# ORIE 4580/5580: Simulation Modeling and Analysis

# ORIE 5581: Monte Carlo Simulation

unit 4: generating random numbers

Sid Banerjee

School of ORIE, Cornell University

# generating uniformly distributed random variables

random number: a sample from $U[0, 1]$

**the 'fundamental theorem' of simulation**

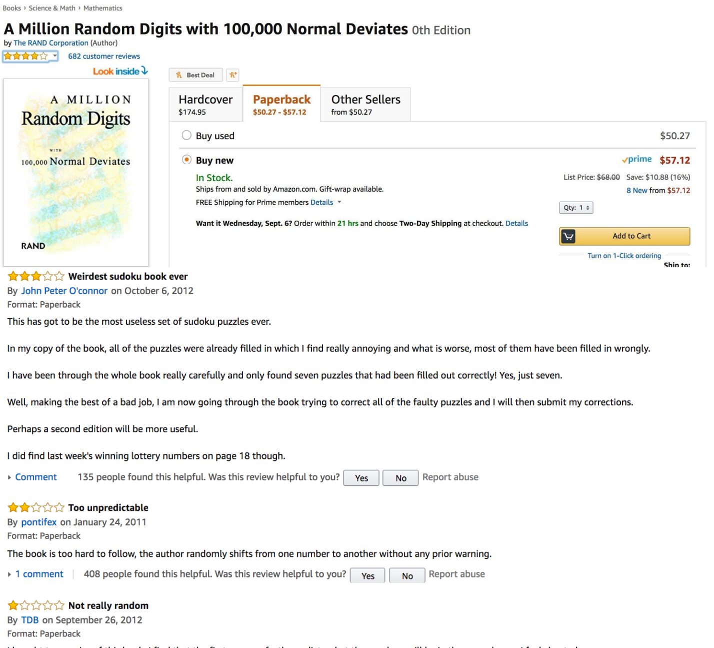
can 'transform' a stream of $U[0, 1]$ to any other random variable

– arbitrary probability distribution

– arbitrary correlations

– complex processes

# where can we find random numbers?



Books › Science & Math › Mathematics

**A Million Random Digits with 100,000 Normal Deviates** 0th Edition

by The RAND Corporation (Author)

682 customer reviews

Look inside

A MILLION Random Digits WITH 100,000 Normal Deviates RAND

Best Deal

| Hardcover | Paperback | Other Sellers |
|---|---|---|
| $174.95 | $50.27 - $57.12 | from $50.27 |

Buy used $50.27

Buy new prime $57.12

In Stock.

Ships from and sold by Amazon.com. Gift-wrap available.

FREE Shipping for Prime members Details

List Price: ~~$68.00~~ Save: $10.88 (16%)

8 New from $57.12

Qty: 1

**Want it Wednesday, Sept. 6?** Order within **21 hrs** and choose **Two-Day Shipping** at checkout. Details

Add to Cart

Turn on 1-Click ordering

Ship to:

**Weirdest sudoku book ever**

By John Peter O'connor on October 6, 2012

Format: Paperback

This has got to be the most useless set of sudoku puzzles ever.

In my copy of the book, all of the puzzles were already filled in which I find really annoying and what is worse, most of them have been filled in wrongly.

I have been through the whole book really carefully and only found seven puzzles that had been filled out correctly! Yes, just seven.

Well, making the best of a bad job, I am now going through the book trying to correct all of the faulty puzzles and I will then submit my corrections.

Perhaps a second edition will be more useful.

I did find last week's winning lottery numbers on page 18 though.

Comment 135 people found this helpful. Was this review helpful to you? Yes No Report abuse

**Too unpredictable**

By pontifex on January 24, 2011

Format: Paperback

The book is too hard to follow, the author randomly shifts from one number to another without any prior warning.

1 comment | 408 people found this helpful. Was this review helpful to you? Yes No Report abuse

**Not really random**

By TDB on September 26, 2012

Format: Paperback

# physical Methods

manual methods: coin toss, dice throw, drawing from an urn

objects that "appear" random: computer clock

physical devices: circuit noise, gamma-ray detectors

**advantage**

"true" random numbers (critical for cryptographic applications)

- for example, check out Radiolab podcast on launching a cryptocurrency

**drawbacks**

- **slow** (if generated as needed)
  **expensive** (if precomputed and stored in memory)
- **bias** may exist in the device
  for example, see Persi Diaconis on coin-tossing
- **hard to replicate** the random input sequence

## ‘pseudo-random’ generators

- mid-square method (von Neumann, 1949)

$$
\begin{aligned}
8234 \times 8234 &= 67(7987)56 \\
7987 \times 7987 &= 63(7921)69 \\
7921 \times 7921 &= 62(7422)41 \\
7422 \times 7422 &= 55(0860)84 \quad \ldots .
\end{aligned}
$$

