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# MA0301 Elementary discrete mathematics Spring 2018

**Solutions — exercise 5**

- 2 Let  $Y := \{1, 2, 3, 4, \dots, 600\}$ . Use the inclusion-exclusion principle to find the number of positive integers  $Y$  that are not divisible by 3, 5 or 7.

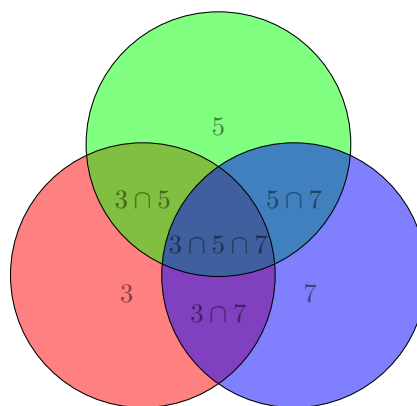


Figure 1

While not relevant for the exam, the Venn diagram displayed in figure 1 can shed some light on why the inclusion-exclusion principle works. We are interested in finding cardinality<sup>1</sup> of the union  $3 \cup 5 \cup 7$ , where the numbers are now interpreted as sets e.g.  $3 = \{3n \mid n \in Y\}$  is the subset of  $Y$  consisting of all numbers divisible by 3 and so forth. The following argument might be a bit hard to digest, so try to read it while carefully studying figure 1.

Simply adding the number of elements in each set will give us an over-estimate. As an example 15 is counted twice, once in the set with numbers divisible by 3, and once in the set consisting of numbers divisible by 5. Thus  $|3 \cup 5 \cup 7| \neq |3| + |5| + |7|$ . To remedy this we can remove every union of two integers:  $|3 \cap 5| + |3 \cap 7| + |5 \cap 7|$ . However, this creates another problem. Take  $105 = 3 \cdot 5 \cdot 7$ . This number is included three times as it lies in 3, 5 and 7, alas it is also removed three times since it lies in  $3 \cap 5$ ,  $3 \cap 7$  and  $5 \cap 7$ . Thus, we need to include back every number that is a multiple of 3, 5 and 7. This gives us our final expression,

$$|3 \cup 5 \cup 7| = |3| + |5| + |7| - |3 \cap 5| - |3 \cap 7| - |5 \cap 7| + |3 \cap 5 \cap 7|$$

<sup>1</sup>In mathematics, the cardinality of a set is a measure of the "number of elements of the set".

which is precisely the inclusion-exclusion principle. The number of integers that are a multiple of  $k$  beneath  $n$  are precisely  $\lfloor n/k \rfloor$ . Every number divisible by 3, 5 or 7 is thus

$$\begin{aligned} |3 \cup 5 \cup 7| &= |3| + |5| + |7| - |3 \cap 5| - |3 \cap 7| - |5 \cap 7| + |3 \cap 5 \cap 7| \\ &= \lfloor \frac{600}{3} \rfloor + \lfloor \frac{600}{5} \rfloor + \lfloor \frac{600}{7} \rfloor - \lfloor \frac{600}{3 \cdot 5} \rfloor - \lfloor \frac{600}{3 \cdot 7} \rfloor - \lfloor \frac{600}{5 \cdot 7} \rfloor + \lfloor \frac{600}{3 \cdot 5 \cdot 7} \rfloor \\ &= 200 + 120 + 85 - 40 - 28 - 17 + 5 = 325 \end{aligned}$$

Thus, there are  $|Y| - |3 \cup 5 \cup 7| = 275$  numbers not divisible by 3, 5 or 7 amongst the first 600 integers.

[3] Use the principle of induction to show that for all natural numbers  $n$ ,

$$4 \sum_{i=1}^n i(i+2)(i+4) = n(n+1)(n+4)(n+5)$$

**Base case:** For  $n = 1$

$$LHS = 4 \sum_{i=1}^1 i(i+2)(i+4) = 4 \cdot 1 \cdot 3 \cdot 5 = 60, \quad RHS = 1(1+1)(1+4)(1+5) = 60$$

So the hypothesis holds for the base case.

