



# ORIE 4580/5580: Simulation Modeling and Analysis

## Unit 10: intro to Markov chains

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# rest of course: the ‘simulation hierarchy’

**aim:** simulate some **complex real-world system** to **measure/optimize/control**  
– can do so in 3 ways of **decreasing complexity**

## **discrete-event simulations**

- most general framework
- allows detailed modeling, general distributions
- complex; takes time to code/execute

## **Markovian models**

- need **inter-event times to be memoryless** (i.e., Exponentially distributed)
- easier to simulate (no event-list needed)
- can give spurious insights, hide critical issues

## **closed-form solutions**

- e.g. queueing models
- have formulas for steady-state performance measures
- restrictive assumptions

**today: intro to Markov chains + Markovian simulation models**

# random process

## random process

indexed collection of rvs  $X_t \in \mathcal{S}$ , one for each  $t \in \mathcal{T}$

–  $\mathcal{S}$ : state space,  $\mathcal{T}$ : index set

## Markov property

random process  $X_t$  has the Markov property if the probability of moving to a future state **only depends on the present state** and not on past states

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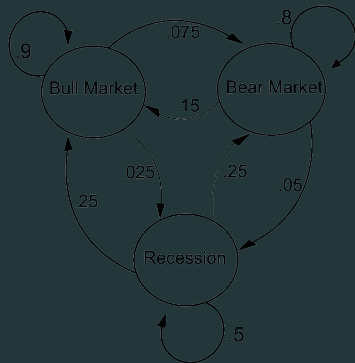
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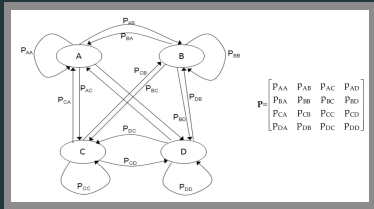
## four types

- $\mathcal{S}$  discrete,  $\mathcal{T}$  discrete: **discrete-time Markov chain** (DTMC)
  - **random walk on integers**
- $\mathcal{S}$  discrete,  $\mathcal{T}$  continuous: **continuous-time Markov chain** (CTMC)
  - **Poisson process**
- $\mathcal{S}$  continuous,  $\mathcal{T}$  discrete:
  - **random walk on the reals**
- $\mathcal{S}$  continuous,  $\mathcal{T}$  continuous: Markov process
  - **Brownian motion**

## Markov chains: basic definition



# Markov chains: transition-diagram and transition matrix



## example: coin tosses and geometric rv.

recall the **Geometric rv**  $p(k) = q^{k-1}(1-q) \forall k \in \{1, 2, \dots\}$

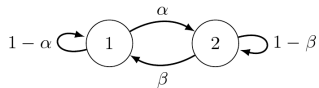
we can view this as a Markov chain as follows:

## example: the coupon collector

a brand of cereal always distributes a baseball card in every cereal box, chosen randomly from a set of  $n$  distinct cards

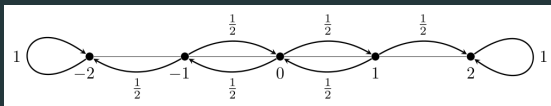
Markov chain model for number of cards owned by a collector:

## Markov chains: two viewpoints

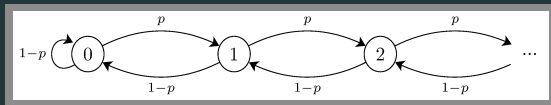


## Markov chains: long-term behavior

# Markov chains: absorbing chain

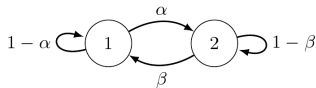


## Markov chains: transient/recurrent chain

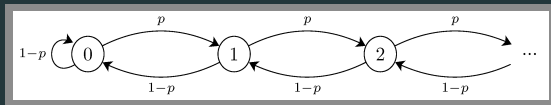


# Markov chains: steady-state behavior

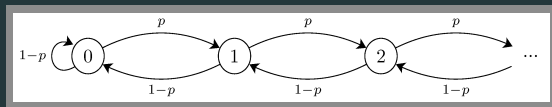
example:



## Markov chains: steady-state for infinite chains



# Markov chains: the ergodic theorem



## DTMC: applications and problems

## from discrete to continuous time

### Markov property

random process  $X_t$  has the Markov property if the probability of moving to a future state **only depends on the present state** and not on past states

- $\mathcal{S}$  discrete,  $\mathcal{T}$  discrete: **discrete-time Markov chain** (DTMC)
  - **random walk**
- $\mathcal{S}$  continuous,  $\mathcal{T}$  discrete: **continuous-time Markov chain** (CTMC)
  - **Poisson process**

# exponential distribution cheat sheet

can extend DTMCs to continuous time using special properties of the exponential distribution/poisson process

suppose  $T \sim \text{exponential}(\lambda)$ , then:

- pdf:  $f_T(t) =$
- cdf:  $F_T(t) = \mathbb{P}[T \leq t] =$
- (**memorylessness**): cdf of  $T$  knowing it is bigger than  $t$ ?

$$\mathbb{P}[T \leq t + x | T > t] =$$

# understanding exponential distributions

suppose  $T_1, T_2, \dots, T_n$  are all exponentially distributed, with  $T_i \sim \text{exponential}(\lambda_i)$ .

