ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 6: generating random vectors

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review and roadmap

generating random variables

we have seen how to:

- generate pseudorandom $U[0, 1]$ samples
- transform $U[0, 1]$ samples to another rv using
  - inversion
  - acceptance-rejection

two special cases

- multivariate Normal rvs
  - generating correlated vectors
- Exponential rvs and the Poisson process
  - generating time-indexed processes

generating Gaussians

generating normal random variables

- method 1: inversion
  - no closed form for $\phi^{-1}(x)$
  - inversion done numerically
- method 2: via the central limit theorem.
  - generate $U_1, U_2, \ldots$
  - scale and center appropriately
  - not exact!
- generalized AR (using $Exp(1)$)
- Box-Muller

the Box-Muller method

generates a pair of $\mathcal{N}(0,1)$ rvs

- $N_1 \sim \mathcal{N}(0,1), \quad N_2 \sim \mathcal{N}(0,1), \qquad N_1 \perp\!\!\!\perp N_2$
- the point (N_1, N_2) can be expressed in polar coordinates as

$$(N_1, N_2) = (R\cos\theta, R\sin\theta)$$

the Box-Muller method

$$(N_1, N_2) = (R\cos\theta, R\sin\theta)$$

- $\theta \sim U[0, 2\pi]$, and is independent of R.
- $R = \sqrt{N_1^2 + N_2^2} = \sqrt{2X}$, where $X \sim Exp(1)$

Box-Muller Algorithm

1. generate $U_1 \sim U[0,1]$, $U_2 \sim U[0,1]$.
2. set

   $R =$ $\theta =$

3. set

   $N_1 =$ $N_2 =$

generating correlated rvs

variance and covariance

- variance: $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- X and Y are independent if $\mathbb{P}[X \leq x, Y \leq y] = F_X(x)F_Y(y)$ for all x, y
- covariance: $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \Rightarrow Cov(X, Y) = 0$ (independent implies uncorrelated)
- however, uncorrelated rvs can be dependent

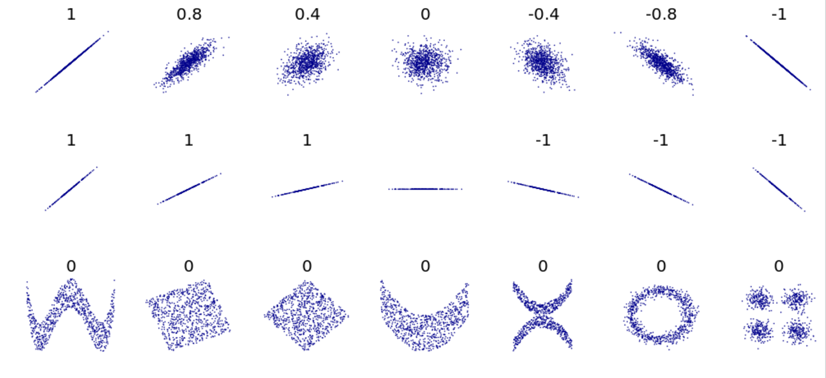
correlation

for any rvs X, Y, their correlation coefficient is

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

properties:

correlation: examples



multivariate normal rvs

- given a sample $X \sim \mathcal{N}(0,1)$, can generate $Y \sim \mathcal{N}(\mu, \sigma^2)$ as

- *d*-dimensional multivariate normal $\iff$ vector $(X_1, X_2, \ldots, X_d)$
  - each $X_i \sim \mathcal{N}(\mu_i, \sigma_{ii})$ (i.e., normally distributed with mean μ_i and variance σ_{ii})
  - covariance between X_i and X_j is $Cov(X_i, X_j) = \sigma_{ij}$
  - covariance between X_j and X_i is $Cov(X_j, X_i) = \sigma_{ji}$
  - conditions:

random vectors and covariance

consider any random vector $(X_1, X_2, \ldots, X_n)$

- vector of means $\mu = (\mu_1, \mu_2, \ldots, \mu_d)^T$
- covariance matrix: $\Sigma = \mathbb{E}[(X-\mu)(X-\mu)^T]$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{bmatrix}$$

- Σ always positive definite

multivariate Normal rvs via linear combinations

multivariate Normal rvs via linear combinations

multivariate Normal rvs via linear combinations

correlated Normal random variables

- $\mu = (\mu_1, \mu_2, \ldots, \mu_d)^T$ (column vector)
- $\Sigma =$ positive semidefinite covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{bmatrix}$$

- want to generate samples of such *correlated* random variables.

bivariate Normal rvs

- start with the case $d = 2$.
- $\sigma_1^2 = \sigma_{11} = Var(X_1), \quad \sigma_2^2 = \sigma_{22} = Var(X_2),$

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{\sigma_{21}}{\sigma_1 \sigma_2}.$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

- want to generate samples of X_1 and X_2, where

$$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), \; X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2),$$

$$Cov(X_1, X_2) = \sigma_{12}.$$

generating correlated bivariate Normal rvs

- take $N_1, N_2 \sim \mathcal{N}(0,1)$ and independent.
- set $X_1 = \mu_1 + \sigma_1 N_1$,
- set $X_2 = \mu_2 + aN_1 + bN_2$
- we need to have

$$\sigma_2^2 = Var(X_2) = a^2 Var(N_1) + b^2 Var(N_2) =$$
$$\sigma_{12} = Cov(X_1, X_2) = Cov(\mu_1 + \sigma_1 N_1, \mu_2 + aN_1 + bN_2) =$$

- $a^2 + b^2 = \sigma_2^2, a\sigma_1 = \rho\sigma_1\sigma_2 \implies (a, b) = \left(\frac{\sigma_{12}}{\sigma_1}, \sigma_2\sqrt{1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}}\right)$

generating correlated bivariate Normal rvs

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ \frac{\sigma_{12}}{\sigma_1} & \sigma_2\sqrt{1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

$$X = \mu + L N$$

generating correlated multivariate Normal rvs

- this method works when $d > 2$ as well
- write X as $X = \mu + LN$, where N is a d-dimensional vector whose components are independent $\mathcal{N}(0,1)$.
- to connect the matrix L to Σ, observe that

$$\Sigma = \mathbb{E}\left[(X-\mu)(X-\mu)^T\right] =$$

generating correlated Normal rvs

- L is a "square root" of Σ.
- writing Σ as

$$\Sigma = LL^T$$

  is called the Cholesky factorization of Σ.
- once we can compute the Cholesky factor of Σ, we are done!

correlated rvs beyond multivariate Normal

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