Norwegian University of Science and Technology
Department of Mathematics

Solutions — exercise 6

5 Use the alternative principle of induction to show that if u_n is defined recursively by the rules $u_1 = 1$, $u_2 = 5$ and for all $n > 1$, $u_{n+1} = 5u_n - 6u_{n-1}$, then $u_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Base case : For $n = 1, 2$ we have

$$u_1 = 1 = 3^1 - 2^1 \qquad \text{and} \qquad u_2 = 5 = 3^2 - 2^2,$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statement

$$u_k = 3^k - 2^k$$

holds for all k, $k < n$. Wish to show that it then holds for $k = n + 1$.

$$\begin{aligned} u_{k+2} = 5u_{k+1} - 6u_k &= 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k) \\ &= (5 \cdot 3^{k+1} - 5 \cdot 2^{k+1}) - (2 \cdot 3^{k+1} - 3 \cdot 2^{k+1}) \\ &= 3 \cdot 3^{k+1} - 2 \cdot 2^{k+1} = 3^{k+2} - 2^{k+2} \end{aligned}$$

which is what we wanted to show. To see the second equality a bit clearer we have $6 \cdot 3^k = 2 \cdot 3 \cdot 3^k = 2 \cdot 3^{k+1}$ and similarly for $6 \cdot 2^k$.

6 **a)** Guess a formula for $\sum_{i=1}^{n} bi + c$, where b, c are given numbers, and prove it using the principle of induction.

The sum of the first n natural numbers, is $n(n+1)/2$ thus,

$$\sum_{i=1}^{n} bi + c = b\sum_{i=1}^{n} i + \sum_{i=1}^{n} c = b\frac{n(n+1)}{2} + cn.$$

Base case: For $n = 1$ we have

$$LHS = \sum_{i=1}^{1} bi + c = b + c\,, \qquad RHS = b\frac{1(1+1)}{2} + c \cdot 1 = b + c.$$

so the hypothesis holds for the base case.

Inductive step: Assume that the statement holds for $n = k$, in other words

$$\sum_{i=1}^{k} bi + c = b\frac{k(k+1)}{2} + ck.$$

Need to show that this implies that the statement holds for $n = k + 1$

$$\begin{aligned}
RHS &= \frac{(k+1)(k+2)}{2} + c(k+1) \\
LHS &= \sum_{i=1}^{k+1} bi + c \\
&= b(k+1) + c + \sum_{i=1}^{k} bi + c \\
&= b(k+1) + c + b\frac{k(k+1)}{2} + ck = b\frac{(k+1)(k+2)}{2} + c(k+1)
\end{aligned}$$

As $3(2k+1) + k(2k-1) = 2k^2 + 5k + 3 = (k+1)(2k+3)$ either by inspection or the quadratic formula. The rest follows now by induction, and thus concludes the proof.

b) Use $6\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)$ and the result of step **a)** to write down a formula for $\sum_{i=1}^{n} ai^2 + bi + c$, where a,b,c are given numbers.

Splitting the sum and using step **a)** immediately gives

$$\begin{aligned}
\sum_{i=1}^{n} ai^2 + bi + c &= a\sum_{i=1}^{2} i^2 + \sum_{i=1}^{n} bi + c \\
&= a\frac{n(n+1)(2n+1)}{6} + b\frac{n(n+1)}{2} + cn = \frac{n(n+1)(2an+a+3b)}{6} + cn.
\end{aligned}$$

Section 5.1

9 Complete the proof of Theorem 1

Theorem 0.1. *For any sets $A, B, C \subseteq \mathscr{U}$:*

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

The book has proven item a), thus we need to prove the remaining three parts.

Proof of Theorem 1 **b)**.

$$\begin{aligned}
A \times (B \cup C) &= \{(x,y) \mid x \in A \text{ and } y \in (B \cup C)\} \\
&= \{(x,y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C)\} \\
&= \{(x,y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ or } y \in C)\} \\
&= \{(x,y) \mid x \in A \text{ and } y \in B\} \cup \{(x,y) \mid x \in A \text{ and } y \in C\} \\
&= (A \times B) \cup (A \times C)\,.
\end{aligned}$$
□

Proof of Theorem 1 **c)**.

