ORIE 4580/5580: Simulation Modeling and Analysis
ORIE 5581: Monte Carlo Simulation

unit 1: probability review

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given the circle $x^2 + y^2 = 1$, how can we sample a uniform random chord?

(a) pick 2 endpoints (X, Y) *uniformly at random* (u.a.r) on the circumference of the circle, and connect

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(b) pick angle $\theta \in [0, 2\pi]$ u.a.r and draw radius at angle θ, and then draw chord perpendicular to this radius at distance $R \in [0, r]$ picked u.a.r
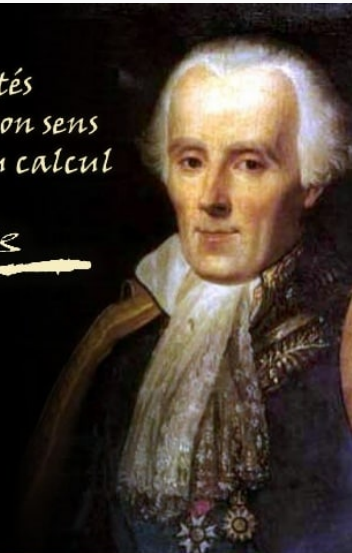
18

(c) pick point (Z, W) u.a.r inside the circle, connect it to the center, and draw chord perpendicular to this line

4

(d) all of these are the same

65

(a)

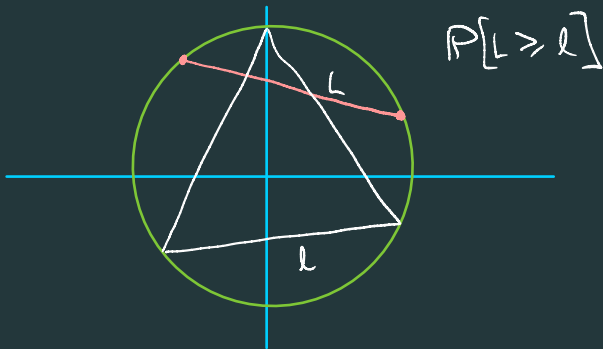La théorie des probabilités n'est, au fond, que le bon sens réduit au calcul

Laplace

"probability theory is common sense reduced to calculation"

not quite. . .

Bertrand's ~~problem~~ Paradox
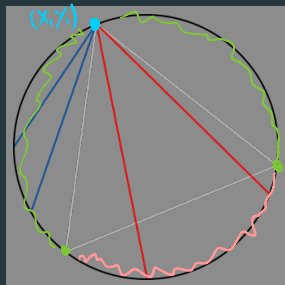
given equilateral triangle inscribed in a circle and a random chord C, what is the $\mathbb{P}$[length of C exceeds the side of the triangle]?

$\mathbb{P}\left[L > \ell\right]$

not quite. . .

Bertrand's problem

given equilateral triangle inscribed in a circle and a random chord C, what is the $\mathbb{P}[$length of C exceeds the side of the triangle$]$?
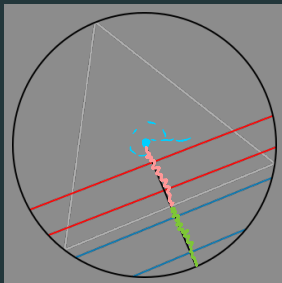
$^1/_3$ $^1/_2$ $^1/_4$
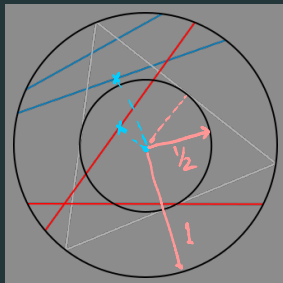
not quite...

Bertrand's problem

given equilateral triangle inscribed in a circle and a random chord C, what is the $\mathbb{P}[\text{length of } C \text{ exceeds the side of the triangle}]$?
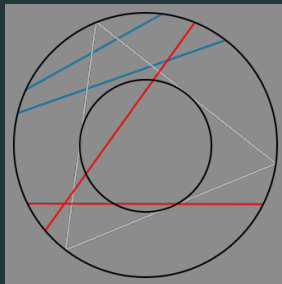

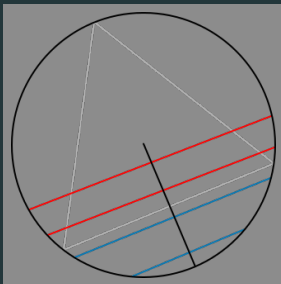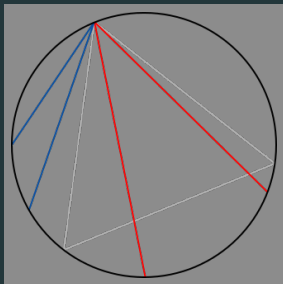**the moral (for this course... and for life)**

be very precise about defining experiments/random variables/distributions

also see Wikipedia article on Bertrand's paradox

Grinstead & Snell - Ch 1, 2

these are the things you must be comfortable with

- random variables (rv) + cumulative distribution fn (cdf) } today   next Tuesday!
- expectation and variance of random variables
- independence (and dependence – mutual exclusivity, conditioning, Bayes rule)
- common rvs (Bernoulli, Binomial, Geometric, Gaussian, Exponential, Poisson)

in 2 classes

law of large numbers
central limit thm } → Simulate!

