MA0301 Elementary discrete mathematics Spring 2018

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Exercise 5

$\boxed{2}$ Let $Y := \{1, 2, 3, 4, \dots, 600\}$. Use the inclusion-exclusion principle to find the number of positive integers Y that are not divisible by 3, 5 or 7.

$\boxed{3}$ Use the principle of induction to show that for all natural numbers n,

$$4\sum_{i=1}^{n} i(i+2)(i+4) = n(n+1)(n+4)(n+5).$$

$\boxed{7}$ Use the laws of set theory to show for arbitrary sets A, B, C that:

a) If $(A \cup B) \subseteq (A \cap B)$ then $A = B$.

b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Section 4.1

$\boxed{8}$ Prove each of the following for all $n \geq 1$ by the Principle of Mathematical induction

a) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$.

b) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$.

c) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

$\boxed{12}$ **a)** Prove that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i \sin 2\theta$ where $i \in \mathbb{C}$ and $i^2 = -1$.

b) Using induction, prove that for all $n \in \mathbb{Z}^+$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

c) Verify that $1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$, and compute $(1 + i)^{100}$.

$\boxed{16}$ **a)** For $n = 3$ let $X_3 = 1, 2, 3$. Now consider the sum

$$s_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} = \sum_{\emptyset \neq A \subseteq X_3} \frac{1}{p_A},$$

where p_A denotes the product of all elements in a non-empty subset A of X_3. Note that the sum is taken over all the non-empty subsets of X_3. Evaluate this sum

b) Repeat the calculation in step **a)** for s_2 and s_4.

c) Conjecture a general result suggested by the calculations from steps **a)** and **b)**. Prove your conjecture using the Principle of Mathematical induction.

$\boxed{17}$ For $n \in \mathbb{Z}^+$. We define the n'th harmonic number H_n as the sum of the first n reciprocals (of the natural integers), $H_n = \sum_{i=1}^{n} 1/i$.

a) For all $n \in \mathbb{N}$ prove that $1 + \frac{n}{2} \leq H_{2^n}$.

b) Prove that for all $n \in \mathbb{Z}^+$,

$$\sum_{j=1}^{n} jH_j = \left[\frac{n(n+1)}{2}\right] H_{n+1} - \left[\frac{n(n+1)}{4}\right].$$