



ORIE 4580/5580: Simulation Modeling and Analysis

Unit 10: intro to Markov chains

Sid Banerjee

School of ORIE, Cornell University

rest of course: the ‘simulation hierarchy’

aim: simulate some complex real-world system to measure/optimize/control
– can do so in 3 ways of decreasing complexity

discrete-event simulations

- most general framework
- allows detailed modeling, general distributions
- complex; takes time to code/execute

Markovian models

- need inter-event times to be memoryless (i.e., Exponentially distributed)
- easier to simulate (no event-list needed)
- can give spurious insights, hide critical issues

closed-form solutions

- e.g. queueing models
- have formulas for steady-state performance measures
- restrictive assumptions

today: intro to Markov chains + Markovian simulation models

random process

random process

indexed collection of rvs $X_t \in \mathcal{S}$, one for each $t \in T$

– \mathcal{S} : state space, T : index set

Markov property

random process X_t has the Markov property if the probability of moving to a future state **only depends on the present state** and not on past states

random process

random process

indexed collection of rvs $X_t \in \mathcal{S}$, one for each $t \in T$

- \mathcal{S} : state space, \mathcal{T} : index set

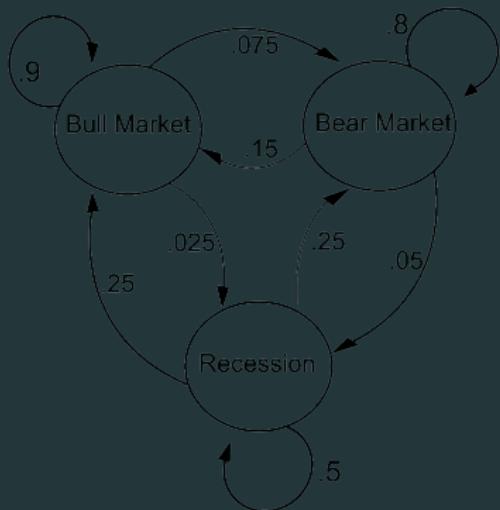
Markov property

random process X_t has the Markov property if the probability of moving to a future state **only depends on the present state** and not on past states

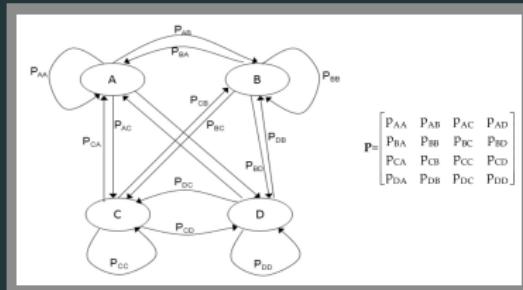
four types

- \mathcal{S} discrete, \mathcal{T} discrete: **discrete-time Markov chain** (DTMC)
 - **random walk on integers**
- \mathcal{S} discrete, \mathcal{T} continuous: **continuous-time Markov chain** (CTMC)
 - **Poisson process**
- \mathcal{S} continuous, \mathcal{T} discrete:
 - **random walk on the reals**
- \mathcal{S} continuous, \mathcal{T} continuous: **Markov process**
 - **Brownian motion**

Markov chains: basic definition



Markov chains: transition-diagram and transition matrix



$$P = \begin{bmatrix} P_{AA} & P_{AB} & P_{AC} & P_{AD} \\ P_{BA} & P_{BB} & P_{BC} & P_{BD} \\ P_{CA} & P_{CB} & P_{CC} & P_{CD} \\ P_{DA} & P_{DB} & P_{DC} & P_{DD} \end{bmatrix}$$

example: coin tosses and geometric rv.

recall the Geometric rv $p(k) = q^{k-1}(1 - q) \forall k \in \{1, 2, \dots\}$

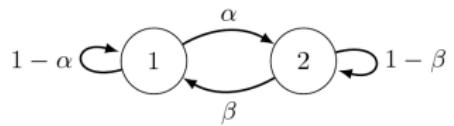
we can view this as a Markov chain as follows:

example: the coupon collector

a brand of cereal always distributes a baseball card in every cereal box, chosen randomly from a set of n distinct cards

