ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 9: generating random processes

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review and roadmap

generating random variables

we have seen how to:

- generate pseudorandom $U[0,1]$ samples
- transform $U[0,1]$ samples to another rv using
  - inversion
  - acceptance-rejection
- generate random vectors
  - understanding correlation
  - multivariate Normal rvs

and now for the grand finale!!

- generating time-indexed random processes
  - (discrete time) Markov chains
  - exponential rvs and the Poisson process
  - and beyond (Brownian motion...)

random processes

random process

random process

indexed collection of rvs $X_t \in \mathcal{S}$, one for each $t \in T$

– $\mathcal{S}$: state space

– $\mathcal{T}$: index set

four types

- $\mathcal{S}$ discrete, $\mathcal{T}$ discrete: discrete-time Markov chain (DTMC)
  – random walk
- $\mathcal{S}$ continuous, $\mathcal{T}$ discrete: discrete-time Markov process
- $\mathcal{S}$ discrete, $\mathcal{T}$ continuous: continuous-time Markov chain (CTMC)
  – Poisson process
- $\mathcal{S}$ continuous, $\mathcal{T}$ continuous: Markov process
  – Brownian motion

discrete-time Markov chain

the random walk

$X_0 = 0$, $Y_k \sim \text{Bernoulli}(p)$ iid, and

$$X_t = \sum_{i=0}^{t} Y_k$$

counting process visualized

counting process

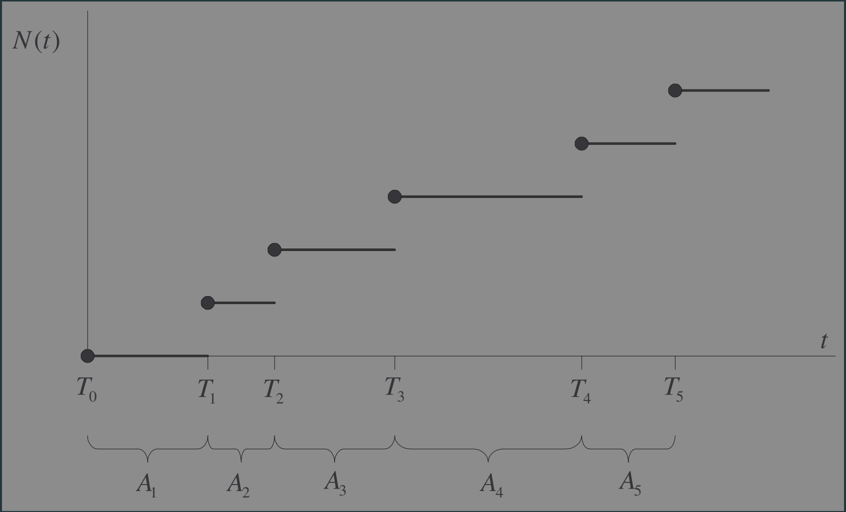
non-negative integer-valued stochastic process $[N(t) : t \geq 0]$

- $N(0) = 0$
- $N(t) = \#$ of arrivals during time interval $(0, t]$

- $[N(t) : t \geq 0]$ increases by jumps
- T_n = time of the n-th arrival, $T_0 = 0$
- A_n = be the interarrival time for the n-th arrival.

$$A_n = T_n - T_{n-1}$$

counting process



a desired (?) properties of inter-arrival times?

the Exponential distribution

suppose $T \sim Exp(\lambda)$, then:

- pdf: $f_T(t) =$
- cdf: $F_T(t) = \mathbb{P}[T \leq t] =$

why is $Exp(\lambda)$ special?

memorylessness

cdf of T given that T bigger than t?

$\mathbb{P}[T \leq t + x | T > t] =$

the exponential distribution: properties

suppose $T_1, T_2, \ldots, T_n$ are all exponentially distributed, with $T_i \sim Exp(\lambda_i)$.

- (minimum of exponentials): let $T_{\min} = \min\{T_i | i \in \{1, 2, \ldots, n\}\}$
  distribution of $T_{\min}$?

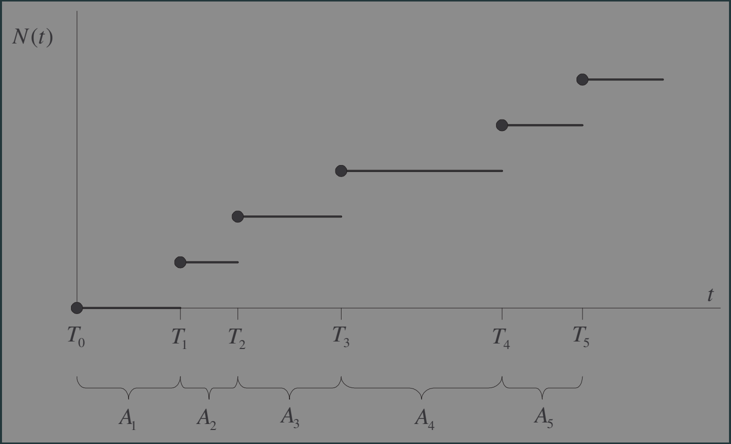
  $T_{\min} \sim$

- (first arrival): let $I_{\min} = \arg\min\{T_i | i \in \{1, 2, \ldots, n\}\}$
  distribution of $I_{\min}$?

