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Christian Wassermann | Numerik 1 | Abgabe bis: 18.6.2018 | Blatt

1)

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

a)

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\arctan(x)\right]_0^1 = \arctan(1) \approx 0{,}7853981634$$

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2)

$x_0, \dots, x_n \in [-1, 1]$

$\beta_0, \dots, \beta_n \in \mathbb{R}$

$$\int_{-1}^{1} p(x)\,dx = \sum_{i=0}^{n} \beta_i\, p(x_i)$$

$\int_{-1}^{1} p(x) = \int_{-1}^{1}$ Interpolation-Polynom durch x_i

$\hookrightarrow$ Lagrange-Polynome (existiert, da Stützstellen x_i paarweise verschieden sind)

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} \sum_{i=0}^{n} p(x_i) \cdot L_i(x)\,dx$$

$$= \sum_{i=0}^{n} p(x_i) \cdot \int_{-1}^{1} L_i(x)\,dx$$

$$= \sum_{i=0}^{n} p(x_i) \cdot \beta_i \qquad \text{mit } \beta_i = \int_{-1}^{1} L_i(x)\,dx$$

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4)
$$\int_0^h f(t)\,dt$$

$l_0 = h$:
$$T_0 = \frac{h}{2}\left(f(0) + f(h)\right)$$

$l_1 = \frac{h}{2}$:
$$T_1 = \frac{h/2}{2}\left(f(0) + 2 f(\tfrac{h}{2}) + f(h)\right)$$
$$= \frac{h}{4}\left(f(0) + 2 f(\tfrac{h}{2}) + f(h)\right)$$

Extrapolation:
$$A \cdot \alpha = e$$

$$\begin{pmatrix} 1 & 1 \\ l_0^2 & l_1^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ h^2 & \frac{h^2}{4} & 0 \end{array}\right) \downarrow -h^2 \cdot I$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & \frac{h^2}{4} - h^2 & -h^2 \end{array}\right)$$

$$\left(\frac{h^2}{4} - h^2\right) \alpha_2 = -h^2$$
$$-\frac{3}{4} h^2 \alpha_2 = -h^2$$
$$\alpha_2 = \frac{4}{3}$$
$$\alpha_1 + \frac{4}{3} = 1$$
$$\alpha_1 = -\frac{1}{3}$$

$$\alpha = \begin{pmatrix} -1/3 \\ 4/3 \end{pmatrix}$$

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$$T = -\frac{1}{3} T_0 + \frac{4}{3} \cdot T_1$$

$$= -\frac{1}{3} \frac{h}{2} (f(0) + f(h)) + \frac{4}{3} \left(\frac{h}{4} \left(f(0) + 2 f\left(\frac{h}{2}\right) + f(h) \right) \right)$$

$$= -\frac{h}{6} f(0) - \frac{h}{6} f(h) + \frac{h}{3} f(0) + \frac{2h}{3} f\left(\frac{h}{2}\right) + \frac{h}{3} f(h)$$

$$= \frac{h}{6} f(0) + \frac{2h}{3} f\left(\frac{h}{2}\right) + \frac{h}{6} f(h)$$

$$= h \left(\frac{1}{6} f(0) + \frac{2}{3} f\left(\frac{h}{2}\right) + \frac{1}{6} f(h) \right)$$

$$= (h-0) \left(\frac{1}{6} f(0) + \frac{4}{6} f\left(\frac{0+h}{2}\right) + \frac{1}{6} f(h) \right)$$

$$= (b-a) \left(\frac{1}{6} f(a) + \frac{4}{6} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f(b) \right)$$

$\hookrightarrow$ Simpson-Regel

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6)

a)

Nach Satz gilt für den Fehler der Näherung mit Trapez-Regel T:

$$E = \int_{a_i}^{b_i} f(x)\,dx - T = -\frac{1}{12} h^3 f''(\xi) \qquad \xi \in [a_i; b_i]$$
$$h = b_i - a_i$$

ξ f'' nicht bekannt

Trapez-Regel auf beiden Hälften anwenden

$$m_i = \frac{a_i + b_i}{2}$$

$$E_l = \int_{a_i}^{m_i} f(x)\,dx - T_l = -\frac{1}{12}\left(\frac{h}{2}\right)^3 f''(\xi_l) \qquad \xi_l \in [a_i; m_i]$$

$$E_r = \int_{m_i}^{b_i} f(x)\,dx - T_r = -\frac{1}{12}\left(\frac{h}{2}\right)^3 f''(\xi_r) \qquad \xi_r \in [m_i; a_i]$$

neue genauere Näherung: $\tilde{T} = T_l + T_r$ für $\int_{a_i}^{b_i} f(x)\,dx$

$$\tilde{E} = \int_{a_i}^{b_i} f(x)\,dx - \tilde{T} = E_l + E_r$$

$$= -\frac{1}{12}\left(\frac{h}{2}\right)^3 \left(f''(\xi_l) + f''(\xi_r)\right)$$