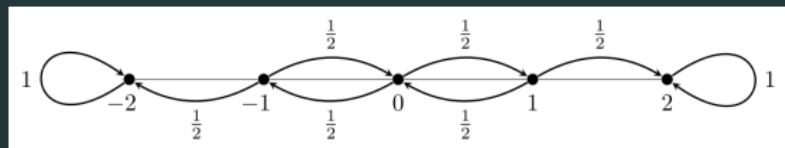
Markov chain model for number of cards owned by a collector:

Markov chains: two viewpoints

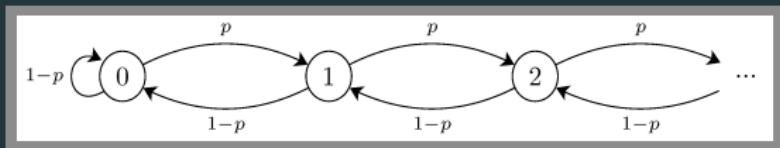


Markov chains: long-term behavior

Markov chains: absorbing chain

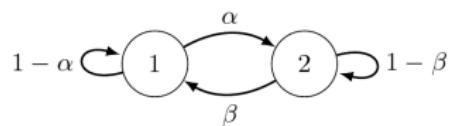


Markov chains: transient/recurrent chain

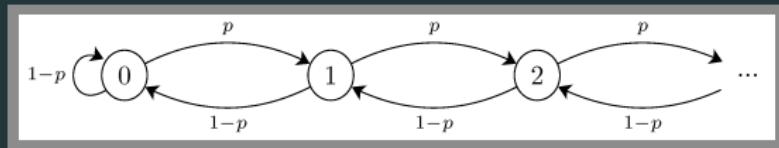


Markov chains: steady-state behavior

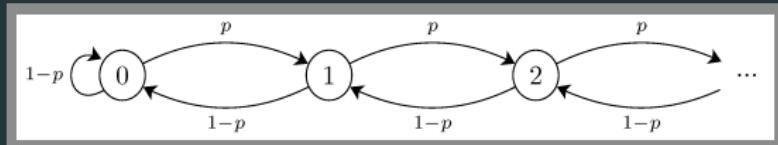
example:



Markov chains: steady-state for infinite chains



Markov chains: the ergodic theorem



DTMC: applications and problems

from discrete to continuous time

Markov property

random process X_t has the Markov property if the probability of moving to a future state **only depends on the present state** and not on past states

- \mathcal{S} discrete, \mathcal{T} discrete: **discrete-time Markov chain** (DTMC)
 - random walk
- \mathcal{S} continuous, \mathcal{T} discrete: **continuous-time Markov chain** (CTMC)
 - Poisson process

exponential distribution cheat sheet

can extend DTMCs to continuous time using special properties of the exponential distribution/poisson process

suppose $T \sim \text{exponential}(\lambda)$, then:

- pdf: $f_T(t) =$
- cdf: $F_T(t) = \mathbb{P}[T \leq t] =$
- (memorylessness): cdf of T knowing it is bigger than t ?

$$\mathbb{P}[T \leq t + x | T > t] =$$

understanding exponential distributions

suppose T_1, T_2, \dots, T_n are all exponentially distributed, with $T_i \sim \text{exponential}(\lambda_i)$.

