ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

Unit 9: generating random processes

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review and roadmap

generating random variables

we have seen how to:

- generate pseudorandom $U[0,1]$ samples
- transform $U[0,1]$ samples to another rv using
  - inversion
  - acceptance-rejection
- generate random vectors
  - understanding correlation
  - multivariate Normal rvs

and now for the grand finale!!

- generating time-indexed random processes
  - (discrete time) Markov chains - next week
  - exponential rvs and the Poisson process - today
  - and beyond (Brownian motion...)

random processes

random process

infinite dimensional - $\mathbb{R}^\alpha$

finite dimensional - $\mathbb{R}^n$

random process (random vector ++)

indexed collection of rvs $\underline{\underline{X_t}} \in \mathcal{S}$, one for each $t \in \mathcal{T} = \mathbb{N}$

Subscripted r.v.s — set of subscripts/indices

– $\mathcal{S}$: state space

– $\mathcal{T}$: index set

$\mathbb{N} = \{1, 2, \ldots\}$

four types

easy to code (for loop ...)

- $\underline{\mathcal{S} \text{ discrete}, \mathcal{T} \text{ discrete}}$: discrete-time Markov chain (DTMC)
  – random walk $X_t = \sum_{i=1}^{t} Y_i, \; Y_i \sim \text{Ber}(p), \text{iid}$ | $T = \mathbb{N}$, $S = \mathbb{N} \cup \{0\}$
- $\mathcal{S}$ continuous, $\mathcal{T}$ discrete: discrete-time Markov process
  $X_t = \sum_{i=1}^{t} Y_i, \; Y_i \sim N(0,1)$ | $T = \mathbb{N}$, $S = \mathbb{R}$
- $\mathcal{S}$ discrete, $\mathcal{T}$ continuous: $\boxed{\text{continuous-time Markov chain}}$ (CTMC)
  – Poisson process $= X_t, \; t \in \mathbb{R}, \; X_t \in \{\ldots -2, -1, 0, 1, 2, \ldots\}$
- $\mathcal{S}$ continuous, $\mathcal{T}$ continuous: Markov process
  – Brownian motion $X_t, \; t, X_t \in \mathbb{R}$