$$\begin{aligned}
(A \cap B) \times C &= \{(x,y) \mid x \in (A \cap B) \text{ and } y \in C\} \\
&= \{(x,y) \mid (x \in A \text{ and } x \in B) \text{ and } y \in C\} \\
&= \{(x,y) \mid (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in C)\} \\
&= \{(x,y) \mid x \in A \text{ and } y \in C\} \cap \{(x,y) \mid x \in B \text{ and } y \in C\} \\
&= (A \times C) \cap (B \times C)\,.
\end{aligned}$$
□

Proof of Theorem 1 **d)**.

$$\begin{aligned}
(A \cup B) \times C &= \{(x,y) \mid x \in (A \cup B) \text{ and } y \in C\} \\
&= \{(x,y) \mid (x \in A \text{ or } x \in B) \text{ and } y \in C\} \\
&= \{(x,y) \mid (x \in A \text{ and } y \in C) \text{ or } (x \in B \text{ and } y \in C)\} \\
&= \{(x,y) \mid x \in A \text{ and } y \in C\} \cup \{(x,y) \mid x \in B \text{ and } y \in C\} \\
&= (A \times C) \cup (B \times C)\,.
\end{aligned}$$
□

$\boxed{11}$ For $A, B, C \subset \mathscr{U}$, prove that

$$A \times (B - C) = (A \times B) - (A \times C)$$

Proof. $B - C$ means everything in B except everything in C. The proof is done as before

$$\begin{aligned} A \times (B - C) &= \{(x, y) \mid x \in A \text{ and } y \in B - C\} \\ &= \{(x, y) \mid x \in A \text{ and } (y \in B \text{ and } y \notin C)\} \\ &= \{(x, y) \mid (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C)\} \\ &= \{(x, y) \in A \times B\} \cap \{(x, y) \notin A \times C\} \\ &= (A \times B) - (A \times C)\,. \end{aligned}$$

□

Section 7.2

$\boxed{6}$ For sets A, B and C, consider relations $\mathscr{R}_1 \subseteq A \times B$, $\mathscr{R}_2 \subseteq B \times C$, and $\mathscr{R}_3 \subseteq B \times C$. Prove that:

a) $\mathscr{R}_1 \circ (\mathscr{R}_2 \cup \mathscr{R}_3) = (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_1 \circ \mathscr{R}_3)$,

As usual we prove the inclusion both ways.

$\mathscr{R}_1 \circ (\mathscr{R}_2 \cup \mathscr{R}_3) \subseteq (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_1 \circ \mathscr{R}_3)$: Let $(x, z) \in \mathscr{R}_1 \circ (\mathscr{R}_2 \cup \mathscr{R}_3)$, then there exists some $y \in B$, $(x, y) \in \mathscr{R}_1$, $(y, z) \in \mathscr{R}_2 \cup \mathscr{R}_3$. By splitting this implies that for some $y \in B$, $((x, y) \in \mathscr{R}_1, (y, z) \in \mathscr{R}_2)$ or $((x, y) \in \mathscr{R}_1, (y, z) \in \mathscr{R}_3)$. So $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$ or $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$. Which is the same as $(x, z) \in (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_1 \circ \mathscr{R}_2)$, and this proves the inclusion.

$(\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_1 \circ \mathscr{R}_3) \subseteq \mathscr{R}_1 \circ (\mathscr{R}_2 \cup \mathscr{R}_3)$: Let $(x, y) \in (\mathscr{R}_1 \circ \mathscr{R}_2) \cup (\mathscr{R}_1 \circ \mathscr{R}_3)$. Then, $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$ or $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_3$. Assume without loss of generality that $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$. Then there exists an element $y \in B$ so that $(x, y) \in \mathscr{R}_1$ and $(y, z) \in \mathscr{R}_2$. But $(y, z) \in \mathscr{R}_2$ means that $(y, z) \in \mathscr{R}_2 \cup \mathscr{R}_3$, so $(x, z) \in \mathscr{R}_1 \circ (\mathscr{R}_2 \cup \mathscr{R}_3)$.

b) $\mathscr{R}_1 \circ (\mathscr{R}_2 \cap \mathscr{R}_3) \subseteq (\mathscr{R}_1 \circ \mathscr{R}_1) \cap (\mathscr{R}_1 \circ \mathscr{R}_3)$.