- (minimum of exponentials): let $T_{\min} = \min\{T_i | i \in \{1, 2, \dots, n\}\}$; distribution of T_{\min} ?

$$T_{\min} \sim$$

- (first arrival): let $T_{\min} = \arg \min_i \min\{T_i | i \in \{1, 2, \dots, n\}\}$; distribution of T_{\min} ?

$$T_{\min} \sim$$

Poisson process cheat sheet

given Poisson processes $X(t) \sim PP(\lambda)$

- (inter-arrival times): let $\{A_1, A_2, A_3 \dots\}$ be the arrival times of the agents; then

$$T_i = A_i - A_{i-1} \sim$$

- (splitting): suppose we probabilistically split arrivals from $X(t)$ to $Y(t)$ with probability p , else to $Z(t) = X(t) - Y(t)$

$$Y(t) \sim$$

$$Z(t) \sim$$

- (time-varying rate): a time-varying rate of $\lambda(t) \in [0, \lambda^*]$ is equivalent to a $PP(\lambda^*)$ for which arrivals at time t are thinned with probability $p(t) = \lambda(t)/\lambda^*$

Poisson process cheat sheet (contnd)

given independent Poisson processes

$$X_1(t) \sim PP(\lambda_1), X_2(t) \sim PP(\lambda_2), X_3(t) \sim PP(\lambda_3)$$

- (**superposition**): suppose $S(t) = X_1(t) + X_2(t) + X_3(t)$

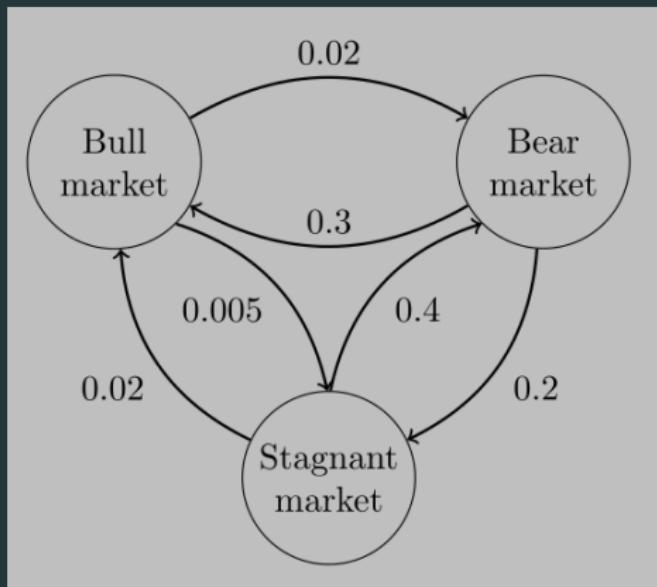
$$S(t) \sim$$

- (**first arrival**): let I_{\min} = the identity (i.e., $\{1, 2, 3\}$) of the first arrival among the three processes; distribution of I_{\min} ?

$$I_{\min} \sim$$

continuous-time Markov chains

CTMC: continuous-time Markov processes on discrete state-space
eg. modeling the financial market in continuous time



CTMCs

used as a model for many applications:

- queueing and service systems
- transportation networks
- epidemiology
- communications and computer networks
- agent choice models

advantages

- easy to analyze (in some cases)
- easier to simulate than general discrete-event simulation

problems

- need all inter-event times to be exponentially distributed
- can give spurious insights, hide critical issues

simulating a CTMC

example: queueing

the single-server M/M/1 queue

- number of servers: 1
- capacity: infinite
- service discipline: first-come-first-served (FCFS or FIFO)
- interarrival times: exponential(λ)
- service times: exponential(μ)
(independent interarrival and service times)

example: simulating an $M/M/1$ queue

example: epidemics

- want to model the spread of the latest influenza strain among the population
- there is a population of n people
 - at any time t , each person i is either **susceptible** (denoted S or 0) or **infected** (denoted I or 1)
 - infected people **get cured on average after time τ** , becoming susceptible to future infection.
 - each **pair of people (i, j) independently meet** each other with **average rate λ**
 - when a susceptible person meets an infected person, the susceptible person becomes infected
 - when two susceptible people or two infected people meet, nothing happens

example: SIS epidemic (contnd)

what assumptions do we need to make this Markovian?

example: SIS Epidemic (contnd)

how do we simulate it?

example: SIS Epidemic (contnd)

