Norwegian University of Science and Technology
Department of Mathematics

MA0301 Elementary discrete mathematics Spring 2018

Solutions — exercise 3

$\boxed{7}$ Use a truth table to show that $\big((a \wedge b) \longrightarrow c\big) \Leftrightarrow \big((a \longrightarrow c) \vee (b \longrightarrow c)\big)$.

a	b	c	$(a \to c)$	$(b \to c)$	$(a \to c) \vee (b \to c)$	$(a \wedge b) \to c$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	T	T	T	T
T	T	F	F	F	F	F
T	T	T	T	T	T	T

Section 3.1

$\boxed{6}$ Consider the following six subsets of $\mathbb{Z}$.

$A = \{2m+1 \mid m \in \mathbb{Z}\}$ $\quad$ $B = \{2n+3 \mid n \in \mathbb{Z}\}$
$C = \{2p-3 \mid p \in \mathbb{Z}\}$ $\quad$ $D = \{2r+1 \mid r \in \mathbb{Z}\}$
$E = \{3s+1 \mid s \in \mathbb{Z}\}$ $\quad$ $F = \{3t-2 \mid t \in \mathbb{Z}\}$

Which of the following statements are true, and which are false?

a) $A = B$ **True** $\quad$ **b)** $A = C$ **True** $\quad$ **c)** $B = C$ **True**
d) $D = E$ **False** $\quad$ **e)** $D = F$ **True** $\quad$ **f)** $E = F$ **False**

Section 3.2

$\boxed{6}$ Prove each of the following results without using Venn diagrams or membership tables.

a) If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$ and $A \cup C \subseteq B \cup D$.

Assume $x \in A \cap C$, then $x \in A$ and $x \in C$. Now $x \in A \Rightarrow x \in B$ since $A \subseteq B$, similarly $x \in C \Rightarrow x \in D$ since $C \subseteq D$. As $x \in B$ and $x \in D$ then $x \in B \cap D$. Thus $A \cap C \subseteq B \cap D$. To see that the reverse implication fails, choose x such that $x \in (B/A) \cap (D/C)$ then $x \in B \cap D$, but $x \notin A \cap C$. This proves $A \cap C \subseteq B \cap D$.

Let $x \in A \cup C$. Then $x \in A \vee x \in C$. If $x \in A$ then $x \in B$ since $A \subseteq B$, similarly if $x \in C$ then $x \in D$ since $C \subseteq D$. This shows that $x \in B \cup D$. For the reverse implication choose $x \in (B/A) \cup (C/D)$, then $x \in B \cup D$, however $x \notin A \cup C$. This proves $A \cup C \subseteq B \cup D$.

$\boxed{7}$ Prove or disprove each of the following:

b) For sets $A, B, C \subseteq \mathscr{U}$, $A \cup C = B \cup C \implies A = B$.

Let $A \neq B$ and $C = \mathscr{U}$, then $A \cup C = B \cup C$.
More concretely let $\mathscr{U} = C = \mathbb{N}$, $A = \{p\}$, $B = \{q\}$, $p, q \in \mathbb{N}$. Then $A \cup C = B \cup C$, however $A \neq B$ when $p \neq q$.

d) For sets $A, B, C \subseteq \mathscr{U}$, $A \,\Delta\, C = B \,\Delta\, C \implies A = B$.

Assume that $x \in A$, then either x lies in C or not in C. If $x \in C$ then $x \notin A\Delta C \Rightarrow \notin B\Delta C \Rightarrow x \in B$. Else if $x \notin C$ then $x \in A\Delta C \Rightarrow B\Delta C \Rightarrow x \in B$, as $x \notin C$. Thus, $A \subseteq B$.

The other direction is shown in precicely the same way. Assume that $x \in B$, then either x lies in C or outside C. If $x \in C$, then $x \notin B\Delta C \Rightarrow x \notin A\Delta C \Rightarrow x \in A$. Else if $x \notin C$ then $x \in B\Delta C \Rightarrow A\Delta C \Rightarrow x \in A$, as $x \notin C$. Thus, $B \subseteq A$.

As $A \subseteq B$ and $B \subseteq A$, then $A = B$, which is what was to be proven.

$\boxed{16}$ Provide the justifications for the steps that are needed to simplify the set

$$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))]$$

where $A, B, C, D \subseteq \mathscr{U}$.