discrete-time Markov chain

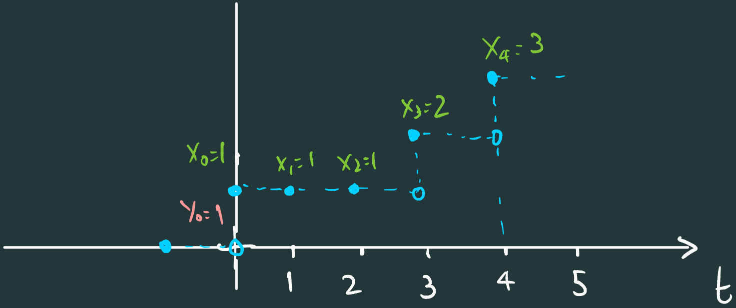
the random walk

$X_0 = 0$, $Y_k \sim \text{Bernoulli}(p)$ iid, and

$$X_t = \sum_{i=0}^{t} Y_k \quad , \quad X_{-1} = 0$$

Y_0

Eg. $Y = (1, 0, 0, 1, 1 \ldots)$



counting process visualized

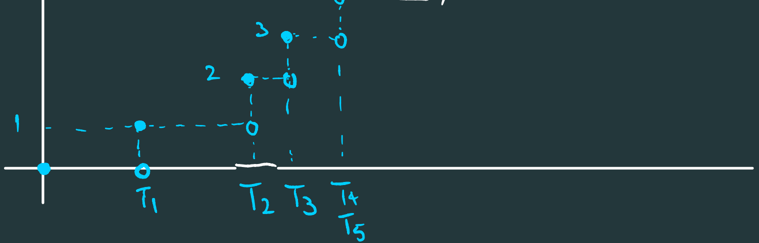
counting process

non-negative integer-valued stochastic process $[N(t) : t \geq 0]$

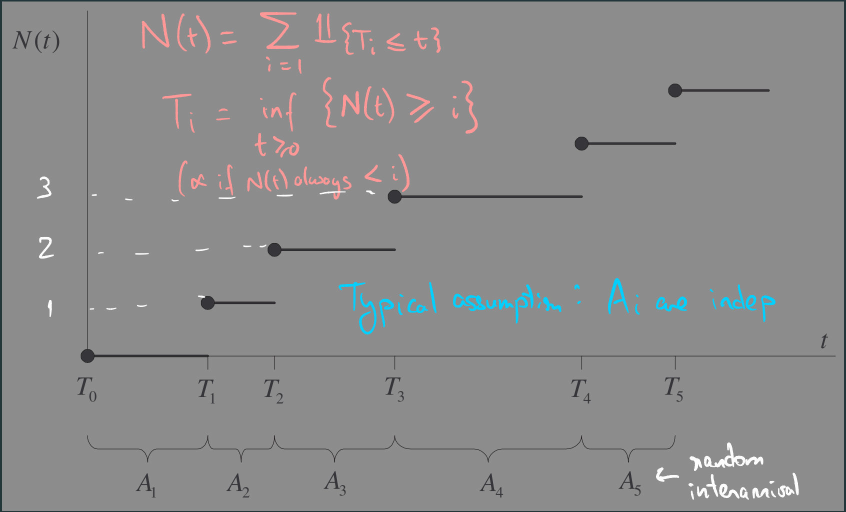
- $N(0) = 0$ (convention)
- $N(t) = \#$ of arrivals during time interval $(0, t]$

- $[N(t) : t \geq 0]$ increases by jumps
- $T_n =$ time of the n-th arrival, $T_0 = 0$ $\left(T_1 \leq T_2 \leq T_3 \leq \ldots\right)$
- $A_n =$ be the interarrival time for the n-th arrival.

$\boxed{A_n = T_n - T_{n-1}}$, $A_n \geq 0$



counting process



$$N(t) = \sum_{i=1}^{\infty} \mathbb{1}\{T_i \le t\}$$

$$T_i = \inf_{t \ge 0} \{N(t) \ge i\}$$

(∞ if $N(t)$ always $< i$)

Typical assumption: A_i are indep

$\leftarrow$ random interarrival times

a desired (?) properties of inter-arrival times?

- Suppose you are told $N(10.2) = 3$ and you know that $A_i \sim U[0,1]$

- Q: When is next arrival, ie

$$\mathbb{E}\left[T_4 \mid N(10.2) = 3\right] = \underbrace{\overbrace{10.1}^{}+\overset{\mathbb{E}[A_i]}{0.5} = 10.7}_{\text{NOT TRUE...}}$$

(What if I tell you $N(8) = 3$) ← INCONSISTENT !!

- What minimal extra info do you need? Need T_3 $\Rightarrow \mathbb{E}[T_4 \mid T_3] = T_3 + 0.5$
  (ie, need time since last arrival ...)

Still complex. $\mathbb{E}\left[T_4 \mid N(10.1) = 3, T_3 = 9.9\right] = ?$ (10.5)

the Exponential distribution

suppose $T \sim Exp(\lambda)$, then:

$\lambda =$ 'RATE' of the Exp (unit $= s^{-1}$)
$E[T] = 1/\lambda$ (unit s)

- pdf: $f_T(t) = \lambda e^{-\lambda t}$
- cdf: $F_T(t) = \mathbb{P}[T \leq t] = 1 - e^{-\lambda t}$ $\left(\mathbb{P}[T > t] = e^{-\lambda t}\right)$

why is $Exp(\lambda)$ special?

memorylessness (you can 'restart' a random exp stopwatch anytime)

cdf of T given that T bigger than t?

$\mathbb{P}[T \leq t + x | \underline{T > t}] = F_T(x) = 1 - e^{-\lambda x}$

ie, $T \sim Exp(1)$ AND $T > t \Rightarrow T \sim t + Exp(1)$

$T \sim Exp(1)$ AND $\underset{\text{ie, not happened till } t_1 \ldots}{(T > t_1)}$ AND $(T > t_2) \Rightarrow T \sim \max(t_1, t_2) + Exp(1)$

the exponential distribution: properties

independently

suppose $T_1, T_2, \ldots, T_n$ are all exponentially distributed, with $T_i \sim Exp(\lambda_i)$.