- **objection**: random numbers are not random at all!
  - this criticism applies to all pseudo-random number generators
  - need tests to determine if algorithm produces “valid” outputs

# linear congruential generators

$$X_{n+1} = (aX_n + c) \bmod m,$$

- (fixed) parameters: modulus $m$, multiplier $a$, increment $c$
- seed: $X_0$ (the first input) is typically supplied by the user
- each $X_n$ is an integer in the set $\{0, 1, \ldots, m-1\}$.
- to get pseudorandom number $U_n \in (0, 1)$, set:

## LCG: example

$m = 32, \quad a = 11, \quad c = 0,$ different seeds.

| $n$ | $X_n$ | $X_n$ | $X_n$ |
|---|---|---|---|
| 0 | 1 | 2 | 4 |
| 1 | 11 | 22 | 12 |
| 2 | 25 | 18 | 4 |
| 3 | 19 | 6 | 12 |
| 4 | 17 | 2 | 4 |
| 5 | 27 | 22 | 12 |
| 6 | 9 | 18 | 4 |
| 7 | 3 | 6 | 12 |
| 8 | 1 | 2 | 4 |
| 9 | 11 | 22 | 12 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

## LCG: properties

- an LCG produces periodic output. period $\leq m$
  1. if period $= m$ with a seed $X_0$, then period $= m$ *for any* seed
  2. if period $< m$, then it may depend on the seed
- full period is desirable:
  1. one should never use the whole period of a LCG, otherwise dependencies between the random numbers will occur.
  2. not have full period $\implies$ gaps in the output sequence.
- full period $\implies$ granularity $= 1/m$.
  not a problem when $m$ is large.

# LCG: theory

the period of LCGs is well understood
(the following results are for demonstration; you do not need to know them)

**Theorem: LCGs with full period**

an LCG$(m, a, c)$ has full period if all of the following are true

1. $m$ and $c$ are relatively prime.
2. if $q$ is a prime number that divides $m$, then $q$ divides $a - 1$.
3. if 4 divides $m$, then 4 divides $a - 1$.

**Corollary:**
an LCG with $m = 2^b$ has full period if $c$ is odd and 4 divides $a - 1$.

*example* – $m = 8, \quad a = 5, \quad c = 3$

# multiplicative generators

$$X_{n+1} = (aX_n) \bmod m$$

- $X_n = 0 \implies X_n = X_{n+1} = X_{n+2} = \ldots = 0$
- for an MG, full period $\implies \{1, \ldots, m-1\}$.

**Theorem: MGs with full period**

an MG has full period if all of the following are true

1. $m$ is prime
2. $a^{m-1} - 1$ is divisible by $m$.
3. there is no $j < m-1$ such that $a^j - 1$ is divisible by $m$.

**Theorem: Sufficient conditions for good MGs**

the largest possible period for a MG with $m = 2^b$ is $m/4$

this is achieved when $X_0$ is odd and $a$ is of the form:

$$a = 3 + 8k \quad \text{or} \quad a = 5 + 8k,$$

for some positive integer $k$.

# deficiencies of LCGs

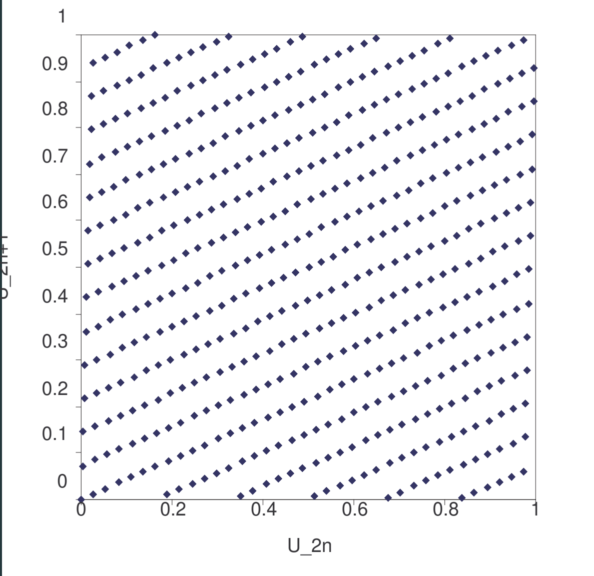
- LCG's possess **theoretical deficiencies**
  (any deterministic generator must have deficiencies.)
- if $U_0, U_1, \ldots$ are iid $U[0,1]$, then the points

$$(U_0, U_1),\ (U_2, U_3),\ (U_4, U_5), \ldots$$

  should lie uniformly over the square $[0,1] \times [0,1]$.
- suppose $U_0, U_1, \ldots$ be generated by a LCG:
  how do the points $(U_0, U_1),\ (U_2, U_3),\ (U_4, U_5), \ldots$ behave?