**inductive step:** Assume that for all  $k$  we have

$$4 \sum_{i=1}^k i(i+2)(i+4) = k(k+1)(k+4)(k+5)$$

We wish to prove that this implies that the hypothesis holds for  $k+1$ .

$$\begin{aligned} RHS &= (k+1)(k+2)(k+5)(k+6) \\ LHS &= 4 \sum_{i=1}^{k+1} i(i+2)(i+4) \\ &= 4(k+1)(k+3)(k+5) + 4 \sum_{i=1}^k i(i+2)(i+4) \\ &= 4(k+1)(k+3)(k+5) + k(k+1)(k+4)(k+5) \\ &= [4(k+3) + k(k+4)](k+1)(k+5) \\ &= (k+1)(k+2)(k+5)(k+6) \end{aligned}$$

As  $LHS = RHS$  the rest follows by induction. The steps performed in the calculation above were respectively: (1)

$$\sum_{i=1}^{n+1} a_i = a_1 + a_2 + \cdots + a_n + a_{n+1} = (a_1 + a_2 + \cdots + a_n) + a_{n+1} = a_{n+1} + \sum_{i=1}^n a_i.$$

(2) use the induction hypothesis. (3) To see this clearer we have  $4(k+3) + k(k+4) = k^2 + 8k + 12 = (k+6)(k+2)$  either by the quadratic formula, or inspection (as  $6+2=8$  and  $6 \cdot 2 = 12$ ).

7 Use the laws of set theory to show for arbitrary sets  $A, B, C$  that:

a) If  $(A \cup B) \subseteq (A \cap B)$  then  $A = B$ .

I got stuck on this one, lol! To prove that  $A = B$  we show that  $A \subseteq B$  and  $B \subseteq A$  when the conditions holds

$$\begin{aligned} A &\subseteq A \cup B \subseteq A \cap B \subseteq B \\ B &\subseteq A \cup B \subseteq A \cap B \subseteq A, \end{aligned}$$

and we are done.

b)  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

So we are to prove De Morgan's first law, which suddenly is different in set notation compared to logic... Again we prove both inclusions.

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ : Assume that  $x \in \overline{A \cap B}$ . Then  $x \notin A \cap B$ , so either  $x \notin A$  or  $x \notin B$ . Which, is the same as to say that  $x \in \overline{A}$  or  $x \in \overline{B}$ . In either case  $x \in \overline{A} \cup \overline{B}$ .

$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ : Let  $x \in \overline{A} \cup \overline{B}$ . Then we know that either  $x \notin A$  or  $x \notin B$ . Then it is not in the intersection either  $x \notin A \cap B$ , which is to say  $x \in \overline{A \cap B}$  and we are done.

c)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

As before we prove both inclusions

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ : Assume that  $x \in A \cap (B \cup C)$ . This implies that  $x \in A$  and  $x \in B \cup C$ . The last expression is the same as saying that  $x \in B$  or  $x \in C$ . If  $x \in B$  then  $x \in A \cap B$  (as  $x \in A$ ), otherwise if  $x \in C$  then  $x \in A \cap C$ . As  $x$  is in either of those we at least have  $x$  in the union  $x \in (A \cap B) \cup (A \cap C)$ .

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ : Assume now that  $x \in (A \cap B) \cup (A \cap C)$ , as such  $x \in (A \cap B)$  or  $x \in (A \cap C)$ . In the first case  $x \in A$  and  $x \in B$ , in the latter  $x \in A$  and  $x \in C$ . Regardless,  $x \in A$ . This leaves us with two cases either  $x \in B$  or  $x \in C$ . In either case,  $x$  has to lie in the union  $x \in B \cup C$ . This proves the inclusion  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$  and we are done.