Annahme: $f''(\xi_l) \approx f''(\xi_r) \approx f''(\xi)$

$$\approx -\frac{1}{12} \cdot \frac{1}{4} \cdot h^3 f''(\xi)$$

$$= \frac{1}{4} E$$

$$\tilde{T} - T = E - \tilde{E} \approx \frac{3}{4} E \approx 3\tilde{E}$$

$$\Longrightarrow \quad F = \frac{4}{3} |\tilde{T} - T| \approx E$$

bzw. $$\tilde{F} = \frac{1}{3} \cdot |\tilde{T} - T| \approx \tilde{E}$$

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b) $$\int_1^5 \frac{1}{x}\,dx = \left[\ln(x)\right]_1^5 = \ln(5) - \ln(1) = \ln(5)$$

$$= 1,60943791243$$

$$T_1^5 = \frac{4}{2}\left(1 + \frac{1}{5}\right) = 2 \cdot \frac{6}{5} = \frac{12}{5} = 2,4$$

$$\tilde{F}_1^5 = \frac{\left|\frac{4}{4}\left(1 + 2\cdot\frac{1}{3} + \frac{1}{5}\right) - \frac{12}{5}\right|}{3} = \frac{\left|\frac{28}{15} - \frac{36}{15}\right|}{3} = \frac{8}{15} \cdot \frac{1}{3} = \frac{8}{45} = 0,1\overline{7}$$

$$T_1^3 = \frac{2}{2}\left(1 + \frac{1}{3}\right) = \frac{4}{3}$$

$$T_3^5 = \frac{2}{2}\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{8}{15}$$

$$\tilde{F}_1^3 = \frac{\left|\frac{2}{4}\left(1 + 2\cdot\frac{1}{2} + \frac{1}{3}\right) - \frac{4}{3}\right|}{3} = \frac{\left|\frac{7}{6} - \frac{4}{3}\right|}{3} = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} = 0,0\overline{5}$$

$$\tilde{F}_3^5 = \frac{\left|\frac{2}{4}\left(\frac{1}{3} + 2\cdot\frac{1}{4} + \frac{1}{5}\right) - \frac{8}{15}\right|}{3} = \frac{\left|\frac{1}{2}\left(\frac{10}{30} + \frac{15}{30} + \frac{6}{30}\right) - \frac{8}{15}\right|}{3} = \frac{\left|\frac{31}{60} - \frac{32}{60}\right|}{3}$$

$$= \frac{1}{60} \cdot \frac{1}{3} = \frac{1}{180} = 0,00\overline{5}$$

$$\tilde{F}_3^5 < 0,02$$

$$T_1^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

$$T_2^3 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

$$\tilde{F}_1^2 = \frac{\left|\frac{1}{4}\left(1 + 2\cdot\frac{1}{1,5} + \frac{1}{2}\right) - \frac{3}{4}\right|}{3} = \frac{\left|\frac{1}{4}\left(1 + \frac{4}{3} + \frac{1}{2}\right) - \frac{3}{4}\right|}{3} = \frac{\left|\frac{17}{24} - \frac{18}{24}\right|}{3}$$

$$= \frac{1}{24} \cdot \frac{1}{3} = \frac{1}{72} = 0,013\overline{8}$$

$$\tilde{F}_1^2 < 0,02$$

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$$\tilde{F}_2^3 = \frac{\left| \frac{1}{4}\left(\frac{1}{2} + 2 \cdot \frac{1}{2{,}5} + \frac{1}{3}\right) - \frac{5}{12} \right|}{3}$$

$$= \frac{\left| \frac{1}{4} \cdot \left(\frac{1}{2} + \frac{4}{5} + \frac{1}{3}\right) - \frac{5}{12} \right|}{3}$$

$$= \frac{\left| \frac{1}{4} \cdot \left(\frac{15}{30} + \frac{24}{30} + \frac{10}{30}\right) - \frac{5}{12} \right|}{3}$$

$$= \frac{\left| \frac{49}{120} - \frac{50}{120} \right|}{3} = \frac{1}{120} \cdot \frac{1}{3} = \frac{1}{360} = 0{,}002\overline{7}$$

$$\tilde{F}_2^3 < 0{,}02$$

$$\int_1^5 \frac{1}{x}\,dx \approx T_1^2 + T_2^3 + T_3^5$$

$$= \frac{3}{4} + \frac{5}{12} + \frac{8}{15} = \frac{17}{10} = 1{,}7$$

Fehler zu exakter Lösung: 0,0305620875 7