MA0301 Elementary discrete mathematics Spring 2018

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Solutions — exercise 2

Section 2.2

19 Provide the steps and reasons, as in Exercise 18, to establish the following logical equivalences

c) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \;\Leftrightarrow p \wedge q$

	Reasons
$[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$	**Reasons**
$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$	Material implication $a \rightarrow b \Leftrightarrow \neg a \vee b$
$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$	DeMorgan's Laws and Law of Double Negation
$\Leftrightarrow (p \wedge q)$	Absorption laws and associative properties of $\vee$.

20 Simplify each of the networks shown in figure 1.

For figure 1a notice that the first block $r \vee t \vee \neg r$ is true when r is either true or false, in other words the block is always true. Similarly $\neg r \vee q \vee \neg q$ is always true. Thus, the diagram simplifies to $p \vee \neg q$ as shown in figure 2a.

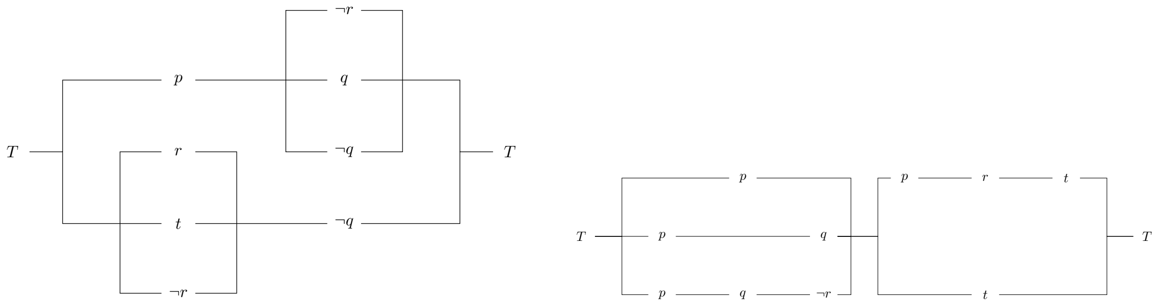
Imagine walking from T_0 to T_1 in figure 1b, there is no way to do this without passing through both p and q. All the other boolean variables can be avoided and thus, are not neccceccary. The resulting diagram is shown in figure 2b.

Section 2.3

2 Use truth tables to verify that each of the following is a logical implication

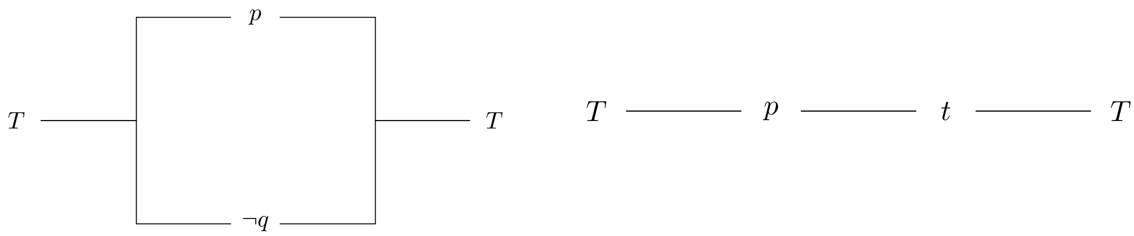
b) $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$

d) $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$



(a) (b)

Figure 1



(a) (b)

Figure 2

p	q	$\neg q$		$(p \to q) \wedge \neg q$	$[(p \to q) \wedge \neg q] \to \neg p$
F	F	T	T	T	T
F	T	F	T	F	T
T	F	T	F	F	T
T	T	F	T	F	T

p	q	r	$p \to r$	$q \to r$	$(p \vee q) \to r$	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	F	T
T	F	T	T	T	T	T
T	T	F	F	F	F	T
T	T	T	T	T	T	T

