} = \overline{\overline{A}} \cap \overline{\overline{C}} = A \cap C = \{(x, y) \mid y = 2x + 1 \text{ and } x - y = 7\}.$$

Adding these two equations gives $x = 2x + 8$, thus $x = -8$ and $y = 2x + 1 = -15$. As such we have $\overline{\overline{A} \cup \overline{C}} = \{(-8, -15)\}$ which is what we wanted to find.

d) $\overline{B} \cup \overline{C}$

Again by De Morgans laws we have $\overline{B} \cup \overline{C} = \overline{B \cap C}$ and now as before

$$B \cap C = \{(x, y) \mid y = 3x \text{ and } x - y = 7\}$$

Insertion gives $x - 3x = 7$ implying $x = -7/2$ and $y = -21/2$. Since $B \cap C = \{(-7/2, -21/2)\}$ we have $\overline{B} \cup \overline{C} = \mathbb{R}^2 - \{(-7/2, -21/2)\}$.

$\boxed{8}$ Determine whether each of the following statements is true or false. If the statement is false, provide a counterexample

a) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in \mathbb{Z}$.

True: $\lfloor a \rfloor = a$ and $\lceil a \rceil = a$ for every whole number $a \in \mathbb{Z}$.

b) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in \mathbb{R}$.

False: Let $a = n + \frac{1}{n+1}$, where $n \in \mathbb{N}$ then $\lfloor n + \frac{1}{n+1} \rfloor = n$ but $\lceil n + \frac{1}{n+1} \rceil = n + 1$.

c) $\lfloor a \rfloor = \lceil a \rceil - 1$ for all $a \in \mathbb{R} - \mathbb{Z}$.

True: If $a \in \mathbb{R} - \mathbb{Z}$, then a has to have a non-zero remainder (otherwise it would be an integer), as such we can write $a = b + r$, where $0 < r < 1$ and $b \in \mathbb{R}$. Then, $\lfloor a \rfloor = \lfloor b + r \rfloor = b$ and $\lceil a \rceil - 1 = \lceil b + r \rceil - 1 = (b + 1) - 1 = b$.

d) $-\lceil a \rceil = \lceil -a \rceil$ for all $a \in \mathbb{R}$.

False: Let once again $a \in \mathbb{R} - \mathbb{Z}$ meaning $a = b + r$ with $0 < r < 1$. Then, $-\lceil a \rceil = -\lceil b + r \rceil = -(b+1) = -b - 1$, but $\lceil -a \rceil = \lceil -(b+r) \rceil = -b$. You might want to use $a = 1.25$ or some other fraction to see this clearer.

$\boxed{9}$ Find all the real numbers x such that

a) $7\lfloor x \rfloor = \lfloor 7x \rfloor$

Let $x = b + r$ where $b \in \mathbb{Z}$ and $0 \le r < 1$. Then $7\lfloor x \rfloor = 7\lfloor b + r \rfloor = 7b + 7\lfloor r \rfloor$, on the other side $\lfloor 7x \rfloor = \lfloor 7b + 7r \rfloor = 7b + \lfloor 7r \rfloor$. So we need $7\lfloor r \rfloor = \lfloor 7r \rfloor$ for the equality to hold. Since $0 \le r < 1$ we have $7\lfloor r \rfloor = 0$ for all r. In order for $\lfloor 7r \rfloor = 0$ as well, we need to enforce the restriction $0 \le r < 1/7$.

So the equality holds for all $x = b + r$ with $b \in \mathbb{Z}$ and $0 \le r < 1/7$. We could write

$$x \in \bigcup_{k \in \mathbb{Z}} [k, k + 1/7) = \cdots \cup [-1, -6/7] \cup [0, 1/7) \cup [1, 8/7) \cup \cdots, \tag{1}$$

if we wanted to showoff our mathematical prowess ...

b) $\lfloor 7x \rfloor = 7$

We can assume without loss of generality that $x = b + r$ where $0 \le r < 1$ and $b \in \mathbb{Z}$. Then, $\lfloor 7x \rfloor = \lfloor 7b + 7r \rfloor = 7b + \lfloor 7r \rfloor$. In order for this to be equal to 7, we need that $b = 1$ and $0 \le r < 1/7$. Thus the equality $\lfloor 7x \rfloor = 7$ holds for all $1 \le x < 1 + 1/7$, or $x \in [1, 8/7)$.

c) $\lfloor x + 7 \rfloor = x + 7$

Let $x \in \mathbb{N}$, then $\lfloor x + 7 \rfloor = \lfloor x \rfloor + 7 = x + 7$ as wanted.

d) $\lfloor x + 7 \rfloor = \lfloor x \rfloor + 7$

Holds for all $x \in \mathbb{R}$, as integers are unaffected by the ceil and floor-functions.