## Section 4.1

8 Prove each of the following for all  $n \geq 1$  by the Principle of Mathematical induction

a)  $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

**Base case:** For  $n = 1$  we have  $(2 \cdot 1 - 1)^2 = 1$ , while the right hand-side becomes  $(1 \cdot 1 \cdot 3)/3 = 1$ .

**Inductive step:** Assume that the statement holds for  $n = k$ , in other words

$$\sum_{i=1}^n (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Need to show that this implies that the statement holds for  $n = k + 1$ .

$$\begin{aligned} RHS &= \frac{(k+1)(2k+1)(2k+3)}{3} \\ LHS &= \sum_{i=1}^{k+1} (2i-1)^2 \\ &= (2k+1)^2 + \sum_{i=1}^k (2i-1)^2 \\ &= (2k+1)^2 + \frac{k(2k-1)(2k+1)}{3} \\ &= \frac{(2k+1)[3(2k+1) + k(2k-1)]}{3} \\ &= \frac{(k+1)(2k+1)(2k+3)}{3} \end{aligned}$$

As  $3(2k+1) + k(2k-1) = 2k^2 + 5k + 3 = (k+1)(2k+3)$  either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

b)  $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Wow, so fun solving a thousand similar exercises..

**Base case:**  $n(n+2) = 1(1+2) = 3$  and  $(1 \cdot 2 \cdot 9)/6 = 3$ , so it holds for the base case.

**Inductive step:** Assume that the statement holds for  $n = k$ , that is

$$\sum_{i=1}^k i(i+2) = \frac{k(k+1)(2k+7)}{6}$$

Wish to show that this implies that the statement holds for  $n = k + 1$ .

$$\begin{aligned}
 RHS &= \frac{(k+1)(k+2)(2k+9)}{6} \\
 LHS &= \sum_{i=1}^{k+1} i(i+2) \\
 &= (k+1)(k+3) + \sum_{i=1}^k i(i+2) \\
 &= (k+1)(k+3) + \frac{k(k+1)(2k+7)}{6} \\
 &= \frac{(k+1)[(6(k+3) + k(2k+7))]}{6} \\
 &= \frac{(k+1)(k+2)(2k+9)}{6}
 \end{aligned}$$

As  $6(k+3) + k(2k+7) = 2k^2 + 13k + 18 = (k+2)(2k+9)$  either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

$$\text{c) } \sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$$

While the question *clearly* states that the proof has to be done by induction, I can not help myself to show the following proof as the series is telescoping

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \frac{1}{i} - \frac{1}{i+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

**Base case:** For  $n = 1$  we have  $\sum_{i=1}^1 1/(i(i+1)) = 1/2$  and  $n/(n+1) = 1/2$ .

**Inductive step:** Assume that the statement holds for  $n = k$ , that is

$$\sum_{i=1}^k \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Wish to show that this implies that the statement holds for  $n = k + 1$

$$\begin{aligned}
 RHS &= \frac{k+1}{k+2} \\
 LHS &= \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \\
 &= \frac{1}{(k+1)(k+2)} + \sum_{i=1}^k \frac{1}{i(i+1)} \\
 &= \frac{1}{(k+1)(k+2)} + \frac{k}{k+1} \\
 &= \frac{1 + k(k+2)}{(k+2)(k+1)} = \frac{(1+k)^2}{(k+2)(k+1)} = \frac{1+k}{k+2}
 \end{aligned}$$

As  $1 + k(k + 2) = k^2 + 2k + 1^2 = (k + 1)^2$  either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

- 12 a) Prove that  $(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta$  where  $i \in \mathbb{C}$  and  $i^2 = -1$ .

By De Moivre's laws we have

$$(\cos \theta + i \sin \theta)^2 = (e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i \sin 2\theta$$

However, this might be considered cheating as we are supposed to prove this theorem later. Expanding the square gives the same results

$$\begin{aligned} (\cos \theta + i \sin \theta)^2 &= \cos^2 \theta + 2i \cos \theta \sin \theta + i^2 \sin^2 \theta \\ &= (\cos^2 \theta - \sin^2 \theta) + i(2 \cos \theta \sin \theta) = \cos(2\theta) + i \sin(2\theta) \end{aligned}$$

- b) Using induction, prove that for all  $n \in \mathbb{Z}^+$ ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

much easier to prove by other means by w/e.