Steps	**Reasons**
$(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \overline{D}))]$	
$=(A \cap B) \cup [B \cap (C \cap (D \cup \overline{D}))]$	Distributive Laws
$=(A \cap B) \cup [B \cap (C \cap \mathscr{U})]$	Inverse Laws
$=(A \cap B) \cup (B \cap C)$	Domination Laws
$=(B \cap A) \cup (B \cap C)$	Commutative Laws
$=B \cap (A \cup C)$	Distributive Laws

Section 2.5

$\boxed{8}$ **a)** Let $p(x)$, $q(x)$ be open statements in the variable x, with a given universe. Prove that

$$\forall x\; p(x) \vee \forall x\; q(x) \implies \forall x[p(x) \vee q(x)]$$

Universe $\mathscr{U}$. Assume that $\forall x\; p(x) \vee \forall x\; q(x)$ is true $x \in \mathscr{U}$. Assume that $\forall x\; p(x)$ is true, then there exists some $c \in \mathscr{U}$ such that $p(c)$ is true, and thus $p(c) \vee q(c)$ is true. Since c can be choosen arbitarily, we have shown that $\forall x[p(x) \vee q(x)]$, which is what we wanted to show. If we instead assume that $\forall x\; q(x)$ is true, the exact same argument can be made.

b) Find a counterexample to the converse in part **a)**. That is, find open statements $p(x)$, $q(x)$, and a universe such that $\forall x[p(x) \vee q(x)]$ is true, while $\forall x\; p(x) \vee \forall x\; q(x)$ is false.

Let $\mathscr{U}$ be some universe such that there exists non-empty disjunctive subsets A, B such that $(A \cap B = \emptyset)$ and $\mathscr{U} = A \cup B$.

Let $p(x) : x \in A$, and similarly $q(x) : x \in B$. If $y \in B$, then $p(y)$ is false (as $A \cap B = \emptyset$), thus $p(x)$ can not hold for all x in other words $\forall x\; p(x)$ is false. Similarly, let $y \in A$ then $q(y)$ is false (again since $A \cap B = \emptyset$), thus $\forall x\; q(x)$ is false. However, $\forall x[p(x) \vee q(x)]$ is true as for every $x \in \mathscr{U} = A \cup B$.

For a concrete example let $\mathscr{U} = \mathbb{N}$, $n \in \mathbb{N}$, and let $p(n)$: n is odd, $q(n)$: n is even. Then, $p(n)$ is not true for every n as there exists even numbers, and similarly $q(n)$ is not true for every n as there exists odd numbers. However, every n is either even or odd.

10 Provide the missing reasons for the steps verifying the following argument:

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \exists x\ \neg p(x) \\ \forall x\ [\neg q(x) \vee r(x)] \\ \underline{\forall x\ [s(x) \rightarrow \neg r(x)]} \\ \therefore \exists x\ \neg s(x) \end{array}$$

Steps		**Reasons**
1)	$\forall x\ [p(x) \vee q(x)]$	Premisse
2)	$\exists x\ \neg p(x)$	Premisse
3)	$\neg p(a)$	Step (2) and the definition of truth for $\exists x\ p(x)$. The reason for this step is also referred to as the *Rule of Existential Specification*
4)	$p(a) \vee q(a)$	Step (1) and the *Rule of Universal Specification*
5)	$q(a)$	Steps (3) and (4) and the Rule of Disjunctive Syllogism
6)	$\forall x\ [\neg q(x) \vee r(x)]$	Premisse
7)	$\neg q(a) \vee r(a)$	Step (6) and the Rule of Universal Specification
8)	$q(a) \rightarrow r(a)$	Step (7) and the rule of Material Implication ($P \rightarrow Q \Leftrightarrow \neg P \vee Q$).
9)	$r(a)$	Modus Ponens on Steps (5) and (8).
10)	$\forall x\ [s(x) \rightarrow \neg r(x)]$	Premisse
11)	$s(a) \rightarrow \neg r(a)$	Step (10) and the Rule of Universal Specification
12)	$r(a) \rightarrow \neg s(a)$	Transposition ($P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$) and step (11)
13)	$\neg s(a)$	Modus Tollens on steps (9) and (12)
14)	$\therefore\ \exists x\ \neg s(x)$	Step (13) and the definition of the truth for $\exists x\ \neg s(x)$. The reason for this step is also referred to as the *Rule of Existential Generalization*.