MA0301 Elementary discrete mathematics
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Solutions — exercise 9

Section 1, Supplementary Exercises

16 **b)** How many distinct terms are there in the complete expansion of

$$\left(\frac{x}{2} + y - 3z\right)^5?$$

The Multinomial Theorem states that

$$\left(\sum_{i=1}^{k} x_i\right)^n = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} x_1^{n_1} \ldots x_k^{n_k}$$

where

$$\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! \ldots n_k!}.$$

So the number of terms in the expansion is equal to the number of non-negative solutions to the equation $n_1 + \cdots + n_k = n$, which is $\binom{n+k-1}{n}$ as is proved using the stars and bars technique. The number of distinct solutions is thus

$$\binom{5+3-1}{5} = \binom{7}{5} = \frac{7 \cdot 6}{2} = 21$$

c) What is the sum of all coefficients in the complete expansion?

The sum of the coefficients of a polynomial is just the value that such polynomial takes in 1, hence:

$$p(1,1,1) = \left(\frac{1}{2} + 1 - 3\right)^5 = -\left(\frac{3}{2}\right)^5 = -7 - \frac{19}{32}$$

18 **b)** Determine the number of non-negative integer solutions to the pair of equations

$$x_1 + x_2 + x_3 \leq 6, \qquad x_1 + x_2 + \cdots + x_5 \leq 15$$

where $x_i \geq 0$ and $1 \leq i \leq 5$.

Let $0 \leq k \leq 6$. For $x_1 + x_2 + x_3 = k$ there are $\binom{3+k-1}{k} = \binom{k+2}{k}$ solutions. Since $x_1 + x_2 + x_3 + x_4 + x_5 \leq 15$ this means that the remaining two numbers must satisfy $x_4 + x_5 \leq 15 - k$, consider $x_4 + x_5 = 15 - k$, $x_4, x_5 \geq 0$. Now there are $\binom{2+15-k-1}{15-k} = \binom{16-k}{15-k}$ solutions. Summing for $0 \leq k \leq 6$ gives the total number of solutions

$$\sum_{k=0}^{6} \binom{k+2}{k}\binom{16-k}{15-k} = 6132$$

$\boxed{28}$ **b)** In how many ways can one travel in the xy-plane from $(1,2)$ to $(5,9)$ if each move is one of the following types:

(R): $(x,y) \to (x+1,y)$

(U): $(x,y) \to (x,y+1)$

(D): $(x,y) \to (x+1,y+1)$

Since a diagonal moves takes the place of one horizontal move and one vertical move, the number of diagonal moves is $0 \leq D \leq 4$. The resulting cases are

$$\begin{array}{lr}
(0D):\ (4R):\ (7U): & 11!/(0!4!7!) \\
(1D):\ (3R):\ (6U): & 10!/(1!3!6!) \\
(2D):\ (2R):\ (5U): & 9!/(2!2!5!) \\
(3D):\ (1R):\ (5U): & 8!/(3!1!4!) \\
(4D):\ (0R):\ (3U): & 7!/(4!0!3!)
\end{array}$$

The answer is the sum of these five possibilities

$$\sum_{i=0}^{4} \frac{(11-i)!}{i!(4-i)!(7-i)!} = 2241$$

Section 2, Supplementary Exercises

$\boxed{7}$ **a)** For primitive statements p, q, find the dual of the statement

$$(\neg p \wedge \neg q) \vee (T_0 \wedge p) \vee p\,.$$

Forming the dual just wants you to replace p by $\neg p$ for each literal p, $\vee$ by $\wedge$ and vice versa and T_0 by F_0. This gives

$$(p \vee q) \wedge (F_0 \vee \neg p) \wedge \neg p.$$

b) Use the laws of logic to show that your result from part **a)** is logically equivalent to

$$p \wedge \neg q.$$

By the Laws of Logic we have

$$\begin{aligned}
(p \vee q) \wedge (F_0 \vee \neg p) \wedge \neg p &\Leftrightarrow (p \vee q) \wedge \neg p \wedge \neg p && \text{Identity laws} \\
&\Leftrightarrow (p \vee q) \wedge \neg p && \text{Absorption laws} \\
&\Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p) && \text{Distributive laws:} \\
&\Leftrightarrow F_0 \vee (q \wedge \neg p) && \text{Inverse laws} \\
&\Leftrightarrow q \wedge \neg p && \text{Identity laws}
\end{aligned}$$

this is sort of what we wanted to show. The reason this diverges with the statement above is that the book does not switch p to $\neg p$, and q to $\neg q$ when finding the dual, which is stupid.

10 Establish the validity of the argument

$$[(p \to q) \wedge [(q \wedge r) \to s] \wedge r] \to (p \to s)\,.$$

Just thinking about creating a truth table for this gives me an headache

$$\begin{aligned}
& [(p \to q) \wedge [(q \wedge r) \to s] \wedge r] \to (p \to s) && \textbf{Reasons} \\
\Leftrightarrow\ & \neg[(\neg p \vee q) \wedge [\neg(q \wedge r) \vee s] \wedge r] \vee (\neg p \vee s) && \text{Material implication } a \to b \Leftrightarrow \neg a \vee b \\
\Leftrightarrow\ & \neg[(\neg p \vee q) \wedge [\neg q \vee \neg r \vee s] \wedge r] \vee (\neg p \vee s) && \text{DeMorgans Laws } \neg(p \wedge q) = \neg p \vee \neg q \\
\Leftrightarrow\ & [\neg(\neg p \vee q) \vee \neg[\neg q \vee \neg r \vee s] \vee \neg r] \vee (\neg p \vee s) && \text{DeMorgans Laws } \neg(p \wedge q) = \neg p \vee \neg q \\
\Leftrightarrow\ & (p \wedge \neg q) \vee [q \wedge r \wedge \neg s] \vee \neg r \vee (\neg p \vee s) && \text{DeMorgans Laws } \neg(p \vee q) = \neg p \wedge \neg q \\
\Leftrightarrow\ & (p \wedge \neg q) \vee [q \wedge \neg s] \vee \neg r \vee (\neg p \vee s) && \text{Absorption laws} \\
\Leftrightarrow\ & (p \wedge \neg q) \vee q \vee s \vee \neg r \vee \neg p && (q \wedge \neg s) \vee s = q \vee s \\
\Leftrightarrow\ & p \vee q \vee s \vee \neg r \vee \neg p && (p \wedge \neg q) \vee q = p \vee q \\
\Leftrightarrow\ & T_0 \vee q \vee s \vee \neg r && \text{Inverse Laws } p \vee \neg p \Leftrightarrow T_0 \\
\Leftrightarrow\ & T_0 && \text{Domination laws}
\end{aligned}$$

Section 3, Supplementary Exercises

4 **a)** For positive integers m, n, r, with $r \leq \min(m, n)$, show that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \tag{1}$$

Equation (1) is know as the Chu-Vandermonde Identity. Let us briefly state two proofs for this interesting identity.

Algebraic proof. Recall that for every $x, y \in \mathbb{R}$ we have that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{2}$$

for all $n \in \mathbb{Z}^+$. This is know as the *binomial theorem*. Now we consider the binomial expansion of $(1+x)^{m+n}$

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

Another way to expand the binomial is by first using $(1+x)^{m+n} = (1+x)^m(1+x)^n$ then

$$\begin{aligned}
(1+x)^{m+n} &= (1+x)^m(1+x)^n \\
&= \left(\sum_{i=0}^{m} \binom{m}{i}\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right) \\
&= \left(\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots\right) \cdot \left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots\right) \\
&= \left(\binom{m}{0}\binom{n}{0}\right) x^0 + \left(\binom{m}{0}\binom{n}{1} + \binom{m}{1}\binom{n}{0}\right) x^1 \\
&+ \left(\binom{m}{0}\binom{n}{2} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{0}\right) x^2 + \cdots
\end{aligned}$$

Thus, we can conclude that the coefficient of x^k in the above expansion is

$$\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}$$

Therefore, by comparing the coefficients of x^k

$$\sum_{r=0}^{k} \binom{m}{k}\binom{n}{k-r} = \binom{m+n}{k}$$

which is what we wanted to show. □

Combinatorial Proof. Suppose there are m boys and n girls in a class and you're asked to form a team of k pupils out of these $m+n$ students, with $0 \leq k \leq m+n$. You can do this in $\binom{m+n}{k}$ ways. But, now we count in rather a different manner. To form the team, you can choose i boys and $k-i$ girls for some fixed k. There are $\binom{m}{i}\binom{n}{k-1}$ ways to do this. Now, either you can have 0 boys and k girls, or 1 boy and $k-1$ girls, or 2 boys and $k-2$ girls, or .... That is, there are $\sum_{r=0}^{k} \binom{m}{r}\binom{n}{n-r}$ ways to form the team.

Thus, we derive at our result

$$\sum_{r=0}^{k} \binom{m}{k}\binom{n}{k-r} = \binom{m+n}{k}$$ □

b) For n a positive integer, show that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Note that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ such that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{n+n}{n} = \binom{2n}{n}$$

where part **a)** was used in the second equality.