random variables

random variables

sample space Ω: set of all possible outcomes of random expmt

random variable: any function from $\Omega \rightarrow \mathbb{R}$

$\Omega \equiv$ the <u>data structure</u> you need to code your expt

Bertrand's : Method 1 - Random end-pts
<u>Problem</u>

$$\Omega = \left\{ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^4 \,\middle|\, x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1 \right\}$$

$$\approx \left\{ (x_1, x_2) \in [-1, 1]^2, (s_1, s_2) \in \{+1, -1\} \right\}$$

'easy'

trickier → $y_1 = s_1 \left| \sqrt{1 - x_1^2} \right|$, $y_2 = s_2 \left| \sqrt{1 - x_2^2} \right|$

+ 2 random bits

Vol = 4

Meter Theorem   independent
- Given a way to generate $U \sim \text{Unif}[0,1]$,
  I can simulate everything!

Method 2 - Random angle + length

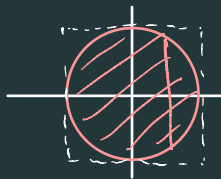$$\Omega = \{(\theta, l) \mid \theta \in [0, 2\pi], l \in [0, 1]\}$$

Vol = $2\pi \approx 6.3$

Method 3 - Random center

$$\Omega = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

$$\approx \{(\tfrac{x}{2}, \tfrac{y}{2}) \mid x^2 + y^2 \leq 4\}$$

Vol = $\pi \approx 3.14$

random variables

sample space Ω: set of all possible outcomes of random expmt

random variable: any function from $\Omega \to \mathbb{R}$

example: Youtube's ad algo (or so I sometimes feel...) pick a random number of ads between 0 and 2 (inclusive), and a random length between 0 and 30s for each ad. Let T = Total length of ads on video
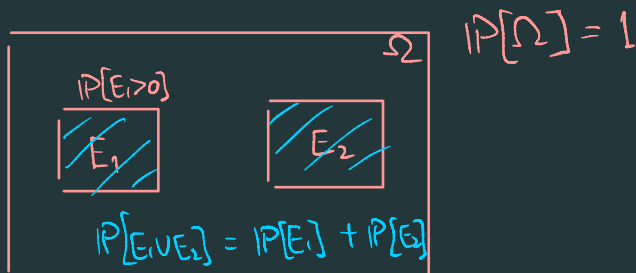
↳ non-negative integer

$$\Omega = \{0,1,2\} \times \{0,1,2,\ldots,30\} \times \{0,1,2,\ldots,30\}$$

↑
Cartesian Product

$\{0,1,2,\ldots,30\}^2$

↑
ideally, the 'minimal' data-structure/info you need

probabilities

Probability $\mathbb{P}(E)$: 'number' for 'every' subset $E \subseteq \Omega$, such that:

- $\mathbb{P}(E) \geq 0$ for all $E \subset \Omega$
- $\mathbb{P}(\Omega) = 1$ (i.e., probs 'summed over all outcomes' adds to 1)
- $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$ if $E_1 \wedge E_2 = \phi$
  (i.e., probs add for mutually exclusive events $E_1 \cap E_2 = \varphi$)

$\Omega = \mathbb{R}$

ALERT!!

always try to think of probability and rvs through the cdf

- for any rv X (discrete or continuous), its probability distribution is defined by its cumulative distribution function (cdf)

$$F(x) = \mathbb{P}\left[X \leq x\right]$$

lower case letter $\Leftrightarrow$ 'realization'

$\{z \in \mathbb{R} \mid z \leq x\} = (-\infty, z]$

upper case   letters $\Leftrightarrow$ random variables

- using the cdf we can compute probabilities — convention

$$\mathbb{P}[a < X \leq b] = F(b) - F(a)$$

$(a, b]$

visualizing a cdf

- RCLL / càdlàg

the plot of a cdf obeys **3 essential rules** + **one convention**

example: consider an rv $\in [-2, 5]$ with a **jumps** at 1 and 2

because of ← convention
⇒ upper circle colored

$y = 1$

②
$\lim_{x \to \infty} F(x) = 1$
→

①
$\lim_{x \to -\infty} F(x) = 0$
←

δ ← $\mathbb{P}[x = y] = \delta \iff \text{'jump'}$

③
non-decreasing

y

x_1 x_2

$y = 0$

$\mathbb{P}[x \leq x_1] \leq \mathbb{P}[x \leq x_2]$