  $I_{\min} \sim$

Poisson process

$A_1, A_2, \ldots$ i.i.d. $Exp(\lambda) \implies$ Poisson process of rate λ
– denoted $PP(\lambda)$



Poisson process: properties

note: $N(t+s) - N(t) = \#$ arrivals in time interval $(t, t+s]$

- Exponential interrarival times: $A_n = T_n - T_{n-1} \sim Exp(\lambda)$, i.e.

$$\mathbb{P}[A_n \leq t] = 1 - e^{-\lambda t}$$

- independent increments: $N(t+s) - N(t) \perp\!\!\!\perp N(t) - N(0)$
  more generally, for $t_1 \leq t_2 \leq t_3 \leq t_4$, $N(t_4) - N(t_3) \perp\!\!\!\perp N(t_2) - N(t_1)$

- Poisson arrivals: $N(t+s) - N(t) \sim \; Poisson(\lambda s)$, i.e.,

$$\mathbb{P}\left[N(t+s) - N(t) = k\right] = \frac{e^{-\lambda s}\ (\lambda s)^k}{k!}$$

  moreover, $\mathbb{E}[\text{Arrivals in interval of length } s] = \lambda s$.

Poisson process computations

these properties of PP are useful for computations; Eg. $\mathbb{P}[\text{no arrivals in } [0,t]]$?

– using Exponential interarrival times:

$\mathbb{P}\left[\mathcal{N}(t) - \mathcal{N}(0) = 0\right] =$

– using Poisson arrivals:

$\mathbb{P}\left[\mathcal{N}(t) - \mathcal{N}(0) = 0\right] =$

Poisson process: formal definition

Poisson process $PP(\lambda)$

an arrival process $[N(t) : t \geq 0]$ is a Poisson process with rate λ if

1. $N(t+s) - N(t)$ independent of $N(t) - N(0)$.

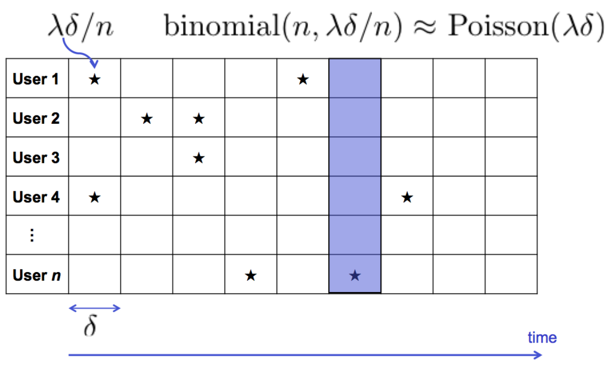
2. $\mathbb{P}[N(t+s) - N(t) = 1] = \lambda s + o(s)$, where $o(s)$ denotes a function $g(\cdot)$ satisfying: $\lim_{s \to 0} \frac{g(s)}{s} = 0$.

3. $\mathbb{P}[N(t+s) - N(t) \geq 2] = o(s)$.

example: $P(N(t+s) - N(t) = 1) = e^{-\lambda s}(\lambda s)$

why Poisson process?

- easy to simulate!
- behavioral justifications: arrivals are modeled as PP because
  - memorylessness of interarrival times
  - the Palm-Khintchine theorem



generating samples of Poisson processes

1. set the arrival counter $n = 0$. Set $T_0 = 0$.
2. increment n by 1
let A_n be a sample from exponential distribution with parameter λ.
3. advance time
$$T_n = T_{n-1} + A_n.$$
4. return to Step 2.

thinning and superposition

two more important properties of Poisson processes

superposition

let $N_1(t) \sim PP(\lambda_1)$ and $N_2(t) \sim PP(\lambda_2)$ be two independent Poisson processes
then $N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$

thinning

given $N(t) \sim PP(\lambda_1)$, let $N_1(t)$ be the process generated by retaining each arrival of $N(t)$ independently with probability p, and let $N_2(t) = N(t) - N_1(t)$ be the rejected points
then $N_1(t) \sim PP(\lambda p)$ and $N_2(t) \sim PP(\lambda(1-p))$
moreover, $N_1(t) \perp\!\!\!\perp N_2(t)$!