- (minimum of exponentials): let $T_{\min} = \min\{T_i | i \in \{1, 2, \ldots, n\}\}$
  distribution of $T_{\min}$?

independence

$$T_{\min} \sim \mathbb{P}\left[\min(T_1, T_2, \ldots, T_n) \geq t\right] = \mathbb{P}\left[T_1 \geq t, T_2 \geq t, \ldots T_n \geq t\right] = \prod_{i=1}^{n} \mathbb{P}[T_i \geq t]$$

$$= \prod_{i=1}^{n} e^{-\lambda_i t}$$

$$= e^{-(\sum \lambda_i) t} \sim Exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

- (first arrival): let $I_{\min} = \arg\min\{T_i | i \in \{1, 2, \ldots, n\}\}$
  distribution of $I_{\min}$?

$$I_{\min} \sim \mathbb{P}[I_{\min} = i] = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

(multinomial distn)

$$\cdot \quad \mathbb{P}[I_{min} = i] = \mathbb{P}[T_i < T_j \;\; \forall j \neq i]$$

do not need $\leq$ as $\mathbb{P}[T_i = T_j] = 0$

independence

$$= \int_0^\infty \mathbb{P}[T_i = t] \left(\prod_{j \neq i} \mathbb{P}[T_j > t] \right) dt$$

$$= \int_0^\infty \lambda_i e^{-\lambda_i t} \left(\prod_{j \neq i} e^{-\lambda_j t} \right) dt$$

$$= \lambda_i \int_0^\infty e^{-\left(\sum_{j=1}^{n} \lambda_j\right) t} \, dt = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

class poll: parallel simulations

we use 10 GPUs in parallel for simulating a machine learning model
– each GPU finishes one simulation in independent *Exp*(1) time (and then stops)
– we want to get 3 replications to be confident of our model
what is the expected time this will take?

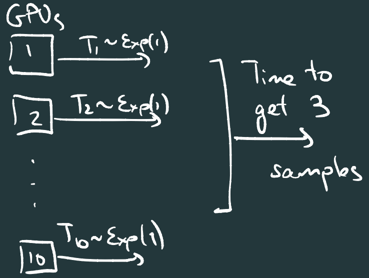
(a) $3 \times \frac{1}{10}$

(b) $\frac{1}{10} + \frac{1}{9} + \frac{1}{8}$

(c) $\frac{1}{10^3}$

(d) $\frac{1}{30}$

Note - Each GPU does at most 1 simulation (does not restart after finishing)



Soln

Given $T_1, T_2, \ldots, T_{10} \sim$ iid $\mathcal{E}xp(1)$, let

$T_{(n)} < T_{(n-1)} < \ldots < T_{(1)}$ be the sorted times

using min of exp

- Claim - $T_{(n)} = \min\{T_i\}_{i=1}^{n} \sim \mathcal{E}xp(10)$

$\Rightarrow \mathbb{E}[T_{(n)}] = 1/10$

- Q: What is $T_{(n-1)} \overset{\text{memorylessness}}{=} T_{(n)} + \underbrace{\begin{Bmatrix}\text{min of } 9 \\ \text{indep } \mathcal{E}xp(1)\end{Bmatrix}}_{\mathcal{E}xp(9)}$

$\Rightarrow \mathbb{E}[T_{(n-1)}] = \mathbb{E}[\mathcal{E}xp(10)] + \mathbb{E}[\mathcal{E}xp(9)] = \frac{1}{10} + \frac{1}{9}$

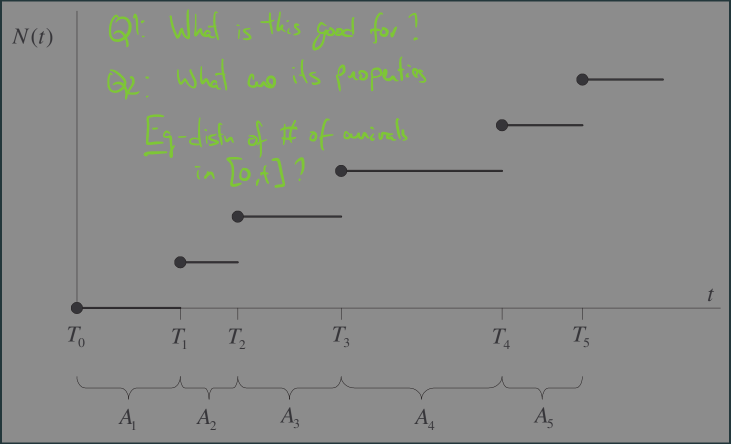
- Similarly - $T_{(n-k)} = \mathcal{E}xp(n) + \mathcal{E}xp(n-1) + \ldots + \mathcal{E}xp(n-k)$

$\Rightarrow \mathbb{E}[T_{(n-k)}] = \frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{n-k}$

Poisson process $PP(\lambda)$

$\boxed{A_1, A_2, \ldots \text{ i.i.d. } Exp(\lambda)} \implies$ Poisson process of rate λ