- (**minimum of exponentials**): let  $T_{\min} = \min\{T_i | i \in \{1, 2, \dots, n\}\}$ ; distribution of  $T_{\min}$ ?

$$T_{\min} \sim$$

- (**first arrival**): let  $T_{\min} = \arg \min_i \min\{T_i | i \in \{1, 2, \dots, n\}\}$ ; distribution of  $T_{\min}$ ?

$$T_{\min} \sim$$

# Poisson process cheat sheet

given Poisson processes  $X(t) \sim PP(\lambda)$

- (**inter-arrival times**): let  $\{A_1, A_2, A_3 \dots\}$  be the arrival times of the agents; then

$$T_i = A_i - A_{i-1} \sim$$

- (**splitting**): suppose we probabilistically split arrivals from  $X(t)$  to  $Y(t)$  with probability  $p$ , else to  $Z(t) = X(t) - Y(t)$

$$Y(t) \sim$$

$$Z(t) \sim$$

- (**time-varying rate**): a time-varying rate of  $\lambda(t) \in [0, \lambda^*]$  is equivalent to a  $PP(\lambda^*)$  for which arrivals at time  $t$  are thinned with probability  $p(t) = \lambda(t)/\lambda^*$

## Poisson process cheat sheet (contnd)

given independent Poisson processes

$$X_1(t) \sim PP(\lambda_1), X_2(t) \sim PP(\lambda_2), X_3(t) \sim PP(\lambda_3)$$

- (**superposition**): suppose  $S(t) = X_1(t) + X_2(t) + X_3(t)$

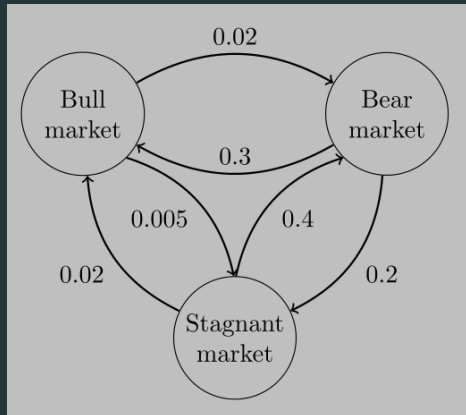
$$S(t) \sim$$

- (**first arrival**): let  $I_{\min}$  = the identity (i.e.,  $\{1, 2, 3\}$ ) of the first arrival among the three processes; distribution of  $I_{\min}$ ?

$$I_{\min} \sim$$

## continuous-time Markov chains

**CTMC**: continuous-time Markov processes on discrete state-space  
eg. modeling the financial market in continuous time



# CTMCs

used as a model for many applications:

- queueing and service systems
- transportation networks
- epidemiology
- communications and computer networks
- agent choice models

## advantages

- easy to analyze (in some cases)
- **easier to simulate** than general discrete-event simulation

## problems

- **need all inter-event times to be exponentially distributed**
- can give spurious insights, hide critical issues

simulating a CTMC

## example: queueing

### the single-server M/M/1 queue

- number of servers: 1
- capacity: infinite
- service discipline: first-come-first-served (FCFS or FIFO)
- interarrival times:  $\text{exponential}(\lambda)$
- service times:  $\text{exponential}(\mu)$   
(independent interarrival and service times)

example: simulating an  $M/M/1$  queue

## example: epidemics

- want to model the spread of the latest influenza strain among the population
- there is a population of  $n$  people
  - at any time  $t$ , each person  $i$  is either **susceptible** (denoted  $S$  or 0) or **infected** (denoted  $I$  or 1)
  - infected people **get cured on average after time  $\tau$** , becoming susceptible to future infection.
  - each **pair of people  $(i, j)$  independently meet** each other with **average rate  $\lambda$**
  - when a susceptible person meets an infected person, the susceptible person becomes infected
  - when two susceptible people or two infected people meet, nothing happens

## example: SIS epidemic (contnd)

what assumptions do we need to make this Markovian?

## example: SIS Epidemic (contnd)

how do we simulate it?

## example: SIS Epidemic (contnd)