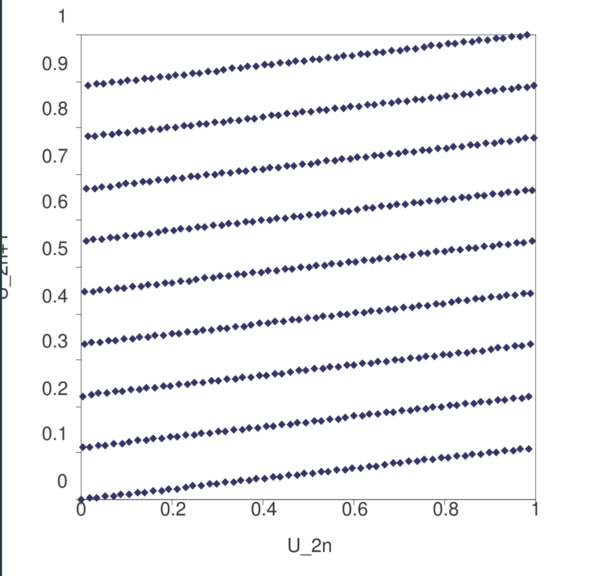
# deficiencies of LCGs

$$m = 2^{10},\ a = 37,\ c = 1$$



# deficiencies of LCGs

$$m = 2^{10},\ a = 57,\ c = 1$$



# deficiencies of LCGs

- the points

$$(U_0, U_1),\ (U_2, U_3),\ (U_4, U_5), \ldots$$

lie on a relatively small number of parallel lines!

- in general, the points

$$(U_0, U_1, U_2, \ldots, U_{d-1}),\ (U_d, U_{d+1}, U_{d+2}, \ldots, U_{2d-1}), \ldots$$

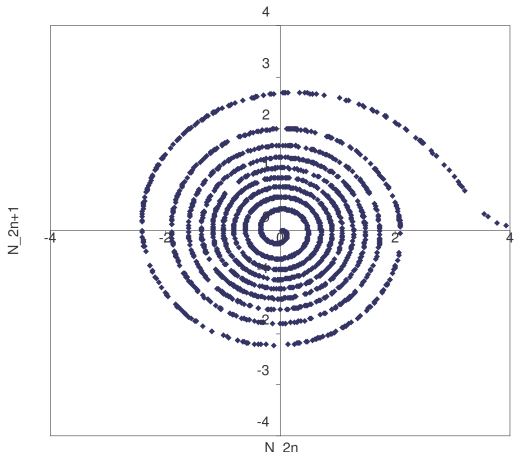
lie on parallel $(d-1)$-dimensional hyperplanes!

- problematic in simulations of geometric phenomena.
  OK for discrete-event simulations.

# deficiencies of LCGs for generating other rvs

let $N_0, N_1, N_2, \ldots$ be samples from $\mathcal{N}(0,1)$ generated using the Box-Muller method using $U_0, U_1, U_2, \ldots$ from an LCG

– then the pairs $(N_0, N_1)$, $(N_2, N_3)$, $(N_4, N_5), \ldots$ lie on a spiral in two-dimensional space. E.g., $a = 9, m = 2^{21}, c = 1$



## combining generators

- $m = 2^{31} - 1$ is popular, but period is only about 2 billion.
- not sufficient! e.g. traffic simulators need lots of random numbers. (10s of 000s of vehicles, 1000s of random disturbances, lots of replications).
- shouldn't use anywhere near the full period - maybe $\leq 1\%$
- to generate longer period, take two MG's

$$X_{n+1} = (a_1 X_n) \bmod m_1 \qquad , \qquad Y_{n+1} = (a_2 Y_n) \bmod m_2$$

  and set

$$Z_n = (X_n + Y_n) \bmod m_3.$$

- period can be on the order of $m_1 m_2$; e.g., set $a_1 = 40014,\ a_2 = 40692,\ m_1 = 2147483563,\ m_2 = 2147483399$ and $m_3 = m_1$.
- can combine more than two.

# streams and substreams

- useful to divide PRNG output into streams and substreams
- stream = simulation replication
  substream = source of randomness
- useful for debugging and for variance-reduction techniques

# tl;dr

- hyperplane/spiral problems are well understood (and avoided)
- current generators have been carefully tested, pass lots of statistical tests (but must fail at least one test...)

**the last word**

modern PRNGs are

- random enough for your sim answer to be correct
- deterministic enough (by setting seed) for your sim to be repeatable