ORIE 4580/5580: Simulation Modeling and Analysis

ORIE 5581: Monte Carlo Simulation

unit 2: mean, variance, and tails

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expectations and independence

expected value (mean, average)

let X be a random variable, and $g(\cdot)$ be any real-valued function

- If X is a discrete rv with $\Omega = \mathbb{Z}$ and pmf $p(\cdot)$, then

  $$\mathbb{E}[X] =$$

  $$\mathbb{E}[g(X)] =$$

- if X is a continuous rv with $\Omega = \mathbb{R}$ and pdf $f(\cdot)$, then

  $$\mathbb{E}[X] =$$

  $$\mathbb{E}[g(X)] =$$

variance and standard deviation

- definition: $Var(X) =$ $\sigma(X) =$
- (alternate formula for computing variance)

  $Var(X) =$

independence

what do we mean by "random variables X and Y are independent"?
(denoted as $X \perp\!\!\!\perp Y$; similarly, $X \not\perp\!\!\!\perp Y$ for 'not independent')

intuitive definition: knowing X gives no information about Y

formal definition:

- one measure of independence between rv is their covariance

$$Cov(X, Y) = \qquad \text{(formal definition)}$$

$$= \qquad \text{(for computing)}$$

independence and covariance

how are independence and covariance related?

- X and Y are independent, then they are uncorrelated

  in notation: $X \perp\!\!\!\perp Y \quad \Rightarrow \quad Cov(X, Y) = 0$

- however, uncorrelated rvs can be dependent

  in notation: $Cov(X, Y) = 0 \quad \not\Rightarrow \quad X \perp\!\!\!\perp Y$

- $Cov(X, Y) = 0 \Rightarrow X \perp\!\!\!\perp Y$ only for multivariate Gaussian rv
  (this though is confusing; see this Wikipedia article)

linearity of expectation

for any rvs X and Y, and any constants $a, b \in \mathbb{R}$

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

note 1: no assumptions! (in particular, does not need independence)

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note 1: no assumptions! (in particular, does not need independence)

note 2: does not hold for variance in general

- for general X, Y

  $$Var(aX + bY) =$$

- when X and Y are independent

  $$Var(aX + bY) =$$

using linearity of expectation

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on average, how many students get their own hw?

using linearity of expectation

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on average, how many students get their own hw?

Let $X_i = \mathbb{1}_{[\text{student i gets her hw}]}$ (indicator rv)
$N =$ number of students who get their own hw $= \sum_{i=1}^{10} X_i$
then we have:

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$$
$$= \sum_{i=1}^{n} \mathbb{E}[X_i]$$
$$= \sum_{i=1}^{n} \mathbb{P}[X_i = 1] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

variance of linear combinations

normal distribution

normal distribution

rv X is said to be normally distributed with mean μ and variance σ^2 (in notation, $X \sim \mathcal{N}(\mu, \sigma^2)$) if its pdf $f(\cdot)$ is given by

$$f(x) = \frac{1}{\sigma\ \sqrt{2\pi}}\ \exp\left[-\frac{1}{2}\ \left(\frac{x-\mu}{\sigma}\right)^2\right],\quad -\infty < x < \infty.$$

the pdf looks like

properties of the normal distribution

1. the pdf is symmetric around the mean μ: if $X \sim \mathcal{N}(\mu, \sigma^2)$

   $\mathbb{P}[X \leq \mu - a] =$

2. (Linear transformation) If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

   $aX + b \ \sim$

   $\dfrac{X - \mu}{\sigma} \ \sim$

3. If $X \sim \mathcal{N}(\mu_1, \sigma_1^2), \ \ Y \sim \mathcal{N}(\mu_2, \sigma_2^2), \ $ and $X \perp\!\!\!\perp Y$, then

   $X + Y \sim$

cdf of normal distribution

if $X \sim \mathcal{N}(\mu, \sigma^2)$, then its cdf is given by

$\mathbb{P}[X \leq x] =$

- knowing cdf for $\mathcal{N}(0,1)$ is enough to find cdf for any normally distributed rv
- for $X \sim \mathcal{N}(0,1)$: cdf denoted $\Phi(x)$, available in most computing packages

  closely related to the error function $erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt$; in particular,

$$\Phi(x) = \frac{1}{2}(1 + erf(\frac{x}{\sqrt{2}}))$$

sums of independent rv

sums and averages of independent rv

- $X_1, X_2, \ldots$ are independent rv that are uniformly distributed over the interval $[0, 1]$; $\mathbb{E}[X_1] = 1/2$, $Var(X_1) = 1/12$.
- the pdf of X_1 looks like
- what about the pdf of $X_1 + X_2$ and $(X_1 + X_2)/2$?

sums and averages of independent rv

$$S_n = X_1 + \ldots + X_n \qquad \bar{X}_n = \frac{1}{n}\left[X_1 + \ldots + X_n\right]$$

- $\mathbb{E}[S_n] =$

  $\mathbb{E}[\bar{X}_n] =$

  $Var(S_n) =$

  $Var(\bar{X}_n) =$

- (roughly) sum of n i.i.d. random variables is $\sqrt{n}$ times as variable as any one of the random variables
- average of n i.i.d. random variables is $1/\sqrt{n}$ times as variable as any one of the random variables

law of large numbers

let $X_1, X_2, \dots$ be a sequence of independent rvs with $\mathbb{E}[X_i] = \mu$ for all i
then, "almost" always

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow \mu \quad , \quad \text{as } n \to \infty$$

note: for any finite n, $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ is still a random variable

central limit theorem

let $X_1, X_2, \ldots$ be a sequence of independent rvs with

$$\mathbb{E}[X_i] = \mu, Var(X_i) = \sigma^2 < \infty \text{ for all } i$$

then,

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} \sigma\mathcal{N}(0,1) = \mathcal{N}(0,\sigma^2) \quad , \quad \text{as } n \to \infty$$

this suggests the following approximations for large n,

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{D}{\approx}$$

$$S_n = \sum_{i=1}^{n} X_i \overset{D}{\approx}$$