**Base case:** The case  $n = 0$  holds as both sides are 1, while the case  $n = 1$  has been handled above.

**Inductive step:** Assume that the statement holds for  $n = k$  that is

$$(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta,$$

want to prove that this implies that the statement holds for  $n = k + 1$ . Now

$$\begin{aligned} RHS &= \cos(k + 1)\theta + i \sin(k + 1)\theta \\ LHS &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\ &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\ &= (\cos \theta \cos k\theta - \sin \theta \sin k\theta) + i(\sin k\theta \cos \theta + \sin \theta \cos k\theta) \\ &= \cos(k + 1)\theta + i \sin(k + 1)\theta \end{aligned}$$

since  $RHS = LHS$  the statement follows by induction.

- c) Verify that  $1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$ , and compute  $(1 + i)^{100}$ .

As  $\cos \pi/4 = \sin \pi/4 = 1/\sqrt{2}$ , the first part is trivial

$$\sqrt{2}(\cos \pi/4 + i \sin \pi/4) = \sqrt{2} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 1 + i$$

and so  $(1 + i)^{100} = (\sqrt{2})^{100}(\cos \pi/4 + i \sin \pi/4)^{100} = 2^{50}(\cos 25\pi + i \sin 25\pi) = -2^{50}$

- 16 a) For  $n = 3$  let  $X_3 = 1, 2, 3$ . Now consider the sum

$$s_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} = \sum_{\emptyset \neq A \subseteq X_3} \frac{1}{p_A}$$

where  $p_A$  denotes the product of all elements in a non-empty subset  $A$  of  $X_3$ . Note that the sum is taken over all the non-empty subsets of  $X_3$ . Evaluate this sum

Finding the common denominator a straight forward computation yields

$$\begin{aligned} s_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} \\ &= \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2} \right) + \left( \frac{1}{3} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} \right) \\ &= \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2} \right) + \left( 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2} \right) \frac{1}{3} = 2 + (1 + 2) \frac{1}{3} = 3 \end{aligned}$$

- b) Repeat the calculation in ?? a) for  $s_2$  and  $s_4$ .

By “accident”  $s_2$  has already been computed

$$s_2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2} = 2$$

$s_4$  is done in the same matter, albeit a bit more work has to be done

$$\begin{aligned} s_4 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} \\ &\quad + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 4} + \frac{1}{1 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \\ &= \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} \right) \\ &\quad + \left[ \frac{1}{4} + \frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \right] \\ &= \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} \right) \\ &\quad + \left[ 1 + \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} \right) \right] \cdot \frac{1}{4} \\ &= s_3 + [1 + s_3] \frac{1}{4} = 3 + (1 + 3) \frac{1}{4} = 4 \end{aligned}$$

- c) Conjecture a general result suggested by the calculations from ?? a)?? b). Prove your conjecture using the Principle of Mathematical induction.

Interesting problem! I conjecture that for every  $n \in \mathbb{N}$ ,  $s_n = n$ .

**Base case:** For  $n = 1$  we simply have  $s_n = \frac{1}{1} = 1$ , and this proves the base case.

**Inductive step:** We assume that for every  $k$  we have

$$s_k = \sum_{\emptyset \neq A \subset X_k} \frac{1}{p_A}$$

wish to prove that this implies that the conjecture holds for  $s_{k+1}$ , then

$$s_{k+1} = \sum_{\emptyset \neq A \subset X_{k+1}} \frac{1}{p_A} = \sum_{\emptyset \neq A \subset X_k} \frac{1}{p_B} + \sum_{\{k+1\} \subseteq A \subset X_{k+1}} \frac{1}{p_C}$$