To see that equality can not hold in the relation above let $A = B = C = \{1, 2, 3\}$ with $\mathscr{R}_1 = (1, 2), (1, 1)$, $\mathscr{R}_2 = (2, 3)$, $\mathscr{R}_3 = (1, 3)$. Then $\mathscr{R}_1 \circ (\mathscr{R}_1 \circ \mathscr{R}_3) = \mathscr{R}_1 \circ \emptyset = \emptyset$, but $(\mathscr{R}_1 \circ \mathscr{R}_2) \circ (\mathscr{R}_1 \circ \mathscr{R}_3) = (1, 3) \neq \emptyset$. The proof is nearly identical to the proof above

$\mathscr{R}_1 \circ (\mathscr{R}_2 \cap \mathscr{R}_3) \subseteq (\mathscr{R}_1 \circ \mathscr{R}_1) \cap (\mathscr{R}_1 \circ \mathscr{R}_3)$: Let $(x, z) \in \mathscr{R}_1 \circ (\mathscr{R}_2 \cap \mathscr{R}_3)$, then there exists some $y \in B$, $(x, y) \in \mathscr{R}_1$, $(y, z) \in \mathscr{R}_2 \cap \mathscr{R}_3$. This implies that for some $y \in B$, $(x, y \in \mathscr{R}_1, (y, z) \in \mathscr{R}_2)$ *and* $((x, y) \in \mathscr{R}_1, (y, z) \in \mathscr{R}_3)$. So $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$ *and* $(x, z) \in \mathscr{R}_1 \circ \mathscr{R}_2$. Which is the same as $(x, z) \in (\mathscr{R}_1 \circ \mathscr{R}_2) \cap (\mathscr{R}_1 \circ \mathscr{R}_2)$, and this proves the inclusion.

15 **a)** Draw the digraph $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d, e, f\}$ and $E_1 = \{(a, b), (a, d), (b, c), (b, e), (d, b), (d, e), (e, c), (e, f), (f, d)\}$.

Just connecting the different points immediately gives figure 1

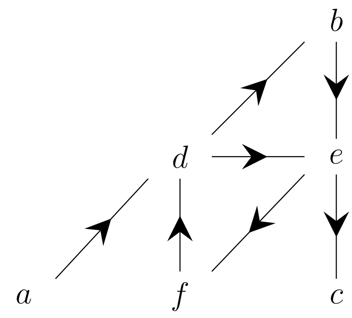

Figure 1

b) Draw the undirected graph $G_1 = (V_2, E_2)$ where $V_2 = \{s, t, u, v, w, x, y, z\}$ and

$$E_2 = \{\{s,t\}, \{s,u\}, \{s,x\}, \{t,u\}, \{t,w\}, \{u,w\}, \{u,x\}, \{v,w\}, \{v,x\}, \{v,y\}, \{w,z\}, \{x,y\}\}$$

Sorting and connecting the different labels immediately gives figure 2

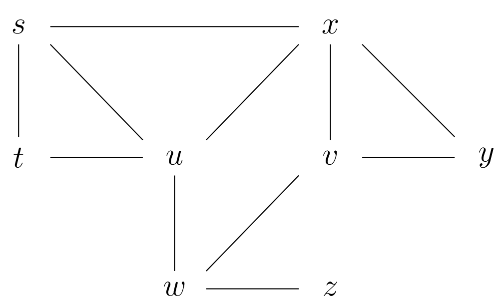

Figure 2

$\boxed{18}$ For $A = \{v, w, x, y, z\}$, each of the following is the $(0,1)$-matrix for a relation $\mathscr{R}$ on A. Here the rows and the columns are indexed in the order v, w, x, y, z. Determine the relation $\mathscr{R} \subset A \times A$ in each case, and draw the undirected graph G associated with $\mathscr{R}$

a) $M(\mathscr{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\mathscr{R} = \{(v,w),(v,x),(w,v),(w,x),(w,y),(w,z),(x,z),(y,z)\}$$

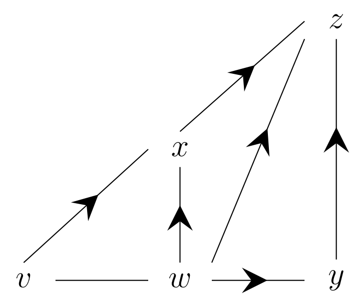

Figure 3

b) $M(\mathscr{R}) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

$$\mathscr{R} = \{(v,w),(v,x),(v,y),(w,v),(w,x),(x,v),(x,w),(x,z),(y,v),(y,z),(z,x),(z,y)\}$$

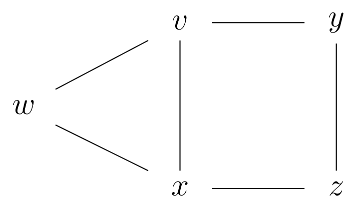

Figure 4