– denoted $PP(\lambda)$



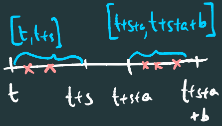
Poisson process: properties

$N(t) \equiv$ Count of the process at time t

note: $N(t+s) - N(t) = \#$ arrivals in time interval $(t, t+s]$

- Exponential interrarival times: $A_n = T_n - T_{n-1} \sim Exp(\lambda)$, i.e.

$$\mathbb{P}[A_n \leq t] = 1 - e^{-\lambda t}$$



Using just this fact, we get the following

- independent increments: $N(t+s) - N(t) \perp\!\!\!\perp N(t) - N(0)$ (for non-overlapping intervals)
  more generally, for $t_1 \leq t_2 \leq t_3 \leq t_4$, $N(t_4) - N(t_3) \perp\!\!\!\perp N(t_2) - N(t_1)$

- Poisson arrivals: $\underline{N(t+s) - N(t)} \sim Poisson(\lambda s)$, i.e.,

$$\mathbb{P}\left[N(t+s) - N(t) = k\right] = \frac{e^{-\lambda s}\,(\lambda s)^k}{k!}$$

moreover, $\mathbb{E}[\text{Arrivals in interval of length } s] = \underbrace{\lambda s}$.

<u>Note</u> - doubles if length of interval doubles
- doubles if λ doubles

- $\text{Poisson}(\alpha) \equiv$ distn on $\{0, 1, 2, \ldots\}$

  s.t. $\mathbb{P}[X = k] = p_\alpha(k) = \dfrac{e^{-\alpha}\alpha^k}{k!}$

  (no closed form for the CDF)

- $\mathbb{E}[X] = \alpha$

Poisson process computations

these properties of PP are useful for computations; Eg. $\mathbb{P}$[no arrivals in $[0, t]$]?

– using Exponential interarrival times:

$\mathbb{P}[\mathcal{N}(t) - \mathcal{N}(0) = 0] =$ $\mathbb{P}[A_1 \sim \mathcal{E}xp(\lambda) > t] = e^{-\lambda t}$

– using Poisson arrivals:

$\mathbb{P}[\mathcal{N}(t) - \mathcal{N}(0) = 0] =$ $\mathbb{P}[Poi(\lambda t) = 0] = \dfrac{e^{-\lambda t}(\lambda t)^0}{0!}$

$= e^{-\lambda t}$

thinning and superposition (parallel of min of Expos)

two more important properties of Poisson processes

superposition

let $N_1(t) \sim PP(\lambda_1)$ and $N_2(t) \sim PP(\lambda_2)$ be two independent Poisson processes
then $N_1(t) + N_2(t)$ is a Poisson process with rate $(\lambda_1 + \lambda_2)$

Pf: First arrival is $\min(Exp(\lambda_1), Exp(\lambda_2))$

thinning

given $N(t) \sim PP(\lambda_1)$, let $N_1(t)$ be the process generated by retaining each arrival of $N(t)$ independently with probability p, and let $N_2(t) = N(t) - N_1(t)$ be the rejected points

then $N_1(t) \sim PP(\lambda p)$ and $N_2(t) \sim PP(\lambda(1-p))$

moreover, $\boxed{N_1(t) \perp\!\!\!\perp N_2(t)!}$

these are very useful for discrete-event simulation!

generating samples of Poisson processes

1. set the arrival counter $n = 0$. Set $T_0 = 0$.
2. increment n by 1
let A_n be a sample from exponential distribution with parameter λ.
3. advance time
$$T_n = T_{n-1} + A_n.$$
4. return to Step 2.

Poisson process: formal definition

Poisson process $PP(\lambda)$

an arrival process $[N(t) : t \geq 0]$ is a Poisson process with rate λ if

1. $N(t+s) - N(t)$ independent of $N(t) - N(0)$.

2. $\mathbb{P}[N(t+s) - N(t) = 1] = \lambda s + o(s)$, where $o(s)$ denotes a function $g(\cdot)$ satisfying: $\lim_{s \to 0} \frac{g(s)}{s} = 0$.

3. $\mathbb{P}[N(t+s) - N(t) \geq 2] = o(s)$.

example: $P(N(t+s) - N(t) = 1) = e^{-\lambda s}(\lambda s)$

why Poisson process?

- easy to simulate!
- behavioral justifications: arrivals are modeled as PP because
  - memorylessness of interarrival times
  - the Palm-Khintchine theorem