these are very useful for discrete-event simulation!

nonstationary Poisson processes

- imagine that the arrival rate of the Poisson process is not constant, but changes with time.
- $\lambda(t) =$ arrival rate at time t.
- "time of the day" or "seasonality" effects.

formal definition:

an arrival process $[\mathcal{N}(t) : t \geq 0]$ is called a nonstationary Poisson process with rate function $\lambda(\cdot)$ if

1. $\mathcal{N}(t+s) - \mathcal{N}(t)$ is independent of $\mathcal{N}(t) - \mathcal{N}(0)$.
2. $\mathbb{P}[\mathcal{N}(t+s) - \mathcal{N}(t) = 1] = \lambda(t)s + o(s)$.
3. $\mathbb{P}[\mathcal{N}(t+s) - \mathcal{N}(t) \geq 2] = o(s)$.

nonstationary Poisson process: properties

- $\mathcal{N}(t+s) - \mathcal{N}(t) \sim \mathit{Poisson}\left(\int_t^{t+s} \lambda(u)du\right)$, that is

$$\mathbb{P}\left[\mathcal{N}(t+s) - \mathcal{N}(t) = k\right] = \frac{e^{-\int_t^{t+s} \lambda(u)du} \left(\int_t^{t+s} \lambda(u)\ du\right)^k}{k!}.$$

- $\mathbb{E}\Big[\#$ of arrivals in interval $(t, t+s]\Big] = \int_t^{t+s} \lambda(u)du.$
- distribution of number of arrivals in $(t, t+s]$ depends on t

example

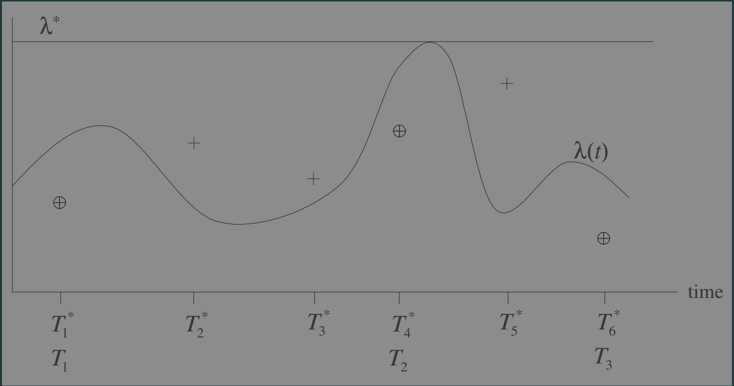
let $[\mathcal{N}(t) : t \geq 0]$ have arrival rate function

$$\lambda(t) = \begin{cases} 5 + 5t & \text{if } 0 \leq t \leq 3 \\ 20 & \text{if } 3 \leq t \leq 5 \\ 20 - 2(t-5) & \text{if } 5 \leq t \leq 9. \end{cases}$$

- the number of arrivals between $t = 0.5$ and $t = 1.5$ has Poisson distribution with parameter
$$\int_{1/2}^{3/2} (5 + 5t) dt = 10.$$
- the probability of having 7 customer arrivals between $t = 0.5$ and $t = 1.5$ is
$$\frac{e^{-10}\ 10^7}{7!}.$$

generating nonstationary PP via acceptance-rejection

- $\lambda^* = \max\left[\lambda(t) : t \geq 0\right].$
- generate a sample of a stationary $PP(\lambda^*)$
- suppose arrival times we obtain are $T_1^*, T_2^*, \ldots$
  accept each arrival time T_i^* with probability $\mathbb{P}[Accept] = \frac{\lambda(T_i^*)}{\lambda^*}$



generating nonstationary PP via acceptance-rejection

1. set $\lambda^* \geq \max[\lambda(t) : t \geq 0]$
2. set arrival counter $n = 0,\; T^* = 0,\; T_0 = 0$
3. generate $A \sim \; Exp(\lambda^*)$
4. update $\quad T^* = T^* + A.$
5. generate $U \sim U[0, 1]$
6. If $U \leq \frac{\lambda(T^*)}{\lambda^*},$, then increment n by 1 and let $T_n = T^*$
7. return to Step 2

to show this works, need to verify the 3 properties:

1. $\mathcal{N}(t+s) - \mathcal{N}(t)$ is independent of $\mathcal{N}(t) - \mathcal{N}(0)$
2. $\mathbb{P}[\mathcal{N}(t+s) - \mathcal{N}(t) = 1] = \lambda(t)s + o(s).$
3. $\mathbb{P}[\mathcal{N}(t+s) - \mathcal{N}(t) \geq 2] = o(s)$

generating nonstationary Poisson processes via AR

the main thing we need to check is property 2: