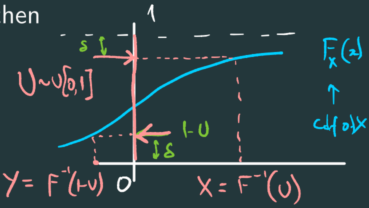
class poll: U and $1 - U$ (simplest example of variance reduction)

Recorded lecture = inversion method

$U \sim U[0,1]$ and F is a cdf

set $X = F^{-1}(U)$, and also set $Y = F^{-1}(1-U)$; then



(a) X has cdf F but Y does not.

(b) Y has cdf F but X does not.

(c) both X and Y have cdf F, and X and Y are independent

✓ (d) both X and Y have cdf F, and X and Y are dependent

(in fact, they 'negatively correlated')

ie, if X is big, Y is small (and vice versa)

Note

CDF of U is the same as CDF of 1-U

$\Rightarrow$ $F^{-1}(U)$ and $F^{-1}(1-U)$ have the same dist^n but they are dependent!

· It turns out that

$$Z = \frac{X+Y}{2}$$

then $E[Z] = E[X]$, but $Var(Z) \leq Var(X)$

Last few classes

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

- How to get CIs : Std recipe - given $X_1, X_2, \ldots, X_n$ iid st $\mu = \mathbb{E}[X_i]$

  95% CI for μ $= \left[\bar{X}_n \mp \underbrace{1.96}_{\approx 2 \text{ (via CLT approx)}} \sqrt{\underbrace{\mathrm{Var}(\bar{X}_n)}_{\approx \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}}\right]$

- Replace 2 by $\frac{1}{\sqrt{0.05}} \approx 5$ to get 95% CI without CLT approx (via Chebyshev's)
- Combine multiple CIs via union bound

(recorded lecture) - PRNGs (pseudo random number generators)

- basic idea : These are deterministic fns $X_n = f(X_{n-1})$ such that the sequence $X_0, X_1, X_2, \ldots X_n$ appear to be independent $U[0,1]$

example - LCG : $X_{n+1} = (aX_n + b) \bmod m$

for some chosen a, b, m, and seed X_0

Benefit of PRNGs - Can get repeatable simulations (by setting sim 'seed')

ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 5: generating non-uniform random variables

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random variable generation

random number: a sample from $U[0, 1]$

modern PRNGs (like `np.random.rand()`) are

- random enough for your simulation to be correct
- deterministic enough (by setting seed) for your simulation to be repeatable

the 'fundamental theorem' of simulation

can 'transform' a stream of i.i.d. $U[0, 1]$ into

- a random variable with any given cdf
- a random vector with any given correlation matrix
- any stochastic process

Intermediate Question - How do we get $U[0,1]$?

- pseudo randomness

generating rvs with arbitrary distributions

aim: "transform" $U[0,1]$ rv to another rv with given probability distribution.

monte carlo sampling techniques

basic methods

- inversion – basic method (see recorded lecture)
- acceptance-rejection ← most important general technique for generating random numbers
- distribution-specific techniques (Box-Muller for Gaussians)
- advanced techniques (adaptive rejection sampling, SIR) ← importance sampling

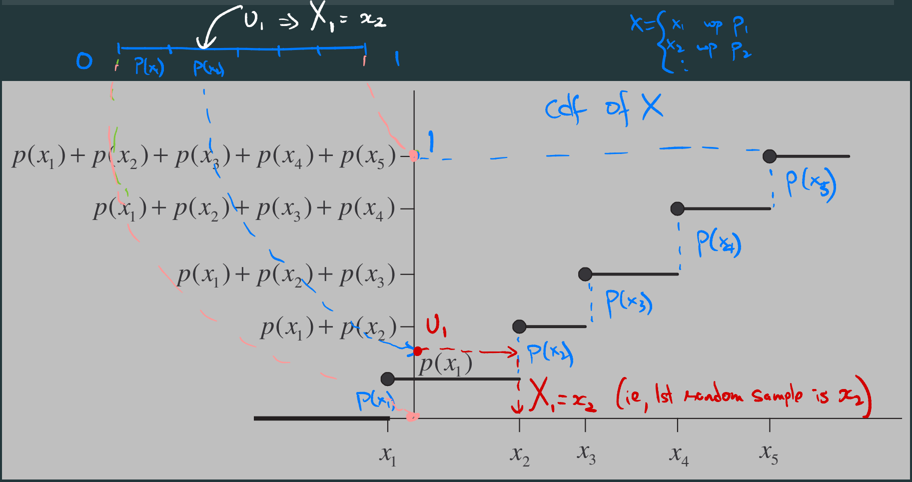
markov-chain monte carlo (MCMC)

- Metropolis-Hastings
- Gibbs sampling
- advanced techniques (slice sampling, Hamiltonian MC)

inversion

warm-up: simulating discrete rv

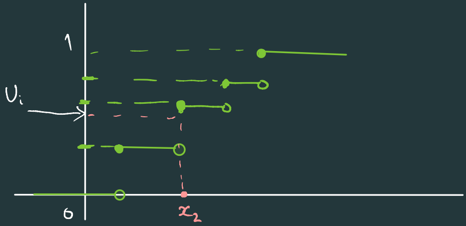
X takes values $x_1 \leq x_2 \leq \ldots \leq x_5$, $\mathbb{P}[X = x_i] = p(x_i)$



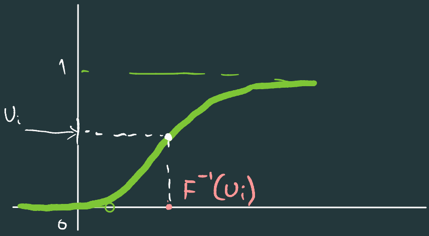
example (contnd) (intuition)

Given stream $U_1, U_2, \ldots \sim U[0,1]$

'bucketing' $\xrightarrow{\text{'limit'}}$ inversion



$X_i = x_2$

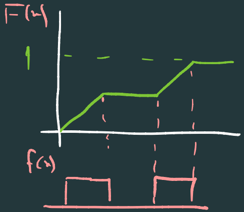


$X_i = F^{-1}(U_i)$

the inversion method

X continuous r.v. with pdf f and c.d.f. $F(\cdot)$

- want to generate samples of X.
- $F(\cdot)$ non-decreasing $\implies$ can define inverse $F^{-1}(\cdot)$
- $F(x) = u \iff F^{-1}(u) = x$



inversion method

given desired cdf F (continuous, increasing), generate sample $X_0 \sim F$ as:

non-decreasing

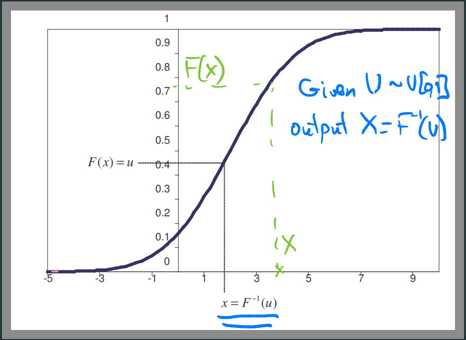
1. generate $U \sim U[0,1]$.
2. return $X_o = F^{-1}(U)$.

- If X is not cont (discrete, or mixture of discrete + cont) use 'generalized' inverse (every cdf has one!)

$\forall u \in [0,1] : F^{-1}(u) = \min\{x \in \mathbb{R} \text{ s.t. } F(x) \geq u\}$

intuition/proof for inversion method

$x = F^{-1}(u) \longleftrightarrow u = F(x)$



main idea: always consider cdf

- $U =$ np.random.rand( )

  $X = F^{-1}(U)$

- $\mathbb{P}[X \le y]$ for any $y \in \mathbb{R}$

  $= \mathbb{P}[F^{-1}(U) \le y]$ ∵ F is non-dec

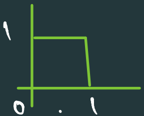
  $= \mathbb{P}[F(F^{-1}(U)) \le F(y)]$

  $= \mathbb{P}[U \le \underbrace{F(y)}_{\in [0,1]\ \forall y \in \mathbb{R}}]$ (by defn)

  $= F(y)$

U[0,1]

$f_U(x)$

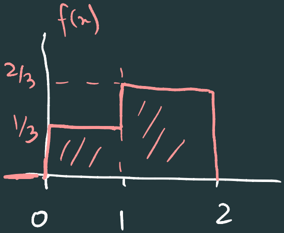


$$F_U(x) = \begin{cases} 0 & ; x < 0 \\ x & ; x \in [0,1] \\ 1 & ; x > 1 \end{cases}$$

example

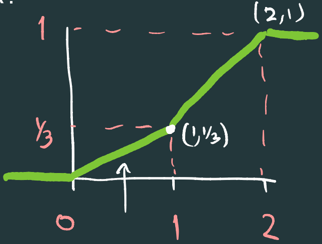
example – the pdf of X is given by

$$f(x) = \begin{cases} 1/3 & \text{if } 0 \le x \le 1 \\ 2/3 & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$



develop an inversion method to generate samples of X.

$F(x) =$ see figure



To generate X

- Sample $U \sim U[0,1]$
- output $X = F^{-1}(u)$

$F(x) = x/3$ for $x \in [0,1]$

$\Rightarrow F^{-1}(u) = 3u$ for $u \in [0, 1/3]$

example (contnd)

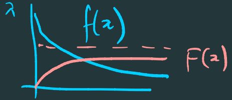
$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ x/3 & ; \quad x \in [0,1] \\ \frac{2x-1}{3} & ; \quad x \in [1,2] \\ 1 & ; \quad x > 2 \end{cases}$$

$$\Rightarrow \quad F^{-1}(u) = \begin{cases} 3u & ; \quad u \in [0, 1/3] \\ \frac{3u+1}{2} & ; \quad u \in [1/3, 1] \end{cases}$$

example (exponential rv)

generate samples of an exponential r.v. with parameter λ, with cdf

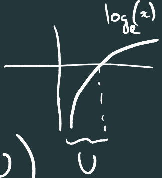
$$f(x) = \lambda e^{-\lambda x},\ x \geq 0 \Rightarrow \boxed{F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}}$$



$$\Rightarrow F^{-1}(u) = -\frac{1}{\lambda} \ln (1 - u)$$ for $u \in [0,1]$

$\Rightarrow 1 - u \in [0,1]$

- To generate $X \sim Exp(\lambda)$
  - sample $U \sim U[0,1]$
  - output $X = -\frac{1}{\lambda} \ln (1 - U)$



(alternate - output $Y = -\frac{1}{\lambda} \ln (U)$)

drawback of inversion method

- inversion method may be computationally expensive. Eg - $X \sim N(0,1)$
- computing $F^{-1}(\cdot)$ may require numerical search.

ie, $F_X(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z} dz$

$= \Phi(z)$

no closed form $\Rightarrow$ what is $F^{-1}(\cdot)$

example – the pdf of X is given by

$$f(x) = \begin{cases} 60x^3(1-x)^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{F(x) = 15x^4 - 24x^5 + 10x^6} \quad \text{for } 0 \le x \le 1.$$

generate samples of X by using the inversion method.

$F^{-1}(y) \Rightarrow$ Find x s.t $y = 15x^4 - 24x^5 + 10x^6$

Problem - There is no formula for 'solving' a polynomial of degree 5 and above (Abel, Galois)

acceptance-rejection

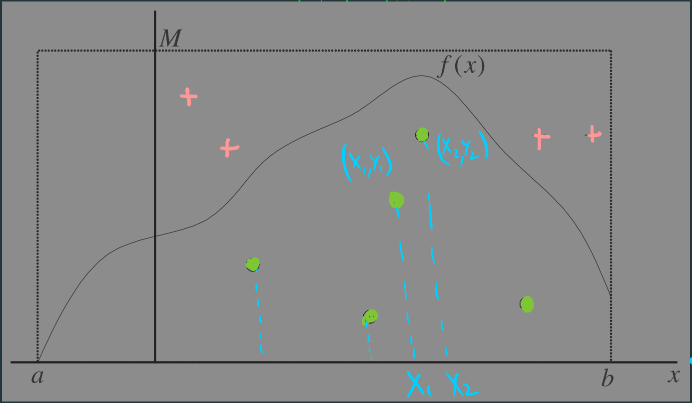
acceptance-rejection (vanilla / basic AR)

want to generate samples of a rv X

- pdf $f(\cdot)$ of X takes positive values only over $[a, b]$
- M is an upper bound on pdf of X, i.e., $M \geq \max_{x\in[a,b]} f(x)$

$\Rightarrow$ can enclose pdf in the rectangle

$[a, b] \times [0, M]$



Idea

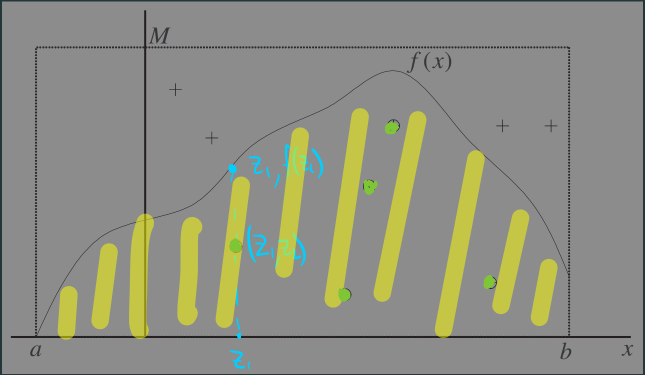
- Sample pts uniformly in rectangle
- 'accept' pts which lie under curve $y=f(x)$
- Output 'x-coordinate' of accepted point

Claim - X_i's dist^n is $F(\cdot)$

acceptance-rejection

Easy claim - Accepted pts (Z_1, Z_2) are uniformly distributed 'between $y=0$ and $y=f(x)$'

want samples of a rv $X \in [a, b]$, with pdf $f(x) \leq M$



acceptance-rejection sampling

1. generate $U_1, U_2 \sim U[0, 1]$
2. set $Z_1 = a + (b - a)U_1$, $Z_2 = MU_2$
3. if $Z_2 \leq f(Z_1)$, return $X_o = Z_1$; else, reject and repeat

$Z_1 \sim \text{Unif}[a,b]$

$Z_2 \sim \text{Unif}[0,M]$

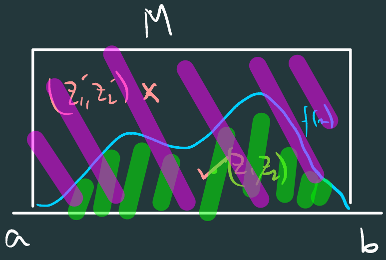
$(Z_1, Z_2) \sim$ Unif in rectangle

AR sampling: proof of correctness

let X_o denote the output of the AR method for cdf F (main idea = look at CDF of accepted pts)

- $F_{X_o}(x) = \mathbb{P}[X_o \leq x] = \mathbb{P}[Z_1 \leq x \mid (Z_1, Z_2) \text{ are accepted}]$

$$= \frac{\mathbb{P}[(Z_1, Z_2) \text{ accepted AND } Z_1 \leq x]}{\mathbb{P}[(Z_1, Z_2) \text{ accepted}]}$$



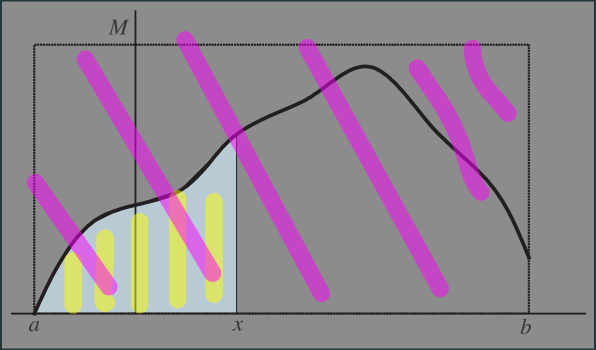
- $\mathbb{P}[(Z_1, Z_2) \text{ accepted}] = \dfrac{\text{Area of green}}{\text{Area of purple}}$

$$= \frac{1}{M(b-a)}$$

AR sampling: proof of correctness

$\mathbb{P}[(Z_1, Z_2) \text{ accepted AND } Z_1 \le x]$

observe: $\mathbb{P}[Z_1 \le x,\ Z_2 \le f(Z_1)] = \mathbb{P}[(Z_1, Z_2) \in \text{shaded region in figure}]$



$= \dfrac{\text{Area of}}{\text{Area of}}$

$= \dfrac{\int_a^x f(z)\,dz = F(x)}{M(b-a)}$

- $\mathbb{P}[Z_1 \le x \mid (Z_1, Z_2) \text{ accepted}] = \dfrac{F(x)/M(b-a)}{1/M(b-a)} = F(x)$