Where the first sum is taken over all non-empty subsets  $B$  of  $X_k$  and the second sum over all subsets  $C$  of  $X_{k+1}$  that contain  $k+1$ . In other words, we do as before and isolate all the terms that contain  $k+1$ . As seen before we can factor out the  $1/(k+1)$  term giving

$$s_{k+1} = s_k + \frac{1}{1+k}(1 + s_k) = k + \frac{1}{1+k}(1+k) = 1+k$$

Why the second sum equals  $(1 + s_k)/(1+k)$  has been shown in excruciating detail in ?? a)?? b). The proof for the conjecture follows now follows induction.

17 For  $n \in \mathbb{Z}^+$ . We define the  $n$ 'th harmonic number  $H_n$  as the sum of the first  $n$  reciprocals (of the natural integers),  $H_n = \sum_{i=1}^n 1/i$ .

**a)** For all  $n \in \mathbb{N}$  prove that  $1 + \frac{n}{2} \leq H_{2^n}$ .

As is with this exercise we use the Principle of Mathematical Induction ad nauseam.

**Base case:**  $n = 1$ . Then  $1 + n/2 = 1 + 1/2$  and  $H_{2^1} = H_2 = 1 + 1/2$ . Thus, the hypothesis holds for the base case.

**Inductive step:** Assume that the statements holds for  $n = k$  that is

$$1 + \frac{n}{2} \leq H_{2^{n+1}}$$

wish to show that this implies that the inequality holds for  $n = k+1$ . Then,  $LHS = 1 + \frac{k+1}{2}$  and

$$\begin{aligned} RHS = H_{2^{k+1}} &= H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k} \\ &\geq H_{2^k} + \frac{1}{2^k+2^k} + \frac{1}{2^k+2^k} + \cdots + \frac{1}{2^k+2^k} \\ &= H_{2^k} + \frac{2^k}{2 \cdot 2^k} = H_{2^k} + \frac{1}{2} \geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2} \end{aligned}$$

as  $LHS = RHS$  the rest follows by induction.



b) Prove that for all  $n \in \mathbb{Z}^+$ ,

$$\sum_{j=1}^n jH_j = \left\lfloor \frac{n(n+1)}{2} \right\rfloor H_{n+1} - \left\lfloor \frac{n(n+1)}{4} \right\rfloor$$

**Base case:** If  $n = 0$  then  $RHS$ , and  $LHS$  are zero. This proves the base case.

**Inductive step:** Assume that

$$\sum_{j=1}^k jH_j = \left\lfloor \frac{k(k+1)}{2} \right\rfloor H_{k+1} - \left\lfloor \frac{k(k+1)}{4} \right\rfloor$$

holds for every  $n = k$ . Wish to prove that this implies that the statement holds for  $n = k + 1$ . Now

$$\begin{aligned} LHS &= \left\lfloor \frac{(k+1)(k+2)}{2} \right\rfloor H_{k+2} - \left\lfloor \frac{(k+1)(k+2)}{4} \right\rfloor \\ RHS &= \sum_{j=1}^{k+1} jH_j = (k+1)H_{k+1} + \sum_{j=1}^k jH_j \\ &= (k+1)H_{k+1} + \left\lfloor \frac{k(k+1)}{2} \right\rfloor H_{k+1} - \left\lfloor \frac{k(k+1)}{4} \right\rfloor \\ &= \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{k(k+1)}{4} \\ &= \frac{(k+1)(k+2)}{2} \left[ H_{k+2} - \frac{1}{k+2} \right] - \frac{k(k+1)}{4} \\ &= \frac{(k+1)(k+2)}{2} H_{k+2} - \frac{2(k+1)}{4} - \frac{k(k+1)}{4} \\ &= \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{(k+1)(k+2)}{4} \end{aligned}$$

As  $LHS = RHS$  the proof follows by induction. It was REALLY hard to see that  $H_{k+1} = H_{k+2} - 1/(k+2)$  haha.