§ 2. Bayesian Modeling

Components of Bayesian models

eg: heights $n=15$ men y_i is height of the man whose num is 'i'

$y_i = \mu + \varepsilon_i \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2), i = 1, \cdots, n$ (μ is the average)

$\Rightarrow y_i \overset{iid}{\sim} N(\underset{\smile}{\mu, \sigma^2})$ unknown constant

(y_i comes from a normal distribution independent and identically distributed with the normal distribution) $\Rightarrow$ treating μ, σ as random variables with their own probability distributions

3 key components of Bayesian models:

① likelihood: the probabilistic model for the data eg. $P(y|\theta)$

② the prior eg: $P(\theta)$

$\Big\} P(y,\theta) = P(\theta) P(y|\theta)$

③ the posterior eg: $P(\theta|y) = \dfrac{P(\theta, y)}{P(y)} = \dfrac{P(\theta, y)}{\int P(\theta, y)\, d\theta} = \dfrac{P(\theta) P(y|\theta)}{\int P(\theta) P(y|\theta)\, d\theta}$

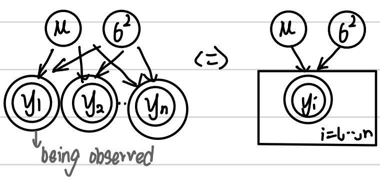
Model Specification

e.g: $y_i | \mu, \sigma^2$ (y_i is height for person i) $\overset{iid}{\sim} N(\mu, \sigma^2)$, $i = 1, \cdots, n$

$P(\mu, \sigma^2) = P(\mu) P(\sigma^2)$

suppose $\mu \sim N(\mu_0, \sigma_0^2)$ $\sigma^2 \sim IG(\nu_0, \beta_0)$

graphical representation:



(show how we could hypothetically simulate data from this model)

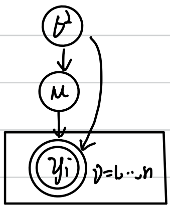
Posterior derivation

example of a hierachical model:

$$\left.\begin{aligned} y_i \mid \mu, \sigma^2 &\overset{iid}{\sim} N(\mu, \sigma^2) \\ \mu \mid \sigma^2 &\sim N\left(\mu_0, \frac{\sigma^2}{w_0}\right) \end{aligned}\right\} \text{joint distribution of } y \text{ and } \mu \text{ given } \sigma^2$$

⇒ we need to complete the model with the prior 4 σ^2

$$\sigma^2 \sim IG(\nu_0, \beta_0)$$



$$\begin{aligned} P(y_1,\dots,y_n,\mu,\sigma^2) &= P(y_1,\dots,y_n \mid \mu,\sigma^2)\,P(\mu,\sigma^2) \\ &= P(y_1,\dots,y_n \mid \mu,\sigma^2)\,P(\mu \mid \sigma^2)\,P(\sigma^2) \\ &= \left[\prod_{i=1}^{n} N(y_i \mid \mu,\sigma^2)\right] N\left(\mu \mid \mu_0, \frac{\sigma^2}{w_0}\right) IG(\sigma^2 \mid \nu_0, \beta_0) \end{aligned}$$

$$\left(\because\ y_i \mid \mu,\sigma^2 \overset{iid}{\sim} N(\mu,\sigma^2) \qquad P(\theta \mid y) = \frac{P(y \mid \theta)P(\theta)}{\underbrace{\int P(y \mid \theta)P(\theta)\,d\theta}_{\text{constant number}}} \propto P(y \mid \theta)P(\theta)\right)$$

$$\boxed{\propto}\ P(\mu,\sigma^2 \mid y_1,\dots,y_n)$$

the constant number is missing

Non-conjugate models

an example of a one parameter model that is not conjugate:

$n=10$ (companies)

$y_i|\mu \overset{iid}{\sim} N(\mu,1)$ (the percentage changed, μ is unknown)

$\mu \sim t(0,1,1)$ ⇒ unknown

$$P(\mu|y_1,\cdots,y_n) \propto \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}(y_i-\mu)^2]\right]\frac{1}{\pi(1+\mu^2)}$$

constant number

$$\propto \exp[-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2]\frac{1}{1+\mu^2}$$

$$\propto \exp[-\frac{1}{2}(\sum_{i=1}^{n}y_i^2-2\mu\sum_{i=1}^{n}y_i+n\mu^2)]\frac{1}{1+\mu^2}$$

$$\propto \frac{\exp[n(\bar{y}\mu-\frac{\mu^2}{2})]}{1+\mu^2}$$