$\boxed{9}$ Let $A, B, C \in \mathscr{U}$. Prove that

$$(A \cap B) \cup C = A \cap (B \cup C)$$

if and only if $C \subseteq A$.

First assume that $C \not\subseteq A$ this means there exists some element in $x \in C$ such that $x \notin A$. As $x \in C$ then $x \in (A \cap B) \cup C$, however $x \notin [A \cap (B \cup C)]$, since $x \notin A$.

Assume now that $C \subseteq A$, this means for *every* $y \in C$ then $y \in A$. Pick some $x \in C$. Then $x \in B \cup C$ as $x \in C$, and $x \in [A \cap (B \cup C)]$ since $x \in A$. On the other hand, $x \in C$ so $x \in [(A \cap B) \cup C]$ and we are done.

Section 4, Supplementary Exercises

$\boxed{6}$ For $n \in \mathbb{Z}^+$ define the sum s_n by the formula

$$s_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(n-1)}{n!} + \frac{n}{(n+1)!}$$

d) Conjecture a formula for the sum of the terms in s_n and verify your conjecture for all $n \in \mathbb{Z}^+$ by the Principle of Mathematical Induction.

Testing a few values we see that

$$s_4 = \frac{119}{120} = 1 - \frac{1}{5!}, \quad s_5 = \frac{719}{720} = 1 - \frac{1}{6!}, \quad \text{and} \quad s_6 = \frac{5039}{5040} = 1 - \frac{1}{7!},$$

and based on this calculation I conjecture that

$$s_n = 1 - \frac{1}{(1+n)!} \qquad \forall\, n \in \mathbb{Z}^+.$$

Base case: $s_0 = 0$ and $1 - 1/0! = 0$. Perhaps more interestingly is the case $n = 1$, then $s_1 = 1/2!$ and $1 - 1/2! = 1/2!$.
Inductive step: Assume that the statement holds for some $n = k \in \mathbb{Z}^+$ that is

$$s_k = 1 - \frac{1}{(1+k)!}$$

wish to show that this implies that the statement holds for $n = k + 1$. Since we have $1/(k+1)! = (k+2)/[(k+2)(k+1)!] = (k+2)/(k+2)!$, some elementary algebra gives

$$\begin{aligned} s_{k+1} = s_k + \frac{k+1}{(k+2)!} &= 1 - \frac{1}{(1+k)!} + \frac{k+1}{(k+2)!} \\ &= 1 - \frac{k+2}{(k+2)} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}\,, \end{aligned}$$

and the rest now follows by the Principle of Mathematical Induction.

$\boxed{7}$ For all $n \in \mathbb{Z}$, $n \geq 0$ prove that

d) $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Every number is either divisible by 3, one away from being divisible by 3 or two away from being divisible by 3. We can not be 3 away from being divisible by 3, as 3 is divisible by 3. This can be written mathematically as

$$n = 3k, \quad n = 3k+1, \quad \text{or} \quad n = 3k+2$$

Testing each of these cases gives

$$\begin{aligned} (3k)^3 + (3k+1)^3 + (3k+1)^3 &= 9(9k^3 + 9k^2 + 5k + 1) \\ (3k+1)^3 + (3k+2)^3 + (3k+3)^3 &= 9(9k^3 + 18k^2 + 14k + 4) \\ (3k+2)^3 + (3k+3)^3 + (3k+4)^3 &= 9(9k^3 + 27k^2 + 29k + 11) \end{aligned}$$

from which we can see that the expression on the right is always divisible by 3. Let us for completeness sake also prove the statement using induction.
Base case: For $n = 0$ we have $0^3 + (0+1)^3 + (0+2)^3 = 0 + 1 + 8 = 9$, and this proves the base case.
Inductive step: Assume that the statement holds for some $n = k \in \mathbb{Z}^+$ that is

$$k^3 + (k+1)^3 + (k+2)^3 \text{ is divisible by 9.}$$

Wish to show that this implies that the statement holds for $n = k + 1$

$$\begin{aligned} (k+1)^3 + (k+2)^3 + [(k+3)^3] &= (k+1)^3 + (k+2)^3 + [k^3 + 9k^2 + 27k + 27] \\ &= [k^3 + (k+1)^3 + (k+2)^3] + 9(k^2 + 3k + 3) \end{aligned}$$

by the induction hypothesis we know that the 9 divides the expression in the squared brackets, similarly it is trivial to see that 9 divides $9(k^2 + 3k + 3)$. By the principle of induction 9 divides $n^3 + (n+1)^3 + (n+2)^3$ for all $n \in \mathbb{Z}^+$.