$\boxed{8}$ Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

Steps		**Reasons**
1)	$\neg s \wedge \neg u$	Premisse
2)	$\neg u$	Step (1) and the rule of Conjunctive Simplification
3)	$\neg u \rightarrow \neg t$	Premisse
4)	$\neg t$	Step (3) and Modus Potens
5)	$\neg s$	Step (1) and the rule of Conjunctive Simplification
6)	$\neg s \wedge \neg t$	Steps (4) and (5) and the Rule of Conjunction
7)	$r \rightarrow (s \vee t)$	Premisse
8)	$\neg(s \vee t) \rightarrow \neg r$	Step (8) and Contraposition $a \rightarrow b \Leftrightarrow \neg b \rightarrow \neg a$
9)	$(\neg s \wedge \neg t) \rightarrow \neg r$	Step (9) and DeMorgan's Laws
10)	$\neg r$	Steps (6) and (9) and Modus Potens
11)	$(\neg p \vee q) \rightarrow r$	Premisse
12)	$\neg r \rightarrow \neg(\neg p \vee q)$	Step (11) and Contraposition $a \rightarrow b \Leftrightarrow \neg b \rightarrow \neg a$
13)	$\neg r \rightarrow (p \wedge \neg q)$	Step (12) and DeMorgan's Laws
14)	$p \wedge \neg q$	Step (13) and Modus Potens
15)	$\therefore p$	Step (14) and the rule of Conjunctive Simplification

Section 2.4

$\boxed{8}$ Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements.

$$\begin{aligned} p(x)&: \quad x^2 - 8x + 15 = 0 \\ q(x)&: \quad x \text{ is odd} \\ r(x)&: \quad x > 0 \end{aligned}$$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

e) $\exists\, x\ [p(x) \rightarrow q(x)]$

True: As $p(x) = (x-3)(x-5)$ we have $p(x) = 0 \Leftrightarrow x = 3 \wedge x = 5$. Thus, there exists a solution to $p(x)$ such that x is odd.

f) $\forall\, x\ [\neg q(x) \rightarrow \neg p(x)]$

True: $q(x) \vee \neg p(x)$. As every solution to $p(x)$ is odd, every even x implies that x is *not* an solution to $p(x)$. Thus, the statement above holds.

g) $\exists\, x\ [p(x) \to (q(x) \wedge r(x))]$

True: As $p(5)$ is True, there exists some solution to $p(x)$ such that x is positive and odd.

h) $\forall\, x\ [(p(x) \vee q(x)) \to r(x)]$

False: Let x be a negative even number. Then $p(x) \vee q(x)$ is False, but $r(x)$ is True.

$\boxed{12}$ **a)** Let $p(x, y)$ denote the open statement "x divides y", where the universe for each of the variables x, y comprises all integers. Determine the truth value of each of the following statements; if a quantified statement is false, provide an explanation or a counterexample.

iii) $\forall\, y\ p(1, y)$

True: As $y/1 = 1$ for all y.

iv) $\forall\, x\ p(x, 0)$

False: take $x = 0$ as an counterexample $p(0, 0)$ is false as $0/0$ is undefined.

vi) $\forall\, y\ \exists\, x\ p(x, y)$

True: This is "obviously" true? For every y let $x = 1$, then $p(1, y)$ is true for all y.

vii) $\exists y\ \forall\, x\ p(x, y)$

False: Let $x = 0$ then $p(0, y)$ is undefined for all y.

viii) $\forall\, x\ \forall\, y\ [(p(x, y) \wedge p(y, x)) \to (x = y)]$

False: Let $x = y = 0$. Then $x = y$, but $p(x, y)$ is undefined and thus, $p(x, y) \wedge p(y, x)$ is false. If we exclude 0 the statement holds.

c) Let $p(x, y)$ denote the open statement"x divides y“, as before, but let the universe now be compromised of the *positive* integers x, y. Determine the truth value of each of the following statements. If the statement is false, provide an explanation or a counterexample.

i) $\forall\, x\ \exists\, y\ p(x, y)$

True: Let $y = x$, as $x, y > 0$ we can always choose such an y.

ii) $\forall\, y\; \exists\, x\; p(x, y)$

True We could let $x = 1$ here as well, but this states that the x can be different for every y. To illustrate this let $x = y$. Then clearly $p(x, y) = p(y, y)$ is true as y/y. Thus, for every y we can find an x such that $p(x, y)$ is true.

iii) $\exists\, x\; \forall\, y\; p(x, y)$

True: Again let $x = 1$, then $p(x, y) = p(1, y)$ is true, as $y/1 = y$.

iv) $\exists\, y\; \forall\, x\; p(x, y)$

False: This asks whether there exists a single number y that divides every x. Meaning y has to divide 1, 2 and so forth. If such a number where to exist it would have to be the product of all the integers

$$y = 1 \cdot 2 \cdot 3 \cdots = \prod_{i \in \mathbb{N}} i\,.$$

Then clearly y divides x for every integer $x > 0$, however y is now no longer an integer!