Section 5.3

$\boxed{2}$ For each of the following functions $f\colon \mathbb{Z} \to \mathbb{Z}$, determine whether the function is one-to-one and whether it is onto. If the function is not onto, determine the range $f(\mathbb{Z})$.

b) $f(x) = 2x - 3$

This function is one-to-one $x = \big(f(x) + 3\big)/2$. Thus, each x is mapped to an unique $f(x)$ and vice-versa. However it only covers the odd integers, as such it is not onto. The range are the odd integers $x \in 2\mathbb{Z} + 1$.

d) $f(x) = x^2$

This is not one-to-one as $f(-n) = f(n)$, similarly it is not onto as it only targets the square numbers. The range is thus $x \in \mathbb{N}^2$ or more commonly $f = \{0, 1, 4, 9, 16, \ldots\}$

f) $f(x) = x^3$

As $f'(x) = 3x^2 > 0$ our function f is non-decreasing, and as such one-to-one. However again not every element in $\mathbb{Z}$ lies in its range, only the cubic numbers, $\{0, 1, 8, 27, 64, \ldots\}$ as such it is not onto.

$\boxed{3}$ For each of the following functions $g\colon \mathbb{R} \to \mathbb{R}$, determine whether the function is one-to-one and whether it is onto. If the function is not into, determine the range $f(\mathbb{R})$.

b) $f(x) = 2x - 3$

This function is one-to-one and onto. $f\colon \mathbb{R} \to \mathbb{R}$. Let $y \in \mathbb{R}$ then we can find an x such that $f(x) = y$ for every $y \in \mathbb{R}$.

d) $f(x) = x^2$

As before this function not one-to-one as $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Let $y < 0$, then there does not exists an $x \in \mathbb{R}$ such that $f(x) = y$, hence it is not onto. The range is $[0, \infty)$.

f) $f(x) = x^3$

As before this function is non-decreasing and as such one-to-one. For every $y \in \mathbb{R}$ there exists an x such that $f(x) = y$, let $x = \sqrt[3]{y}$. Thus, the function is onto as well.

$\boxed{4}$ Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

a) How many functions are there from A to B? How many of these are one-to-one? How many are onto?

There are $|B|^{|A|} = 6^4 = 7776$ functions from A to B. If the function is one-to-one every element in A is mapped to a different value in B. So the number of functions that are one-to-one are $6 \cdot 5 \cdot 4 \cdot 3 = 6!/2! = 360$. There are more elements in B than in A, thus there are 0 functions that are onto.

b) How many functions are there from B to A? How many of these are onto? How many are one-to-one?

There are $|A|^{|B|} = 4^6 = 4096$ functions from B to A. As there are more elements in B than in A, every element in B can not be mapped to an unique element in A, as such there are no one-to-one functions. The number of onto functions is $4!\left\{ {6 \atop 4} \right\} = 1560$. Where a more throughout explanation on how this number arises is given below.

To find the number of functions that are onto we'll use an inclusion-exclusion argument. In general, if $|\mathbb{A}| = m$ and $|\mathbb{B}| = n$ there are n^m functions of all kinds from $\mathbb{A}$ to $\mathbb{B}$. Obviously if $m < n$, there are no function from $\mathbb{A}$ onto $\mathbb{B}$, so assume that $m \geq n$. If $b \in \mathbb{B}$, there are $(n-1)^m$ functions from $\mathbb{A}$ to $\mathbb{B} \setminus \{b\}$, i.e., functions whose ranges do not include b. We need to subtract these from the original n^m, and we need to do it for each of the n members of $\mathbb{B}$, so a better approximation is $n^m - n(n-1)^m$.

Unfortunately, a function whose range misses *two* members of $\mathbb{B}$ gets subtracted twice in that computation, and it should be subtracted only once. Thus, we have to add back in the functions whose ranges miss at least two points of $\mathbb{B}$. If $b_0, b_1 \in \mathbb{B}$, there are $(n-2)^m$ functions from $\mathbb{A}$ to $\mathbb{B} \setminus \{b_0, b_1\}$, and there are $\binom{n}{2}$ pairs of points of $\mathbb{B}$, so we have to add back in $\binom{n}{2}(n-2)^m$ to get

$$n^m - n(n-1)^m + \binom{n}{2}(n-2)^m ,$$

which can be expressed more systematically as

$$\binom{n}{0}n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m .$$

Unfortunately, this over-corrects in the other direction, by adding back in too much. The final result, when the entire inclusion-exclusion computation is made, is

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m ,$$

which can also be written

$$n!\left\{ {m \atop n} \right\} = n! \cdot S(m,n) ,$$

where $S(m,n) = \left\{ {m \atop n} \right\}$ is a Stirling number of the second kind. The Stirling number gives the number of ways of dividing up $\mathbb{A}$ into n non-empty pieces, and the $n!$ then gives the number of ways of assigning those pieces to the n elements of $\mathbb{B}$.